Tangent plane equation:-

$a = x_0$, $b = y_0$

$$\boxed{T(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$$

In other form:-

$$T(x,y) = \underbrace{f(a,b)}_{z_0} + \underbrace{\begin{pmatrix} f_x(a,b) & f_y(a,b) \end{pmatrix}}_{\Delta f(a,b)} \begin{pmatrix} (x-a) \\ (y-b) \end{pmatrix}$$

$$\boxed{z - z_0 = \nabla f(x_0, y_0) \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}}$$

Example: $f(x,y) = \sin\left(\frac{x}{1+y}\right)$ at $(\pi, 2)$, and approximate $f(3,2)$

$$f(\pi, 2) = \sin\left(\frac{\pi}{3}\right) = \frac{2}{\sqrt{3}}$$

$$f_x(x,y) = \cos\left(\frac{x}{1+y}\right) \cdot \frac{1}{1+y}$$

$$f_x(\pi, 2) = \cos\left(\frac{\pi}{3}\right) \cdot \frac{1}{3} = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$$

$$f_y(x,y) = \cos\left(\frac{x}{1+y}\right) \cdot \frac{-x}{(1+y)^2}$$

$$f_y(\pi, 2) = \cos\left(\frac{\pi}{3}\right) \cdot \frac{-\pi}{(3)^2} = -\frac{\pi}{18}$$

$$T(x,y) = f(a,b) + \Delta f(a,b) \begin{pmatrix} x-a \\ y-b \end{pmatrix} = \frac{2}{\sqrt{3}} + \frac{1}{6}(x-\pi) - \frac{\pi}{18}(y-2) \Rightarrow \frac{1}{6}x - \frac{\pi}{18}y + \left(\frac{2}{\sqrt{3}} - \frac{\pi}{18}\right)$$

approx. of $f(x,y)$ at $(3,2)$

$$T(3,2) = \frac{3}{2} - \frac{\pi}{18}(2) + \frac{2}{\sqrt{3}} - \frac{\pi}{18}$$

$x^3+4xy^2+y^3=107$ at $P(2,3)$ find tangent line: (implicit differention)

$y-y_1=m(x-x_1)$

$$\frac{d}{dx}\left[x^3+4xy^2+y^3=107\right] \Rightarrow 3x^2+4y^2+4x(2y)\frac{dy}{dx}+(3y^2)\frac{dy}{dx}=0$$

$$\Rightarrow 3(2)^2+4(3)^2+4(2)(2(3))\frac{dy}{dx}+(3(3)^2)\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{12+36}{48+27} \Rightarrow \frac{dy}{dx}=m=-\frac{48}{75}=\frac{-16}{25}$$

$$y-(3)=\left(-\frac{16}{25}\right)(x-2) \Rightarrow 25y-75=-16x+32$$

$$\Rightarrow \boxed{25y+16x=107}$$

Chain Rule

upgrading to MV

Case 1

Suppose $z=f(x,y)$ is a differentiable function of x and y, where $x=g(t)$ and $y=h(t)$ are both differentiable functions of t.

Then z is a differentiable function of t and:

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}$$

z is a function of one variable only (t)

Case 2

Suppose $z=f(x,y)$ is a differentiable function of x and y, where $x=g(s,t)$ and $y=h(s,t)$ are differentiable functions of s and t

Then z is a differentiable function of s and t, and its partial derivatives are given by:

$$\frac{\partial z}{ds}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

now z is a function of more than one variable

Example

$z=e^x\sin y$ where $x=st^2$ and $y=s^2t$

find the partial derivatives $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$

Solution:

Chain rule:

to complete the partial derivatives - on what variables of the function z does s depend?

$$\frac{\partial z}{ds}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$=e^x\sin y\cdot t^2+(-e^x\cos y)\cdot 2st$$

$$=e^{st^2}\sin(s^2t)\cdot t^2+(-e^{st^2}\cos(s^2t))\cdot 2st$$

$$=te^{st^2}\left(t\sin(s^2t)-2s\cos(s^2t)\right)$$

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$=e^x\sin y\cdot 2st+e^x\cos y\cdot s^2$$

$$=se^{st^2}\left(2t\sin(s^2t)+s\cos(s^2t)\right)$$

Directional Derivatives:-

Theorem:

If f is a differentiable function of x and y, then f has a directional derivative in the direction of a vector $\overrightarrow{v}$ given by:

$$D_u f = \nabla f \cdot \overrightarrow{u}$$

$$\text{where } \overrightarrow{u} = \frac{1}{|\overrightarrow{v}|}\overrightarrow{v}$$

$$\overbrace{D_u f}^{\text{directional derivatives}} = \underbrace{\begin{pmatrix} f_x & f_y \end{pmatrix}}_{\nabla f} \cdot \vec{u}$$

* The maximum value of the directional derivative $D_u f$ is $|\nabla f| = \sqrt{f_x^2(a,b) + f_y^2(a,b)}$, and it occurs when $\vec{u}$ has the same direction as gradient vector ∇f.

The angle between

$$D_u f = \nabla f \cdot \vec{u} = |\nabla f|\,|\vec{u}| \cos\theta = |\nabla f| \cos\theta$$

maximum when $\theta = 0 \rightarrow \cos(0) = 1$

From the book $\boxed{14.6}$

4) $f(x,y) = x^3y^4 + x^4y^3$ at $(1,1)$, $\theta = \pi/6$

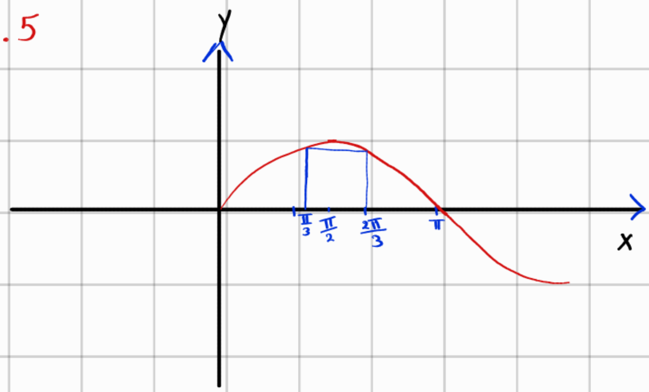
$f_x = 3y^4x^2 + 4y^3x^3 \Rightarrow f_x(1,1) = 3 + 4 = 7$

$f_y = 4x^3y^3 + 3x^4y^2 \Rightarrow f_y(1,1) = 4 + 3 = 7$

$\vec{u} = \begin{pmatrix} \cos(\pi/6) \\ \sin(\pi/6) \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}$

$D_u f = \nabla f \cdot \vec{u} = \begin{pmatrix} 7 & 7 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} \\ 1/2 \end{pmatrix} = \frac{7\sqrt{3}}{2} + \frac{7}{2} = \frac{7(1+\sqrt{3})}{2} \approx 9.5$

5) $f(x,y) = ye^{-x}$, at $(0,4)$, $\theta = 2\pi/3$



$f_x = -ye^{-x} \Rightarrow f_x(0,4) = -4e^0 = -4$

$f_y = e^{-x} \Rightarrow f_y(0,4) = e^0 = 1$

$\vec{u} = \begin{pmatrix} \cos(2\pi/3) \\ \sin(2\pi/3) \end{pmatrix} = \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}$

$D_u f = \nabla f \cdot \vec{u} = \begin{pmatrix} 1 & -4 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} \\ \sqrt{3}/2 \end{pmatrix} = -\frac{1}{2} - \frac{4\sqrt{3}}{2} = \frac{-1-4\sqrt{3}}{2} \approx -4$

a) Find gradient f.

b) Evaluate the gradient at Point P.

C) Find the rate of change of f at P in the direction of the vector $\vec{u}$.

8) $f(x,y) = y^2/x$, $P(1,2)$, $u = \frac{1}{3}(2i + \sqrt{5}\,j)$

$f_x = \frac{-y^2}{x^2} \Rightarrow f_x(1,2) = \frac{-(2)^2}{(1)^2} = -4$

$f_y = 2\frac{y}{x} \Rightarrow f_y(1,2) = 2\,\frac{2}{1} = 4$

$\vec{u} = \begin{pmatrix} 2/3 \\ \frac{\sqrt{5}}{3} \end{pmatrix}$

$D_u f(x,y) = \nabla f \cdot \vec{u} = \left(\frac{-y^2}{x^2} \quad \frac{2y}{x}\right)\begin{pmatrix} 2/3 \\ \frac{\sqrt{5}}{3} \end{pmatrix} = \frac{-2y^2}{3x^2} + \frac{2\sqrt{5}\,y}{3x}$

$D_u f(1,2) = -4\left(\frac{2}{3}\right) + 4\left(\frac{\sqrt{5}}{3}\right) = \frac{4\sqrt{5}-8}{3}$

Find the directional derivative of the function at the given point in the direction of $\vec{v}$:-

11) $f(x,y) = e^x \sin y$, $(0, \pi/3)$, $v = \langle -6, 8 \rangle$

$f_x = e^x \sin y \Rightarrow f_x(0,\pi/3) = e^0 \sin(\pi/3) = \frac{\sqrt{3}}{2}$

$f_y = e^x \cos y \Rightarrow f_y(0,\pi/3) = e^0 \cos(\pi/3) = \frac{1}{2}$

$|\vec{v}| = \sqrt{(-6)^2+(8)^2} = 10$

$\vec{u} = \frac{1}{10}\vec{v}$

$D_u f = \nabla f \cdot \vec{u} = \left(\frac{\sqrt{3}}{2} \quad \frac{1}{2}\right)\begin{pmatrix} -\frac{3}{5} \\ \frac{4}{5} \end{pmatrix} = \frac{-3\sqrt{3}}{10} + \frac{4}{10} = \frac{4-3\sqrt{3}}{10}$

* Find the maximum rate of change of f :

21) $f(x,y) = 4y\sqrt{x}$, $(4,1)$

$f_x = \frac{2y}{\sqrt{x}} \Rightarrow f_x(4,1) = \frac{2(1)}{\sqrt{4}} = 1$

$f_y = 4\sqrt{x} \Rightarrow f_y(4,1) = 4\sqrt{4} = 8$

$\nabla f = (1 \quad 8)$, $|\nabla f| = \sqrt{8^2+1^2} = \sqrt{65}$

$D_u f = |\nabla f| \cdot \underbrace{\cancel{|u|}}_{1} \underbrace{\cancel{\cos\theta}}_{1}$ for maximum $D_u f$. $\Rightarrow D_u f(4,1) = |\nabla f| = \sqrt{65}$

Local extreme:-

1- Find critical points of $f(x,y)$

by setting $\left.\begin{array}{l} f_x(x,y)=0 \\ \& \ f_y(x,y)=0 \end{array}\right\}$ two equations with two unknows $(x \& y)$

2- Qualifying critical points (extrema)

$\hookrightarrow D = f_{xx} \cdot f_{yy} - (f_{yx})^2 \longrightarrow$

$$D = \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} \end{array} \begin{array}{c} \\ x \\ y \end{array}$$

(3D)

$$D = \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} \end{array} \begin{array}{c} \\ x \\ y \\ z \end{array}$$

(4D)

if $D<0 \longrightarrow$ saddle point

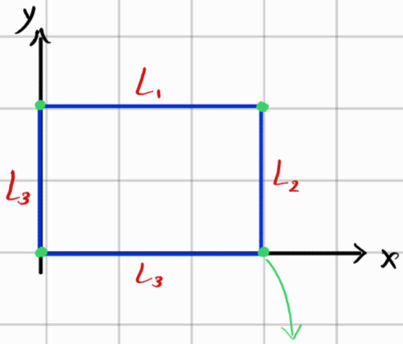
if $D>0 \begin{cases} f_{xx}<0 \longrightarrow \text{Local maximum} \\ f_{xx}>0 \longrightarrow \text{Local minimum} \end{cases}$

Global extreme:-

1- Find critical points of $f(x,y)$

by setting $\left.\begin{array}{l} f_x(x,y)=0 \\ \& \ f_y(x,y)=0 \end{array}\right\}$ two equations with two unknows $(x \& y)$

2- Draw The domain in x-y plane



3- Find all critical points on each line of the domain & all edges

4- Make a table & plug all points on the original function to find global extreme.