Separable Variables

In this chapter we will discuss the methods for the solution of 1st order differential equations. The first order diff eq can be characterized by the type and method of solution.

1. Separable Variables
2. Homogeneous diff eqs
3. Exact diff eqs
4. Linear diff eqs
5. Bernoulli diff eq.

SEPARABLE EQUATION

A differential equation of the form

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

is said to be separable or to have separable variables.

Method of Solution

$$h(y)\, dy = g(x)\, dx$$

Integrate both sides to get rid of differentials

$$\int h(y)\, dy = \int g(x)\, dx$$

$$y = f(x) + c.$$

Q $$\frac{dx}{dy} = \frac{1+2y^2}{y\sin x}$$

$$\int \sin x \, dx = \int \frac{1+2y^2}{y} dy$$

$$-\cos x = \int \frac{1}{y} dy + \int 2y \, dy$$

$$-\cos x = \ln y + y^2 + c$$

↓

General solution.

Q $$y \ln x \frac{dx}{dy} = \left(\frac{y+1}{x}\right)^2$$

$$x^2 \ln x \, dx = \frac{(y+1)^2}{y} dy$$

$$\underbrace{\int x^2 \ln x \, dx}_{\text{Integration by parts}} = \int \frac{y^2+2y+1}{y} dy$$

Let $\ln x = t \Rightarrow \frac{1}{x} dx = dt \Rightarrow dx = e^t dt$

$x = e^t$

$$\int e^{3t} t \, dt = t \frac{e^{3t}}{3} - \int \frac{e^{3t}}{3} dt = t\frac{e^{3t}}{3} - \frac{1}{9} e^{3t}$$

$$= \ln x \frac{x^3}{3} - \frac{1}{9} x^3$$

$$\frac{x^3}{3} \ln x - \frac{x^3}{9} = \int y \, dy + \int 2 \, dy + \int \frac{1}{y} dy$$

$$\frac{x^3}{3} \ln x - \frac{x^3}{9} = y^2/2 + 2y + \ln y + c$$

Q $$\frac{dP}{dt} = P - P^2$$

$$\int \frac{1}{P-P^2}\, dP = \int dt$$

$$\underbrace{\int \frac{1}{P(1-P)}\, dP}_{} = t + c$$

Using partial fraction for integrating

$$\frac{1}{P(1-P)} = \frac{A}{P} + \frac{B}{1-P} \Rightarrow \int \frac{1}{P(1-P)}\, dP = \int \frac{1}{P}\, dP + \int \frac{1}{1-P}\, dP$$

$$1 = A(1-P) + B(P) \qquad = \ln P - \ln|1-P|$$

$$\boxed{1 = B} \qquad = \ln\left(\frac{P}{1-P}\right)$$

$$\boxed{1 = A}$$

$$\boxed{\ln\left(\frac{P}{1-P}\right) = t + c}$$

$$e^{\ln\left(\frac{P}{1-P}\right)} = e^{t+c}$$

$$\frac{P}{1-P} = e^t e^c$$

$$\boxed{\frac{P}{1-P} = c_1 e^t}$$

Q $$\frac{dy}{dx} = \frac{xy + 2y - x - 2}{xy - 3y + x - 3}$$

$$\frac{dy}{dx} = \frac{xy - x + 2y - 2}{xy + x - 3y - 3}$$

$$= \frac{x(y-1) + 2(y-1)}{x(y+1) - 3(y+1)}$$

$$\frac{dy}{dx} = \frac{(y-1)(x+2)}{(y+1)(x-3)}$$

$$\int \frac{y+1}{y-1}\, dy = \int \frac{x+2}{x-3}\, dx$$

$$y-1 \overline{\big)\, y+1\,}^{\;1} \quad \pm y \mp 1 \quad 2$$

$$\int 1 + \frac{2}{y-1}\, dy = \int 1 + \frac{5}{x-3}\, dx$$

$$x-3 \overline{\big)\, x+2\,}^{\;1} \quad \pm x \mp 3 \quad 5$$

$$\int dy + 2\int \frac{1}{y-1}\, dy = \int dx + 5\int \frac{1}{x-3}\, dx$$

$$y + 2\ln(y-1) = x + 5\ln(x-3)$$

$$y + \ln(y-1)^2 = x + \ln(x-3)^5$$

$$y - x = \ln(x-3)^5 - \ln(y-1)^2 + c$$

$$y - x = \ln\left(\frac{(x-3)^5}{(y-1)^2}\right) + c$$

$$e^{y-x} = \frac{(x-3)^5}{(y-1)^2}\, e^{c} \Rightarrow e^{y-x} = \frac{(x-3)^5}{(y-1)^2}\, c$$

Q Solve the differential equation subject to the indicated initial condition.

$$\frac{dx}{dy} = 4(x^2+1) \quad , \quad x(\pi/4) = 1$$

$$\frac{1}{x^2+1}\, dx = 4\, dy$$

$$\int \frac{1}{1+x^2}\, dx = \int 4\, dy$$

$$\tan^{-1}x = 4y + c \longrightarrow \text{General Solution}$$

Using initial conditions $x = 1$ at $y = \pi/4$

$$\boxed{\tan^{-1} 1 = 4\, \pi/4 + c} \longrightarrow \text{~~General Solution~~}$$

$$\pi/4 = \cancel{4}\, \pi/\cancel{4} + c$$

$$c = \pi/4 - \pi = \frac{\pi - 4\pi}{4} = -3\pi/4$$

$$\boxed{\tan^{-1}x = 4y - 3\pi/4} \longrightarrow \text{Particular Solution.}$$