Chapter Three
Determinants

The determinant of a matrix is a scalar representation of matrix. Only square matrices have determinants. The value of determinant of the 2×2 matrix can be obtained as:

$$|\boldsymbol{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Determinant of A, denoted by |**A**|, is a number and can be evaluated as:

* The determinant of a matrix A is denoted by |**A**| (or det (**A**)).

* Determinants exist only for square matrices.

$$det(AB) = det(A)det(B)$$

$$det(A + B) \neq det(A) + det(B)$$

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

$$\boldsymbol{det(A)} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Example1:

Evaluate the value of the following determinant:

$$\begin{vmatrix} 2 & 3 \\ 5 & -4 \end{vmatrix}$$

Solution:

$$|A| = 2\times(-4) - 3\times5 = -23$$

On the other hand, the value of 3 × 3 matrix can be obtained as:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example 2:

Evaluate the value of the following determinant:

$$\begin{vmatrix} 2 & 3 & -2 \\ -5 & -4 & 6 \\ 4 & 5 & -3 \end{vmatrix}$$

Solution:

$$|A| = 2\begin{vmatrix} -4 & 6 \\ 5 & -3 \end{vmatrix} - 3\begin{vmatrix} -5 & 6 \\ 4 & -3 \end{vmatrix} + (-2)\begin{vmatrix} -5 & -4 \\ 4 & 5 \end{vmatrix} = -153$$

Effect of Elementary Operations

Use them to compute determinant of a matrix A by reducing it to a simpler matrix (like triangular matrices).

Theorem:

Let A, B be two square matrices of same size. If B is obtained by interchanging two rows of A, then\

$$|\mathbf{B}| = -|A|$$

If B is obtained by adding a scalar multiple of a row of A to another row of A, then

$$|\mathbf{B}| = |A|$$

If B is obtained by multiplying a row of A by a scalar c, then

$$|\mathbf{B}| = \mathbf{c}\,|A|$$

Properties of Determinants

Determinants have several mathematical properties which are useful in matrix manipulations.

1. The of a determinant A is equal to the value of its transpose, that is

$$|A| = |A'|.$$

Example 3:

$$A = \begin{vmatrix} 1 & -2 & 5 \\ 0 & 1 & -3 \\ 2 & 3 & 7 \end{vmatrix} \quad \Rightarrow \quad |A| = 18$$

$$A' = \begin{vmatrix} 1 & 0 & 2 \\ -2 & 1 & 3 \\ 5 & -3 & 7 \end{vmatrix} \quad \Rightarrow \quad |A'| = 18$$

2. If a row or column of A is 0, then

$$|A| = 0.$$

Example 4:

$$A = \begin{vmatrix} 0 & -2 & 5 \\ 0 & 1 & -3 \\ 0 & 3 & 7 \end{vmatrix} \quad \Rightarrow \quad |A| = 0$$

3. If every value in a row or column is multiplied by k, then

$$|A| = k\,|A|.$$

Example 5:

$$A = \begin{vmatrix} 4 & 8 & 12 \\ 2 & 1 & 3 \\ 5 & 3 & 7 \end{vmatrix} \Rightarrow |A| = 4\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 5 & 3 & 7 \end{vmatrix}$$

4. If two rows (or columns) are interchanged the sign, but not value, of $|A|$ changes.

Example 6:

$$A = \begin{vmatrix} 2 & 1 & -1 \\ -5 & 2 & 3 \\ 2 & 3 & 0 \end{vmatrix} \quad B = \begin{vmatrix} -5 & 2 & 3 \\ 2 & 1 & -1 \\ 2 & 3 & 0 \end{vmatrix}$$

$$|A| = -|B|$$

5. If two rows or columns are identical, then

$$|A| = 0.$$

Example 7:

$$A = \begin{vmatrix} 4 & -2 & 8 \\ 0 & 1 & -3 \\ 4 & -2 & 8 \end{vmatrix} \Rightarrow |A| = 0$$

6. If two rows or columns are linear combination of each other, one row (or column) is a scalar multiple of another row (or column) then

$$|A| = 0$$

Example 8:

$$A = \begin{vmatrix} 1 & -2 & 5 \\ 4 & 6 & 14 \\ 2 & 3 & 7 \end{vmatrix}$$

We notice that:

Second Row = 2(Third Row) $\Rightarrow$ $|A| = 0$

7. |A| remains unchanged if each element of a row or each element multiplied by a constant is added to any other row. .

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + ka_2 & b_1 + kb_2 & c_1 + kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Example 9:

$$\begin{vmatrix} 2 & 1 & -1 \\ 5 & 2 & 3 \\ 2 & 3 & 0 \end{vmatrix} = \begin{vmatrix} 2+10 & 1+4 & -1+6 \\ 5 & 2 & 3 \\ 2 & 3 & 0 \end{vmatrix} = -23$$

8. The value of a determinant of product A and B is equal to the value A multiplied by the value of determinant B.

$$|AB| = |A|\,|B|$$

Example 10:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 4 & -4 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} = -7 \text{ and } |B| = \begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix} = 4$$

$$|\boldsymbol{AB}| = \begin{vmatrix} 2 & 5 \\ 4 & -4 \end{vmatrix} = \mathbf{-28}$$

$$|AB| = |A|\,|B|$$

9. The determinant of a diagonal matrix A is equal to the product of the diagonal elements, that is

$$A = \begin{bmatrix} k & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & M \end{bmatrix} \Rightarrow |A| = KLM$$

Example 11:

$$\begin{vmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{vmatrix} = -6$$

Higher-order determinants are reduced to those of lower order, that is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \\ a_4 & c_4 & d_4 \end{vmatrix} + \cdots$$

From these formulas, one sees that a determinant of order n is a sum of terms, each of which is f 1 times a product of n factors, one each from the n columns of the array and one each from the n rows of the array.

$$a_1b_2c_3 - a_1b_3c_2 - b_1a_2c_3 + b_1a_3c_2 + c_1a_2b_3 - c_1a_3b_2.$$

We now state six rules for determinants:

Example 12:

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{vmatrix} = 1 \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} - 2 \begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} + 3 \begin{vmatrix} 0 & 4 \\ 1 & 0 \end{vmatrix}$$

$$= 24 + 10 - 12 = 22$$

10. Rows and columns can be interchanged, that is

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$$

Hence in every rule the words row and column can be interchanged.

Example 13:

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & -1 \\ 1 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 4 & 0 \\ 3 & -1 & 3 \end{vmatrix} = -2$$

11. Interchanging two rows (or columns) multiply the determinant by -1, that is

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Example 14:

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & -1 \\ 1 & 0 & 3 \end{vmatrix} = -\begin{vmatrix} 0 & 4 & -1 \\ 1 & 2 & 3 \\ 1 & 0 & 3 \end{vmatrix} = -(-2) = 4$$

12. A common factor of any row (or column) can be placed before the determinant, that is

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Example 15:

$$\begin{vmatrix} 2 & 4 & 6 \\ 0 & 4 & -1 \\ 1 & 0 & 3 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & -1 \\ 1 & 0 & 3 \end{vmatrix} = 2 \times (-2) = -4$$

13- If two rows (or columns) are proportional, the determinant equals 0. That is,

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$$

Example 16:

$$\begin{vmatrix} 2 & 6 & -2 \\ 1 & 3 & -1 \\ 1 & 0 & 3 \end{vmatrix} = 0$$

Notice that the first row = 2 (second row)

14- Determinants differing in only one row (or column) can be added by adding corresponding elements in that row and leaving the other elements unchanged, that is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} A_1 & b_1 & c_1 \\ A_2 & b_2 & c_2 \\ A_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + A_1 & b_1 & c_1 \\ a_2 + A_2 & b_2 & c_2 \\ a_3 + A_3 & b_3 & c_3 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & -1 \\ 1 & 0 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 2 & 3 \\ 2 & 4 & -1 \\ 1 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 4 & 2 & 3 \\ 2 & 4 & -1 \\ 2 & 0 & 3 \end{vmatrix}$$

-2+ 10= 8

15. The value of a determinant is unchanged if the elements of one row are multiplied by the same quantity k and added to the corresponding elements of another row, that is,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + ka_2 & b_1 + kb_2 & c_1 + kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

By a suitable choice of k, one can use this rule to introduce zeros; by repetition of the process, one can reduce all elements but one in a chosen row to 0. This procedure is basic for numerical evaluation of determinants.

From Rule 11 one deduces that a succession of an even number of interchanges of rows (or of columns) leaves the determinant unchanged, whereas an odd number of interchanges reverses the sign.

In each case we end up with a permutation of the rows (or columns) which we term even or odd according to the number of interchanges.

From an arbitrary determinant, one obtains others, called minors or the given one, by deleting k rows and k columns.

There is a similar expansion by minors of the first column or by minors of any chosen row or column.

Solution of a Linear System of Equations Using Cramer's Rule

Example 17:

Solve the following system of equations using Cramer's Rule.

$$2x + 4y = 1$$
$$-x + 2y = 4$$

Solution

The determinant of coefficients is

$$\Delta = \begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} = 2\times 2 - 4(-1) = 4 + 4 = 8$$

Replacing the first columns in Δ by the constant terms to obtain Δ_1, we get

$$\Delta_1 = \begin{vmatrix} 1 & 4 \\ 4 & 2 \end{vmatrix} = 1\times 2 - 4\times 4 = 2 - 16 = -14$$

Replacing the second columns in Δ by the constant terms to obtain Δ_2, we get

$$\Delta_2 = \begin{vmatrix} 2 & 1 \\ -1 & 4 \end{vmatrix} = 2\times4-1(-1)=8+1=9$$

Now, we can find the values for x and y as follows

$$x=\frac{\Delta_1}{\Delta}=-\frac{16}{8}=-2 \text{ , and } y=\frac{9}{8}$$

Example 18:

Solve the following system of equations using Cramer's Rule

$3x - y + 2z = -1$

$2x + y - z = 5$

$x + 2y + z = 4$

Solution:

The determinant of coefficients is

$$\Delta = \begin{vmatrix} 3 & -1 & 2 \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix}$$

Expanding with first row, we obtain

$$\Delta = 3\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= 3\,(1+2) + 1(2+1) + 2\,(4-1)$$
$$= 3(3) + 1(3) + 2(3)$$
$$= 9 + 3 + 6 = 18$$

Replacing the first columns in Δ by the constant terms to obtain Δ_1, we get

$$\Delta_1 = \begin{vmatrix} -1 & -1 & 2 \\ 5 & 1 & -1 \\ 4 & 2 & 1 \end{vmatrix}$$

Using the first row to obtain the value of Δ_1 as follows:

$$\Delta_1 = -1\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 5 & -1 \\ 4 & 1 \end{vmatrix} + 2\begin{vmatrix} 5 & 1 \\ 4 & 2 \end{vmatrix}$$
$$= -1\,(1+2) + 1\,(5+4) + 2\,(10-4)$$
$$= -1\,(3) + 1(9) + 2\,(6)$$
$$= -3 + 9 + 12 = 18$$

Replacing the second columns in Δ by the constant terms to obtain Δ_2, we get

$$\Delta_2 = \begin{vmatrix} 3 & -1 & 2 \\ 2 & 5 & -1 \\ 1 & 4 & 1 \end{vmatrix}$$

Using the first row to obtain the value of Δ_2 as follows:

$$\Delta_2 = 3\begin{vmatrix} 5 & -1 \\ 4 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix}$$

$= 3\,(5 + 4) + 1\,(2 + 1) + 2\,(8 - 5)$
$= 3\,(9) + 1\,(3) + 2\,(3)$
$= 27 + 3 + + 6 = 36$

Replacing the third columns in Δ by the constant terms to obtain Δ_3, we get

$$\Delta_3 = \begin{vmatrix} 3 & -1 & -1 \\ 2 & 1 & 5 \\ 1 & 2 & 4 \end{vmatrix}$$

Using the first row to obtain the value of Δ_3 as follows:

$$\Delta_3 = 3\begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} + (-1)\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

$= 3\,(4 - 10) + 1\,(8 - 5) - 1\,(4 - 1)$
$= 3\,(-6) + 1\,(3) - 1\,(3)$
$= -18 + 3 - 3 = -18$

Now, we can find the values for x, y and z as follows

$$x = \frac{\Delta_1}{\Delta} = \frac{18}{18} = 1\ ,\quad y = \frac{\Delta_2}{\Delta} = \frac{36}{18} = 2 \text{ and } z = \frac{\Delta_3}{\Delta} = -\frac{18}{18} = -1$$

Notes:

(i) If $\Delta = 0$

The system of equation has a unique solution.

(ii) If $\Delta = 0$ *and* $\Delta_1 = \Delta_2 = \Delta_3 = 0$

The system of equation has an infinite number of solutions.

(iii) If $\Delta = 0$ *and* $\Delta_1 \neq 0$ *or* $\Delta_2 \neq 0$ *or* $\Delta_3 \neq 0$

The system of equation has no solution.

Exercise Three

1. Find the value of each of the following determinants:

(a) $\begin{vmatrix} 1 & -2 \\ 3 & -4 \end{vmatrix}$

(b) $\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}$

(c) $\begin{vmatrix} -5 & 1 \\ \cdot 2 & -3 \end{vmatrix}$

(d) $\begin{vmatrix} -1 & 6 & -3 \\ 2 & -5 & 1 \\ -2 & 3 & 4 \end{vmatrix}$

(e) $\begin{vmatrix} 5 & 3 & -1 \\ 4 & 7 & 1 \\ 2 & -2 & 7 \end{vmatrix}$

2. Solve the following systems of equations using Cramer's Rule:

(a) $2x + 2y = 1$

$2x - \ \ y = 3$

(b) $2x - 5y = 8$

$3y + 7x = -13$

(c) $4x + 5y - 14 = 0$

$3y = 7 - x$

3. Solve the following systems of equations using Cramer's Rule:

(a) $2x - y + 3z = 4$

$x + 3y - z = 5$

$6x - 3y + 9z = 10$

(b) $4x + 2y - 6z = 7$

$3x - y + 2z = 12$

$6x + 3y - 9z = 10$

(c) $x + 2y - z = 2$

$2x - 3y + 4z = -4$

$3x + y + z = 0$