Quantum Error Correction - Task 2

QOSF

Abdellah Tounsi

February 22, 2021

0.1 Error-Free Quantum Circuit

The subject quantum circuit creates Bell states by applying the following operator:

$$CX(0,1)(H \otimes I) \tag{1}$$

on two qubit system. Hence, the final state $|\psi\rangle$ of this circuit is the Bell state since:

$$\begin{aligned}
|\psi\rangle &= CX(0,1)(H \otimes I)\,|00\rangle && (2)\\
&= \frac{1}{\sqrt{2}} CX(0,1)(|00\rangle + |10\rangle) && (3)\\
&= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) && (4)
\end{aligned}$$

In the basis of X eigenvectors, this final state is read as:

$$\begin{aligned}
|\psi\rangle_X &= (H \otimes H)\,|\psi\rangle && (5)\\
&= \frac{1}{\sqrt{2}}(H \otimes H)(|00\rangle + |11\rangle) && (6)\\
&= \frac{1}{2\sqrt{2}}\left((|0\rangle + |1\rangle)(|0\rangle + |1\rangle) + (|0\rangle - |1\rangle)(|0\rangle - |1\rangle)\right) && (7)\\
&= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) && (8)
\end{aligned}$$

0.2 Introducing Noise

We suppose random bit-flips or sign-flips before applying CX gate. For instance, if bit-flip happens in the second qubit, the final state will be:

$$\begin{aligned}
|\psi'\rangle &= CX(0,1)(I \otimes X)(H \otimes I)\,|00\rangle && (9)\\
&= \frac{1}{\sqrt{2}} CX(0,1)(I \otimes X)(|00\rangle + |10\rangle) && (10)\\
&= \frac{1}{\sqrt{2}} CX(0,1)(|01\rangle + |11\rangle) && (11)\\
&= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) && (12)
\end{aligned}$$

which is different state since $\langle\psi'|\psi\rangle \neq 1$.

Introducing sign-flip to the first qubit

$$
\begin{aligned}
|\psi''\rangle &= CX(0,1)(Z \otimes I)(H \otimes I)\,|00\rangle && (13)\\
&= \frac{1}{\sqrt{2}} CX(0,1)(Z \otimes I)(|00\rangle + |10\rangle) && (14)\\
&= \frac{1}{\sqrt{2}} CX(0,1)(|00\rangle - |10\rangle) && (15)\\
&= \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle) && (16)
\end{aligned}
$$

The state $|\psi''\rangle$ is orthogonal to $|\psi\rangle$ and that appears clearly in the basis of X eigenvectors.

0.3 Error Correction

To correct bit-flips and sign-flips, we use Shor's encoding states:

$$
|0\rangle_S = \frac{1}{\sqrt{8}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) \quad (17)
$$

$$
|1\rangle_S = \frac{1}{\sqrt{8}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle) \quad (18)
$$

Applying Hadamard gate on $|0\rangle_S$ and $|1\rangle_S$ states should yield:

$$
H_S\,|0\rangle_S = \frac{1}{\sqrt{2}}(|0\rangle_S + |1\rangle_S) \quad (19)
$$

$$
H_S\,|1\rangle_S = \frac{1}{\sqrt{2}}(|0\rangle_S - |1\rangle_S) \quad (20)
$$

To do that, we can create a controlling ancilla and perform the following calculation:

$$
\begin{aligned}
H_S\,|0\rangle\,|0\rangle_S &= X_S C X_S (C X_i)\,|0\rangle\,|0\rangle_S && (21)\\
&= C X_S \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\,|0\rangle_S && (22)\\
&= \frac{1}{\sqrt{2}}(|0\rangle\,|0\rangle_S + |1\rangle\,|1\rangle_S) && (23)\\
&= \frac{1}{\sqrt{2}}(|1\rangle\,|0\rangle_S + |0\rangle\,|1\rangle_S) && (24)
\end{aligned}
$$

$$
\begin{aligned}
H_S \left|1\right\rangle \left|0\right\rangle_S &= X_S CX_S (CX_i) \left|0\right\rangle \left|1\right\rangle_S && (25)\\
&= CX_S \frac{1}{\sqrt{2}} (\left|0\right\rangle - \left|1\right\rangle) \left|1\right\rangle_S && (26)\\
&= \frac{1}{\sqrt{2}} (\left|0\right\rangle \left|1\right\rangle_S - \left|1\right\rangle \left|0\right\rangle_S) && (27)\\
&= \frac{1}{\sqrt{2}} (\left|0\right\rangle \left|0\right\rangle_S - \left|1\right\rangle \left|1\right\rangle_S) && (28)
\end{aligned}
$$

where CX_i controls the ancilla with arbitrary i'th qubit from Shor block, while CX_S controls Shor block with the ancilla. Here:

$$
X_S = Z_i Z_j Z_k \tag{29}
$$

where $i = 1, 2, 3,\ j = 4, 5, 6$ and $k = 7, 8, 9$