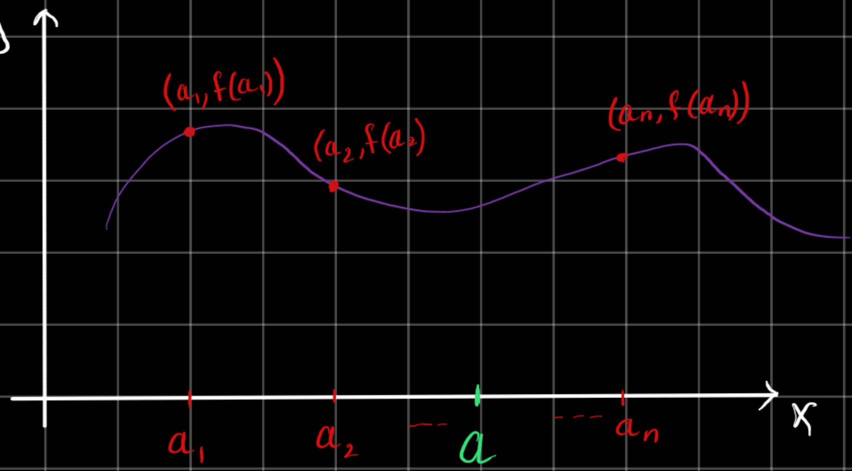
Numerical Differentiation:

{ Lec 10 & 11 }



Finite difference formula: (Linear combination of data values)

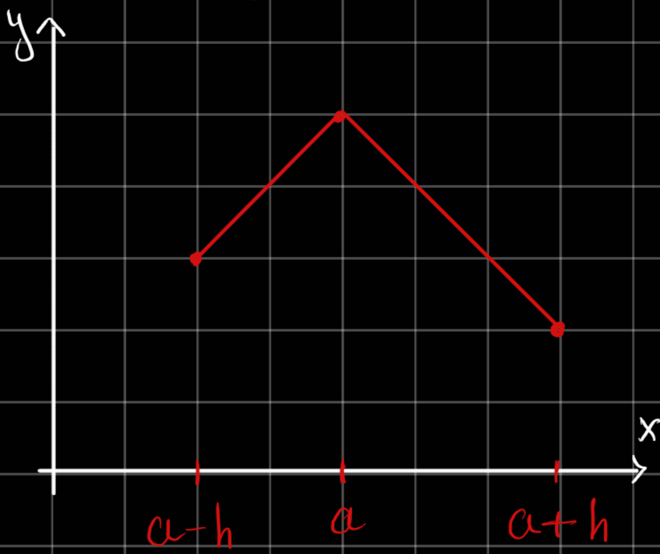
$$\boxed{f'(a) \approx C_1 f(a_1) + C_2 f(a_2) + \cdots + C_n f(a_n)}$$

How do we find the coefficients $C_1, C_2, \ldots, C_n$ depending on $a_1, a_2, a_3, \ldots, a_n$, and a?

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$ for functions

$$f'(a) \approx \frac{f(a+h) - f(a)}{h}$$ for discrete points

* getting rid of limit process, introduces a Truncation Error



$$\boxed{f'(a) \approx \frac{f(a+h) - f(a)}{h}}$$ Forward difference

$$\boxed{f'(a) \approx \frac{f(a-h) - f(a)}{-h}}$$ Backward difference

$$f'(a) \approx \frac{1}{2}\left(\text{Forward difference} + \text{Backward difference}\right) \rightarrow \boxed{f'(a) \approx \frac{f(a+h) - f(a-h)}{2h}}$$ Centered difference

How good are these formulas?

Taylor series help!

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

* Use this for forward difference:

$$f(\underbrace{a+h}_{x}) = f(a) + f'(a)\cdot \underbrace{h}_{(a+h-a)} + f''(a)\cdot h^2 + f'''(a)\cdot h^3 + \cdots$$

Dominant term of the truncation error.

$$f'(a) \cong \frac{f(a+h)-f(a)}{h} = f'(a) + \underbrace{\frac{f''(a)}{2!}h + \frac{f'''(a)}{3!}h^2 + \cdots}_{\text{Truncation Error}}$$

Differentiation via Lagrange Interpolation:

$$f_n^{(m)} :\approx \frac{d^m}{dx^m} P_n(x)$$

defined as

Lagrange polynomial

$$\binom{n}{k} = \frac{n!}{(n-k)! \cdot k!}$$

for case $n = m$: $\approx \sum_{i=0}^{n} y_i \frac{(-1)^{n-i}}{h^n} \binom{n}{i}$

$$h = \frac{x_n - x_0}{n} = \frac{b-a}{n}$$

biggest x-value

The least value

for case $n > m$ $\approx \frac{d^m}{dx^m} \sum_{i=0}^{n} y_i \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$

Assuming: 3 data points, 1st derivative

$f'(x_0) = \rightarrow$ Forward difference

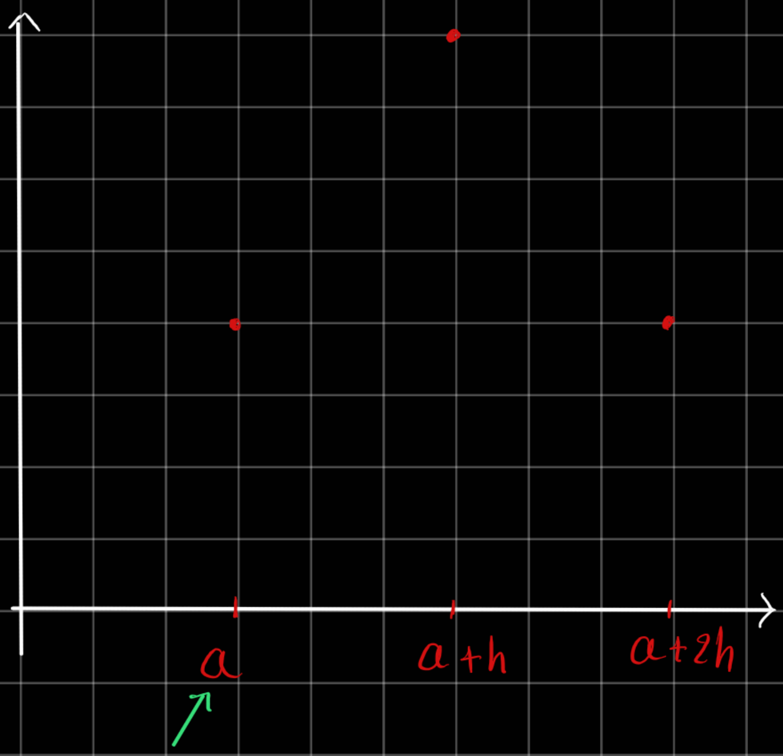
$f'(x_1) = \rightarrow$ Central diff.

$f'(x_2) = \rightarrow$ backward diff.

assume: Aequidistant nodes

usually $x_0, x_n, \& n$ are Know, from that we can get the step size (h), and then we can get all (any) $x_i = x_0 + i \cdot h$

When we need only one specific finite difference formula, the following method using Taylor series is often easier & faster than using Lagrange Interpolation.



Want derivative at a

$f'(a) \approx C_1 f(a) + C_2 f(a+h) + C_3 f(a+2h)$

Goal: Approximate with the highest possible order.

Definition

The Tylor series of f about a is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

A Tylor series about 0,

$$f(0) + f'(0)\cdot x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

is called Maclawrin series

Workseet 5:

Q1: $f(x) = e^x$

Tylor series around $(x=0)$:

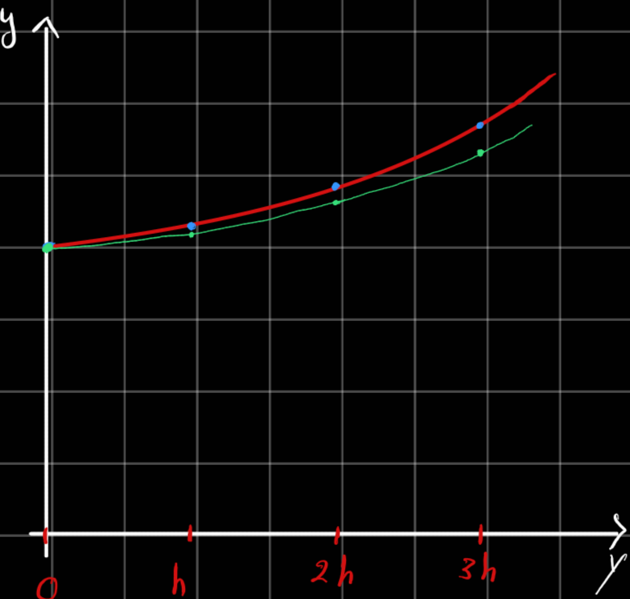
$$F(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

a) compute the divided difference $f'(x)$ using $[0, h, 2h, 3h]$ for $(h = 0.1)$

b) compare the computed & the actual value.

$$F'(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

$$f'(x) = e^x$$



x	$F'(x)$	$f'(x)$	$f'(x) - F'(x)$
0	1	1	0
0.1	1.105170833	1.105170918	8.474×10^{-8}
0.2	1.2214	1.2214027758	2.75816×10^{-6}
0.3	1.34984	1.34986	2.13×10^{-5}

b) comparing $F'(x)$ & $f'(x)$ we can see that the high the (h) goes, the bigger the error we have between the actual & approximated functions.

Question 2:

Consider the function

$f(x) = x + \sin(x)$.

a) Use the central difference formula with a step size of h to approximate the first derivative of f(x) at x = $\pi/3$. Calculate the approximation for h = 0.1 and h = 0.01.

b) Compare the approximations obtained in part (a) with the exact value of $f'(\pi/3) = 1 + \cos(\pi/3)$.

Centered difference formula

$$f'(a) \approx \frac{f(a+h) - f(a-h)}{2h} \qquad a = \frac{\pi}{3}$$

at $h = 0.1 \longrightarrow f'\left(\frac{\pi}{3}\right) \approx \frac{\left(\left(\frac{\pi}{3}+0.1\right) + \sin\left(\frac{\pi}{3}+0.1\right)\right) - \left(\left(\frac{\pi}{3}-0.1\right) + \sin\left(\frac{\pi}{3}-0.1\right)\right)}{2(0.1)}$

≈ 1.499167083

at $h = 0.01 \longrightarrow f'\left(\frac{\pi}{3}\right) \approx \frac{\left(\left(\frac{\pi}{3}+0.01\right) + \sin\left(\frac{\pi}{3}+0.01\right)\right) - \left(\left(\frac{\pi}{3}-0.01\right) + \sin\left(\frac{\pi}{3}-0.01\right)\right)}{2(0.01)}$

≈ 1.499991667

b) actual value $f'\left(\frac{\pi}{3}\right) = 1 + \cos\left(\frac{\pi}{3}\right) = 1.5$

The smaller the step size, the higher the accuracy (smaller error)

TU4 → Q5

$f(x) = \sin(x)$

$f'(x) = \cos(x)$

$f'(1) = \cos(1) = 0.5403023$

a) $f'(a) \approx \dfrac{f(a+h) - f(a-h)}{2h}$

$f'(1) \approx \dfrac{\sin(1+h) - \sin(1-h)}{2h}$

b)

at $h = 0.1 \rightarrow f'(1) \approx 0.5394$

abs err $= 9.00537 \times 10^{-4}$

relative $= 0.166\%$

at $h = 0.01 \longrightarrow f'(1) \approx 0.540293301$

abs err $= 9.005 \times 10^{-6}$

relative err $\approx 1.6667 \times 10^{-3}\%$

Q1) $n =$ nodes $- 1 = 4 - 1 = 3$

$f(x) = \sin(2x)$

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$
y	0	1	0	-1

$$L(x) = \sum_{i=0}^{3} y_i \cdot \prod_{\substack{j=0 \\ j \neq i}}^{3} \frac{x - x_j}{x_i - x_j}$$

$$L(x) = 0 + 1 \cdot \frac{x - 0}{\frac{\pi}{4} - 0} \cdot \frac{x - \frac{\pi}{2}}{\frac{\pi}{4} - \frac{\pi}{2}} \cdot \frac{x - \frac{3\pi}{4}}{\frac{\pi}{4} - \frac{3\pi}{4}} + 0$$

$$+ (-1) \cdot \frac{x - 0}{\frac{3\pi}{4} - 0} \cdot \frac{x - \frac{\pi}{4}}{\frac{3\pi}{4} - \frac{\pi}{4}} \cdot \frac{x - \frac{\pi}{2}}{\frac{3\pi}{4} - \frac{\pi}{2}}$$

$$L(x) = \frac{-32}{\pi^3}\left(x^3 - \frac{5\pi}{4}x^2 + \frac{3\pi^2}{8}x\right) - \frac{32}{3\pi^3}\left(x^3 - \frac{3\pi}{4}x^2 + \frac{\pi^2}{8}x\right)$$

$$\boxed{L(x) = -\frac{128}{3\pi^3}x^3 + \frac{48}{\pi^2}x^2 - \frac{40}{3\pi}x}$$

$L(2.5) \approx -1.714997402$

$f(2.5) = -0.9589243$

relative error $= \dfrac{|f(2.5) - L(2.5)|}{|f(2.5)|} \approx 0.78846$