

## Worksheet 4 (Tutorial & Lab)

**Exercise 1** *The Law of Large Numbers states that as the sample size increases, the sample mean will get closer to the population mean.*

*Write a code in order to visualize this concept.*

**Exercise 2**  *$N$  darts are thrown randomly into a square with sides of length 2, the square centered at  $O$  contains a circle with radius 1 (centered at  $O$ ). This circle is tangent to the four sides of the square. We assume that each dart thrown at random hits a point may be:*

- *Inside the circle, if the distance from the dart to the center  $O$  is less than 1,*
  - *Outside the circle, if the distance from the dart to the center  $O$  is equal to 1.*
1. *Estimate the value of  $\pi$  using Monte Carlo method, by counting the number  $n$  of darts in the circle.*
  2. *Write a program that estimates  $\pi$  by simulation, 10000 random throws are made.*

**Exercise 3** *Return to Exercise 3 in the previous series, where we simulated the daily demand for a product over a two-week period.*

*To improve the accuracy, use the variance reduction method “antithetic variables method.” Do we really see a reduction in variance?*

**Exercise 4** *We wish to estimate*

$$I = \int_{-2}^{+\infty} e^{(x - \frac{x^2}{2})} dx.$$

1. a) *Propose an estimator  $\hat{I}_1(N)$  of  $I$  by Monte Carlo method based on  $N$  random variables of distribution  $\mathcal{N}(0, 1)$ .*  
b) *Write a Python program whose output is a realization of  $\hat{I}_1(N)$  and an estimate of the variance of  $\hat{I}_1(N)$ . Specify the numerical results obtained for  $N = 10^6$ .*
2. a) *Propose an estimator  $\hat{I}_2(N)$  of  $I$  by Importance sampling based on  $N$  random variables of distribution  $\mathcal{N}(1, 1)$ .*  
b) *Write a Python program whose output is a realization of  $\hat{I}_2(N)$  and an estimate of the variance of  $\hat{I}_2(N)$ . Specify the numerical results obtained for  $N = 10^6$  and compare them with those obtained in question 1.b).*  
c) *Compare the theoretical variances of  $\hat{I}_2(N)$  and  $\hat{I}_1(N)$ . Comment.*

3. a) Express  $I$  as a function of

$$J = \int_1^2 e^{-\frac{(x-1)^2}{2}} dx.$$

- b) From the previous question, propose an estimator  $\hat{I}_3(N)$  of  $I$  using a control variate.
- c) Write a Python program whose output is a realization of  $\hat{I}_3(N)$  and an estimate of the variance of  $\hat{I}_3(N)$ . Specify the results obtained for  $N = 10^6$ . Comment.
- d) Propose an estimator  $\hat{I}_4(N)$  of  $I$  by Monte Carlo Method based on  $N$  random variables of uniform distribution.
- e) Write a Python program whose output is a realization of  $\hat{I}_4(N)$  and an estimate of the variance of  $\hat{I}_4(N)$ . Specify the results obtained for  $N = 10^6$ . Comment and compare with previous results.