PROBLEMS

1. Use graphical method to find the approximate root of the following equations:

   (a) $x^3 - x + 1 = 0$, (b) $x^4 + 3x - 1 = 0$, (c) $e^x - 3x = 0$

2. (a) The cubic equation $x^3 - 2x - 5 = 0$ has one real root that is near to $x = 2$. The equation can be rewritten in the following manner:

   (i) $x = \frac{1}{2}(x^3 - 5)$ (ii) $x = \frac{5}{x^2 - 2}$ (iii) $x = (x^2 + 5)^{\frac{1}{3}}$

   Choose the form which satisfies the condition $|\Phi'(x)| < 1$ and find the root correct to 4 dp.

   (b) The cubic equation $x^3 - 3x - 20 = 0$, has one real root that is near to $x_0 = 0.3$. The equation can be rewritten in the following manner:

   (i) $x = \frac{1}{3}(x^3 - 20)$ (ii) $x = \frac{20}{x^2 - 3}$ (iii) $x = \sqrt{3 + \frac{20}{x}}$ (iv) $x = (3x + 20)^{\frac{1}{3}}$

   Choose the form which satisfies the condition $|\Phi'(x)| < I$ and find the root correct to 4 dp. Which of them gives rise to very rapid convergence?

   (c) Given the following variations of the equation, $x^4 + x^2 - 80 = 0$,

   (i) $x = (80 - x^2)^{\frac{1}{4}}$ (ii) $x = \sqrt{80 - x^4}$ (iii) $x = \sqrt{\frac{80}{1 + x^2}}$

   Which of them gives rise to a convergent sequence? Find the real root of the equation correct to 4 dp. Take $x_0 = 3$.

3. (a) To locate the root of $e^{-x} - \cos x = 0$ that is near to 1.29, using iteration, we could rewrite the equation as,

   (i) $x = \cos^{-1}(e^{-x})$ (ii) $x = -\log\cos x = \log\sec x$

   (iii) $x = x - 0.01(e^{-x} - \cos x)$

   Which of these three forms (if any) would yield a convergence iteration scheme? Which would converge the fastest?

(c) Determine which of the following iterative functions, $\Phi(x)$, can be used to locate the zeros of the equation $x^3 + 2x - 1 = 0$ on the interval $\left[\frac{1}{4}, \frac{1}{2}\right]$:

(i) $\frac{1}{2}\left(1-x^3\right)$ (ii) $\frac{(1-2x)}{x^2}$ (iii) $\frac{x^3}{(1-x)}$ (iv) $(1-2x)^{\frac{1}{3}}$

(v) $x - 0.2(x^3 + 2x - 1)$ (vi) $\frac{x^3 + 2x - 1}{3x^3 + 2}$

(d) (i) Starting at $x_0 = 0$, use the simple iterative method to find the first five approximations for the solution of $x^4 - x + 0.12 = 0$.

(ii) Starting at $x_0 = 1$, use the simple iterative method to find the first five approximations for the solution of $x^3 - x\sqrt{x} - 4 = 0$.

(iii) Compute a solution, correct to 6 dp, of $e^{-3x} - \cot x = 0$ by Newton's method starting at $x_0 = 1$.

(iv) Compute a solution, correct to 6 dp, of $x^3 - x \sin x = 0$ by Newton's method starting at $x_0 = 1$.

(v) Find a rearrange of the equation $e^x - 3x - 1 = 0$, which will converge to the unique positive root when the simple iterative method is applied. Take $x_0 = 2$.

(e) The cubic equation $2x^3 + 3x^2 - 3x - 5 = 0$ has a root near $x = 1.25$. Show that the equation can be rearranged into any of the following three forms suitable for the simple (fixed-point) iterative method:

(i) $$x = \left\{\frac{(5 - 3x - 3x^2)}{2}\right\}^{\frac{1}{3}}$$

(ii) $$x = ((5+32)/(22+3))^{\frac{1}{2}}$$

(iii) $$x = \frac{(2x^3 + 3x^2 - 5)}{3}$$

Use simple iterative method on the rearranged equation (i) with an initial guess of $x_0 = 1.2$ in order to find the root to 4 dp.

Repeat part (b) for the rearrangement (ii) using $x_0 = 1.2$. Which method converges faster? Why?

Try a few iterations using rearrangement(iii). What goes wrong?

4. Use Newton-Raphson method to obtain a root of each of the following equations correct to 3 dp:

(a)	$x^3 - 2x + 2 = 0;$	with $x_0 = 0.2$
(b)	$x^3 - 3x - 3 = 0;$	with $x = 2$
(c)	$x^6 - x - 1 = 0;$	with $x_0 = 0.5$
(d)	$\sin x - 5x + 2 = 0;$	with $x_0 = 0.4$
(e)	$\cos x - x = 0;$	with $x_0 = 0.74$
(f)	$e^{-x} - x = 0;$	with $x_0 = 0$
(g)	$e^x - 3x^2 = 0;$	with $x_0 = 1$
(h)	$\sin x - x + 1 = 0;$	with $x_0 = 1.5$
(i)	$\tan x - 0.5x = 0;$	with $x_0 = 4.0$
(j)	$x^2 = e^x;$	with $x_0 = -1$
(k)	$x^4 + x^2 = 80;$	with $x_0 = 3$
(l)	$x \sin x = 1;$	with $x_0 = 1.11$
(m)	$x \operatorname{lin} x = 3;$	with $x_0 = 2$
(n)	$x^3 - 2x^2 + x - 3;$	with $x_0 = 4$

5. (a) By applying Newton-Raphson method to the function defined by $f(x) = 1 - \frac{10}{x^2}$, develop an iterative formula for calculating $\sqrt{10}$. Hence, using 2 as an initial approximation to $\sqrt{10}$, calculate $\sqrt{10}$ correct to 2 dp. Show that if x_n, the nth approximation to $\sqrt{10}$, has a small error e_n, then the correct approximation e_{n+1} has an error of magnitude about $0.5e_n^2$.

(b) Use the following iterative formula for $\frac{1}{\sqrt{a}}$ to find $\frac{1}{\sqrt{5}}$ to 4 dp:

$$x_{n+1} = \frac{1}{2} x_n (3 - a x_n^2).$$

(c) Show that the curve $f(x) = x^3 - 2x - 1$ crosses the x-axis between $x = 1$ and $x = 2$. Use a recurrence relation of the form,

$$x_{n+1} = x_n - \frac{f(x_n)}{m}$$

where (i) $m = 5$ and (ii) $m = f'(x) = 3x^2 - 2$, to find the value of the root to 3 dp. Take $x_0 = 2$ in both cases.

7. Use bisection method to find correct to 4 dp, the solutions of the following equations:

(a) $\sin x - \frac{1}{2}x = 0$; in the interval $\left(\frac{\pi}{2}, \pi\right)$.

(b) $x^3 - 9.0x + 1.0 = 0$; $x_1 = 2$, $x_2 = 4$.

(c) $9x^3 + 4x^2 + 5x - 8 = 0$; $x_1 = -5$, $x_2 = 5$.

(d) $8x^3 + 8x - 5 = 0$; $x_0 = 0.3$, $x_1 = 0.6$

(e) $x \sin x - 1 = 0$; $x_0 = 0$, $x_1 = 2.0$

8. Use secant method to find, correct to 4 dp, the solutions of the following equations:

(a) $x^3 - 9x + 1 = 0$; $x_0 = 3$, and $x_1 = 4$

(b) $\sin x - 5x + 2 = 0$; $x_0 = 0.4$, and $x_1 = 0.5$

(c) $x^3 - 5 = 0$; $x_0 = 0$, and $x_1 = 3.0$

(d) $x^3 = x - 2$ $x_0 = 2.6$ and $x_1 = -2.4$

(e) $x^3 - 3.23x^2 - 5.54x + 9.84 = 0$; $x_0 = 0.9$ and $x_1 = 1.0$

9. Use Regula Falsi method to find, correct to 4 dp, the solutions of the following equations:

(a) $x^6 = x + 1$; $x_0 = 1$, $x_1 = 1.2$

THEOREM : If α be a root of $f(x) = 0$ which is equivalent to $x = \phi(x)$, I, be any interval containing the point $x = \alpha$ and $|\phi'(x)| < 1 \ \forall \ x \in I$, then the sequence of approximations $x_0, x_1, x_2, \ldots, x_n$ will converge to the root α provided the initial approximation x_0 is chosen in I.

PROOF : Since α is a root of $x = \phi(x)$, we have

$$\alpha = \phi(\alpha)$$

If x_{n-1} and x_n be two successive approximations to α, we have

$$x_n = \phi(x_{n-1})$$

$$\therefore \quad x_n - \alpha = \phi(x_{n-1}) - \phi(\alpha) \tag{3.2}$$

By Mean Value theorem,

$$[\phi(x_{n-1}) - \phi(\alpha)]/[x_{n-1} - \alpha] = \phi'(\xi), \text{ where } x_{n-1} < \xi < \alpha$$

Hence, Eqn (3.2) becomes

$$x_n - \alpha = (x_{n-1} - \alpha)\,\phi'(\xi) \tag{3.3}$$

Let k be the maximum absolute value of $\phi'(x)$ over the interval I. Then from Eqn (3.3),

$$|x_n - \alpha| \le k\,|x_{n-1} - \alpha| \tag{3.4}$$

Similarly, $$|x_{n-1} - \alpha| \le k\,|x_{n-2} - \alpha|$$

$$\therefore \quad |x_n - \alpha| \le k^2\,|x_{n-2} - \alpha|$$

Proceeding on,

$$|x_n - \alpha| \le k^n\,|x_0 - \alpha| \tag{3.5}$$

Now, if $k < 1$ over the entire interval, as n increases the RHS of Eqn (3.5) becomes small and therefore,

$$\underset{n\to\infty}{Lt}|x_n - \alpha| = 0, \quad \text{i.e. } \underset{n\to\infty}{Lt}\, x_n = \alpha$$

That is, the sequence of approximations converges to α if $k < 1$,

i.e. $$|\phi'(x)| < 1 \ \forall \ x \in I.$$

Note 1) Smaller the value of $\phi'(x)$, more rapid will be the convergence.
2) This method of iteration is particularly useful for finding the real roots

3.5 NEWTON'S ITERATION METHOD

This method, also known as *Newton-Raphson method* and is a particular form of the iteration method discussed in Section 3.3. When an approximate value of a root of an equation is given, a better and closer approximation to the root can be found using this method.

It can be derived as follows :

Let x_0 be an approximation of a root of the given equation $f(x) = 0$, which may be algebraic or transcendental.

Let $x_0 + h$ be the exact value or the better approximation of the corresponding root, h being a small quantity. Then $f(x_0 + h) = 0$.

Expanding it by Taylor's theorem, we get

$$f(x_0 + h) = f(x_0) + h f' x_0 + h^2/2!\, f''(x_0) + \ldots = 0$$

Since h is small, we can neglect second, third and higher degree terms in h and thus we get

$$f(x_0) + h f'(x_0) = 0$$

or $$h = -\frac{f(x_0)}{f'(x_0)};\ f'(x_0) \neq 0$$

Hence, $$x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Now substituing x_1 for x_0 and x_2 for x_1 the next better approximations are given by $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$ and $x_3 = x_2 - \dfrac{f(x_n)}{f'(x_n)}$

Proceeding in the same way n times, we get the general formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \text{ for } n = 0; 1, 2, \ldots \tag{3.8}$$

which is known as Newton–Raphson formula.

3.7 CONVERGENCE OF NEWTON–RAPHSON METHOD

In this section, we will see the condition for convergence of Newton–Raphson method. Newton–Raphson formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

is an iteration method where

$$x_{n+1} = \phi(x_n)\,;\ \phi(x_n) = x_n - \frac{f(x)}{f'(x)}$$

In general, $x = \phi(x)$, where $\phi(x) = x - \dfrac{f(x_n)}{f'(x_n)}$.

We know that the iteration method converges if

$$|\phi'(x)| < 1, \text{ i.e. } \left|1 - \frac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2}\right| < 1$$

or
$$\left|\frac{f(x)f''(x)}{[f'(x)]^2}\right| < 1$$

i.e.
$$|f(x)\, f''(x)| < [f'(x)]^2$$

The interval containing the root α of $f(x) = 0$ should be selected in which the above is satisfied.

3.8 RATE OF CONVERGENCE OF NEWTON–RAPHSON METHOD

Let x_n and x_{n+1} be two successive approximations to the actual root α of $f(x) = 0$. If ε_n and ε_{n+1} are the corresponding errors, we have

$$x_n - \alpha = \varepsilon_n \text{ and } x_{n+1} - \alpha = \varepsilon_{n+1}$$

$$\therefore \varepsilon_{n+1} - \varepsilon_n = x_{n+1} - x_n$$

$$= x_{n+1} + x_n$$

$$= \frac{f(x_n)}{f'(x_n)} \quad \text{(using Newton–Raphson formula)}$$

$$= \frac{f(\alpha+\varepsilon_n)}{f'(\alpha+\varepsilon_n)}$$

$$= -\frac{f(\alpha)+\varepsilon_n f'(\alpha)+1/2[\varepsilon_n^2 f''(\alpha)]+\cdots}{f'(\alpha)+\varepsilon_n f''(\alpha)+1/2[\varepsilon_n^2 f'''(\alpha)]+\cdots}$$

(by Taylor's Theorem)

$$= -\frac{\varepsilon_n f'(\alpha)+1/2[\varepsilon_n^2 f''(\alpha)]+\cdots}{f'(\alpha)+\varepsilon_n^2 f''(\alpha)+\cdots} \quad (\because f(\alpha)=0)$$

$$\therefore \varepsilon_{n+1} = \varepsilon_n - \frac{\varepsilon_n f'(\alpha)+1/2[\varepsilon_n^2 f''(\alpha)]}{f'(\alpha)+\varepsilon_n f''(\alpha)}$$

(by omitting derivatives of order higher than two)

$$= \frac{1/2\varepsilon_n^2 f''(\alpha)}{f'(\alpha)+\varepsilon_n f''(\alpha)}$$

$$= \frac{\varepsilon_n^2 f''(\alpha)}{2f'(\alpha)}\left[1+\frac{\varepsilon_n f''(\alpha)}{f'(\alpha)}\right]^{-1} = \frac{\varepsilon_n^2 f''(\alpha)}{2f'(\alpha)}$$