University of BRISTOL

COMS 30115

Stochastic Raytracing

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Last Time

- Clarification of transport problem
- Area formulation of Rendering Equation
- Radiosity

Today

- Approximate integration
- Monte Carlo Methods

Material

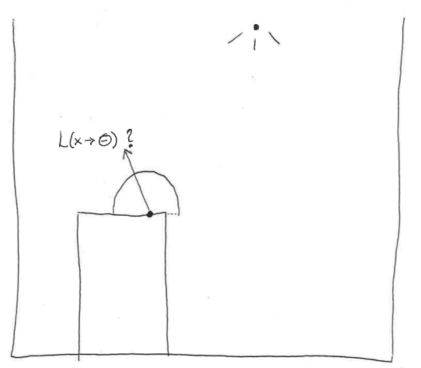
- Thesis Appendix of Wojciech Jarosz
- Equation Compendium
- Global Illumination Resources

Light Transport

Transport of Light

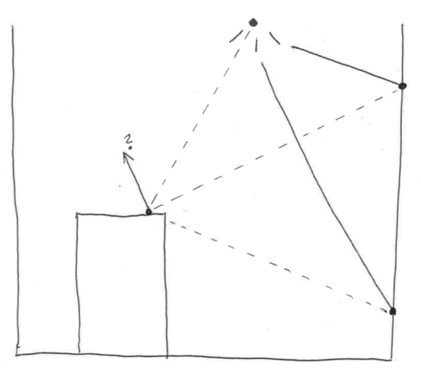
$$L(x \to \Theta) = L_e(x \to \Theta) + \mathcal{T}(L(x \to \Theta))$$
$$\mathcal{T}(L(x \to \Theta)) = \int_{\Omega_x} f_r(x, \Psi \to \Theta) L(x \leftarrow \Psi) cos(\mathbf{n}_x, \Psi) d\omega_{\Psi}$$

Transport of Light



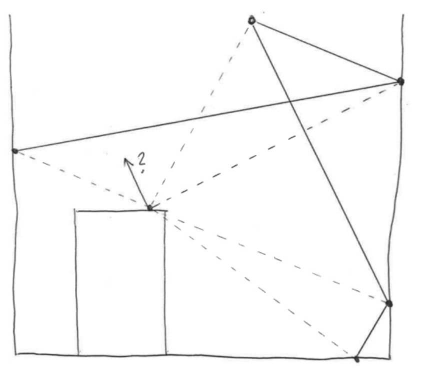
$$L(x \to \Theta) = L_e(x \to \Theta) + \ldots$$

Transport of Light



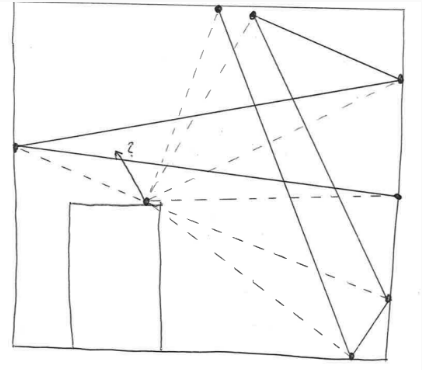
$$L(x \to \Theta) = L_e(x \to \Theta) + \langle T, L_e \rangle + \ldots$$

Transport of Light



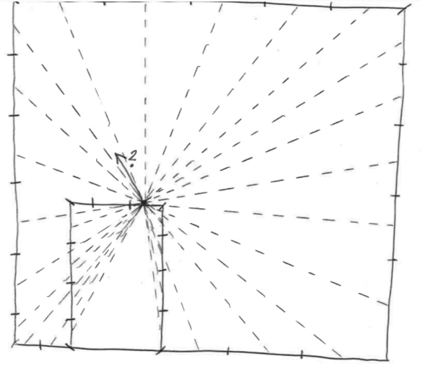
$$L(x \to \Theta) = L_e(x \to \Theta) + \langle T, L_e \rangle + \langle T, TL_e \rangle + \ldots$$

Transport of Light



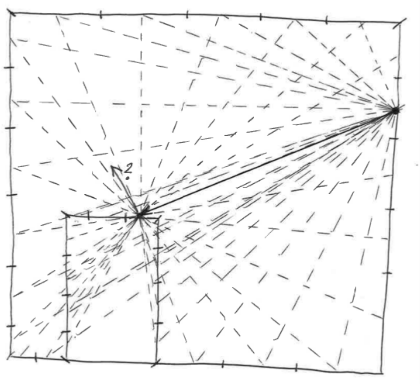
$$L(x \to \Theta) = L_e(x \to \Theta) + \langle T, L_e \rangle + \langle T, TL_e \rangle + \langle T, TTL_e \rangle + \dots$$

Transport of Light



$$L(x \to \Theta) = L_e(x \to \Theta) + \langle T, L_e \rangle + \langle T, TL_e \rangle + \langle T, TTL_e \rangle + \dots$$

Transport of Light

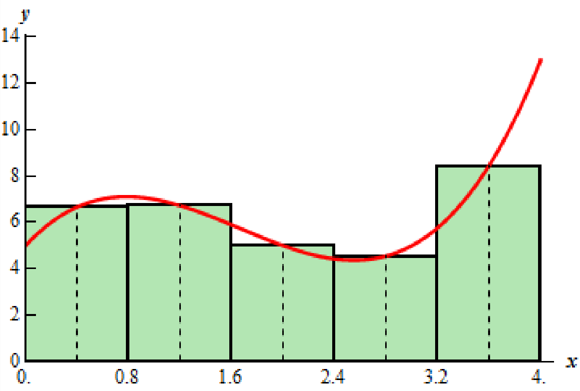


$$L(x \to \Theta) = L_e(x \to \Theta) + \langle T, L_e \rangle + \langle T, TL_e \rangle + \langle T, TTL_e \rangle + \dots$$

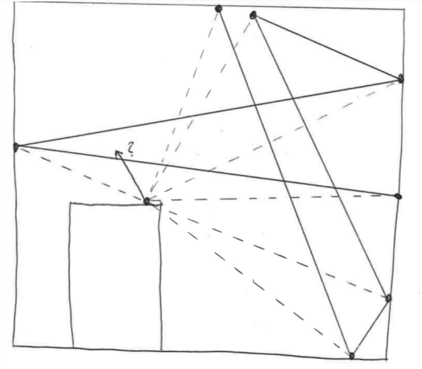
That Integral

$$\mathcal{T}(L(x \to \Theta)) = \int_{\Omega_x} f_r(x, \Psi \to \Theta) L(x \leftarrow \Psi) cos(\mathbf{n}_x, \Psi) d\omega_{\Psi}$$

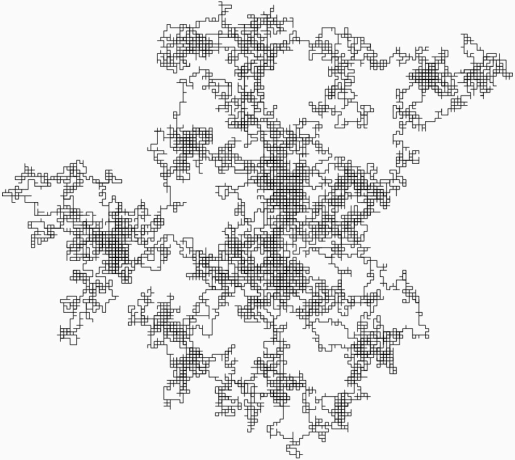
Approximate Integration (Stochastic)



$$\mathcal{T}(L(x \to \Theta)) = \int_{\Omega_x} f_r(x, \Psi \to \Theta) L(x \leftarrow \Psi) cos(\mathbf{n}_x, \Psi) d\omega_\Psi$$

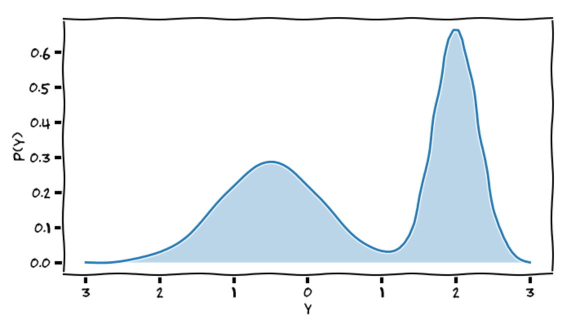


Random Walks



Sampling

Random Variable



- Random variable, is a stochastic variable that follows a distribution
- Random does not mean max entropy

Expected Value

Expected Value

$$\mathbb{E}[x] = \int x p(x) \mathrm{d}x$$

Ex: Fair dice

$$\begin{aligned} \mathbf{X} &= \{1, 2, 3, 4, 5, 6\} \\ p(x_i) &= \frac{1}{6} \\ \mathbb{E}[x] &= \sum_{x_i \in \mathbf{X}} x_i p(x_i) = \frac{1}{6} = \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = 3.5 \end{aligned}$$

Expected Value

Variance

$$\sigma^2(x) = \mathbb{E}\left[(x - \mathbb{E}[x])^2\right]$$

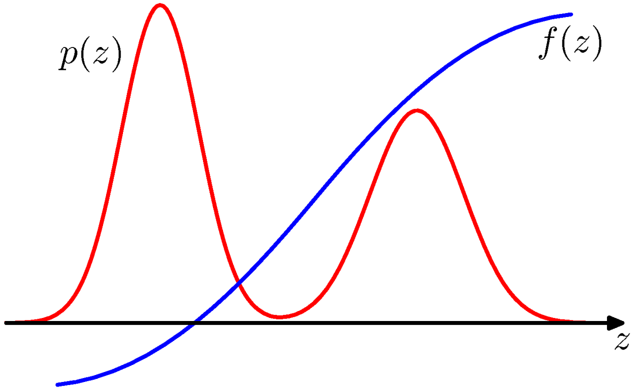
Ex: Fair dice

$$\begin{aligned}
\mathbf{X} &= \{1, 2, 3, 4, 5, 6\} \\
p(x_i) &= \frac{1}{6} \\
\sigma^2(x) = \mathbb{E}\left[(x - \mathbb{E}[x])^2\right] &= \sum_{x_i \in \mathbf{X}} (x_i - 3.5)^2 p(x_i) \\
&= \frac{1}{6}((1 - 3.5)^2 + (2 - 3.5)^2 + \dots) = 2.91
\end{aligned}$$

Expected Value

- The expected value is characteristic of a distribution
- It is a location parameters
- Also referred to as *the mean*
- The expected variation of the expected value is the variance

Expected Value



$$\mathbb{E}_{p(z)}[f] = \int f(z)p(z)\mathrm{d}z \approx \frac{1}{L}\sum_{l=1}^{L} f(z^{(l)})$$

$$\mathbf{z}^{(l)} \sim p(\mathbf{z})$$

Sampling

$$
\begin{aligned}
\hat{f} &= \frac{1}{L}\sum_{l=1}^{L} f(z^{(l)}) \\
z^{(l)} &\sim p(z) \\
\mathbb{E}[\hat{f}] &= \mathbb{E}[f] \\
\mathrm{var}[\hat{f}] &= \frac{1}{L}\mathbb{E}\left[(f(z) - \mathbb{E}[f])^2\right]
\end{aligned}
$$

- Approximation not dependent on dimensionality of z
- Variance of estimator shrinks with number of samples

Basic Sampling

$$z^{(l)} \sim p(z)$$

- Lets assume that we can get uniformly random numbers $z \sim \text{Uniform}(0, 1)$
- A computer cannot, but lets assume it could
- Idea: can we transform this uniform distribution to something interesting
- If we could then we could use samples from z

Basic Probabilities

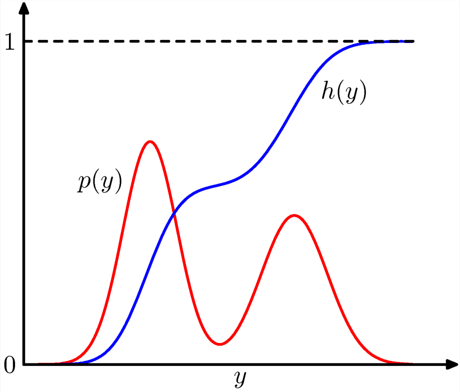
$$z \sim \text{Uniform}(0, 1)$$

- We have access to a uniformly distributed variable z
- Change of variable

$$y = f(z)$$

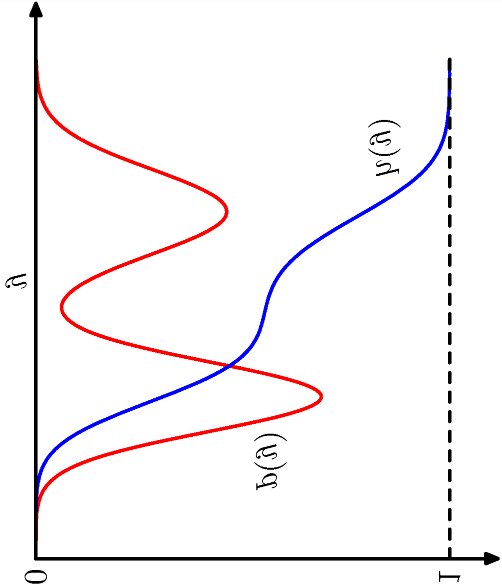
- Idea: can we find $f(z)$ such that it induces $p(y)$ to be the distribution that we want?

Basic Probabilities

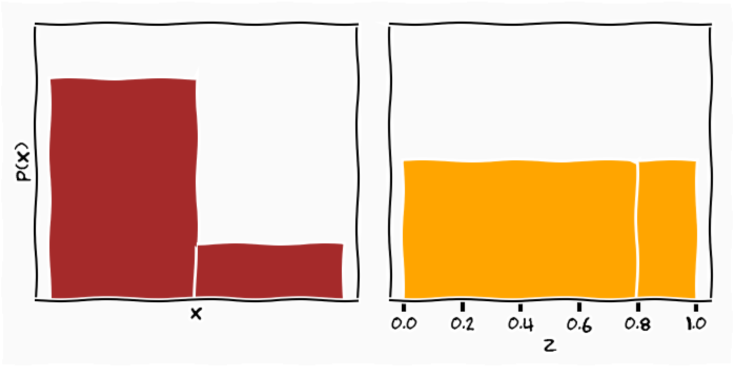


$$z = f^{-1}(y) = \int_{-\infty}^{y} p(y)\mathrm{d}y$$

Change of Variables



Change of Variables



Sampling

- We know how to transform samples from uniform to any distribution we can formulate the cummulative distribution
- We do not know the distribution to sample from
- Can we sample from distrubitons we do not know the form of?

Importance Sampling

$$\mathbb{E}_{p(\mathbf{z})}[f] = \int f(\mathbf{z}) p(\mathbf{z}) \mathrm{d}\mathbf{z}$$

Importance Sampling

$$\mathbb{E}_{p(\mathbf{z})}[f] = \int f(\mathbf{z})p(\mathbf{z})\mathrm{d}\mathbf{z} = \int f(\mathbf{z})p(\mathbf{z})\frac{q(\mathbf{z})}{q(\mathbf{z})}\mathrm{d}\mathbf{z}$$

Importance Sampling

$$\begin{aligned}\mathbb{E}_{p(\mathbf{z})}[f] &= \int f(\mathbf{z})p(\mathbf{z})\mathrm{d}\mathbf{z} = \int f(\mathbf{z})p(\mathbf{z})\frac{q(\mathbf{z})}{q(\mathbf{z})}\mathrm{d}\mathbf{z} \\ &= \int f(\mathbf{z})\frac{p(\mathbf{z})}{q(\mathbf{z})}q(\mathbf{z})\mathrm{d}\mathbf{z}\end{aligned}$$

Importance Sampling

$$
\begin{aligned}
\mathbb{E}_{p(\mathbf{z})}[f] &= \int f(\mathbf{z})p(\mathbf{z})\mathrm{d}\mathbf{z} = \int f(\mathbf{z})p(\mathbf{z})\frac{q(\mathbf{z})}{q(\mathbf{z})}\mathrm{d}\mathbf{z} \\
&= \int f(\mathbf{z})\frac{p(\mathbf{z})}{q(\mathbf{z})}q(\mathbf{z})\mathrm{d}\mathbf{z} = \mathbb{E}_{q(\mathbf{z})}\left[f(\mathbf{z})\frac{p(\mathbf{z})}{q(\mathbf{z})}\right]
\end{aligned}
$$

Importance Sampling

$$
\begin{aligned}
\mathbb{E}_{p(\mathbf{z})}[f] &= \int f(\mathbf{z})p(\mathbf{z})\mathrm{d}\mathbf{z} = \int f(\mathbf{z})p(\mathbf{z})\frac{q(\mathbf{z})}{q(\mathbf{z})}\mathrm{d}\mathbf{z} \\
&= \int f(\mathbf{z})\frac{p(\mathbf{z})}{q(\mathbf{z})}q(\mathbf{z})\mathrm{d}\mathbf{z} = \mathbb{E}_{q(\mathbf{z})}\left[f(\mathbf{z})\frac{p(\mathbf{z})}{q(\mathbf{z})}\right] \\
&\approx \frac{1}{L}\sum_{l=1}^{L} f(\mathbf{z}^{(l)})\frac{p(\mathbf{z}^{(l)})}{q(\mathbf{z}^{(l)})}
\end{aligned}
$$

Importance Sampling

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\begin{aligned}
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&\approx \frac{1}{L}\sum_{l=1}^{L} f(\mathbf{z}^{(l)})\frac{p(\mathbf{z}^{(l)})}{q(\mathbf{z}^{(l)})} \\
&= \frac{1}{L}\sum_{l=1}^{L} r_l \cdot f(\mathbf{z}^{(l)})
\end{aligned}
$$

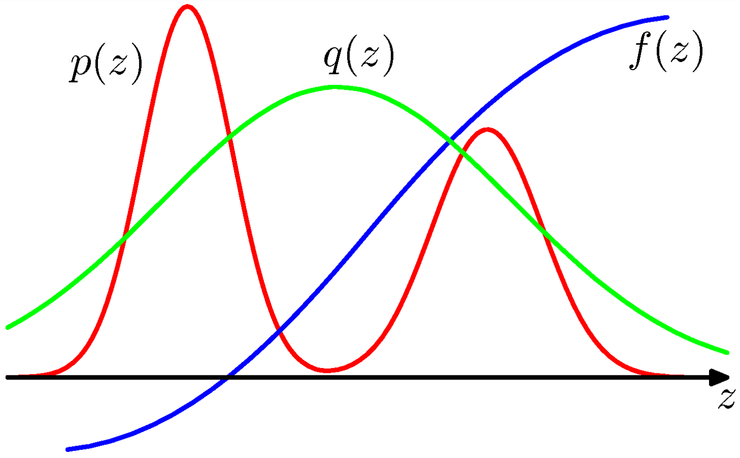
Importance Sampling

$$\mathbb{E}_{p(\mathbf{z})}[f] \approx \frac{1}{L} \sum_{l=1}^{L} r_l \cdot f(\mathbf{z}^{(l)})$$

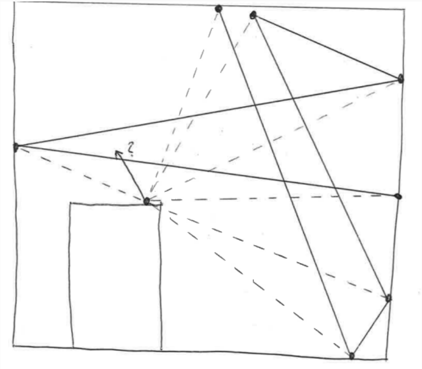
$$\mathbf{z}^{(l)} \sim q(\mathbf{z}), \qquad r_l = \frac{p(\mathbf{z}^{(l)})}{q(\mathbf{z}^{(l)})}$$

- Directly approximate expectation
- Accepts all samples
- r_l corrects bias in sampling from wrong distribution

Importance Sampling



Random Walks



Markov Chain Monte Carlo

- Sample from a proposal distribution
- Remembers the state and samples from a conditional
- Can lead to much better exploration of the space

Metropolis Sampling

1. start with state $\mathbf{z}^{(0)}$

Markov Chain Monte Carlo

Metropolis Sampling

1. start with state $\mathbf{z}^{(0)}$
2. sample from conditional proposal distribution $q(\mathbf{z}^*|\mathbf{z}^{(0)})$

Markov Chain Monte Carlo

Metropolis Sampling

1. start with state $\mathbf{z}^{(0)}$
2. sample from conditional proposal distribution $q(\mathbf{z}^*|\mathbf{z}^{(0)})$
3. compute acceptance probability

$$A(\mathbf{z}^*, \mathbf{z}^{(0)}) = \min\left(1, \frac{\tilde{p}(\mathbf{z}^*)}{\tilde{p}(\mathbf{z}^{(0)})}\right)$$

Metropolis Sampling

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$$A(\mathbf{z}^*, \mathbf{z}^{(0)}) = \min\left(1, \frac{\tilde{p}(\mathbf{z}^*)}{\tilde{p}(\mathbf{z}^{(0)})}\right)$$

4. Draw uniform random number $u \sim \text{Uniform}(0, 1)$

Metropolis Sampling

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4. Draw uniform random number $u \sim \text{Uniform}(0, 1)$
   - if $A(\mathbf{z}^*, \mathbf{z}^{(0)}) > u \rightarrow \mathbf{z}^{(1)} = \mathbf{z}^*$

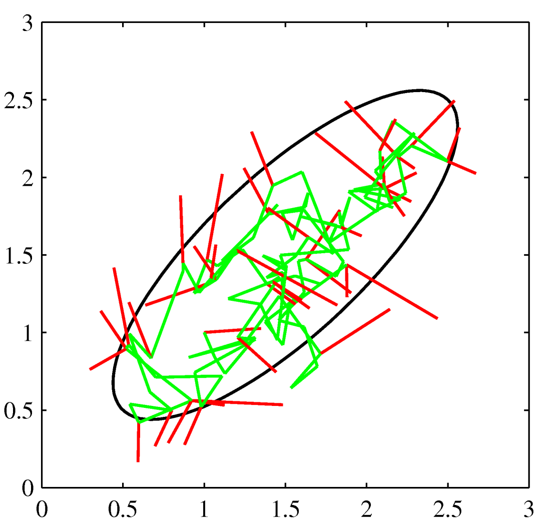
Metropolis Sampling

1. start with state $\mathbf{z}^{(0)}$
2. sample from conditional proposal distribution $q(\mathbf{z}^*|\mathbf{z}^{(0)})$
3. compute acceptance probability

$$A(\mathbf{z}^*, \mathbf{z}^{(0)}) = \min\left(1, \frac{\tilde{p}(\mathbf{z}^*)}{\tilde{p}(\mathbf{z}^{(0)})}\right)$$

4. Draw uniform random number $u \sim \text{Uniform}(0, 1)$
   - if $A(\mathbf{z}^*, \mathbf{z}^{(0)}) > u \rightarrow \mathbf{z}^{(1)} = \mathbf{z}^*$
   - otherwise reject $\mathbf{z}^*$ and start over

Metropolis Gaussian

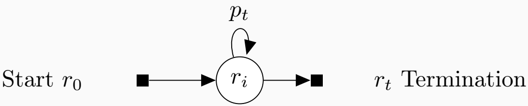


Monte Carlo Path Tracing

Path Tracing

- Path Tracing is the recursive way to solve the rendering equation
- Just keep shooting rays and the image will look better
- Will eventually get you "everything" but it will take time

Markov Chain



Step 1 Create a random particle in state i with probabilit p_i^0

Step 2 With probability $p_i^* = 1 - \sum_{j=0} p_{ij}$ terminate in state i

- if terminate go to Step 1

Step 3 Randomly select new state j according to transition probability p_{ij} and go to Step 2

Monte Carlo Path Tracer

1. Choose a ray that goes through the pixel
   - $\lambda = 1.0$
2. Find intersection
   - $\lambda = \rho(\cdot)$ (reflectance function)
   - choose if returning emitted light or calculating reflected

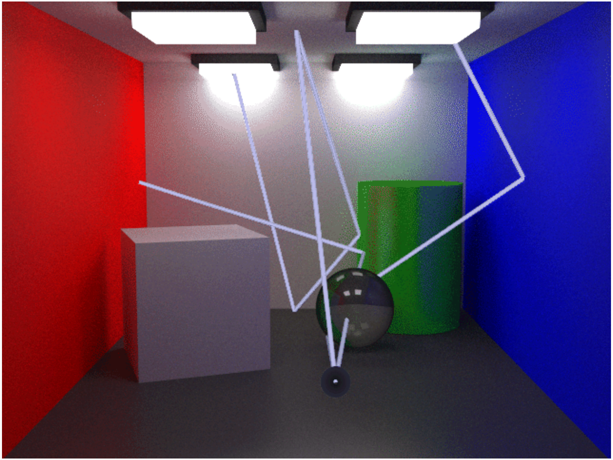
Emitted `return` $\lambda \cdot \frac{1}{q} L_e$

Reflected `return` $\lambda \cdot \frac{1}{q}$ TraceRay

3. Pixel value is average of each ray

Markov Chain

Paths



Bias

$$\hat{f} = \frac{1}{L}\sum_{l=1}^{L} f(z^{(l)})$$

$$z^{(l)} \sim p(z)$$

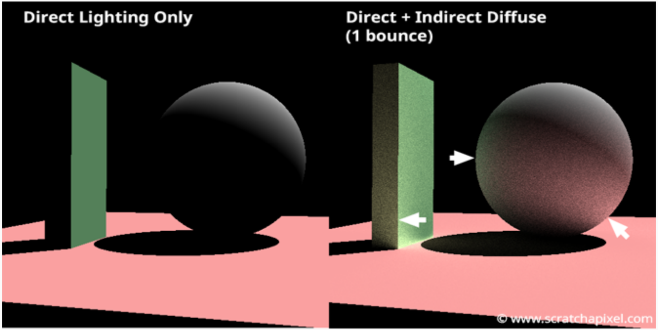
$$\mathbb{E}[\hat{f}] = \mathbb{E}[f]$$

$$\text{var}[\hat{f}] = \frac{1}{L}\mathbb{E}\left[(f(z) - \mathbb{E}[f])^2\right]$$

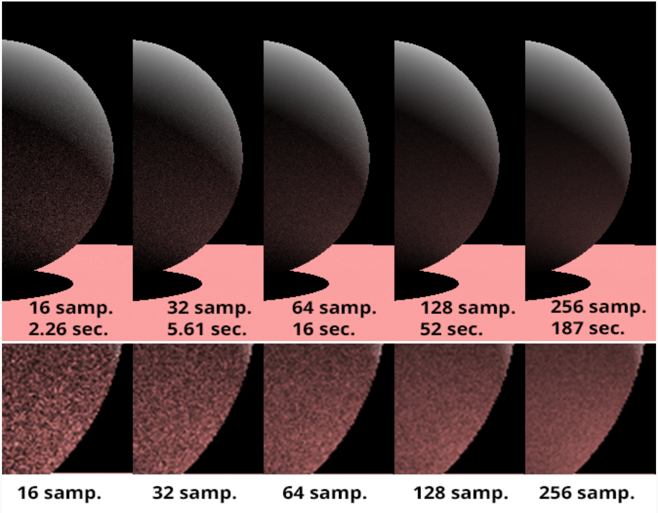
- We want to have an unbiased estimate of the light transport
- Path tracing is unbiased as in the limit $\hat{f} = f$

Bias can be good

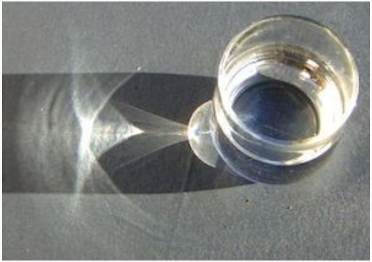
Sampling

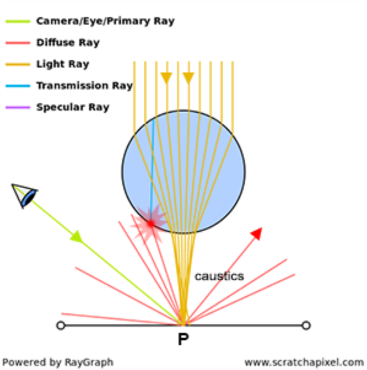

Direct Lighting Only
Direct + Indirect Diffuse
(1 bounce)
© www.scratchapixel.com

Sampling


16 samp.
2.26 sec.
32 samp.
5.61 sec.
64 samp.
16 sec.
128 samp.
52 sec.
256 samp.
187 sec.
16 samp.
32 samp.
64 samp.
128 samp.
256 samp.

Caustics





Camera/Eye/Primary Ray
Diffuse Ray
Light Ray
Transmission Ray
Specular Ray
caustics
P
Powered by RayGraph
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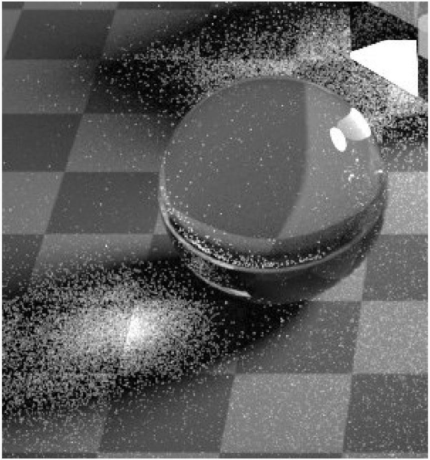
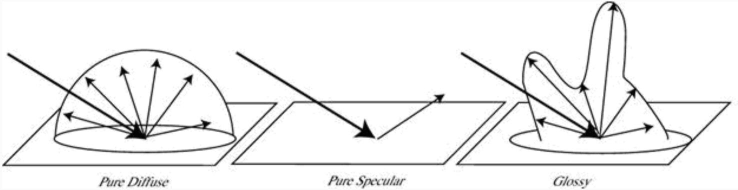
Caustics



Bi-directional Path tracing

$$\leftrightarrow$$

BRDF



Summary

Summary

- A lot of you have already done this as extensions
- Just shoot rays recursively and you will get a better image
- Path tracing is a very brutal method but will get the right results eventually
- You can get it to converge quicker by using clever sampling

Next Time

Lecture Thursday 16th of April

- Photon Mapping (Caustics)

Lecture Monday 27th of April

- Last lecture
- Unit Summary

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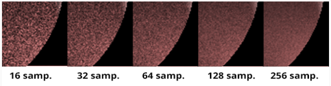
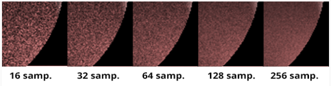
Appendix

Monte-Carlo Integration



$$I = \int f(x)\mathrm{d}x$$

Monte-Carlo Integration



$$
\begin{aligned}
I &= \int f(x)\mathrm{d}x \\
&\approx \int \mathrm{d}x \frac{1}{|\mathbf{X}|} \sum_{x_i \in \mathbf{X}} f(x_i)
\end{aligned}
$$