FUNDAMENTALS OF STATISTICS (4ECON013C)

TUTORIAL 4 ANSWERS

Continuous Probability Distributions

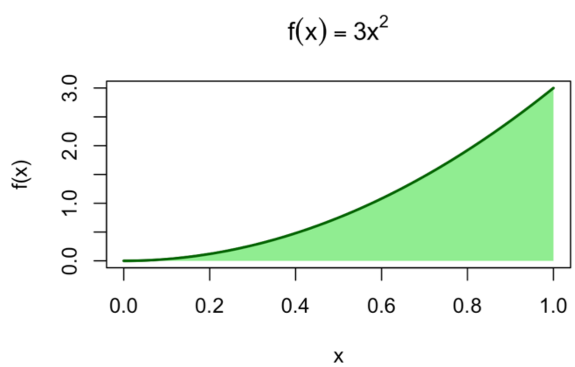
TASKS

1. Decide whether the statement makes sense (or is clearly true) or does not make sense (or is clearly false). Explain clearly.

- **a.** The probability that a continuous random variable X equals exactly 5 is greater than 0. False: $P(X = 5) = 0$
- **b.** The mean and median of a uniform distribution *U(a,b)* are always the same. True
- **c.** Scores on a statistics test are normally distributed with a mean of 75 and a standard deviation of 75. False
- **d.** Birth weights (in grams) in Uzbekistan are normally distributed with a mean of 3.32 kg and a standard deviation of 400 g. True
- **e.** Scores on a standard test of depth perception are normally distributed with two different modes. False
- **f.** SAT scores are normally distributed with a mean of 1518 and a standard deviation of 325. False

2. You are given the following function:

$$f(x) = \begin{cases} 3x^2, & 0 \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}$$



- **a.** Prove that this is pdf.

$$\int_0^1 3x^2 dx = x^3 \Big|_0^1 = 1$$

- **b.** Find the expected value of x.

$$E(x) = \int_0^1 x * 3x^2 dx = \frac{3}{4} x^4 \Big|_0^1 = \frac{3}{4}(1 - 0) = \frac{3}{4}$$

- **c.** Find the variance of x.

$$E(x^2) = \int_0^1 x^2 * 3x^2 dx = \frac{3}{5} x^5 \Big|_0^1 = \frac{3}{5}$$

$$\sigma^2 = \frac{3}{5} - \left(\frac{3}{4}\right)^2 = 0.0375$$

d. Find the probability that X is between 0.2 and 0.8.

P(0.2 < X < 0.8) = $\int_{0.2}^{0.8} 3x^2 dx = \; x^3 |_{0.2}^{0.8} = 0.504$

3. An unprincipled used-car dealer sells a car to an unsuspecting buyer, even though the dealer knows that the car will have a major breakdown within the next 6 months. The dealer provides a warranty of 45 days on all cars sold. Let x represent the length of time until the breakdown occurs. Assume that x is a uniform random variable with values between 0 and 6 months. Calculate the probability that the breakdown occurs while the car is still under warranty (assume there are 30 days in a month).

P(X < 1.5) = $\frac{1.5-0}{6-0}$ = 0.25

4. According to the Insurance Institute of America, a family of four spends between $400 and $3,800 per year on all types of insurance. Suppose the money spent is uniformly distributed between these amounts.

a. If we select a family at random, what is the probability they spend less than $2,000 per year on insurance?

P(X < 2000) = $\frac{2000-400}{3800-400}$ = 0.47

b. What is the 90th percentile of the family spending on insurance?

P(X < k) = 0.90

$\frac{k-400}{3800-400} = 0.9$ => k = $3460

c. If you are told that the family spent more than $1,500, what is the probability that this family spent $2500 at most?

P(X < 2500 | X > 1500) = $\frac{P(1500 < X < 2500)}{P(X > 1500)} = \frac{2500-1500}{3800-1500} = 0.435$

5. The annual income of full-time employees in a certain town is normally distributed with a mean $11,000 and standard deviation $1,500. Given that there are 150,000 people in full-time employment in this town, estimate the number of people whose annual income is between $8,000 and $16,000.

P(8000 < X < 16000) = P(X < 16000) – P(X < 8000) = P(z < 3.33) – P(z < -2.0) = 0.977

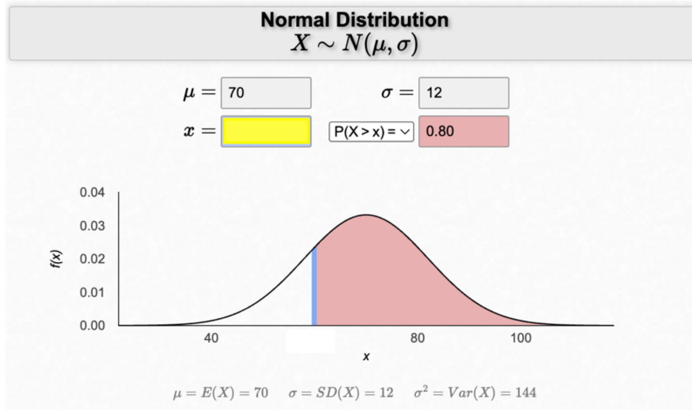
0.977*150000 = 146550

6. Find the value of x.

P(X < x) = 0.20

z ≈ -0.84

x = 70-0.84*12 = 59.92



7. The following website provides the average IQ scores by country for 2024:

https://worldpopulationreview.com/country-rankings/average-iq-by-country

The IQ scores follow a normal distribution and the standard deviation is approximately 15.

a. If a person in Uzbekistan scores 125 or higher on an IQ test, they are considered *highly intelligent*. Find the percentage of people in Uzbekistan who belong to this category.

$\mu = 96.8$

P(X ≥ 125) = P(z ≥ $\frac{125-96.8}{15}$) = P(z ≥ 1.88) = 0.03, 3%

b. Suppose a national scholarship program in Uzbekistan accepts only the top 5% of students based on IQ test performance. What is the minimum IQ cutoff for eligibility?

z = 1.645 for P(X < x) = 0.95

x = 96.8 + 1.645*15 = 121.48

c. If Uzbekistan's average IQ rose from by 2 points, how would the percentage of people above 120 change?

$\mu = 96.8$: P(X > 120) = 0.061

$\mu = 98.8$: P(X > 120) = 0.079

It increases by 1.8 percentage points.

d. Suppose X is an IQ score of an individual in a specific country where X ~ N($\mu, 15^2$).

If P(X > 99) = 0.6554, then which country is this? μ = 105 – Singapore.

8. X is distributed normally, P(X ⩾ 59.1) = 0.0281 and P(X ⩾ 29.2) = 0.9345. Find the mean and standard deviation of the distribution.

P(z > 1.91) = 0.0281 and P(z > -1.51) = 0.9345

$\frac{59.1-\mu}{\sigma} = 1.91$ *and* $\frac{29.2-\mu}{\sigma} = -1.51$

$$\begin{cases} 59.1 - \mu = 1.91\sigma \\ 29.2 - \mu = -1.51\sigma \end{cases}$$

$\sigma = \frac{59.1-29.2}{1.91+1.51} = 8.74$

$\mu = 59.1 - 1.91 * 8.74 = 42.41$

HOMEWORK

9. Suppose 8-week old babies' smiling times, in seconds, follow a uniform distribution between zero and 23 seconds.

a. What is the probability a randomly chosen 8-week-old baby smiles between 2 and 18 seconds?

$P(2 < X < 18) = \frac{18-2}{23-0} = 0.6957$

b. Find the 80th percentile for an eight-week-old baby's smiling time.

$P(X < k) = 0.80$

$k = 23*0.8 = 18.4$

c. Find the probability that a random eight-week-old baby smiles more than 12 seconds knowing that the baby smiles more than eight seconds.

$P(X > 12 \mid X > 8) = \frac{P(X>12)}{P(X>8)} = \frac{23-12}{23-8} = 0.7333$

10. Ace Heating and Air Conditioning Service finds that the amount of time a repairman needs to fix a furnace is uniformly distributed between 1.5 and four hours. Let x = the time needed to fix a furnace. Find the 30th percentile of furnace repair times.

$P(X < k) = 0.30$

$\frac{k-1.5}{4-1.5} = 0.30 \Rightarrow k = 2.25$

11. The amount of time, in minutes, that a person must wait for a bus is uniformly distributed between zero and 15 minutes.

a. What is the probability that a person waits less than 12.5 minutes? (0.833)

b. On average, how long must a person wait? Find the mean, μ, and the standard deviation, σ. ($\mu = 7.5$ $\sigma = 4.3$)

c. Ninety percent of the time, the time a person must wait falls below what value? (13.5 minutes).

12. The number of newspapers sold daily at a kiosk is normally distributed with a mean of 350 and a standard deviation of 30.

a. Find the probability that fewer than 300 newspapers are sold on a particular day. $P(x < 300) = P(z < -1.67) = 0.05$

b. Find the probability that the number of newspapers sold is between 300 and 360 on a particular day. $P(300 < x < 360) = P(z < 0.33) - P(z < -1.67) = 0.63 - 0.05 = 0.58$

c. How many newspapers should the newsagent stock each day such that the probability of running out on any particular day is 5%?

$P(X < k) = 0.95$

$1.645 = \frac{k-350}{30}$ => k = 399.35. If 400 newspapers are stocked, then this probability will be slightly lower than 0.05.

13. Assume that the hourly cost to operate a commercial airplane follows the normal distribution with a mean of \$2,500 per hour and a standard deviation of \$300. What is the operating cost for the lowest 15% of the airplanes?

2500 – 300*1.04 (z score for 0.15) = \$2188

14. The annual commissions earned by sales representatives of Machine Products Inc., a manufacturer of light machinery, follow the normal probability distribution. The mean yearly amount earned is \$40,000 and the standard deviation is \$5,000.

a. What percent of the sales representatives earn more than \$42,000 per year? (34.5%)

b. What percent of the sales representatives earn less than \$34,000 per year? (11.5%)