MILP Formulation: Multi-Vehicle Routing with Time Windows and Budget Constraints

Sets:

$$\begin{aligned} A &: \text{Set of locations (0 is the depot)} \\ V &: \text{Set of vehicles} \\ T &: \text{Set of time periods (e.g., days)} \end{aligned}$$

Parameters:

$$\begin{aligned} t_{ij}^v &: \text{Travel time from node } i \text{ to } j \text{ by vehicle } v \\ tv_j &: \text{Service time at node } j \\ Q &: \text{Vehicle capacity (time or load)} \\ [a_i, b_i] &: \text{Time window for node } i \\ \text{cost\_rate}_v &: \text{Travel cost rate for vehicle } v \\ \text{Available}_v &: \text{Seat capacity for vehicle } v \\ \text{shared}_v &: \text{Indicator if vehicle } v \text{ is shared (boolean)} \\ n_{\text{passengers}} &: \text{Number of passengers per ride} \\ \text{Budget} &: \text{Total cost budget} \\ q_i &: \text{Demand parameter for MTZ (e.g., 3)} \end{aligned}$$

Decision Variables:

$$\begin{aligned} x_{ijvt} &\in \{0,1\} \quad \text{1 if vehicle } v \text{ travels from } i \text{ to } j \text{ at time } t \\ t_{ivt} &\geq 0 \quad \text{Arrival time at node } i \text{ by vehicle } v \text{ at time } t \\ y_t &\in \{0,1\} \quad \text{1 if day } t \text{ is used} \\ u_{iv} &\in \mathbb{R} \quad \text{MTZ variable for subtour elimination} \end{aligned}$$

Objective:

$$\max \sum_{i \in A} \sum_{j \in A, j \neq i} \sum_{v \in V} \sum_{t \in T} \frac{x_{ijvt}}{t_{ij}^v}$$

Constraints:

$$\sum_{t\in T}\sum_{v\in V} x_{iivt} = 0 \qquad \forall i \in A \quad \text{(no loops)} \tag{1}$$

$$\sum_{i\in A}\sum_{v\in V} x_{ijvt} = \sum_{i\in A}\sum_{v\in V} x_{jivt} \qquad \forall j \in A, \forall t \in T \quad \text{(flow conservation)} \tag{2}$$

$$\sum_{i\in A}\sum_{t\in T}\sum_{v\in V} x_{ijvt} \le 1 \qquad \forall j \in A \setminus \{0\} \quad \text{(visit each node at most once)} \tag{3}$$

$$\sum_{j\in A\setminus\{0\}}\sum_{v\in V} x_{0jvt} = y_t \qquad \forall t \in T \quad \text{(link tour to day usage)} \tag{4}$$

$$\sum_{i\in A}\sum_{j\in A}\sum_{v\in V} x_{ijvt}(tv_j + t^v_{ij}) \le Q \qquad \forall t \in T \quad \text{(capacity)} \tag{5}$$

$$y_t \le y_{t+1} \qquad \forall t \in T \setminus \{\max(T)\} \quad \text{(non-decreasing usage)} \tag{6}$$

$$y_0 = 1 \qquad \text{(first day must be used)} \tag{7}$$

$$a_i \le t_{ivt} \le b_i \qquad \forall i \in A, v \in V, t \in T \quad \text{(time windows)} \tag{8}$$

$$\sum_{i,j\in A}\sum_{v\in V} x_{ijvt} \cdot \text{cost\_rate}_v \cdot t^v_{ij} \cdot \delta_v \le \text{Budget} \qquad \forall t \in T \quad \text{(daily budget)} \tag{9}$$

$$x_{ijvt} \cdot n_{\text{passengers}} \le \text{Available}_v \qquad \forall i, j \in A, v \in V, t \in T \quad \text{(seating)} \tag{10}$$

$$t_{ivt} + tv_i + t^v_{ij} - M(1 - x_{ijvt}) \le t_{jvt} \qquad \forall i, j \in A, j \ne 0, v, t \quad \text{(time propagation)} \tag{11}$$

$$u_{0v} = 0 \qquad \forall v \in V \quad \text{(MTZ depot fix)} \tag{12}$$

$$q_i \le u_{iv} \le Q \qquad \forall i \in A \setminus \{0\}, v \in V \tag{13}$$

$$u_{iv} - u_{jv} + Qx_{ijvt} \le Q - q_j \qquad \forall i \ne j, i, j \in A \setminus \{0\}, v, t \quad \text{(MTZ subtour)} \tag{14}$$

Where:

$$\delta_v = \begin{cases} n_{\text{passengers}}, & \text{if vehicle } v \text{ is private} \\ 1, & \text{if vehicle } v \text{ is shared} \end{cases}$$