# Definitions

## What is the determinant?

We saw determinants before in the definition of the cross product

Start with some properties

Calculation formulae come up later

Only applies for **square** matrices

Determinant is an important quantity that provides a "measure" of a matrix

Computed from all $n^2$ elements of the matrix (A is an $n \times n$ matrix)

Mathematically denoted:

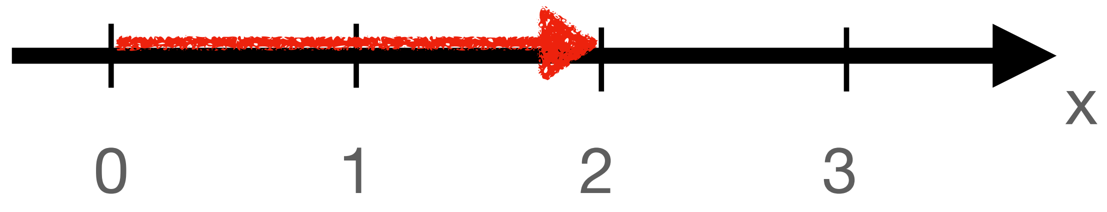
$\det A$ or $\det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ or $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

It's a sort of "area" or "volume" measure, but with a direction

**n=1**

With A=(2)

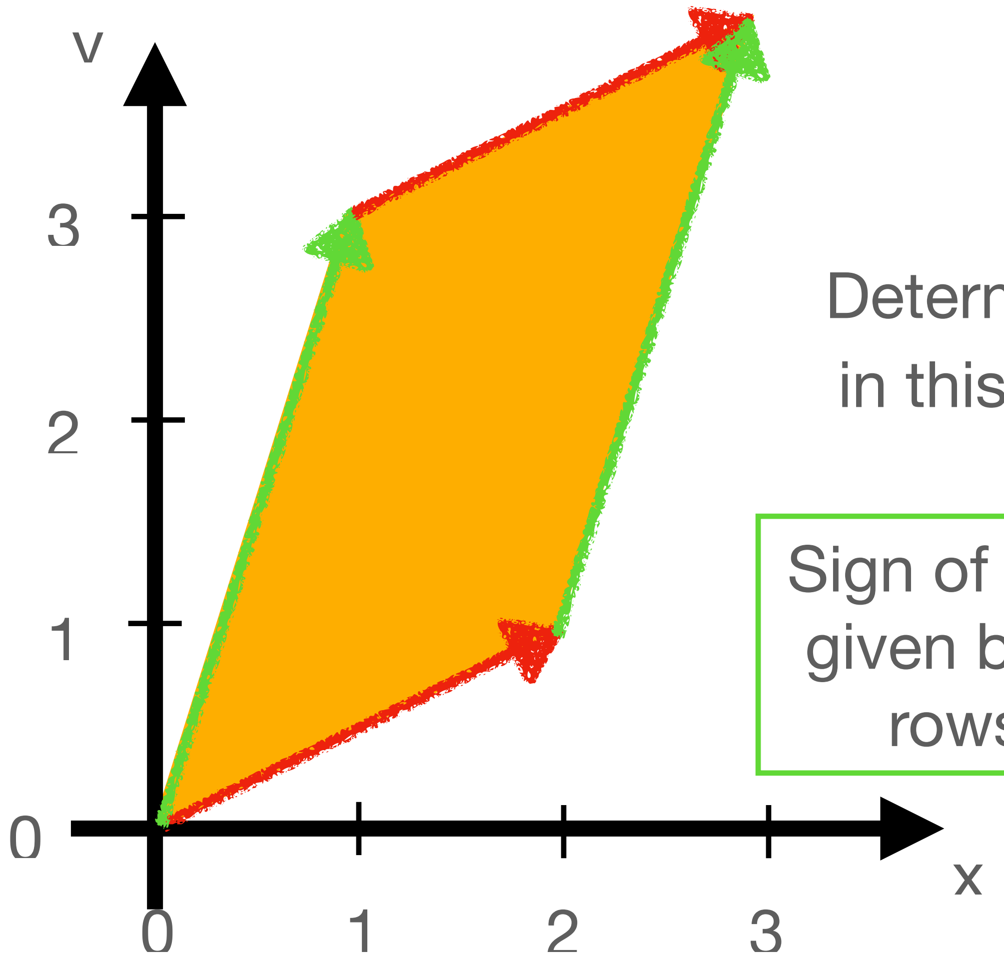
$1 \times 1$




$\det(2) = 2$ NB: oriented length. $\det(-3) = -3$

**n=2**

With $A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$

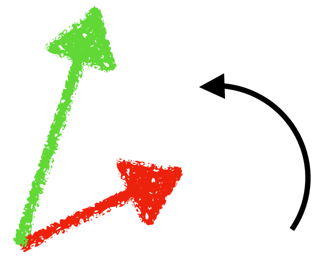



Determinant gives area in this case $\det A = 5$

Sign of the determinant is given by the order of the rows of the matrix.

$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$

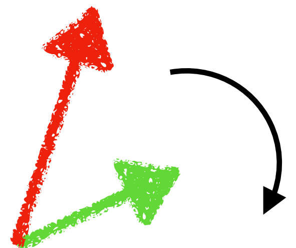
$\det A = +5$



+ve

$A = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$

$\det A = -5$



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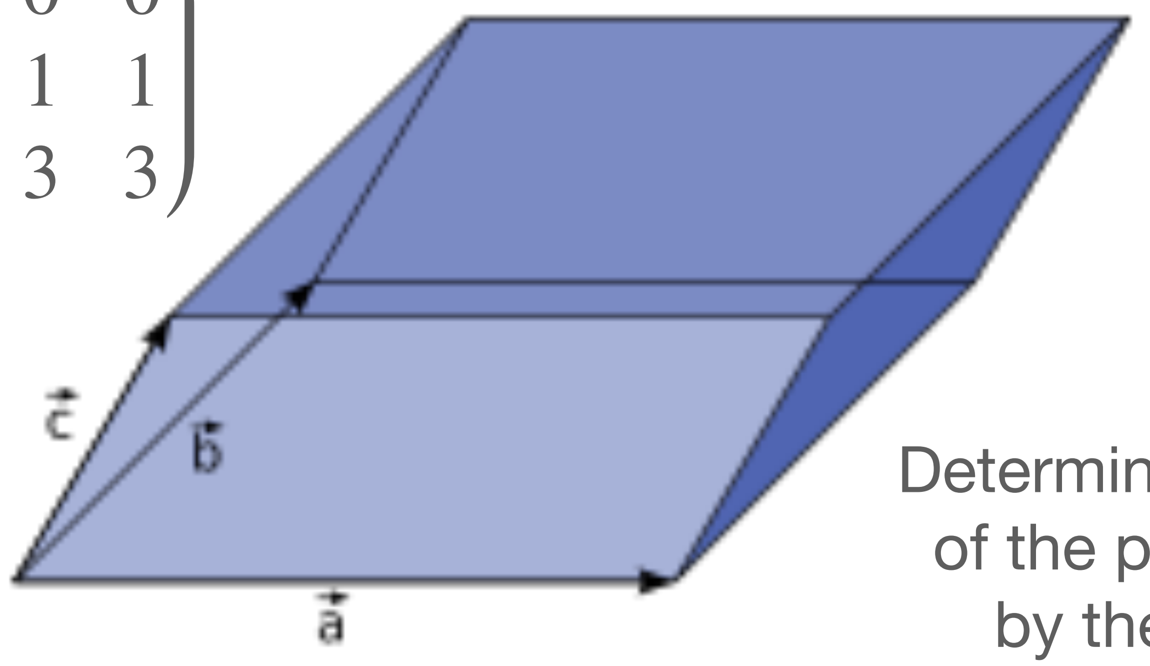
**n=3**

With $A = \begin{pmatrix} 5 & 0 & 0 \\ 3 & 1 & 1 \\ 1 & 3 & 3 \end{pmatrix}$




Determinant is now the volume of the parallelepiped defined by the three row vectors

Sign of the determinant is given by the RH rule.
Thumb = $\vec{a}$, Forefinger = $\vec{b}$, Middle finger = $\vec{c}$

**n>3**

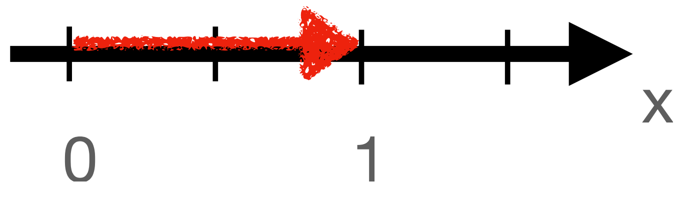
Let your imagination run wild.

Determinant is now the "hypervolume" defined by the row vectors.

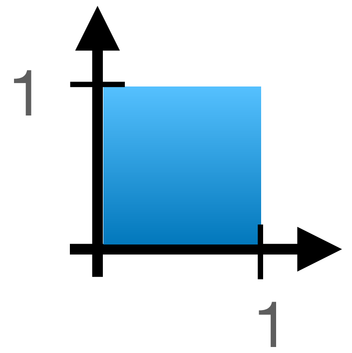
# Properties

1. $\det I_n = 1$ $\quad \forall n$ for all

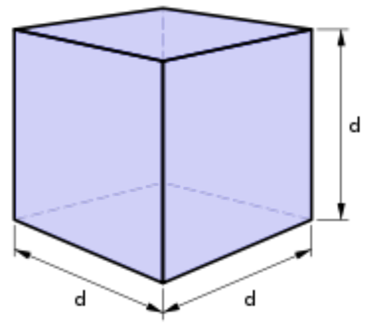
n=1: $\det(1) = 1$




n=2: $\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$




n=3: $\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1$

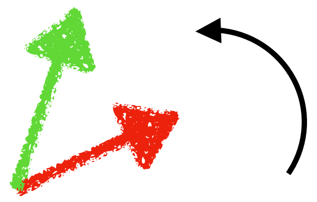


n>3: later

2. Each switch of rows in the identity matrix will swap the orientation and thus swap the sign of the determinant.

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\det A = +1$



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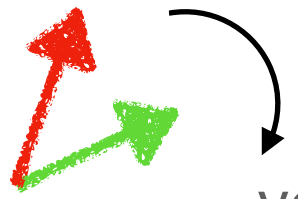
$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

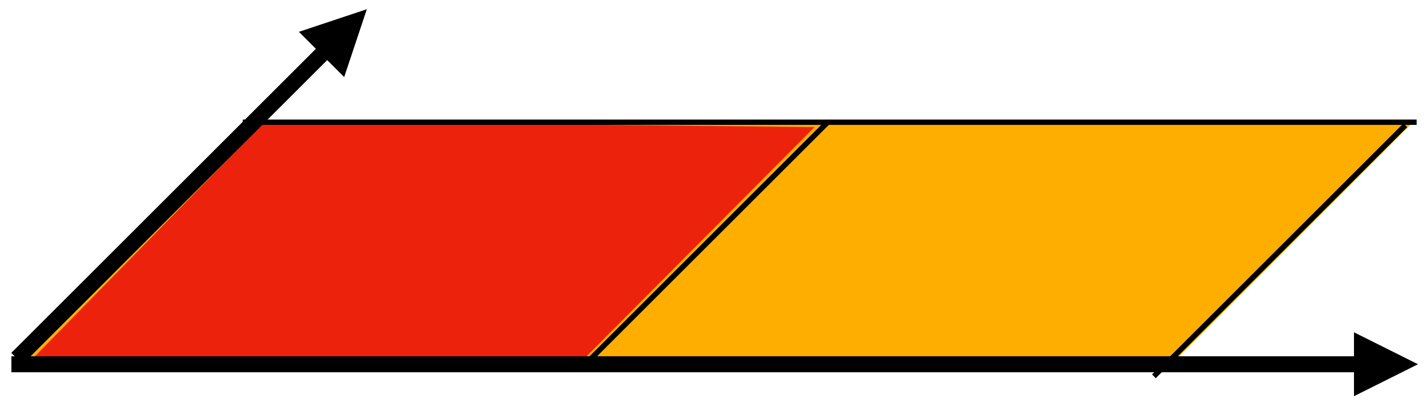
$\det A = -1$



-ve

3a. A common factor can be factorised out of a row

$\boxed{\det\begin{pmatrix} 6 & 8 \\ 1 & 2 \end{pmatrix}} = \det\begin{pmatrix} 2\cdot 3 & 2\cdot 4 \\ 1 & 2 \end{pmatrix} = 2\cdot \boxed{\det\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}}$ $= 2\cdot(3\cdot 2 - 4\cdot 1) = 2\cdot(2) = 4$



NB: $\begin{pmatrix} 6 & 8 \\ 1 & 2 \end{pmatrix} \neq 2\cdot\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$

In both 3a and 3b, only change (any) one row at a time. The other rows are unchanged in each operation.

3b. A row can be split into a sum

$\det\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} = \det\begin{pmatrix} 1+2 & 3+1 \\ 5 & 6 \end{pmatrix} = \det\begin{pmatrix} 1 & 3 \\ 5 & 6 \end{pmatrix} + \det\begin{pmatrix} 2 & 1 \\ 5 & 6 \end{pmatrix}$

This comes from the way the determinant is calculated ("product.sum" over all elements)

NB: $\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} \neq \begin{pmatrix} 1 & 3 \\ 5 & 6 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 5 & 6 \end{pmatrix}$

The Determinant is said to be "llinear" in every row of the matrix, or "multilinear" overall.

Example of property 3b):

$$\det\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} = \det\begin{pmatrix} 3 & 4 \\ 1+1+1+2 & 2+2+2+0 \end{pmatrix} = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}$$

$$+ \begin{vmatrix} 3 & 4 \\ 2 & 0 \end{vmatrix}$$

$$\Rightarrow 3\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 2 & 0 \end{vmatrix} = 3(2) + (-8) = -2$$

# Permutation Matrices

A matrix that is the result of (repeatedly) switching rows of the identity matrix is called a **permutation matrix**.

n=1: only one permutation possible

n=2: only two possibilities

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The determinant changes sign each time we swap the rows around.

Swapping is called permutation.

Shorthand: $\begin{array}{cc} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{matrix} \color{red}{1} \\ \color{blue}{2} \end{matrix} \end{array} \rightarrow 1 \quad 2$ $\qquad$ $\begin{array}{cc} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \begin{matrix} \color{blue}{2} \\ \color{red}{1} \end{matrix} \end{array} \rightarrow 2 \quad 1$

n=3: more possibilities

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

123 $\quad$ 132 $\quad$ 213 $\quad$ 231 $\quad$ 312 $\quad$ 321

6 permutation matrices

NB: $6 = 3 \cdot 2 \cdot 1$

n=4: even more possibilities

Using the shorthand:

| | | | |
|---|---|---|---|
| 1234 | 2134 | 3124 | 4123 |
| 1243 | 2143 | 3142 | 4132 |
| 1324 | 2314 | 3214 | 4213 |
| 1342 | 2341 | 3241 | 4231 |
| 1423 | 2413 | 3412 | 4312 |
| 1432 | 2431 | 3421 | 4321 |

24 possibilities here

$$= 4 \cdot (3 \cdot 2 \cdot 1)$$

n=5: starting to get silly

$$= 5 \cdot 4 \cdot (3 \cdot 2 \cdot 1) = 120$$

In general, the number of permutations of n elements:

$$\begin{aligned} \text{perm}(n) &= n \cdot \text{perm}(n-1) \\ &= n \cdot (n-1) \cdot \text{perm}(n-2) \\ &\text{etc} \\ &= n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1 \\ &= n! \end{aligned}$$

The determinant of a permutation matrix will always be 1 or -1

Changes sign with every permutation

if we did an even number of permutations (changes), then....

If perm(n) is an even number, the permutation is called "even", and $\det = 1$

If perm(n) is an odd number, the permutation is called "odd", and $\det = -1$

Doesn't matter how you get to final result. Keep track of permutations. Even or odd will always stay the same for any final form of the permutation matrix.

**Example.2**

Consider the elements in Matrix $P_2$ containing 1s

Columns

$$\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\times\begin{array}{ccc}\text{1st}\to\text{2nd}&\text{2nd}\to\text{1st}&\text{3rd}=\text{3rd}\\ \hline \end{array}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}=\begin{bmatrix}2&1&3\\5&4&6\\8&7&9\end{bmatrix}$$

Matrix $A$    Matrix $P_2$    Matrix $B$

The elements in the Matrix $P_1$ containing 1's can be used to determine the column positions in Matrix B.

$p_{12}$ Column 1 (matrix A) → Column 2 (matrix B)
$p_{21}$ Column 2 (matrix A) → Column 1 (matrix B)
$p_{33}$ Column 3 (matrix A) → Column 3 (matrix B)

*NB*: $A \times P = B$ Creates a **column permutation.**

**Example.5**

Consider the elements in Matrix $P_5$ containing 1s

rows

$$\begin{array}{r}\text{1st}\to\text{3rd}\\ \text{2nd}=\text{2nd}\\ \text{3rd}\to\text{1st}\end{array}\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\times\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}=\begin{bmatrix}7&8&9\\4&5&6\\1&2&3\end{bmatrix}$$

Matrix $P_5$    Matrix A    Matrix $B$

The elements in the Matrix $P_1$ containing 1's can be used to determine the column positions in Matrix B.

$p_{13}$ Row 3 (matrix A) → Row 1 (matrix B)
$p_{22}$ Row 2 (matrix A) → Row 2 (matrix B)
$p_{31}$ Row 1 (matrix A) → Row 3 (matrix B)

*NB*: $P \times A = B$ Creates a **row permutation,** with **reverse** row ordering.

## Permutation Summary

Column permutation - switch columns

$B = A \times P$

$$\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\times\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}=\begin{bmatrix}3&2&1\\6&5&4\\9&8&7\end{bmatrix}$$

Matrix A    Matrix $P$    Matrix $B$

$p_{13}$
column 1(matrix A) →
column 3 (matrix B)

Row permutation - switch rows

$B = P \times A$

$$\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\times\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}=\begin{bmatrix}7&8&9\\4&5&6\\1&2&3\end{bmatrix}$$

Matrix P    Matrix A    Matrix B

$p_{13}$
row 3 (matrix A) →
Row 1 (matrix B)

# Properties 1

**So far, we know:**

1. $\det I_n = 1$
2. swap two rows, swap sign of determinant
3. determinants are multilinear
   (can extract factors or split as summations)

**These properties define the determinant uniquely.**

**Other properties of the determinant follow from them.**

4. If the matrix A contains a row of zeros, $\det A = 0$

Proof

$$\text{Let } A = \begin{pmatrix} - & a_1 & - \\ & \vdots & \\ - & a_k & - \\ 0 & \ldots 0 \ldots & 0 \end{pmatrix}$$

$$\text{then } \det A = \begin{pmatrix} - & a_1 & - \\ & \vdots & \\ - & a_k & - \\ 0 & \ldots 0 \ldots & 0 \end{pmatrix} \overset{\text{Rule 3}}{=} 0 \cdot \det \begin{pmatrix} - & a_1 & - \\ & \vdots & \\ - & a_k & - \\ 0 & \ldots 0 \ldots & 0 \end{pmatrix}$$

$\det A = 0 \cdot \det A$

can only be true if $\det A = 0$ QED

5. If the matrix A contains two identical rows, $\det A = 0$

Proof

$$\text{Let } A = \begin{pmatrix} a & b & c \\ d & e & f \\ a & b & c \end{pmatrix}$$

$$\text{Then } \det A = \begin{pmatrix} a & b & c \\ d & e & f \\ a & b & c \end{pmatrix} \overset{\text{Rule 2}}{=} -\det \begin{pmatrix} a & b & c \\ d & e & f \\ a & b & c \end{pmatrix} = -\det A$$

$\det A = -\det A$

can only be true if $\det A = 0$ QED

6. Addition of one row to another does not change $\det A$

Proof

$$\text{Let } A = \begin{pmatrix} - & a_1 & - \\ & \vdots & \\ - & a_n & - \end{pmatrix} \text{ and } B = \begin{pmatrix} - & a_1 & - \\ & \vdots & \\ - & a_{j-1} & - \\ - & a_j + k \cdot a_i & - \\ - & a_{j+1} & - \\ & \vdots & \\ - & a_n & - \end{pmatrix}$$

$$\text{Then } \det B = \det \begin{pmatrix} - & a_1 & - \\ & \vdots & \\ - & a_{j-1} & - \\ - & a_j + k \cdot a_i & - \\ - & a_{j+1} & - \\ & \vdots & \\ - & a_n & - \end{pmatrix} = \det \begin{pmatrix} - & a_1 & - \\ & \vdots & \\ - & a_{j-1} & - \\ - & a_j & - \\ - & a_{j+1} & - \\ & \vdots & \\ - & a_n & - \end{pmatrix} + \det \begin{pmatrix} - & a_1 & - \\ & \vdots & \\ - & a_{j-1} & - \\ - & k \cdot a_i & - \\ - & a_{j+1} & - \\ & \vdots & \\ - & a_n & - \end{pmatrix} \quad \text{Rule 3 b}$$

$$\det B = \det A + k \cdot \det \begin{pmatrix} - & a_1 & - \\ & \vdots & \\ - & a_i & - \\ & \vdots & \\ - & a_i & - \\ & \vdots & \\ - & a_n & - \end{pmatrix} = \det A + k \cdot 0 = \det A$$

Rule 3 a Rule 5 QED

# Calculation

**Finally** well, for 2x2 matrices, anyway…

With the properties listed so far, we can work out a calculation formula for the determinant

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

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Then $\det A = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$= \det \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} + \det \begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$ Rule 3

$= \det \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} + \det \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} + \det \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} + \det \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$ Rule 3

$= ac \cdot \det \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + ad \cdot \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + bc \cdot \det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + bd \cdot \det \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$

Rule 5 =0   Rule 1 =1   Rule 2 =-1   Rule 5 =0

$= ac \cdot 0 + ad \cdot 1 + bc \cdot (-1) + bd \cdot 0$

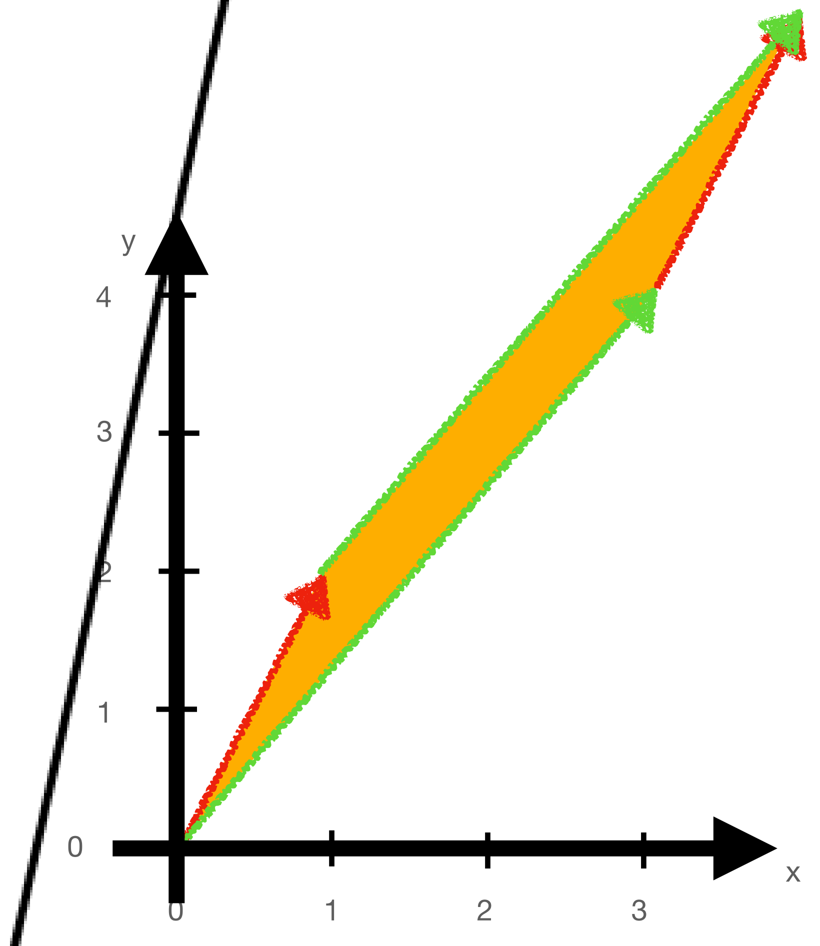
$= ad - bc$

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

↘ positive

↙ negative

$\det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 1 \cdot 4 - 2 \cdot 3 = -2$ The area of the parallelogram. Done.




~~To do this geometrically, we'd have to:~~

~~1. Find the length of the green vector (base of parallelogram)~~

$\left|\left|\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right|\right| = \sqrt{3^2 + 4^2} = 5$

~~2. Find the component of the red vector on the green one~~

$$\text{COMP}_{\vec{a}}\vec{b} = \left(\frac{\vec{b} \cdot \vec{a}}{\| \vec{a} \|}\right) = \frac{\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix}}{5} = \frac{1 \cdot 3 + 2 \cdot 4}{5} = \frac{11}{5}$$

Don't need this

~~3. Use Pythagoras to find the height of the parallelogram~~

$$h = \sqrt{||\vec{b}||^2 - (\text{COMP}_{\vec{a}}\vec{b})^2} = \sqrt{5 - \frac{121}{25}} = \frac{1}{5}\sqrt{125 - 121} = \frac{2}{5}$$

~~4. Multiply base x height to find area~~

$$A_{parallel} = \text{base} \times \text{height} = 5 \times \frac{2}{5} = 2$$

NB: not oriented (just an area)
To work out orientation, look at order of vectors

Much easier with determinant, hey?

# Determinant vs Gauß

**How does GE affect the determinant?**

GE 1. Swap two rows Rule 2

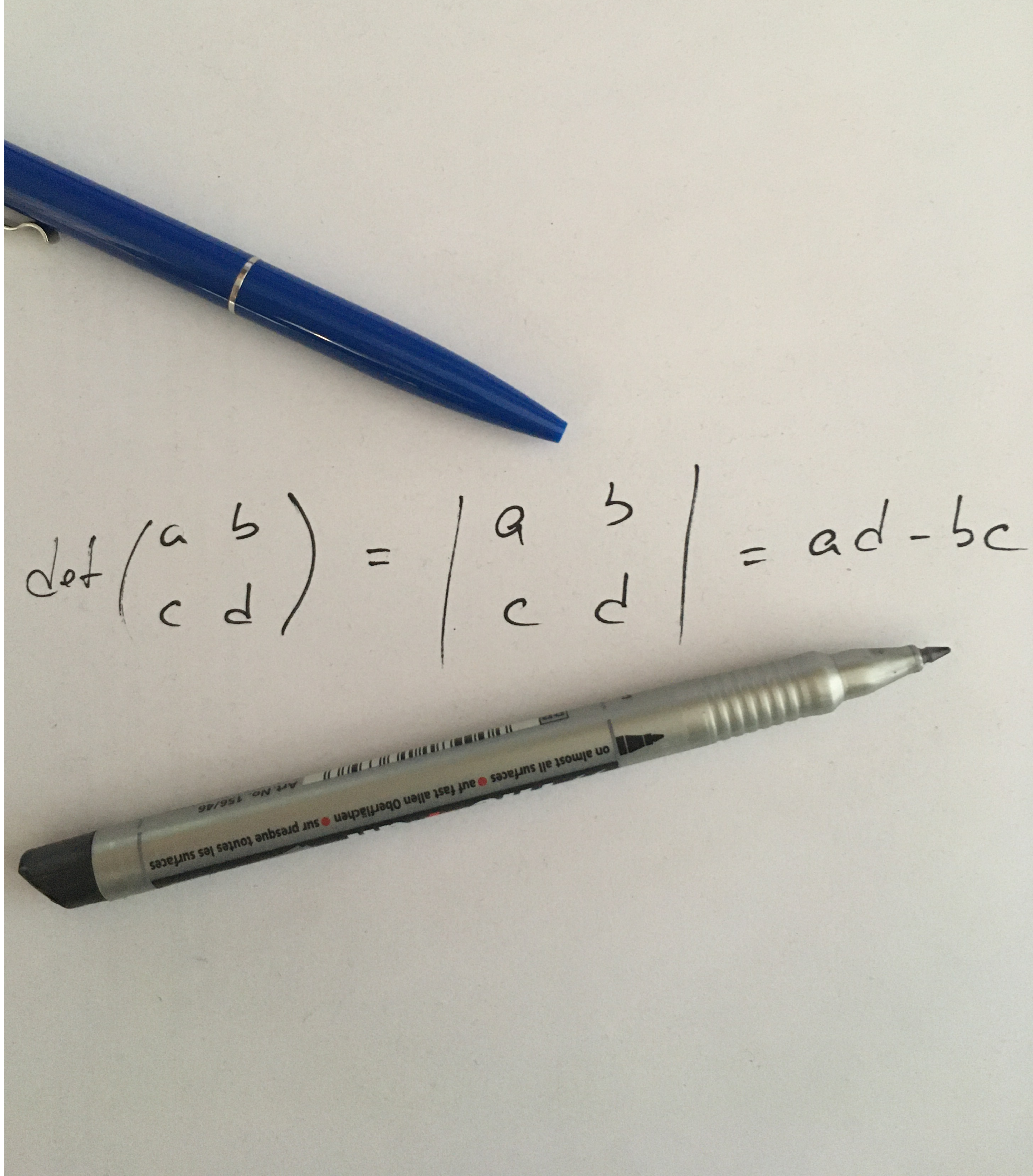
-> Change the sign

GE 2. Multiply a row by a non-zero constant Rule 3

-> Multiplies the determinant by the constant

GE 3. Add a row to another Rule 6

-> No change



# Properties 2

## So far, we know:

1. $\det I_n = 1$
2. swap two rows, swap sign of determinant
3. determinants are multilinear
   (can extract factors or split as summations)
4. If the matrix A contains a row of zeros, $\det A = 0$
5. If the matrix A contains two identical rows, $\det A = 0$
6. Addition of one row to another does not change $\det A$

## New this slide:

7. Given an upper triangular matrix, $\boxed{\det A = \prod_{i=1}^{n} a_{ii}}$

ie. the product of the elements on the main diagonal

$$\det \begin{pmatrix} a_{11} & \dots & a_{1n} \\ 0 & & \vdots \\ 0 & 0 \dots & a_{nn} \end{pmatrix} = a_{nn} \cdot \det \begin{pmatrix} a_{11} & \dots & \dots & a_{1n} \\ & \vdots & \vdots & \\ 0 & \dots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & \dots & 0 & 1 \end{pmatrix} \quad \text{Rule 3}$$

$$= a_{nn} \cdot \det \begin{pmatrix} a_{11} & \dots & a_{1,n-1} & 0 \\ & \vdots & \vdots & 0 \\ 0 & \dots & a_{n-1,n-1} & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix} \quad \text{Rule 6} \quad \text{n-1 times}$$

repeat process: $(1 \times$ Rule 3 $+ (n_i - 1) \times$ Rule 6$)$

$$= a_{nn} \cdot a_{n-1,n-1} \cdot \dots \cdot a_{11} \cdot \det I_n$$

$$= a_{nn} \cdot a_{n-1,n-1} \cdot \dots \cdot a_{11}$$ QED

Remember we showed that GE's don't affect the determinant

## Calculating $\det A$

Use the properties and GE to compute the determinant of any matrix

Example

NB: This was the matrix we tried to invert earlier, and found we couldn't.

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad \begin{matrix} \text{II-4I} \\ \text{III-7I} \end{matrix}$$

$$= \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix} \quad \text{III-2II}$$

$$= \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix} = 0 \quad \text{Rule 4}$$

or

$$\text{Rule 3} \quad = (-3)(-6) \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} = 0 \quad \text{Rule 5}$$

Example

$$\det \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \quad \begin{matrix} \text{II-I} \\ \text{III-I} \end{matrix}$$

$$= \det \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 8 \end{pmatrix} \quad \text{III-2II}$$

$$= \det \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{Rule 7} \quad = 1 \cdot 1 \cdot 2$$

Example

$$\det\begin{pmatrix} 9 & 13 & 5 & 2 \\ 1 & 11 & 7 & 6 \\ 3 & 7 & 4 & 1 \\ 6 & 0 & 7 & 10 \end{pmatrix} \begin{matrix} \\ \\ \mathrm{III}-3\mathrm{II} \\ \mathrm{IIII}-6\mathrm{II} \end{matrix} = \det\begin{pmatrix} 9 & 13 & 5 & 2 \\ 1 & 11 & 7 & 6 \\ 0 & -26 & -17 & -17 \\ 0 & -66 & -35 & -26 \end{pmatrix} \begin{matrix} \\ \mathrm{II}-\frac{1}{9}\mathrm{I} \\ \\ \\ \end{matrix} = \det\begin{pmatrix} 9 & 13 & 5 & 2 \\ 0 & 86/9 & 58/9 & 52/9 \\ 0 & -26 & -17 & -17 \\ 0 & -66 & -35 & -26 \end{pmatrix} \begin{matrix} \\ \\ \mathrm{III}+\frac{117}{43}\mathrm{II} \\ \mathrm{IIII}+\frac{297}{43}\mathrm{II} \end{matrix}$$

$$\Rightarrow \det\begin{pmatrix} 9 & 13 & 5 & 2 \\ 0 & \frac{86}{9} & \frac{58}{9} & \frac{52}{9} \\ 0 & 0 & \frac{23}{43} & -\frac{55}{43} \\ 0 & 0 & \frac{409}{43} & \frac{598}{43} \end{pmatrix} \begin{matrix} \\ \\ \\ \mathrm{IIII}-\frac{409}{23}\mathrm{III} \end{matrix} = \det\begin{pmatrix} 9 & 13 & 5 & 2 \\ 0 & \frac{86}{9} & \frac{58}{9} & \frac{52}{9} \\ 0 & 0 & \frac{23}{43} & -\frac{55}{43} \\ 0 & 0 & 0 & \frac{843}{23} \end{pmatrix}$$

$$\Rightarrow \det\begin{pmatrix} 9 & 13 & 5 & 2 \\ 0 & \frac{86}{9} & \frac{58}{9} & \frac{52}{9} \\ 0 & 0 & \frac{23}{43} & -\frac{55}{43} \\ 0 & 0 & 0 & \frac{843}{23} \end{pmatrix} = 9 \cdot \frac{86}{9} \cdot \frac{23}{43} \cdot \frac{843}{23} = 1686$$

Why? according to rule no. 7

$$\det\begin{pmatrix} 9 & 13 & 5 & 2 \\ 0 & \frac{86}{9} & \frac{58}{9} & \frac{52}{9} \\ 0 & 0 & \frac{23}{43} & -\frac{55}{43} \\ 0 & 0 & 0 & \frac{843}{23} \end{pmatrix} = \left(\frac{843}{23}\right) \cdot \det\begin{pmatrix} 9 & 13 & 5 & 2 \\ 0 & \frac{86}{9} & \frac{58}{9} & \frac{52}{9} \\ 0 & 0 & \frac{23}{43} & -\frac{55}{43} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} \mathrm{I}-2\mathrm{IIII} \\ \mathrm{II}-\frac{52}{9}\mathrm{IIII} \\ \mathrm{III}+\frac{55}{43}\mathrm{IIII} \\ \\ \end{matrix}$$

$$\Rightarrow \left(\frac{843}{23}\right) \cdot \left(\frac{23}{43}\right) \det\begin{pmatrix} 9 & 13 & 5 & 0 \\ 0 & \frac{86}{9} & \frac{58}{9} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} \mathrm{I}-5\mathrm{III} \\ \mathrm{II}-\frac{58}{9}\mathrm{III} \\ \\ \\ \end{matrix} = \left(\frac{843}{23}\right)\left(\frac{23}{43}\right)\left(\frac{86}{9}\right) \cdot \det\begin{pmatrix} 9 & 13 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} \mathrm{I}-13\mathrm{II} \\ \\ \\ \\ \end{matrix}$$

$$\Rightarrow \left(\frac{843}{23}\right)\left(\frac{23}{43}\right)\left(\frac{86}{9}\right)(9) \cdot \det\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{843}{23} \cdot \frac{23}{43} \cdot \frac{86}{9} \cdot 9 \cdot 1$$

$$\Rightarrow \det\begin{pmatrix} 9 & 13 & 5 & 2 \\ 1 & 11 & 7 & 6 \\ 3 & 7 & 4 & 1 \\ 6 & 0 & 7 & 10 \end{pmatrix} = 1686$$

# Properties

**So far, we know:**

1. $\det I_n = 1$
2. swap two rows, swap sign of determinant
3. determinants are multilinear
   (can extract factors or split as summations)
4. If the matrix A contains a row of zeros, $\det A = 0$
5. If the matrix A contains two identical rows, $\det A = 0$
6. Addition of one row to another does not change $\det A$
7. Given an upper triangular matrix, $\det A = \prod_{i=1}^{n} a_{ii}$

**New this slide:**

8a. $\det A \neq 0 \implies A$ is invertible
8b. $\det A = 0 \implies A$ is not invertible

Proof:

if A is invertible:

$$A \xrightarrow{\text{GE}} \begin{pmatrix} a_{11} & & * \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}$$

matrix is “full rank”: has a pivot in every column

By definition a pivot is non-zero, so Rule 7 says $\det A \neq 0$ **QED**

if A is not invertible:

$$A \xrightarrow{\text{GE}} \begin{pmatrix} a_{11} & & * & \\ & \ddots & & \\ 0 & & a_{n-1,n-1} & \\ 0 & \dots & 0 & 0 \end{pmatrix}$$

Rule 4

row of zeros -> $\det A = 0$ **QED**

9. $\det(AB) = \det A \cdot \det B$ Proof is too long, so just trust me on this one.

Consequences:

(i) $\det(AB) = \det(BA)$ even though $AB \neq BA$

Proof $\det(AB) = \det A \cdot \det B$
$\det(BA) = \det B \cdot \det A$

these are all real numbers, so commutative. **QED**

(ii) $\det(A^{-1}) = \dfrac{1}{\det(A)}$

Proof $I_n = AA^{-1}$
$1 = \det I_n = \det(AA^{-1}) = \det A \cdot \det A^{-1}$

rearrange and **QED**

10. $\det A = \det A^T$ Another long proof. Trust me again?

Consequences: Everything that applied to “rows” up to now also applies to “columns”

# Leibniz's Rule

# Properties of the Determinant

**Quite a collection**

1. $\det I_n = 1$
2. swap two rows, swap sign of determinant
3. determinants are multilinear (can extract factors or split as summations)
4. If the matrix A contains a row of zeros, $\det A = 0$
5. If the matrix A contains two identical rows, $\det A = 0$
6. Addition of one row to another does not change $\det A$
7. Given an upper triangular matrix, $\det A = \prod_{i=1}^{n} a_{ii}$
8. $\det A \neq 0 \implies A$ is invertible and
   $\det A = 0 \implies A$ is not invertible
9. $\det(AB) = \det A \cdot \det B$
10. $\det A = \det A^T$

We'll use these now to calculate the determinant for matrices of any size. The formula we'll develop is called Leibniz' Rule.

## The Determinant of a 3x3 Matrix

Start with $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Rule 3

$$= \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{11} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

This process can be expanded to work with matrices of any size

All n! permutation matrices will show up

Rule 6

$$= \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{11} \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{vmatrix}$$

Rule 3

$$= \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ a_{31} & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{11} \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{11} \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{vmatrix}$$

Rule 6

$$= \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{11} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{11} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{vmatrix}$$

Rule 3

$$= a_{11} \cdot a_{22} \cdot a_{33} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + a_{11} \cdot a_{23} \cdot a_{32} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} + a_{12} \cdot a_{21} \cdot a_{33} \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} + a_{12} \cdot a_{23} \cdot a_{31} \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} + a_{13} \cdot a_{21} \cdot a_{32} \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} + a_{13} \cdot a_{22} \cdot a_{31} \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$$

| | | | | | | |
|---|---|---|---|---|---|---|
| Permutations of $I_n$: | 0 | 1 | 2 | 3 | 4 | 5 |
| sign $\det A =$ | +1 | -1 | +1 | -1 | +1 | -1 |

$$= a_{11} \cdot a_{22} \cdot a_{33} \cdot (+1) + a_{11} \cdot a_{23} \cdot a_{32} \cdot (-1) + a_{12} \cdot a_{21} \cdot a_{33} \cdot (+1) + a_{12} \cdot a_{23} \cdot a_{31} \cdot (-1) + a_{13} \cdot a_{21} \cdot a_{32} \cdot (+1) + a_{13} \cdot a_{22} \cdot a_{31} \cdot (-1)$$

We saw this before in the context of the cross product

This only works for 3x3 matrices!

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

add the red diagonal products;
subtract the blue backdiagonal products

# Permutations

## determine the Determinant…

Recall: this messy thing:

$$= a_{11} \cdot a_{22} \cdot a_{33} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + a_{11} \cdot a_{23} \cdot a_{32} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} + a_{12} \cdot a_{21} \cdot a_{33} \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} + a_{12} \cdot a_{23} \cdot a_{31} \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} + a_{13} \cdot a_{21} \cdot a_{32} \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} + a_{13} \cdot a_{22} \cdot a_{31} \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$$

is a systematic way of calculating the determinant using our simple rules.

By recognising what's going on here, we can develop a formula for the Determinant of any size matrix.

Let's look first in more detail at Permutations:

For a general $n \times n$ matrix, we have $n!$ summands

For a $4 \times 4$ matrix, that means 24 terms in the messy thing above.

A $5 \times 5$ matrix would involve 120 of them.

Quickly becomes a lot of fun, no doubt.

For the $a_{ij}$'s in each summand (term) above,

row indices are always 1 2 3

while column indices correspond to the names of the permutation matrices

$a_{11}\ a_{23}\ a_{32}$

↳ 1 3 2 $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$

Notation: Permutation symbolised as $\sigma(i \in [1,n]) =$ position of i in string

eg above: $\sigma(3) = 2$

or: given $\sigma = 1365247$, $\sigma(2) = 5$

The signum of $\sigma$ is written sign $\sigma$ and defined as:

$$\text{sign } \sigma = \begin{cases} +1 & \text{if } \sigma \text{ even} \\ -1 & \text{if } \sigma \text{ odd} \end{cases}$$

To determine the parity, count the number of swaps

for $\sigma = 1365247$:

| | |
|---|---|
| 1234567 | +1 |
| 1324567 | -1 |
| 1364527 | +1 |
| 1365427 | -1 |
| 1365247 | +1 |

$\sigma(2) =$ even

Now can generate a formula to calculate the Determinant of a square matrix of any size:

Leibniz Formula for the Determinant

$$\det \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} = \sum a_{1\sigma(1)} \cdot a_{2\sigma(2)} \cdot \ldots \cdot a_{n\sigma(n)} \cdot \text{sign } (\sigma)$$

for n=3, see the mess above

for n=2:

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} \cdot (+1) + a_{12}a_{21} \cdot (-1)$$

$\sigma = 12$ $\qquad \sigma = 21$

sign $\sigma = 1$ $\qquad$ sign $\sigma = -1$

# Laplace Expantion

# Generalising

Consider the 3x3 matrix from before:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

From Leibnitz or Sarrus, we had:

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Look more closely at the pattern

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \overbrace{a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}}^{+\, a_{11}\cdot\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}} - \overbrace{a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31}}^{-\, a_{12}\cdot\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}} + \overbrace{a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}}^{+\, a_{13}\cdot\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}$$

The determinant of the 3x3 matrix is a linear combination of the determinants $M_{ij}$ of the **minor matrices.**

This is called the **Laplace Expansion of the Determinant**

Works for square matrices of any size

We're not limited to the first row, as in the example.
This works for any choice of row or column.
The final answer will be the same.

eg: take the 2nd row

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= -a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{22}(a_{11}a_{33} - a_{13}a_{31}) - a_{23}(a_{11}a_{32} - a_{12}a_{31})$$

$$= -a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} + a_{22}a_{11}a_{33} - a_{22}a_{13}a_{31} - a_{23}a_{11}a_{32} + a_{23}a_{12}a_{31}$$

$$= +a_{22}a_{11}a_{33} - a_{23}a_{11}a_{32} + a_{23}a_{12}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{22}a_{13}a_{31}$$

identically the same as

## Generalising Further

$$A = \begin{pmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{pmatrix}$$

$M_{ij}$ is the **minor** resulting from the elimination of the ith row and jth column

$C_{ij} = (-1)^{i+j}M_{ij}$ is the **cofactor** or signed minor needed to calculate the determinant

Laplace Expansion formulae for the:

- $i$ th row $\det A = \sum_{j=1}^{n} a_{ij}\cdot C_{ij} = \sum_{j=1}^{n} a_{ij}\cdot(-1)^{i+j}\cdot M_{ij}$
- $j$ th column $\det A = \sum_{i=1}^{n} a_{ij}\cdot C_{ij} = \sum_{i=1}^{n} a_{ij}\cdot(-1)^{i+j}\cdot M_{ij}$

# Examples

➜ $i$ th row $\quad \det A = \sum_{j=1}^{n} a_{ij} \cdot C_{ij} = \sum_{j=1}^{n} a_{ij} \cdot (-1)^{i+j} \cdot M_{ij}$

Find $\det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

Try 2nd row

$$= 4 \cdot (-1)^{2+1} \cdot \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} + 5 \cdot (-1)^{2+2} \cdot \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} + 6 \cdot (-1)^{2+3} \cdot \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix}$$

$$= -4 \cdot (2 \cdot 9 - 8 \cdot 3) + 5 \cdot (-1)^{2+2} \cdot (1 \cdot 9 - 3 \cdot 7) - 6 \cdot (1 \cdot 8 - 2 \cdot 7)$$

$$= -4 \cdot (-6) + 5 \cdot (-12) - 6 \cdot (-6)$$

$$= 24 - 60 + 36 = 0$$

To remember the signs of the cofactors, note the alternating pattern of +'s and -'s

$$\begin{pmatrix} (-1)^{1+1} & (-1)^{1+2} & (-1)^{1+3} \\ (-1)^{2+1} & (-1)^{2+2} & (-1)^{2+3} \\ (-1)^{3+1} & (-1)^{3+2} & (-1)^{3+3} \end{pmatrix} = \begin{pmatrix} (-1)^{2} & (-1)^{3} & (-1)^{4} \\ (-1)^{3} & (-1)^{4} & (-1)^{5} \\ (-1)^{4} & (-1)^{5} & (-1)^{6} \end{pmatrix} = \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Find $\det \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \begin{matrix} \\ \mathrm{II}-\mathrm{I} \\ \mathrm{III}-\mathrm{I} \end{matrix} = \det \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 8 \end{pmatrix} = 1 \cdot \begin{vmatrix} 1 & 3 \\ 2 & 8 \end{vmatrix} = 8 - 6 = 2$

Try 3rd column

$$= 1 \cdot (-1)^{1+1} \cdot \begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} + 1 \cdot (-1)^{2+1} \cdot \begin{vmatrix} 1 & 1 \\ 3 & 9 \end{vmatrix} + 1 \cdot (-1)^{3+1} \cdot \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix}$$

$$= 1 \cdot (2 \cdot 9 - 3 \cdot 4) - 1 \cdot (1 \cdot 9 - 3 \cdot 1) + 1 \cdot (1 \cdot 4 - 2 \cdot 1)$$

$$= 1 \cdot (6) - 1 \cdot (6) + 1 \cdot (2)$$

$$= 2$$

---

Works systematically, but easiest to calculate if choose an easy column or row

Find $\det \begin{pmatrix} 1 & 3 & 2 & 4 \\ -1 & 0 & 1 & 1 \\ 2 & 1 & -1 & 0 \\ 5 & 0 & 1 & 2 \end{pmatrix} = 3 \cdot \det \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & 0 \\ 5 & 1 & 2 \end{pmatrix} + 1 \cdot \det \begin{pmatrix} 1 & 2 & 4 \\ -1 & 1 & 1 \\ 5 & 1 & 2 \end{pmatrix} \begin{matrix} \mathrm{I}+\mathrm{II} \\ \\ \\ \end{matrix} \qquad 1 \cdot \det \begin{pmatrix} 0 & 3 & 5 \\ -1 & 1 & 1 \\ 5 & 1 & 2 \end{pmatrix}$

$= 3\left(1\begin{vmatrix} 2 & -1 \\ 5 & 1 \end{vmatrix} + 2\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix}\right) + \left(-1\begin{vmatrix} 3 & 5 \\ 1 & 2 \end{vmatrix} + 5\begin{vmatrix} 3 & 5 \\ 1 & 1 \end{vmatrix}\right) = 3(2+5) + 3(2)(1-2) - 1(6-5) + 5(3-5)$

$= 21 - 6 - 1 - 10 = 4$

Try 1st row (systematic)

$$= 1 \cdot (-1)^{1+1} \cdot \begin{vmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 2 \end{vmatrix} + 3 \cdot (-1)^{1+2} \cdot \begin{vmatrix} -1 & 1 & 1 \\ 2 & -1 & 0 \\ 5 & 1 & 2 \end{vmatrix} + 2 \cdot (-1)^{1+3} \cdot \begin{vmatrix} -1 & 0 & 1 \\ 2 & 1 & 0 \\ 5 & 0 & 2 \end{vmatrix} + 4 \cdot (-1)^{1+4} \cdot \begin{vmatrix} -1 & 0 & 1 \\ 2 & 1 & -1 \\ 5 & 0 & 1 \end{vmatrix}$$

Try 2nd column (clever)

$$= 3 \cdot (-1)^{1+2} \cdot \begin{vmatrix} -1 & 1 & 1 \\ 2 & -1 & 0 \\ 5 & 1 & 2 \end{vmatrix} + 0 \cdot * \cdot |\,*\,| + 1 \cdot (-1)^{3+2} \cdot \begin{vmatrix} 1 & 2 & 4 \\ -1 & 1 & 1 \\ 5 & 1 & 2 \end{vmatrix} + 0 \cdot * \cdot |\,*\,|$$

Not done yet, but a lot less calculation!

# Combine with GE

**Efficiency in Mathematics**

To reduce the number of computations, start with some Gauss steps

$$\begin{vmatrix}1&3&2&4\\-1&0&1&1\\2&1&-1&0\\5&0&1&2\end{vmatrix}\ \begin{matrix}\\ \text{II+I}\\ \text{III-2I}\\ \text{IV-5I}\end{matrix} = \begin{vmatrix}1&3&2&4\\0&3&3&5\\0&-5&-5&-8\\0&-15&-9&-18\end{vmatrix}$$

det Prop 3

$$= (-3)\cdot\begin{vmatrix}1&3&2&4\\0&3&3&5\\0&-5&-5&-8\\0&5&3&6\end{vmatrix}$$

Laplace 1st column

other cofactors all zero

$$= (-3)(1)\cdot\begin{vmatrix}3&3&5\\-5&-5&-8\\5&3&6\end{vmatrix} + (0\cdot|*|+0\cdot|*|+0\cdot|*|)$$

III+II

$$= (-3)\cdot\begin{vmatrix}3&3&5\\-5&-5&-8\\0&-2&-2\end{vmatrix}\ \text{II+2I}\quad = 6\cdot\begin{vmatrix}3&3&5\\1&1&2\\0&1&1\end{vmatrix}\ \text{I-3II}$$

$$= 6\cdot\begin{vmatrix}0&0&-1\\1&1&2\\0&1&1\end{vmatrix}$$

Laplace 3rd column

$$= 6\cdot(-1)\cdot\begin{vmatrix}1&1\\0&1\end{vmatrix}$$

$$= 6\cdot(-1)\cdot(1-0) = -6$$

## Summary: Laplace Expansion

$$A = \begin{pmatrix}a_{11}&\cdots&a_{1j}&\cdots&a_{1n}\\ \vdots&&\vdots&&\vdots\\ a_{i1}&\cdots&a_{ij}&\cdots&a_{in}\\ \vdots&&\vdots&&\vdots\\ a_{n1}&\cdots&a_{nj}&\cdots&a_{nn}\end{pmatrix}$$

$M_{ij}$ is the **minor** resulting from the elimination of the ith row and jth column

$C_{ij} = (-1)^{i+j}M_{ij}$ is the **cofactor** or signed minor needed to calculate the determinant

Laplace Expansion formulae for the:

$i$ th row $$\det A = \sum_{j=1}^{n} a_{ij}\cdot C_{ij} = \sum_{j=1}^{n} a_{ij}\cdot(-1)^{i+j}\cdot M_{ij}$$

$j$ th column $$\det A = \sum_{i=1}^{n} a_{ij}\cdot C_{ij} = \sum_{i=1}^{n} a_{ij}\cdot(-1)^{i+j}\cdot M_{ij}$$

### Advantages

Laplace provides a simple method (easier to remember)

Laplace is less work that Leibniz

# Application #1

## Determinants vs Matrix Inversion

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

Cofactors: $C_{ij} = (-1)^{i+j} \det M_{ij}$

Remember: $M_{ij}$ is the result of deleting row i and column j in A

Laplace Expansion Rule

$$\sum_{j=1}^{n} a_{ij} C_{ij} = \det A \qquad \text{(ith row)}$$

Consider the matrix product:

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{pmatrix} = ?$$

What does the result look like?

Diagonals

$$(1,1) : a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n} \quad = \det A \qquad \text{Laplace}$$

$$(n,n) : a_{n1}C_{n1} + a_{n2}C_{n2} + \ldots + a_{nn}C_{nn} \quad = \det A \qquad \text{General row}$$

Off-diagonals

$$(k,i) : a_{k1}C_{i1} + a_{k2}C_{i2} + \ldots + a_{kn}C_{in} = \sum_{j=1}^{n} a_{kj} C_{ij} \quad = \det S$$

Obviously a determinant, but of what matrix?

$$S = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{i-1,1} & \dots & a_{i-1,n} \\ a_{k1} & \dots & a_{kn} \\ a_{i+1,1} & \dots & a_{i+1,n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \longleftarrow \quad i\text{th row}$$

S has two identical rows (k and i are both row k of A), so $\det S = 0$

So now we know:

$$A \cdot \begin{pmatrix} C_{11} & \dots & C_{n1} \\ \vdots & & \vdots \\ C_{1n} & \dots & C_{nn} \end{pmatrix} = \begin{pmatrix} \det A & & 0 \\ & \ddots & \\ 0 & & \det A \end{pmatrix}$$

$$C = \begin{pmatrix} C_{11} & \dots & C_{1n} \\ \vdots & & \vdots \\ C_{n1} & \dots & C_{nn} \end{pmatrix}$$

$$AC^T = \det A \cdot I_n$$

if A is invertible, then $\det A \neq 0$, so

$$\frac{1}{\det A} AC^T = I_n$$

$$A \cdot \underbrace{\left( \frac{1}{\det A} C^T \right)}_{= A^{-1}} = I_n$$

# Application #1

## Invert A using Co-Factors

1. Compute $\det A$. Check if A is invertible
2. Compute minors $\det M_{ij}$, put them in a matrix
3. Change the signs in the checkerboard pattern
4. Transpose the matrix, multiply by $\dfrac{1}{\det A}$

In short: $A^{-1} = \dfrac{1}{\det A} \cdot C^T$

Example

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$$

1. $\det A = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 3 & 9 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 4 \\ 1 & 0 & 9 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \end{vmatrix}$

$$\det A = 1 \cdot (2 \cdot 9 - 4 \cdot 3) - 1 \cdot (1 \cdot 9 - 4 \cdot 1) + 1 \cdot (1 \cdot 3 - 2 \cdot 1)$$
$$\det A = 6 - 5 + 1 = 2$$

2. $M = \begin{pmatrix} 6 & 5 & 1 \\ 6 & 8 & 2 \\ 2 & 3 & 1 \end{pmatrix}$ using laplace with each element

3. $C = \begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}$

4. $A^{-1} = \boxed{\dfrac{1}{\det A} \cdot C^T} = \dfrac{1}{2} \begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ -2.5 & 4 & -1.5 \\ 0.5 & -1 & 0.5 \end{pmatrix}$

# Application #2

## Extra super cool… Cramer's Rule

SLEs made easy

Task: Solve $A\overrightarrow{x} = \overrightarrow{b}$

Requirements: A must be square and invertible

Let $A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$ $\overrightarrow{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $\overrightarrow{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

Define $B_j = \begin{pmatrix} a_{11} & \dots & a_{1,j-1} & b_1 & a_{1,j+1} & \dots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \dots & a_{n,j-1} & b_n & a_{n,j+1} & \dots & a_{nn} \end{pmatrix}$

replace column j of A with $\overrightarrow{b}$

Then $x_j = \dfrac{\det B_j}{\det A}$

Example

Solve: $\begin{matrix} x_1 + 3x_2 = 0 \\ 2x_1 + 4x_2 = 6 \end{matrix}$ using Cramer's rule

1. $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$, $\det A = 1 \cdot 4 - 2 \cdot 3 = -2$

$\det A \neq 0$ -> A invertible -> unique solution

2.1. $B_1 = \begin{pmatrix} 0 & 3 \\ 6 & 4 \end{pmatrix}$, $\det B_1 = 0 \cdot 4 - 6 \cdot 3 = -18$

so $x_1 = \dfrac{\det B_1}{\det A} = \dfrac{-18}{-2} = 9$

2.2. $B_2 = \begin{pmatrix} 1 & 0 \\ 2 & 6 \end{pmatrix}$, $\det B_1 = 1 \cdot 6 - 2 \cdot 0 = 6$

so $x_2 = \dfrac{\det B_2}{\det A} = \dfrac{6}{-2} = -3$

3. unique solution is $\overrightarrow{x} = \begin{pmatrix} 9 \\ -3 \end{pmatrix}$

Notes:

1. impractical for large matrices (determinant becomes tedious to calculate)
2. helpful for computing a single component of the solution (don't have to solve the whol
3. big theoretical value