Name: ______________________________

Problem 5: (11 points) Find

$$\int_{-\infty}^{\boxed{\text{infin}}\text{ity}} \frac{e^x}{e^{2x}+1}\,dx = \int_{-\infty}^{0} \frac{e^x}{e^{2x}+1}\,dx + \int_{0}^{\infty} \frac{e^x}{e^{2x}+1}\,dx$$

$$\Rightarrow \lim_{t\to-\infty} \int_{t}^{0} \frac{e^x}{1+(e^x)^2}\,dx + \lim_{n\to\infty} \int_{0}^{n} \frac{e^x}{1+(e^x)^2}\,dx$$

$$= \lim_{t\to-\infty} \Big[\arctan(e^x)\Big]_{t}^{0} + \lim_{n\to\infty} \Big[\arctan(e^x)\Big]_{0}^{n}$$

$$= \lim_{t\to-\infty} \Big[\arctan(1) - \arctan(e^t)\Big] + \lim_{n\to\infty} \Big[\arctan(e^n) - \arctan(1)\Big]$$

$$= \cancel{\arctan(1)} - \cancel{\arctan(1)} + \arctan\left(\lim_{n\to\infty} e^n\right) - \arctan\left(\lim_{t\to-\infty} e^t\right)$$

$$= \arctan(\infty) - \arctan(0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

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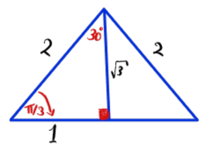
Problem 6: (8 points) Find the directional derivative of

$$f(x,y) = 4x^3 - 3xy^2 + \boxed{4}$$

at the point $(1,2)$ in the direction that has an angle of $\pi/3$ with the positive x-axis.

$$D_u f = (f_x \quad f_y)\cdot \vec{u} \qquad \vec{u} = \begin{pmatrix}\cos(\pi/3)\\ \sin(\pi/3)\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}$$



$$f_x(x,y) = 12x^2 - 3y^2 \Rightarrow f_x(1,2) = 12(1)^2 - 3(2)^2 = 0$$

$$f_y(x,y) = -6xy \Rightarrow f_y(1,2) = -6(1)(2) = -12$$

$$D_u f = (0 \quad -12)\begin{pmatrix}\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix} = \frac{1}{2}(0) + \frac{\sqrt{3}}{2}(-12) = -6\sqrt{3}$$

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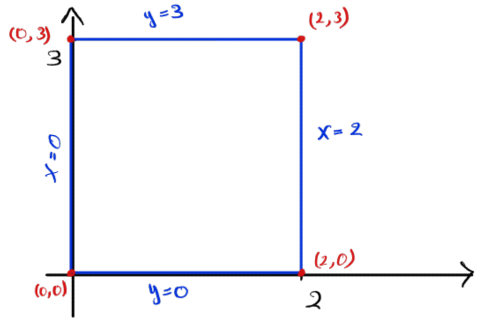
Problem 7: (20 Points) Find the absolute maxima and absolute minima of

$$f(x,y) = x^2 + 2y^2 - 2x - 4y + \boxed{2}$$

on the domain

$$D = \{(x,y) \mid 0 \le x \le 2\,, \quad 0 \le y \le 3\}\,.$$



$f_x = 2x - 2$

$f_x = 0 \Longrightarrow \boxed{x=1}$

$f_y = 4y - 4 \quad , f_y = 0 \Longrightarrow \boxed{y=1}$

$P(1,1)$ critical point

at $x=0$ $f(0,y) = 2y^2 - 4y + 2$

$f'(0,y) = 4y - 4 = 0 \Rightarrow y = 1 \Rightarrow (0,1)$

at $y=0$, $f(x,0) = x^2 - 2x + 2$

$f'(x,0) = 0 \Rightarrow x = 1 \Rightarrow (1,0)$

at $x=2$, $f(2,y) = 4 + 2y^2 - 4 - 4y + 2 = 2y^2 - 4y + 2$

$f'(2,y) = 0 \Rightarrow y = 1 \Rightarrow (2,1)$

at $y=3$, $f(x,3) = x^2 + 18 - 2x - 12 + 2 = x^2 - 2x + 8$

$f'(x,3) = 0 \Rightarrow x = 1 \Rightarrow (1,3)$

x	y	$f(x,y)$
1	1	−1
0	1	0
1	0	1
2	1	0
1	3	7
0	0	2
2	0	2
0	3	8
2	3	8

We have global minimum at $(1,1)$

& two global maxima at $(0,3)$ & $(2,3)$

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Problem 8: (4+4 + [Bonus 5] Points) Let N be the random variable of work-related accidents at a company in one year with $P(N = n) = C \cdot (\boxed{\frac{1}{5}})^n$ for $n = 0, 1, 2, \ldots$.

a) Compute C.

b) Find $P(2 \le W \le 4)$.

c) [Bonus] Compute the expected value. Hint: Differentiate the geometric series formula

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z}\,, \quad \text{for} \quad -1 < z < 1\,.$$

a) Since our random variable is discrete, we know that the sum of all our probabilities must equal one

$$\sum_{n=0}^{\infty} P(N=n) = \sum_{n=0}^{\infty} C \cdot \left(\tfrac{1}{5}\right)^n = 1 = \underbrace{\frac{C}{1-\frac{1}{5}}}_{\text{for a geometric series with } |r| = \frac{1}{5} < 1}$$

$$1\left(1-\tfrac{1}{5}\right) = C \Rightarrow \boxed{C = \tfrac{4}{5}}$$

b) $P(2 \le W \le 4) = P(2) + P(3) + P(4)$

$$= \frac{4}{5}\sum_{n=2}^{4}\left(\frac{1}{5}\right)^n = \frac{4}{5}\left(\frac{1}{25}+\frac{1}{125}+\frac{1}{625}\right) = \frac{4}{5}\left(\frac{31}{625}\right) = \frac{124}{3125}$$