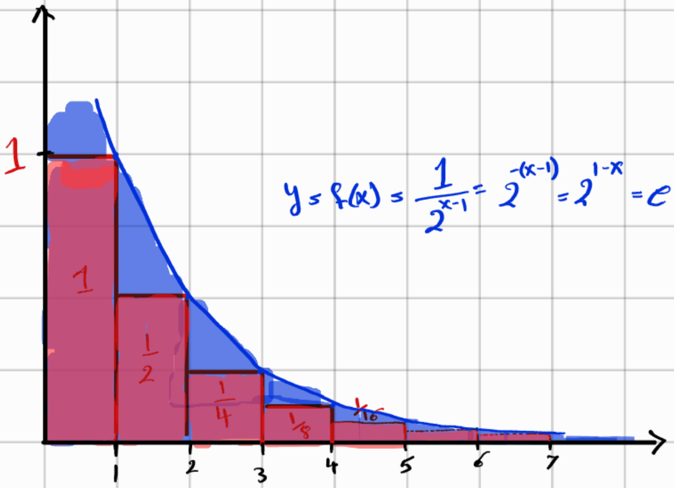
Series:

Example: $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\dots\dots = 2$



$$y = f(x) = \frac{1}{2^{x-1}} = 2^{-(x-1)} = 2^{1-x} = e^{\ln(2^{1-x})} = e^{(1-x)\cdot\ln 2}$$

If the blue area (the improper integral) is finite,

then the red area is finite as well.

Area under blue curve:

$$\int_0^\infty e^{(1-x)\ln 2}\,dx = \lim_{b\to\infty}\int_0^b e^{(1-x)\ln 2}\,dx = \lim_{b\to\infty}\left[\frac{-1}{\ln 2}e^{(1-x)\ln 2}\right]_0^b$$

$$\lim_{b\to\infty}\left[\frac{-1}{\ln 2}e^{(1-b)\ln 2} + \frac{1}{\ln 2}e^{\ln 2}\right] = \frac{-1}{\ln 2}\cdot\cancel{\lim_{b\to\infty} e^{(1-b)\ln 2}}_{0} + \frac{2}{\ln 2}$$

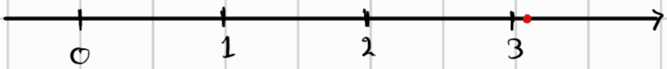
$$\Longrightarrow \int_0^\infty e^{(1-x)\ln 2}\,dx = \frac{2}{\ln(2)} \approx 2.885$$

$\therefore$ The red area is finit as well & less than $\frac{2}{\ln(2)}$

Another Example

$$\frac{(3)^{0.5+0.5(-1)^{x-1}} + 1\left(0.5+0.5(-1)^{x-1}\right)}{10^{x-1}}$$

$\pi = \underline{3.14159}\dots\dots = 3+\frac{1}{10}+\frac{4}{100}+\frac{1}{1000}+\frac{5}{10000}+\dots\dots$



IT is possible to add infinitely many numbers & still obtain a finite sum.

Definition:

$$a_1 + a_2 + a_3 + \dots\dots = \sum_{K=1}^{\infty} a_K \qquad \text{is called an ifinite series.}$$

$$\underbrace{\sum_{K=1}^{\infty} a_K}_{\text{Infinite Series}} = \underbrace{\lim_{n\to\infty}}_{\text{Limit Process}} \underbrace{\sum_{K=1}^{n} a_K}_{\text{Finite sum}}$$

(we check a finite sums, & see what happens to those sums when we let the numbers of summands go to infinity

$$\lim_{n\to\infty}\underbrace{(a_1+a_2+a_3\dots a_n)}_{\text{Partial sum}}$$

If this limit exists (as a number) then the infinte series Converges. Otherwise the series Diverges.

In case the series converges, this limit is called the sum of the series.

It is also written as $\sum_{K=1}^{\infty} a_K$.

The sum of a Series is the limit of it's partial sums.

Back to the introductory example:

$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$ Partial sums : $S_n = \frac{2^{n+1}-1}{2^n}$

$\underbrace{\text{Sequence}}_{\text{list}}$ of summands $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \cdots$

	Summands	Partial sums	
0	1	1	$= 1$
1	$\frac{1}{2} = 0.5$	$1 + 0.5 = 1.5$	$= \frac{3}{2}$
2	$\frac{1}{4} = 0.25$	$1.5 + 0.25 = 1.75$	$= \frac{7}{4}$
3	$\frac{1}{8} = 0.125$	$1.75 + 0.125 = 1.875$	$= \frac{15}{8}$
4	$\frac{1}{16} = 0.0625$	$1.875 + 0.0625 = 1.9375$	$= \frac{31}{16}$
$\vdots$	$\vdots$	$\vdots$	
		2	

Very useful Table

$S_0 = \frac{2-1}{1} = 1$

$S_1 = \frac{4-1}{2} = \frac{3}{2}$

$S_2 = \frac{8-1}{4} = \frac{7}{4}$

$\vdots$

limit of partial sums:

$$\lim_{n\to\infty} \frac{2^{n+1}-1}{2^n} = \lim_{n\to\infty} \frac{\cancel{2^n} \cdot 2^1}{\cancel{2^n}} - \lim_{n\to\infty} \frac{1}{2^n} = 2 - 0 = 2$$

Divergence Test for series:

$1, 1, 1, 1, 1, \cdots$ sequence

$1+1+1+1+1+\cdots$ Series

$\lim_{n\to\infty} a_n = 0 \implies$ series may converge

a_n → sequence (summands)

Telescoping series:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots$$

	Summands	Partial sums
0	$\frac{1}{2} = 0.5$	$\frac{1}{2} = 0.5$
1	$\frac{1}{6}$	$\frac{2}{3}$
2	$\frac{1}{12}$	$\frac{3}{4}$
3	$\frac{1}{20}$	$\frac{4}{5}$
4	$\frac{1}{30}$	$\frac{5}{6}$
⋮	⋮ 0	⋮

It may converge

Educated geuss: partial sums $S_n = \frac{n}{n+1}$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{\cancel{n}\cdot 1}{\cancel{n}(1+\frac{1}{n})} = \frac{1}{1+0} = \boxed{1}$$

Limit of the partial sums is the limit of the series.
Thus, if the geuss is correct, the series converges.

Let's confirm our geuss about the partial sums:

$$\boxed{\sum_{n=1}^{K} \frac{1}{n(n+1)} = \frac{K}{K+1}}$$

Integration is similar to summation, so we try partial fration decomposition

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} \Longrightarrow 1 = A(n+1) + B\cdot n$$

by comparing coefficients: $A=1$, $B=-1$

$$\therefore \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

Telescoping

$$\therefore \sum_{n=1}^{K} \frac{1}{n(n+1)} = \sum_{n=1}^{K}\left(\frac{1}{n} - \frac{1}{(n+1)}\right) = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right) \cdots$$
$$+ \left(\cancel{\frac{1}{K}} - \frac{1}{K+1}\right)$$
$$= 1 - \frac{1}{K+1} = \frac{K+1}{K+1} - \frac{1}{K+1} = \boxed{\frac{K}{K+1}}$$

Since the educated geuss of the partial sums is correct, & its limit is 1, then the series converges.

Geometric series:

Introductory Example: $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

$\cdot\frac{1}{2}$ $\cdot\frac{1}{2}$ $\cdot\frac{1}{2}$

here it's $\frac{1}{2}$

The ratio of neighboring summands is constant: $a_{n+1} = q \cdot a_n$

Definition:

A Geometric series has the form:

$$a + a\cdot q + a\cdot q^2 + a\cdot q^3 + \cdots = \sum_{n=0}^{\infty} a\cdot q^n$$

Simpler: $a\underbrace{(1+q+q^2+q^3+\cdots)}_{\text{Always start with 1 by factoring } a}$

$1+q+q^2+q^3+\cdots$

n	Summands	partial sums
	1	1
	q	$1+q$
	q^2	$1+q+q^2$
	q^3	$1+q+q^2+q^3$
	q^4	$1+q+q^2+q^3+q^4$

$\vdots$

$\longrightarrow 0$

To have a chance to converge

We need $|q|<1$

$\therefore\ -1<q<1$

The geometric series Diverges For $q<-1$ or $q>+1$

Because in these cases: $\lim_{n\to\infty} q^n \neq 0$

Do we get convergence for $|q|<1$?

Examine the partial sums:

$$S_n = 1+q+q^2+q^3+\cdots+q^{n-1}+q^n$$

$$q\cdot S_n = q+q^2+q^3+q^4+\cdots+q^{n-1}+q^n+q^{n+1}$$

$$S-q\cdot S_n = 1-q^{n+1} \Longrightarrow S_n(1-q)=1-q^{n+1}$$

$$\Longrightarrow \boxed{S_n=\frac{1-q^{n+1}}{1-q}},\ q\neq 1$$

Know by Heart

$$\boxed{\text{Finite Geometric sum: } 1+q+q^2+q^3+\cdots+q^n = \begin{cases} \dfrac{1-q^{n+1}}{1-q}, & q\neq 1 \\ n+1, & q=1 \end{cases}}$$

With this formula for the partial sums, we get:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{1-q^{n+1}}{1-q} = \frac{1-\overbrace{\lim_{n\to\infty} q^{n+1}}^{\to 0 \text{ When } |q|<1}}{1-q} = \frac{1}{1-q}$$

By Heart

For Geometric series, IF $|q|<1 \longrightarrow$ then it converges

$$1+q+q^2+q^3+q^4+q^5+\cdots = \frac{1}{1-q},\quad |q|<1.$$

& Diverges For $|q|\geq 1$

Examples:

(i) First example we took: $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$ is A Geometric series

With $q=\frac{1}{2}$, so it converges to $\frac{1}{1-q}=\frac{1}{1-\frac{1}{2}}=2$

(ii) $0.343434\cdots = \frac{34}{100}+\frac{34}{10000}+\frac{34}{1000000}+\cdots$

$$=\frac{34}{100}\left(1+\frac{1}{100}+\frac{1}{10000}+\frac{1}{1000000}+\cdots\right)$$

since $q=\frac{1}{100}<1$, $\Rightarrow \frac{34}{100}\cdot\frac{1}{1-q}=\frac{34}{100}\cdot\frac{1}{1-\frac{1}{100}}=\frac{34}{100-1}=\frac{34}{99}$

(iii) $5-\frac{10}{3}+\frac{20}{9}-\frac{40}{27}+\frac{80}{81}-\frac{160}{243}+\cdots$

$$5\left(1-\frac{2}{3}+\frac{4}{9}-\frac{8}{27}+\frac{16}{81}-\frac{32}{243}+\cdots\right)=\sum_{n=0}^{\infty}a\cdot q^n=a\cdot\frac{1}{1-q}=\sum_{n=0}^{\infty}5\left(-\frac{2}{3}\right)^n=\frac{5}{1-\left(-\frac{2}{3}\right)}=3$$

(iv) $$\sum_{n=1}^{\infty}2^{2n}\cdot 3^{(1-n)}=\sum_{n=1}^{\infty}4^n\cdot 3^{-n}\cdot 3=\sum_{n=1}^{\infty}3\cdot\left(\frac{4}{3}\right)^n=\sum_{n=0}^{\infty}3\cdot\left(\frac{4}{3}\right)^{n+1}$$

$\Longrightarrow \sum_{n=0}^{\infty}\cancel{3}\cdot\frac{4}{\cancel{3}}\left(\frac{4}{3}\right)^n$, Geometric series with $a=4$ & $q=\frac{4}{3}>1$

$\Longrightarrow$ The series Diverges.

Alternating series:

$$\underbrace{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+-\cdots}_{\text{Alternating Harmonic series}}$$

Pattern $+-+-+-+-$ "Alternating"

or $-+-+-+-+$

Alternating series Test:

If the Alternating series

$b_1-b_2+b_3-b_4+b_5-+\cdots$

Satisfies:

(i) $\lim_{n\to\infty} b_n=0$ (Summands must approach 0)

(ii) $b_1\geq b_2\geq b_3\geq b_4\geq b_5\geq\cdots\geq 0$

} then it converges

Moreover, the partial sums provide upper & lower bounds for the sum.

Eample:

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\cdots \qquad \text{For all } x \in R}$$

(But only efficient when x close to 0) Requires Radians!

Let's do $x = 1$ $(\approx 60^\circ)$

$$\cos 1 = 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots\cdots = 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720}$$

Summands	Partial sums
1	1
$-\frac{1}{2}$	$\frac{1}{2}$
$+\frac{1}{24}$	$0.541\overline{6}$
$-\frac{1}{720}$	$0.5402\overline{7}$

$\Rightarrow \frac{1}{2} < \cos 1 < 1$

$0.5 < \cos 1 < 0.541\overline{6}$

$0.541\overline{6} < \cos 1 < 0.5402\overline{7}$

Playing around with series:

$$\frac{1}{19} \approx \frac{1}{20} = \frac{5}{100} = 0.05$$

geometric series

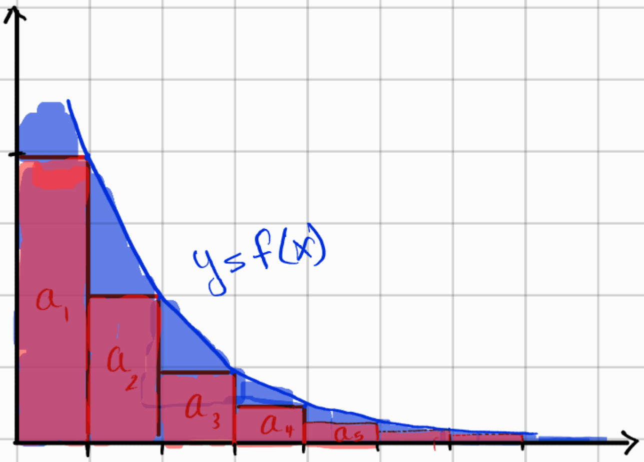
$$\frac{1}{19} = \frac{1}{20-1} = \frac{1}{20\left(1-\frac{1}{20}\right)} = \frac{1}{20}\left(1 + \left(\frac{1}{20}\right) + \left(\frac{1}{20}\right)^2 + \left(\frac{1}{20}\right)^3 + \cdots\right)$$

$$\Rightarrow 0.05\left(1 + 0.05 + (0.05)^2 + (0.05)^3 + \cdots\right) = 0.05 + 0.0025 + 0.000125 + \cdots$$

$$\approx 0.0526\ldots\ldots$$

Often we use about three summands to obtain a good aproximation.

Integral Test compares series with improper integrals.



Improper integral

$$\int_a^\infty f(x)\,dx$$

non-negative Function, $f(x) \geq 0$

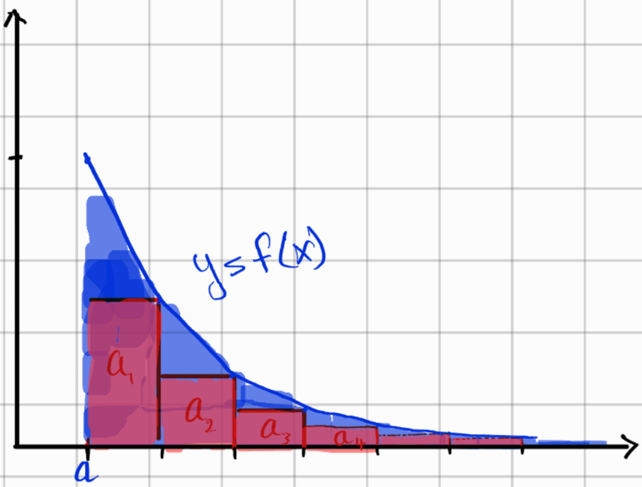
Converges if Area is finite

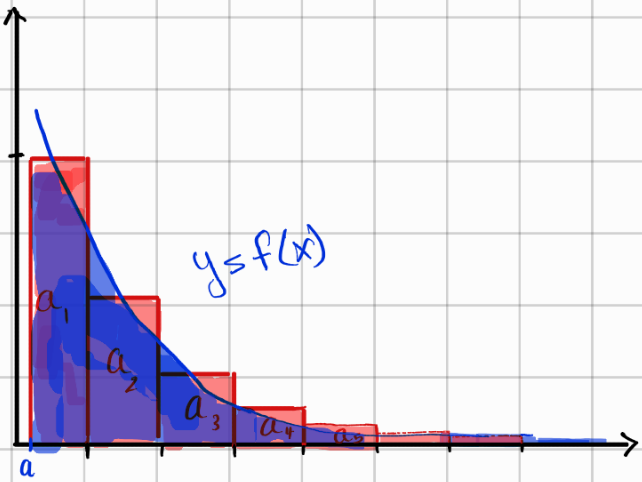
$$a_1 + a_2 + a_3 + a_4 + \cdots = \sum_{n=1}^{\infty} a_n$$

needs non-negative summands, $a_n \geq 0$

Converges if Area is finite

Two possible comparisons





Rectangles are inside the function area from some a on.

If the blue area is finite, then the Red area is finite too.

$$\sum a_n \leq \int_a^\infty f(x)\,dx < \infty$$

Function Area is inside the rectangles From a on.

If the blue area is infinite, then the red area is infinite too.

$$\sum a_n \geq \int_a^\infty f(x)\,dx = \infty$$

Let's consider P-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

Consider the improper integral:

$$\int_1^{\infty} \frac{1}{x^p}dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^p}dx \quad \text{if } p \neq 1 \rightarrow \lim_{b\to\infty} \left[\frac{1}{1-p}\cdot x^{1-p}\right]_1^b$$

$$\lim_{b\to\infty}\left[\frac{1}{1-p}\left(\cancel{b^{1-p}} - \cancel{1^{1-p}}^{\,1}\right)\right] = \frac{-1}{1-p} = \frac{1}{p-1} \qquad \text{Converges if } p>1$$

0 if $p>0$

$$\boxed{\text{Hence } \sum_{n=1}^{\infty}\frac{1}{n^p} \text{ converges if } p>1}$$

Case: $p=1$

$$\int_1^{\infty}\frac{1}{x}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}dx = \lim_{b\to\infty}\left[\ln|x|\right]_1^b = \lim_{b\to\infty}\ln|b| = +\infty \quad \text{Diverges}$$

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots \geq \int_1^{\infty}\frac{1}{x}dx = \infty$$

We see again why harmonic series diverges!

Case: $0<p<1$ (roots)

$$\boxed{\text{P-series } \sum_{n=1}^{\infty}\frac{1}{n^p} \text{ Diverges for } p<1}$$ $\Rightarrow$ Same reason of the last case using integral test.

For $p\leq 0$ Test for Divergence applies.

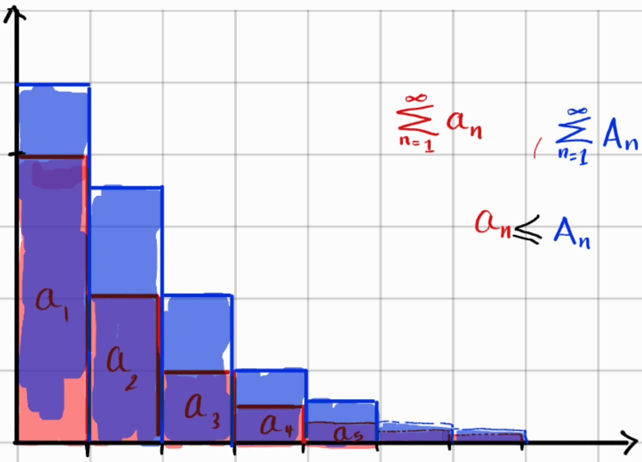
Comparison Test:

Suppose that $\sum a_n$ & $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent & $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.

sufficient: for all $n \geq N$

(ii) If $\sum b_n$ is Divergent & $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.



$\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} A_n$

$a_n \leq A_n$

(i) If the blue area is finite, so is the red one. If $\sum_{n=1}^{\infty} A_n$ converges, so does $\sum_{n=1}^{\infty} a_n$

(ii) If the red area is infinite, so is the blue one.

If $\sum_{n=1}^{\infty} a_n$ diverges, so does $\sum_{n=1}^{\infty} A_n$.

* If the Blue area is infinite, that doesn't tell us anything about the red area.

* The same if the red are is finite.

We often use A p-series or A Geometric series for Comparison.

$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \begin{array}{l} \text{Converges if } p>1 \\ \text{Diverges if } p \leq 1 \end{array}}$

$\boxed{1+q+q^2+q^3+\cdots \quad \begin{array}{l} \text{Converges to } \frac{1}{1-q} \text{ if } |q|<1 \\ \text{Diverges if } |q| \geq 1 \end{array}}$

Example: $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ (We see some p-series in this fourmula)

$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges.

& we have $\frac{5}{2n^2+4n+3} < \frac{5}{2n^2}$

By comparison, $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ converges as well.

* $\sum_{K=1}^{\infty} \frac{\ln K}{K}$

Summands	Partial sums
$\frac{\ln 1}{1} = 0$	0
$\frac{\ln 2}{2} = \frac{0.69\ldots}{2}$	$0.34\ldots$
$\frac{\ln 3}{3} = \frac{1.098\ldots}{3}$	$0.712\ldots$
$\frac{\ln 4}{4} = \frac{1.38\ldots}{4}$	$\ldots$

Harmonic series

$\sum_{n=1}^{\infty} \frac{1}{K}$ Diverges

(also p-series with $p \leq 1$)

$\frac{1}{K} < \frac{\ln K}{K}$ For all $K \geq 3$

By comparison, $\sum_{K=1}^{\infty} \frac{\ln K}{K}$ diverges as well.

$*\ \sum_{n=1}^{\infty} \frac{1}{2^n-1} = \frac{1}{1} + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \frac{1}{31} + \cdots\cdots$

Geometric series $\sum_{n=1}^{\infty} 2\cdot\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{2}{2^n}$ since $|r|<1$ it converges

$\frac{1}{2^n-1} \leq \frac{2}{2^n}$ for all n

By comparison, $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ converges as well

The limit comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = C$, where C is a finite number & $C>0$ [i.e. $C \neq 0$]

Then either both series converges or both diverges.

again with $\sum_{n=1}^{\infty} \frac{1}{2^n-1} = \sum_{n=1}^{\infty} a_n$

& $\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} b_n$ which converges.

Try $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{2^n-1} \cdot \frac{2^n}{1} = \lim_{n\to\infty} \frac{2^n}{2^n-1} = \lim_{n\to\infty} \frac{\cancel{2^n}\cdot 1}{\cancel{2^n}(1-\underbrace{\tfrac{1}{2^n}}_{0})} = 1 > 0$

By limit comparizon, $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ converges.

$*\ \sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}} = \sum_{n=1}^{\infty} a_n$

Consider the dominating terms in numerator and denominator

$\sum_{n=1}^{\infty} \frac{2n^2}{\sqrt{n^5}} = \sum_{n=1}^{\infty} 2n^{2}\cdot n^{-\frac{5}{2}} = \sum_{n=1}^{\infty} \frac{2}{n^{\frac{1}{2}}} = \sum_{n=1}^{\infty} b_n$ P-series diverges because $P \leq 1$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+3n}{\sqrt{5+n^5}} \cdot \frac{\sqrt{n^5}}{2n^2} = \lim_{n\to\infty} \frac{(1+\underbrace{\tfrac{3}{2n}}_{0})\cdot \cancel{n^{\frac{5}{2}}}}{\cancel{n^{\frac{5}{2}}}\sqrt{\underbrace{\tfrac{5}{n^5}}_{0}+1}} = \frac{1}{\sqrt{1}} = 1$

By limit comparizon test, $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ Diverges.

From the book, section 11.2

17) $3-4+\frac{16}{3}-\frac{64}{9}+\cdots = \sum_{n=0}^{\infty} 3\cdot\left(-\frac{4}{3}\right)^n$

$\cdot\frac{-4}{3}$ $\cdot\frac{-4}{3}$ $\cdot\frac{-4}{3}$

Diverges because $\left|-\frac{4}{3}\right| \geq 1$

18) $\frac{1}{8}-\frac{1}{4}+\frac{1}{2}-1+\cdots = \sum_{n=0}^{\infty} \frac{1}{8}(-2)^n$

(-2) (-2) (-2)

Diverges because $|-2| \geq 1$

19) $10-2+\frac{4}{10}-\frac{8}{100}+\cdots = \sum_{n=0}^{\infty} 10\left(-\frac{2}{10}\right)^n = 10\cdot\frac{1}{1-\left(\frac{-2}{10}\right)} = \frac{100}{12} = \frac{25}{3}$

$\frac{-2}{10}$ $\frac{-2}{10}$ $\frac{-2}{10}$

Converges because $\left|\frac{-2}{10}\right| < 1$

20) $1+\frac{4}{10}+\frac{16}{100}+\frac{64}{1000}+\cdots = \sum_{n=0}^{\infty}\left(\frac{4}{10}\right)^n = \frac{1}{1-\frac{2}{5}} = \frac{5}{3}$

$\frac{4}{10}$ $\frac{4}{10}$ $\frac{4}{10}$

Converges because $\left|\frac{4}{10}\right| < 1$

21) $\sum_{n=1}^{\infty} 6(0.9)^{n-1} = 6\cdot\frac{1}{1-\frac{9}{10}} = 60$ Converges because $\frac{9}{10} < 1$

22) $\sum_{n=1}^{\infty}\frac{10^n}{(-9)^{n-1}} = \sum_{n=1}^{\infty}\frac{10^n}{(-9)^n}\cdot\frac{1}{(-9)^{-1}} = \sum_{n=1}^{\infty} -9\left(-\frac{10}{9}\right)^n = \sum_{n=1}^{\infty}(-9)\left(-\frac{10}{9}\right)\left(-\frac{10}{9}\right)^{n-1}$

Diverges because $\left|-\frac{10}{9}\right| \geq 1$

23) $\sum_{n=1}^{\infty}\frac{(-3)^{n-1}}{4^n} = \sum_{n=1}^{\infty}\cancel{(-3)^{-1}}\left(\frac{-3}{4}\right)\left(\frac{-3}{4}\right)^{n-1} = \sum_{n=1}^{\infty}\frac{1}{4}\left(\frac{-3}{4}\right)^{n-1} = \frac{1}{4}\cdot\frac{1}{1+\frac{3}{4}} = \frac{1}{7}$ Converges because $\left|\frac{-3}{4}\right| < 1$

24) $\sum_{n=0}^{\infty}\frac{1}{(\sqrt{2})^n} = \sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^n = \frac{1}{1-\sqrt{2}} = \frac{1+\sqrt{2}}{1-2} = -(1+\sqrt{2})$ Converges because $\left|\frac{1}{\sqrt{2}}\right| < 1$

25) $\sum_{n=0}^{\infty}\frac{\pi^n}{3^{n+1}} = \sum_{n=0}^{\infty}\frac{1}{3}\left(\frac{\pi}{3}\right)^n$ Diverges because $\left|\frac{\pi}{3}\right| \geq 1$

26) $\sum_{n=1}^{\infty}\frac{e^n}{3^{n-1}} = \sum_{n=1}^{\infty}\frac{1}{\cancel{(3)^{-1}}}\cdot\frac{e}{\cancel{3}}\left(\frac{e}{3}\right)^{n-1} = \frac{e}{1-\frac{e}{3}} = \frac{3e}{3-e}$ Converges because $\left|\frac{e}{3}\right| < 1$

27) $\frac{1}{3}+\frac{1}{6}+\frac{1}{9}+\frac{1}{12}+\frac{1}{15}+\cdots = \sum_{n=1}^{\infty}\frac{1}{3}\left(\frac{1}{n}\right)$ Harmonic serices diverges

28) $\frac{1}{3}+\frac{2}{9}+\frac{1}{27}+\frac{2}{81}+\frac{1}{243}+\frac{2}{729}+\cdots = \sum_{n=1}^{\infty}\left(\frac{1}{3^{2n-1}}+\frac{2}{3^{2n}}\right)$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{9^n\cdot(3)^{-1}}+\sum_{n=1}^{\infty}\frac{2}{9^n} = 3\sum_{n=1}^{\infty}\left(\frac{1}{9}\right)^n+2\sum_{n=1}^{\infty}\left(\frac{1}{9}\right)^n$$

$$\Rightarrow \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{1}{9}\right)^{n-1}+\frac{2}{9}\sum_{n=1}^{\infty}\left(\frac{1}{9}\right)^{n-1}$$

both series are Geometric & converges because $\left|\frac{1}{9}\right|<1$

$$\Rightarrow \frac{1}{3}\cdot\frac{1}{1-\frac{1}{9}}+\frac{2}{9}\cdot\frac{1}{1-\frac{1}{9}} = \frac{1}{\cancel{3}}\cdot\frac{\cancel{9}^{3}}{8}+\frac{2}{\cancel{9}}\cdot\frac{\cancel{9}}{8} = \frac{5}{8}$$

29) $\sum_{n=1}^{\infty}\frac{n-1}{3n-1}$

Divergence test: $\lim_{n\to\infty}\frac{n-1}{3n-1} = \lim_{n\to\infty}\frac{\cancel{n}(1-\cancel{\frac{1}{n}}^{0})}{\cancel{n}(3-\cancel{\frac{1}{n}}_{0})} = \frac{1}{3}\neq 0$

This series diverges

30) $\sum_{K=1}^{\infty}\frac{K(K+2)}{(K+3)^2}$ (When we have fraction of two polynomials with the same order, it Diverges)

Diverges by the Test for Divergence, since $\lim_{K\to\infty} a_K = \lim_{K\to\infty}\frac{K^2+2}{K^2+6K+9} = 1 \neq 0$

31) $\sum_{n=1}^{\infty}\frac{1+2^n}{3^n} = \sum_{n=1}^{\infty}\left(\frac{1}{3^n}+\frac{2^n}{3^n}\right) = \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^{n-1}+\frac{2}{3}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n-1}$

Both Geometric series converges, because $\left|\frac{1}{3}\right|$ and $\left|\frac{2}{3}\right|$ are less than 1

$$\Rightarrow \frac{1}{3}\cdot\frac{1}{1-\frac{1}{3}}+\frac{2}{3}\cdot\frac{1}{1-\frac{2}{3}} = \frac{1}{2}+2 = \frac{5}{2}$$

32) $\sum_{n=1}^{\infty}\frac{1+3^n}{2^n} = \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n+\underline{\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n}$

Diverges because one of the series is Divergent $\left|\frac{3}{2}\right|\geqslant 1$

33) $\sum_{n=1}^{\infty}\sqrt[n]{2} = \sum_{n=1}^{\infty}(2)^{\frac{1}{n}}$

Diverges by Test for Divergence,

since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(2)^{\frac{1}{n}} = 2^0 = 1 \neq 0$

34) $\sum_{n=1}^{\infty}\left[(0.8)^{n-1}-(0.3)^{n}\right] = \sum_{n=1}^{\infty}\left(\frac{4}{5}\right)^{n-1} - \frac{3}{10}\sum_{n=1}^{\infty}\left(\frac{3}{10}\right)^{n-1}$ Sum of two Convergent Geometric series

$$\Rightarrow \frac{1}{1-\frac{4}{5}} - \frac{3}{10}\cdot\frac{1}{1-\frac{3}{10}} = 5 - \frac{3}{\cancel{10}}\cdot\frac{\cancel{10}}{7} = \frac{32}{7}$$

35) $\sum_{n=1}^{\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right)$ Diverges by the Test of divergence,

since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(\frac{\cancel{n^2}(1+\frac{1}{n^2})}{\cancel{n^2}(2+\frac{1}{n^2})}\right) = \ln\left(\frac{1}{2}\right) \neq 0$

36) $\sum_{n=1}^{\infty} \frac{1}{1+\left(\frac{2}{3}\right)^n}$ Diverges by the Test for Divergence,

Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{1+\left(\frac{2}{3}\right)^n} = \frac{1}{1+0} = 1 \neq 0$

37) $\sum_{K=0}^{\infty}\left(\frac{\pi}{3}\right)^K$ A Divergent Geometric series, since $\left|\frac{\pi}{3}\right| \geq 1$

38) $\sum_{K=1}^{\infty} (\cos 1)^K = \cos(1)\cdot\sum_{K=1}^{\infty} (\cos 1)^{K-1}$ A Convergent Geometric series, since $|\cos(1)| < 1$

$$\Rightarrow = \frac{\cos 1}{1-\frac{1}{\cos 1}} = \frac{(\cos 1)^2}{\cos(1)-1}$$

39) $\sum_{n=1}^{\infty} \arctan(n)$ Diverges by the Test for Divergence, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \arctan(n) = \frac{\pi}{2} \neq 0$

40) $\sum_{n=1}^{\infty}\left(\frac{3}{5^n}+\frac{2}{n}\right) = \frac{3}{5}\sum_{n=1}^{\infty}\left(\frac{1}{5}\right)^{n-1} + 2\underline{\sum_{n=1}^{\infty}\frac{1}{n}}$

Diverges because one of the sumed series is a Divergent Harmonic series.

41) $\sum_{n=1}^{\infty}\left(\frac{1}{e^n}+\frac{1}{n(n+1)}\right) = \frac{1}{e}\cdot\sum_{n=1}^{\infty}\left(\frac{1}{e}\right)^{n-1} + \sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ check example 7, or telescoping series $\longrightarrow \frac{n}{n+1}$

Sum of 2 Convergent series, Geomelric series with $a = \frac{1}{e}$ & $q = \left|\frac{1}{e}\right| < 1$, and Telescoping Series.

$$\Rightarrow \frac{1}{e}\cdot\frac{1}{1-\frac{1}{e}} + \lim_{n\to\infty}\frac{n}{n+1} = \frac{\cancel{e}\cdot 1}{\cancel{e}(e-1)} + \cancel{\lim_{n\to\infty}}\frac{\overset{1}{\cancel{n}}}{\cancel{n}(1+\frac{1}{n})} = \frac{e}{(e-1)} + 1$$

42) $\sum_{n=1}^{\infty} \frac{e^n}{n^2} = \frac{2.7183\ldots}{1} + \frac{7.39}{4} + \frac{20.08\ldots}{8} + \frac{54.6}{16} + \cdots$

It is very clear that e^n is growing much faster than n^2

$\therefore$ The series diverges by Divergence Test,

since $\lim_{n\to\infty} \frac{e^n}{n^2} = \infty \neq 0$

43) $\sum_{n=2}^{\infty} \frac{2}{n^2-1}$

$$\frac{2}{n^2-1} = \frac{2}{(n-1)(n+1)} = \frac{A}{(n-1)} + \frac{B}{(n+1)}$$

$$2 = A(n+1) + B(n-1)$$

$\boxed{A=1}$, $\boxed{B=-1}$

$$\Rightarrow \sum_{n=2}^{K} \frac{2}{n^2-1} = \sum_{n=2}^{K}\left(\frac{1}{(n-1)} - \frac{1}{(n+1)}\right) = \left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right)$$

$$+\left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}\right) + \cdots + \left(\frac{1}{K-1} - \frac{1}{K+1}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{K-1} - \frac{1}{K+1} = \frac{3}{2} + \frac{2}{(K^2-1)}$$

$$S_K = \frac{3}{2} + \frac{2}{K^2-1}$$

$\lim_{K\to\infty} S_K = \frac{3}{2}$ $\quad \therefore$ Converges as $\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \frac{3}{2}$

44) $\sum_{n=1}^{\infty} \ln \frac{n}{n+1} = \sum_{n=1}^{\infty} (\ln(n) - \ln(n+1))$

Partial sum $S_K = \sum_{n=1}^{K} (\ln(n) - \ln(n+1)) = (\ln(1) - \cancel{\ln(2)}) + (\cancel{\ln(2)} - \cancel{\ln(3)}) + (\cancel{\ln(3)} - \cancel{\ln(4)}) + \cdots + (\cancel{\ln(K)} - \ln(K+1))$

$$S_K = \ln(1) - \ln(K+1)$$

$$\lim_{K\to\infty} S_K = \lim_{K\to\infty} \ln(1) - \lim_{K\to\infty} \ln(K+1) = 1 - \infty = -\infty$$

The series Diverges

45) $\sum_{n=1}^{\infty} \frac{3}{n(n+3)} \Rightarrow \frac{3}{n(n+3)} = \frac{A}{n} + \frac{B}{n+3} \Rightarrow 3 = A(n+3) + B\cdot n$

By comparing Coeffiecents $\boxed{A=1}$, $\boxed{B=-1}$

$\sum_{n=1}^{\infty} \frac{3}{n(n+3)} = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+3}\right) \Rightarrow$ Partial sums $S_K = \sum_{n=1}^{K}\left(\frac{1}{n} - \frac{1}{n+3}\right) = \left(\frac{1}{1} - \cancel{\frac{1}{4}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{5}}\right) + \left(\frac{1}{3} - \cancel{\frac{1}{6}}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{7}\right)$

$$+\left(\frac{1}{5} - \frac{1}{8}\right) + \cdots + \left(\cancel{\frac{1}{K}} - \frac{1}{K+3}\right)$$

$$S_K = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{K+3} = \frac{11}{6} + \frac{1}{K+3}$$

Thus, $\sum_{n=1}^{\infty} \frac{3}{n(n+3)} = \lim_{K\to\infty} S_K = \lim_{K\to\infty}\left(\frac{11}{6} + \frac{1}{K+3}\right) = \frac{11}{6}$ converges

46) $\sum_{n=1}^{\infty}\left(\cos\left(\frac{1}{n^2}\right)-\cos\left(\frac{1}{(n+1)^2}\right)\right) = \left(\cos(1)-\cos\left(\frac{1}{4}\right)\right)+\left(\cos\left(\frac{1}{4}\right)-\cos\left(\frac{1}{9}\right)\right)+\cdots$

$$S_K=\sum_{n=1}^{K}\left(\cos\left(\frac{1}{n^2}\right)-\cos\left(\frac{1}{(n+1)^2}\right)\right) = \left(\cos(1)-\cancel{\cos\left(\frac{1}{4}\right)}\right)+\left(\cancel{\cos\left(\frac{1}{4}\right)}-\cancel{\cos\left(\frac{1}{9}\right)}\right)+\left(\cancel{\cos\left(\frac{1}{9}\right)}-\cancel{\cos\left(\frac{1}{16}\right)}\right)+\cdots$$
$$+\left(\cos\left(\frac{1}{K^2}\right)-\cos\left(\frac{1}{(K-1)^2}\right)\right) = \cos 1 - \cos\left(\frac{1}{(K+1)^2}\right)$$

$$\lim_{K\to\infty} S_K = \lim_{K\to\infty}\left(\cos 1-\cos\left(\frac{1}{(K+1)^2}\right)\right) = \cos 1 - \cos\left(\frac{1}{\lim_{K\to\infty}(K+1)^2}\right) = \cos 1 - \cos 0 = \cos 1 - 1 = -0.4597$$

$\therefore$ The Telescoping series $\sum_{n=1}^{\infty}\left(\cos\left(\frac{1}{n^2}\right)-\cos\left(\frac{1}{(n+1)^2}\right)\right)$ Converges to -0.4597

47) $\sum_{n=1}^{\infty}\left(e^{\frac{1}{n}}-e^{\frac{1}{(n+1)}}\right)$

$$S_K=\sum_{n=1}^{K}\left(e^{\frac{1}{n}}-e^{\frac{1}{n+1}}\right) = \left(e^{1}-\cancel{e^{\frac{1}{2}}}\right)+\left(\cancel{e^{\frac{1}{2}}}-\cancel{e^{\frac{1}{3}}}\right)+\left(\cancel{e^{\frac{1}{3}}}-\cancel{e^{\frac{1}{4}}}\right)+\cdots+\left(\cancel{e^{\frac{1}{K}}}-e^{\frac{1}{K+1}}\right) = e - e^{\frac{1}{K+1}}$$

$$\sum_{n=1}^{\infty}\left(e^{\frac{1}{n}}-e^{\frac{1}{n+1}}\right) = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(e-e^{\frac{1}{n+1}}\right) = e - e^{0} = e-1 \approx 1.7183$$

This telescoping series Converges to 1.7183

48) $\sum_{n=2}^{\infty}\frac{1}{n^3-n}$, $\frac{1}{n^3-n}=\frac{1}{n(n^2-1)}=\frac{1}{n(n-1)(n+1)}=\frac{A}{n}+\frac{B}{(n+1)}+\frac{C}{(n-1)}$

$\Longrightarrow 1 = A(n+1)(n-1) + B\cdot n(n-1) + C\cdot n(n+1)$

By comparing coefficients $\boxed{A=-1}$, $\boxed{B=\frac{1}{2}}$, $\boxed{C=\frac{1}{2}}$

$\Rightarrow \frac{1}{n^3-n}=\frac{-1}{n}+\frac{1}{2(n+1)}+\frac{1}{2(n-1)}$, $-\frac{1}{n}=\frac{-1}{2n}-\frac{1}{2n}$

$$\sum_{n=2}^{K}\left(\frac{1}{n^3-n}\right)=\sum_{n=2}^{K}\left(\frac{-1}{2n}+\frac{1}{2(n+1)}-\frac{1}{2n}+\frac{1}{2(n-1)}\right)=\left(-\frac{1}{4}+\cancel{\frac{1}{6}}-\cancel{\frac{1}{4}}+\frac{1}{2}\right)+\left(-\cancel{\frac{1}{6}}+\cancel{\frac{1}{8}}-\cancel{\frac{1}{6}}+\cancel{\frac{1}{4}}\right)$$
$$+\left(-\cancel{\frac{1}{8}}+\cancel{\frac{1}{10}}-\cancel{\frac{1}{8}}+\cancel{\frac{1}{6}}\right)+\cdots\left(-\frac{1}{2K}+\frac{1}{2(K+1)}-\cancel{\frac{1}{2K}}+\cancel{\frac{1}{2(K-1)}}\right) = -\frac{1}{4}+\frac{1}{2}-\frac{1}{2K}+\frac{1}{2(K+1)}$$

$$S_K=\frac{1}{4}+\frac{\cancel{K}-\cancel{K}-1}{2K(K+1)}=\frac{1}{4}-\frac{1}{2K(K+1)}$$

The series converges to $\lim_{K\to\infty} S_K = \lim_{K\to\infty}\left(\frac{1}{4}-\frac{1}{2K(K+1)}\right)=\frac{1}{4}$

$$\therefore \sum_{n=2}^{\infty}\left(\frac{1}{n^3-n}\right)=\frac{1}{4}$$

$\lim_{n\to\infty} S_n = S$

> **2** **Remainder Estimate for the Integral Test** Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geqslant n$ and $\Sigma\, a_n$ is convergent. If $R_n = s - s_n$, then
>
> $$\int_{n+1}^{\infty} f(x)\,dx \leqslant R_n \leqslant \int_{n}^{\infty} f(x)\,dx$$

since $R_n = S - S_n$

$$\int_{n+1}^{\infty} f(x)\,dx \leqslant S - S_n \leqslant \int_{n}^{\infty} f(x)\,dx$$

$$\therefore \quad \boxed{S_n + \int_{n+1}^{\infty} f(x)\,dx \leqslant S \leqslant S_n + \int_{n}^{\infty} f(x)\,dx}$$

Section $\boxed{11.3}$ from the book:

3) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}} = \sum_{n=1}^{\infty} (n)^{\frac{-1}{5}}$ $f(x) = x^{\frac{-1}{5}}$ is a continuous, positive, & decreasing function on $[1,\infty)$, then integral test applies.

$$\int_1^{\infty} f(x) = \lim_{t\to\infty} \int_1^t x^{\frac{-1}{5}}\,dx = \lim_{t\to\infty}\left[\frac{5}{4}x^{\frac{4}{5}}\right]_1^t = \lim_{t\to\infty}\left[\frac{5}{4}t^{\frac{4}{5}} - \frac{5}{4}\right] = \infty$$

since $\int_1^{\infty} f(x)$ diverges, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$ diverges as well

also because it's a p-series with $P = -\frac{1}{5} \leqslant 1$.

5) $\sum_{n=1}^{\infty} \frac{1}{(2n+1)^3}$, The function $f(x) = \frac{1}{(2x+1)^3}$ is continuous, positive, & decreasing on $[1,\infty)$, then integral test applies.

$$\int_1^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_1^t (2n+1)^{-3}\,dx = \lim_{t\to\infty}\left[\frac{1}{2}\cdot\frac{-1}{2}(2n+1)^{-2}\right]_1^t = \lim_{t\to\infty}\left[\frac{-1}{4(2t+1)^2} + \frac{1}{4(2+1)^2}\right] = \frac{1}{4(9)} = \frac{1}{36}$$

Since the improper integral is convergent, the series $\sum_{n=1}^{\infty}\left(\frac{1}{(2n+1)^3}\right)$ is also convergent by integral test.

7) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$, the function $f(x) = \frac{x}{x^2+1}$ is continuous, positive, & decreasing on $[1,\infty)$ then integral test applies.

$$\int_1^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_1^t \frac{x}{x^2+1}\,dx = \lim_{t\to\infty}\left[\frac{1}{2}\ln|x^2+1|\right]_1^t = \lim_{t\to\infty}\left[\frac{1}{2}\ln|t^2+1| - \frac{1}{2}\ln|2|\right] = \infty$$

Since the improper integral is divergent, the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges also by the integral test.

8) $\sum_{n=1}^{\infty} n^2 \cdot e^{-n^3}$, The function $f(x) = x^2 e^{-x^3}$ is Continuous, Positive, decreasing, then integral test applies

$$\int_1^{\infty} f(x)dx = \lim_{t\to\infty} \int_1^t x^2 e^{-x^3} dx = \lim_{t\to\infty} \left[-\frac{1}{3} e^{-x^3}\right]_1^t = \lim_{t\to\infty} \left[-\frac{1}{3}e^{-t^3} + \frac{1}{3e}\right] = 0 + \frac{1}{3e} \approx 0.1226\ldots$$

Since the improper integral is convergent, the series $\sum_{n=1}^{\infty} n^2 \cdot e^{-n^3}$ is also convergent by integral test.

9) $\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}}$ P-series with $p \leq 1 \Longrightarrow$ The series diverges

10) same answer

11) $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^3}$ Convergent p-series, because $P > 1$

12) $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} n^{-1} \cdot n^{-\frac{1}{2}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$

(2^3 3^3 4^3 5^3)

Convergent P-series with $p = \frac{3}{2} > 1$

13) $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2n-1}$

The function $f(x) = \frac{1}{2x-1}$ is continuous, positive, & decreasing on for $x \in [1, \infty)$, then integral test applies

$$\int_1^{\infty} f(x)dx = \lim_{t\to\infty} \int_1^t \frac{1}{2x-1} dx = \lim_{t\to\infty} \left[\frac{1}{2} \ln|2x-1|\right]_1^t = \lim_{t\to\infty} \left[\frac{1}{2}\ln|2t-1| - \frac{1}{2}\ln|2-1|\right] = \infty - 0 = \infty$$

Since the improper integral diverges, the series $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ diverges as well by integral test

14) $\frac{1}{5} + \frac{1}{8} + \frac{1}{11} + \frac{1}{14} + \frac{1}{17} + \cdots = \sum_{n=2}^{\infty} \frac{1}{3n-1}$ Same as the previous, diverges.

15) $\sum_{n=1}^{\infty} \frac{\sqrt{n} + 4}{n^2} = \sum_{n=1}^{\infty} \left(\frac{n^{\frac{1}{2}}}{n^2} + \frac{4}{n^2}\right) = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}$

Sum of two convergents p-series, first with $p = \frac{3}{2} > 1$, & second with $p = 2 > 1$

$\therefore \sum_{n=1}^{\infty} \frac{\sqrt{n} + 4}{n^2}$ is convergent

16) $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1} = \sum_{n=1}^{\infty} a_n$, very close to $\sum_{n=1}^{\infty} \frac{n^2}{n^3} = \sum_{n=1}^{\infty} b_n$ which is a divergent p-series with $p = 1$

limit comparisson: $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\cancel{n^2}}{n^3+1} \cdot \frac{n^3}{\cancel{n^2}} = \lim_{n\to\infty} \frac{\cancel{n^3} \cdot 1}{\cancel{n^3}(1+\frac{1}{n^3})} = 1$

by limit comparison test, $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ diverges

19) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^3}$, $f(x) = \frac{\ln(x)}{x^3}$ is continuous, positive, & decrease function on $x \in [2,\infty)$; thus integral test applies.

$$\int_2^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_2^t \frac{\ln(x)}{x^3}\,dx = \lim_{t\to\infty} \left[-\frac{1}{2}x^{-2}\cdot \ln(x) - \frac{1}{4}x^{-2}\right]_2^t$$

$u = \ln(x)$ $\quad dv = x^{-3}\,dx$

$du = \frac{1}{x}$ $\quad v = \frac{-1}{2}x^{-2}$

$$\Rightarrow \lim_{t\to\infty}\left[-\frac{\ln(t)}{2t^2} - \frac{1}{4t^2} + \frac{1}{4}\right] = \lim_{t\to\infty}\left[\cancelto{0}{\frac{-1}{4t^2}}\left(\ln(t) + 1\right) + \frac{1}{4}\right] = \frac{1}{4}\cdot\frac{1}{16}$$

Since the improper integral converges, then the series $\sum_{n=1}^{\infty}\frac{\ln(n)}{n^3}$ is convergent.

20) $\sum_{n=1}^{\infty}\frac{1}{n^2+6n+13}$, very similar to the convergent p-series with $p=2>1$ $\quad \sum_{n=1}^{\infty}\frac{1}{n^2}$

$$\frac{1}{n^2} > \frac{1}{n^2+6n+13}$$

By composizon Test, $\sum_{n=1}^{\infty}\frac{1}{n^2+6n+13}$ is convergent as well.

21) $\sum_{n=2}^{\infty}\frac{1}{n\ln(n)}$, $f(x) = \frac{1}{x\ln(x)}$ is a continuous, positive, decreasing function on $x\in[2,\infty)$, then integral test applies.

$$\int_2^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_2^t \frac{1}{x\ln(x)}\,dx = \lim_{t\to\infty}\left[\ln(\ln(x))\right]_2^t = \lim_{t\to\infty}\left[\ln(\ln(t)) - \ln(\ln(2))\right] = \infty$$

since the improper integral is divergent, then the series $\sum_{n=2}^{\infty}\frac{1}{n\ln(n)}$ is Divergent as well, by integral test.

22) $\sum_{n=2}^{\infty}\frac{1}{n(\ln(n))^2}$

Same as 21, but convergent because $\int_2^{\infty} f(x) = \lim_{t\to\infty}\left[\frac{1}{\ln(t)} - \frac{1}{\ln(2)}\right] = -\frac{1}{\ln(2)}$

23) $\sum_{n=1}^{\infty}\frac{e^{1/n}}{n^2}$, $f(x) = \frac{e^{\frac{1}{x}}}{x^2}$ is a continuous, positive, decreasing function on $[1,\infty)$, then integral test applies.

$$\int_1^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_1^t e^{\frac{1}{x}}\cdot x^{-2}\,dx = \lim_{t\to\infty}\left[-e^{\frac{1}{x}}\right]_1^t = \lim_{t\to\infty}\left[-e^{\cancelto{0}{\frac{1}{t}}} + e^1\right] = e^1 - e^0 = e - 1$$

since the improper integral converges, then the series $\sum_{n=1}^{\infty}\frac{e^{1/n}}{n^2}$ Converges as well by integral test.

24) $\sum_{n=3}^{\infty}\frac{n^2}{e^n}$, $f(x) = \frac{x^2}{e^x}$ is a continuous, positive, & decreasing function on $[3,\infty)$, thus integral test applies.

$$\int_3^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_3^t x^2 e^{-x}\,dx = \lim_{t\to\infty}\left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_3^t = \lim_{t\to\infty}\left[-e^{-x}(x^2+2x+2)\right]_3^t$$

u		dv
x^2	+	e^{-x}
$2x$	−	$-e^{-x}$
2	+	e^{-x}
0		$-e^{-x}$

0.8463801623

$$\Rightarrow \lim_{t\to\infty}\left[-\cancelto{0}{e^{-t}}(t^2+2t+2) + e^{-3}((3)^2+2(3)+2)\right] =$$

Since improper integral converges, the series $\sum_{n=3}^{\infty}\frac{n^2}{e^n}$ converges as well by integral Test.

25) $\sum_{n=1}^{\infty} \frac{1}{n^2+n^3} = \sum_{n=1}^{\infty} \frac{1}{n^3(1+\frac{1}{n})}$ very close to the convergent p-series with $p=3>1$ $\sum_{n=1}^{\infty} \frac{1}{n^3}$

$\because \frac{1}{n^3} \geqslant \frac{1}{n^3(1+\frac{1}{n})}$ for all n

By comparizon test, the series $\sum_{n=1}^{\infty} \frac{1}{n^2+n^3}$ converges.

26) $\sum_{n=1}^{\infty} \frac{n}{n^4+1} = \sum_{n=1}^{\infty} a_n$, using limit comparizon test with the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^3} = \sum_{n=1}^{\infty} b_n$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n^4+1} \cdot \frac{n^3}{1} = \lim_{n\to\infty} \frac{\cancel{n^4} \cdot 1}{\cancel{n^4}(1+\underbrace{\frac{1}{n^4}}_{0})} = 1$ Both series converges

By limit comparizon Test, the series $\sum_{n=1}^{\infty} \frac{n}{n^4+1}$ converges

---

Section $\boxed{11.4}$ from the book:

The Comparison Test Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms.

(i) If $\Sigma\, b_n$ is convergent and $a_n \leqslant b_n$ for all n, then $\Sigma\, a_n$ is also convergent.

(ii) If $\Sigma\, b_n$ is divergent and $a_n \geqslant b_n$ for all n, then $\Sigma\, a_n$ is also divergent.

The Limit Comparison Test Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

3) $\sum_{n=1}^{\infty} \frac{n}{2n^3+1}$

$\frac{n}{2n^3+1} < \frac{n}{2n^3} = \frac{1}{2n^2} < \frac{1}{n^2}$ for all $n \geq 1$,

thus $\sum_{n=1}^{\infty} \frac{n}{2n^3+1}$ is convergent by comparizon with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges because it's a p-series with $p=2>1$

4) $\sum_{n=2}^{\infty} \frac{n^3}{n^4-1}$, $\frac{n^3}{n^4-1} > \frac{n^3}{n^4} = \frac{1}{n}$ for all $n \geq 2$, so $\sum_{n=2}^{\infty} \frac{n^3}{n^4-1}$ is divergent by comparizon with $\sum_{n=2}^{\infty} \frac{1}{n}$, which also diverges because it's a harmonic series.

5) $\sum_{n=1}^{\infty} \frac{n+1}{n\cdot\sqrt{n}}$, $\frac{n+1}{n^{\frac{3}{2}}} > \frac{n}{n^{\frac{3}{2}}} = \frac{1}{n^{\frac{1}{2}}}$ for all $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}$ is divergent by comparizon with $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$, which diverges because it's a p-series with $P = \frac{1}{2} \leq 1$

6) $\sum_{n=1}^{\infty} \frac{n-1}{n^2\sqrt{n}}$, $\frac{n-1}{n^2\sqrt{n}} < \frac{\cancel{n}}{n^{\cancel{2}}\sqrt{n}} = \frac{1}{n^{\frac{3}{2}}}$ for all $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{n-1}{n^2\sqrt{n}}$ converges by comparizon with $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$, which converges because it's a p-series with $P = \frac{3}{2} > 1$

7) $\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$, $\frac{9^n}{3+10^n} < \frac{9^n}{10^n} = \left(\frac{9}{10}\right)^n$ for all $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$ converges by comparizon with $\sum_{n=1}^{\infty} \left(\frac{9}{10}\right)^n$ which also converges because it's a geometric series with $|q| = \frac{9}{10} < 1$

8) $\sum_{n=1}^{\infty} \frac{6^n}{5^n - 1}$, $\frac{6^n}{5^n - 1} > \frac{6^n}{5^n} = \left(\frac{6}{5}\right)^n$ for all $n \geq 1$, thus $\sum_{n=1}^{\infty} \frac{6^n}{5^n - 1}$ diverges by comparizon with $\sum_{n=1}^{\infty} \left(\frac{6}{5}\right)^n$ which also diverges because it's a geometric series with $|q| = \frac{6}{5} \geq 1$

9) $\sum_{K=1}^{\infty} \frac{\ln K}{K}$, $\frac{\ln K}{K} > \frac{1}{K}$ for all $K \geq 3$, so $\sum_{K=1}^{\infty} \frac{\ln(K)}{K}$ diverges by comparizon with $\sum_{K=3}^{\infty} \frac{1}{K}$ which diverges because it a harmonic series.

10) $\sum_{n=1}^{\infty} \frac{K\sin^2 K}{1+K^3}$, $\sin^2 K \leq 1 \Rightarrow \frac{K\cdot\sin^2 K}{1+K^3} \leq \frac{K}{K^3} = \frac{1}{K^2}$ for all $K \geq 1$, so $\sum_{K=1}^{\infty} \frac{K\cdot\sin^2 K}{1+K^3}$ diverges by Comparizon with $\sum_{K=1}^{\infty} \frac{1}{K^2}$ which converges because it's a p-series with $P = 2 > 1$

11) $\sum_{K=1}^{\infty} \frac{\sqrt[3]{K}}{\sqrt{K^3+4K+3}}$, $\frac{\sqrt[3]{K}}{\sqrt{K^3+4K+3}} < \frac{\sqrt[3]{K}}{\sqrt{K^3}} = \frac{1}{K^{\frac{7}{6}}}$ for all $K \geq 1$, so $\sum_{K=1}^{\infty} \frac{\sqrt[3]{K}}{\sqrt{K^3+4K+3}}$ converges by comparizon with $\sum_{K=1}^{\infty} \frac{1}{K^{7/6}}$ which converges because it's p-series with $p = \frac{7}{6} > 1$

13) $\sum_{n=1}^{\infty} \frac{\arctan(n)}{n^{1.2}}$, $\frac{\arctan(n)}{n^{1.2}} \leq \frac{\pi/2}{n^{1.2}}$ for all $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{\arctan(n)}{n^{1.2}}$ Converges by comparizon with $\sum_{n=1}^{\infty} \frac{1}{n^{1.2}}$ which converges because it's a p-series with $p = 1.2 > 1$

14) $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1}$ diverges like number 4, $P = \frac{1}{2} < 1$

15) $\sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n-2}$, $\frac{4^{n+1}}{3^n-2} \geqslant \frac{4^{n+1}}{3^n} = \frac{16}{3}\left(\frac{4}{3}\right)^{n-1}$ for all $n \geqslant 1$, so $\sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n-2}$ diverges by comparizon test with $\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^{n-1}$ which diverges because it's geometric series with $|q| = \frac{4}{3} \geqslant 1$

16) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{3n^4+1}}$, $\frac{1}{\sqrt[3]{3n^4+1}} < \frac{1}{\sqrt[3]{3} \cdot n^{4/3}}$ for all $n \geqslant 1$, so $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{3n^4+1}}$ Converges by comparizon Test

with $\sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$ which also converges because it's a p-series where $p = \frac{4}{3} > 1$

17) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ using limit comparizon test where $a_n = \frac{1}{\sqrt{n^2+1}}$ & $b_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{\cancel{n} \cdot 1}{\cancel{n}\sqrt{1+\underbrace{\frac{1}{n^2}}_{0}}} = 1 > 0$, Since $\sum_{n=1}^{\infty} b_n$ diverges as a harmonic series

So does $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$!

18) $\sum_{n=1}^{\infty} \frac{1}{2n+3}$ diverges as 17

19) $\sum_{n=1}^{\infty} \frac{1+4^n}{1+3^n}$ using limit comparizon Test with $a_n = \frac{1+4^n}{1+3^n}$ & $b_n = \frac{4^n}{3^n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1+4^n}{1+3^n} \cdot \frac{3^n}{4^n} = \lim_{n\to\infty} \frac{\cancel{4^n}(\overbrace{\frac{1}{4^n}}^{0}+1)}{\cancel{3^n}(\underbrace{\frac{1}{3^n}}_{0}+1)} \cdot \frac{\cancel{3^n}}{\cancel{4^n}} = 1 > 0$, since $\sum_{n=1}^{\infty} b_n$ diverges because

it's a geometric series with $|q| = \frac{4}{3} \geqslant 1$, so does $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1+4^n}{1+3^n}$

(We could use also divergence Test easily.)

20) $\sum_{n=1}^{\infty} \frac{n+4^n}{n+6^n}$ using limit comparizon test with $a_n = \frac{n+4^n}{n+6^n}$ & $b_n = \frac{4^n}{6^n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\cancel{4^n}(\frac{n}{4^n}+1)}{\cancel{6^n}(\frac{n}{6^n}+1)} \cdot \frac{\cancel{6^n}}{\cancel{4^n}} = \frac{0+1}{0+1} = 1 > 0$, since $\sum_{n=1}^{\infty} b_n$ converges because it's a geometric

series with $|q| = \frac{4}{6} < 1$, so does $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n+4^n}{n+6^n}$

21) $\sum_{n=1}^{\infty} \frac{\sqrt{n+2}}{2n^2+n+1}$ using limit comparizon Test with $a_n = \frac{\sqrt{n+2}}{2n^2+n+1}$ & $b_n = \frac{1}{n^{3/2}}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n+2}}{2n^2+n+1} \cdot \frac{n^{\frac{3}{2}}}{1} = \lim_{n\to\infty} \frac{\cancel{n^{\frac{1}{2}}}\sqrt{1+\frac{2}{n}}}{\cancel{n^2}(2+\frac{1}{n}+\frac{1}{n^2})} \cdot \frac{\cancel{n^{\frac{3}{2}}}}{1} = \frac{\sqrt{1+0}}{2+0+0} = \frac{1}{2} > 0$ since $\sum_{n=1}^{\infty} b_n$ is a convergent p-series with $p = \frac{3}{2} > 1$, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\sqrt{n+2}}{2n^2+n+1}$ is also convergent.

☆

22) $\sum_{n=1}^{\infty} \frac{n+2}{(n+1)^3}$ using limit comparizon test where $a_n = \frac{n+2}{(n+1)^3} = \frac{n+2}{n^3+3n^2+3n+1}$ & $b_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\cancel{n}(1+\frac{2}{n})}{\cancel{n^3}(1+\frac{3}{n}+\frac{3}{n^2}+\frac{1}{n^3})} \cdot \frac{1 \cdot \cancel{n^2}}{1} = \frac{1+0}{1+0+0+0} = 1 > 0$ since $\sum_{n=1}^{\infty} b_n$ Converges as a p-series with $p=2>1$ so deos $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n+2}{(n+1)^2}$

23) $\sum_{n=1}^{\infty} \frac{5+2n}{(1+n^2)^2}$ using limit comparizon test with $a_n = \frac{5+2n}{(1+n^2)^2}$ & $b_n = \frac{1}{n^3}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n(\frac{5}{n}+2)}{\cancel{n^4}(\frac{1}{n^2}+1)^2} \cdot \frac{\cancel{n^3}}{1} = \frac{0+2}{(0+1)^2} = 2 > 0$ since $\sum_{n=1}^{\infty} b_n$ converges as a p-series with $p=3>1$ so does $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{5+2n}{(1+n^2)^2}$

24) $\sum_{n=1}^{\infty} \frac{n^2-5n}{n^3+n+1}$ Diverges by limit comparizon test with $b_n = \frac{1}{n}$

25) $\sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{n^3+n^2}$ using limit Comparizon Test with $a_n = \frac{\sqrt{n^4+1}}{n^3+n^2}$ & $b_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n^4+1}}{n^3+n^2} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{\cancel{n}\sqrt{1+\frac{1}{n^4}} \cdot \cancel{n}}{\cancel{n^3}(1+\frac{1}{n})} = \frac{\sqrt{1+0}}{1+0} = 1 > 0$ since $\sum_{n=1}^{\infty} b_n$ diverges as a harmonic series so does $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{n^3+n^2}$

26) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$ Converges by limit ComparizonTest with $b_n = \frac{1}{n^2}$

27) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^2 e^{-n}$ using limit comparizon Test with $a_n = (1+\frac{1}{n})^2 e^{-n}$ & $b_n = \frac{1}{e^n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(1+\frac{1}{n})^2}{\cancel{e^n}} \cdot \frac{\cancel{e^n}}{1} = (1+0)^2 = 1 > 0$ since $\sum_{n=1}^{\infty} b_n$ converges because it's a geometric series with $|r| = \frac{1}{e} < 1$ So does $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (1+\frac{1}{n})^2 e^{-n}$

28) $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$ Diverges by limit comparizon Test with $b_n = \frac{1}{n}$

29) $\sum_{n=1}^{\infty} \frac{1}{n!}$ Clearly $n! = n(n-1)(n-2)\cdots(3)(2) \geq 2\cdot 2\cdot 2\cdots 2\cdot 2 = 2^{n-1}$

so $\frac{1}{n!} \leq \frac{1}{2^{n-1}}$ for all $n \geq 1$

Since $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ is a convergent geometric series ($|q| = \frac{1}{2} < 1$),
thus $\sum_{n=1}^{\infty} \frac{1}{n!}$ is also convergent by comparizon test.

30) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$, $\frac{n!}{n^n} \leq \frac{19^{n-1}}{20^n}$ for $n \geq 20$, so $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges by comparizon Test

31) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ using limit comparizon Test with $a_n = \sin\left(\frac{1}{n}\right)$ & $b_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{1/n} = \lim_{x\to 0} \frac{\sin x}{x} = 1 > 0$ Since $\sum_{n=1}^{\infty} b_n$ is a divergent harmonic series
$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ is also divergent.

32) $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}} = \sum_{n=1}^{\infty} \frac{1}{n \cdot n^{1/n}}$ using limit comparizon Test with $a_n = \frac{1}{n^{1+1/n}}$ & $b_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\cancel{n} \cdot n^{1/n}} \cdot \frac{\cancel{n}}{1} = \frac{1}{n^0} = 1 > 0$ Since $\sum_{n=1}^{\infty} b_n$ is a divergent harmonic series
$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ is also divergent.

Section $\boxed{11.5}$ from the book:

Alternating Series Test If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad b_n > 0$$

satisfies

(i) $b_{n+1} \leq b_n$ for all n

(ii) $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

8) $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n}{\sqrt{n^3+2}}$, $f(x) = \frac{x}{\sqrt{x^3+2}}$

$$f'(x) = \frac{1\cdot\sqrt{x^3+2} - \frac{3}{2}x^3 \frac{1}{\sqrt{x^3+2}}}{(\sqrt{x^3+2})^2} * \frac{\sqrt{x^3+2}}{\sqrt{x^3+2}} = \frac{x^3+2-\frac{3}{2}x^3}{(x^3+2)^{\frac{3}{2}}}$$

$$f'(x) = \frac{2-\frac{1}{2}x^3}{(x^3+2)^{\frac{3}{2}}}$$

$2-\frac{1}{2}x^3 < 0$ ⟶ when f(x) is decreasing

$x^3 > 4 \Rightarrow x > \sqrt[3]{4} \approx 1.6$

$\sum_{n=1}^{\infty} (-1)^n \cdot b_n$

f(x) decreasing on the interval $(\sqrt[3]{4}, \infty)$

(i) $b_{n+1} < b_n$ for all $n > \sqrt[3]{4}$

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\cancel{n}\cdot 1}{n^{\frac{3}{2}}\sqrt{1+\frac{2}{n^3}}} = \lim_{n\to\infty} \frac{1}{\sqrt{n}\cdot\sqrt{1+\frac{2}{n^3}}} = 0$

Since both conditions are satisfied, the series $\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+2}}$ is convergent by Alternating series Test.

11) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4} = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot b_n$

$f(x) = \frac{x^2}{x^3+4}$, $f'(x) = \frac{2x(x^3+4) - 3x^2\cdot x^2}{(x^3+4)^2} = \frac{8x - x^4}{(x^3+4)^2} = \frac{x(8-x^3)}{(x^3+4)^2}$

f(x) is decreasing when $8-x^3 < 0$

$\therefore x > \sqrt[3]{8} = 2$

f(x) is decreasing in the interval $(2, \infty)$

(i) $\therefore$ $b_{n+1} < b_n$ for all $n > 2$

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^3+4} = 0$

Since (i) & (ii) are satisfied, the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4}$ is convergent by Alternating Series Test.

27) $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \approx -0.45969\ldots\ldots$

n	Sumands	Partial sums
1	$-\frac{1}{2}$	$-\frac{1}{2}$
2	$\frac{1}{24}$	$-\frac{11}{24} = -0.45833$
3	$\frac{-1}{720}$	$-0.459722\ldots$
4	$\frac{1}{40320}$	-0.4596974
5	$\frac{-1}{3628800}$	$-0.459697\ldots\ldots$
6	$\vdots$	
	0	

2) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6} \approx 0.9855\ldots$

n	Sumands	Partial sums
1	1	1
2	$\frac{-1}{64}$	$\frac{63}{64} = 0.984375$
3	$\frac{1}{729}$	$0.98574\ldots$
4	$\frac{-1}{4096}$	$0.985502\ldots$
5	$\frac{1}{15625}$	$0.985566\ldots$
6	$\frac{-1}{46656}$	$0.985545\ldots\ldots$
	$\vdots$	
	0	

29) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2}{10^n} = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot b_n \approx 0.06761\ldots$

n	Sumands	Partial sums
1	$\frac{1}{10}$	$\frac{1}{10} = 0.1$
2	$-\frac{4}{100}$	0.06
3	$\frac{9}{1000}$	0.069
4	$\frac{-16}{10^4}$	0.0674
5	$\frac{25}{10^5}$	0.06765
6	$\frac{-36}{10^6}$	0.067614

$b_6 = \frac{6^2}{10^6} = 0.000036$

30) $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n \cdot n!} = \sum_{n=1}^{\infty} (-1)^n \cdot b_n \approx -0.28347\ldots$

n	Sumands	Partial sums
1	$-\frac{1}{3 \cdot 1!} = -\frac{1}{3}$	$-\frac{1}{3} = -0.33333$
2	$\frac{1}{18}$	-0.27777
3	$\frac{-1}{162}$	$-0.2839506\ldots$
4	$\frac{1}{1944}$	-0.283436
5	$\frac{-1}{29160}$	$-0.28347\ldots$

$b_5 = 0.00003429\ldots$

Section $\boxed{11.6}$ from the book:

1 Definition A series $\Sigma\, a_n$ is called **absolutely convergent** if the series of absolute values $\Sigma\, |a_n|$ is convergent.

2 Definition A series $\Sigma\, a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

3 Theorem If a series $\Sigma\, a_n$ is absolutely convergent, then it is convergent.

The Ratio Test

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\Sigma\, a_n$.

2) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$ let $a_n = \frac{(-2)^n}{n^2}$, using ration test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}}{(n+1)^2}\cdot\frac{n^2}{(-2)^n}\right|$

$\Rightarrow \lim_{n\to\infty}\left|\frac{(-2)\cdot \cancel{n^2}}{\cancel{n^2}(1+\frac{1}{n})^2}\right| = \frac{2}{(1+0)^2} = 2 > 1$ $\therefore \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\frac{(-2)^n}{n^2}$ diverges by Ratio test.

3) $\sum_{n=1}^{\infty}\frac{n}{5^n}$ $\sum_{n=1}^{\infty}\left|\frac{n}{5^n}\right| = \sum_{n=1}^{\infty}\frac{n}{5^n}$, $\frac{n}{5^n} < \frac{4^n}{5^n}$ for all $n \geq 1$.

$\therefore$ The series $\sum_{n=1}^{\infty}\frac{n}{5^n}$ is absolutely convergent by comparizon Test with $\sum_{n=1}^{\infty}\left|\frac{4^n}{5^n}\right| = \sum_{n=1}^{\infty}\left(\frac{4}{5}\right)^n$ which is a convergent geometric series with $|q| = \frac{4}{5} < 1$

Another approach (Ratio test): $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{5^{n+1}}\cdot\frac{5^n}{n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{n}(1+\frac{1}{n})}{5\cdot\cancel{n}}\right| = \left|\frac{1+0}{5}\right| = \frac{1}{5} < 1$

$\therefore$ By Ratio test, the series $\sum_{n=1}^{\infty}\frac{n}{5^n}$ is absolutely convergent.

4) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+4} = \sum_{n=1}^{\infty} a_n$, $|a_n| > 0$ for all $n \geq 1$, $|a_n|$ is decreasing for all $n \geq 1$, and $\lim_{n\to\infty}|a_n| = 0$, So $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+4}$ is Convergent by Alternating series Test.

To determine absolute convergence:

$\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{n}{n^2+4}$ using limit comparizontest with $|a_n| = \frac{n}{n^2+4}$ & $|b_n| = \frac{1}{n}$

$\lim_{n\to\infty}\frac{|a_n|}{|b_n|} = \lim_{n\to\infty}\frac{n}{n^2+4}\cdot\frac{n}{1} = \lim_{n\to\infty}\frac{\cancel{n^2}\cdot 1}{\cancel{n^2}(1+\frac{4}{n^2})} = \frac{1}{1+0} = 1 > 0$ since $\sum_{n=1}^{\infty}|b_n|$ is a divergent harmonic series, so is $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{n}{n^2+4}$

Thus $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+4}$ is Conditionally Convergent.

5) $\sum_{n=0}^{\infty} \frac{(-1)^n}{5n+1} = \sum_{n=0}^{\infty} a_n$, $a_n = \frac{1}{5n+1} > 0$ for all $n \geq 0$, b_n is decreasing for $n \geq 0$ and $\lim_{n\to\infty} a_n = 0$, so $\sum_{n=0}^{\infty} \frac{(-1)^n}{5n+1}$ converges by alternating series test.

To determine absolute convergence: let $b_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{5n+1} = \lim_{n\to\infty} \frac{\cancel{n}\cdot 1}{\cancel{n}(5+\frac{1}{n})} = \frac{1}{5} > 0$, so $\sum_{n=0} \frac{1}{5n+1}$ diverges by limit Comparizon Test with a divergent harmonic series. Thus $\sum_{n=0}^{\infty} \frac{(-1)^n}{5n+1}$ is conditionally convergent.

6) $\sum_{n=0}^{\infty} \frac{(-3)^n}{(2n+1)!} = \sum_{n=0}^{\infty} a_n$, $|a_n| > 0$ for $n \geq 0$, $|a_n|$ is decreasing for $n > 0$, & $\lim_{n\to\infty} |a_n| = 0$

So $\sum_{n=0}^{\infty} \frac{(-3)^n}{(2n+1)!}$ is convergent by Alternating series test.

To determine absolute convergence: $(2n+1)! \geq (4)^n$ for $n \geq 0$

$\therefore |a_n| = \frac{(3)^n}{(2n+1)!} \leqslant \frac{(3)^n}{(4)^n}$ for $n \geq 0$ Thus $\sum_{n=0}^{\infty} \frac{3^n}{(2n+1)!}$ is Convergent by comparizon Test with a convergent geometric series with $|q| = \frac{3}{4} < 1$

Thus $\sum_{n=0}^{\infty} \frac{(-3)^n}{(2n+1)!}$ is absolutely convergent. (We could have only done the second part)

7) $\sum_{K=1}^{\infty} K\left(\frac{2}{3}\right)^K = \sum_{K=1}^{\infty} a_K$ using Ratio Test $\lim_{K\to\infty} \left|\frac{a_{K+1}}{a_K}\right| = \lim_{K\to\infty} \left|\frac{(K+1)\left(\frac{2}{3}\right)^{K+1}}{K\left(\frac{2}{3}\right)^K}\right| = \lim_{K\to\infty} \left|\frac{\cancel{K}(1+\frac{1}{K})\cancel{\left(\frac{2}{3}\right)^K}\left(\frac{2}{3}\right)}{\cancel{K}\cdot\cancel{\left(\frac{2}{3}\right)^K}\cdot 1}\right|$

$= \frac{(1+0)}{1} \cdot \frac{2}{3} = \frac{2}{3} < 1$ Thus, By Ratio Test $\sum_{K=1}^{\infty} K\left(\frac{2}{3}\right)^K$ is absolutely convergent.

8) $\sum_{n=1}^{\infty} \frac{n!}{100^n} = \sum_{n=1}^{\infty} a_n$ using Ratio test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!}{100^{n+1}} \cdot \frac{100^n}{n!}\right| = \lim_{n\to\infty} \left|\frac{\cancel{n!}(n+1)\cdot \cancel{100^n}}{\cancel{100^n}(100)\cdot \cancel{n!}}\right| = \infty$

Thus $\sum_{n=1}^{\infty} \frac{n!}{100^n}$ diverges by Ratio test.

9) $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{(1.1)^n}{n^4}$ $f(x) = \frac{(1.1)^x}{x^4} \Rightarrow f'(x) = \frac{(1.1)^x \ln(1.1) x^4 - 4x^3 (1.1)^x}{x^8} = \frac{\cancel{x^3}(1.1)^x(\ln(1.1)\cdot x - 4)}{x^{\cancel{8}5}}$

$(1.1)^x\left(x - \frac{4}{\ln(1.1)}\right) < 0$

$(1.1)^x \neq 0 \longrightarrow x < \frac{4}{\ln(1.1)}$

$= \sum_{n=1}^{\infty} (-1)^n a_n$ $a_n > 0$ for $n \geq 1$, a_n is increasing for $n > \frac{4}{\ln(1.1)}$

$\therefore$ By Alternating Test $\sum_{n=1}^{\infty} (-1)^n \frac{(1.1)^n}{n^4}$ diverges.

Another approach: Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(1.1)^{n+1}}{(n+1)^4} \cdot \frac{n^4}{(1.1)^n} = \lim_{n\to\infty} \frac{1.1\cancel{(1.1)^n}}{\frac{(n+1)^4}{n^4}} \cdot \frac{1}{\cancel{(1.1)^n}} = \lim_{n\to\infty} \frac{1.1}{(1+\frac{1}{n})^4} = \frac{1.1}{(1+0)^4} = 1.1 > 1$

$\therefore$ The series diverges by Ratio test.

10) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+2}} = \sum_{n=1}^{\infty} (-1)^n \cdot a_n$, $a_n > 0$ for $n \geq 1$, a_n is decreasing for $n \geq 1$, and $\lim_{n\to\infty} a_n = 0$

So $\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+2}}$ is Convergent by Alternating series Test.

$\sum_{n=1}^{\infty} \left| (-1)^n \cdot \frac{n}{\sqrt{n^3+2}} \right| = \sum_{n=1}^{\infty} |b_n|$ using limit comparizon Test with $|b_n| = \frac{n}{\sqrt{n^3+2}}$ & $d_n = \frac{1}{n^{\frac{1}{2}}}$

$\lim_{n\to\infty} \frac{|b_n|}{d_n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^3(1+\frac{2}{n^3})}} \cdot \frac{n^{\frac{1}{2}}}{1} = \lim_{n\to\infty} \frac{\cancel{n^{\frac{3}{2}}} \cdot 1}{\cancel{n^{\frac{3}{2}}} \sqrt{1+\frac{2}{n^3}}} = \frac{1}{\sqrt{1+0}} = 1 > 0$ so $\sum_{n=1}^{\infty} |b_n|$ diverges by limit comparizon

Test with a divergent p-series ($P = \frac{1}{2} \leq 1$). Thus the series $\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+2}}$ is Conditionally Convergent.

11) $\sum_{n=1}^{\infty} \frac{(-1)^n e^{1/n}}{n^3} = \sum_{n=1}^{\infty} (-1)^n \cdot a_n$ $a_n > 0$ for $n \geq 1$, a_n is decreasing for $n > 1$, and $\lim_{n\to\infty} a_n = 0$

So $\sum_{n=1}^{\infty} (-1)^n \frac{e^{1/n}}{n^3}$ is Convergent by Alternating Test.

We only need this Part if it's conditionally convergent!

$\sum_{n=1}^{\infty} \left| (-1)^n \frac{e^{1/n}}{n^3} \right| = \sum_{n=1}^{\infty} a_n$ $0 \leq \frac{e^{1/n}}{n^3} \leq \frac{e}{n^3}$ for $n \geq 1$, since $\sum_{n=1}^{\infty} \frac{e}{n^3}$ is a convergent p-series ($P = 3 > 1$),

so $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^3}$ also converges by comparizon Test. This part is enough.

Thus $\sum_{n=1}^{\infty} \frac{(-1)^n e^{1/n}}{n^3}$ is absolutely convergent

12) $\sum_{n=1}^{\infty} \frac{\sin(4n)}{4^n}$ $|\sin(4n)| \leq 1$ $\therefore 0 \leq \left| \frac{\sin(4n)}{4^n} \right| \leq \frac{1}{4^n}$ for all $n \geq 1$ since $\sum_{n=1}^{\infty} \frac{1}{4^n}$ is a convergent geometric series ($|q| = \frac{1}{4} < 1$)

$\therefore$ Thus $\sum_{n=1}^{\infty} \frac{\sin(4n)}{4^n}$ is absolutely convergent.

13) $\sum_{n=1}^{\infty} \frac{10^n}{(n+1) \cdot 4^{2n+1}} = \sum_{n=1}^{\infty} a_n$ using Ratio test $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{10^{n+1}}{(n+2)\, 4^{2n+3}} \cdot \frac{(n+1) \cdot 4^{2n+1}}{10^n} \right| = \lim_{n\to\infty} \frac{10\,(1+\frac{1}{n})}{4^2\,(1+\frac{2}{n})} = \frac{10\,(1+0)}{16\,(1+0)} = \frac{5}{8} < 1$

So the Series $\sum_{n=1}^{\infty} a_n$ is Absolutely convergent by Ratio Test.

14) $\sum_{n=1}^{\infty} \frac{n^{10}}{(-10)^{n+1}} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^{10}}{(10)^{n+1}} = \sum_{n=1}^{\infty} a_n$ using Ratio Test $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)^{10}}{(10)^{n+2}} \cdot \frac{(10)^{n+1}}{n^{10}}$

$\Longrightarrow = \lim_{n\to\infty} \frac{\frac{(n+1)^{10}}{n^{10}} \cdot \cancel{(10)^{n+1}}}{\cancel{(10)^{n+1}} \cdot 10} = \lim_{n\to\infty} \frac{(1+\frac{1}{n})^{10}}{10} = \frac{1+0}{10} = 0.1 < 1$

Thus, the series $\sum_{n=1}^{\infty} \frac{n^{10}}{(-10)^{n+1}}$ is absolutely convergent.

15) $\sum_{n=1}^{\infty} \frac{(-1)^n \arctan(n)}{n^2} = \sum_{n=1}^{\infty} a_n$ $\arctan(n) \leq \frac{\pi}{2}$, $0 \leq |a_n| = \frac{\arctan(n)}{n^2} \leq \frac{2}{n^2}$ for $n \geq 1$

Since $\sum_{n=1}^{\infty} 2 \cdot \frac{1}{n^2}$ is a convergent p-series ($P = 2 > 1$), thus $\sum_{n=1}^{\infty} a_n$ is absolutely convergent by Comparizon Test.

16) $\sum_{n=1}^{\infty} \frac{3-\cos(n)}{n^{2/3}-2} = \sum_{n=1}^{\infty} a_n$ $\quad 2 \leqslant 3-\cos(n) \leqslant 4 \Rightarrow |a_n| = \left|\frac{3-\cos(n)}{n^{2/3}-2}\right| \geqslant \frac{1}{n^{2/3}}$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ diverges because it's a p-series with $p=\frac{2}{3}<1$,

Thus, the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{3-\cos(n)}{n^{2/3}-2}$ diverges as well by comparizon Test.

17) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)} = \sum_{n=1}^{\infty} a_n$ $\quad$ Since $|a_n|$ is decreasing, & $\lim_{n\to\infty} |a_n| = 0$ so $\sum_{n=2} a_n$ is convergent by Alternating Series Test.

$|a_n| = \frac{1}{\ln(n)} > \frac{1}{n}$ for $n \geqslant 2$, since $\sum_{n=2}^{\infty} \frac{1}{n}$ is a divergent harmonic series, so $\sum_{n=2}^{\infty} |a_n|$ diverges as well.

Thus $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ is Conditionally convergent.

18) $\sum_{n=1}^{\infty} \frac{n!}{n^n} = \sum_{n=1}^{\infty} a_n$ $\quad$ using Ratio Test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n\to\infty} \frac{\cancel{n!}\,\cancel{(n+1)} \cdot n^n}{(n+1)^n \cdot \cancel{(n+1)} \cdot \cancel{n!}} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n}$

$\Rightarrow \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = e^{-1} = \frac{1}{e} < 1$ $\qquad \left(\lim_{x\to\infty} \left(\frac{x}{x+K}\right)^x = e^{-K}\right)$

By Ratio test the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ is Absolutely convergent, & therefore convergent.

19) $\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n!} = \sum_{n=1}^{\infty} a_n$ $\quad \frac{|\cos(n\pi/3)|}{n!} \leqslant \frac{1}{n!}$ for all $n \geq 1$

if we used Ratio Test with $b_n = \frac{1}{n!} \Rightarrow \lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1.$

Since $\sum_{n=1}^{\infty} \frac{1}{n!}$ Converges by Ratio Test, The series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n!}$ is absolutely convergent by comparizon Test.

20) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n} = \sum_{n=1}^{\infty} a_n$ $\quad$ Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-2)^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{(-2)^n}\right| = \lim_{n\to\infty} \frac{2n^n}{(n+1)^n(n+1)} = 2\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1} = e^{-1} \cdot 0 = 0 < 1$

By Ratio test, the series $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$ is absolutely convergent.

21) $\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n = \sum_{n=1}^{\infty} \left|\left(\frac{n^2+1}{2n^2+1}\right)^n\right|$ $\quad$ using limit comparizon Test with $a_n = \left(\frac{n^2+1}{2n^2+1}\right)^n$ & $b_n = \left(\frac{1}{2}\right)^n$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n \cdot \frac{2^n}{1} = \lim_{n\to\infty} \left(\frac{\cancel{n^2}(1+\frac{1}{n^2})}{\cancel{n^2}(2+\frac{1}{n^2})}\right)^n \cdot \frac{2^n}{1} = \lim_{n\to\infty} \left(\frac{1+0}{2+0}\right)^n \cdot \frac{2^n}{1} = \lim_{n\to\infty} \frac{1}{\cancel{2^n}} \cdot \frac{\cancel{2^n}}{1} = 1$

Since $\sum_{n=1}^{\infty} b_n$ is a convergent geometric series with $(|r| = \frac{1}{2} < 1)$, thus the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} |a_n|$

Converges by Limit comparizon Test.

Power Series:

Polynomials: Finite Linear combination of powers of x

$$\underbrace{C_0, C_1, C_2, C_3, \dots, C_n}_{\text{coefficients}} \qquad C_0 + C_1 x + C_2 x^2 + C_3 x^3 \cdots + C_n x^n$$

Extend to infinite sums:

$$C_0 + C_1 x + C_2 x^2 + \cdots = \sum_{k=0}^{\infty} C_k x^k$$

Definition:

$$C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^2 + \cdots = \sum_{k=0}^{\infty} C_k (x-a)^k$$

Is a power series with center a,

and coefficients $C_0, C_1, C_2, \dots$

Example: The Geometric series

$$1 + x + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k$$

Convergence for $|x| < 1$, so it defines a function with the domain $(-1, 1)$

and the values:

$$f(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

Example: $\displaystyle\sum_{k=1}^{\infty} \frac{(x-3)^k}{k} = (x-3) + \frac{(x-3)^2}{2} + \frac{(x-3)^3}{3} + \cdots$

For which x does this series converges?

$$x=3: \sum_{k=1}^{\infty} \frac{(3-3)^k}{k} = 0 + 0 + 0 + \cdots = 0$$

Every power series converges at its center.

$$\sum_{n=0}^{\infty} C_n (x-a)^n \xrightarrow{x=a} \sum_{n=0}^{\infty} C_n (a-a)^n = C_0$$

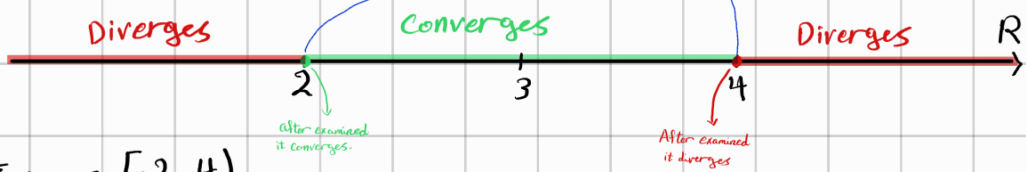
$$x=4: \sum_{k=1}^{\infty} \frac{(4-3)^k}{k} = \sum_{k=1}^{\infty} \frac{1}{k}$$ Diverges (Harmonic series)

$$x=2: \sum_{k=1}^{\infty} \frac{(2-3)^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$ (Negative alternating harmonic series) Converges.

Ratio Test: $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n}\right| = \lim_{n\to\infty} |x-3| \frac{n}{n+1} = |x-3| \cdot 1 = |x-3|$

$$|x-3| \begin{cases} < 1 & \text{Converges} \\ > 1 & \text{Diverges} \end{cases}$$

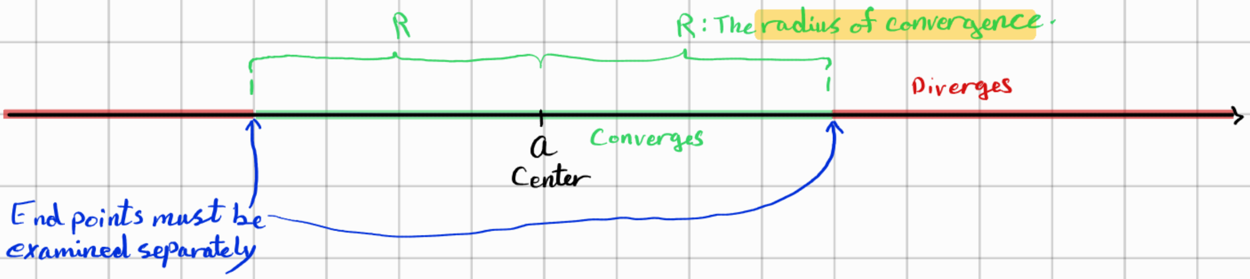
End points must be examined separately.

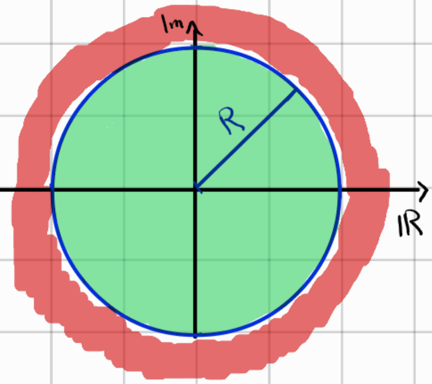


So $\displaystyle\sum_{k=1}^{\infty} \frac{(x-3)^k}{k}$ defines a function on $[2, 4)$

Interval of convergence

This Example is typical of all power series:

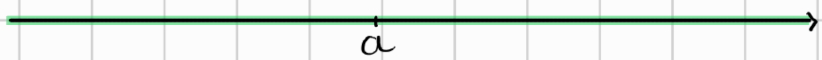




Two extreme cases for the radius of convergence are possible.

$R = 0$ Power series only converges at center.

$R = \infty$ Power series converges everywhere.



A Power Series with center a and radius of convergence R define a function defined on $(a-R, a+R)$
(sometimes one or both end points can be included.

Power Series form a whole class of functions with very nice properties:

- Easy domain, interval (Symmetric about center, except maybe at end points).
- Continuous on the whole domain.
- Infinitely many times differentiable $\longrightarrow \frac{d}{dx}\left[C_0 + C_1(x-a) + C_2(x-a)^2 + \cdots\right] = C_1 + 2C_2(x-a) + 3C_3(x-a)^2 + \cdots$
  as well
  With same radius of convergence.
- Integration is easy too. $\longrightarrow \int\left[C_0 + C_1(x-a) + C_2(x-a)^2 + \cdots\right]dx = C + C_0(x-a) + C_1\frac{(x-a)^2}{2} + C_2\frac{(x-a)^3}{3} + \cdots$
  With the same radius of convergence.
- The function values can be computed within any desired tolerence using partial sums.
  These Partial Sums are polynomials and require only addition, Substraction, and Multiplication.
- Many important functions can be written as Power Series. ☆
  (e.g. Sine, Cosine, Tangent, Arctangent, some Root, Exponential, Logarithmic, many engineering functions, for which there is no key on the calculator)
  [e.g. Bessel Func]

Series 2 Function:

Geometric Series:

$$1+x+x^2+x^3+\cdots=\sum_{K=0}^{\infty}x^K=\frac{1}{1-x}\quad \text{for } |x|<1$$

Differentiating Both sides:

$$0+1+2x+3x^2+4x^3+\cdots=\sum_{\substack{K=0\\K=1}}^{\infty}Kx^{K-1}=\frac{0(1-x)-(1)(-1)}{(1-x)^2}=\frac{1}{(1-x)^2}$$

$$\therefore 1+2x+3x^2+\cdots=\sum_{K=1}^{\infty}Kx^{K-1}=\frac{1}{(1-x)^2}\quad \text{for } |x|<1$$

Alternatively, let's Integerate the geometric series formula

$$C+x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+\cdots=\sum_{K=0}^{\infty}\frac{1}{K+1}x^{K+1}=-\ln|1-x|\quad \text{for } |x|<1$$

Sufficient on one side of the equation.

$$=\sum_{K=1}^{\infty}\frac{x^K}{K}=\ln\left|\frac{1}{1-x}\right|\quad \text{for } |x|<1$$

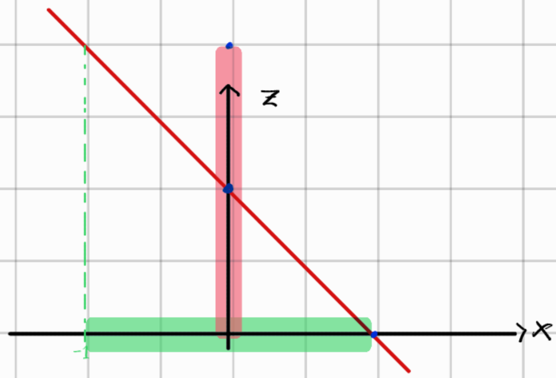
Compute C: plug in the center $x=a=0$

$$C+0+0+0+\cdots=-\ln|1|=0\Longrightarrow \boxed{C=0}$$

$$\therefore \ln|1-x|=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\cdots\quad \text{for } |x|<1$$

Substitution $z=1-x$
$\rightarrow x=1-z$

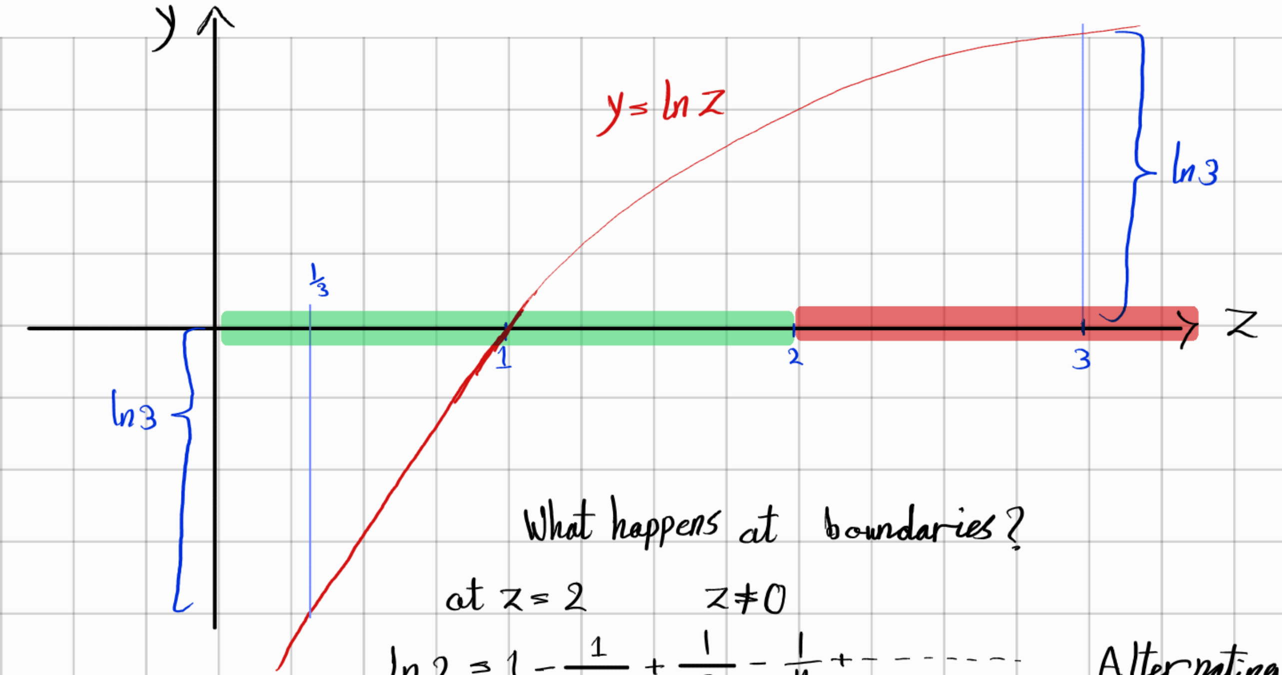
$-1<x<1\Longrightarrow 0<z<2$



$$\therefore \ln z=-(1-z)-\frac{(1-z)^2}{2}-\frac{(1-z)^3}{3}-\frac{(1-z)^4}{4}-\cdots\quad \text{for } 0<z<2$$

$$\boxed{\ln z=(z-1)-\frac{(z-1)^2}{2}+\frac{(z-1)^3}{2}-\frac{(z-1)^4}{4}+\cdots\quad \text{for } 0<z<2}$$

$$=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(z-1)^n}{n}$$



What happens at boundaries?

at $z=2$ $\qquad z \neq 0$

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Alternating harmonic series

Converges

How to compute $\ln 3$?

$$\ln 3 = \ln\left(\tfrac{1}{3}\right)^{-1} = -\ln\left(\tfrac{1}{3}\right)$$

$\ln\left(\frac{1}{3}\right)$ inside our radius of convergence.

To find ($\ln 3$) with a tolerance (approximation of $\ln 3$), we use partial sums.

$$\ln 3 = \frac{2}{3} + \frac{1}{2}\left(\frac{2}{3}\right)^2 + \frac{1}{3}\left(\frac{2}{3}\right)^3 + \frac{1}{4}\left(\frac{2}{3}\right)^4 + \cdots$$

n	Sumands	Partial sums
1	$\frac{2}{3} = 0.666$	0.6666...
2	$\frac{2}{9} = 0.2222$	0.8888...
3	$\frac{8}{81} = 0.09876...$	0.9876...
4	$\frac{4}{81} = 0.0493827$	1.037037...
5	$\vdots$	$\vdots$
	$\to 0$	$\to \ln 3$

When all summands are positive, the partial sums are Lower Bounds for the sum of the series.

$$1.037 < \ln 3 < ???$$

$1.037 + 0.08 = 1.117$
≈ 1.12

$$\ln 3 = \underbrace{\frac{2}{3} + \frac{1}{2}\left(\frac{2}{3}\right)^2 + \frac{1}{3}\left(\frac{2}{3}\right)^3 + \frac{1}{4}\left(\frac{2}{3}\right)^4}_{\substack{\text{Fourth partial sum} \\ \approx 1.037}} + \underbrace{\frac{1}{5}\left(\frac{2}{3}\right)^5 + \frac{1}{6}\left(\frac{2}{3}\right)^6 + \frac{1}{7}\left(\frac{2}{3}\right)^7 + \cdots}_{\text{Reminder}}$$

$$\text{Reminder} < \frac{1}{5}\left(\frac{2}{3}\right)^5 + \frac{1}{5}\left(\frac{2}{3}\right)^6 + \frac{1}{5}\left(\frac{2}{3}\right)^7 + \cdots$$

$$// \quad < \frac{1}{5}\left(\frac{2}{3}\right)^5 \left[1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots\right]$$ → geometric series

$$// \quad < \frac{1}{5}\left(\frac{2}{3}\right)^5 \cdot \frac{1}{1-\frac{2}{3}} = 0.079\ldots \approx \underline{0.08}$$

↳ we round up to make sure it's an upper bounds

Example: Use of Series in Integration:

$$\int_0^{1/2} \frac{1}{1+x^7}\,dx = \int_0^{1/2} \frac{1}{1-(-x^7)}dx = \int_0^{1/2}\left(1+(-x^7)+(-x^7)^2+(-x^7)^3+\cdots\right)dx \quad \text{for } -1<x<1$$

$$= \left[x-\frac{x^8}{8}+\frac{x^{15}}{15}-\frac{x^{22}}{22}+\cdots\right]_0^{\frac{1}{2}} = \underline{\left(\frac{1}{2}-\frac{1}{8}\left(\frac{1}{2}\right)^8+\frac{1}{15}\left(\frac{1}{2}\right)^{15}-\frac{1}{22}\left(\frac{1}{2}\right)^{22}+\cdots\right)} - 0$$

approximate it using Partial sums.

Summands	partial sums
$\frac{1}{2}=0.5$	0.5
$-\frac{1}{8(2)^8}=-\frac{1}{1024}$ ≈ 0.0005	≈ 0.4995

$$0.4995 < \int_0^{1/2}\frac{1}{1+x^7}dx < 0.5$$

960

+64

next value is very small $\Rightarrow$ $\int_0^{1/2}\frac{1}{1+x^7}dx \approx 0.4995$

Example: Arctangent

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

Geometric series $1+z+z^2+\cdots = \frac{1}{1-z}$ for $|z|<1$

Substitution: $z=-x^2$ for $-1<x<1$

$$\frac{1}{1-(-x^2)} = 1-x^2+(-x^2)^2+(-x^2)^3+(-x^2)^4+\cdots$$

$$= 1-x^2+x^4-x^6+x^8-+\cdots = \sum_{n=0}^{\infty}(-1)^n x^{2n} \quad \text{for } |x|<1$$

Integerat both sides:

$$\arctan x = C + x - \frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots \quad \text{for } -1<x<1$$

Find C by pluging in $x=0$:

$$0 = C+0+0+\cdots \Longrightarrow \boxed{C=0}$$

Hence

$$\boxed{\text{Arctan}\, x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots \quad \text{for } |x|<1}$$

Important Aspect:

$\boxed{\text{Power Series Allow us to approximate Complicated function by simpler Partial sums (= Polynomials)}}$

Taylor Series:

How can we find power Series Systematically?

Start with some function f. It must be infinitely many times differentiable to allow a power series representation

$$f(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + \cdots$$

Can we determine the coefficients systematically?

Plug in the center a:

$$f(a) = C_0 + 0 + 0 + 0 + \cdots \Rightarrow \boxed{C_0 = f(a)}$$

$$f(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + \cdots$$

Differentiate:

$$f'(x) = C_1 + 2C_2(x-a) + 3C_3(x-a)^2 + \cdots$$

Plug in a: $\boxed{f'(a) = C_1}$

Differentiate once more

$$f''(x) = 2C_2 + 3\cdot 2C_3(x-a) + 4\cdot 3C_4(x-a)^2 + \cdots$$

Plug in a: $f''(a) = 2C_2$

Differentiate once more

$$f'''(x) = 3\cdot 2C_3 + 4\cdot 3\cdot 2C_4(x-a) + 5\cdot 4\cdot 3C_5(x-a)^2 + \cdots$$

Plug in a: $\boxed{f'''(a) = 3\cdot 2C_3}$

Generally: $\boxed{C_n = \dfrac{f^{(n)}(a)}{n!}}$ Note $0! = 1$

Good will hunting! Movie.

Definition

The Tylor series of f about a is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

A Tylor series about 0,

$$f(0) + f'(0)\cdot x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

is called Maclaurin Series

If a function f has a power series representation with center a, then it's given by the Tylor series.

Problem 1: One of the derivatives may not exist.

Prolem 2: Some Tylor series have radius of convergence 0.

Problem 3: The Tylor series may exist and may have positive radius of convergence, but need not converge to the original function.

For example:

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

f(x) is infinitely many times differentiable at 0. The Taylor series has coefficients $\dfrac{f^{(n)}(a)}{n!} = 0$.

It is the constant zero function (with infinite radius of convergence) and differs from f at every $x \neq 0$.

Luckily, In engineering such examples never occur.
It is safe to use:

Engineering assumption:

If the tylor series exists and has positive radius of Convergence R, then

$$f(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \quad \text{for all } a-R<x<a+R$$

I.E. we assume the tylor series converges to the original function.

Example: $f(x)=e^x$, find Maclaurin Series:

$f(0) = e^0 = 1$ Since $\frac{f^{(n)}(a)}{n!} = e^a$

$f'(x) = e^x$

$f''(x) = e^x$

$\therefore$ all coefficients $\frac{f^{(n)}(0)}{n!} = \frac{e^0}{n!} = \frac{1}{n!}$

$\therefore$ Tylor series about 0 (Maclaurin series)

$$f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \quad \text{for all } a-R<x<a+R$$

$$R<x<R$$

$$\Rightarrow 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{k=0}^{\infty}\frac{x^k}{k!}$$

Radius of convergence: Ratio Test:

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{x^{k+1}}{(k+1)!}\cdot\frac{k!}{x^k}\right| = \lim_{k\to\infty}\left|\frac{x\cdot \cancel{k!}}{\cancel{k!}(k+1)}\right| = \lim_{k\to\infty}\frac{|x|}{k+1} = 0<1 \quad \text{Regardless of } x$$

This series converges for all x, I.E. it's Radius of convergence is $R=\infty$.

By Engineering assumption we obtain:

Know by heart

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty}\frac{x^k}{k!} \quad \text{for } -\infty<x<\infty}$$

In particular:

$$e = e^1 = \underbrace{\underbrace{1 + 1 + \frac{1}{2!}}_{2.5} + \frac{1}{3!}}_{2.666} + \frac{1}{4!} + \cdots\cdots = 2.71815$$

Euler's number.

Neater

$$e^{-1} = \frac{1}{e} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots\cdots$$

Summands	Partiale sums
1	1
-1	0
$\frac{1}{2}$	$\frac{1}{2}$
$-\frac{1}{6}$	$\frac{1}{3}$

$\frac{1}{3} < \frac{1}{e} < \frac{1}{2} \Rightarrow 2 < e < 3$

continue till desired accuricy

Example:

$$\int_0^1 e^{-x^2} dx \qquad e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$= \int_0^1 \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - + \cdots\right) dx$$

$$= \left[x - \frac{x^3}{3} + \frac{1}{5}\frac{x^5}{2!} - \frac{1}{7}\frac{x^7}{3!} + \cdots\right]_0^1$$

$$= \left[\left(x - \frac{1}{3\cdot 1} + \frac{1}{5\cdot 2} - \frac{1}{7\cdot 6} + \frac{1}{9\cdot 24} - + \cdots\right) - 0\right] = 1 - \frac{1}{3\cdot 1} + \frac{1}{5\cdot 2} - \frac{1}{7\cdot 6} + \frac{1}{9\cdot 24} - + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)n!}$$

Summands	Partial sums
1	1
$-\frac{1}{3}$	$\frac{2}{3}$
$+\frac{1}{10}$	$0.7666\ldots$
$-\frac{1}{42}$	

$0.6666 < \int_0^1 e^{-x^2} dx < 0.7666$

Formulas:

Require Radian measurement

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \text{for all } x$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \text{for all } x$$

(Do not use them for Large x; No problem, we only need them till $\frac{\pi}{4}$.)

Binomial series

$$(1+x)^K = 1 + \frac{K}{1}x + \frac{K(K-1)}{2\cdot 1}x^2 + \frac{K\cdot(K-1)(K-2)}{3\cdot 2\cdot 1}x^3 + \cdots \quad \text{for } -1<x<1$$

(for K = positive integer this is a finite sum)

$$\sqrt{1+x} = (1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\cdot(-\frac{1}{2})}{2\cdot 1}\cdot x^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3\cdot 2\cdot 1}x^3 + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{K=0}^{\infty}\frac{x^K}{K!} \quad \text{for } -\infty < x < \infty$$

$$\ln z = (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \frac{(z-1)^4}{4} + \cdots = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(z-1)^n}{n} \quad \text{for } 0<z<2$$

$$\arctan x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$$

Know by heart

Reminder: We use these series to prove Euler's formula:

$$e^{ix} = \cos x + i\sin x$$

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots$$

Then seperate into real and imaginary part.

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

From section $\boxed{11.8}$ in the book

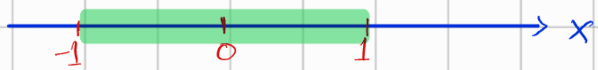
3) $\sum_{n=1}^{\infty} (-1)^n n x^n$

Ratio test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)x^{n+1}}{n x^n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{n}(1+\frac{1}{n})\cancel{x^n}\cdot x}{\cancel{n}\cancel{x^n}\cdot 1}\right| = \frac{1+0}{1}|x| = |x| < 1$

$-1 < x < 1$

The radius of convergence $R = 1$

Check boundaries



at $x = -1$ $\sum_{n=1}^{\infty} (-1)^n n (-1)^n = \sum_{n=1}^{\infty} n$

diverges by divergence Test, since $\lim_{n\to\infty} n = \infty$.

at $x = 1$ $\sum_{n=1}^{\infty} (-1)^n n (1)^n = \sum_{n=1}^{\infty} (-1)^n n$ // // // //

Then the interval of convergence for x is $(-1, 1)$

4) $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}}$

RatioTest $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^{\frac{3}{2}}}\cdot\frac{n^{\frac{3}{2}}}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{(1+\frac{1}{n})^{\frac{3}{2}}}\right| = \frac{|x|}{1+0} = |x| < 1$

$-1 < x < 1$

Radius of convergence $R = 1$

Check boundaries;

at $x = -1$ $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ A convergent p-series $(P = \frac{3}{2} > 1)$

at $x = 1$ $\sum \frac{(-1)^n}{n^{\frac{3}{2}}}$ $\left|\frac{(-1)^n}{n^{\frac{3}{2}}}\right| = \frac{1}{n^{\frac{3}{2}}}$ a convergent p-series $(P = \frac{3}{2} > 1)$

$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}}$ Converges at the interval $x \in [-1, 1]$, with $R = 1$

5) $\sum_{n=1}^{\infty} \frac{x^n}{2n-1}$ Ratio Test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{2n+1}\cdot\frac{2n-1}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x(2-\frac{1}{n})}{(2+\frac{1}{n})}\right| = \frac{2-0}{2+0}|x| = |x| < 1$

By Ratio test, the series $\sum_{n=1}^{\infty} \frac{x^n}{2n-1}$ converges when $|x| < 1$, So the Radius of convergence $R = 1$

check end points

at $x = 1$ $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ $\frac{1}{2n-1} > \frac{1}{2n}$ for all $n \geq 1$

since $\frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent harmonic series

$\sum_{n=1}^{\infty} \frac{1}{2n-1}$ diverges by divergence Test

at $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} = \sum_{n=1}^{\infty} b_n$ $\quad \lim_{n\to\infty} |b_n| = 0$, and $|b_n|$ is decreasing for $n > 1$

$\therefore \sum_{n=1}^{\infty} b_n$ converges by Alternating series Test.

Thus, The interval of convergence is $I = [-1, 1)$

6) $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}$ $\quad$ Ratio test $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{(1+\frac{1}{n})^2} = \frac{|x|}{(1+0)^2} = |x| < 1$

By Ratio test, the series $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}$ converges when $|x| < 1$, so radius of convergence is $R = 1$

at end points:

for both $x = -1$ & $x = 1$ the absolute value of the series is a p-series with $(P = 2)$

Therefore, the interval of convergence $I = [-1, 1]$

7) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\quad$ Ratio test $\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{|n+1|} = 0$

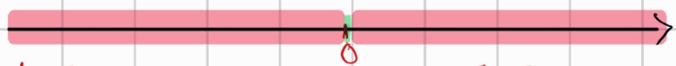
By Ratio test, the series $\sum_{n=0} \frac{x^n}{n!}$ converges regardless of the value of x

So, The radius of convergence $R = \infty$

Thus, The interval of convergence for x, $I = (-\infty, \infty)$

8) $\sum_{n=1}^{\infty} n^n x^n$ $\quad$ Ratio test $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{n^{n+1} x^{n+1}}{n^n x^n} \right| = \lim_{n\to\infty} |n x| = \infty \cdot |x|$

By Ratio test, the series $\sum_{n=1}^{\infty} n^n x^n$ diverges everywhere except for the center, So $R = 0$



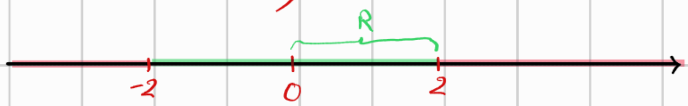
Since the center $a = 0$; The interval of convergence $I = [a] \Rightarrow I = [0]$

9) $\sum_{n=1}^{\infty} (-1)^n \frac{n^2 x^n}{2^n}$ $\quad$ Ratio Test $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2 x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{n^2 \cdot x^n} \right| = \lim_{n\to\infty} \left| \frac{x(1+\frac{1}{n})^2}{2} \right| = \frac{|x|(1+0)^2}{2} = \frac{|x|}{2} < 1$

$|x| < 2$

By Ratio Test, the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^2 x^n}{2^n}$ converges when $|x| < 2$, So $R = 2$

Checking the Boundaries



when $x = 2$ $\quad \sum_{n=1}^{\infty} (-1)^n \frac{n^2 \cancel{2^n}}{\cancel{2^n}}$

when $x = -2$ $\quad \sum_{n=1}^{\infty} \frac{n^2 \cancel{2^n}}{\cancel{2^n}}$

$\Big\}$ Both diverges by divergence test $\lim_{n\to\infty} a_n = \infty$

Thus, the interval of convergence $I = (-2, 2)$

10) $\sum_{n=1}^{\infty} \frac{10^n x^n}{n^3}$ Ratio test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{10^{n+1}x^{n+1}}{(n+1)^3}\cdot\frac{n^3}{10^n x^n}\right| = \lim_{n\to\infty}\frac{|10x|}{(1+\frac{1}{n})^3} = |10x| < 1$

$|x| < \frac{1}{10}$

By Ratio Test, the series $\sum_{n=1}^{\infty}\frac{10^n x^n}{n^3}$ converges when $|x|<\frac{1}{10}$, so the radius of convergence $R=\frac{1}{10}$

at the end points

at $x=\frac{1}{10}$ $\sum_{n=1}^{\infty}\frac{\cancel{10^n}\cdot 1}{\cancel{10^n}n^3}$

at $x=-\frac{1}{10}$ $\sum_{n=1}^{\infty}(-1)^n\frac{\cancel{10^n}}{\cancel{10^n}n^3}$

Both absolute value of the series are convergent p-series (with $p=3>1$)

Thus, the interval of convergence $I=[-\frac{1}{10},\frac{1}{10}]$

11) $\sum_{n=1}^{\infty}\frac{(-3)^n}{n\sqrt{n}}x^n = \sum_{n=1}^{\infty}(-1)^n\frac{3^n\cdot x^n}{n^{\frac{3}{2}}}$ Ratio test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{3^{n+1}|x^{n+1}|}{(n+1)^{3/2}}\cdot\frac{n^{3/2}}{3^n|x^n|} = \lim_{n\to\infty}\frac{3|x|}{(1+\frac{1}{n})^{3/2}} = 3|x|<1$

$|x|<\frac{1}{3}$

By Ratio test the series $\sum_{n=1}^{\infty}\frac{(-3)^n x^n}{n^{3/2}}$ converges when $|x|<\frac{1}{3}$, so the radius of convergence $R=\frac{1}{3}$

check end points

when $x=\frac{1}{3}$ $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{3/2}}$

when $x=-\frac{1}{3}$ $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$

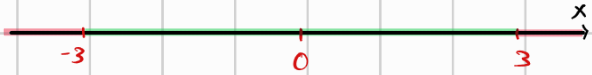
Both absolute values of the series are convergent p-series ($P=\frac{3}{2}>1$)

Thus the interval of convergence $I=[-\frac{1}{3},\frac{1}{3}]$

12) $\sum_{n=1}^{\infty}\frac{x^n}{n3^n}$ Ratio test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)3^{n+1}}\cdot\frac{n3^n}{x^n}\right| = \lim_{n\to\infty}\frac{|x|}{3(1+\frac{1}{n})} = \frac{|x|}{3(1+0)} = \frac{|x|}{3}<1 \Rightarrow |x|<3$

By Ratio test, the series $\sum_{n=1}^{\infty}\frac{x^n}{n3^n}$ converges when $|x|<3$, so radius of convergence $R=3$

check end points



when $x=3$ $\sum_{n=1}^{\infty}\frac{1}{n}$ A divergent harmonic series

when $x=-3$ $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ Alternating harmonic series $\rightarrow$ converges

Thus, the interval of convergence is $I=[-3,3)$

14) $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}a_n$ Ratio test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x^{2n+3}|}{(2n+3)!}\cdot\frac{(2n+1)!}{|x^{2n+1}|} = \lim_{n\to\infty}\frac{|x^2|}{(2n+3)(2n+2)} = 0<1$

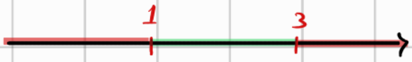
By Ratio Test, the series $\sum_{n=0}^{\infty}a_n$ converges regardless of x, so the radius of convergence $R=\infty$

Thus, the Interval of convergence $I=(-\infty,\infty)$.

15) $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1} = \sum_{n=0}^{\infty} a_n$ Ratio test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)^2+1}\cdot\frac{n^2+1}{\cancel{(x-2)^n}}\right| = \lim_{n\to\infty}\frac{|x-2|(1+\frac{1}{n^2})}{(1+\frac{1}{n})^2+\frac{1}{n^2}} = |x-2| < 1$

By Ratio test, the series $\sum_{n=0}^{\infty} a_n$ converges when $|x-2|<1 \Rightarrow |x|<3$, so $R=1$

checking end points $1 < x < 3$



when $x=3$ $\sum_{n=0}^{\infty}\frac{1}{n^2+1}$ $\frac{1}{n^2+1} < \frac{1}{n^2}$ for all $n \geq 1$ since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent p-series ($P=2>1$)

$\therefore \sum_{n=1}^{\infty}\frac{1}{n^2+1}$ converges by comparizon Test

when $x=1$ $\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2+1}$ $\left|\frac{(-1)^n}{n^2+1}\right| < \frac{1}{n^2}$ for all $n \geq 1$ since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent p-series ($P=2>1$)

Thus, $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}$ converges by comparizon Test.

Therefor, the interval of convergence $I=[1,3]$

16) $\sum_{n=0}^{\infty}(-1)^n\frac{(x-3)^n}{2n-1}$ Ratio Test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|(x-3)^{n+1}|}{2n+1}\cdot\frac{2n-1}{|(x-3)^n|} = \lim_{n\to\infty}\frac{|x-3|(2-\frac{1}{n})}{(2+\frac{1}{n})} = |x-3| < 1$

$|x-a| < R$

By Ratio test, the series $\sum_{n=0}^{\infty} a_n$ Converges when $|x-3|<1$, so $R=1$

check end points $2<x<4$

when $x=4$ $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n-1} = \sum_{n=0}^{\infty} b_n$ since $\lim_{n\to\infty}|b_n| = 0$, & $|b_n|$ is decreasing for $n \geq 1$,

thus, the series $\sum_{n=1}^{\infty} b_n$ converges by Alternating series test.

When $x=2$ $\sum_{n=0}^{\infty}\frac{1}{2n-1} = \sum_{n=0}^{\infty} C_n$ $\frac{1}{2n-1} > \frac{1}{2n}$ for $n>1$, since $\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$ is a divergent harmonic series time a constant.

Thus, the series $\sum_{n=1}^{\infty} C_n$ diverges by Comparizon test.

Thus, The interval of convergence is $I=(2,4]$

17) $\sum_{n=1}^{\infty}\frac{3^n(x+4)^n}{n^{1/2}} = \sum_{n=1}^{\infty} a_n$ Ratio test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}(x+4)^{n+1}}{(n+1)^{1/2}}\cdot\frac{n^{1/2}}{3^n(x+4)^n}\right| = \lim_{n\to\infty}\frac{3|x+4|}{(1+\frac{1}{n})^{1/2}} = 3|x+4| < 1 \Rightarrow |x+4| < \frac{1}{3}$

$-R < x+a < R \Rightarrow -\frac{13}{3} < x < -\frac{11}{3}$

By Ratio Test, The Series $\sum_{n=1}^{\infty} a_n$ converges when $|x+4|<\frac{1}{3}$, so $R=\frac{1}{3}$

Checking end Points $-\frac{13}{3} < x < -\frac{11}{3}$

when $x=\frac{-11}{3}$ $\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$ Divergent p-series ($P=\frac{1}{2}\leq 1$)

When $x=-\frac{13}{3}$ $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{1/2}} = \sum_{n=1}^{\infty} b_n$ Since $\lim_{n\to\infty} b_n = 0$, & $|b_n|$ is decreasing for $n>1$;

Thus $\sum_{n=1}^{\infty} b_n$ converges by Alternating Series Test.

Therefor, the interval of convergence is $I=(-\frac{13}{3}, -\frac{11}{3}]$

18) $\sum_{n=1}^{\infty} \frac{n}{4^n}(x+1)^n = \sum_{n=1}^{\infty} a_n$ Ratio test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x+1)^{n+1}}{4^{n+1}}\cdot\frac{4^n}{n(x+1)^n}\right| = \lim_{n\to\infty}\frac{|x+1|(1+\frac{1}{n})}{4} = \frac{1}{4}|x+1|<1 \Rightarrow |x+1|<4$

By Ratio test, the series $\sum_{n=1}^{\infty} a_n$ converges when $|x+1|<4$, so $R=4$

check end points $-5<x<3$

when $x=3$ $\sum_{n=1}^{\infty} n$ diverges by divergence Test since $\lim_{n\to\infty} n = \infty$

when $x=-5$ $\sum_{n=1}^{\infty}(-1)^n n$ since $\lim_{n\to\infty} n=\infty$, the series $\sum_{n=1}^{\infty}(-1)^n * n$ diverges by alternating series test.

Thus, The interval of convergence is $I=(-5,3)$

19) $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^n} = \sum_{n=1}^{\infty} a_n$ Ratio test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{(x-2)^n}\right| = \lim_{n\to\infty}\frac{|x-2|}{(n+1)(1+\frac{1}{n})^n} = 0<1$

By Ratio test, the series $\sum_{n=1}^{\infty} a_n$ converges always regardless of x value, $R=\infty$

Thus, the interval of convergence $I=(-\infty,\infty)$

20) $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty} a_n$ Ratio test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2x-1)^{n+1}}{5^{n+1}(n+1)^{\frac12}}\cdot\frac{5^n n^{\frac12}}{(2x-1)^n}\right| = \lim_{n\to\infty}\frac{|2x-1|}{(1+\frac{1}{n})^{\frac12}} = |2x-1|<1 \Rightarrow |x-\frac12|<\frac12$

By Ratio test, the series $\sum_{n=1}^{\infty} a_n$ converges when $|x-\frac12|<\frac12$, so $R=\frac12$

Check end points $0<x<1$

when $x=0$ $\sum_{n=1}^{\infty}\frac{(-1)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty} C_n$, Since $\lim_{n\to\infty} C_n = 0$, & $|C_n|$ decreases for all $n\geq 1$, so $\sum_{n=1}^{\infty} C_n$ converges by alternating series test.

when $x=1$ $\sum_{n=1}^{\infty}\frac{1}{5^n\sqrt{n}} = \sum_{n=1}^{\infty} b_n$ $\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\left|\frac{5^n n^{\frac12}}{5^{n+1}(n+1)^{\frac12}}\right| = \lim_{n\to\infty}\frac{1}{5(1+\frac{1}{n})^{\frac12}} = \frac{1}{5(1+0)} = \frac15 < 1$

Thus $\sum_{n=1}^{\infty} b_n$ converges by ratio test.

Thus, the interval of convergence is $I=[0,1]$

From section $\boxed{11.9}$ in the book

25) $\int \frac{t}{1-t^8}\,dt$

$t\cdot\frac{1}{1-t^8} = t\sum_{n=0}^{\infty}(t^8)^n$ for $|t^8|<1$

$= \sum_{n=0}^{\infty} t^{8n+1}$

$\int t\cdot\frac{1}{1-t^8}\,dt = \int\sum_{n=0}^{\infty} t^{8n+1}\,dt$

$= C + \sum_{n=0}^{\infty}\frac{t^{8n+2}}{8n+2}$ for all $|t|<1$

Thus, radius of convergence $R=1$

26) $\int \frac{t}{1+t^3}\,dt$ $t\cdot\frac{1}{1-(-t^3)} = t\sum_{n=0}^{\infty}(-t^3)^n = \sum_{n=0}^{\infty}(-1)^n t^{3n+1}$ for $|-t^3|<1 \Longrightarrow |t|<1$

$\int t\cdot\frac{1}{1-(-t^3)}\,dt = \int\sum_{n=0}^{\infty}(-1)^n t^{3n+1}\,dt = \sum_{n=0}^{\infty}\left[(-1)^n\cdot\int t^{3n+1}\,dt\right] = C+\sum_{n=0}^{\infty}(-1)^n\frac{t^{3n+2}}{3n+2}$ for $|t|<1=R$

Thus, $\int\frac{t}{1+t^3}\,dt = C+\sum_{n=0}^{\infty}(-1)^n\frac{t^{3n+2}}{3n+2}$ with a radius of convergence $R=1$

27) $\int x^2 \ln(1+x)\,dx$

$$\ln(z) = (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - + \cdots$$

$$\therefore \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

$$x^2 \ln(1+x) = x^3 - \frac{x^4}{2} + \frac{x^5}{3} - + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{n+2}}{n}$$

$$\int x^2 \ln(1+x)\,dx = \int \left(\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{n+2}}{n}\right) dx = \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{n} \cdot \int x^{n+2}\,dx\right] = C + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{n+3}}{n(n+3)}$$

Ratio test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x^{n+4}|}{(n+1)(n+4)} \cdot \frac{n(n+3)}{|x^{n+3}|} = \lim_{n\to\infty} \frac{|x|\,(1+\frac{3}{n})}{(1+\frac{1}{n})(1+\frac{4}{n})} = |x| < 1$

By Ratio test, the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{n+3}}{n(n+3)}$ converges only when $|x|<1$, so the radius of convergence $R=1$

& the interval of convergence $I=[-1,1]$ where the end points are included because in both cases

The absolute value of the series with be convergent by comparizon test with $\frac{1}{n^2}$ which is also convergent p-series ($P=2>1$)

28) $\int \frac{\tan^{-1}x}{x}\,dx$

$$\frac{d \tan^{-1}x}{dx} = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n \quad \text{for } |x^2|<1$$

$$\therefore \tan^{-1}x = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx = \sum_{n=0}^{\infty} \left[(-1)^n \cdot \int x^{2n}\,dx\right] = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

Plug in $x=0$

$\tan^{-1}(0) = C + 0 + 0 + 0 + \cdots \Rightarrow C = 0$

$$\frac{\tan^{-1}x}{x} = \frac{\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n}$$

$$\int \frac{\tan^{-1}x}{x}\,dx = \int \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{2n+1}\,dx = \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{2n+1} \cdot \int x^{2n}\,dx\right] = C + \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)^2}$$

Ratio test with $a_n = (-1)^n \frac{x^{2n+1}}{(2n+1)^2}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x^{2n+3}|}{(2n+3)^2} \cdot \frac{(2n+1)^2}{|x^{2n+1}|} = \lim_{n\to\infty} \frac{|x^2| \cdot (2+\frac{1}{n})^2}{(2+\frac{3}{n})^2} = \frac{x^2 (2+0)^2}{(2+0)^2} = x^2 < 1 \Rightarrow -1 < x < 1$$

Thus $\int \frac{\tan^{-1}x}{x}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)^2}$, and the radius of convergence is 1 ($R=1$)

29) $\int_0^{0.2} \frac{1}{1+x^5}\,dx$ (use power series to approximate the series)

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$$

$$\frac{1}{1-(-x^5)} = 1 - x^5 + x^{10} - x^{15} + x^{20} - + \cdots$$

$$\int_0^{0.2} \frac{1}{1-(-x^5)}\,dx = \int_0^{0.2} (1 - x^5 + x^{10} - x^{15} + \cdots)\,dx$$

$$= \left[x - \frac{x^6}{6} + \frac{x^{11}}{11} - \frac{x^{16}}{16} + \cdots\right]_0^{0.2} = \left[(0.2 - \frac{(0.2)^6}{6} + \frac{(0.2)^{11}}{11} - \frac{(0.2)^{16}}{16} + \cdots) - 0\right]$$

Summands	Partial sums
$0.2 = \frac{2}{10}$	0.2
$\frac{-\frac{64}{10^6}}{6} = -0.00001067$	0.1999893
$\frac{\frac{2048}{10^{11}}}{11} \simeq 0.2 \times 10^{-8}$	$\simeq 0.1999893352$

$$0.1999893 < \int_0^{0.2} \frac{1}{1+x^5}\,dx < 0.1999893352$$

$$\int_0^{0.2} \frac{1}{1+x^5}\,dx \simeq 0.1999893$$

30) $\int_0^{0.4} \ln(1+x^4)\,dx$

$$\ln z = (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \frac{(z-1)^4}{4} + \cdots$$

$$\ln(1+x^4) = x^4 - \frac{(x^4)^2}{2} + \frac{(x^4)^3}{3} - \frac{(x^4)^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{4n}}{n}$$

$$\int_0^{0.4} \ln(1+x^4)\,dx = \int_0^{0.4} \sum_{n=1}^{\infty} (-1)^n \frac{x^{4n}}{n}\,dx = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \int_0^{0.4} x^{4n}\,dx = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left[\frac{x^{4n+1}}{4n+1}\right]_0^{0.4} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left[\frac{(0.4)^{4n+1}}{4n+1} - 0\right]$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n (0.4)^{4n+1}}{n(4n+1)} = \sum_{n=1}^{\infty} (-1)^n b_n$$

Since $\lim_{n\to\infty} b_n = 0$, & b_n is decreasing for $n > 1$

The series $\sum_{n=1}^{\infty} (-1)^n b_n$ converges by Alternating series Test.

31) $\int_0^{0.1} x \arctan(3x)\,dx$

$$\tan(z) = \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1} \Rightarrow \tan(3x) = \sum_{n=1}^{\infty} (-1)^n \frac{(3x)^{2n+1}}{2n+1}$$

$$x\tan(3x) = 3 \sum_{n=1}^{\infty} (-1)^n \frac{3^{2n} x^{2(n+1)}}{2n+1}$$

$$\Rightarrow 3\int_0^{0.1} \sum_{n=1}^{\infty} (-1)^n \frac{3^{2n} x^{2n+2}}{2n+1}\,dx = 3\sum_{n=1}^{\infty} \frac{(-1)^n 3^{2n}}{2n+1} \int_0^{1} x^{2n+2}\,dx = 3\sum_{n=1}^{\infty} \frac{(-1)^n (3)^{2n}}{2n+1} \left[\frac{x^{2n+3}}{2n+3}\right]_0^{0.1} = \sum_{n=1}^{\infty} \frac{(-1)^n (3)^{2n+1} (0.1)^{2n+3}}{(2n+1)(2n+3)}$$

$$0.01 \sum_{n=1}^{\infty} \frac{(-1)^n (0.3)^{2n+1}}{(2n+1)(2n+3)}$$

This series diverges by divergence Test

32) $\int_0^{0.3} \frac{x^2}{1+x^4}\,dx$

$$\frac{1}{1-(-x^4)} = \sum_{n=0}^{\infty} (-1)^n (x^4)^n \Rightarrow \frac{x^2}{1-(-x^4)} = \sum_{n=0}^{\infty} (-1)^n x^{4n+2}$$

$$\Rightarrow \int_0^{0.3} \sum_{n=0}^{\infty} (-1)^n x^{4n+2}\,dx = \sum_{n=0}^{\infty} (-1)^n \left[\frac{x^{4n+3}}{4n+3}\right]_0^{0.3} = \sum_{n=0}^{\infty} (-1)^n \frac{(0.3)^{4n+3}}{4n+3} = \sum_{n=0}^{\infty} (-1)^n b_n$$

Since $\lim_{n\to\infty} b_n = 0$, & b_n is decreasing for $n > 1$, then the series $\sum_{n=0}^{\infty} (-1)^n b_n$ converges by Alternating series Test.

$$\boxed{\text{* Important: } \lim_{n\to\infty} \frac{x^n}{n!} = 0 \text{ for every real number } x}$$

$$\lim_{n \to 0} \frac{1 - \cos x}{1 + x - e^{x}}$$

$$\lim_{n \to \infty} \frac{1 - \cos x}{1 + x - e^{x}}$$

$$\int x^{2} \ln(1 + x)\, dx$$