To remember:

Matrix:

$$\boxed{A\vec{v}=\lambda I_n \vec{v}}$$

ODE:-

2nd-order: $r_{1,2}=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

Case 1: $r_1 \neq r_2$ → linearly independent solutions

$y_h = C_1 \underbrace{e^{r_1 x}}_{y_1(x)} + C_2 \underbrace{e^{r_2 x}}_{y_2(x)}$ $\quad C_1, C_2 \in \mathbb{R}$

Case 2: $r_1 = r_2$ →

$y_1(x)=e^{rx}$, $y_2(x)=xe^{rx}$

$y_h = C_1 e^{rx} + C_2 x e^{rx}$ $\quad C_1, C_2 \in \mathbb{R}$

Case 3: $r_{1,2} = \alpha \pm \beta i$ →

$y_h = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$

$C_1, C_2 \in \mathbb{R}$

Inhomogeneous:

$f(x)$	Assumed y_p
Cons. (5, 7, -2, ...)	A
$3x-2$	$Ax+B$
$7x^2$ $3x^2+2x-1$	Ax^2+Bx+C
num. ← $5e^{\alpha x}$	$Ae^{\alpha x}$
$3xe^{\alpha x}$	$(Ax+B)e^{\alpha x}$
$2\sin(\alpha x)$	$A\sin(\alpha x)+B\cos(\alpha x)$
$5\cos(\alpha x)$	$A\sin(\alpha x)+B\cos(\alpha x)$

Example 1: $f(x)=x\cos 3x$

Let

$y_p=(Ax+B)(C\cos(3x)+D\sin(3x))$

Example 2: $f(x)=xe^x+\cos 2x$

Let

$y_{p_1}=(Ax+B)e^x \xrightarrow{\text{Solve}} y_{p_1}''+y_{p_1}'+y_{p_1}=xe^x$

$y_{p_2}=(C\cos 2x+D\sin 2x) \xrightarrow{\text{Solve}} y_{p_2}''+y_{p_2}'+y_{p_2}=\cos 2x$, then $\underline{y_p=y_{p_1}+y_{p_2}}$

Multivariate function:

Tangent plane equ.: $\boxed{T(x,y)=f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)}$

In other form:

$$T(x,y)=\underbrace{f(a,b)}_{z_0}+\underbrace{\begin{pmatrix} f_x(a,b) & f_y(a,b)\end{pmatrix}}_{\Delta f(a,b)}\begin{pmatrix}(x-a)\\(y-b)\end{pmatrix}$$

$$\boxed{z-z_0=\nabla f(x_0,y_0)\begin{pmatrix}x-x_0\\y-y_0\end{pmatrix}=\begin{pmatrix} f_x(x_0,y_0) & f_y(x_0,y_0)\end{pmatrix}\begin{pmatrix}x-x_0\\y-y_0\end{pmatrix}}$$

Directional derivative:

diretional derivatives

$$D_u f = \underbrace{\begin{pmatrix} f_x & f_y \end{pmatrix}}_{\nabla f} \cdot \vec{u}$$

The angle between

$$D_u f = \nabla f \cdot \vec{u} = |\nabla f|\,|\vec{u}| \cos\theta = |\nabla f| \cos\theta$$

maximum when $\theta = 0 \rightarrow \cos(0) = 1$

Local extreme:

2- Qualifying critical points (extrema)

$\hookrightarrow D = f_{xx} \cdot f_{yy} - (f_{yx})^2 \longrightarrow$

$$D = \begin{array}{c|cc|c} & x & y & \\ \hline & f_{xx} & f_{xy} & x \\ & f_{yx} & f_{yy} & y \end{array}$$

(3D)

$$D = \begin{array}{c|ccc|c} & x & y & z & \\ \hline & f_{xx} & f_{xy} & f_{xz} & x \\ & f_{yx} & f_{yy} & f_{yz} & y \\ & f_{zx} & f_{zy} & f_{zz} & z \end{array}$$

(4D)

if $D < 0 \longrightarrow$ saddle point

if $D > 0 \begin{cases} f_{xx} < 0 \longrightarrow \text{Local maximum} \\ f_{xx} > 0 \longrightarrow \text{Local minimum} \end{cases}$

Series:

Formulas:

Require Radian measurement

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \text{for all } x$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \text{for all } x$$

(Do not use them for large x; No problem, we only need them till $\frac{\pi}{4}$.)

Binomial series

$$(1+x)^K = 1 + \frac{K}{1}x + \frac{K(K-1)}{2 \cdot 1}x^2 + \frac{K\cdot(K-1)(K-2)}{3 \cdot 2 \cdot 1}x^3 + \cdots \quad \text{for } -1 < x < 1$$

(for K = positive integer this is a finite sum)

$$\sqrt{1+x} = (1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\cdot(-\frac{1}{2})}{2 \cdot 1} \cdot x^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3 \cdot 2 \cdot 1} x^3 + \cdots$$

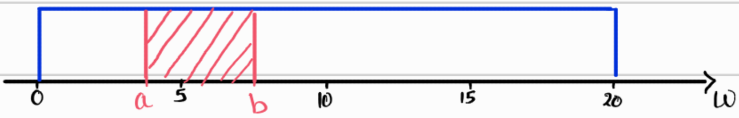
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{K=0}^{\infty} \frac{x^K}{K!} \quad \text{for } -\infty < x < \infty$$

$$\ln z = (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \frac{(z-1)^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(z-1)^n}{n} \quad \text{for } 0 < z < 2$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

Probability:

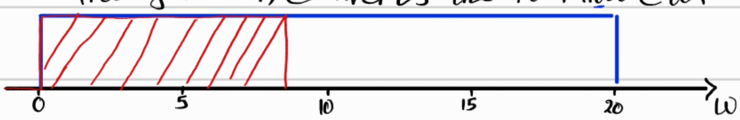
* Probability Density Function (PDF):

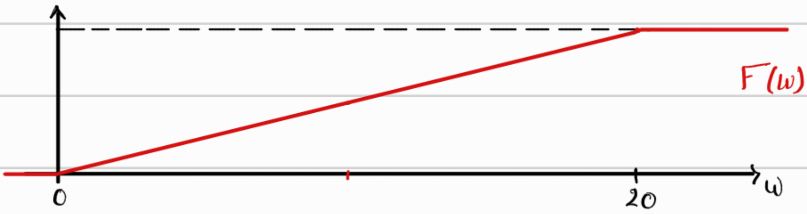


The probability is given by the area

$$P(a \le w \le b) = \underbrace{\int_a^b \underbrace{f(w)}_{\text{Pdf}}\, dw}_{\text{Probability}}$$

- Integeration converts the Pdf into Cdf: $F(w) = \int_{-\infty}^{w} \underbrace{f(x)}_{\text{Pdf}} dx$





$F(w) = P(W = w)$

A Pdf $f(x)$ must satisfy:

(i) $f(x) \geq 0$ for all x

(ii) $\int_{-\infty}^{+\infty} f(x)\, dx = 1$

Both properties are directly related to Kolmogorov's Axioms.

Extpected value (= mean):

$E[X] = \sum_x x \cdot P(X = x)$, if x is a Discrete R.V.

$E[x] = \int_{-\infty}^{\infty} x\, f(x)\, dx$, if x is continuous with pdf $f(x)$.

$$\boxed{E[X + Y] = E[X] + E[Y]}$$

Linear Transformation

$$\boxed{E[a \cdot X + b] = a \cdot E[X] + b}$$

Valid for both discrete & continuous R.V.

Non-Linear Transformation

$$E[g(x)] = \begin{cases} \sum_x g(x) \cdot P(X = x) \\ \int_{-\infty}^{\infty} g(x) \cdot f(x)\, dx \end{cases}$$

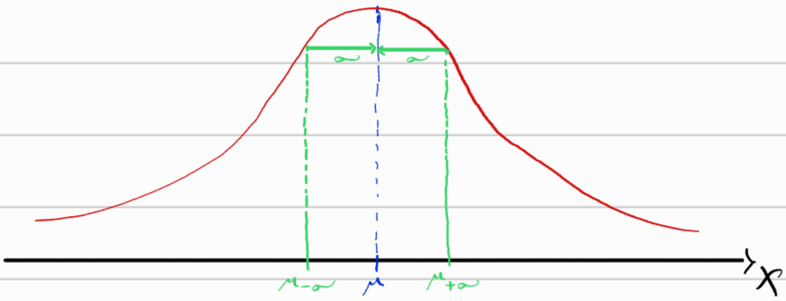
$$(\text{Variance} = \sigma^2) = Var(X) = E[(X - \mu)^2] = \boxed{E[X^2] - \underbrace{\mu^2}_{(E[X])^2}}$$

with

$$\boxed{E[X^2] = \begin{cases} \sum_x x^2 \cdot P(X = x) & \text{If } x \text{ is Discrete R.V.} \\ \int_{-\infty}^{+\infty} x^2 f(x)\, dx & \text{If } x \text{ is Continuous R.V. with Pdf } f(x) \end{cases}}$$

A Random variable with a normal distribution is continuous with

Pdf $f(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ for $-\infty < X < +\infty$

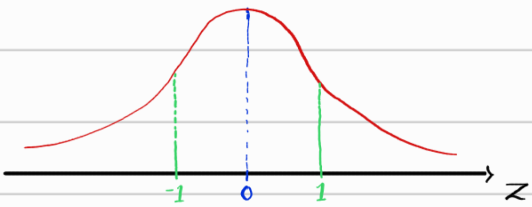


- Maximum at $\mu = E[X]$
- Symmetric about μ
- Points of inflection at $\mu \pm \sigma$
- everywhere positive
- Horizontal asymptote at $y = 0$

$$\boxed{Z = \frac{X - \mu_x}{\sigma_x}}$$

R.V. with standard Normal distribution is usually denoted by Z :-

Pdf $\phi(z) = \dfrac{1}{\sqrt{2\pi}} \cdot e^{-\frac{z^2}{2}}$



cdf $\Phi(z) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{z} e^{-\frac{t^2}{2}} \, dt$