Chapter Four
Linear Transformation

The general concept of linear transformation from a vector space into a vector space is A mapping L from a vector space V into a vector space W, fromR^n to R^m.

A linear transformation is a function ***L*** that maps a vector space ***V*** into another vector space ***W***:

Definition:

Let V and W be two vector spaces. A function $L : V \to W$ is called a linear transformation from V to W if it satisfies the following two conditions, for all vectors u, v in V and for all scalars c.

$a. L(u + v) = L(u) + L(v)$ for any two vectors u and v in V.

$b. L(cu) = cL(u)$ for any scalar c and vector u in V.

If V and W are the same, we call a linear transformation from V to V a linear operator.

Theorem1:

A function $L : V \to W$ is a linear transformation if and only if for allvectorsv_1, v_2 in V and for all scalarsk_1, k_2 we have

$$L(k_1v_1 + k_2v_2) = k_1L(v_1) + k_2L(v_2)$$

Theorem 2:

Basic properties of linear transformations:

If L is a linear transformation then

a) $L(0) = 0$

b) $L(-v) = -L(v)$

c) $L(u - v) = L(u) - L(v)$

Zero transformation

$$L : V \rightarrow W L(v) = 0, \forall \text{v} \in \text{V}$$

Identity transformation

$$L : V \rightarrow V L(v) = v, \forall \text{v} \in \text{V}$$

Theorem 3:

If $L : V \rightarrow W$ is a linear transformation, $S = \{v_1, v_2, \dots v_n\}$ is a basis in V, then for any vector v in V we can evaluate $L(v)$ by

$$L(v) = k_1 L(v_1) + k_2 L(v_2) + \cdots + k_n L(v_n) \text{ , where}$$

$$v = k_1 v_1 + k_2 v_2 + \cdots + k_n v_n.$$

Example 1:

Mapping a vector space from R^n to R^m can be expressed as an $m \times n$ matrix.

Thus the transformation can be written as

$$L(x_1, x_2, x_3) = (x_1 + 2x_2, 3x_2 + 4x_3)$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Example 2:

Define L: $R^3 \to R^2$ by $L(x_1, x_2, x_3) = (x_3 - x_1, x_1 + x_2)$

a. Compute $L(e_1)$, $L(e_2)$ and $L(e_3)$

b. Show L is a linear transformation.

c. Show $L(x_1, x_2, x_3) = x_1 L(e_1) + x_2 L(e_2) + x_3 L(e_3)$

Solution

a. $L(e_1) = L(1,0,0) = (-1,1)$, $L(e_2) = L(0,1,0) = (0,1)$

$L(e_3) = L(0,0,1) = (1,0)$.

b. $L(x + y) = L(x_1 + y_1, x_2 + y_2, x_3 + y_3)$

$= ((x_3 + y_3) - (x_1 + y_1), (x_1 + y_1) + (x_2 + y_2))$

$= (x_3 - x_1, x_1 + x_2) + (y_3 - y_1, y_1 + y_2$

$= L(x) + L(y)$

$L(cx) = L(cx_1, cx_2, cx_3) = (cx_3 - cx_1, cx_1 + cx_2)$

$= c(x_3 - x_1, x_1 + x_2) = c\, L(x)$

Thus L satisfies conditions 1 and 2 of Definition, and it is a linear transformation.

$$\begin{aligned} \text{c. } L(x_1, x_2, x_3) &= L(x_1e_1 + x_2e_2 + x_3e_3) \\ &= L(x_1e_1) + L(x_2e_2) + L(x_3e_3) \\ &= x_1L(e_1) + x_2L(e_2) + x_3L(e_3) \end{aligned}$$

Notice that c implies that once L (e_k), k = 1,2,3, are known, the fact that L is a linear transformation completely determines $L(x)$ for any vector x in R^3.

We collect a few facts about linear transformations in the next theorem.

Theorem 4:

Let L be a linear transformation from a vector space V into a vector space W. Then

1. $L(0) = 0$
2. $L(-x) = -L(x)$
3. $L(\sum_{k=1}^{N} a_k\, x_k) = \sum_{k=1}^{N} a_k L(\, x_k)$

Proof.

1. Let x be any vector in V. Then $L(x) = L(x + 0) = L(x) + L(0)$Adding $-L(x)$to both sides, we have $(0) = 0$, where the zero vector on the left-hand side is in V while the zero vector on the right-hand side is in W.
2. $0 = L(0) = L(x - x) = L(x) + L(-x)$,Thus

$$L(-x) = -L(x)$$

3. We show that this formula is true for n = 3 and leave the details of an induction argument to the reader.

$$L(a_1x_1 + a_2x_2 + a_3x_3) = L(a_1x_1 + a_2x_2) + L(a_3x_3)$$
$$= L(a_1x_1 + L(a_2x_2) + L(a_3x_3)$$
$$= a_1L(x_1) + a_2L(x_2) + a_3L(x_3)$$

Example 3:

Let $L: R^2 \to R^2$ be a linear transformation. Suppose we know that

$$L(1,1) = (0,2) \text{ and } L(1,-1) = (2,0)$$

a. Compute $L(1,4)$

b. Compute $L(-2,1)$

Solution:

a. $(1,4) = a(1,1) + b(1,-1)$

$$= (a,a) + (b,-b) = (a+b, a-b)$$

$a + b = 1$, $a - b = 4$, $a = 2.5$, $b = -1.5$

$(1,4) = 2.5(1,1) - 1.5(1,-1)$, so

$$L(1,4) = 2.5\, L(1,1) - 1.5\, L(1,-1)$$
$$= 2.5\,(0,2) - 1.5(2,0) = (-3,5)$$

b. $(-2,1) = a(1,1) + b(1,-1)$

solving $(-2,1) = -.5(1,1) - 1.5(1,-1)$

so,

$$L(-2,1) = -.5\, L(1,1) - 1.5\, L(1,-1)$$
$$= -.5\,(0,2) - 1.5\,(2,0) = (-3,-1)$$

Example 4:

Let $L: R^3 \rightarrow R^4$ be a linear transformation. Suppose we know that $L(1,0,1) = (-1,1,0,2)$,

$L(0,1,1) = (0,6,-2,0)$, and $L(-1,1,1) = (4,-2,1,0)$

Determine $L(1,2,-1)$

Solution:

The trick is to realize that the three vectors for which we know L form a basis F of R^3. Thus, all we need to do is find the coordinates of (1, 2, −1) with respect to F, and then use 3 of Theorem 4.

The change of basis matrix P below is such that

$$[x]^T F = P[x]^T S.$$

$$P = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

Using this matrix to find the coordinates of (1,2,−1) with respect to F, we have

$$[1,2,-1]^T F = P\,[1,2,-1]^T S$$

$$= \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \\ -4 \end{bmatrix}$$

Thus

$$(1,2,-1) = -3\,(1,0,1) + 6(0,1,1) + (-4)\,(-1,1,1)$$

and

$$L(1,2,-1) = -3\,L(1,0,1) + 6\,L(0,1,1) +$$

$$(-4)L(-1,1,1) = -3(-1,1,0,2) + 6(0,6,-2,0) +$$
$$(-4)(4,-2,1,0) = (-13,41,-16,-6)$$

A standard method of defining a linear transformation from R^n to R^m is by matrix multiplication. Thus, if $X = (x_1, x_2, \dots x_n)$ is any vector in R^n and $A = [\,a_{ij}]$ is an $m \times n$ matrix, define $L(x) = Ax^T$. Then $L(x)$ is an $m \times 1$matrix that we think of as a vector in R^m. The various properties of matrix multiplication that were proved in Theorem are just the statements that L is a linear transformation from R^n to R^m.

Example 5:

Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$. If L is the linear transformation defined by A, compute the following:

a. $L\,(x_1, x_2, x_3)$
b. L (1, 0, 0), L (0,1,0), L (0,0,1)

Solution

$$L\,(x_1, x_2, x_3) = \begin{bmatrix} 1 & -1 & 2 \\ 4 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 + x_3 \\ 4x_1 + x_2 + 3x_3 \end{bmatrix}$$

$L(1,0,0) = (1,4)^T L(0,1,0) = (-1,1)^T L(0,0,1) = (2,3)^T$

The reader should note that L(e1) is the first column of A, L(e2) is the second column of A, and L(e 3) is the third column.

In general, if A is an m×n matrix and L(x) = Ax, then $L(e_k)$ will be the k^{th} column of the matrix A.

Example 6:

Let $A = \begin{bmatrix} -1 & 2 & 1 & 3 & 4 \\ 0 & 0 & 2 & -1 & 0 \end{bmatrix}$,

Let L: $R^5 \rightarrow R^2$ be the linear transformation

$$T(x) = Ax.$$

a) Compute L (1, 0, −1, 3, 0).

b) Compute preimage, under L, of (−1, 8).

Solution:

a) $L(1, 0, -1, 3, 0) = \begin{bmatrix} -1 & 2 & 1 & 3 & 4 \\ 0 & 0 & 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \end{bmatrix}$

b) Compute preimage, under L, of (−1, 8).

The preimage consists of the solutions of the linear system

$$\begin{bmatrix} -1 & 2 & 1 & 3 & 4 \\ 0 & 0 & 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \end{bmatrix}$$

The augmented matrix of this system

is $\begin{bmatrix} -1 & 2 & 1 & 3 & 4 & -1 \\ 0 & 0 & 2 & -1 & 0 & 8 \end{bmatrix}$

The Gauss-Jordan form is

$$\begin{bmatrix} 1 & -2 & 0 & -3.5 & -4 & 5 \\ 0 & 0 & 1 & -.5 & 0 & 4 \end{bmatrix}$$

We use parameters $x_2 = t$, $x_4 = s$, $x_5 = u$ and the solutions are given by

$x_1 = 5 + 2t + 3.5s + 4u$, $x_2 = t$, $x_3 = 4 + .5s$, $x_4 = s$, $x_5 = u$

So, the preimage

$L^{-1}(-1, 8) = \{(5+2t+3.5s+4u, t, 4+.5s, s, u): t, s, u \in R\}$.

Kernel and range of a linear transformation

Definition:

Let $L : V \to W$ is a linear transformation.

The set of all vectors v in V for which $L(v) = 0$,is called the kernel of L.

We denote the kernel of L by ker (L).

The set of all outputs (images) $L(v)$ of vectors in V via the transformation L is called the range of L.

We denote the range of L by R (L).

The range space of a transformation L: X → Y is the set of all vectors that can be reached by the transformation$R(L) = \{y = L(x): x \in X\}$

The null space of the transformation is the set of all vectors in X that are transformed to the null vector in Y.

$N(L) = \{L(x) = 0 : x \in X\}$

Theorem: If $L : V \to W$ is a linear transformation, then ker(L) is a subspace of V, while R(L) is a subspace of W.

Definition:

If V and W are finite dimensional vector spaces and $L : V \to W$is a linear transformation, then we call

dim ker (L) = nullity of L   dim R(L) = rank of L

Theorem 5:

If V and W are finite dimensional vector spaces and $L : V \to W$is a linear transformation, then rank (L) + nullity (L) = dim (V)

One-to-one and onto functions

Definition (one-to-one function):

A function f: X → Y is called one-to-one if to distinct inputs it assigns distinct outputs. More precisely, f is 1-1 means: if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$. This is logically equivalent to saying that if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

Definition (onto function):

A function $f: X \to Y$ is called onto if every element in Y is an output off. More precisely, f is onto if for every y in Y there is at least one x in X such that $f(x) = y$. Linear transformations are functions, so being one-to-one or onto applies (makes sense) for them as well.

Definition:

Let $L: V \to W$. The kernel of L is the set of vectors x xx in V for which $L(x) = 0$. Letting ker (L) represent the kernel of L, we have ker $(L) = \{x: L(x) = 0\}$.

Example 7:

Let $A = \begin{bmatrix} 2 & -6 & 4 \\ 1 & -1 & 2 \end{bmatrix}$ be the matrix representation of L.

Find the kernel K of this linear transformation.

Solution: Since A is a 2×3 matrix, $A: R^3 \to R^2$. We are asked to find those $\boldsymbol{x} = (x_1, \ x_2, \ x_3)$ such that

$$Ax = \begin{bmatrix} 2 & -6 & 4 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - 6x_2 + 4x_3 \\ x_1 - x_2 + 2x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, x is in the kernel of A if and only if

$2x_1 - 6x_2 + 4x_3 = 0, x_1 - x_2 + 2x_3 = 0$.

Hence K = $\{(x_1, x_2, x_3):\ x_1 + 2x_3 = 0 = x_2\}$.

The kernel is just the solution set of a homogeneous system of linear equations.

Example 8:

Consider the following system of equations:

$$\begin{aligned} -x_1 + 2x_2 + \qquad\quad 3x_4 &= b_1 \\ 2x_1 + 3x_2 + 7x_3 + 8x_4 &= b_2 \\ 4x_1 - 2x_2 + 6x_3 \qquad\quad &= b_3 \end{aligned}$$

Find the kernel and range of the coefficient matrix of the above system of equations.

Solution:

The coefficient matrix A equals

$A = \begin{bmatrix} -1 & 2 & 0 & 3 \\ 2 & 3 & 7 & 8 \\ 4 & -2 & 6 & 0 \end{bmatrix}$, and is row equivalent to the matrix $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Thus, x is a solution to the homogeneous system, i.e., x is in ker (A) if and only if

$$x_1 = -2x_3 - x_4 \quad and \quad x_2 = -x_3 - 2x_4$$

Thus,

Ker (A)= $\{(x_1, x_2, x_3, x_4): x_1 = -2x_3 - x_4, x_2 = -x_3 - 2x_4\}$

A basis for ker (A) is {(−2, −1, 1, 0), (−1, −2, 0, 1)}.

Thus, dim (ker (A)) = 2.

The augmented matrix of

$$\begin{bmatrix} -1 & 2 & 0 & 3 & b_1 \\ 2 & 3 & 7 & 8 & b_2 \\ 4 & -2 & 6 & 0 & b_3 \end{bmatrix},$$

is row equivalent to

$$\begin{bmatrix} -1 & 2 & 0 & 3 & b_1 \\ 0 & 1 & 1 & 2 & (2b_1 + b_2)/7 \\ 0 & 0 & 0 & 0 & (26b_1 - 2\,b_2 + 7\,b_3)/14 \end{bmatrix}.$$

has a solution if and only if $26b_1 - 2\,b_2 + 7\,b_3 = 0$

Thus, Rg (A) = $\{ (b_1, b_2, b_3) : 26b_1 - 2\,b_2 + 7\,b_3 = 0\}$.

A basis for Rg (A) is {(1, 13, 0), (7, 0, −26)} and

dim (Rg (A)) = 2.

Example 9:

Define $L: R^3 \to R^2$ be a linear transformation such that

$L(1,0,0) = (2,-1,4)$

$L(0,1,0) = (1,5,-2)$

$L(0,0,1) = (0,3,1)$.

Find $L(2,3,-2)$

Solution

$(2,3,-2) = 2(1,0,0) + 3(0,1,0) - 2(0,0,1)$

$L(2,3,-2) = 2L(1,0,0) + 3L(0,1,0) - 2L(0,0,1)$

$L(2,3,-2) = 2(2,-1,4) + 3(1,5,-2) - 2(0,3,1)$

$= (7,7,0)$

Exercise Four

1. Let $L(x_1, x_2, x_3) = x_1 - x_2 + x_3$.

a. Show that L is a linear transformation from R^3 to R.

b. Find a 1×3 matrix A such that $L(x) = Ax^T$ for every x in R^3.

c. Compute $L(e_k)$ for k = 1, 2, 3.

d. Find a basis for the subspace $K = \{x: Ax^T = 0\}$.

2. Let L be a linear transformation from R^3 to R^2 such that $L(e_1) = (-1,6)$, $L(e_2) = (0,2)$, $L(e_3) = (8,1)$.

a. $L(1,2,-6) = ?$

b. $L(x_1, x_2, x_3) = ?$

c. Find a matrix A such that $L(x) = Ax^T$.

3. Let $L(x_1, x_2) = (3x_1 + 6x_2, -2x_1 + x_2)$

a. Find the matrix representation of L using the standard bases.

b. Find the matrix representation of L using the basis F = $\{(-4, 1), (2, 3)\}$.

4- Find the matrix representation of $L: R^3 \to R^4$,

$$L\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3x + 2y + z \\ x + y + z \\ x - 3y \\ 2x + 3y + z \end{bmatrix}$$

5- Define the linear transformation

$$L: R^3 \to R^2,\ L\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x - y + 5z \\ -4x + 2y - 10z \end{bmatrix}$$

Compute the preimages, $L^{-1}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $L^{-1}\begin{bmatrix} 4 \\ -8 \end{bmatrix}$

6. Let$L(v_1, v_2, v_3) = (2v_1 + v_2, 2v_2 - 3v_1, v_1 - v_3)$

a. Compute $L(-4, 5, 1)$.

b. Compute the preimage of w = (4, 1, −1).

7. Let T: L: $R^3 \to R^3$ be a linear transformation such that

$L(1,0,0) = (2,4,-1),\ L(0,1,0) = (1,3,-2)L(0,0,1)$
$= (0,-2,2).$

Compute$L(-2,4,-1)$.

8. Let L $(x, y, z) = (5x - 3y + z, 4y + 2z, 5x + 3y)$.

What is the standard matrix of L?

9. Let L $(x, y, z) = (2x + y, 3y - z)$.

Write down the standard matrix of L and use it to find $L(0, 1, -1)$

10. If $L: R^2 \to R^3$ satisfies $L\left(\begin{bmatrix}2\\3\end{bmatrix}\right) = \begin{bmatrix}2\\2\\1\end{bmatrix}$ and $L\left(\begin{bmatrix}3\\4\end{bmatrix}\right) = \begin{bmatrix}-1\\0\\2\end{bmatrix}$, find the matrix representation of L.

11. Let $L(x, y, z) = (3x - 2y + z, 2x - 3y, y - 4z)$.

a. Write down the standard matrix of L.

b. Compute L (2, −1, −1).

12. For each of the matrices below determine the range and kernel

$$a. \begin{bmatrix}1 & 0 & -1\\0 & 0 & 4\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix} \qquad b. \begin{bmatrix}1 & 2 & -1 & 3\\1 & -1 & 1 & -1\\0 & 1 & 0 & 1\\1 & 0 & 1 & 0\\1 & 1 & 0 & 0\end{bmatrix}$$

13. Find the kernel of

$$A = \begin{bmatrix}1 & 2 & 5 & 0\\1 & -1 & -4 & 0\\-1 & 0 & 1 & 1\end{bmatrix}$$

14. Consider the following system of equations:

$$\begin{aligned}-x_1 + 2x_2 + \qquad\quad 3x_4 &= b_1\\2x_1 + 3x_2 + 7x_3 + 8x_4 &= b_2\\4x_1 - 2x_2 + 6x_3 \qquad\quad &= b_3\end{aligned}$$

Find the kernel and range of the coefficient matrix of the above system of equations.