Name: ______________________________

Problem 1: (3+3+8 points) Consider the matrices

$$A = \underbrace{\begin{pmatrix} \boxed{0} & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}}_{4\times 4}, \quad B = \underbrace{\begin{pmatrix} 1 & 2 \\ 3 & -1 \\ 0 & 2 \\ 4 & 1 \end{pmatrix}}_{4\times 2}, \quad C = \underbrace{\begin{pmatrix} 3 & -2 & 0 & 5 \\ 1 & 0 & -3 & 4 \end{pmatrix}}_{2\times 4}$$

a) Does $D = ABC$ exist? If so, compute its entry d_{24}. Otherwise, explain why it does not.

b) Does $E = BAC$ exist? If so, compute its entry e_{22}. Otherwise, explain why it does not.

c) Find A^{-1}, if it exists. Otherwise explain, why it does not.

a) yes. $AB = \begin{pmatrix} 7 & 2 \\ 5 & 5 \\ 8 & 2 \\ 4 & 3 \end{pmatrix}$, $d_{24} = 5\times 5 + 4\times 5 = 45$

b) No, because $\underset{4\times\boxed{2}}{B}$ $\underset{\boxed{4}\times 4}{A}$, The number of columns of B $\neq$ the num. of rows in A.

c)
$$(A \mid I_4) = \left(\begin{array}{cccc|cccc} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \mathrm{I}+\mathrm{II} \\ \mathrm{II}+\mathrm{III} \\ \mathrm{III}+\mathrm{IV} \\ \end{array}$$

$$= \left(\begin{array}{cccc|cccc} 1 & 1 & 2 & 2 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 2 & 0 & 1 & 1 & 0 \\ 2 & 2 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \mathrm{II}-\mathrm{III} \\ \mathrm{III}-2\mathrm{I} \\ \mathrm{IV}+\mathrm{III} \end{array} \rightarrow \left(\begin{array}{cccc|cccc} 1 & 1 & 2 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & -3 & -3 & -2 & -2 & 1 & 1 \\ 3 & 3 & 2 & 1 & 0 & 0 & 1 & 2 \end{array}\right) \begin{array}{l} \\ \\ \mathrm{III}/-3 \\ \mathrm{IV}-3\mathrm{I} \end{array}$$

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 2 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & \frac{2}{3} & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & -4 & -5 & -3 & -3 & 1 & 2 \end{array}\right) \begin{array}{l} \\ \\ \\ \mathrm{IV}+4\mathrm{III} \end{array} \rightarrow \left(\begin{array}{cccc|cccc} 1 & 1 & 2 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & \frac{2}{3} & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & -1 & -\frac{1}{3} & -\frac{1}{3} & \frac{7}{3} & \frac{10}{3} \end{array}\right) \begin{array}{l} \mathrm{I}+2\mathrm{IV} \\ (\mathrm{II}+\mathrm{IV})(-1) \\ \mathrm{III}+\mathrm{IV} \\ -\mathrm{IV} \end{array} \rightarrow$$

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 2 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{14}{3} & \frac{20}{3} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & \frac{-2}{3} & -\frac{7}{3} & -\frac{7}{3} \\ 0 & 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{8}{3} & \frac{11}{3} \\ 0 & 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{-7}{3} & -\frac{10}{3} \end{array}\right) \begin{array}{l} \mathrm{I}-\mathrm{II}-2\mathrm{III} \\ \\ \\ \\ \end{array} \Longrightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -\frac{2}{3} & \frac{1}{3} & \frac{5}{3} & \frac{5}{3} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & \frac{-2}{3} & -\frac{7}{3} & -\frac{7}{3} \\ 0 & 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{8}{3} & \frac{11}{3} \\ 0 & 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{-7}{3} & -\frac{10}{3} \end{array}\right) = (I_4 \mid A^{-1})$$

$$(A) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{array}{l} \\ \mathrm{II}-\mathrm{III} \\ \mathrm{III}-\mathrm{IV} \\ \mathrm{IV}-\mathrm{I} \end{array} \qquad A^{-1} = \frac{1}{3}\begin{pmatrix} -2 & 1 & 5 & 5 \\ 1 & -2 & -7 & -7 \\ 1 & 1 & 8 & 11 \\ 1 & 1 & -7 & -10 \end{pmatrix}$$

$$(A) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{array}{l} \mathrm{I}+\mathrm{II} \\ \\ \\ \\ \end{array}$$

$$\det(A) = -\begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = +1\begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}$$

$$= 1 + 1 - (-1) = 3$$

$$A = \begin{pmatrix} 0 & 0 & 2 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \qquad M = \begin{pmatrix} 0 & 0 & -2 & 1 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 3 & 0 & 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 & -2 & -1 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ -3 & 0 & 0 & 0 \end{pmatrix} \Longrightarrow C^T = \begin{pmatrix} 0 & 0 & 0 & -3 \\ 0 & -3 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ -1 & 0 & -2 & 0 \end{pmatrix}$$

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Problem 2: (8 Points)
Solve the IVP
$$\frac{dy}{dx} = \frac{x \cdot \cos x}{\boxed{y}}\ , \quad y(0) = -4\ .$$

Seperable,

$$\int y\,dy = \int x \cos x\,dx$$

$u = x \qquad dv = \cos x\,dx$

$du = dx \qquad v = \sin x$

$$y = x \sin x - \int \sin x\,dx$$

$$y(x) = x \sin x + \cos x + C$$

$$y(0) = (0)\sin(0) + \cos(0) + C = -4$$

$$C + 1 = -4 \Rightarrow \boxed{C = -5}$$

$$\boxed{y(x) = x \sin x + \cos x - 5}$$

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Problem 3: (4+8+4 Points)

a) Use a power series to compute $\int \frac{e^x - 1}{\boxed{x}} \, dx$.

b) Compute the first four nonzero summands of the Taylor series of $f(x) = \sqrt[3]{x}$ about $a = \boxed{8}$. Write the coefficients as fractions.

c) For which x does $\sum_{n=1}^{\infty} (x + \boxed{2})^n$ converge. For those x compute the sum.

a) $\int \frac{e^x - 1}{x} \, dx$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots$$

$$e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!} = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\frac{e^x - 1}{x} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots$$

$$\int \frac{e^x - 1}{x} dx = \int \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} dx = C + \sum_{n=1}^{\infty} \frac{x^n}{n(n!)}$$

b) $f(8) = \sqrt[3]{8} = 2$

$f'(x) = \frac{1}{3} x^{-\frac{2}{3}} \Rightarrow f'(8) = \frac{1}{3} \frac{1}{\sqrt[3]{8^2}} = \frac{1}{12}$

$f''(x) = -\frac{2}{9} x^{-\frac{5}{3}} \Rightarrow f''(8) = -\frac{2}{9} \frac{1}{(2)^5} = \frac{-1}{144}$

$f'''(x) = \frac{10}{27} x^{-\frac{8}{3}} \Rightarrow f'''(8) = \frac{10}{27} \frac{1}{(2)^8} = \frac{5}{3456}$

1280 2560

640, 256 } 896

$f''''(x) = \frac{-80}{81} x^{-\frac{11}{3}} \Rightarrow f''''(8) = \frac{\cancel{-80}\ 10}{81} \cdot \frac{1}{(2)^{\cancel{11}\,8}} = \frac{-10}{81(256)} = \frac{-10}{20736}$ → we don't need

$$f(x) = f(a) + \frac{f'(a)}{1}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \quad \text{for } |x-a| < 1$$

$$f(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 + \frac{5}{20736}(x-8)^3 + \cdots \quad \text{for } |x-8| < 1$$

$7 < x < 9$

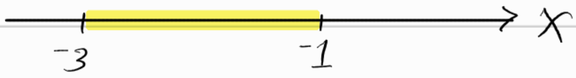
Radius of Convergence $R = 1$

$\sum_{n=1}^{\infty} (x+2)^n$ as a Geometric series $|x+2|<1$

$-3<x<-1$

$x \in (-3,-1)$

$$\sum_{n=1}^{\infty} (x+2)^n = \frac{1}{1-(x+2)}$$ for $-3<x<-1$



$$= \frac{1}{-1-x}$$

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Problem 4: (5+3+2+3 Points) A test for contamination of water samples is *positive*, if it is suspected to be contaminated. If a sample is really contaminated then 99% of the tests will be positive. If a sample is not contaminated, then still 5% of the tests will be positive. Only $\boxed{2}$% of the water samples are contaminated.

a) What are the relevant events? Introduce names for them and express the information from the text in proper symbolic notation.

b) Plot a suitable probability tree.

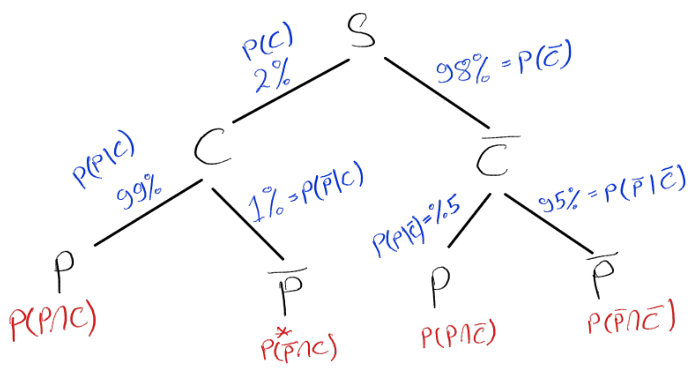
c) Find the probability that a random water sample tests *negative*.

d) If a sample tests *positive*, what is the probability that it actually is contaminated?

a) C = "An event that randomly chosen sample of water is Contaminated"

P = "An event that a randomly chosen sample of water is tested positive"

b)



c) $P(\bar{P}) = P(\bar{P}\cap C) + P(\bar{P}\cap \bar{C}) = P(C)\cdot P(\bar{P}|C) + P(\bar{C})\cdot P(\bar{P}|\bar{C})$

$= 0.02\,(0.01) + 0.98\,(0.95)$

$= 0.9312$

d) $$P(C|P) = \frac{P(C\cap P)}{P(P)} = \frac{P(C\cap P)}{P(P\cap C)+P(P\cap \bar{C})} = \frac{P(C)\cdot P(P|C)}{P(C)\cdot P(P|C)+P(\bar{C})\cdot P(P|\bar{C})} = \frac{0.02\,(0.99)}{\underbrace{0.02(0.99)}_{0.0198}+\underbrace{0.05(0.98)}_{0.049}}$$

$$= \frac{0.0198}{0.0688} = \frac{99}{344}$$