CHAPTER 7

Central Difference Interpolation Formulae

7.1 INTRODUCTION

Newton's forward and backward Interpolation formulae which have been discussed in the previous chapter are best suited for interpolation near the beginning and the end of a difference table. But, to interpolate near the middle (centre) of a difference table, the following central difference interpolation formulae are most suitable.

Let $y = f(x)$ be the functional relation between x and y. If x takes the values $x_0 - 2h$, $x_0 - h$, x_0, $x_0 + h$ and $x_0 + 2h$, and the corresponding values of y are y_{-2}, y_{-1}, y_0, y_1 and y_2 then we can write the difference table in the two notations ··· as follows using the operator $\Delta = \delta E^{1/2}$, i.e. $\delta = \Delta E^{1/2}$, known as central difference table. ✓

x	y	First difference	Second difference	Third difference	Fourth difference
$x_0 - 2h$	y_{-2}				
		$\Delta y_{-2} (= \delta y_{-3/2})$			
$x_0 - h$	y_{-1}		$\Delta^2 y_{-2} (= \delta^2 y_{-1})$		
		$\Delta y_{-1} (= \delta y_{-1/2})$		$\Delta^3 y_{-2} (= \delta^3 y_{-1/2})$	
x_0 ……	y_0	…….	$\Delta^2 y_{-1} (= \delta^2 y_0)$	…….	$\Delta^4 y_{-2} (= \delta^4 y_0)$
		$\Delta y_0 (= \delta y_{1/2})$		$\Delta^3 y_{-1} (= \delta^3 y_{1/2})$	
$x_0 + h$	y_1		$\Delta^2 y_0 (= \delta^2 y_1)$		
		$\Delta y_1 (= \delta y_{3/2})$			
$x_0 + 2h$	y_2				

7.2 GAUSS'S FORWARD INTERPOLATION FORMULA

The Gregory–Newton forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots \quad (7.1)$$

where $p = \dfrac{x - x_0}{h}$. $\Delta^2 E y_{-1} = \Delta^2 y_{-1+1} = \Delta^2 y_0$

We have,

$$\Delta^2 y_0 = \Delta^2 E y_{-1} = \Delta^2(1 + \Delta)y_{-1} = \Delta^2 y_{-1} + \Delta^3 y_{-1} \quad (7.2)$$

$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1} \quad (7.3)$$

$$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1} \quad (7.4)$$

Similarly,

$$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2} \quad (7.5)$$

$$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2} \text{ etc.} \quad (7.6)$$

Substituting for $\Delta^2 y_0, \Delta^3 y_0, \Delta^4 y_0 \cdots$ from Eqns (7.2)–(7.6), in Eqn (7.1), we get

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{p(p-1)(p-2)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-1})$$
$$+ \frac{p(p-1)(p-2)(p-3)}{4!}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \cdots$$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \left\{\frac{p(p-1)}{2!} + \frac{p(p-1)(p-2)}{3!}\right\}\Delta^3 y_{-1}$$
$$+ \left\{\frac{p(p-1)(p-2)}{3!} + \frac{p(p-1)(p-2)(p-3)}{4!}\right\}\Delta^4 y{-1} + \cdots$$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$
$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-1} + \frac{(p+1)p(p-1)(p-2)(p-3)}{5!}\Delta^5 y_{-1}$$
$$+ \cdots$$

$$= y_0 + p\Delta y_0 + {}^{p}C_2\,\Delta^2 y_{-1} + {}^{(p+1)}C_3\,\Delta^3 y_{-1} + {}^{(p+1)}C_4\,\{\Delta^4 y_{-2} + \Delta^5 y_{-2}\}$$
$$+ {}^{(p+1)}C_5\{\Delta^5 y_{-2} + \Delta^6 y_{-2}\} + \cdots \quad \text{[using Eqn (7.6) etc.]}$$

$$= y_0 + {}^{p}C_1\,\Delta y_0 + {}^{p}C_2\,\Delta^2 y_{-1} + {}^{(p+1)}C_3\,\Delta^3 y_{-1} + {}^{(p+1)}C_4\,\Delta^4 y_{-2}$$
$$+ \{{}^{(p+1)}C_5 + {}^{(p+1)}C_5\}\,\Delta^5 y_{-2} + \cdots$$

$$\therefore y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$

$$+\frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\Delta^5 y_{-2}$$

$$+\ldots \tag{7.7}$$

which is called *Gauss's forward interpolation formula.*

In the central differences notation, this formula will be

$$y_p = y_0 + p\delta y_{1/2} + \frac{p(p-1)}{2!}\delta^2 y_0 + \frac{(p+1)p(p-1)}{3!}\delta^3 y_{1/2}$$

$$+\frac{(p+1)p(p-1)(p-2)}{4!}\delta^4 y_0$$

$$+\frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\delta^5 y_{1/2} + \cdots \tag{7.8}$$

Note: This formula, Eqn (7.7), involves odd differences below the central line ($x = x_0$) and even differences on the line as shown below.

$$\begin{array}{ccccccccccccc} y_0 & \cdots & & \cdots & \Delta^2 y_{-1} & \cdots & & \cdots & \Delta^4 y_{-2} & \cdots & & \cdots & \Delta^6 y_{-3} \;\cdots \\ & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\ & & \Delta y_0 & & & & \Delta^3 y_{-1} & & & & \Delta^5 y_{-2} & & & & \Delta^7 y_{-3} \end{array}$$

It is used to interpolate the values of y for $0 < p < 1$.

7.3 GAUSS'S BACKWARD INTERPOLATION FORMULA

Gregory–Newton's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots \tag{7.9}$$

where $p = \dfrac{x - x_0}{h}$.

We have,

$$\Delta y_0 = \Delta E y_{-1} = \Delta(1+\Delta) y_{-1} = \Delta y_{-1} + \Delta^2 y_{-1} \tag{7.10}$$

Similarly,

$$\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1} \tag{7.11}$$

$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1} \text{ etc.} \tag{7.12}$$

Also,

$$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2} \tag{7.13}$$

$$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2} \text{ etc.} \tag{7.14}$$

Substituting for Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0 \ldots$ from Eqns (7.10) – (7.12) in Eqn (7.9), we get

$$y_p = y_0 + p\,(\Delta y_{-1} + \Delta^2 y_{-1}) + \frac{p(p-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{p(p-1)(p-2)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-1}) + \frac{p(p-1)(p-2)(p-3)}{4!}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \cdots$$

$$= y_0 + p\,\Delta y_{-1} + \left\{p + \frac{p(p-1)}{2!}\right\}\Delta^2 y_{-1} + \left\{\frac{p(p-1)}{2!} + \frac{p(p-1)(p-2)}{3!}\right\}\Delta^3 y_{-1} + \left\{\frac{p(p-1)(p-2)}{3!} + \frac{p(p-1)(p-2)(p-3)}{4!}\right\}\Delta^4 y_{-1} + \cdots$$

$$= y_0 + p\,\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-1} + \cdots$$

$$= y_0 + p\,\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\{\Delta^3 y_{-2} + \Delta^4 y_{-2}\} + \frac{(p+1)p(p-1)(p-2)}{4!}\{\Delta^4 y_{-2} + \Delta^5 y_{-2}\} + \cdots$$

[using Eqns (7.13) and (7.14)]

$$\therefore\ y_p = y_0 + p\,\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \cdots \tag{7.15}$$

Eqn (7.15) is called *Gauss's backward interpolation formula*.

In central differences notation, this formula can be written as

$$y_p = y_0 + p\delta y_{-1/2} + \frac{(p+1)p}{2!}\delta^2 y_0 + \frac{(p+1)p(p-1)}{3!}\delta^3 y_{-1/2} + \frac{(p+2)(p+1)p(p+1)}{4!}\delta^4 y_0 + \cdots \tag{7.16}$$

Note : Eqn (7.15) involves odd differences above the central line and even differences on the central line as shown below:

$$\begin{array}{ccccccccccc}
 & & \Delta y_{-1} & & & \Delta^3 y_{-1} & & & \Delta^5 y_{-3} & & \\
 & \nearrow & & \searrow & \nearrow & & \searrow & \nearrow & & \searrow & \\
y_0 & \cdots & & \Delta^2 y_{-1} & \cdots & & \Delta^4 y_{-2} & \cdots & & \Delta^6 y_{-3} & \cdots \text{ central line}
\end{array}$$

It is useful when $-1 < p < 0$.

7.4 STIRLING'S FORMULA

Taking the mean of Gauss's forward interpolation and backward interpolation formulae, that is, Eqns (7.7) and (7.15), we get the following equation:

$$y_p = y_0 + p\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{1}{2}\left[\frac{p(p-1)}{2!} + \frac{(p+1)p}{2!}\right]\Delta^2 y_{-1}$$

$$+\frac{(p+1)p(p-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right]$$

$$+\frac{1}{2}\left[\frac{(p+1)p(p-1)(p-2)}{4!} + \frac{(p+2)(p+1)p(p-1)}{4!}\right]\Delta^4 y_{-2} + \cdots$$

or
$$y_p = y_0 + p\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right]$$

$$+\frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots \tag{7.17}$$

The above equation is called *Stirling's formula*.

In the central differences notation, Eqn (7.17) takes the form

$$y_p = y_0 + p\mu\delta y_0 + \frac{p^2}{2!}\delta^2 y_0 + \frac{p(p^2-1^2)}{3!}\mu\delta^3 y_0$$

$$+\frac{p^2(p^2-1^2)}{4!}\delta^4 y_0 + \cdots \tag{7.18}$$

where $\frac{1}{2}(\Delta y_0 + \Delta y_{-1}) = \frac{1}{2}(\delta y_{1/2} + \delta y_{-1/2}) = \mu\delta y_0$

$\frac{1}{2}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) = \frac{1}{2}(\delta^3 y_{1/2} + \delta^3 y_{-1/2}) = \mu\delta^3 y_0$ etc.

Note: This formula involves the means of the odd differences just above and below the central line and even differences on the central line as shown below:

$$y_0 \cdots \begin{bmatrix}\Delta y_{-1}\\ \Delta y_0\end{bmatrix} \cdots \Delta^2 y_{-1} \cdots \begin{bmatrix}\Delta^3 y_{-2}\\ \Delta^3 y_{-1}\end{bmatrix} \cdots \Delta^4 y_{-2} \cdots \begin{bmatrix}\Delta^5 y_{-3}\\ \Delta^5 y_{-2}\end{bmatrix} \cdots \Delta^6 y_{-3} \cdots$$

This is useful when $-\frac{1}{2} < p < \frac{1}{2}$ and a good estimate when $-\frac{1}{4} < p < \frac{1}{4}$.

7.5 BESSEL'S FORMULA

Once again, consider Gauss's forward interpolation fomula given by Eqn (7.7). We know that

$$y_1 - y_0 = \Delta y_0 \quad \therefore \; y_0 = y_1 - \Delta y_0 \tag{7.19}$$

$$\Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1} \quad \therefore \; \Delta^2 y_{-1} = \Delta^2 y_0 - \Delta^3 y_{-1} \tag{7.20}$$

similarly,

$$\Delta^4 y_{-2} = \Delta^4 y_{-1} - \Delta^5 y_{-2} \text{ etc.} \tag{7.21}$$

So, Eqn (7.7) can be written as

$$y_p = \left(\frac{y_0}{2} + \frac{y_0}{2}\right) + p\Delta y_0 + \frac{1}{2}\,\frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{1}{2}\,\frac{p(p-1)}{2!}\Delta^2 y_{-1}$$

$$+ \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \dots \tag{7.22}$$

Substituting Eqns (7.19) and (7.20) in Eqn (7.22), we get

$$y_p = \frac{y_0}{2} + \frac{1}{2}(y_1 - \Delta y_0) + p\Delta y_0 + \frac{1}{2}\,\frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{1}{2}\,\frac{p(p-1)}{2!}$$

$$(\Delta^2 y_0 - \Delta^3 y_{-1}) + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \cdots$$

$$= \frac{1}{2}(y_0 + y_1) + \left(p - \frac{1}{2}\right)\Delta y_0 + \frac{1}{2}\,\frac{p(p-1)}{2!}(\Delta^2 y_{-1} + \Delta^2 y_0)$$

$$+ \frac{p(p-1)}{2!}\left(-\frac{1}{2} + \frac{p+1}{3!}\right)\Delta^3 y_{-1} + \cdots$$

$$= \frac{1}{2}(y_0 + y_1) + \left(p - \frac{1}{2}\right)\Delta y_0 + \frac{p(p-1)}{2!}\left[\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right]$$

$$+ \frac{\left(p - \frac{1}{2}\right)p(p-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\left[\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right] + \cdots \tag{7.23}$$

Eqn (7.23) is known as *Bessel's formula.*

7.8 ADVANTAGES OF CENTRAL DIFFERENCE INTERPOLATION FORMULAE

The coefficients in the central difference formulae are smaller and converge faster than those in Gregory–Newton's formulae. The coefficients in the Stirling's formula decrease more rapidly than those of the Bessel's formula. And, in Bessel's formula the coefficients decrease more rapidly than those of Gregory–Newton's formulae. The right choice of an interpolation formula, however, depends on the position of the value to be interpolated in the given data. The following rules will be useful in selecting the interpolation formulae.

1. If interpolation is required near the beginning of the table then use Gregory–Newton forward interpolation formula.
2. If interpolation is required near the end of the table then use Gregory–Newton backward interpolation formula.
3. To interpolate near the centre of the table, use eigher Stirling's or Bessel's or Laplace–Everett's formula.
   Stirling's formula is to be preferred if $-\frac{1}{4} < p < \frac{1}{4}$.
   Bessel's or Everett's formulae are better suited for the condition $\frac{1}{4} < p < \frac{3}{4}$.

Example 7.1 Using Gauss's forward interpolation formula, find the value of log 337.5 from the following table :

x	310	320	330	340	350	360
y	2.4914	2.5051	2.5185	2.5315	2.5441	2.5563

x_0 = [illegible]0, [illegible] $\frac{x-330}{10}$ since $h = 10$. Now the central [illegible] table is as follows:

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$	$\Delta^5 y_p$
310	-2	2.4914					
			0.0138				
320	-1	2.5052		-0.0005			
			0.0133		0.0002		
330	0	2.5185		-0.0003		-0.0003	
			0.0130		-0.0001		0.0004
340	1	2.5315		-0.0004		0.0001	
			0.0126		0.0000		
350	2	2.5441		-0.0004			
			0.0122				
360	3	2.5563					

Gauss's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$

$$+\frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \frac{(p+2)(p+1)p(p-1)(p+2)}{5!}\Delta^5 y_{-2}$$

$$+\frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2}$$

$$+\frac{(p+2)(p+1)p(p-1)(p+2)}{5!}\Delta^5 y_{-2} + \cdots$$

From the central difference table we get the following values:

$$x = 337.5,\ p = \frac{337.5 - 330}{10} = 0.75,\ y_0 = 2.5185$$

$\Delta y_0 = 0.0130,\ \Delta^2 y_{-1} = -0.0003,\ \Delta^3 y_{-1} = -0.0001,\ \Delta^4 y_{-2} = -0.0003,$
$\Delta^5 y_{-2} = 0.0004$

$$\therefore\ y_{0.75} = 2.5185 + (0.75)(0.0130) + \frac{(0.75)(0.75-1)}{2!}(-0.0003) +$$

$$+\frac{(0.75+1)(0.75)(0.75-1)}{3!}(-0.0001)$$

$$+\frac{(0.75+1)(0.75)(0.75-1)(0.75-2)}{4!}(-0.0003)$$

$$+\frac{(0.75+2)(0.75+1)(0.75)(0.75-1)(0.75-2)}{5!}(0.0004)$$

$$= 2.5185 + 9.75\times 10^{-3} + 2.8125\times 10^{-5} + 5.46875\times 10^{-6}$$

$$- 5.1269531\times 10^{-6} + 3.7597656\times 10^{-6}$$

$$= 2.5282822$$

$\therefore \log 337.5 = 2.5283$

Example 7.2 Interpolate by means of Gauss's backward interpolation formula, the sales of a concern for the year 1976, given that

Year	1940	1950	1960	1970	1980	1990
Sales (in lakhs of Rs)	17	20	27	32	36	38

Solution Taking 1970 as the origin and $h = 10$ years as one unit, the sales is to be found for $p = \dfrac{x - 1970}{10}$. The central difference table will be as follows:

x	p	y	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$	$\Delta^5 y_p$
1940	–3	17					
			3				
1950	–2	20		4			
			7		–6		
1960	–1	27		–2		7	
			5		–1		–9
1970	0	32		–1		–2	
			4		–1		
1980	1	36		–2			
			2				
1990	2	38					

Gauss's backward interpolation formula is

$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2}$$

$$+\frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\Delta^5 y_{-3}$$

$$+\cdots$$

where $x = 1976$, $p = \frac{1976-1970}{10} = 0.6$, $y_0 = 32$, $\Delta y_{-1} = 5$, $\Delta^2 y_{-1} = -1$,
$\Delta^3 y_{-2} = 1$, $\Delta^4 y_{-2} = -2$, $\Delta^5 y_{-3} = -9$

$$\therefore y_{0.6} = 32 + (0.6)(5) + \frac{(0.6+1)(0.6)}{2!}(-1) + \frac{(0.6+1)(0.6)(0.6-1)}{3!}(1)$$

$$+\frac{(0.6+2)(0.6+1)(0.6)(0.6-1)}{4!}(-2)$$

$$+\frac{(0.6+2)(0.6+1)(0.6)(0.6-1)(0.6-2)}{5!}(-9)$$

$$= 32 + 3 - 4.8 - 0.064 + 0.0832 - 0.104832$$

$$= 30.114368$$

Therefore, the sales in the year 1976 is Rs 30.114368 lakhs.

Example 7.3 Use Stirling's formula to find y_{35} given that $y_{10} = 600$, $y_{20} = 512$, $y_{30} = 439$, $y_{40} = 346$, $y_{50} = 243$ *(Mysore B.E., 1987)*

Solution Take $x_0 = 30$, $h = 10$ $\therefore p = \frac{x-30}{10}$

Now the central difference table is

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$
10	−2	600				
			−88			
20	−1	512		15		
			−73		−35	
30	0	439		−20		145
			−93		10	
40	1	346		−10		
			−103			
50	2	243				

Thus, at $x = 35$, $p = \dfrac{35-30}{10} = 0.5$, $y_0 = 439$, $\Delta y_0 = -93$, $\Delta y_{-1} = -73$,

$\Delta^2 y_{-1} = -20$, $\Delta^3 y_{-1} = 10$, $\Delta^3 y_{-2} = -35$ and $\Delta^4 y_{-2} = 145$

Substituting the above values in Eqn (7.17) (Stirling's formula), we get

$$y_{0.5} = 435 + (0.5)\left[\frac{(-93)+(-73)}{2}\right] + \frac{(0.5)^2}{2!}(-20)$$

$$+ \frac{(0.5)[(0.5)^2-1]}{3!}\cdot\left[\frac{10+(-35)}{2}\right] + \frac{(0.5)^2[(0.5)^2-1]}{4!}(145)$$

$$= 435 - 41.5 - 2.5 + 0.78125 - 1.1328125$$

$$= 390.64844$$

$$\therefore y_{35} \simeq 390.648$$

Example 7.4 Apply Bessel's formula to obtain y_{25} given that $y_{20} = 2854$, $y_{24} = 3162$, $y_{28} = 3544$, $y_{32} = 3992$. *(Mysore B.E., 1987)*

Solution Taking $x_0 = 24$, $p = \dfrac{x-24}{4}$, where $h = 4$. The central difference table is

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$
20	−1	2854			
			308		
24	0	3162		74	
			382		−8
28	1	3544		66	
			448		
32	2	3992			

From the table we get the following values:

$$x = 25,\ p = \frac{25-24}{4} = 0.25,\ y_0 = 3162,\ \Delta y_0 = 382$$

$\Delta^2 y_{-1} = 74,\ \Delta^2 y_0 = 66,\ \Delta^3 y_{-1} = -8.$

Putting the above values in Eqn (7.23) (Bessel's formula) we get the following equation.

$$y_{0.25} = 3162 + (0.25)(382) + \frac{(0.25)(0.25-1)}{2!}\left\{\frac{74+66}{2}\right\}$$

$$+ \frac{(0.25-0.5)(0.25)(0.25-1)}{3!}(-8)$$

$$= 3162 + 95.5 - 0.65625 - 0.0625 = 3256.7813$$

$\therefore y_{25} = 3256.7813$

Example 7.5 Use Laplace–Everett's formula to obtain $f(1.15)$ given that $f(1) = 1.000, f(1.10) = 1.049, f(1.20) = 1.096\ f(1.30) = 1.140$

Solution Let $x_0 = 1.10, h = 0.1, p = \frac{x - x_0}{h} = \frac{x - 1.10}{0.1}$

The central difference table is

x	p	$y_p = f(x)$	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$
1	−1	1.000			
			0.49		
1.10	0	1.049		−0.002	
			0.047		−0.001
1.20	1	1.096		−0.003	
			0.044		
1.30	2	1.140			

Laplace–Everett's formula is

$$y_p = qy_0 + \frac{q(q^2-1^2)}{3!}\Delta^2 y_{-1} + \cdots + py_1 + \frac{p(p^2-1^2)}{3!}\Delta^2 y_0 + \cdots$$

Here, $p = \frac{1.15-1.10}{0.1} = 0.5$ since $x = 1.15$

$\therefore q = 1-p = 1-0.5 = 0.5,\ y_0 = 1.049,\ y_1 = 1.096,\ \Delta^2 y_{-1} = 0.002,$
$\Delta^2 y_0 = -0.003$

$$\therefore y_{0.5} = (0.5)(1.049) + \frac{(0.5)(0.25-1)}{3!}(-0.002) + (0.5)(1.096)$$

$$+ \frac{(0.5)(0.25-1)}{3!}(0.003)$$

$$= 0.5245 + 1.25 \times 10^{-4} + 0.548 + 1.875 \times 10^{-4}$$

$$= 1.0728125$$

$$\therefore f(1.15) = 1.0728125$$

EXERCISE 7.1

1. The values of annuities for certain ages are given for the following ages. Find the annuity at age 27½ using Gauss's forward interpolation formula.

Age	25	26	27	28	29
Annuity	16.195	15.919	15.630	15.326	15.006

2. Using Gauss's forward interpolation formula, find y at $x = 1.7489$ given that

x	1.72	1.73	1.74	1.75	1.76	1.77	1.78
y	0.1791	0.1773	0.1775	0.1738	0.1720	0.1703	0.1686

3. Find $\sqrt{12516}$ using Gauss's backward interpolation foumula given that $\sqrt{12500} = 111.8033$, $\sqrt{12510} = 111.8481$, $\sqrt{12520} = 111.8928$ and $\sqrt{12530} = 111.9374$.

4. Find sin 45° using Gauss's backward interpolation formula given that sin 20° = 0.342, sin 30° = 0.502, sin 40° = 0.642, sin 50° = 0.766, sin 60°= 0.866, sin 70° = 0.939, sin 80° = 0.984.

5. Employ Bessel's formula to find the value of y at $x = 1.95$ given that

x	1.7	1.8	1.9	2.0	2.1	2.2	2.3
y	2.979	3.144	3.283	3.391	3.463	3.997	4.491

6. Use Bessel's formula to find the value of y when $x = 3.75$ given the table:

x	2.5	3.0	3.5	4.0	4.5	5.0
y	24.145	22.043	20.225	18.644	17.262	16.047

7. Given cos (0.8050) = 0.6931, cos (0.8055) = 0.6928, cos (0.8060) = 0.6924, cos (0.8065) = 0.6920, cos (0.8070) = 0.6917, cos (0.8075) = 0.6913, and cos (0.8080) = 0.6909, find cos (0.806595) using Stirling's formula.

8. Apply Stirling's formula to find a polynomial of degree four which takes

x	1	2	3	4	5
y	1	–1	1	–1	1

9. Use Laplace–Everett's formula to find log 337.5 given that log 310 = 2.4913, log 320 = 2.5051, log 330 = 2.5185, log 340 = 2.5315, log 350 = 2.5441 and log 360 = 2.5563.

10. Find y at $x = 34$ using Laplace – Everett's formula given the table.

x	20	25	30	35	40
y	11.4699	12.7834	13.7648	14.4982	15.0463

11. The following table gives the values of the probability integral $f(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}dx$ for certain values of x. Find the value of their integral at $x = 0.5437$ using (i) Stirling's formula, (ii) Bessel's formula, and (iii) Everett's formula.

x	0.51	0.52	0.53	0.54	0.55
y	0.5292437	0.5378987	0.5464641	0.5549392	0.5633233

x	0.56	0.57
y	0.5716157	0.5798158

ANSWERS

1. 15.480
2. 0.1739
3. 111.8749
4. 0.707
5. 3.347
6. 19.4074
7. 0.6919
8. ½ $\{2(x-3)^4 - 8(x-3)^2 + 3\}$
9. 2.5283
10. 14.3684
11. 0.55805196, 0.55805196, 0.55805195.