Canonical Forms

- It is useful to specify Boolean functions in a form that:
  - Allows comparison for equality.
  - Has a correspondence to the truth tables
- Canonical Forms in common usage:
  - Sum of Minters (SOM)
  - Product of Maxterms (POM)
- Any boolean function that is expressed as a sum of minterms or as a product of maxterms

Minterms

- A minterm is a special product of literals, in which each input variable appears exactly once. each binary variable may appear normal (e.g., x) or complemented (e.g.,$\overline{x}$) .
- A function with n variables has 2^n minterms (since each variable can appear complemented or not)
- A three-variable function, such as f(x,y,z), has $2^3 = 8$ minterms:

x'y'z' x'y'z x'yz' x'yz
xy'z' xy'z xyz' xyz

- Each minterm is true for exactly one combination of inputs:

Minterm	Is true when…	Shorthand
x’y’z’	x=0, y=0, z=0	m_0
x’y’z	x=0, y=0, z=1	m_1
x’yz’	x=0, y=1, z=0	m_2
x’yz	x=0, y=1, z=1	m_3
xy’z’	x=1, y=0, z=0	m_4
xy’z	x=1, y=0, z=1	m_5
xyz’	x=1, y=1, z=0	m_6
xyz	x=1, y=1, z=1	m_7

Sum of minterms form

- Every function can be written as a sum of minterms, which is a special kind of sum of products form
- The sum of minterms form for any function is *unique*
- If you have a truth table for a function, you can write a sum of minterms expression just by picking out the rows of the table where the function output is 1 (1-minterm).

$$\begin{aligned} f &= x'y'z' + x'y'z + x'yz' + x'yz + xyz' \\ &= m_0 + m_1 + m_2 + m_3 + m_6 \\ &= \Sigma(0,1,2,3,6) \end{aligned}$$

$$\begin{aligned} f' &= xy'z' + xy'z + xyz \\ &= m_4 + m_5 + m_7 \\ &= \Sigma(4,5,7) \end{aligned}$$

f' contains all the minterms not in f

x	y	z	f(x,y,z)	f′(x,y,z)
0	0	0	1	0
0	0	1	1	0
0	1	0	1	0
0	1	1	1	0
1	0	0	0	1
1	0	1	0	1
1	1	0	1	0
1	1	1	0	1

- Example
- $F = x + yz$, how to express this in the sum of minterms?

$$= x(y + y')(z + z') + (x + x')yz$$

$$= xyz + xyz' + xy'z + xy'z' + xyz + x'yz$$

$$= x'yz + xy'z' + xy'z + xyz' + xyz$$

$$= \Sigma(3,4,5,6,7)$$

or, convert the expression into truth-table and then read the minterms from the table

Maxterms

- **A maxterm** (OR terms) is a *sum* of literals, in which each input variable appears exactly once.
- A function with n variables has 2^n maxterms
- The maxterms for a three-variable function f(x,y,z):

$$x' + y' + z' \qquad x' + y' + z \qquad x' + y + z' \qquad x' + y + z$$
$$x + y' + z' \qquad x + y' + z \qquad x + y + z' \qquad x + y + z$$

- Each maxterm is *false* for exactly one combination of inputs:

Maxterm	Is *false* when…	Shorthand
x + y + z	x=0, y=0, z=0	M_0
x + y + z’	x=0, y=0, z=1	M_1
x + y’ + z	x=0, y=1, z=0	M_2
x + y’ + z’	x=0, y=1, z=1	M_3
x’ + y + z	x=1, y=0, z=0	M_4
x’ + y + z’	x=1, y=0, z=1	M_5
x’ + y’ + z	x=1, y=1, z=0	M_6
x’ + y’ + z’	x=1, y=1, z=1	M_7

Product of maxterms form

- Every function can be written as a *unique* **product of maxterms**
- If you have a truth table for a function, you can write a product of maxterms expression by picking out the rows of the table where the function output is 0 (**0-maxterm**).

$$f = (x' + y + z)(x' + y + z')(x' + y' + z')$$
$$= M_4\ M_5\ M_7$$
$$= \prod(4,5,7)$$

$$f' = (x + y + z)(x + y + z')(x + y' + z)$$
$$(x + y' + z')(x' + y' + z)$$
$$= M_0\ M_1\ M_2\ M_3\ M_6$$
$$= \prod(0,1,2,3,6)$$

f' contains all the maxterms not in f

x	y	z	f(x,y,z)	f′(x,y,z)
0	0	0	1	0
0	0	1	1	0
0	1	0	1	0
0	1	1	1	0
1	0	0	0	1
1	0	1	0	1
1	1	0	1	0
1	1	1	0	1

- $F = x'y' + xz$, how to express this in the product of maxterms?

$$= (x'y' + x)(x'y' + z)$$

$$= (x' + x)(y' + x)(x' + z)(y' + z)$$

$$= (x + y')(x' + z)(y' + z)$$

$$= (x + y' + zz')(x' + z + yy')(xx' + y' + z)$$

$$= (x + y' + z)(x + y' + z')(x' + y + z)(x' + y' + z)(x + y' + z)(x' + y' + z)$$

$$= (x + y' + z)(x + y' + z')(x' + y + z)(x' + y' + z)$$

$$= \prod(2,3,4,6)$$

or, convert the expression into truth-table and then read the minterms from the table

Minterm and Maxterm Relationship

- Review: DeMorgan's Theorem

$$(xy)' = x' + y' \quad \text{and} \quad (x + y)' = x'y'$$

- Two-variable example:

$$\mathbf{M_2 = \bar{x} + y} \quad \text{and} \quad \mathbf{m_2 = x \cdot \bar{y}}$$

Thus M_2 is the complement of m_2 and vice-versa.

Since DeMorgan's Theorem holds for *n* variables, the above holds for terms of *n* variables

giving:

$$\mathbf{M_i = \overline{m}_i} \text{ and } \mathbf{m_i = \overline{M}_i}$$

Thus M_i is the complement of m_i.

- Any minterm m_i is the *complement* of the corresponding maxterm M_i

Minterm	Shorthand
$x'y'z'$	m_0
$x'y'z$	m_1
$x'yz'$	m_2
$x'yz$	m_3
$xy'z'$	m_4
$xy'z$	m_5
xyz'	m_6
xyz	m_7

Maxterm	Shorthand
$x + y + z$	M_0
$x + y + z'$	M_1
$x + y' + z$	M_2
$x + y' + z'$	M_3
$x' + y + z$	M_4
$x' + y + z'$	M_5
$x' + y' + z$	M_6
$x' + y' + z'$	M_7

- For example, $m_4' = M_4$ because $(xy'z')' = x' + y + z$

Converting between canonical forms

- We can convert a sum of minterms to a product of maxterms

From before $f = \Sigma(0,1,2,3,6)$

and $f' = \Sigma(4,5,7)$

$= m_4 + m_5 + m_7$

complementing $(f')' = (m_4 + m_5 + m_7)'$

so $f = m_4'\, m_5'\, m_7'$ [DeMorgan's law]

$= M_4\, M_5\, M_7$

$= \prod(4,5,7)$

$$f = \Sigma(0,1,2,3,6) = \prod(4,5,7)$$

In general, just replace the minterms with maxterms, using maxterm numbers that don’t appear in the sum of minterms:

The same thing works for converting from a product of maxterms to a sum of minterms

Standard Forms

- Any boolean function that is expressed as a sum of products (SOP) or as a product of sums (POS), where each product-term or sum-term may require fewer than (n-1) operations, is said to be in its standard form.
- Standard forms are not unique, there can be several different SOPs and POSs for a given function.

$$f(x,y,z) = xy + x'yz + xy'z$$

$$f(x,y,z) = (x' + y')(x + y' + z')(x' + y + z')$$

A SOP expression contains:

- Only OR (sum) operations at the “outermost” level
- Each term (implicant) must be a product of literals

A POS expression contains:

- Only AND (product) operations at the “outermost” level
- Each term (implicate) must be a sum of literals

Canonical Forms

- Any boolean function that is expressed as a sum of minterms or as a product of maxterms is said to be in its canonical form.

Sum of minterms: practise

- F = x + yz, how to express this in the sum of minterms?

$= x(y + y')(z + z') + (x + x')yz$

$= xyz + xyz' + xy'z + xy'z' + xyz + x'yz$

$= x'yz + xy'z' + xy'z + xyz' + xyz$

$= \Sigma(3,4,5,6,7)$

or, convert the expression into truth-table and then read the minterms from the table

Product of maxterms: practice

- F = x'y' + xz, how to express this in the product of maxterms?

$$
\begin{aligned}
&= (x'y' + x)(x'y' + z) \\
&= (x' + x)(y' + x)(x' + z)(y' + z) \\
&= (x + y')(x' + z)(y' + z) \\
&= (x + y' + zz')(x' + z + yy')(xx' + y' + z) \\
&= (x + y' + z)(x + y' + z')(x' + y + z)(x' + y' + z)(x + y' + z)(x' + y' + z) \\
&= (x + y' + z)(x + y' + z')(x' + y + z)(x' + y' + z) \\
&= \prod(2,3,4,6)
\end{aligned}
$$

or, convert the expression into truth-table and then read the minterms from the table