Chapter Five

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Eigenvectors are vectors that point in directions where there is no rotation. Eigenvalues are the change in length of the eigenvector from the original length.

The basic equation in eigenvalue problems is: $Ax = \lambda x$

In words, this deceptively simple equation says that for the square matrix **A**, there is a vector **x** such that the product of **Ax** such that the result is a SCALAR, λ, that, when multiplied by **x**, results in the same product. The multiplication of vector **x** by a scalar constant is the same as stretching or shrinking the coordinates by a constant value.

The eigenvectors **x** and eigenvalues λ of a matrix A satisfy

$$\boxed{\mathbf{Ax}} = \boxed{\lambda \mathbf{x}}$$

$Ax = \lambda x$

If A is an n x n matrix, then **x** is an n x 1 vector, and λ is a constant.

The equation can be rewritten as $(A - \lambda I)\mathbf{x} = 0$, where I is the n x n identity matrix.

Since **x** is required to be nonzero, the eigenvalues must satisfy $\det(A - \lambda I) = 0$ which is called the ***characteristic equation***. Solving it for values of λ gives the eigenvalues of matrix A.

The vector **x** is called an **eigenvector** and the scalar λ is called an **eigenvalue**.

Do all matrices have real eigenvalues?

No, they must be square and the determinant of $\boxed{\mathbf{A} - \lambda \mathbf{I}}$ must equal $\boxed{\text{zero}}$. This is easy to show:

$$\mathbf{Ax} - \lambda\mathbf{x} = 0 \quad \mathbf{x}(\mathbf{A} - \lambda\mathbf{I}) = 0 \quad (A - \lambda I)x = 0$$

This can only be true if det $(\mathbf{A} - \lambda\mathbf{I}) = |\mathbf{A} - \lambda\mathbf{I}| = 0$

Are eigenvectors unique?

No, if $\mathbf{x}$ is an eigenvector, then $\beta\mathbf{x}$ is also an eigenvector and $\beta\lambda$ is an eigenvalue.

$\mathbf{A}(\beta\mathbf{x}) = \beta\mathbf{Ax} = \beta l\mathbf{x} = l\,(\beta\mathbf{x})$

$(A - \lambda I)x = 0$

The procedure is:

1. Compute the determinant of $\boxed{\mathbf{A} - \lambda\mathbf{I}}$
2. Find the roots of the polynomial given by $|\mathbf{A} - \lambda\mathbf{I}| = 0$
3. Solve the system of equations $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$

Some facts:

- The product of the eigenvalues = $\boxed{\text{det } |\mathbf{A}|}$
- The sum of the eigenvalues = $\boxed{\text{trace } (\mathbf{A})}$

We could find the eigenvalues of $\mathbf{A}$ and obtain $\mathbf{A}^{100}$ very quickly using eigenvalues.

Finally, zero is an eigenvalue of A if and only if A is singular and det $(A) = 0$.

$1-\lambda \quad -2$

$3 \quad -4-\lambda$

Example 1:

$$A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \qquad \text{so} \quad A - \lambda I = \begin{bmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{bmatrix}$$

det $(A - \lambda I) = (1 - \lambda)(-4 - \lambda) - (3)(-2)$

$$= \lambda^2 + 3\lambda + 2$$

$\lambda^2 + 3\lambda + 2 = 0$

Set $\lambda^2 + 3\lambda + 2$ to 0

$\lambda^2 + 3\lambda + 2 = 0$

$(\lambda + 2)(\lambda + 1) = 0$

Then $\boxed{\lambda = -1}$ $\boxed{\lambda = -2}$

So the two values of λ are -1 and -2.

Finding the Eigenvectors

Once you have the eigenvalues, you can plug them into the equation $A\mathbf{x} = \lambda\mathbf{x}$ to find the corresponding sets of eigenvectors $\mathbf{x}$.

$(A - \lambda I)x = 0$

$\boxed{x_1 = x_2}$

$2x_1 - 2x_2 = 0$

$3x_1 - 3x_2 = 0$

$\boxed{x_1 = x_2}$

$1 - (-1)$ $1 - -1$

$\boxed{+2}$

$-4 - (-1)$

$-4 + 1 = \boxed{-3}$

At $\lambda = -1$

$(A - \lambda I)\vec{X} = 0$

$$\begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$x_1 - x_2 = 0$ $x_1 = x_2$

$\cancel{2}x_1 - \cancel{2}x_2 = 0$

$3x_1 - 3x_2 = 0$

$x_1 = x_2$

we can represent all eigenvectors for eigenvalue -1 as multiples of a simple basis vector:

$V = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ where t is a parameter

$1 - -2$ $\boxed{3}$

$-4 - -2$

$-4 + 2$

$\boxed{-2}$

At $\lambda = -2$

$$\begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$3x_1 - 2x_2 = 0$

$3x_1 - 2x_2 = 0$

$3x_1 - 2x_2 = 0$

$3x_1 - 2x_2 = 0$

$3x_1 = 2x_2$

$3x_1 - 2x_2 = 0$, so eigenvectors are of the form $t\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Example 2:

Find the eigenvalues and the eigenvectors of A ,

$$\begin{matrix} 11-\lambda & -11 & -7 \\ 7 & -7-\lambda & -5 \\ 3 & -3 & -1-\lambda \end{matrix}$$

$$A = \begin{bmatrix} 11 & -11 & -7 \\ 7 & -7 & -5 \\ 3 & -3 & -1 \end{bmatrix}$$

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Solution:

$$\boxed{A\vec{X} = \lambda\vec{X}}$$

$$A\vec{X} = \lambda I\vec{X}$$

$A\vec{X} - \lambda I\vec{X} = 0$ $\qquad$ $(A - \lambda I)\,\vec{X} = 0$

$(A - \lambda I)\,\vec{X} = 0$ $\qquad$ $(A - \lambda I)x = 0$

$|A - \lambda I| = 0$

value

$$|A - \lambda I| = \begin{vmatrix} 11-\lambda & -11 & -7 \\ 7 & -7-\lambda & -5 \\ 3 & -3 & -1-\lambda \end{vmatrix}$$

$$= (11-\lambda)\begin{vmatrix} -7-\lambda & -5 \\ -3 & -1-\lambda \end{vmatrix} + 11\begin{vmatrix} 7 & -5 \\ 3 & -1-\lambda \end{vmatrix} - 7\begin{vmatrix} 7 & -7-\lambda \\ 3 & -3 \end{vmatrix}$$

$= (11-\lambda)[(-7-\lambda)(-1-\lambda) - 15] + 11[7(-1-\lambda)$
$+15] - 7[-21 - 3(-7-\lambda)]$

$= (11-\lambda)[(7+\lambda)(1+\lambda) - 15] + 11[-7(1+\lambda) +$
$15] - 7[-21 + 3(7+\lambda)]$

$= (77+4\lambda-\lambda^2)(1+\lambda) - 165 + 15\lambda + 11[-7 - 7\lambda + 15] -$
$7[-21+21+3\lambda]$

$= 77+77\lambda + 4\lambda + 4\lambda^2 - \lambda^2 - \lambda^3 - 165 + 15\lambda +$
$11[-7\lambda + 8] - 7[3\lambda]$

$= -88+77\lambda + 4\lambda + 4\lambda^2 - \lambda^2 - \lambda^3 + 15\lambda + 88 - 77\lambda - 21\lambda$

$= -\lambda^3 + 3\lambda^2 - 2\lambda = \lambda^3 - 3\lambda^2 + 2\lambda = \lambda(\lambda-2)(\lambda-1)$

$$\lambda(\lambda-2)(\lambda-1) = 0$$

$\lambda = 0$, $\lambda = 2$, $\lambda = 1$

The eigenvectors

At $\lambda = 0$

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$(A - \lambda I)\vec{X} = 0$

$$\begin{bmatrix} 11-0 & -11 & -7 \\ 7 & -7-0 & -5 \\ 3 & -3 & -1-0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix

$$\begin{bmatrix} 11 & -11 & -7 & 0 \\ 7 & -7 & -5 & 0 \\ 3 & -3 & -1 & 0 \end{bmatrix} \begin{matrix} R_1 \leftrightarrow R_3 \\ \frac{1}{3} R_1 \end{matrix} \sim \begin{bmatrix} 1 & -1 & -\frac{1}{3} & 0 \\ 7 & -7 & -5 & 0 \\ 11 & -11 & -7 & 0 \end{bmatrix} \begin{matrix} R_2 - 7R_1 \\ R_3 - 11R_1 \end{matrix} \sim$$

$$\begin{bmatrix} 0 & 0 & -\frac{8}{3} & 0 \\ 0 & 0 & -\frac{10}{3} & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

<u>At $\lambda = 2$</u>

$$\begin{bmatrix} 11-2 & -11 & -7 \\ 7 & -7-2 & -5 \\ 3 & -3 & -1-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 9 & -11 & -7 \\ 7 & -9 & -5 \\ 3 & -3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix

$$\begin{bmatrix} 9 & -11 & -7 & 0 \\ 7 & -9 & -5 & 0 \\ 3 & -3 & -3 & 0 \end{bmatrix} \begin{matrix} R_1 \leftrightarrow R_3 \\ \frac{1}{3}R_1 \end{matrix} \sim \begin{bmatrix} 1 & -1 & -1 & 0 \\ 7 & -9 & -5 & 0 \\ 9 & -11 & -7 & 0 \end{bmatrix} \begin{matrix} R_2 - 7R_1 \\ R_3 - 9R_1 \end{matrix} \sim$$

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & -2 & 2 & 0 \end{bmatrix} \begin{matrix} \frac{1}{2}R_2 \\ \frac{1}{2}R_3 \end{matrix} \sim \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

$x_1 - x_2 - x_3 = 0$

$-x_2 + x_3 = 0$

$x_2 = x_3 = t$

$x_1 - t - t = 0$

$x_1 = 2t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

<u>At $\lambda = 1$</u>

$$\begin{bmatrix} 11-1 & -11 & -7 \\ 7 & -7-1 & -5 \\ 3 & -3 & -1-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 10 & -11 & -7 \\ 7 & -8 & -5 \\ 3 & -3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix

$$\begin{bmatrix} 9 & -11 & -7 & 0 \\ 7 & -8 & -5 & 0 \\ 3 & -3 & -2 & 0 \end{bmatrix} \begin{matrix} R_1 \leftrightarrow R_3 \\ \frac{1}{3} R_1 \end{matrix} \sim \begin{bmatrix} 1 & -1 & -\frac{2}{3} & 0 \\ 7 & -8 & -5 & 0 \\ 9 & -11 & -7 & 0 \end{bmatrix} \begin{matrix} R_2 - 7R_1 \\ R_3 - 9R_1 \end{matrix} \sim \begin{bmatrix} 1 & -1 & -\frac{2}{3} & 0 \\ 0 & -1 & -\frac{1}{3} & 0 \\ 0 & -2 & -1 & 0 \end{bmatrix}$$

$x_1 - x_2 - \frac{2}{3} x_3 = 0$

$-x_2 - \frac{1}{3} x_3 = 0$

$x_2 = -\frac{1}{3} x_3$

$x_1 = \frac{1}{3} x_3$

$x_3 = t, \quad x_2 = -\frac{1}{3}t, \quad x_1 = \frac{1}{3}t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

Example 3:

Given $A = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$ and $\lambda = 10$ find $\vec{x}$ (eigenvector)

Solution:

$A\vec{X} - \lambda I\vec{X} = 0$ $\qquad\qquad (A - \lambda I)\,\vec{X} = 0$

$(A - \lambda I)\,\vec{X} = 0$

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$$\begin{bmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda = 10$

$$\begin{bmatrix} 5-10 & 4 & 2 \\ 4 & 5-10 & 2 \\ 2 & 2 & 2-10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 4 & 2 \\ 4 & -5 & 2 \\ 2 & 2 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix

$$\begin{bmatrix} -5 & 4 & 2 & 0 \\ 4 & -5 & 2 & 0 \\ 2 & 2 & -8 & 0 \end{bmatrix} \begin{matrix} R_1 \leftrightarrow R_3 \\ \dfrac{1}{2}R_1 \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 1 & -4 & 0 \\ 4 & -5 & 2 & 0 \\ -5 & 4 & 2 & 0 \end{bmatrix} \begin{matrix} R_2 - 4R_1 \\ R_3 + 5R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & -4 & 0 \\ 0 & -9 & 18 & 0 \\ 0 & 9 & -18 & 0 \end{bmatrix}$$

$$R_3 + R_2 \sim \begin{bmatrix} 1 & 1 & -4 & 0 \\ 0 & -9 & 18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \frac{1}{-9}R_2 \sim \begin{bmatrix} 1 & 1 & -4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 + x_2 - 4x_3 = 0$

$x_2 - 2x_3 = 0$

$x_2 = 2x_3$, we put $x_3 = t$ $x_2 = 2t$

$x_1 = -2t + 4t = 2t$

$$\vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$