Quantum Computing (CS-4084)

Date: February 28th 2024

Course Instructor(s)

Dr. Faisal Aslam

Sessional-I Exam

Total Time: 1 Hours
Total Marks: 43
Total Questions: 07

Semester: SP-2024
Campus: Lahore
Dept: Computer Science

Student Name	Roll No	Section	Student Signature

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1. Consider the quantum circuit shown in Figure 1. If the input to the circuit is $|111\rangle$, determine its output. Please show your calculations at each stage of the circuit. [**7 Marks**]

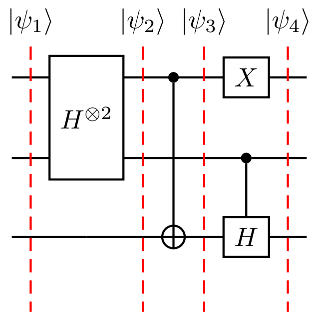


Figure 1: Quantum circuit

$$
\begin{aligned}
|\psi_1\rangle &= |111\rangle \\
|\psi_2\rangle &= |-\rangle\,|-\rangle\,|1\rangle \\
&= \frac{|0\rangle - |1\rangle}{\sqrt{2}}\,\frac{|0\rangle - |1\rangle}{\sqrt{2}}\,|1\rangle \\
&= \frac{1}{2}\Big\{\,|001\rangle - |011\rangle - |101\rangle + |111\rangle\,\Big\} \\
|\psi_3\rangle &= \frac{1}{2}\Big\{\,|001\rangle - |011\rangle - |100\rangle + |110\rangle\,\Big\} \\
|\psi_4\rangle &= \frac{1}{2}\Big\{\,|101\rangle - |11-\rangle - |000\rangle + |01+\rangle\,\Big\}
\end{aligned}
$$

2. What will be the unitary matrix representing the reverse of the quantum circuit shown in Figure 2? [**7 Marks**]

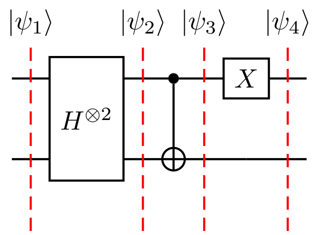


Figure 2: Another Quantum circuit

$$H^{\otimes 2} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$X \otimes I = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Our of original circuit will be:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tfrac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} = \tfrac{1}{2} \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

The matrix of reverse circuit will be conjugate-transpose of it. That is:

$$\tfrac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

3. Prove that the following three qubits are NOT entangled by writing them separately $\frac{\sqrt{3}}{2\sqrt{2}}|000\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|001\rangle + \frac{1}{2\sqrt{2}}|100\rangle + \frac{1}{2\sqrt{2}}|101\rangle$ [**5 Marks**].

   If the qubits are not entangled then we must be able to write them separately. That is:

$$\begin{aligned}\frac{\sqrt{3}}{2\sqrt{2}}|000\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|001\rangle + \frac{1}{2\sqrt{2}}|100\rangle + \frac{1}{2\sqrt{2}}|101\rangle &= (a|0\rangle + b|1\rangle)(c|0\rangle + d|1\rangle)(e|0\rangle + f|1\rangle) \\ &= ace|000\rangle + acf|001\rangle + ade|010\rangle + adf|011\rangle \\ &+ bce|100\rangle + bcf|101\rangle + bde|110\rangle + bdf|111\rangle\end{aligned}$$

   From above we get: $ace = \frac{\sqrt{3}}{2\sqrt{2}}$, $acf = \frac{\sqrt{3}}{2\sqrt{2}}$, $bce = \frac{1}{2\sqrt{2}}$, $bcf = \frac{1}{2\sqrt{2}}$ and rest of them are all equal to zero.

   As $|c|^2 + |d|^2 = 1$ and $d = 0$, thus $c = 1$. So above notation simplifies to.

$$ae = \frac{\sqrt{3}}{2\sqrt{2}} \tag{1}$$

$$af = \frac{\sqrt{3}}{2\sqrt{2}} \tag{2}$$

$$be = \frac{1}{2\sqrt{2}} \tag{3}$$

$$bf = \frac{1}{2\sqrt{2}} \tag{4}$$

   We rewrite Equation 1 and 3 in terms of e as follows:

$$a = \frac{\sqrt{3}}{2e\sqrt{2}} \tag{5}$$

$$b = \frac{1}{2e\sqrt{2}} \tag{6}$$

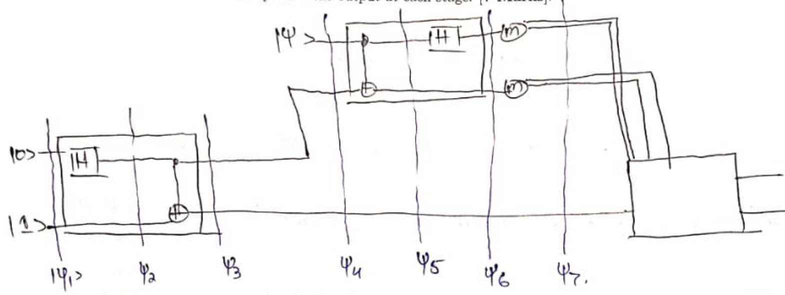
$$|a|^2 + |b|^2 = 1 \tag{7}$$

   Putting values of Equation 5 and 6 in Equation 7 we get:

$$\begin{aligned}e^2 &= \frac{3}{8} + \frac{1}{8} \\ e &= \frac{1}{\sqrt{2}}, f = \frac{1}{\sqrt{2}}, a = \frac{\sqrt{3}}{2}, b = \frac{1}{2}\end{aligned}$$

4. What gates will Alice use to decode the qubit received from Bob in a quantum teleportation scenario where the entanglement is initially created using $|01\rangle$ instead of the typical $|00\rangle$? Please create a circuit diagram and clearly show the output at each stage. [**7 Marks**].

4. What gates will Alice use to decode the qubit received from Bob in a quantum teleportation scenario where the entanglement is initially created using $|01\rangle$ instead of the typical $|00\rangle$? Please create a circuit diagram and clearly show the output at each stage. [**7 Marks**].



$|\psi_1\rangle = |01\rangle$

$|\psi_2\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}(|1\rangle)$

$|\psi_2\rangle = \frac{|01\rangle + |1\rangle}{\sqrt{2}}$

$|\psi_3\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$ ✓

00	$\alpha\|1\rangle + \beta\|0\rangle$	X
10	$\alpha\|1\rangle - \beta\|0\rangle$	XZ
01	$\alpha\|0\rangle + \beta\|1\rangle$	No operation.
11	$\alpha\|0\rangle - \beta\|1\rangle$	Z

7

let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$|\psi_4\rangle = \frac{\alpha|001\rangle + \alpha|010\rangle + \beta|101\rangle + \beta|110\rangle}{\sqrt{2}}$

$|\psi_5\rangle = \frac{\alpha|001\rangle + \alpha|010\rangle + \beta|111\rangle + \beta|100\rangle}{\sqrt{2}}$

$|\psi_6\rangle = \frac{1}{\sqrt{2}}\left(\alpha\frac{|0\rangle+|1\rangle}{\sqrt{2}}(|01\rangle) + \alpha\frac{|0\rangle+|1\rangle}{\sqrt{2}}|10\rangle + \beta\frac{|0\rangle-|1\rangle}{\sqrt{2}}|11\rangle + \beta\frac{|0\rangle+|1\rangle}{\sqrt{2}}|00\rangle\right)$

$|\psi_6\rangle = \frac{1}{2}(\alpha|001\rangle + \alpha|101\rangle + \alpha|010 + \alpha|110\rangle + \beta|011\rangle - \beta|111 + \beta|000\rangle - \beta|100\rangle)$

$|\psi_6\rangle = \frac{1}{2}(|00\rangle(\alpha|1\rangle + \beta|0\rangle) + |10\rangle(\alpha|1\rangle - \beta|0\rangle + |01\rangle(\alpha|0\rangle + \beta|1\rangle + |11\rangle(\alpha|0\rangle - \beta|1\rangle)$ ↑

5. Given the states $|\alpha\rangle = \frac{i}{\sqrt{5}}|0\rangle + \frac{2}{\sqrt{5}}|1\rangle$ and $|\beta\rangle = \frac{1}{\sqrt{3}}|0\rangle - i\sqrt{\frac{2}{3}}|1\rangle$:

a) Compute $|\alpha\rangle\langle\beta|$. Provide result in Dirac's notation and also express it in matrix form. [**2+1 Marks**]

First I write bra-β: $\langle\beta| = \frac{1}{\sqrt{3}}\langle 0| + i\sqrt{\frac{2}{3}}\langle 1|$
Hence,

$$
\begin{aligned}
|\alpha\rangle\langle\beta| &= \left(\frac{i}{\sqrt{5}}|0\rangle + \frac{2}{\sqrt{5}}|1\rangle\right)\left(\frac{1}{\sqrt{3}}\langle 0| + i\sqrt{\frac{2}{3}}\langle 1|\right) \\
&= \frac{i}{\sqrt{15}}|0\rangle\langle 0| - \sqrt{\frac{2}{15}}|0\rangle\langle 1| + \frac{2}{\sqrt{15}}|1\rangle\langle 0| + 2i\sqrt{\frac{2}{15}}|1\rangle\langle 1| \\
&= \begin{pmatrix} \frac{i}{\sqrt{15}} & -\sqrt{\frac{2}{15}} \\ \frac{2}{\sqrt{15}} & 2i\sqrt{\frac{2}{15}} \end{pmatrix}
\end{aligned}
$$

b) Calculate $\langle\alpha|\beta\rangle$ using Dirac's notation. [**2 Marks**]

First I write bra-β: $\langle\alpha| = \frac{-i}{\sqrt{5}}\langle 0| + \frac{2}{\sqrt{5}}\langle 1|$
Thus,

$$
\begin{aligned}
\langle\alpha|\beta\rangle &= \left(\frac{-i}{\sqrt{5}}\langle 0| + \frac{2}{\sqrt{5}}\langle 1|\right)\left(\frac{1}{\sqrt{3}}|0\rangle - i\sqrt{\frac{2}{3}}|1\rangle\right) \\
&= \frac{-i}{\sqrt{15}} - 2i\sqrt{\frac{2}{15}} \\
&= -i\frac{1+2\sqrt{2}}{\sqrt{15}}
\end{aligned}
$$

c) Determine $\langle\alpha|\langle\beta|$ using Dirac's notation. Additionally, provide the result in matrix form. [**2+1 Marks**]

$$
\begin{aligned}
\langle\alpha| &= \frac{-i}{\sqrt{5}}\langle 0| + \frac{2}{\sqrt{5}}\langle 1| \\
\langle\beta| &= \frac{1}{\sqrt{3}}\langle 0| + i\sqrt{\frac{2}{3}}\langle 1| \\
\langle\alpha|\langle\beta| &= \frac{-i}{\sqrt{15}}\langle 0|\langle 0| + \sqrt{\frac{2}{15}}\langle 0|\langle 1| + \frac{2}{\sqrt{15}}\langle 1|\langle 0| + i2\sqrt{\frac{2}{15}}\langle 1|\langle 1| \\
&= \begin{bmatrix} \frac{-i}{\sqrt{15}} & \sqrt{\frac{2}{15}} & \frac{2}{\sqrt{15}} & i2\sqrt{\frac{2}{15}} \end{bmatrix}
\end{aligned}
$$

6. What is the probability of measuring the last two qubits of the state $|\psi\rangle = \frac{\sqrt{3}}{2\sqrt{2}}|000\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|001\rangle + \frac{1}{2\sqrt{2}}|100\rangle + \frac{1}{2\sqrt{2}}|101\rangle$ as $|01\rangle$? Additionally, what will be the resultant state after the measurement? [**2+2 Marks**]

   The probability of measuring the last two qubits is $\frac{1}{2}$.

   The resultant state will be $\frac{\sqrt{3}}{2}|001\rangle + \frac{1}{2}|101\rangle$

7. Given the qubit $|\alpha\rangle = i\sqrt{\frac{2}{3}}\,|0\rangle + \sqrt{\frac{1}{2}}\,|1\rangle$, measure it in the basis defined by the Y-gate: $\left\{ \begin{pmatrix} 0 \\ i \end{pmatrix}, \begin{pmatrix} -i \\ 0 \end{pmatrix} \right\}$. Must show probabilities of measurements and resultant states. [**5 Marks**]

   NOTE: The question has slight mistake that α is not a unit vector. However, I will still be giving marks based on students general working.

   Let say $\beta = \begin{pmatrix} 0 \\ i \end{pmatrix}$ and $\gamma = \begin{pmatrix} -i \\ 0 \end{pmatrix}$. Then we can rewrite $|\alpha\rangle$ in the basis of Y-gate as:

$$\begin{aligned} |\alpha\rangle &= \langle\alpha|\beta\rangle\,|\beta\rangle + \langle\alpha|\gamma\rangle\,|\gamma\rangle \\ &= -i\sqrt{\frac{1}{2}}\,|\beta\rangle - \sqrt{\frac{2}{3}}\,|\gamma\rangle \end{aligned}$$

   We will measure $|\beta\rangle$ with probability $\frac{1}{2}$ and the resultant state will be $|\beta\rangle$. We will measure $|\gamma\rangle$ with probability $\frac{2}{3}$ and the resultant state will be $|\gamma\rangle$.