Chapter Two
Some Other methods for Inverse of Matrix

Gauss elimination method

In this method, we first place a matrix A and I adjacent to each other, then performing row operations. In each case where we add a multiple of one row to another, The left hand matrix, which started as A, has been transformed into the identity matrix, so the right hand matrix, which started as the identity matrix, has been transformed into A^{-1}

Example 1:

Find the Inverse matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix} \begin{matrix} 1 & 2 \\ 0 & 4 \\ 1 & 0 \end{matrix}$

Solution:

$(24+10)-(0+12)$

$|A|=22$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim R_3 \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \frac{1}{4}R_2 \sim$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{5}{4} \\ 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 - 2R_2 \\ R_3 + 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{5}{4} \\ 0 & 0 & \frac{11}{2} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{4} & 0 \\ -1 & \frac{1}{2} & 1 \end{bmatrix} \frac{2}{11}R_3 \sim$$

$$\left[\begin{matrix}1 & 0 & \frac{1}{2}\\ 0 & 1 & \frac{5}{4}\\ 0 & 0 & 1\end{matrix}\right]\left[\begin{matrix}1 & -\frac{1}{2} & 0\\ 0 & \frac{1}{4} & 0\\ -\frac{2}{11} & \frac{1}{11} & \frac{2}{11}\end{matrix}\right.,\ \begin{matrix}R_2 - -R_3\\ R_1 - -R_3\end{matrix}\ \Bigg]\left[\begin{matrix}\frac{12}{11} & -\frac{6}{11} & -\frac{1}{11}\\ \frac{5}{22} & \frac{3}{22} & -\frac{5}{22}\\ -\frac{2}{11} & \frac{1}{11} & \frac{2}{11}\end{matrix}\right]$$

$$A^{-1} = \begin{bmatrix}\frac{12}{11} & -\frac{6}{11} & -\frac{1}{11}\\ \frac{5}{22} & \frac{3}{22} & -\frac{5}{22}\\ -\frac{2}{11} & \frac{1}{11} & \frac{2}{11}\end{bmatrix}$$

- Find the inverse of the matrix use the Gauss elimination approach

$$A=\begin{bmatrix}1 & 3 & 3\\ 1 & 4 & 3\\ 2 & 7 & 7\end{bmatrix} \longrightarrow A^{-1} = \begin{bmatrix}7 & 0 & -3\\ -1 & 1 & 0\\ -1 & -1 & 1\end{bmatrix}$$

Note: Check your answer !

Row Echelon Form (REF)

A leading entry in a row means the first nonzero entry from the left.

A rectangular matrix is in row echelon form if:

1- All nonzero rows are above any all-zero row.

2- All entries below a leading entry are zeros.

3- In any pair of adjacent nonzero row, say row i and row i+1, the leading entry in row i is to the left of the leading entry of row i+1.

Reduced Row Echelon Form (RREF)

1- All nonzero rows are above any all-zero row.

2- All entries above and below a leading entry are zeros.

3- In any pair of adjacent nonzero row, say row i and row i+1, the leading entry in row i is to the left of the leading entry of row i+1.

All leading entries are equal to 1.

By applying the three types of elementary row operations, we can reduce any rectangular matrix to a matrix in reduced row echelon form (RREF).

(In other words, any matrix is *row equivalent* to a matrix in RREF)

Furthermore, the RREF of a matrix is unique.

A row reduction algorithm

1. Scan the columns from left to right.
2. Start from the 1st column.
3. If this column contains a pivot (a nonzero entry), move the pivot to the top by exchanging rows
4. Make all entries below and above the pivot equal to 0.
5. Move to the next column and try to locate a nonzero entry which is not in any row already containing a pivot.
6. Repeat step 5 until you can find such column.
7. Repeat step 3 to step 6 until we reach the right-most column.
8. Finally, normalize all leading entries to 1.

Example 3:

$$\begin{bmatrix} 1 & 1 & -1 & -2 \\ 2 & -1 & 1 & 5 \\ -1 & 2 & 2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & -3 & 3 & 9 \\ -1 & 2 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & -3 & 3 & 9 \\ -1 & 2 & 2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & -3 & 3 & 9 \\ 0 & 3 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & -3 & 3 & 9 \\ 0 & 3 & 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 3 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 3 & 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 4 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 4 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Solution of a Linear System of Equations Using Matrices

Firstly: By Using Adjoint Matrix

Inverses of matrices have many uses, one of which is in solution of systems of equations. A system of equations can be written in matrix form as **AX = B** . If A is invertible, then A^{-1} exists. Multiplying both sides of the given matrix by A^{-1} on the left, we obtain

$$A^{-1}(AX) = A^{-1}B$$

Using the associative property and simplifying, we can write this as follows.

$$(A^{-1}A)X = A^{-1}B$$

$$IX = A^{-1} = A^{-1}B$$

$$X = A^{-1}B$$

Thus we have obtained an expression for the solution X of the given system of equations.

Example 4:

Solve the following system of equations.

2x + 4y = 1

-x + 2y = 4

Solution

The given system of equations in a matrix form is A X = B, where

$$A = \begin{bmatrix} 2 & 4 \\ -1 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Then A^{-1} (as found in example 15 is given by

$$A^{-1} = \frac{1}{8}\begin{bmatrix} 2 & -4 \\ 1 & 2 \end{bmatrix}$$

It follows that the solution of the system of equations is given by

$$X = A^{-1}B = \frac{1}{8}\begin{bmatrix} 2 & -4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 2-16 \\ 1+8 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -16 \\ 9 \end{bmatrix} = \begin{bmatrix} -\frac{16}{8} \\ \frac{9}{8} \end{bmatrix}.$$

That is, $$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ \frac{9}{8} \end{bmatrix}$$

Therefore $x = -2$, $y = \frac{9}{8}$

Example 5:

Solve the following system of equations.

$$X + 2y + 3z = 3$$
$$2x + 5y + 7z = 6$$
$$3x + 7y + 8z = 5$$

Solution

The given system of equations in a matrix form is
A X = B, where

$$A=\begin{bmatrix}1 & 2 & 3\\ 2 & 5 & 7\\ 3 & 7 & 8\end{bmatrix}, \qquad X=\begin{bmatrix}x\\ y\\ z\end{bmatrix} \text{ and } \qquad B=\begin{bmatrix}3\\ 6\\ 5\end{bmatrix}$$

Then A^{-1}(as found in example 16 is given by

$$-\frac{1}{2}\begin{bmatrix}-9 & 5 & -1\\ 5 & -1 & -1\\ -1 & -1 & 1\end{bmatrix}$$

It follows that the solution of the system of equations is given by

$$X=A^{-1}B=-\frac{1}{2}\begin{bmatrix}-9 & 5 & -1\\ 5 & -1 & -1\\ -1 & -1 & 1\end{bmatrix}\begin{bmatrix}3\\ 6\\ 5\end{bmatrix}$$

$$=-\frac{1}{2}\begin{bmatrix}-27+30-5\\ 15-6-5\\ -3-6+5\end{bmatrix}=-\frac{1}{2}\begin{bmatrix}-2\\ 4\\ -4\end{bmatrix}=\begin{bmatrix}1\\ -2\\ 2\end{bmatrix}$$

That is,

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}1\\ -2\\ 2\end{bmatrix}$$

Therefore x = 1,   y = -2,   z = 2

Example 6:

$$3x_1 - x_2 + x_3 = 2$$
$$2x_1 + x_2 = 1$$
$$x_1 + 2x_2 - x_3 = 3$$

The equations can be expressed as

$$\begin{bmatrix} 3 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

When A^{-1} is computed the equation becomes

$$X = A^{-1}B = \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ -1.0 & 2.0 & -1.0 \\ -1.5 & 3.5 & -2.5 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -7 \end{bmatrix}$$

Secondly: By Using Gauss-Jordan Elimination

A method to solve simultaneous linear equations of the form [A][X]=[C]

Two steps

1- Forward Elimination

2- Back Substitution

The goal of forward elimination is to transform the coefficient matrix into **an upper triangular matrix**

Solve each equation starting from the last equation

Any linear system must have exactly one solution, no solution, or an infinite number of solutions.

Previously we considered the 2×2 case, in which the term consistent is used to describe a system with a unique solution, inconsistent is used to describe a system with no solution, and dependent is used for a system with an infinite number of solutions. In this section we will consider larger systems with more variables and more equations, but the same three terms are used to describe them.

The following matrix representations of three linear equations in three unknowns illustrate the three different cases:

1- Case I: consistent

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

From this matrix representation, you can determine that

$$x = 3, \qquad y = 4, \qquad z = 5$$

2- Case 2: inconsistent

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

From the second row of the matrix, we find that

$$0x + 0y + 0z = 6$$

or

$$0 = 6,$$

an impossible equation. From this, we conclude that there are no solutions to the linear system.

3- Case 3: dependent

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

When there are fewer non-zero rows of a system than there are variables, there will be infinitely many solutions, and therefore the system is dependent.

A matrix is said to be in **reduced row echelon form** or, more simply, in **reduced form**, if

1- Each row consisting entirely of zeros is below any row having at least one non-zero element.

2- The leftmost nonzero element in each row is 1.

3- All other elements in the column containing the leftmost 1 of a given row are zeros.

4- The leftmost 1 in any row is to the right of the leftmost 1 in the row above.

Example 7:

Solve the simultaneous linear equations by Gauss-Jordan Elimination:

$$x + y - z = -2$$

$$2x - y + z = 5$$

$$-x + 2y + 2z = 1$$

Solution:

We begin by writing the system as an augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 2 & -1 & 1 & 5 \\ -1 & 2 & 2 & 1 \end{array}\right]$$

We already have a 1 in the diagonal position of first column. Now we want 0's below the 1. The first 0 can be obtained by multiplying row 1 by –2 and adding the results to row 2:

* Row 1 is unchanged

(–2) times Row 1 is added to Row 2 $-2R_1 + R_2 \to R_2$

Row 3 is unchanged

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 2 & -1 & 1 & 5 \\ -1 & 2 & 2 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & -3 & 3 & 9 \\ -1 & 2 & 2 & 1 \end{array}\right]$$

The second 0 can be obtained by adding row 1 to row 3:

*Row 1 is unchanged

* Row 2 is unchanged

* Row 1 is added to Row 3

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & -3 & 3 & 9 \\ -1 & 2 & 2 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & -3 & 3 & 9 \\ 0 & 3 & 1 & -1 \end{array}\right]$$

Moving to the second column, we want a 1 in the diagonal position (where there was a –3). We get this by dividing every element in row 2 by -3:

* Row 1 is unchanged

* Row 2 is divided by –3 $R_2/3 \to R_2$
* Row 3 is unchanged

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & -3 & 3 & 9 \\ 0 & 3 & 1 & -1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 3 & 1 & -1 \end{array}\right]$$

To obtain a 0 below the 1 , we multiply row 2 by –3 and add it to the third row:

* Row 1 is unchanged
* Row 2 is unchanged
* (–3) times row 2 is added to row 3

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 3 & 1 & -1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 4 & 8 \end{array}\right]$$

To obtain a 1 in the third position of the third row, we divide that row by 4. Rows 1 and 2 do not change.

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 4 & 8 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

We can now work upwards to get zeros in the third column, above the 1 in the third row.

* Add R_3 to R_2 and replace R_2 with that sum
* Add R_3 to R1 and replace R_1 with the sum.

*Row 3 will not be changed.

All that remains to obtain reduced row echelon form is to eliminate the 1 in the first row, 2nd position.

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

To get a zero in the first row and second position, we multiply row 2 by -1 and add the result to row 1 and replace row 1 by that result. Rows 2 and 3 remain unaffected.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

We can now "read" our solution from this last matrix. We have $x = 1$, $\quad y = -1$, $\quad z = 2$.

Written as an ordered triple, we have $(1, -1, 2)$. This is a **consistent** system with a unique solution.

Procedure for Gauss-Jordan Elimination

Step 1. Choose the leftmost nonzero column and use appropriate row operations to get a 1 at the top.

Step 2. Use multiples of the row containing the 1 from step 1 to get zeros in all remaining places in the column containing this 1.

Step 3. Repeat step 1 with the **submatrix** formed by (mentally) deleting the row used in step and all rows above this row.

Step 4. Repeat step 2 with the entire matrix, including the rows deleted mentally. Continue this process until the entire matrix is in reduced form.

Note:

If at any point in this process we obtain a row with all zeros to the left of the vertical line and a nonzero number to the right, we can stop because we will have a contradiction.

The Gauss-Jordan elimination method is a technique for solving systems of linear equations of any size.
The operations of the Gauss-Jordan method are

1- Interchange any two equations.
2- Replace an equation by a nonzero constant multiple of itself.
3- Replace an equation by the sum of that equation and a constant multiple of any other equation.

Example 8:

Solve the following system of the simultaneous linear equations:

$$
\begin{aligned}
2x + 4y + 6z &= 22 \\
3x + 8y + 5z &= 27 \\
-x + y + 2z &= 2
\end{aligned}
$$

Solution

The augmented matrix

$$
\left[\begin{array}{ccc|c} 2 & 4 & 6 & 22 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{array}\right] \frac{1}{2}R_1 \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{array}\right] \begin{array}{c} R_2 - 3R_1 \\ R_3 + R_1 \end{array}
$$

$$\begin{bmatrix} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ 0 & 3 & 5 & 13 \end{bmatrix} \frac{1}{2}R_2 \sim \begin{bmatrix} 1 & 2 & 3 & 11 \\ 0 & 1 & -2 & -3 \\ 0 & 3 & 5 & 13 \end{bmatrix} \begin{matrix} R_1 - 2R_2 \\ R_3 - 3R_2 \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 7 & 17 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 11 & 22 \end{bmatrix} -\frac{1}{11}R_3 \sim \begin{bmatrix} 1 & 0 & 7 & 17 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{matrix} R_1 - 7R_3 \\ R_2 + 2R_3 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$x = 3$, $y = 1$, $z = 2$

Row Reduced Form of the Matrix

* Each row consisting entirely of zeros lies below all rows having nonzero entries.

* The first nonzero entry in each nonzero row is 1 (called a leading 1).

* In any two successive (nonzero) rows, the leading 1 in the lower row lies to the right of the leading 1 in the upper row.

* If a column contains a leading 1, then the other entries in that column are zeros.

Row Operations

1. Interchange any two rows.
2. Replace any row by a nonzero constant multiple of itself.
3. Replace any row by the sum of that row and a constant multiple of any other row.

Notation for Row Operations

Letting R_i denote the i throw of a matrix, we write

Operation 1: $R_i \leftrightarrow R_j$ to mean: Interchange row i with row j.

Operation 2: $c\,R_i$, to mean: replace row i with c times row i.

Operation 3: $R_i + a\,R_j$ to mean: Replace row i with the sum of row i and a times row j.

Use the Gauss-Jordan elimination method to solve the system of equations

$$3x - 2y + 3z = 9$$
$$-2x + 2y + z = 3$$
$$x + 2y - 3z = 8$$

Introduction to Systems of Linear Equations

In two dimensions a line in a rectangular xy-coordinate system can be represented by an equation of the form

$ax + by = c$ (a, b not both 0)

and in three dimensions a plane in a rectangular xyz-coordinate system can be represented by an equation of the form

$ax + by + cz = d$ (a, b ,c not all 0)

$ax + by + cz = 0$, which is called a homogeneous linear equation.

Linear Equations

Observe that a linear equation does not involve any products or roots of variables. All variables occur only to the first power.

A general linear system of m equations in the n unknowns $x_{1,}x_2 \ \dots x_n$ can be written as

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ \dots a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ \dots a_{1n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ \dots a_{mn}x_n = b_m$$

Solutions are found at the intersection of the equations in the system.

A solution of a linear system in n unknowns $x_{1,}x_2 \ \dots x_n$ is a sequence of n numbers $s_{1,}s_2 \ \dots s_n$ for which the substitution $x_1 = s_1,\ x_2 = s_2, \dots, x_n = s_n$, of a linear system in n unknowns can be written as $(s_{1,}s_2 \ \dots s_n)$, which is called an ordered n-tuple. With this notation it is understood that all variables appear in the same order in each equation. If $n = 2$ then the n-tuple is called an ordered pair, and if $n = 3$, then it is called an ordered triple.

Every system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.

1. The lines may be parallel and distinct, in which case there is no intersection and consequently no solution

2. The lines may intersect at only one point, in which case the system has exactly one solution.
3. The lines may coincide, in which case there are infinitely many points of intersection (the points on the common line) and consequently infinitely many solutions.
In general, we say that a linear system is consistent if it has at least one solution and inconsistent if it has no solutions.

Augmented Matrices and Elementary Row Operations
Augmented Matrix is a matrix that is used to solve a system of equations.

The system of equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ \dots a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ \dots a_{2n}x_n = b_2$$
$$\vdots\vdots\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ \dots a_{mn}x_n = b_m$$

We can abbreviate the system by writing only the rectangular array of numbers

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

This is called the augmented matrix for the system. For example, the augmented matrix for the system of equations ich case

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$$\begin{aligned} x_1 + x_2 + 2x_3 &= 9 \\ 2x_1 + 4x_2 - 3x_3 &= 1 \\ 3x_1 + 6x_2 - 5x_3 &= 0 \end{aligned} \quad , \quad is \quad \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right]$$

The basic method for solving a linear system is to perform appropriate algebraic operations on the system that do not alter the solution set and that produce a succession of increasingly simpler systems, until a point is reached where it can be ascertained whether the system is consistent, and if so, what its solutions are.

Typically, the algebraic operations are as follows:

1. Multiply an equation through by a nonzero constant.
2. Interchange two equations.
3. Add a constant times one equation to another.

Since the rows (horizontal lines) of an augmented matrix correspond to the equations in the associated system, these three operations correspond to the following operations on the rows of the augmented matrix:

1. Multiply a row through by a nonzero constant.
2. Interchange two rows.
3. Add a constant times one row to another.

These are called elementary row operations on a matrix.

Example 9:

Using Elementary Row Operations

In the left column we solve a system of linear equations by operating on the equations in the system, and in the right column we solve the same system by operating on the rows of the augmented matrix.

$$x_1 + x_2 + 2x_3 = 9$$

$$2x_1 + 4x_2 - 3x_3 = 1$$

$$3x_1 + 6x_2 - 5x_3 = 0$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add −2 times the first equation to the second to obtain
Add −2 times the first row to the Second to obtain

$$x_1 + x_2 + 2x_3 = 9$$

$$2x_2 - 7x_3 = -17 \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

$$3x_1 + 6x_2 - 5x_3 = 0$$

Add −3 times the first equation to the third to obtain
Add −3 times the first row to the third to obtain

$$x_1 + x_2 + 2x_3 = 9$$

$$2x_2 - 7x_3 = -17 \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

$$3x_2 - 11x_3 = -27$$

Multiply the second equation by $\frac{1}{2}$ to obtain
multiply the second row by $\frac{1}{2}$ to obtain

$$x_1 + x_2 + 2x_3 = 9$$

$$x_2 - \frac{7}{2}x_3 = -\frac{17}{2} \qquad \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

$$3x_2 - 11x_3 = -27$$

Add -3 times the second equation to the third to obtain
Add -3 times the second row to the third to obtain

$$x_1 + x_2 + 2x_3 = 9$$

$$x_2 - \frac{7}{2}x_3 = -\frac{17}{2} \qquad \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

$$-\frac{1}{2}x_3 = -\frac{3}{2}$$

Multiply the third equation by -2 to obtain
Multiply the third row by -2 to obtain

$$x_1 + x_2 + 2x_3 = 9$$

$$x_2 - \frac{7}{2}x_3 = -\frac{17}{2} \qquad \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$x_3 = 3$$

Add −1 times the second equation to the first to obtain
Add −1 times the second row to the first to obtain

$$x_1 + \frac{11}{2}x_3 = \frac{35}{2}$$

$$x_2 - \frac{7}{2}x_3 = -\frac{17}{2} \begin{bmatrix} 1 & 0 & \frac{11}{2} & \frac{35}{2} \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$x_3 = 3$$

Add $-\frac{11}{2}$ times the third equation to the first
Add $-\frac{11}{2}$ times the third row to the first and $\frac{7}{2}$
and $\frac{7}{2}$ times the third equation to the second to obtain
times the third row to the second to obtain

$x_1 = 1$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$x_2 = 2x_3 = 3$$

The solution $x_1 = 1, x_2 = 2, x_3 = 3$ is now evident.

Exercises Two

1. Find the inverse of each of the following matrices

$$A=\begin{bmatrix}6 & 8\\ 2 & 4\end{bmatrix} \quad , \quad B=\begin{bmatrix}-3 & -4\\ -1 & -5\end{bmatrix} \quad , \quad C=\begin{bmatrix}1 & 3\\ 4 & 2\end{bmatrix}$$

2. Find the inverse of each of the following matrices by using **Gauss-Jordan Elimination**:

$$A=\begin{bmatrix}1 & 8 & 5\\ 3 & 4 & 2\\ 9 & -3 & -1\end{bmatrix}$$

$$B=\begin{bmatrix}1 & 3 & -4\\ -5 & -2 & 7\\ 8 & -7 & 3\end{bmatrix}$$

$$C=\begin{bmatrix}0 & 1 & 1\\ 2 & 3 & -1\\ -1 & 2 & 1\end{bmatrix}$$

3. Given

$$A=\begin{bmatrix}2 & 1 & 5\\ 6 & 7 & -3\\ 4 & 3 & -1\end{bmatrix},$$

$$B=\begin{bmatrix}9 & -7 & 3\\ 8 & 6 & -1\\ 5 & -2 & 1\end{bmatrix},$$

$$C=\begin{bmatrix}-2 & 1 & 4\\-4 & 7 & 2\\-1 & 3 & 2\end{bmatrix}$$

Carry out the following operations:

(a) Find AB and BA. State your comment.

(b) Find A', B' and C'

(c) Find A^{-1} , B^{-1} and C^{-1}

(d) Compute $10A-9B-8I$

(e) Prove that:

i. $(A-B)^2=A^2-2AB+B^2$

ii. $(A+B)^2=A^2+2AB+B^2$

iii. $(A+B-C)'=A'+B'-C'$

iv. $(AB)'=B'A'$

vi. $(ABC)'=C'B'A'$

v. $(ABC)^{-1}=C^{-1}B^{-1}A^{-1}$

4. Solve the following systems of equations by using **Gauss-Jordan Elimination**

(a) $2x-3y=1$
$3x+4y=10$

(b) $3x+2y=1$
$2x-y=3$

(c) $4x + 5y = 14$
$2x - 3y = 1$

(d) $2x - y + 3z = -3$
$x + y + z = 2$
$3x + 2y - z = 8$

(e) $x + 2y - z = 1$
$2z - 3x = 2$
$3y + 2z = 5$

(f) $x + 2y - 3z = 1$
$y - 2x + z = 3$
$2z + x - 2 = 0$

5. In parts (a)–(h) determine whether the statement is true or false, and justify your answer.

(a) A linear system whose equations are all homogeneous must be consistent.

Answer: True

(b) Multiplying a linear equation through by zero is an acceptable elementary row operation.

Answer: False

(c) The linear system $x - y = 3$
$2x - 2y = k,$

Cannot have a unique solution, regardless of the value of k.

Answer: True

(d) A single linear equation with two or more unknowns must always have infinitely many solutions.

Answer: True

(e) If the number of equations in a linear system exceeds the number of unknowns, then the system must be inconsistent.

Answer: False

(f) If each equation in a consistent linear system is multiplied through by a constant c, then all solutions to the new system can be obtained by multiplying solutions from the original system by c.

Answer: False

(g) Elementary row operations permit one equation in a linear system to be subtracted from another.

Answer: True

(h) The linear system with corresponding augmented matrix $\begin{bmatrix} 2 & -1 & 4 \\ 0 & 0 & -1 \end{bmatrix}$ is consistent.

Answer: False

6. The Pharmaceutical company produces three types I, II and III of medicine in two different sizes. The production (in thousands) per week at its first plant is

$$\begin{array}{cc} & \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ \text{Size} & \begin{bmatrix} 13 & 27 & 15 \\ 12 & 14 & 24 \end{bmatrix} \end{array},$$

and the weekly production at its second plant is

$$\begin{array}{cc} & \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ \text{Size} & \begin{bmatrix} 20 & 32 & 18 \\ 35 & 24 & 30 \end{bmatrix} \end{array}$$

(a) What the total weekly production at the two plants?

(b) If the production at the first plant is increased by 20 %, what will the total production be at the two plants now?

7. Pharmaceutical company produces three products, A, B and C which require processing by three machines, I, II and III. One unit of A requires 3, 1 and 8 hours of processing on the three machines, whereas 1 unit of B requires 2, 3 and 3, and 1 unit of C requires 2, 4 and 2 hours on the three machines. The machines I, II and III are available for 800, 1200 and 1300 hours, respectively. How many units of each should be produced to make use of all the available time on the machines?

8. In Exercise 12, how many units of A, B and C can be produced if the three machines are available for 900, 1200 and 1500 hours, respectively?

9. A firm produces three products, A, B and C, which require processing by three machines. The time (in hour) required for processing one unite of each product by the three machines is as follows.

$$\begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \text{Machines} & \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 4 \\ 2 & 1 & 1 \end{bmatrix} \end{array}$$

Machine I is available 850 hours, machine II, for 1200 hours, and machine III, for 550 hours. How many units of each product should be produced to make use of all the available time on the machines?

10. In Exercise 14, how many units of A, B and C can be produced if the three machines are available for 950, 1400 and 1700 hours, respectively?