HOCHSCHULE RHEIN-WAAL
Rhine-Waal University of Applied Sciences

Prof. Dr. Alexander Struck
Prof. Dr. Matthias Krauledat

# Numerical Mathematics

## 13) Newton-Raphson

**a)** Use the Newton-Raphson algorithm to solve the system of equations

$$\begin{aligned} e^{-x_1} - x_2 &= 0 \quad (1) \\ x_1 + x_2^2 - 3x_2 &= 0 \quad (2) \end{aligned}$$

using initial guesses $\vec{x}^{(0)} = 0$. Plot the values $x_1^{(k)}, x_2^{(k)}$ as a function of iteration number $k$.

**b)** Modify the program in a) for a switched order of equations and find the solution for the same initial guess. $x_1^{(k)}, x_2^{(k)}$ as a function of iteration number $k$.

## 14) Implicit Euler, but don't bother with Newton

Apply the backward (implicit) Euler method to solve the IVP

$$y' = -y^2, \quad y(0) = 1$$

Calculate $y_i$ for $i = 1, 2, 3$ on paper and plot the graph with more points on the computer. Use $h = 0.1$. Note that you do not have to employ a fixed-point method to solve for $y_{i+1}$, there is an easier way!

## 15) Explicit Euler ?

Consider the ODE

$$\frac{dy}{dx} = \sin\left(x \cos(y)\right) \quad (3)$$

with the IVP

$$y(x = 0) = 0 \quad (4)$$

**a)** Apply the explicit Euler method to the IVP and write down the first three iterations on paper.

**b)** Write a program to solve the IVP, using $h = 0.01$. Examine the solutions for $x \in [0, 2.5]$, for $x \in [0, 250]$ and for $x \in [0, 2500]$. Discuss the results.

**c)** Repeat the task in b) for $h = 0.001$. Compare and discuss the results.

**d)** Repeat b) to d) with the implicit Euler method.

13) a) $\boxed{e^{-x_1} - x_2 = 0} \rightarrow f_1$ use Newton

$\boxed{x_1 + x_2^2 - 3x_2 = 0} \rightarrow f_2$

$$\begin{bmatrix} x_1^K \\ x_2^K \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underbrace{\begin{array}{cc} x_1 & x_2 \\ \frac{d}{dx_1} f(x_1) & \frac{d}{dx_2} f(x_1) \\ \frac{d}{dx_1} f(x_2) & \frac{d}{dx_2} f(x_2) \end{array}}_{\underline{\underline{J}} = \frac{\partial f}{\partial \underline{x}}} \underbrace{\begin{bmatrix} x_1^{K+1} - x_1^K \\ x_2^{K+1} - x_2^K \end{bmatrix}}_{\delta x^K} = \underbrace{- \begin{bmatrix} f_1(x^K) \\ f_2(x^K) \end{bmatrix}}_{-f(x^K)}$$

$$\Rightarrow \underline{\underline{J}} = \begin{bmatrix} -e^{-x_1} & -1 \\ 1 & 2x_2 - 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -e^{-x_1^K} & -1 \\ 1 & 2x_2^K - 3 \end{bmatrix} \begin{bmatrix} x_1^{K+1} - x_1^K \\ x_2^{K+1} - x_2^K \end{bmatrix} = \begin{bmatrix} e^{-x_1^K} - x_2^K \\ x_1^K - x_2^{2K} - 3x_2^K \end{bmatrix}$$

| $K$ | $x_1^K$ | $x_1^{K+1}$ | $x_2^K$ | $x_2^{K+1}$ |
|---|---|---|---|---|
| 0 | 0 | $-\frac{3}{4}$ | 0 | $-\frac{1}{4}$ |
| 1 | $-\frac{3}{4}$ | $-0.9925679292 - (-\frac{3}{4}) \approx -0.242568$ | $-\frac{1}{4}$ | $-0.2657336941 - (-\frac{1}{4}) \approx -0.0157337$ |

$K=0$ $\quad \underline{J}(x^K) \cdot \delta x^K = -f(x^K) \Rightarrow \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1^{K+1} \\ x_2^{K+1} \end{bmatrix} = - \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} -1 & -1 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x_1^{K+1} \\ x_2^{K+1} \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$-4x_2^{K+1} = -1 \Rightarrow x_2^{K+1} = \frac{1}{4}$$

$$\Rightarrow x_1^{K+1} = \frac{3}{4}$$

(14) $y' = -y^2$, $y(0) = 1$, $h = 0.1$

$x_n = 0 \Rightarrow y_n = 1$

Implicit Euler $y_{n+1} = y_n + h\,f(y_{n+1})$

$$\int \frac{dy}{-y^2} = \int dx$$

$$\Rightarrow \frac{1}{y} = x + C$$

at $y(0) = 1 \Rightarrow C = 1 \Rightarrow \boxed{y = \frac{1}{x+1}}$

$$\Longrightarrow y_{n+1} = 1 + 0.1(-y_{n+1}^2)$$

$$0.1 y_{n+1}^2 + y_{n+1} - 1 = 0$$

$y_{n+1} = -5 + \sqrt{35} = 0.91608$ ✓ , $y_{n+1} \approx -10.91608$

it make since that we choose the value which close to $y_n$

| x | $y_n$ | $y_{n+1}$ |
|---|---|---|
| 0 | 1 | 0.91608 |
| 0.1 | 0.91608 | 0.8447239311 |
| 0.2 | 0.844724 | 0.7833588261 |
| 0.3 | 0.783359 | 0.7300600574 |
| 0.4 | 0.73006 | 0.6833617318 |

15) $\frac{dy}{dx} = \sin(x\cos(y))$ $y(0) = 0$

$$y_{n+1} = y_n + h\,f(x_n, y_n)$$

assume: $h = 0.1$

$$y_{n+1} = y_n + h \cdot \sin(x_n \cos(y_n))$$

| x | $y_n$ | $y_{n+1}$ |
|---|---|---|
| 0 | 0 | 0 |
| 0.1 | 0 | $9.98334 \times 10^{-3}$ |
| 0.2 | $9.98334 \times 10^{-3}$ | 0.029849298 |
| 0.3 | | 0.059388 |
| 0.4 | | 0.09826542 |
| 0.5 | | 0.145996158 |

d) Implicit

$$y_{i+1} = y_i + h \cdot \sin(x_{i+1} \cdot \cos(y_{i+1}))$$

too complex to solve, use Newton:

Start Newton

$\boxed{\text{with } y_{i+1}^{(0)} = y_i}$

$$y_{i+1}^{k+1} = y_{i+1}^{k} - \frac{L(y_{i+1}^{k})}{L'(y_{i+1}^{k})}$$

$$L(y_{i+1}) = y_{i+1} - y_i - h \cdot \sin(x_{i+1} \cdot \cos(y_{i+1}))$$

$$L'(y_{i+1}) = 1 - h \cdot \cos(x_{i+1} \cdot \cos(y_{i+1})) \cdot x_{i+1} \cdot (-\sin(y_{i+1}))$$

$$y_{i+1}^{k+1} = y_{i+1}^{k} - \frac{y_{i+1}^{k} - y_i - h \cdot \sin(x_{i+1} \cdot \cos(y_{i+1}^{k}))}{1 - h \cdot \cos(x_{i+1} \cdot \cos(y_{i+1}^{k})) \cdot x_{i+1} \cdot (-\sin(y_{i+1}^{k}))}$$

TU 7)

$$f_1 = x^3 - y + z^2 = 0$$
$$f_2 = x^2 + y^3 - z = 0$$
$$f_3 = x + y + z - 3 = 0$$

$$(x_0, y_0, z_0) = (1, 2, 6)$$

$$J(x^k, y^k, z^k) \cdot \vec{x}^k = -\vec{f}(x^k, y^k, z^k)$$

$$\begin{array}{ccc} x & y & z \end{array}$$
$$\begin{bmatrix} 3x^2 & -1 & 2z \\ 2x & 3y^2 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x^{k+1} - x^k \\ y^{k+1} - y^k \\ z^{k+1} - z^k \end{bmatrix} = - \begin{bmatrix} f_1(x^k, y^k, z^k) \\ f_2(x^k, y^k, z^k) \\ f_3(x^k, y^k, z^k) \end{bmatrix}$$

$K = 1$

$$\begin{bmatrix} 3 & -1 & 12 \\ 2 & 12 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x^{k+1} - 1 \\ y^{k+1} - 2 \\ z^{k+1} - 6 \end{bmatrix} = \begin{bmatrix} -35 \\ -3 \\ -6 \end{bmatrix}$$

$$x^{k+1} = -\frac{14}{3} + 1 = -\frac{11}{3}$$

$$y^{k+1} = \frac{5}{13} + 2 = \frac{31}{13}$$

$$z^{k+1} = -\frac{67}{39} + 6 = \frac{167}{39}$$