Name: ______________________________

Problem 1: (2+7+4 Points) Consider the matrix

$$\begin{pmatrix} -t & \boxed{0} & -2\,t \\ \boxed{0} & -2\,t & 0 \\ 4t & 0 & 5t \end{pmatrix}.$$

a) Plug in

$$t = \begin{cases} d_5 & \text{if } d_5 \neq 0 \\ 3 & \text{if } d_5 = 0 \end{cases}$$

where d_5 is the last digit of your student ID number.

b) Compute all eigenvalues.

c) Compute the eigenvectors for the negative eigenvalue.

$d_5 = 8$

a) $A = \begin{pmatrix} -8 & 0 & -16 \\ 0 & -16 & 0 \\ 32 & 0 & 40 \end{pmatrix}$

b) $A\vec{v} = \lambda I_3 \vec{v} \Longrightarrow \vec{v}(A - \lambda I_3) = \vec{0}$

$$\det(A - \lambda I_3) = \begin{pmatrix} -(\lambda+8) & 0 & -16 \\ 0 & -(\lambda+8) & 0 \\ 32 & 0 & 40-\lambda \end{pmatrix} = 0$$

$$= -(\lambda+8) \cdot \begin{vmatrix} -\lambda-8 & -16 \\ 32 & 40-\lambda \end{vmatrix} = 0 \Longrightarrow (\lambda+8)(\lambda+8)(40-\lambda) - 512(\lambda+8) = 0$$

$$(\lambda+8)\left(-\lambda^2 + 32\lambda + 320 - 512\right) = 0$$

$$-(\lambda+8)(\lambda^2 - 32\lambda + 192) = 0$$

$\lambda_1 = -8$, $\lambda_{2,3} = \frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2}$ $\quad 4 \times 192 = 800 - 32 = 768 \Rightarrow b = -(32+a)$

$= 16 \pm \frac{\sqrt{\cancel{1024 - 768}\ 256}}{2} = 16 \pm \frac{\sqrt{2^8}}{2} = -16 \pm 8$

$\lambda_2 = 8$, $\lambda_3 = 24$

$\therefore -(\lambda+8)(\lambda-8)(\lambda-24) = 0$

Eigen vector for $\lambda = -8$

$$(A - \lambda I)\vec{v} = \begin{pmatrix} 0 & 0 & -16 \\ 0 & 0 & 0 \\ 32 & 0 & 48 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{0}$$

$z = 0$

$x = 0$

let $y = t$, $t \in R$

$$\boxed{\vec{v} = t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} , \ t \in R}$$

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Problem 2: (11 Points) Solve the initial value problem

$$\frac{dy}{dx} = \boxed{\ln x} \cdot \frac{1}{x\,y} \qquad , \quad y(1) = 2 \ .$$

Separable: There is no constant solution

$$\int y\,dy = \int \frac{\ln x}{x}\,dx \implies y = \frac{1}{2}(\ln x)^2 + C$$

$$y(1) = \frac{1}{2}(\underbrace{\cancel{\ln(1)}}_{0})^2 + C = 2 \implies C = 2$$

$$\therefore \boxed{y(x) = \frac{1}{2}(\ln(x))^2 + 2}$$

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Problem 3: (9 Points) Find the interval of convergence of

$$\sum_{n=d_5}^{\infty} \frac{x^{2n}}{(\boxed{-3})^n}$$

where d_5 is the last digit of your student ID number.

$$\sum_{n=8}^{\infty} \frac{x^{2n}}{(-3)^n} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+16}}{(3)^{n+8}} = \sum_{n=0}^{\infty} a_n$$

Ratio Test $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{2n+18}}{(3)^{n+9}} \cdot \frac{(3)^{n+8}}{x^{2n+16}} \right| = \lim_{n\to\infty} \left| \frac{\cancel{x^{2n+16}} \cdot x^2}{\cancel{x^{2n+16}}} \cdot \frac{\cancel{(3)^{n+8}}}{\cancel{(3)^{n+8}} \cdot 3} \right| = \frac{|x^2|}{3} < 1$

$|x^2| < 3$

$|x| < \sqrt{3}$

$R = \sqrt{3}$

$-\sqrt{3} < x < \sqrt{3}$

at end points

at $x = -\sqrt{3}$

$$\frac{\cancel{(-\sqrt{3})^{16}}}{\cancel{3^8}} \sum_{n=0}^{\infty} \frac{(-1)^n (-\sqrt{3})^{2n}}{(3)^n} = \sum_{n=0}^{\infty} (-1)^n \frac{\cancel{(3)^n}}{\cancel{(3)^n}} = \sum_{n=0}^{\infty} (-1)^n$$

diverges by divergence test and also by alternating series test since $\lim_{n\to\infty} (1) = 1 \neq 0$

The exact series will be produced at $x = \sqrt{3}$

Thus The series also diverges at $x = \sqrt{3}$

Therefore, the interval of convergence

$I = (-\sqrt{3}, \sqrt{3})$

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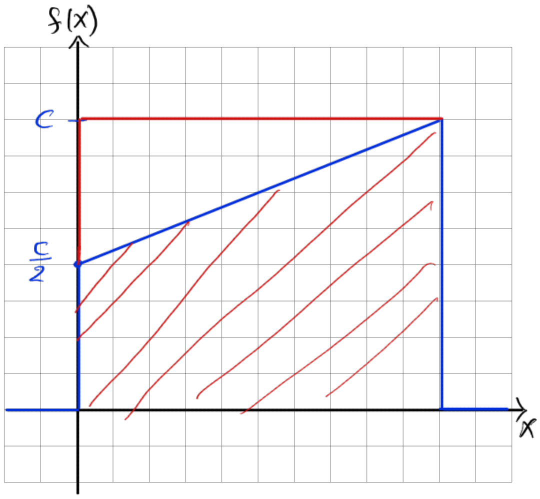
Problem 4: (7+6+3+4 Points) The pdf $f(x)$ of a continuous random variable X is zero for $x < 0$ and also zero for $x > 1$. For $0 \le x \le 1$ the graph of the pdf is the line segment from $f(0) = C/2$ to $f(1) = \boxed{C}$.

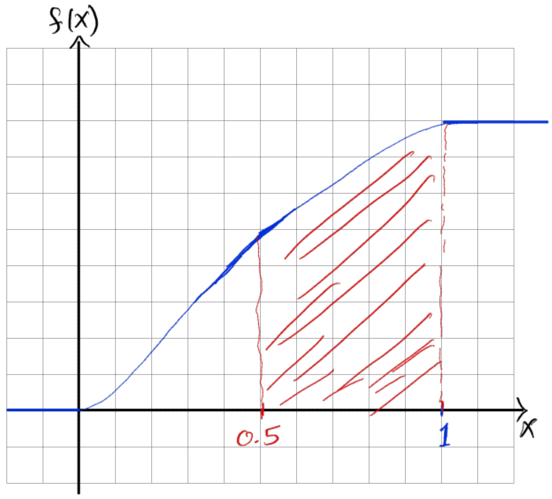
a) Sketch the probability density function, determine C, and find a formula for the pdf.

b) Compute the cumulative distribution function and sketch it.

c) Compute the probability $P(0.5 \le X \le 1)$.

d) Compute the expected valule of X.





a) from the graph of pdf we Know that the red are must be equal to 1.

The red area = The rectangular area − the triangular area

$$= C \times 1 - \frac{1}{2}\left(\frac{C}{2}\right)\cdot(1) = 1$$

$$\boxed{C = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3}}$$

Thus $f(0) = \frac{2}{3}$

$f(1) = \frac{4}{3}$

$$f(x) = \begin{cases} 0 & x < 0 \\ \frac{2}{3}x + \frac{2}{3} & 0 \leq x \leq 1 \\ 0 & 1 < x \end{cases}$$

b)

CDF: $F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{0}^{x} \left(\frac{2}{3}t + \frac{2}{3}\right) dt = \left[\frac{1}{3}t^2 + \frac{2}{3}t\right]_0^x$

$= \frac{1}{3}x^2 + \frac{2}{3}x$

$$F(x) = \begin{cases} 0 & x < 0 \\ x\left(\frac{1}{3}x + \frac{2}{3}\right) & 0 \leq x \leq 1 \\ 1 & 1 < x \end{cases}$$

c) $P(0.5 \leq X \leq 1) = P(X \leq 1) - P(X \leq 0.5) = F(1) - P(0.5)$

$= 1 - \left(\frac{1}{2}\right)\left(\frac{1}{3}\left(\frac{1}{2}\right) + \frac{2}{3}\right) = 1 - \frac{5}{12} = \frac{7}{12}$

d)

$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \frac{2}{3}\int_{0}^{1} x(x+1)\,dx = \frac{2}{3}\left[\frac{1}{3}x^3 + x\right]_0^1 = \frac{2}{3} \cdot \frac{1}{3}(1)^3 + \frac{2}{3}(1) - 0$

$= \frac{2}{9} + \frac{2}{3} = \frac{8}{9}$