1st order example from prev. exams: $y' + \frac{1}{2x} y = \frac{1}{\sqrt{x}}$, $y(1) = 5$

1- Seperable? No

2- homogeneous $S(x) = 0 \rightarrow y' + \frac{1}{2x} y = 0$

$$\frac{dy}{dx} = \frac{-y}{2x}$$

Constant solution at $\boxed{y=0}$

$$\Rightarrow \int \frac{dy}{y} = \frac{-1}{2} \int \frac{dx}{x} \Rightarrow \ln|y| = \frac{-1}{2} \ln|x| + C \quad , \quad C \in \mathbb{R}$$

$$\Rightarrow y = \frac{e^{c}}{\sqrt{x}}$$

$y = K x^{-\frac{1}{2}}$, $K > 0 \Rightarrow y = K x^{-\frac{1}{2}}$, $K \geq 0$ (including the constant solution)

3- Vary the constant of homogeneous solution

$$\Rightarrow y = K(x)\, x^{-\frac{1}{2}}$$

$$y' = K'(x)\, x^{-\frac{1}{2}} - \frac{1}{2x^{3/2}} K(x)$$

$$y'' =$$

Plug in the equation & find $K(x) \Rightarrow$ $K'(x)\, x^{-\frac{1}{2}} \cancel{- \frac{1}{2x^{3/2}} K(x)} \cancel{+ \frac{1}{2x} \cdot \frac{1}{\sqrt{x}} K(x)} = \frac{1}{\sqrt{x}}$

4- I.V.P (Initial Value Problem)

$$\frac{d}{dx} K(x) = 1 \Rightarrow K(x) = x + C_1$$

$$\therefore y = \frac{x + C_1}{\sqrt{x}} \quad , y(1) = 5 \Rightarrow y = \frac{x+5}{\sqrt{x}}$$

5- Check if there is a constant solution

Example: Differential equations, Part 2

$ay'' + by' + cy = S(x) \longrightarrow$ Inhomogeneous

constants

General solution:-

$y(x) = \underbrace{y_h(x)}_{\text{homo}} + \underbrace{y_p(x)}_{\text{Inhomo (Particular solution)}}$

when $S(x) = 0 \rightarrow$ homogeneous

$ay'' + by' + cy = 0$

$e^{rx}(ar^2 + br + c) = 0$

$(rx) \neq -\infty \Leftrightarrow e^{rx} \neq 0$

$\therefore ar^2 + br + c = 0$

Let $y = e^{rx}$, $r \in R$

$y' = re^{rx}$

$y'' = r^2 e^{rx}$

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Case 3 ($r_{1,2} = \alpha \pm i\beta$)

$y_h = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$

$C_1, C_2 \in R$

Case 2 ($r_1 = r_2$)

$y_1(x) = e^{rx}$, $y_2(x) = xe^{rx}$

$y_h = C_1 e^{rx} + C_2 x e^{rx}$ $C_1, C_2 \in R$

Case 1 ($r_1 \neq r_2$)

$y_1(x) = e^{r_1 x}$, $y_2(x) = e^{r_2 x}$

linearly independent solutions

$y_h = C_1 \underbrace{e^{r_1 x}}_{y_1(x)} + C_2 \underbrace{e^{r_2 x}}_{y_2(x)}$ $C_1, C_2 \in R$

Step 1: Find the homogeneous solution ✓

Step 2: Find one solution y_p of the inhomo ODE:

a) Variation of the parameters (same as in 1st-order ODE / beyond our scope)

b) Undetermined coefficients

$f(x)$	Assumed y_p
Cons. (5, 7, −3, ⋯)	A
$3x - 2$	$Ax + B$
$7x^2$ $3x^2 + 2x - 1$	$Ax^2 + Bx + C$
$5e^{\alpha x}$	$Ae^{\alpha x}$
$3xe^{\alpha x}$	$(Ax + B)e^{\alpha x}$
$2\sin(\alpha x)$	$A\sin(\alpha x) + B\cos(\alpha x)$
$5\cos(\alpha x)$	$A\sin(\alpha x) + B\cos(\alpha x)$

num.

Example 1: $f(x) = x\cos 3x$

Let

$y_p = (Ax + B)(C\cos(3x) + D\sin(3x))$

Example 2: $f(x) = xe^x + \cos 2x$

Let

$y_{p_1} = (Ax + B)e^x \xrightarrow{\text{Solve}} y_{p_1}'' + y_{p_1}' + y_{p_1} = xe^x$

$y_{p_2} = (C\cos 2x + D\sin 2x) \xrightarrow{\text{Solve}} y_{p_2}'' + y_{p_2}' + y_{p_2} = \cos 2x$, then $y_p = y_{p_1} + y_{p_2}$