Generalising

Improper Integrals, Type 1

Case a)

If $\int_a^t f(x)\,dx$ exists for every $t \geq a$, then:

define: $\int_a^\infty f(t)\,dt = \lim_{t\to\infty} \int_a^t f(x)\,dx$

if the limit exists

Case b)

If $\int_t^b f(x)\,dx$ exists for every $t \leq b$, then:

define: $\int_{-\infty}^b f(t)\,dt = \lim_{t\to-\infty} \int_t^b f(x)\,dx$

if the limit exists

Case c)

If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ exist, then:

define: $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$

a can be any real number
always check that both integrals exist

Improper integrals of Type 1

Convergent if the limit exists

Divergent otherwise

Example:

$$\int_{-\infty}^{0} x e^{x}\,dx = \lim_{t\to-\infty}\int_t^0 x e^{x}\,dx$$

$u = x \quad dv = e^x dx$

$du = dx \quad v = e^x$

$-\int$

$$\Rightarrow \lim_{t\to-\infty}\left[xe^x - \int e^x dx\right]_t^0 = \lim_{t\to-\infty}\left[xe^x - e^x\right]_t^0 = \lim_{t\to-\infty}\left[(-1) - e^t(t-1)\right] = -1 - 0 = -1$$

$$\therefore \boxed{\int_{-\infty}^{0} x e^{x}\,dx = -1}$$

$$\int_{-\infty}^{\infty} x e^{-x^2}\,dx = \int_{-\infty}^{0} x e^{-x^2}\,dx + \int_0^{\infty} x e^{-x^2}\,dx =$$

$$\int_{-\infty}^{0} x e^{-x^2}\,dx = \lim_{t\to-\infty}\int_t^0 x e^{-x^2}\,dx = \lim_{t\to-\infty}\left[-\frac{1}{2}e^{-x^2}\right]_t^0 = \lim_{t\to-\infty}\left[-\frac{1}{2} + \cancel{\frac{1}{2}e^{-t^2}}^{\,0}\right] = -\frac{1}{2}$$

$$\int_0^{\infty} x e^{-x^2}\,dx = \lim_{t\to\infty}\left[\frac{-1}{2}e^{-x^2}\right]_0^t = \lim_{t\to\infty}\left[\frac{1}{2} - \cancel{\frac{1}{2}e^{-t^2}}^{\,0}\right] = \frac{1}{2}$$

$$\int_{-\infty}^{\infty} x e^{-x^2}\,dx = \frac{1}{2} - \frac{1}{2} = 0$$

Generalising

Improper Integrals, Type 2

Case a)

If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then:

$$\text{define: } \int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

if the limit exists

Case b)

If $f(x)$ is continuous on $(a, b]$ and discontinuous at b, then:

$$\text{define: } \int_a^b f(t)\,dt = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

if the limit exists

Case c)

If $f(x)$ has a discontinuity at c, with $a < c < b$,

and **both** $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ exist, then:

$$\text{define: } \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Improper integrals of Type 2

Convergent if the limit exists
Divergent otherwise

a can be any real number
always check that both integrals exist

Example

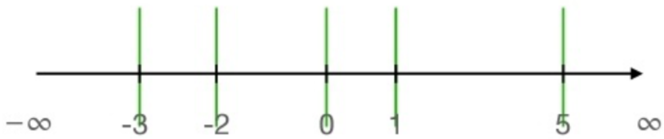
$$\int_{-\infty}^{\infty} \frac{2}{(x-1)(x+2)}\,dx$$

Four problems here:

boundaries at $\pm\infty$ and
discontinuities at $x = 1$ and $x = -2$

one interval should be normal

$$\text{so try: } \int_{-\infty}^{-3} \quad \int_{-3}^{-2} \quad \int_{-2}^{0} \quad \int_{0}^{1} \quad \int_{1}^{5} \quad \int_{5}^{\infty}$$



Problem 30, from section 7.8 in the book:

$$\int_6^8 \frac{4}{(x-6)^3}\,dx = \lim_{t\to 6}\int_t^8 \frac{4}{(x-6)^3}\,dx = \lim_{t\to 6}\left[\frac{\cancel{4}^{\,2}}{\cancel{-2}(x-6)^2}\right]_t^8 = \lim_{t\to 6}\left[\frac{-2}{(8-6)^2}-\left(\frac{-2}{(t-6)^2}\right)\right]$$

$$\Longrightarrow \lim_{t\to 6}\left[\frac{2}{(t-6)^2}-\frac{1}{2}\right] = \lim_{t\to 6}\frac{2}{(t-6)^2}-\lim_{t\to 6}\frac{1}{2}$$

$$\lim_{t\to 6^-}\frac{2}{(t-6)^2} = +\infty \quad , \quad \lim_{t\to 6^+}\frac{2}{(t-6)^2} = +\infty$$

$$\therefore \lim_{t\to 6}\frac{2}{(t-6)^2}-\frac{1}{2} = +\infty - \frac{1}{2} = +\infty$$