WESTMINSTER
International University in Tashkent

Continuous Probability Distributions

By

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Office hours: Tuesday, 09:00 – 11:00 (ATB 216)

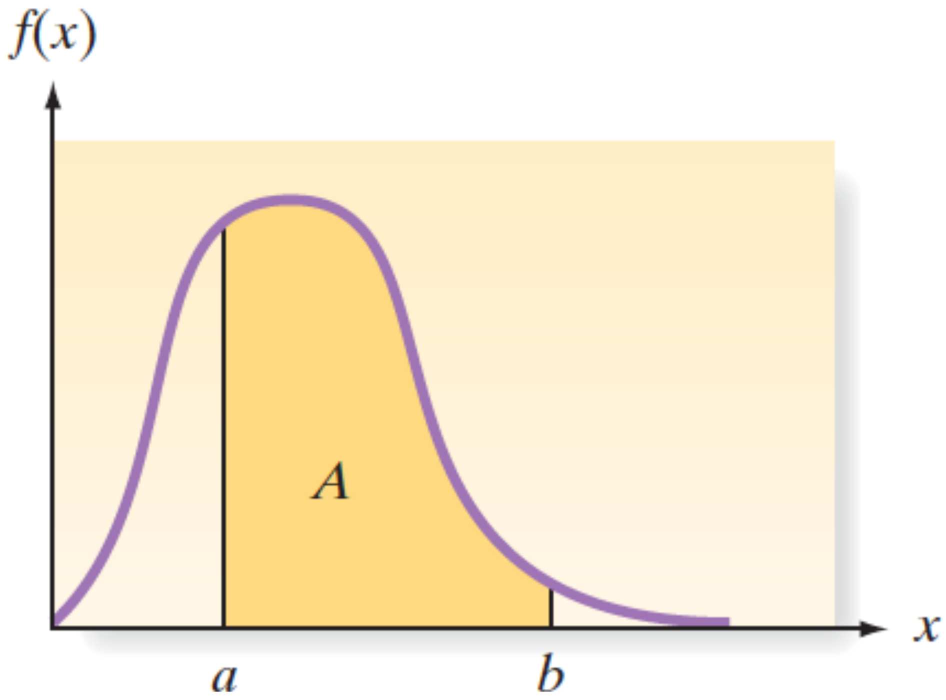
AGENDA

- understand the concept of a continuous random variable.
- be able to find expected value and variance of continuous probability distributions.
- understand the continuous uniform distribution.
- learn about probability density function of Normal (Gaussian) distribution.
- understand Standard Normal (z) distribution.

Continuous random variable

- A continuous random variable can assume any value in an interval on the real line or in a collection of intervals.
- It is not possible to talk about the probability of the random variable assuming a particular value.
- Instead, we talk about the probability of the random variable assuming a value within a given interval.
- If f(x) is a continuous probability distribution where x $\in (-\infty, \infty)$, then $\int_{-\infty}^{\infty} f(x)dx = 1$

Continuous probability distribution



Area: $\mathbf{A} = \int_{a}^{b} f(x)dx$

First moment: $E(x) = \mu = \int_{-\infty}^{\infty} x * f(x)dx$

Second moment: $E(x^2) = \int_{-\infty}^{\infty} x^2 * f(x)dx$

Variance $= \sigma^2 = E(x^2) - \mu^2$

The **probability distribution for a continuous random variable, *x*,** can be represented by a smooth curve—a function of *x*, denoted *f(x)*. The curve is called a **density function** or **frequency function.** The probability that *x* falls between two values, *a* and *b*, i.e., $P(a < x < b)$, is the area under the curve between *a* and *b*.

Example

You are given the following function:

f(x) = $\frac{1}{3}$*x, where $\sqrt{3} \le x \le 3$

a. Is this a probability density function (pdf) ?

b. Find the E(x) and Var(x).

Solution

a. If f(x) is a probability density function then,

$$\int_{-\infty}^{\infty} f(x)dx = 1 \text{ should be true.}$$

Then,

$$\int_{\sqrt{3}}^{3} \frac{1}{3} x dx = \frac{1}{3} * \frac{x^2}{2} \Big|_{\sqrt{3}}^{3} = \frac{3^2}{6} - \frac{3}{6} = 1$$

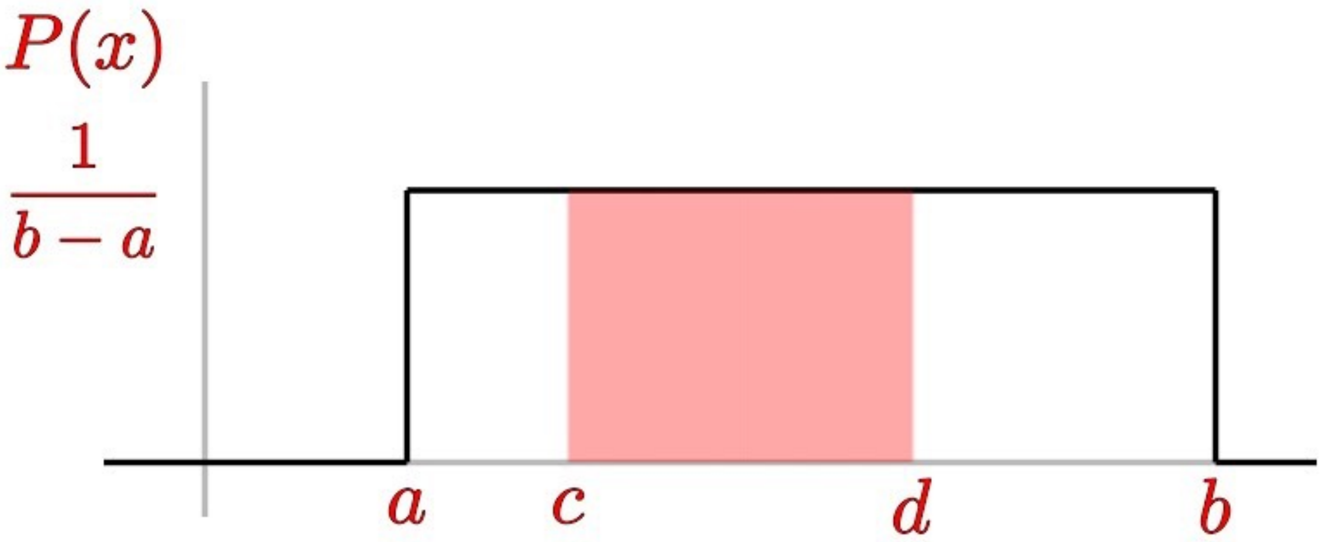
We can conclude that this is a pdf.

b. $E(x) = \int_{\sqrt{3}}^{3} \frac{1}{3} x * x * dx = \int_{\sqrt{3}}^{3} \frac{1}{3} x^2 dx = \frac{1}{3} * \frac{x^3}{3} \Big|_{\sqrt{3}}^{3} = \frac{3^3 - 3\sqrt{3}}{9} \approx 2.42$

$E(x^2) = \int_{\sqrt{3}}^{3} x^2 * \frac{1}{3} x * dx = \int_{\sqrt{3}}^{3} \frac{1}{3} x^3 dx = \frac{1}{3} * \frac{x^4}{4} \Big|_{\sqrt{3}}^{3} = \frac{3^4 - 3^2}{12} = 6$

$Var(x) = 6 - 2.42^2 = 0.1436$

Uniform Distribution



$$Mean: \quad \mu = \frac{a+b}{2}$$

$$S.D.: \quad \sigma = \sqrt{\frac{(b-a)^2}{12}}$$

Probability

$$P(c \leq X \leq d) = \frac{d-c}{b-a}$$

Example

The closest Metro Station to WIUT campus is Amir Temur station. Suppose the wait times of passengers at the station is uniformly distributed between 1 and 10 minutes.

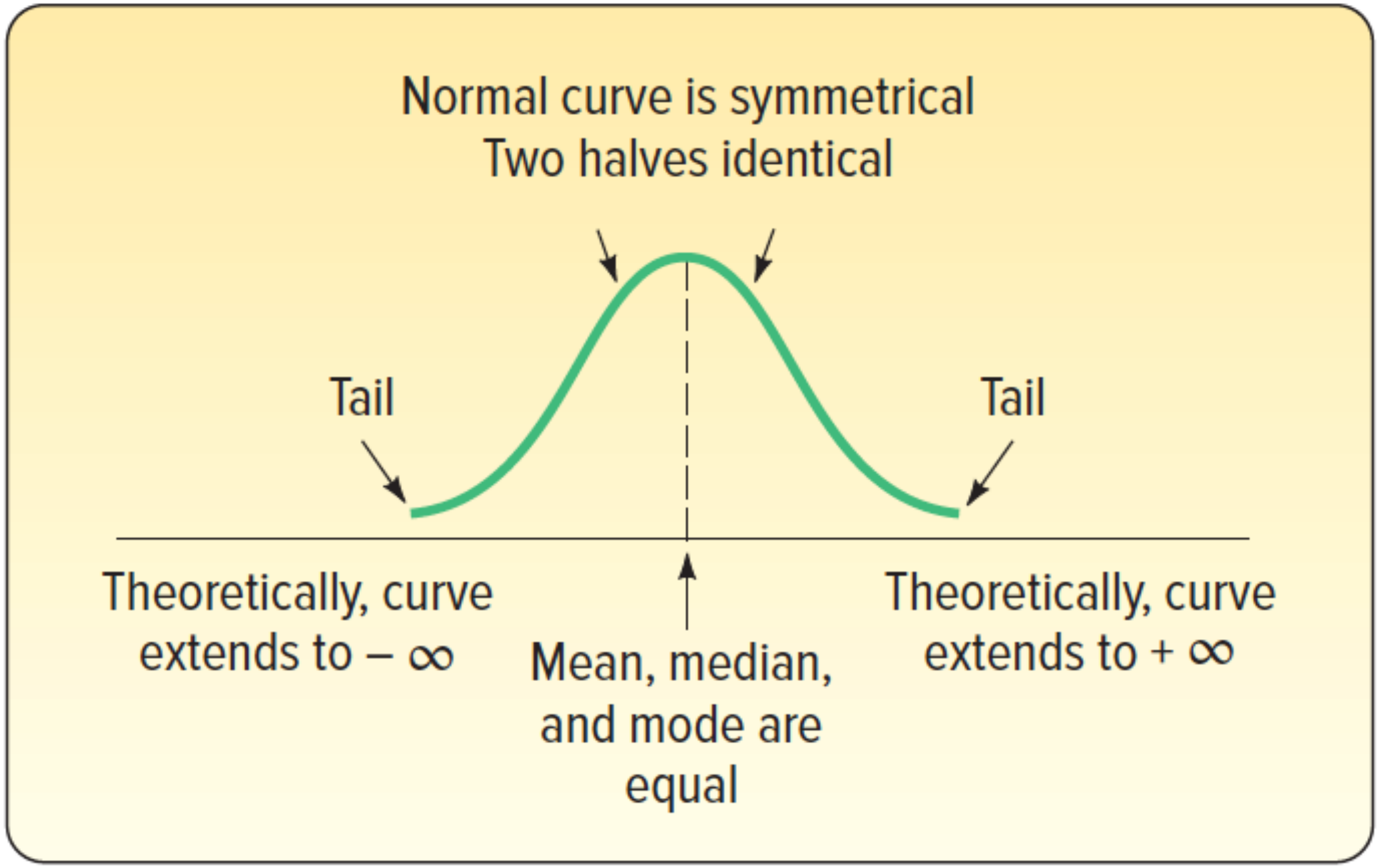
a. What is the probability a student will wait more than 7 minutes?

$$P(X > 7) = \frac{10 - 7}{10 - 1} = \frac{3}{9} = \frac{1}{3}$$

a. What is the probability a student will wait between 3 and 8 minutes?

$$P(3 < X < 8) = \frac{8 - 3}{10 - 1} = \frac{5}{9}$$

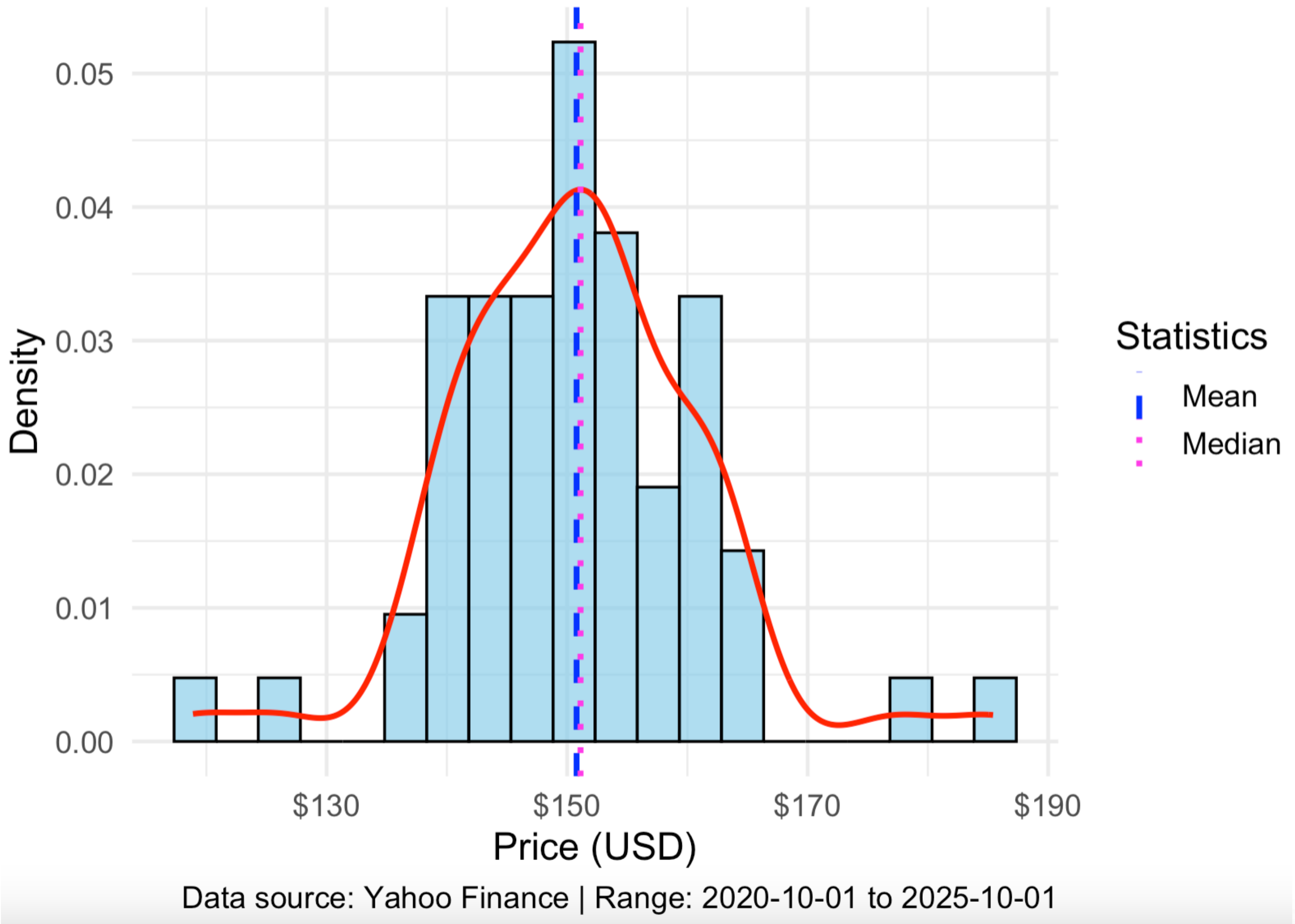
NORMAL (GAUSSIAN) DISTRIBUTION



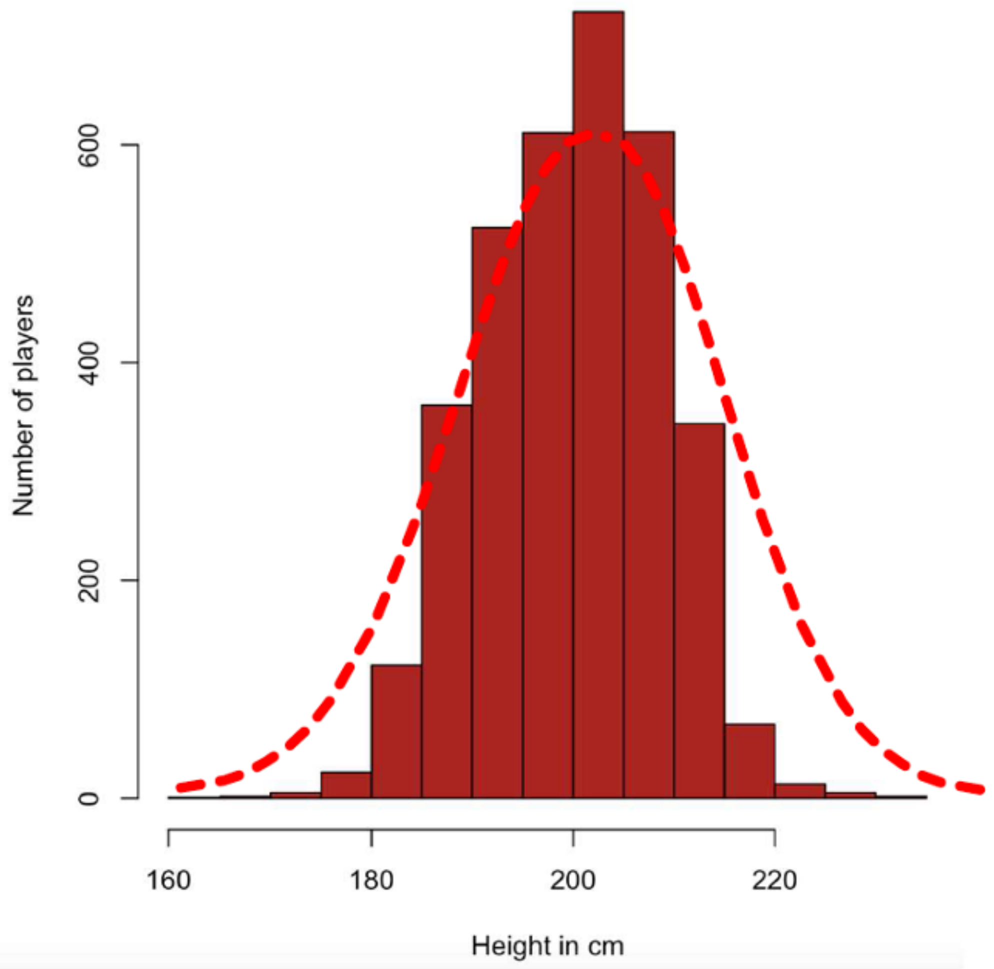
Properties of a Normal Distribution

1. The mean, median, and mode are equal.
2. The normal curve is bell-shaped and symmetric about the mean.
3. The total area under the curve is equal to one.
4. The normal curve approaches, but never touches the *x* axis as it extends farther and farther away from the mean.

Johnson & Johnson stock price (2020 – 2025, monthly)



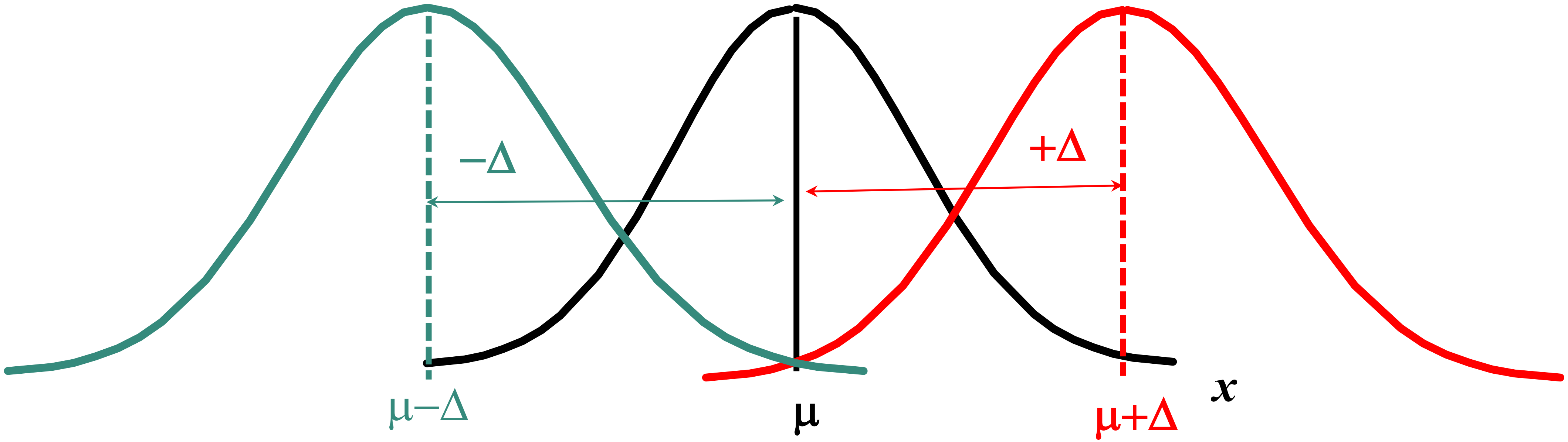
NBA player heights (1955 - 2004)



Normal Distribution

$$x \sim N(\mu,\ \sigma^2)$$

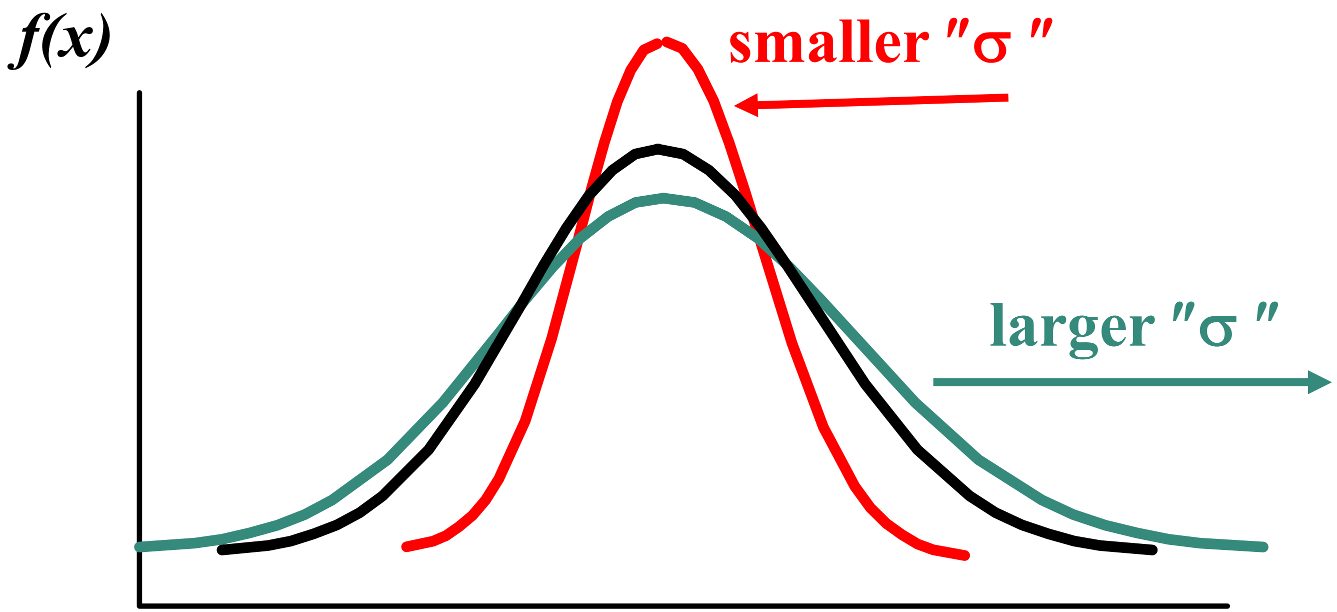
Changing μ shifts the distribution left or right.



Normal Distribution

$$x \sim N(\mu, \ \sigma^2)$$

Changing σ increases or decreases the spread.



Probability density function (pdf)

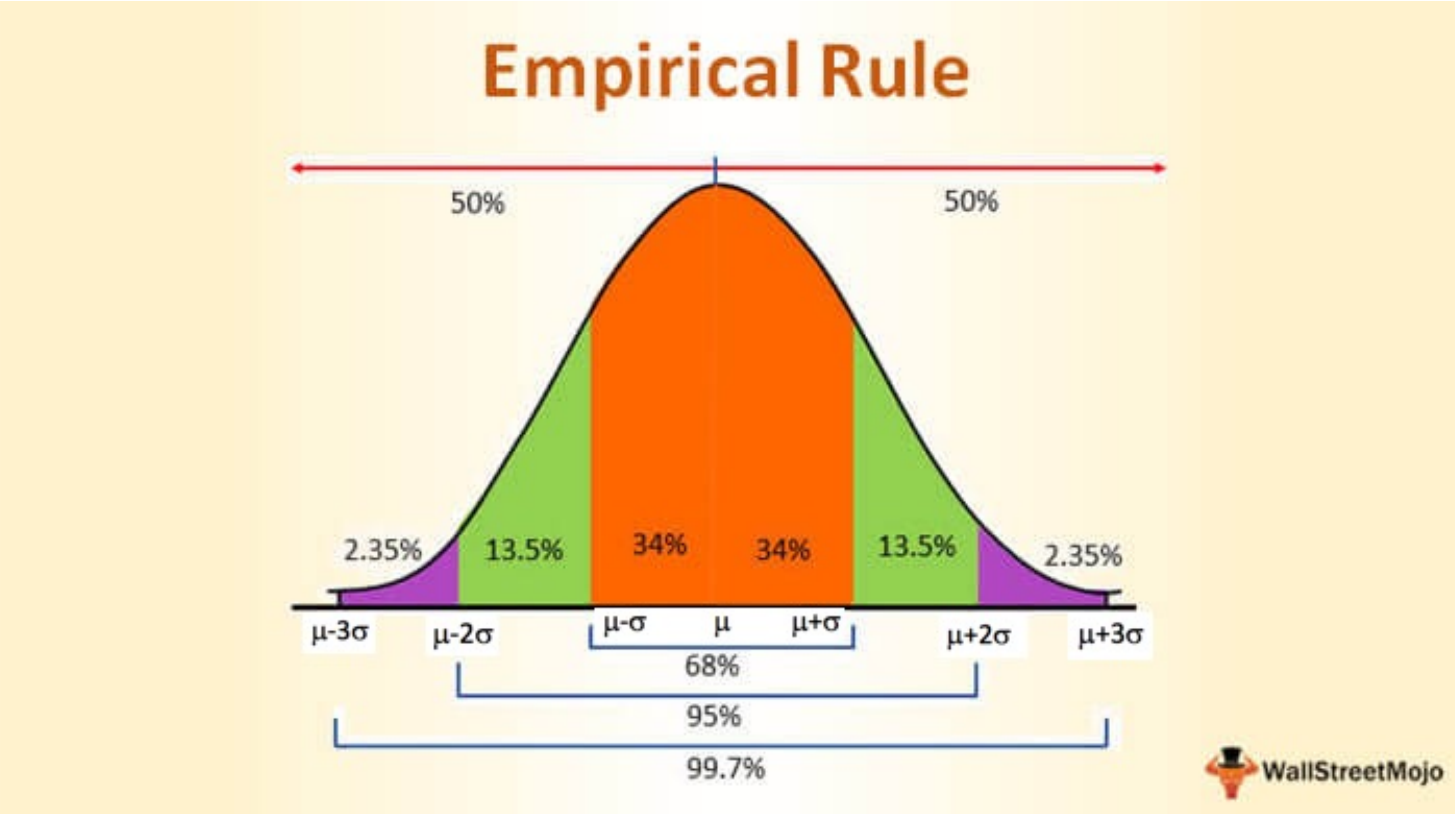
$$f(x) = \frac{1}{\boxed{\sigma}\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\boxed{\mu}}{\sigma}\right)^2}$$

Note constants:

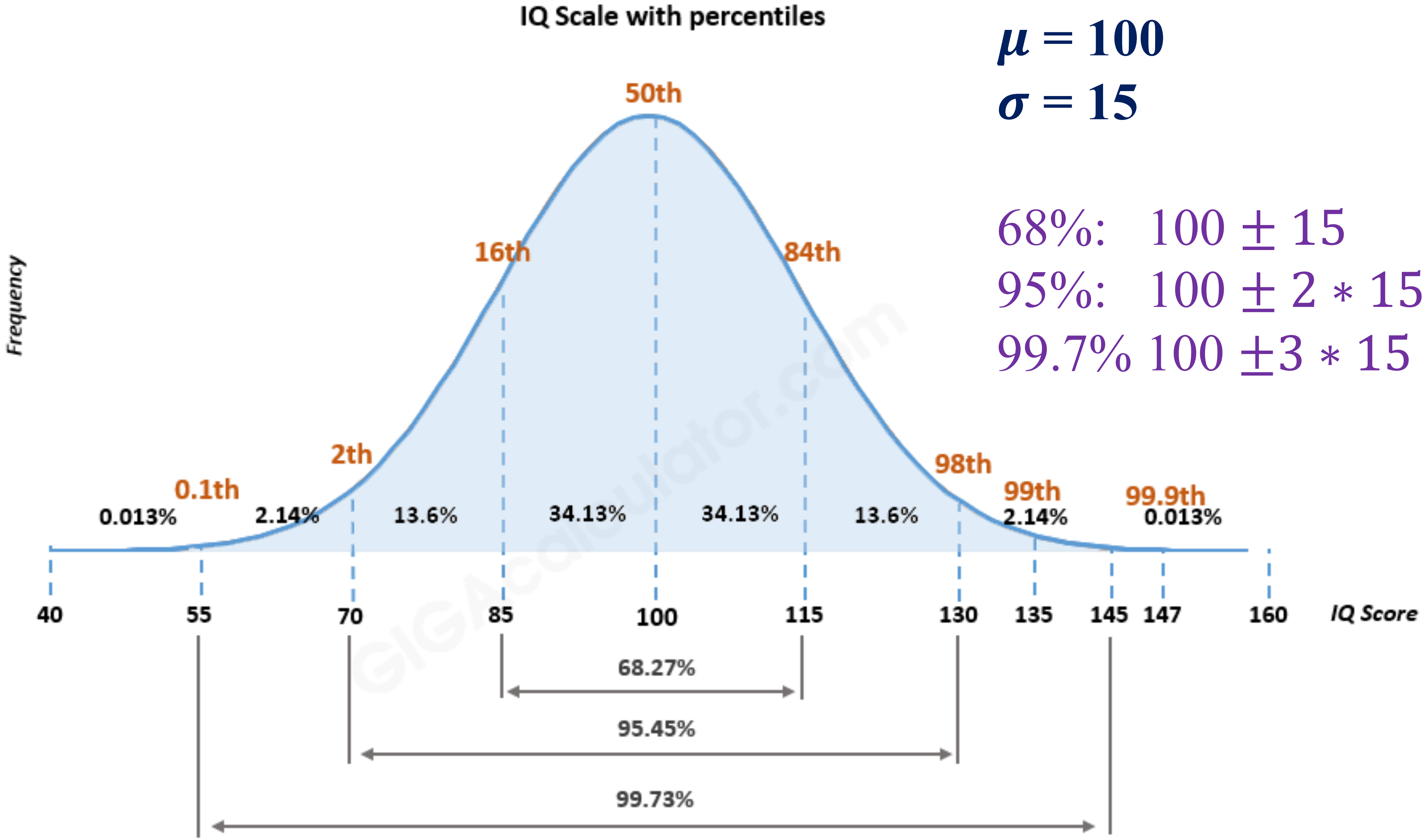
π=3.14159

e=2.71828

This is a bell shaped curve with different centers and spreads depending on μ and σ



Empirical Rule
50%
50%
2.35%
13.5%
34%
34%
13.5%
2.35%
μ-3σ
μ-2σ
μ-σ
μ
μ+σ
μ+2σ
μ+3σ
68%
95%
99.7%
WallStreetMojo



$\mu = 100$

$\sigma = 15$

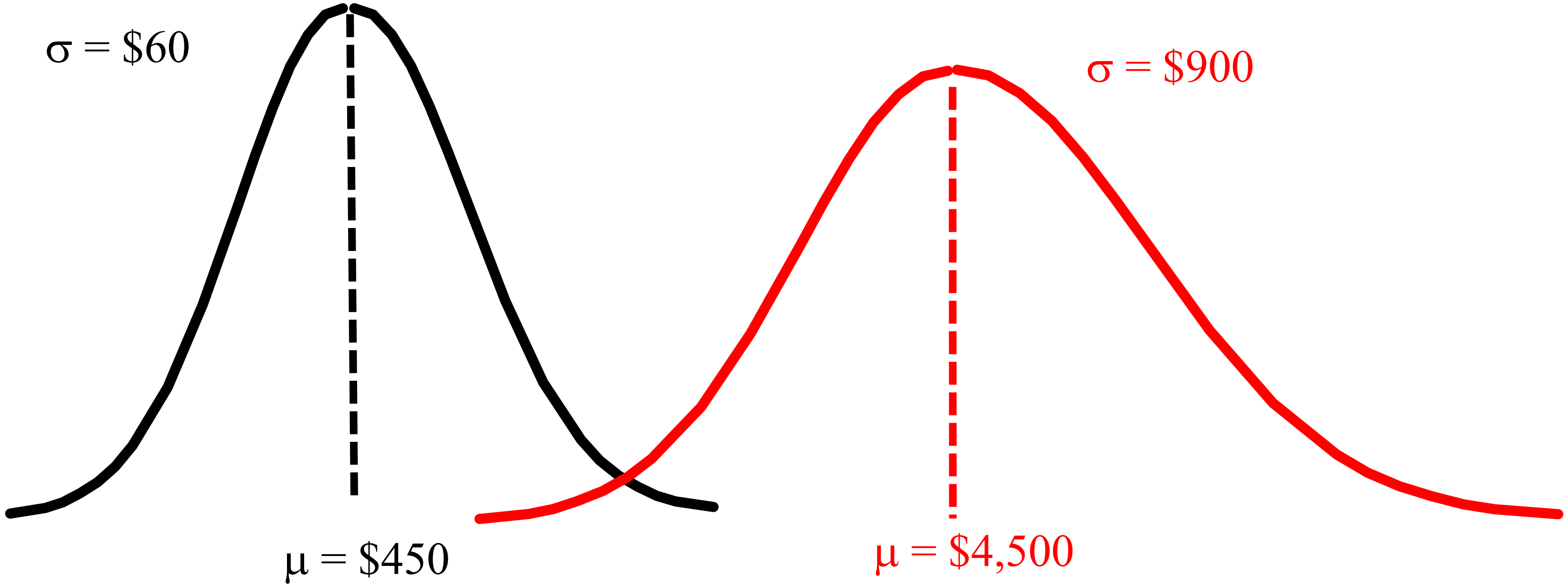
68%: $100 \pm 15 \Rightarrow [85, 115]$

95%: $100 \pm 2 * 15 \Rightarrow [70, 130]$

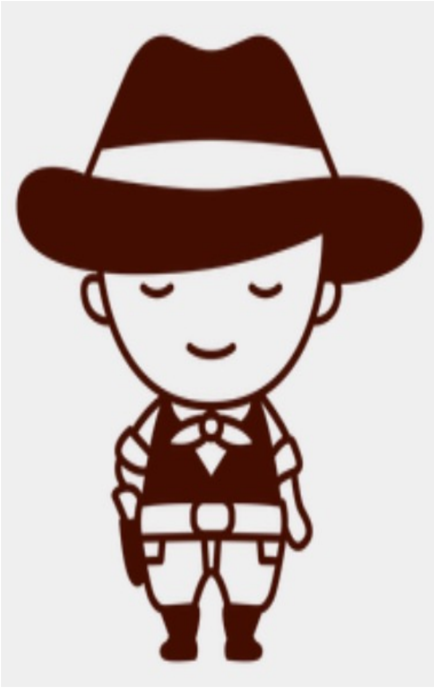
99.7% $100 \pm 3 * 15 \Rightarrow [55, 145]$

Who has higher percentile?

Anwar's income is:
$550



John's income is:
$5,800



The Standard Normal Distribution (Z)

All normal distributions can be converted into the standard normal curve by subtracting the mean and dividing by the standard deviation:

$$Z = \frac{X - \mu}{\sigma}$$

Somebody calculated all the integrals for the standard normal and put them in a table! So we never have to integrate! Even better, computers now do all the integration.

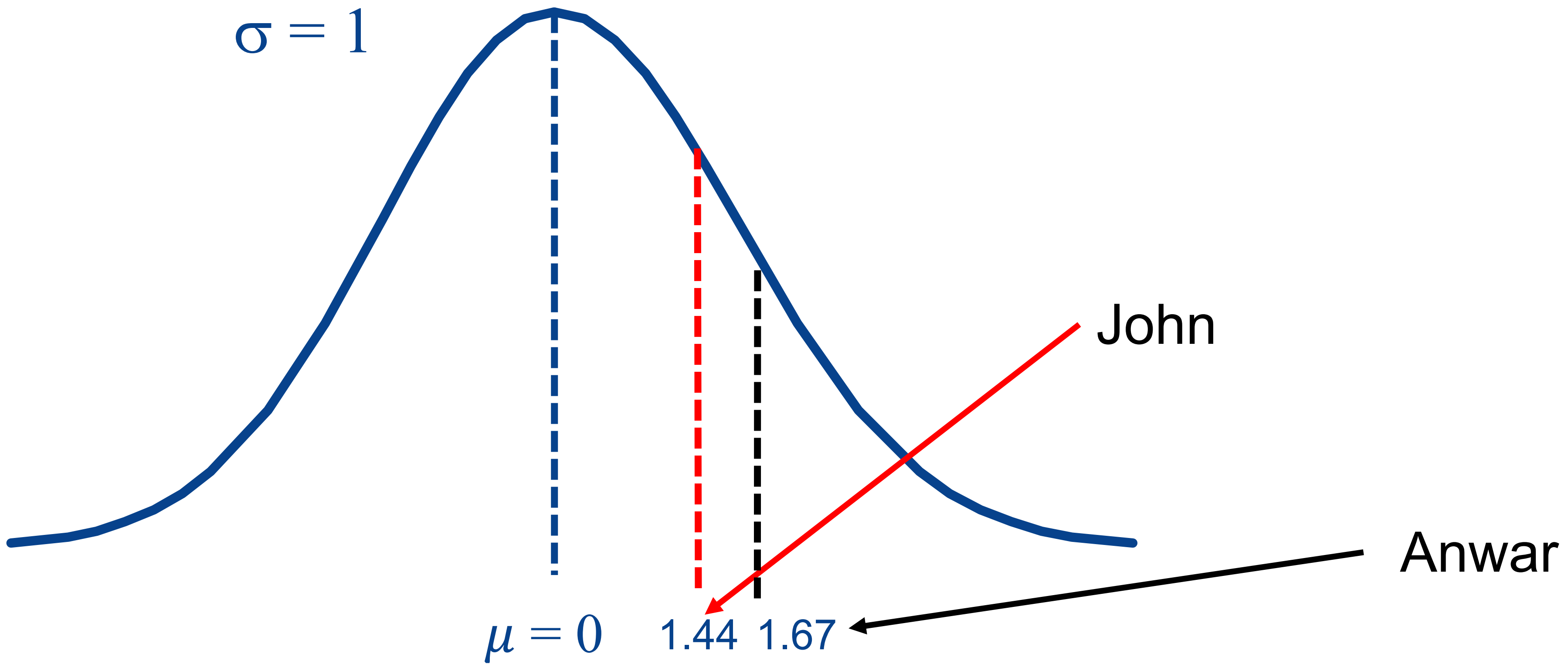
Example:

z-score for Anwar: $\mathbf{z = \frac{550-450}{60} = 1.67}$

z-score for John: $\mathbf{z = \frac{5800-4500}{900} = 1.44}$

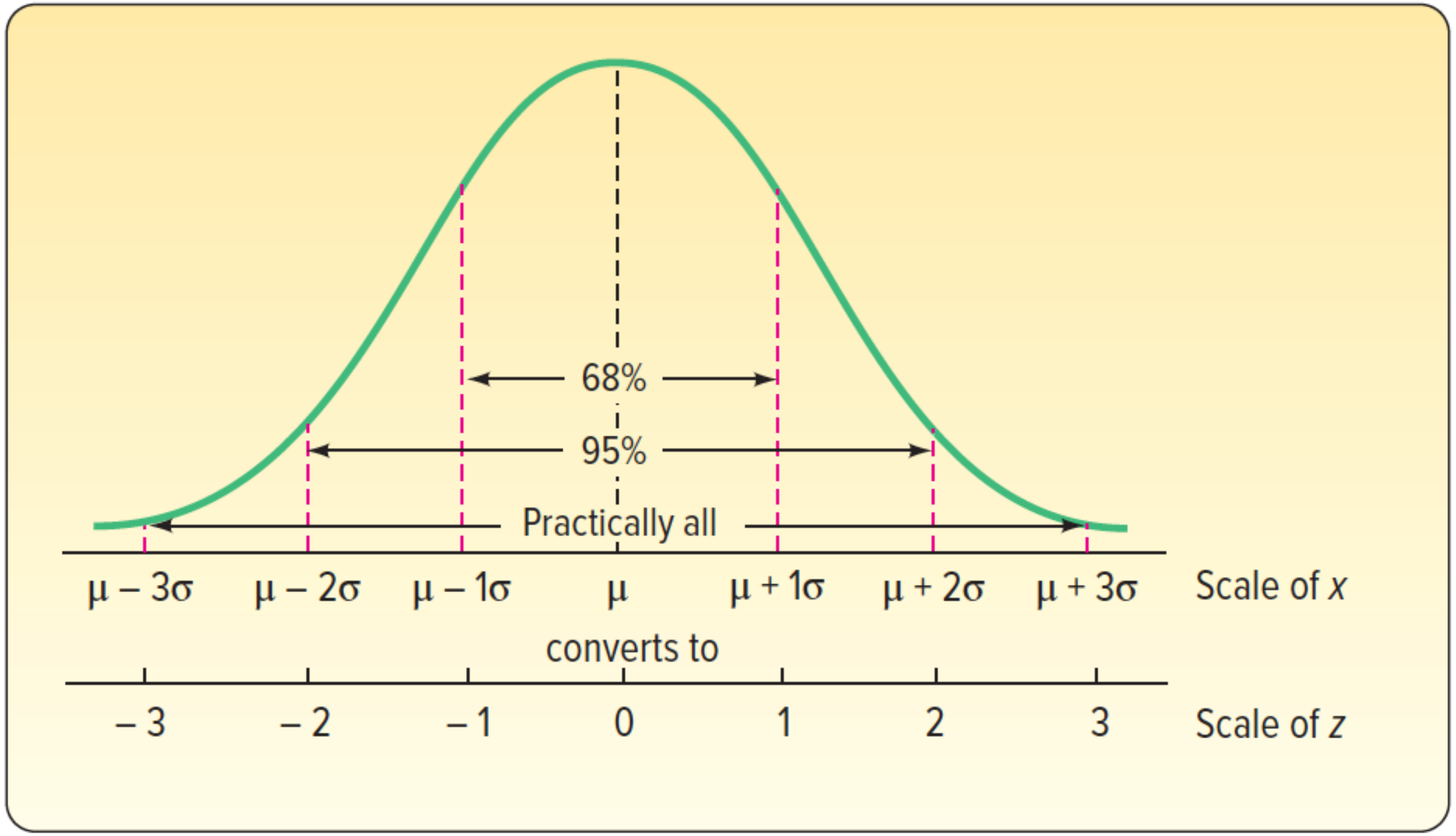
So, Anwar's income percentile in Uzbekistan is higher than John's in the US.

The standard normal distribution



Transformation of Normal to Standard Normal

Standard Normal Distribution (z distribution) is Normal with $\mu = 0$ and $\sigma = 1$: $\mathbf{z \sim \mathit{N}(0, 1)}$



$$p(Z) = \frac{1}{(1)\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{Z-0}{1}\right)^2} = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}(Z)^2}$$

Example

Suppose there are two stocks: **A** and **B**.

Stock A: $\mu_A = 60$, $\sigma_A=10$

Stock B: $\mu_B = 65$, $\sigma_B=25$

Both prices are approximately normally distributed.

We want to know:

a) Which stock is more likely to go below $45?

b) Which stock is more likely to go above $68?

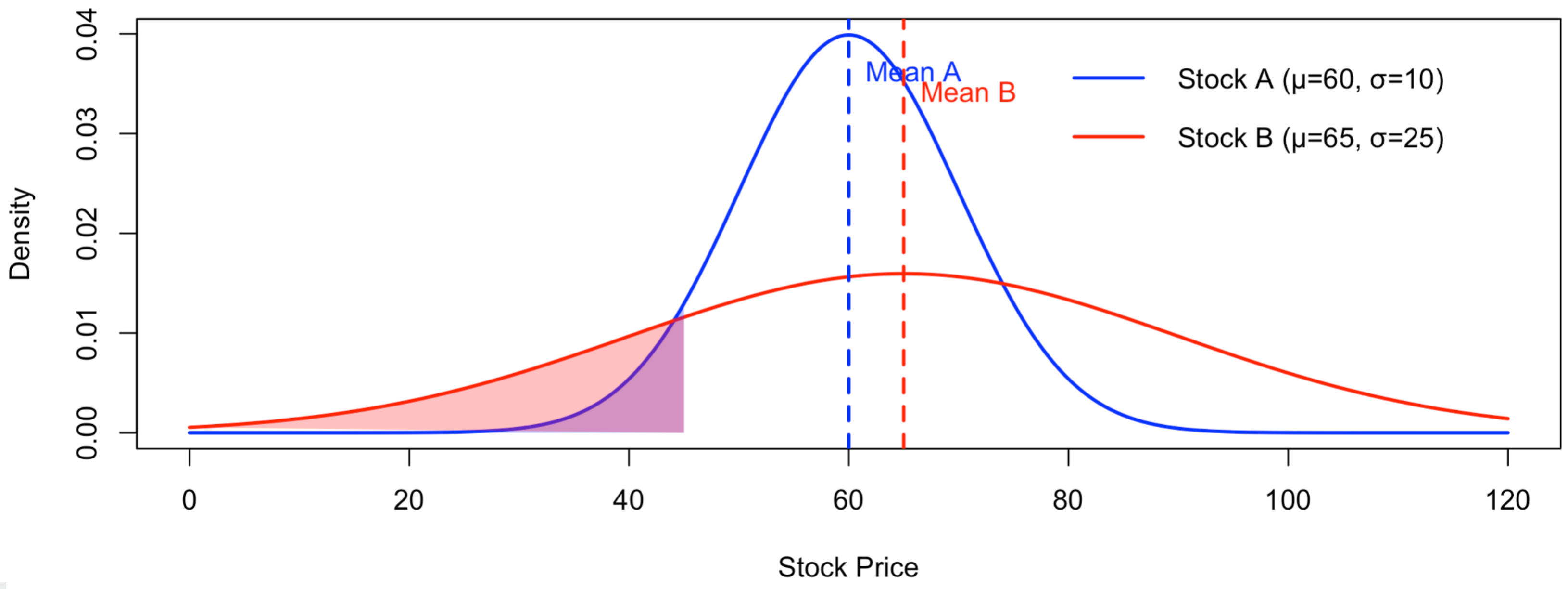
Solution

Let A and B be stock prices, then, $A \sim N(60,\ 10^2)$ and $B \sim N(65,\ 25^2)$.

a) Which stock is more likely to go below $45?

$$P(A < 45) = P\left(z < \frac{45-60}{10}\right) = P(z < -1.5) = P(z > 1.5) = 1 - P(z < 1.5) = 1 - 0.9332 = \mathbf{0.0668}$$

$$P(B < 45) = P\left(z < \frac{45-65}{25}\right) = P(z < -0.8) = P(z > 0.8) = 1 - P(z < 0.8) = 1 - 0.7881 = \mathbf{0.2119}$$

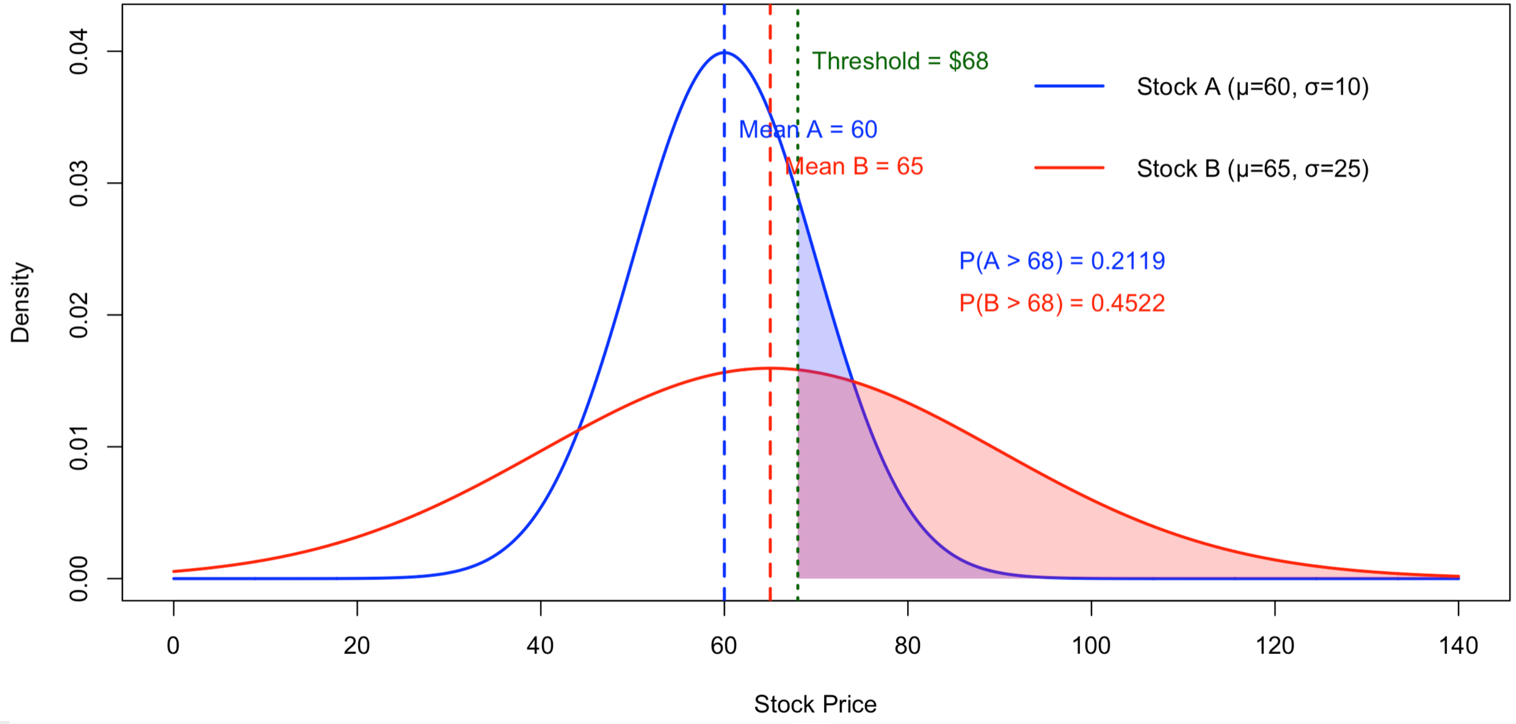


Solution

b) Which stock is more likely to go above $68?

$P(A > 68) = P(z > \frac{68-60}{10}) = P(z > 0.8) = 1 - P(z < 0.8) = 1 - 0.7881 = \mathbf{0.2119}$

$P(B > 68) = P(z > \frac{68-65}{25}) = P(z > 0.12) = 1 - P(z < 0.12) = 1 - 0.5478 = \mathbf{0.4522}$



REFERENCES

1. Textbook (McClave). Chapter 6.
2. Textbook (Lind et al). Chapter 7.
3. Textbook (Ott). Chapter 4.

Thank You!