A probability model requires at least two outcomes.
↳ if only one → it' deterministic model.

These outcomes collected in a set called sample space (S)

Example: * Coin Toss

Possible sample spaces

S = { Heads, edges, tails }
or
S = { Heads, Tails }

your decision.

* Triple Coin Toss

All details (Including order)

S = { HHH, HHT, HTH, THH, TTH, THT, HTT, TTT }

or just the number of "Heads"

S = { 0, 1, 2, 3 }

Model must fit problem!!!

* Exam Grade

S = { Pass, fail }
or
S = { 1,0 ; 1,3 ; --- ; 4,0 ; 5,0 }

Sometimes outcomes are combined into an Event.
→ subset of the sample space

German Grades:

Sehr Gut = { 1,0 ; 1,3 }
Gut = { 1,7 ; 2,0 ; 2,3 }
⋮
Mangelhaft = { 5,0 }

Compound Events } Contains more than one outcome

→ Elementary Event } Contains exactly one outcome

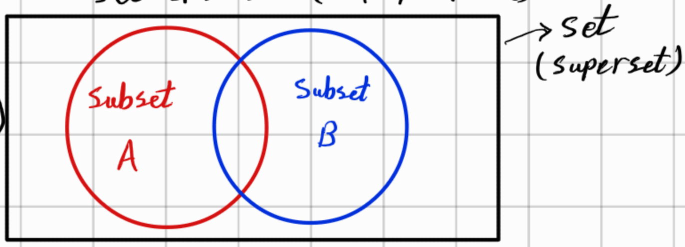
I) Special Events: { } (empty set), also ∅

is called $\boxed{\text{impossible event}}$ because at least one outcome must occur.

II) Special Events: S (whole sample space)

is called $\boxed{\text{Certain event}}$ because whatever outcome occures must belong to the sample space.

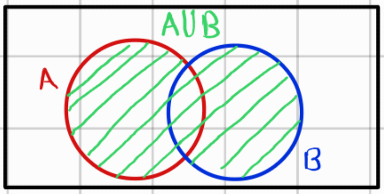
Sets & Subsets (Samplespace & events)



* John venn (venn diagrams)

We "calculate" with events (subsets) using the following operations

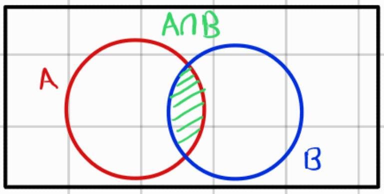
- **Union**: Contains all elements that belonges to A or B.



$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

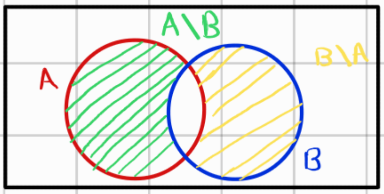
Symbol $\cup$ as in union

- **Intersection**: of A & B Contains all elements that belong to A & B



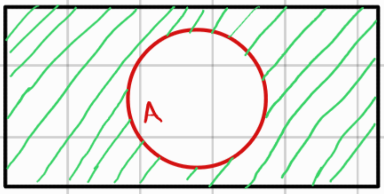
$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

- **Difference**: "A Minus B" contains all elements that belong to A but do not belong to B.



$A \setminus B = A - B = \{x \mid x \in A,\ x \notin B\}$

- **Complement**: of a subset contains all elements from the "superset" that do not belong to the subset.



$\overline{A} = A' = S \setminus A$

$= \{x \in S \mid x \notin A\}$

"Super set" (Sample space)

Two events (subsets) are Disjoint or Mutually exclusive if they do not have a common outcome (elements)

A & B Disjoint $\Longleftrightarrow A \cap B = \{\}$

How do we set up a probability model?

(Step 1: Outcomes, Sample space, and events) ✓

$\boxed{\text{Step 2: Assign probabilities}}$

Answer unclear yet!

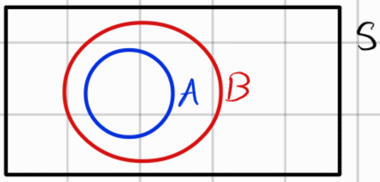
Kolmogorov Axioms

A Function that assigns to each event A of a given sample space, a number $0 \leq P(A) \leq 1$

Can be considered a $\boxed{\text{probability distribution}}$ if:

(i) $A \subseteq B \implies P(A) \leq P(B)$

If the event B contains all the outcomes of event A, then B is at least as likely to occur as A.



(ii) $P(\{\}) = 0$; $P(S) = 1$

(iii) If $A_1, A_2, \cdots$ are Pairwise Disjoint events, $A_i \cap A_j = \{\}$, $i \neq j$,

Then, the probabilities add $P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$

[Finite sum or infinite sum]

Simple consequences

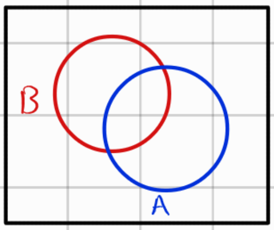
Important rules:

Let A be any event. Then $A \cap \bar{A} = \{\}$ and $A \cup \bar{A} = S$.

Therefore

$$1 = P(S) = P(A \cup \bar{A}) = P(A) + P(\bar{A}) \Longrightarrow P(\bar{A}) = 1 - P(A)$$

Difference of events



$$P(B) = P(P(B \setminus A) \cup P(A \cap B))$$

$$\Longrightarrow \boxed{P(B \setminus A) = P(B) - P(A \cap B)}$$

Probability of unions (not necessary disjoint)

Trick: use Disjoint parts A and $B \setminus A$

$\Longrightarrow P(A \cup B) = P(A \cup (B \setminus A))$

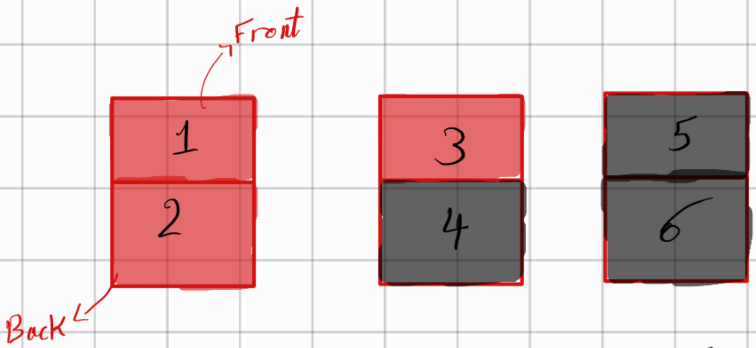
$P(A \cup B) = P(A) + P(B \setminus A)$

$\boxed{P(A \cup B) = PA + P(B) - P(A \cap B)}$ Note: When A & B are disjoint $P(A \cap B) = 0$

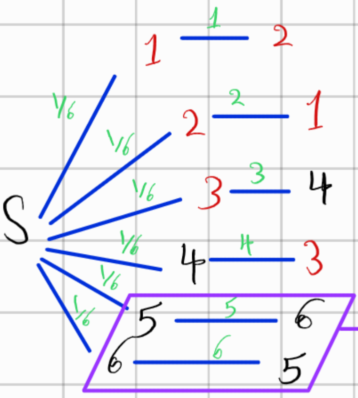
How do we assign concrete probabilities ?

- Symmetry Arguments. (Laplace Assumption)
  There is no reason to believe that the outcome of having Head is more likely to occur than having a Tail when Tossing a Fair Coin.

- Relative Frequencies of past data.
- Subjective choice.



Sample space $S = \{1, 2, 3, 4, 5, 6\}$



1. $P(\text{Front} = \text{Black}) = \frac{3}{6} = 0.5 = 50\%$

2. $P(\text{Back} = \text{Black}) = \frac{3}{6} = 0.5 = 50\%$

3. $P(\text{Back} = \text{Black} \mid \text{Front} = \text{Black})$ = the probability of having a Black back card given that the front is Black.

3) $\longrightarrow P(B = B \mid F = B) = \frac{P(B = B \cap F = B)}{P(F = B)} = \frac{\frac{2}{6}}{\frac{3}{6}} = \frac{4}{6} = \frac{2}{3}$
$= 66.7\%$

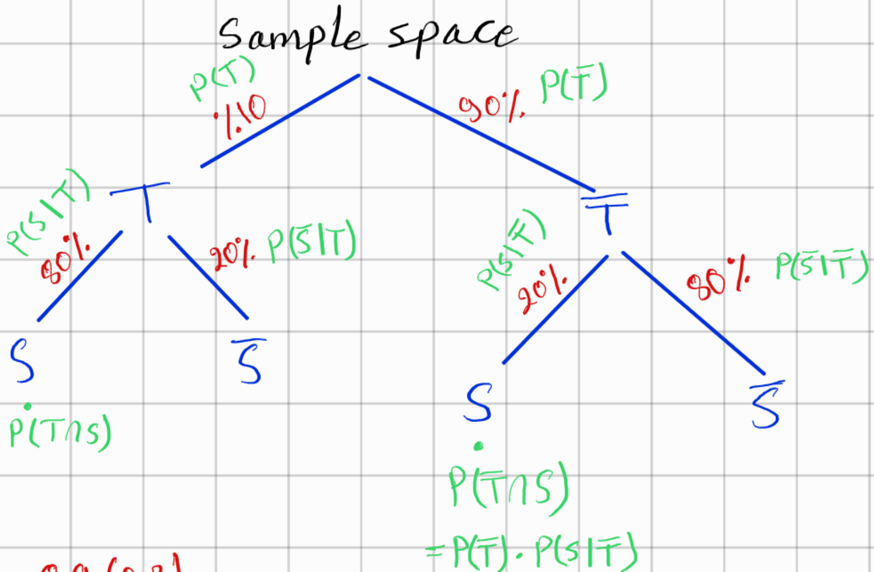
Conditional probability: $\boxed{P(A \mid B) = \frac{P(A \cap B)}{P(B)}}$

Where $P(A)$ & $P(B)$ are unconditional probability

FICTITIOUS EXAMPLE THE MANAGEMENT OF A COMPANY KNOWS FROM HISTORICAL DATA THAT 10% OF THE EMPLOYEES ARE STEALING FROM THE COMPANY. TO FIND OUT WHO ARE THE THIEVES THEY PERFORM A LIE DETECTOR TEST. ACCORDING TO THE MANUAL THE TEST IS 80% RELIABLE. WHAT IS THE PROBABILITY THAT AN HONEST EMPLOYEE IS FALSELY ACCUSED?

T = "The radomly chosen employee is a thief"

S = "The lie detector test suspects the (randomly chosen) employee to be a thief"



$$P(\bar{T}\mid S) = \frac{P(\bar{T}\cap S)}{P(S)}$$

$$= \frac{P(\bar{T})\cdot P(S\mid\bar{T})}{P(\bar{T}\cap S)+P(T\cap S)} = \frac{0.9\,(0.2)}{0.9(0.2)+0.1\,(0.8)}$$

$$\approx 0.6923 = 69.23\ \%$$

Random variable: Describes a random experiment whose outcomes are real numbers.

Example: Roll a 20-sided die:-

Sample space $S=\{1,2,\dots,20\}$

$X =$ "The result of rolling a 20-sided die" is a random variable.

Custom: Use Capital letter for R.V. and corresponding small letters for results. ($x_1=7, x_2=13, \dots$)

Examples: Production of Metal Bolts:-

L: "Length of a produced bolt in mm"

Sample space $S=[49,51]$

All infinitely many values of an interval may be outcomes of a random variable.

Examples: Accidents in a company:-

A: "Number of work-related accidents in a year"

Sample space $S=\{0,1,2,3,4,\dots\}$

Infinitely many outcomes, but with gaps between the outcomes (not an interval).

R.V. (choice not alway obvious)

Discrete	Continuous
Gaps between outcome	All numbers of an interval are outcomes
(Uses Finite sum or Infinite series)	(uses integral)
Related to Counting.	Related to measuring
Uses Finite sums, or Infinite series.	Uses integrals

To Describe a discrete R.V completely, we need the outcomes and their probabilities.

This is Called the Distribution of the R.V

X	P(X=x)	
1	$P(X=1)$	"Success probability"
0	$1-P(X=1)$	"Failure probability"

Example: Continuous R.V.

City of Utopia; Bus arrives every 20 minutes. We arrive "Randomly" at the bus Station. How Long do we have to wait for the next bus?

W = "Waiting time in minutes until the next bus arrives"

Sample space $S = [0, 20]$

Laplace assumption: No reason to believe that any outcome is more likely to happen than the others.

$P(W \leq 5) = 0.25$

$P(5 < W \leq 10) = 0.25$

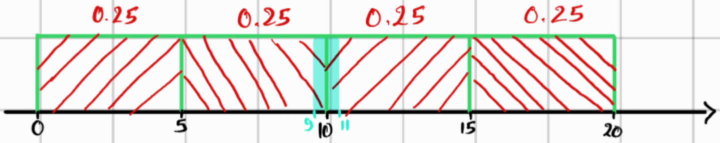
$P(10 < W \leq 15) = 0.25$

$P(15 < W \leq 20) = 0.25$

$\Rightarrow P(W \leq 20) = 1$

$P(W \leq 10) = 0.5$

Grouping & Histogram ➡



$P(9 < W \leq 11) = 0.1$

$P(9.5 < W \leq 10.5) = 0.05$

$P(9.75 < W \leq 10.25) = 0.025$

$P(W = 10) = 0 \Rightarrow$ For Continuous R.V., the probability of an elementry event is zero

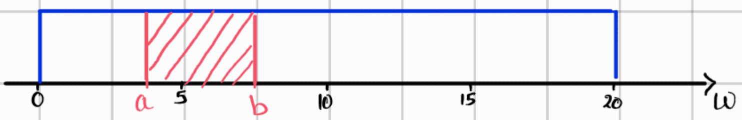
Two ways of visualization for the distribution of a continuous R.V.

1) Probability Density Function (PDF) [Probability Histogram]

2) Cumulative Distribution Function (CDF)

Always Consider both.

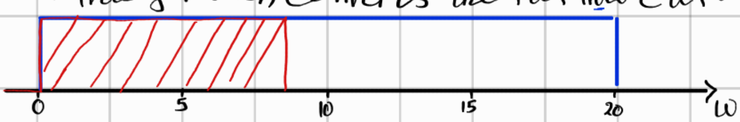
* Probability Density Function (PDF):



The probability is given by the area

$$P(a \leq W \leq b) = \underbrace{\int_a^b \underbrace{f(w)}_{\text{Pdf}}\, dw}_{\text{Probability}}$$

-Integeration converts the Pdf into Cdf: $F(w) = \int_{-\infty}^{w} \underbrace{f(x)}_{\text{Pdf}}\,dx$



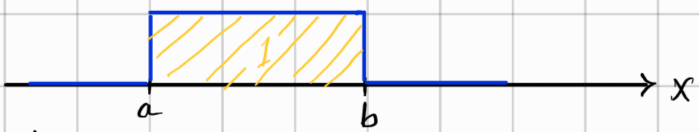
$F(w) = P(W \leq w)$

➡ The Pdf is the derivative of the Cdf:

$$f(w) = F'(w) = \frac{d}{dw} P(W \le w)$$

Also $P(a < W \le b) = \int_a^b f(w)\,dw = \int_{-\infty}^{b} f(w)\,dw - \int_{-\infty}^{a} f(w)\,dw = P(W \le b) - P(W \le a) = F(b) - F(a)$

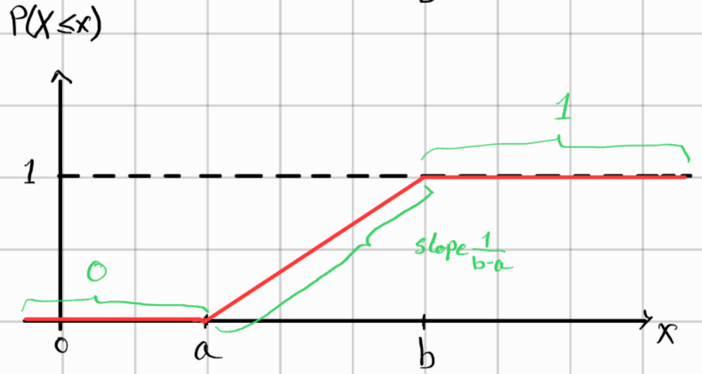
Uniform Continuous Distribution on $[a, b]$



Pdf $\quad f(x) = \begin{cases} \frac{1}{b-a} &, \text{ if } a \le x \le b \\ 0 &, \text{ otherwise} \end{cases}$

↓ Integrate $\qquad$ ↑ Differentiate

Cdf $\quad F(x) = \begin{cases} 0 &, \text{ if } x < a \\ \frac{x-a}{b-a} &, \text{ if } a \le x \le b \\ 1 &, \text{ if } b < x \end{cases}$



* Since $P(X=c) = \int_c^c f(x)\,dx = 0$

➡ $P(a \le X < b) = P(a \le X \le b) = P(a < X \le b) = P(a < X < b)$

A Pdf $f(x)$ must satisfy:

(i) $f(x) \ge 0$ for all x

(ii) $\int_{-\infty}^{+\infty} f(x)\,dx = 1$

Both properties are directly related to Kolmogorov's Axioms.

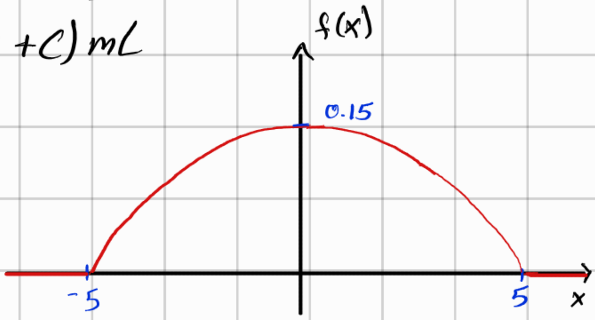
Exercise: A Beverage bottle is filled with $(500 + X)$ mL, where X is a random variable. The pdf of X is $f(x) = 0.006(25 - x^2)$ for $-5 \le x \le 5$ and $f(x) = 0$ otherwise.

a) Check that f is a pdf

b) Find the Cdf and sketch both functions.

c) Find the number c such that there is 95% probability of the bottle having a content between $(500 - c)$ mL and $(500 + c)$ mL

a) $f(x) = \begin{cases} 0.006(25 - x^2) & \text{for } -5 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$



(i) $f(x) \geq 0$ for all x ✓ we can also see it from the graph

(ii) $\int_{-\infty}^{+\infty} f(x)\,dx = 1$ ✓

$$\Rightarrow 0.006\int_{-\infty}^{\infty}(25-x^2)\,dx = 0 + 0.006\int_{-5}^{5}(25-x^2)\,dx = 0.006\left[25x - \frac{x^3}{3}\right]_{-5}^{5} = 0.006\left[125\left(1-\frac{1}{3}\right) - 125\left(\frac{1}{3}-1\right)\right]$$

$$= \frac{\cancel{6}^{\,2}}{\cancel{1000}_{\,4}}\,(\cancel{250})\left(\frac{2}{\cancel{3}}\right) = \frac{4}{4} = 1 ✓$$

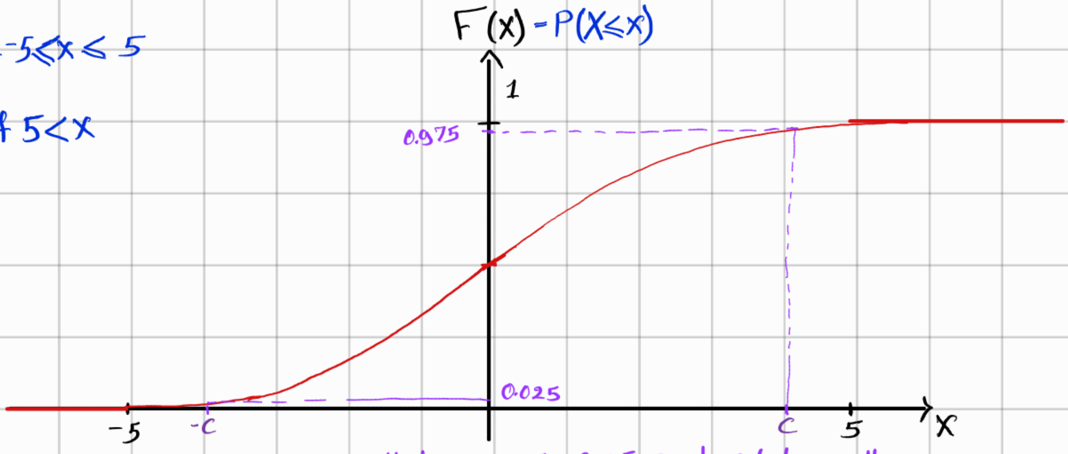
It's a Pdf function.

b) $F(x) = \int_{-\infty}^{x} f(k)\,dk = 0.006\int_{-5}^{x}(25-k^2)\,dk = 0.006\left[25k - \frac{k^3}{3}\right]_{-5}^{x} = 0.006\left(25x - \frac{x^3}{3}\right) + \frac{\cancel{6}^{\,2}}{\cancel{1000}_{\,4}}\cdot\frac{250}{\cancel{3}}$

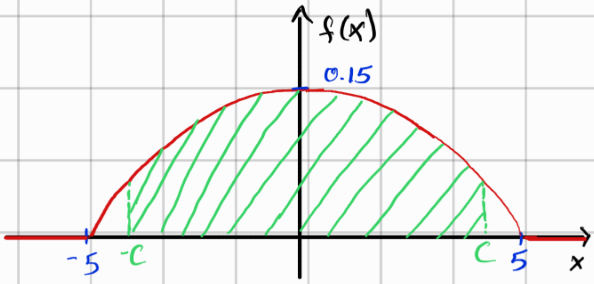
$= 0.006x\left(25 - \frac{x^2}{3}\right) + 0.5$

$$F(x) = \begin{cases} 0 & \text{if } x < -5 \\ 0.006x\left(25-\frac{x^2}{3}\right) + 0.5 & \text{if } -5 \leq x \leq 5 \\ 1 & \text{if } 5 < x \end{cases}$$

Cdf is Always Continuous for a Continuous R.V.



Make sure that 1-0.95 = 0.05 is derided equally for the upper and lower part of cdf graph.



C) From cdf graph and equation

$0.006(25)c - 0.006\frac{c^3}{3} + 0.5 = 0.975$

$-0.002c^3 + 0.15c - 0.475 = 0$

$\Rightarrow 2c^3 - 150c + 475 = 0$

Newton's method:

with initial guess $c_0 = 4$

$f(c=4) = 128 - 600 + 475 = 3$

$f'(c=4) = 6(16) - 150 = -54$

$c_{n+1} = c_0 - \frac{f(c_0)}{f'(c_0)} = 4 + \frac{3}{54} \approx 4.055$

Thus $P(-4.055 \leq X \leq 4.055) = 95\%$

$\int_{-c}^{c} f(x)\,dx = 0.95$

$0.006\left[25x - \frac{x^3}{3}\right]_{-c}^{c} = 0.006\left[\left(25c - \frac{c^3}{3}\right) + \left(25c - \frac{c^3}{3}\right)\right]$

$\frac{\cancel{12}^{\,3}}{\cancel{1000}_{\,10\,\cancel{40}}}\cdot\cancel{25}c - \frac{\cancel{12}^{\,4}}{1000}\cdot\frac{c^3}{\cancel{3}} = 0.95$

$\Rightarrow -0.004c^3 + 0.3c - 0.95 = 0$

The expected value (= The mean) of a R.V. is

$$E[X] = \sum_{x} x \cdot P(X=x) \quad \text{, if } X \text{ is a Discrete R.V.}$$

$$E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx \quad \text{, if } X \text{ is continuous with pdf } f(x).$$

<u>Example:</u> Computing $E[X]$ for Discrete X.

If X has few outcomes use a spreadseet

(Rolling a fair 20-sided die.

x	$P(X=x)$	$x \cdot P(X=x)$
1	$1/20$	$1/20$
2	$1/20$	$2/20$
$\vdots$	$\vdots$	$\vdots$
20	$1/20$	$20/20$

$$E[X] = \frac{1+2+3+\cdots+20}{20}$$

$$= \frac{1}{\cancel{20}} \cdot \frac{\cancel{20}(20+1)}{2} = 10.5$$

$$\sum_{k=1}^{n} k = 1+2+\cdots+n = \frac{n(n+1)}{2}$$

*Expected value doesn't have to be a value of the random variable.

$$\boxed{E[X+Y] = E[X] + E[Y]}$$

Linear Transformation

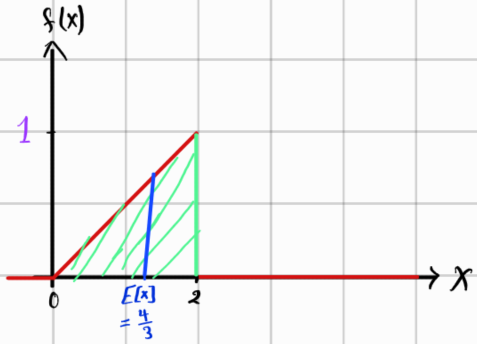
$$\boxed{E[a \cdot X + b] = a \cdot E[X] + b}$$

Valid for both discrete & continuous R.V.

Non-Linear Transformation

$$E[g(x)] = \begin{cases} \sum_{x} g(x) \cdot P(X=x) \\ \int_{-\infty}^{\infty} g(x) \cdot f(x)\, dx \end{cases}$$

Example: Continuous R.V. X with pdf f



in order to have $\int_{-\infty}^{\infty} f(x)dx = 1$

$$f(x) = \begin{cases} \frac{1}{2}x & \text{, if } 0 \leq x \leq 2 \\ 0 & \text{, otherwise} \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\, dx = \frac{1}{2}\int_{0}^{2} x^2 dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 = \frac{1}{2} \cdot \frac{8}{3} = \frac{4}{3}$$

Independant Random Variable:

$P(A|B) = P(A)$ $\longrightarrow$ because A & B are independant events, so the given event B doesn't affect the possibility of event A.

Two events A & B are said to be Independent if:

$$P(A \cap B) = P(A) \cdot P(B)$$

since we always have:

$$P(A \cap B) = P(A|B) \cdot P(B) = P(B|A) \cdot P(A)$$

$P(A) \neq 0$

$P(B) \neq 0$

Independence Implies:

$$P(A|B) = P(A) \quad \text{and} \quad P(B|A) = P(B)$$

Random Variables X and Y are Independent if

$P(X \leqslant x \text{ and } Y \leqslant y) = P(X \leqslant x) \cdot P(Y \leqslant y)$ for all x, y

When Knowing about one event (R.V.) does not tell us anything about the other event (R.V.), then these events (Random variables) are Independent.

Variance: To avoid the balancing we define:

The variance of a R.V. X is

expected value $E[X]$

$\sigma^2 = Var(X) = E[(X - E[X])^2] = E[(X-\mu)^2]$ ("Excepected square deviation")

It's square root is called the Standard Deviation σ (or σ_x).

$$E[(X-\mu)^2] = E[X^2 - 2\mu X + \mu^2] = E[X^2] + E[-2\mu X] + E[\mu^2]$$

$$\Rightarrow E[X^2] \underbrace{-2\mu E[X]}_{-2\mu^2} + \mu^2 = \boxed{E[X^2] - \mu^2}$$

with

$$E[X^2] = \begin{cases} \sum_x x^2 \cdot P(X=x) & \text{If } X \text{ is Discrete R.V.} \\ \int_{-\infty}^{+\infty} x^2 f(x)\,dx & \text{If } X \text{ is Continuous R.V. with Pdf } f(x) \end{cases}$$

$$\sigma^2 = Var(X) = E[X^2] - \mu^2 = E[X^2] - (E[X])^2$$

$$\sigma = \sqrt{Var(X)}$$ Standard Deviation

Formulas for variances:

$Var(X+Y) = Var(X) + Var(Y) + 2\,CoV(X,Y)$

where the covariance of X and Y is

$CoV(X,Y) = E[(X-E[X]) \cdot (Y - E[Y])] = E[X \cdot Y] - \mu_x \cdot \mu_y$

If X and Y are Independent

$Var(X+Y) = Var(X) + Var(Y)$

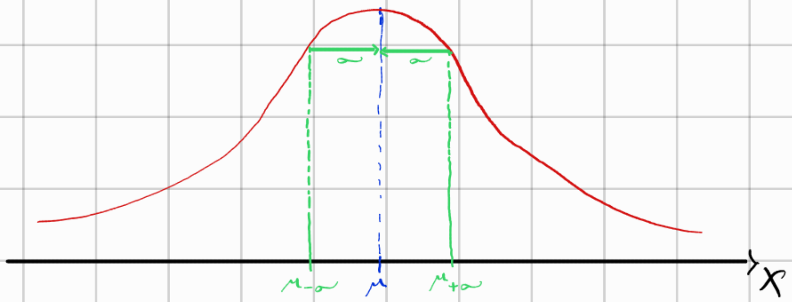
Formula for linear Transformation

$Var(aX+b) = a^2 \cdot Var(X) = a^2 (E[X^2] - (E[X])^2)$

$\sigma_{aX+b} = |a| \cdot \sigma_X$

A Random variable with a normal distribution is continuous with

Pdf $f(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ for $-\infty < x < +\infty$

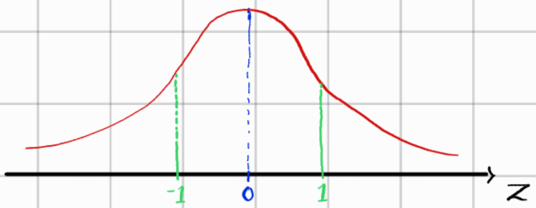


- Maximum at $\mu = E[x]$
- Symmetric about μ
- Points of inflection at $\mu \pm \sigma$
- everywhere positive
- Horizontal asymptote at $y=0$

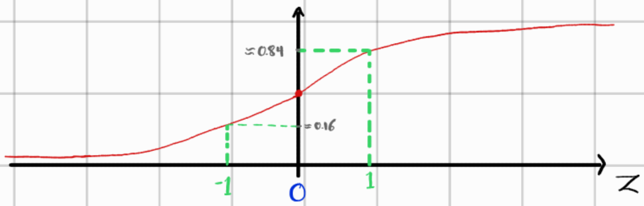
$$\boxed{Z = \frac{x - \mu_x}{\sigma_x}}$$

R.V. with standard Normal distribution is usually denoted by Z :-

Pdf $\phi(z) = \dfrac{1}{\sqrt{2\pi}} \cdot e^{-\frac{z^2}{2}}$



cdf $\Phi(z) = \dfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{t^2}{2}} \, dt$



Cumulative Standard Normal Distribution Function

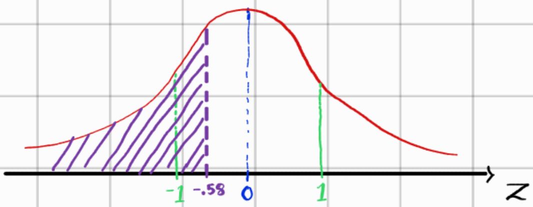
z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

$P(Z \leq 1.38) = \Phi(1.38) = 0.9162$

$P(Z > 0.97) = 1 - P(Z \leq 0.97) = 1 - 0.834$
$= 0.166$

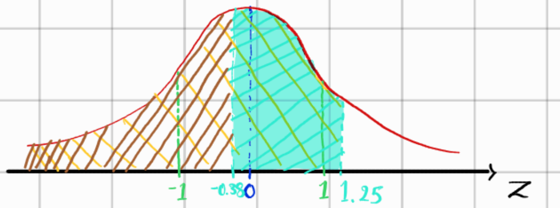
Cumulative Standard Normal Distributon Function

0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.00	z
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-3.9
0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	-3.8
0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	-3.7
0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0002	0.0002	-3.6
0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	-3.5
0.0002	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	-3.4
0.0003	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0005	0.0005	0.0005	-3.3
0.0005	0.0005	0.0005	0.0006	0.0006	0.0006	0.0006	0.0006	0.0007	0.0007	-3.2
0.0007	0.0007	0.0008	0.0008	0.0008	0.0008	0.0009	0.0009	0.0009	0.0010	-3.1
0.0010	0.0010	0.0011	0.0011	0.0011	0.0012	0.0012	0.0013	0.0013	0.0013	-3.0
0.0014	0.0014	0.0015	0.0015	0.0016	0.0016	0.0017	0.0018	0.0018	0.0019	-2.9
0.0019	0.0020	0.0021	0.0021	0.0022	0.0023	0.0023	0.0024	0.0025	0.0026	-2.8
0.0026	0.0027	0.0028	0.0029	0.0030	0.0031	0.0032	0.0033	0.0034	0.0035	-2.7
0.0036	0.0037	0.0038	0.0039	0.0040	0.0041	0.0043	0.0044	0.0045	0.0047	-2.6
0.0048	0.0049	0.0051	0.0052	0.0054	0.0055	0.0057	0.0059	0.0060	0.0062	-2.5
0.0064	0.0066	0.0068	0.0069	0.0071	0.0073	0.0075	0.0078	0.0080	0.0082	-2.4
0.0084	0.0087	0.0089	0.0091	0.0094	0.0096	0.0099	0.0102	0.0104	0.0107	-2.3
0.0110	0.0113	0.0116	0.0119	0.0122	0.0125	0.0129	0.0132	0.0136	0.0139	-2.2
0.0143	0.0146	0.0150	0.0154	0.0158	0.0162	0.0166	0.0170	0.0174	0.0179	-2.1
0.0183	0.0188	0.0192	0.0197	0.0202	0.0207	0.0212	0.0217	0.0222	0.0228	-2.0
0.0233	0.0239	0.0244	0.0250	0.0256	0.0262	0.0268	0.0274	0.0281	0.0287	-1.9
0.0294	0.0301	0.0307	0.0314	0.0322	0.0329	0.0336	0.0344	0.0351	0.0359	-1.8
0.0367	0.0375	0.0384	0.0392	0.0401	0.0409	0.0418	0.0427	0.0436	0.0446	-1.7
0.0455	0.0465	0.0475	0.0485	0.0495	0.0505	0.0516	0.0526	0.0537	0.0548	-1.6
0.0559	0.0571	0.0582	0.0594	0.0606	0.0618	0.0630	0.0643	0.0655	0.0668	-1.5
0.0681	0.0694	0.0708	0.0721	0.0735	0.0749	0.0764	0.0778	0.0793	0.0808	-1.4
0.0823	0.0838	0.0853	0.0869	0.0885	0.0901	0.0918	0.0934	0.0951	0.0968	-1.3
0.0985	0.1003	0.1020	0.1038	0.1056	0.1075	0.1093	0.1112	0.1131	0.1151	-1.2
0.1170	0.1190	0.1210	0.1230	0.1251	0.1271	0.1292	0.1314	0.1335	0.1357	-1.1
0.1379	0.1401	0.1423	0.1446	0.1469	0.1492	0.1515	0.1539	0.1562	0.1587	-1.0
0.1611	0.1635	0.1660	0.1685	0.1711	0.1736	0.1762	0.1788	0.1814	0.1841	-0.9
0.1867	0.1894	0.1922	0.1949	0.1977	0.2005	0.2033	0.2061	0.2090	0.2119	-0.8
0.2148	0.2177	0.2206	0.2236	0.2266	0.2296	0.2327	0.2358	0.2389	0.2420	-0.7
0.2451	0.2483	0.2514	0.2546	0.2578	0.2611	0.2643	0.2676	0.2709	0.2743	-0.6
0.2776	0.2810	0.2843	0.2877	0.2912	0.2946	0.2981	0.3015	0.3050	0.3085	-0.5
0.3121	0.3156	0.3192	0.3228	0.3264	0.3300	0.3336	0.3372	0.3409	0.3446	-0.4
0.3483	0.3520	0.3557	0.3594	0.3632	0.3669	0.3707	0.3745	0.3783	0.3821	-0.3
0.3859	0.3897	0.3936	0.3974	0.4013	0.4052	0.4090	0.4129	0.4168	0.4207	-0.2
0.4247	0.4286	0.4325	0.4364	0.4404	0.4443	0.4483	0.4522	0.4562	0.4602	-0.1
0.4641	0.4681	0.4721	0.4761	0.4801	0.4840	0.4880	0.4920	0.4960	0.5000	0.0



$P(z \leq -0.58) = \Phi(-0.58) = 0.281$

$P(z > 0.58) = 1 - P(z \leq 0.58) = 1 - 0.719 = 0.281$



$P(-0.38 < z < 1.25)$

$= \boxed{P(z \leq 1.25)} - \boxed{P(z \leq -0.38)}$

$= \Phi(1.25) - \Phi(-0.38)$

$= 0.8944 - 0.3520$

$= 0.5424$

Example : X is normaly distributed with $\mu = 5$ & $\sigma = 0.4$

Find $P(4.4 \leq X \leq 5.2)$

$P(4.4 \leq X \leq 5.2) = P(X \leq 5.2) - P(X \leq 4.4)$

$= \Phi\left(\frac{5.2-5}{0.4}\right) - \Phi\left(\frac{4.4-5}{0.4}\right)$

$= \Phi(0.5) - \Phi(-1.5)$

$= 0.6915 - 0.0668$

$\frac{x-5}{0.4}$

$= 0.6247$

1. A jar contains four marbles: three red, one white. Two marbles are drawn with replacement (ie. a marble is randomly selected, the colour noted, the marble replaced in the jar, then a second marble is drawn).

a. Write out, in curly bracket form, a sample space containing four outcomes

b. Write out a sample space with sixteen outcomes

c. What is the probability of each of the four outcomes in (a)?

d. What are the probabilities of the outcomes in (b)?

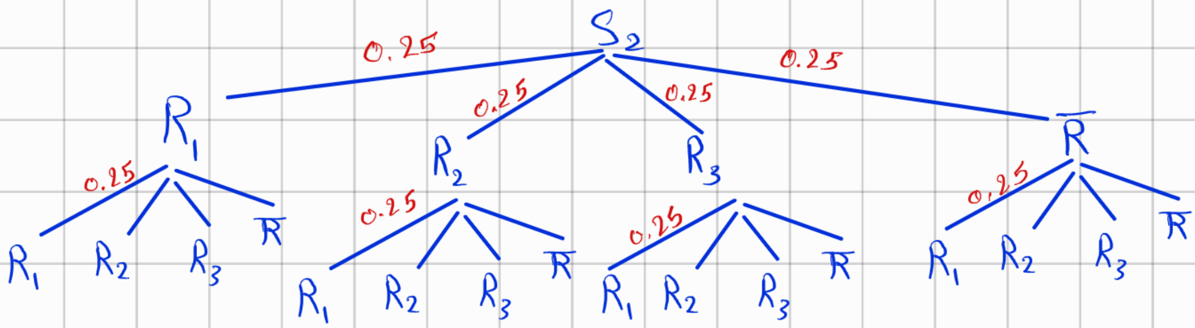
e. What is the probability the colours of the two marbles match?

f. What is the probability the same marble is drawn twice?

R = " The event of randomly pick a red Marble.

a) Sample space$_1$ = $S_1 = \{RR, R\bar{R}, \bar{R}R, \bar{R}\bar{R}\}$

b) $R_{1,2,3}$ = "The event of chosing the 1st, 2nd, or third red marbles randomly"



c) Using Laplace assumption :-

$P(R \cap R) = P(R) \cdot P(R|R) = \frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16}$

$P(R \cap \bar{R}) = \frac{3}{4} \cdot \frac{1}{4} = \frac{3}{16}$

$P(\bar{R} \cap R) = \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{16}$

$P(\bar{R} \cap \bar{R}) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$

d) all of the 16 outcomes are equal to $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$

e) $P(R \cap R) + P(\bar{R} \cap \bar{R}) = \frac{9}{16} + \frac{1}{16} = \frac{10}{16}$

f) $P(R_1 \cap R_1) + P(R_2 \cap R_2) + P(R_3 \cap R_3) + P(\bar{R} \cap \bar{R}) = 4 \times \frac{1}{16} = \frac{1}{4}$

2. Imagine playing with a short deck of cards, as shown at right.
"H" is the event in which the card drawn is a heart.
"D" is the event in which the card drawn is a diamond.
"A" is the event in which the card drawn is an ace.

A♠	A♣	A♥	A♦
2♠	2♣	2♥	2♦
3♠	3♣	3♥	3♦
4♠	4♣	4♥	4♦

a. What are $P(H)$, $P(D)$, and $P(A)$?

b. Find $P(H \cup D)$

c. Find $P(H \cup A)$

d. Find $P(H \cap D)$

e. Find $P(H \cap A)$

f. Are H and D independent events?

g. Are H and A independent events?

Using Laplace assumption:

a) $P(H)=P(D)=P(A)=\frac{4}{16}=\frac{1}{4}=25\%$

b) $P(H \cup D)=P(H)+P(D)=\frac{1}{2}=50\%$

c) $P(H \cup A)=P(H)+P(A)-P(H \cap A)=\frac{1}{4}+\frac{1}{4}-\frac{1}{16}$

$H = \{$ A♥ 2♥ 3♥ 4♥ $\}$

$A = \{$ A♠ A♣ A♥ A♦ $\}$

$D = \{$ A♦ 2♦ 3♦ 4♦ $\}$