HW: Simulating Estimator Properties

1a) unbiasness

Part I: $\hat{\mu}_{MLE}$

$\ln(x) \sim N(\mu, \sigma^2)$

we know that
$E(\ln(x_i)) = \mu$
$\sigma^2(\ln(x_i)) = \sigma^2$

$\hat{\mu}_{MLE} = \frac{1}{n}\sum_{i=1}^{n} \ln(x_i)$

$E[\hat{\mu}_{MLE}] = \mu$

$E\left[\frac{1}{n}\sum_{i=1}^{n} \ln(x_i)\right] = \frac{1}{n} E\left[\sum \ln(x_i)\right] = \frac{1}{n}\sum_{i=1}^{n} E[\ln$

$= \frac{1}{\cancel{n}} \cancel{n}\, E[\ln(x_i)] = \mu$

so $E(\hat{\mu}_{MLE}) = \mu$ is unbiased

Part II: σ^2_{MLE}

- So we know that the MLE for variance is actually bias

$\sigma^2_{MLE} = \frac{1}{n}\sum_{i=1}^{n}\left(\ln(x_i) - \hat{\mu}_{MLE}\right)^2$

$\hat{\mu}_{MLE} = \frac{1}{n}\sum_{i=1}^{n} \ln(x_i)$

$\boxed{Y_i = \ln(x_i) \quad \bar{Y} = \frac{1}{n}\sum \ln(x_i)}$

So the expected value

$E\left[\frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})^2\right] = E\left[\sigma^2_{MLE}\right] = \frac{n-1}{n}\sigma^2$

$\text{Bias}(\sigma^2_{MLE}) = E[\sigma^2_{MLE}] - \sigma^2$

$= \frac{n-1}{n}\sigma^2 - \sigma^2 = \frac{(n-1)\sigma^2 - n\sigma^2}{n}$

$= \sigma^2(\cancel{n} - 1 - \cancel{n}) = -\frac{1}{n}\sigma^2$

So the MLE of σ^2 is biased

$E(\sigma^2_{MLE}) = \sigma^2 \neq -\frac{1}{n}\sigma^2$

it underestimate the true variance slightly because its expected value is not equal to the true variance → it's smaller $-\frac{1}{n}\sigma^2$

1.b) $E(\sigma^2_{MLE})$ is biased

To make unbiased, I need to multiply $\frac{n}{n-1}$

$$E\left[\frac{1}{n}\sum_{i=1}^{n}(Y_i-\bar{Y})^2\right] = E\left[\sigma^2_{MLE}\right] = \frac{n-1}{n}\sigma^2_{Biased}$$

$$\sigma^2_{unbiased} = \frac{n}{n-1} \times \sigma^2_{MLE} = \frac{\cancel{n}}{\cancel{n-1}} \times \frac{\cancel{n-1}}{\cancel{n}}\sigma^2$$

$= \sigma^2$ so σ^2_{MLE} is unbiased!!

1d) Comparison simulated bias of 3 MLE

Theoretical biases

$E(\hat{U}_{MLE}) = \mu \quad \Rightarrow$ Bias $= 0$

$E(\sigma^2_{MLE}) = \frac{n-1}{n}\sigma^2 \quad \Rightarrow$ Bias $= -\frac{1}{n}\sigma^2$

$= -\frac{1}{1000} \times 1 = -0.001$

Simulated Bias

- Bias Mu MLE $= -0.00214$
- Bias Sigma MLE $= -0.09166$
- Bias Sigma Unbiased $= 0.00926$

So The simulated bias closely match the theoritical biases for the MLE estimators.

Efficiency

$$Var(\ln(X_i)) = \sigma^2$$

$$\hat{\mu}_{MLE} = \frac{1}{n} \sum_{i=1}^{n} \ln(X_i)$$

$$Var(\hat{\mu}_{MLE}) = Var\left(\frac{1}{n} \sum_{i=1}^{n} \ln(X_i)\right) = \frac{1}{n^2} Var\left(\sum_{i=1}^{n} \ln(X_i)\right)$$

$$= \frac{1}{n^2}\, n\, Var(\ln(X_i)) = \frac{1}{n} \sigma^2$$

$$\boxed{Var(\hat{\mu}_{MLE}) = \frac{1}{n} \sigma^2}$$

$$L(\hat{\mu}_{MLE}) = \prod_{i=1}^{n} \frac{1}{x\sqrt{2\pi\sigma^2}} e^{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}}$$

$$= \frac{1}{\prod_{i=1}^{n} x\sqrt{2\pi\sigma^2}} e^{\frac{1}{2\sigma^2} \sum_{i=1}^{n} (\ln(x_i)-\mu)^2}$$

$$= \frac{1}{\sum_{i=1}^{n} X_i (2\pi\sigma^2)^{\frac{n}{2}}} e^{\frac{1}{2\sigma^2} \sum_{i=1}^{n} \ln(x_i)^2 - \mu^{2n}}$$

$$\ln(L(\hat{\mu}_{MLE})) = \ln\left(\frac{1}{\sum X_i (2\pi\sigma^2)^{\frac{n}{2}}}\right) \ln e^{\frac{1}{2\sigma^2} \sum \ln(x_i) - \mu}$$

$$= \ln(1) - \ln\left(\sum X_i (2\pi\sigma^2)^{\frac{n}{2}}\right) + b$$

$$= \ln(1) - \ln(\sum X_i$$

Efficiency

Step 1: Define Model

$$X \sim \text{Lognormal}(\mu, \sigma^2) \Rightarrow \ln(X) \sim N(\mu, \sigma^2)$$

Known:

$$\hat{\mu}_{MLE} = \frac{1}{n} \sum_{i=1}^{n} \ln(X_i)$$

Step 2: Variance of MLE

$$Var(\hat{\mu}_{MLE}) = Var\left(\frac{1}{n}\sum_{i=1}^{n} \ln(x_i)\right) = \frac{1}{n^2} Var\left(\sum_{i=1}^{n} \ln(x_i)\right)$$

$$= \frac{1}{n^2} n \, Var(\ln(X_i)) = \frac{1}{n} \sigma^2$$

$$\boxed{Var(\hat{\mu}_{MLE}) = \frac{1}{n} \sigma^2}$$

Step 3: CRLB

$$f(x, \mu, \sigma^2) = \frac{1}{x\sqrt{2\pi\sigma^2}} e^{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}}$$

$$L(\mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{x\sqrt{2\pi\sigma^2}} e^{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}}$$

$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \frac{1}{\prod_{i=1}^{n} X_i} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(\ln(x_i)-\mu)^2}$$

$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \frac{1}{\prod_{i=1}^{n} X_i} e^{-\frac{1}{2\sigma^2}\sum(\ln(x_i)^2 - \mu^{2n}}$$

$$\ln(L(\mu, \sigma^2)) = \ln\left[\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \frac{1}{\prod_{i=1}^{n} X_i} e^{-\frac{1}{2\sigma^2}\sum\sum\ln(x_i) - \mu^{2n}}\right]$$

$$= \ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n + \ln\left(\frac{1}{\prod X_i}\right) + \ln e^{-\frac{1}{2\sigma^2}\sum(\ln x_i \ldots}$$

$$= n\ln(1) - n\ln(\sqrt{2\pi\sigma^2}) + \ln(1) - \ln(\prod X_i) -$$

$$= -\frac{n}{2}\ln(2\pi\sigma^2) - \sum \ln X_i - \frac{1}{2\sigma^2}\sum^{n}(\ln x_i - \mu)$$

$$\boxed{\ln(L(\mu, \sigma^2)) = -\frac{n}{2}\ln(2\pi\sigma^2) - \sum_{i=1}^{n}\ln(x_i) - \frac{1}{2\sigma^2}\sum_{i}^{n}(\ln(x_i) - \mu)}$$

Derivative:

$$\frac{d}{d\mu}\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(\ln(x_i) - \mu)^2\right] =$$

I am going to use chain rule

$$\frac{dl}{d\mu} = -\frac{1}{2\sigma^2}\sum_{i=1}^{n} 2(\ln(x_i) - \mu)(-1) = -\frac{1}{2\sigma^2}\sum_{i=1}^{n} -2\ln(x_i - \mu$$

$$\boxed{\frac{dl}{d\mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(\ln(x_i) - \mu)}$$

$$\frac{1}{\sigma^2}\sum \ln($$

Efficiency

2.a) Step 1:

Since $\ln(X_i) \sim N(\mu, \sigma^2)$, we can use the known Fisher Information for Normal distribution

$$I_n(\mu) = \frac{n}{\sigma^2} \Rightarrow CRLB = \frac{1}{I_n(\mu)} = \frac{\sigma^2}{n}$$

Step 2: Variance of MLE

$$Var(\hat{\mu}_{MLE}) = Var\left(\frac{1}{n}\sum_{i=1}^{n} \ln(X_i)\right) = \frac{1}{n^2} Var\left(\sum_{i=1}^{n} \ln(X_i)\right)$$

$$= \frac{1}{n^2} n \, Var(\ln(X_i)) = \frac{1}{n}\sigma^2$$

$$\boxed{Var(\hat{\mu}_{MLE}) = \frac{1}{n}\sigma^2}$$

We can see that variance of $\hat{\mu}_{MLE}$ equals to CRLB, the MLE estimator efficiency !!

2.c) - My simulated variance of MLE estimator for μ was approximately 0.0985 based on 1000 samples, each size of $n=10$

- The theoretical variance from CRLB:

$$CRLB = \frac{\sigma^2}{n} = \frac{1}{10} = 0.1$$

$$V(\hat{\mu}_{MLE}) = 0.0985 \simeq 0.1$$

$$CRLB = 0.1$$

$$V(\hat{\mu}_{MLE}) \geq CRLB$$

So I concluded that MLE is approximately efficient.

Consistency

3a)

1. We know that $\hat{\mu}_{MLE}$ is unbiased ✓
2. $Var(\hat{\mu}_{MLE}) \to 0$ as $n \to \infty$ ✓

$$P(|\hat{\mu}_{MLE} - \mu| < \varepsilon) \to 1$$

$$Var(\hat{\mu}_{MLE}) = \frac{\sigma^2}{n}$$

as n increase is getting close to 0.
Therefore by chebyshev's inequality
$\hat{\mu}_{MLE}$ is a consistent estimator for μ.

3 b) In 3.a) I approved that $\hat{\mu}_{MLE}$ is unbiased
& the variance is getting close to 0 as
$n \to \infty$ according to chebyshev's inequality
which means the probability that $\hat{\mu}_{MLE}$ is close to
true value μ increase with larger sample size

- In my simulation confirm this behavior
because as I observe the probability
$\hat{\mu}_{MLE}$ falls within 0.01 of true value increase
  - $n=10 \to P \simeq 0.0260$
  - $n=100 \to P \simeq 0.0910$
  - $n=1000 \to P \simeq 0.2480$

So finally the simulation agrees with the theory that $\hat{\mu}_{MLE}$ is consistent

3. d) From simulation I observed that
the probability of $\hat{\mu}_{MM}$ falling 0.01 of
the true value increases which shows me
a consistency. This suggest that $\hat{\mu}_{MM}$
converges too true value of μ as $n \to \infty$

- Although $\hat{\mu}_{MM}$ is not biased and has higher
variance compare to MLE. Therefore I concluded
that $\hat{\mu}_{MM}$ is consistent, but less efficient &
more biased than $\hat{\mu}_{MLE}$.