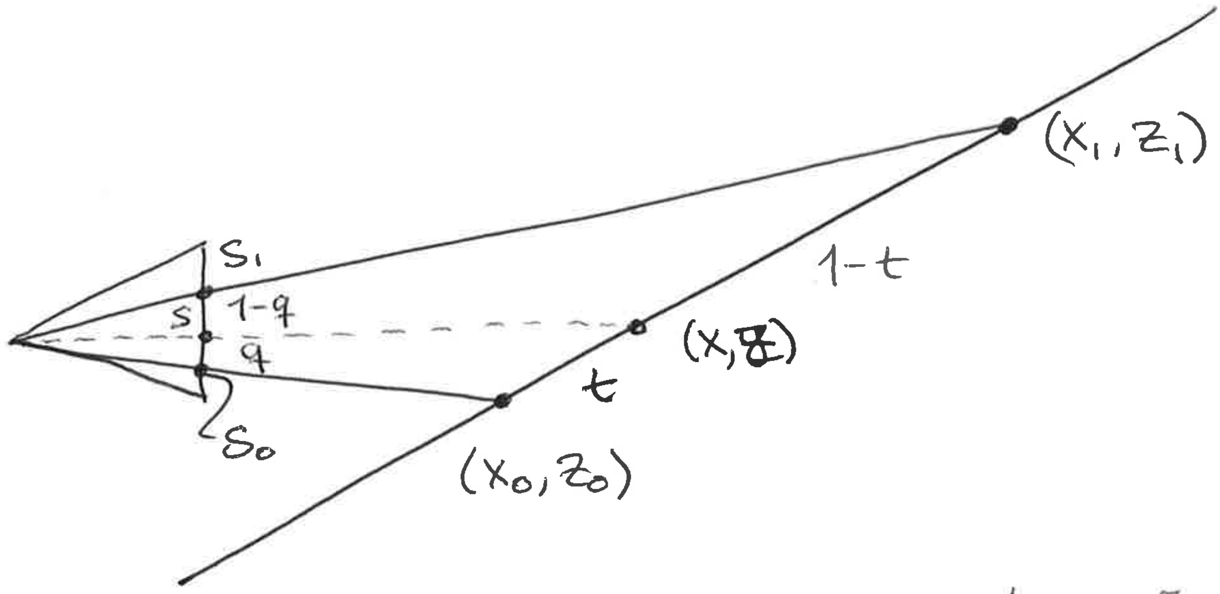
Perspective Projection



$$[x, z]^T = \begin{bmatrix} x_0 \\ z_0 \end{bmatrix} \cdot (1-t) + \begin{bmatrix} x_1 \\ z_1 \end{bmatrix} \cdot t = \begin{bmatrix} x_0 \\ z_0 \end{bmatrix} + t\left(\begin{bmatrix} x_1 \\ z_1 \end{bmatrix} - \begin{bmatrix} x_0 \\ z_0 \end{bmatrix}\right)$$

$$S = S_0 \cdot (1-q) + S_1 \cdot q = S_0 + q(S_1 - S_0)$$

- We want to interpolate in screenspace but move correctly in 3D space.
  $\Rightarrow$ we seek t as a function of q

- S is a projected point

  $$\Rightarrow \quad S = \frac{x}{z}, \quad S_1 = \frac{x_1}{z_1}, \quad S_0 = \frac{x_0}{z_0}$$

$$S = \frac{X}{Z} \Rightarrow Z = \frac{X}{S} = \left\{\text{insert interpolation}\right\} =$$

$$= \frac{X_0 + t(X_1 - X_0)}{S_0 + q(S_1 - S_0)} = \left\{\text{replace } X \text{ with } S\right\} =$$

$$= \left\{X_0 = S_0 \cdot Z_0 ,\ X_1 = S_1 \cdot Z_1\right\} =$$

$$= \frac{S_0 \cdot Z_0 + t(S_1 \cdot Z_1 - S_0 \cdot Z_0)}{S_0 + q(S_1 - S_0)}$$

$$Z = Z_0 + t(Z_1 - Z_0)$$

rewrite in terms of interpolated quantities

$$Z_0 + t(Z_1 - Z_0) = \frac{S_0 \cdot Z_0 + t(S_1 \cdot Z_1 - S_0 \cdot Z_0)}{S_0 + q(S_1 - S_0)}$$

Now we simplify =)

$$z_0 + t(z_1 - z_0) = \frac{s_0 \cdot z_0 + t(s_1 \cdot z_1 - s_0 \cdot z_0)}{s_0 + q(s_1 - s_0)}$$

$$\underbrace{(z_0 + t(z_1 - z_0)) \cdot (s_0 + q(s_1 - s_0))}_{\text{multiply (LHS)}} = s_0 \cdot z_0 + t(s_1 \cdot z_1 - s_0 \cdot z_0)$$

$$\cancel{z_0 \cdot s_0} + z_0 \cdot q(s_1 - s_0) + t \cdot s_0(z_1 - z_0) + tq(z_1 - z_0)(s_1 - s_0) =$$
$$= \cancel{s_0 \cdot z_0} + t(s_1 \cdot z_1 - s_0 \cdot z_0)$$

collect terms with q & t

$$t\left(s_0(z_1 - z_0) + q(z_1 - z_0)(s_1 - s_0) + (s_1 z_1 - s_0 z_0)\right) =$$
$$= -q z_0 (s_1 - s_0)$$

Identify the square

$$t\left(z_1 s_0 - \cancel{z_0 s_0} + q(z_1 - z_0)(s_1 - s_0) - s_1 z_1 + \cancel{s_0 z_0}\right) =$$

$$= t(s_1 - s_0)\left(-z_1 + q(z_1 - z_0)\right) = -q z_0 (s_1 - s_0)$$

flip sign

$$\Rightarrow t\cancel{(s_1 - s_0)}\left(z_1 - q(z_1 - z_0)\right) = q z_0 \cancel{(s_1 - s_0)}$$

$$t\left(z_1 - q(z_1 - z_0)\right) = q z_0$$

$$\Rightarrow \quad t = \frac{q z_0}{z_1 - q(z_1 - z_0)}$$

We want to figure out how z relates to q

$$z = z_0 + t(z_1 - z_0) =$$

$$= z_0 + \frac{q z_0}{z_1 - q(z_1 - z_0)}(z_1 - z_0) = z_0 + \frac{q z_0 (z_1 - z_0)}{z_1 - q(z_1 - z_0)} =$$

$$= z_0 + \frac{q z_1 z_0 - q z_0^2}{q z_0 + z_1(1-q)} =$$

$$= \frac{\cancel{q z_0^2} + z_0 z_1 (1-q) + \cancel{q z_1 z_0} - \cancel{q z_0^2}}{q z_0 + z_1(1-q)} =$$

$$= \frac{z_0 z_1}{q z_0 + z_1(1-q)} = \frac{1}{\frac{q \cancel{z_0}}{\cancel{z_0} z_1} + \frac{\cancel{z_1}(1-q)}{z_0 \cancel{z_1}}} =$$

$$= \frac{1}{\frac{q}{z_1} + \frac{(1-q)}{z_0}} = \frac{1}{\frac{1}{z_1} + q\left(\frac{1}{z_1} - \frac{1}{z_0}\right)} = z$$

$$\frac{1}{Z} = \frac{1}{Z_0} + q\left(\frac{1}{Z_1} - \frac{1}{Z_0}\right)$$

So this means, when you interpolate in screenspace i.e q then Z is going to change as the above relationship.