1.)

a)

$+$
$-$
$+$
$-$
$+$
$-$

b1)

1st: $\det(I_n) = 1$

2nd: Each swiching the row will swap the sign of the determinant.

3rd: we can take a factor of the row and multiply it in the the deteminant

or we can split the row as well

example $\det A = \begin{vmatrix} 2 & 8 \\ 3 & 4 \end{vmatrix} = 2\begin{vmatrix} 1+0 & 2+2 \\ 3 & 4 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} + 2\begin{vmatrix} 0 & 2 \\ 3 & 4 \end{vmatrix}$

b2) If we took a comman factor of the zero row of the detormenant $\Longrightarrow \det. = 0$ (3rd property)

ex. $\det B = \begin{vmatrix} 2 & 1 & 5 \\ 0 & 0 & 0 \\ 3 & 1 & 2 \end{vmatrix} = 0 \cdot \begin{vmatrix} 2 & 1 & 5 \\ 0 & 0 & 0 \\ 3 & 1 & 2 \end{vmatrix} = 0$

c) $A = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 1 & 0 & -2 & 0 \\ 3 & -1 & 1 & -2 \\ 4 & -3 & 0 & 2 \end{pmatrix} \Longrightarrow \det A = \begin{vmatrix} 1 & 2 & 4 & 3 \\ 1 & 0 & 0 & 0 \\ 3 & -1 & 7 & -2 \\ 4 & -3 & 8 & 2 \end{vmatrix}$

$\text{III}+2\text{I}$

$\Rightarrow \det A = -(1)\begin{vmatrix} 2 & 4 & 3 \\ -1 & 7 & -2 \\ -3 & 8 & 2 \end{vmatrix} = -\left(3\begin{vmatrix} -1 & 7 \\ -3 & 8 \end{vmatrix} - (-2)\begin{vmatrix} 2 & 4 \\ -3 & 8 \end{vmatrix} + 2\begin{vmatrix} 2 & 4 \\ -1 & 7 \end{vmatrix}\right)$

$= -\left(3(-8+21) + 2(16+12) + 2(14+4)\right)$

$= -(39 + 56 + 36) = -131$

c) in other way:-

$\text{III}+2\text{I}$

$\det(A) = \begin{vmatrix} 1 & 2 & 2 & 3 \\ 1 & 0 & -2 & 0 \\ 3 & -1 & 1 & -2 \\ 4 & -3 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 4 & 3 \\ 1 & 0 & 0 & 0 \\ 3 & -1 & 7 & -2 \\ 4 & -3 & 8 & 2 \end{vmatrix} = -(1)\left|\begin{array}{ccc|cc} 2 & 4 & 3 & 2 & 4 \\ -1 & 7 & -2 & -1 & 7 \\ -3 & 8 & 2 & -3 & 8 \end{array}\right|$

$= -\left(28 + \cancel{24} - \cancel{24} - (-63) - (-32) - (-8)\right) = -(70 + 61) = -131$

d) $B = \begin{pmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ , $\det B = \begin{vmatrix} 2 & 2 \\ 1 & 0 \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$

$= -2 + 6 - 5 = -1$

$$\text{Adj. of } B = \begin{pmatrix} 1 & -3 & 2 \\ 0 & -1 & +1 \\ -2 & -6 & -5 \end{pmatrix} = C$$

$$C^T = \begin{pmatrix} 1 & 0 & -2 \\ -3 & -1 & -6 \\ 2 & 1 & -5 \end{pmatrix}$$

$$B^{-1} = \frac{1}{\det B} \cdot C^T = \begin{pmatrix} -1 & 0 & 2 \\ 3 & 1 & 6 \\ -2 & -1 & 5 \end{pmatrix}$$

e) charctaristic equation is $C \cdot \vec{v} = \lambda \vec{v}$

$$C\vec{v} - \lambda I_3 \vec{v} = \vec{0} \Rightarrow (C - \lambda I_3)\vec{v} = \vec{0} \quad \text{where } \vec{v} \neq \vec{0}$$

eigen values:

$$\det(C - \lambda I_3) = 0 \Rightarrow \det\begin{pmatrix} 3-\lambda & -1 & 3 \\ -2 & 4-\lambda & 2 \\ 1 & -1 & 5-\lambda \end{pmatrix} = \left(\begin{array}{ccc|cc} 3-\lambda & -1 & 3 & 3-\lambda & -1 \\ -2 & 4-\lambda & 2 & -2 & 4-\lambda \\ 1 & -1 & 5-\lambda & 1 & -1 \end{array}\right) = 0$$

$$\det(C - \lambda I_3) = (3-\lambda)(4-\lambda)(5-\lambda) + (-2) + 6 - 3(4-\lambda) + 2(3-\lambda) - 2(5-\lambda) = 0$$

$$\Rightarrow (3-\lambda)(4-\lambda)(5-\lambda) + (3\lambda - 12) = (3-\lambda)(4-\lambda)(5-\lambda) - 3(4-\lambda) = 0$$

$$\Rightarrow (4-\lambda)((3-\lambda)(5-\lambda) - 3) = (4-\lambda)(\lambda^2 - 8\lambda + 12) = 0$$

$$(4-\lambda)(\lambda-2)(\lambda-6) = 0$$

$\lambda_1 = 2, \lambda_2 = 4, \lambda_3 = 6$

at $\lambda_1 = 2$

$$(C - \lambda_1 I)\vec{v} = \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 1 & -1 & 3 \\ -2 & 2 & 2 \\ 1 & -1 & 3 \end{pmatrix} \end{array} \begin{array}{l} \\ II+2I \\ III-I \end{array} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 3 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{pmatrix} = \vec{0}$$

$z = 0$

let $y = t$

$x - y + 3(0) = 0$

$x = y = t$

$$\boxed{\vec{v}_1 = t\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad t \in R}$$

at $\lambda_2 = 4$

$$(C-\lambda_1 I)\vec{V} = \begin{array}{cccc} x & y & z & \\ \left(\begin{matrix} -1 \\ -2 \\ 1 \end{matrix}\right. & \begin{matrix} -1 \\ 0 \\ -1 \end{matrix} & \left.\begin{matrix} 3 \\ 2 \\ 1 \end{matrix}\right) & \begin{matrix} \\ II+2III \\ III+I \end{matrix} \end{array} = \vec{0}$$

$$\begin{pmatrix} -1 & -1 & 3 \\ 0 & -2 & 4 \\ 0 & -2 & 4 \end{pmatrix} \begin{matrix} \\ II/-2 \\ III+II \end{matrix}$$

$$\begin{pmatrix} -1 & -1 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

Let $z = a$ , $a \in R$

$y - 2a = 0 \Rightarrow y = 2a$

$-x - y + 3z = 0$

$x = 3a - 2a = a$

$$\boxed{\vec{V}_2 = a\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} , \ a \in R}$$

at $\lambda_3 = 6$

$$(C - I\lambda)\cdot\vec{V} = \begin{array}{cccc} x & y & z & \\ \left(\begin{matrix} -3 \\ -2 \\ 1 \end{matrix}\right. & \begin{matrix} -1 \\ -2 \\ -1 \end{matrix} & \left.\begin{matrix} 3 \\ 2 \\ -1 \end{matrix}\right) & \begin{matrix} \\ II/2 - I/3 \\ III + II/2 \end{matrix} \end{array} = \vec{0}$$

$$\begin{pmatrix} -3 & -1 & 3 \\ 0 & -\frac{2}{3} & 0 \\ 0 & -2 & 0 \end{pmatrix}$$

$y = 0$

Let $x = r$ , $r \in R$

$-3x - 0 + 3z = 0 \Rightarrow z = r$

$$\boxed{\vec{V}_3 = r\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} , \ r \in R}$$

2. Series

a) False

b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+6}\, n}{n^2+9} = \sum_{n=1}^{\infty} (-1)^n \cdot a_n$

(i) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\cancel{n}\cdot 1}{n^{\cancel{2}}(1+\underbrace{\frac{9}{n^2}}_{0})} = 0$

(ii) $f(n) = \frac{n}{n^2+9}$, $f'(n) = \frac{n^2+9-2n^2}{(n^2+9)^2} = \frac{9-n^2}{(n^2+9)^2} < 0 \Rightarrow 9-n^2<0 \Rightarrow n>3$

So, the function $f(n)$ or $a_n$ is decreasing for all $n>3$

Therefore, The series $\sum_{n=1}^{\infty} (-1)^n \cdot a_n$ Converges by alternating series Test.

c) $\sum_{n=1}^{\infty} \frac{4n-3}{2n^5} = \sum_{n=1}^{\infty} \frac{4n}{2n^5} - \sum_{n=1}^{\infty} \frac{3}{2n^5} = \frac{4}{2}\sum_{n=1}^{\infty} \frac{1}{n^4} - \frac{3}{2}\sum \frac{1}{n^5}$

Since the series $\sum_{n=1}^{\infty} \frac{4n-3}{2n^5}$ is a combination of two

Convergent p-series (where $P=4>1$ & $P=5>1$) and multiaplied by Coefficients, the the series $\sum_{n=1}^{\infty} \frac{4n-3}{2n^5}$ converges.

d) $f(x) = \sin^2(x)$

Maclaurin series (Tylor series with $a=0$)

$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$

$f(0) = 0$

$f'(x) = 2\sin(x)\cos(x) = f'(0) = 0$

$f''(x) = 2\cos^2(x) - 2\sin^2(x) \Rightarrow f''(0) = 2$

$f'''(x) = -4\cos(x)\sin(x) - 4\cos(x)\sin(x) \Rightarrow f'''(0) = 0$

$f''''(x) = 8\sin^2(x) - 8\cos^2(x) \Rightarrow f''''(0) = -8$

$f'''''(x) = 16\sin x\cos x + 16\sin x\cos x$

$f^{6}(x) = 32\cos^2(x) - 32\sin^2 x \Rightarrow f^{6}(0) = 32$

$f^{8}(x) = 128\sin^2 x - 128\cos^2 x \Rightarrow f^{8}(0) = -128$

$f(x) = \frac{\cancel{2}}{\cancel{2!}}x^2 - \frac{8}{4!}x^4 + \frac{32}{6!}x^6 - \frac{128}{8!}x^6 + \cdots$

$$\boxed{f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2)^{2n-1}}{(2n)!} x^{2n}}$$

3. ODEs

a.)

+

+

—

+

explicit form: ($y'$ is seperated on one side)

$$y' = \frac{2yx + 3\sqrt{x}}{y^3}$$

$$2\sqrt{x}\, y\, y' + 3x = 2$$

↳ Implicit form

b) $\dot{y} + 2y = 4 \qquad y(0) = 10$

$\frac{dy}{dt} = 4 - 2y$ Constan solution $\boxed{y = 2}$

$$\int \frac{dy}{4-2y} = \int dt \Rightarrow -\frac{1}{2}\ln|4-2y| = t + C \quad , C \in \mathbb{R}$$

$$\frac{1}{\sqrt{4-2y}} = e^{t}\cdot e^{c} \Longrightarrow 4 - 2y = \frac{1}{e^{2t}\cdot K^2} \quad , K > 0$$

$$y = 2 - \frac{1}{2K^2 e^{2t}}$$

at $y(0) = 2 - \frac{1}{2K^2 e^0} = 10 \Longrightarrow 2(2-10) = \frac{1}{K^2}$

$\Longrightarrow \boxed{K^2 = -\frac{1}{16}} \Longrightarrow$ K is a complex num. (which shouldn't be the case)

$$\boxed{y = 2 + \frac{8}{e^{2t}}}$$

c) $2xy' + y = 2\sqrt{x}$

homogenuous $2x\frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = \frac{-y}{2x}$ constant solution $\boxed{y = 0}$

$$\int \frac{dy}{y} = -\frac{1}{2}\int \frac{dx}{x} \Longrightarrow \ln|y| = -\frac{1}{2}\ln|x| + C \quad , C \in \mathbb{R}$$

$$\Longrightarrow y = x^{-\frac{1}{2}} \cdot e^{C} \Longrightarrow y = x^{-\frac{1}{2}} \cdot K \quad , K \geq 0 \quad (y = 0 \text{ is included})$$

Varying the constant K $\quad y = x^{-\frac{1}{2}} K(x)$

$$y' = K'(x)\, x^{-\frac{1}{2}} - \frac{1}{2}K(x)\, x^{-\frac{3}{2}}$$

Plugging in $\cancel{2}K'(x)\cancel{x^{\frac{1}{2}}} \cancel{- K(x)x^{-\frac{1}{2}}} \cancel{+ K(x)x^{-\frac{1}{2}}} = \cancel{2}\cancel{\sqrt{x}}$

$\int dK = \int dx \Rightarrow K(x) = x + C_1 \quad , C_1 \in \mathbb{R}$

Plug in K(x) in y

$$\boxed{y = \frac{x + C_1}{\sqrt{x}} = \sqrt{x} + \frac{\sqrt{x}\, C_1}{x}}$$

d) $y'' + 15y' + 56y = e^{-6x}$

homogeneous: $y'' + 15y' + 56y = 0$

let $y_h = e^{\lambda x}$

$y_h' = \lambda e^{\lambda x}$

$y_h'' = \lambda^2 e^{\lambda x}$

$e^{\lambda x}(\lambda^2 + 15\lambda + 56) = 0$

$e^{\lambda x} \neq 0$ $\qquad (\lambda+8)(\lambda+7) = 0$

$\lambda_1 = -8, \lambda_2 = -7$

$\boxed{y_h = C_1 e^{-8x} + C_2 e^{-7x}}$ , $C_1 \& C_2 \in \mathbb{R}$

Particular solution: $y'' + 15y' + 56y = e^{-6x}$

let $y_p = Ae^{-6x}$

$y_p' = -6Ae^{-6x}$

$y_p'' = 36Ae^{-6x}$

$36Ae^{-6x} - 90Ae^{-6x} + 56Ae^{-6x} = e^{-6x}$

$2Ae^{-6x} = e^{-6x}$

$\boxed{A = \frac{1}{2}} \Rightarrow y_p = \frac{1}{2}e^{-6x}$

General solution $y_g = y_p + y_h$

$\boxed{y_g = C_1 e^{-8x} + C_2 e^{-7x} + \frac{1}{2}e^{-6x}}$

4)

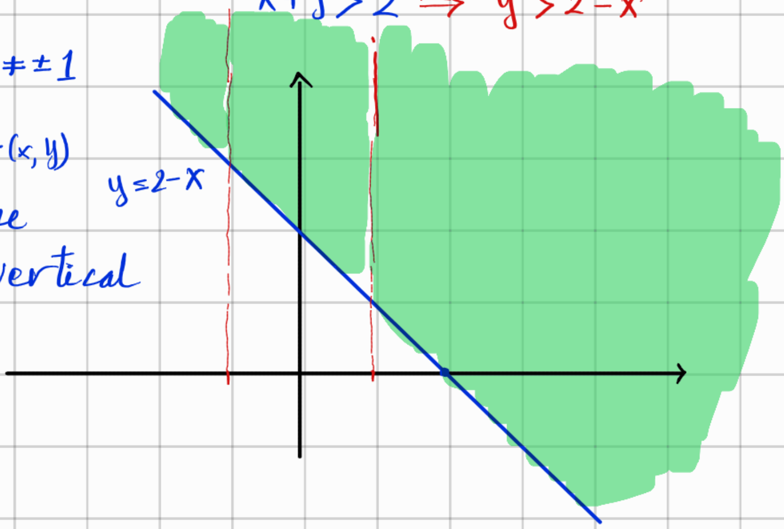
a) $f(x,y) = \dfrac{\ln(x+y-2)}{x^2-1}$

$x + y - 2 > 0$

$x + y > 2 \Rightarrow y > 2 - x$

$x^2 - 1 \neq 0 \Rightarrow x \neq \pm 1$

The domain of the function $f(x,y)$ is everything above the line $y = 2 - x$ except for the vertical asymptotes at $x = 1$ & $x = -1$



b) $f(x,y,z) = 4y^2 \sin^3 x - e^z y^4 + \frac{z^3}{x^2} + 4y - x^{16}$

$f_x(x,y,z) = 12y^2 \sin^2 x \cos x + \frac{-2xz^3}{x^4} - 16x^{15}$

$f_y(x,y,z) = 8y \sin^3 x - 4y^3 e^z + 4$

$f_z(x,y,z) = -y^4 e^z + \frac{3z^2}{x^2}$

c) $z = \ln(2x+y)$ at $P(-1,3)$

$z_0 = \ln(2(-1)+3) = 0$

$f_x(x,y) = \frac{2}{2x+y} \Rightarrow f_x(-1,3) = \frac{2}{2(-1)+3} = 2$

$f_y(x,y) = \frac{1}{2x+y} \Rightarrow f_y(-1,3) = \frac{1}{1} = 1$

Tangentional plane equation:

$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

$z - 0 = 2(x+1) + 1(y-3)$

$$\boxed{z = 2x + y - 1}$$

d) $f(x,y) = xe^{xy} + y$ at $P(2,0)$, $\theta = \frac{2\pi}{3}$

$f_x(x,y) = e^{xy} + xye^{xy} \Rightarrow f_x(2,0) = 1 + 0 = 1$

$f_y(x,y) = x^2 e^{xy} + 1 \Rightarrow f_y(2,0) = (2)^2 + 1 = 5$

$\vec{u} = \begin{pmatrix} \cos(\frac{2\pi}{3}) \\ \sin(\frac{2\pi}{3}) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$

$D_u f = \nabla f \cdot \vec{u} = (1 \quad 5) \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} = \frac{5\sqrt{3}}{2} - \frac{1}{2} = \frac{5\sqrt{3} - 1}{2}$