Q1:

(A) Find the minimum no. of iterations needed by the bisection algorithm to approximate the root $x=3$ of $x^3-6x^2+11x-6=0$

Sol: As, we know that:

$$\frac{b-a}{2^n} \leq \varepsilon$$

$$\Rightarrow \quad 2^n \geq \frac{b-a}{\varepsilon}$$

Taking 'log' on both sides

$$\log 2^n \geq \log\left(\frac{b-a}{\varepsilon}\right)$$

$$n \log 2 \geq \log(b-a) - \log \varepsilon$$

$$\Rightarrow \quad \boxed{n \geq \frac{\log(b-a) - \log \varepsilon}{\log 2}}$$

So, this is the general formula to findout the minimum no. of iterations.

Let $a = 2.9$, $b = 3.1$, $\varepsilon = 0.001$

the

$$\text{minimum iterations} \geq \frac{\log(3.1-2.9) - \log(0.001)}{\log 2}$$

$$\approx 8$$

(B) Develop Newton's method to compute the following.

(a) $\ln(a)$, $a>0$

Sol

Let $\quad x = \ln a$

$$\Rightarrow \quad e^x = a$$

$$\Rightarrow \quad e^x - a = 0$$

So,

$$f(x) = e^x - a$$

$f'(x) = e^x$

By using Newton formula.

$$\boxed{x_{n+1} = x_n - \frac{e^{x_n} - a}{e^{x_n}}}$$

(b) arc Cos a

$x = \cos^{-1} a$

$\Rightarrow \cos x = a$

$\Rightarrow \cos x - a = 0$

So,

$f(x) = \cos x - a$

$\Rightarrow f'(x) = -\sin x$

By using Newton formula.

$$\boxed{x_{n+1} = x_n + \frac{\cos x - a}{\sin x}}$$

(C) Solution.

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT \quad \text{——} \; ①$$

where $P = 1$, $a = 3.592$, $b = 0.04267$

$R = 0.082054$, $T = 300$

Put all these values in Eq ①, We get

$$\left(\frac{V^2 + 3.592}{V^2}\right)(V - 0.04267) - 24.6162 = 0$$

So,

$$f(V) = \left(1 + \frac{3.592}{V^2}\right)(V - 0.04267) - 24.6162$$

$$f'(V) = \left(1 + \frac{3.592}{V^2}\right) + (V - 0.04267)\left(\frac{-0.7184}{V^3}\right)$$

$$V_0 = \frac{RT}{P} = 24.6162$$

$$V_1 = V_0 - \frac{f(V_0)}{f'(V_0)}$$

$= 24.6162 - \frac{0.1029}{0.9941}$

$\boxed{V_1 = 24.5126}$

$V_2 = V_1 + \frac{f(V_1)}{f'(V_1)}$

$= 24.5126 - \frac{0.00001}{1.0048}$

$\boxed{V_2 = 24.5125}$

So, volumn is 24.512.

(D) Find 3rd approximate tue root of the eq. $x^3 - 4x - 9 = 0$ using fixed point iteration process.

Sol

$x^3 - 4x - 9 = 0$

$f(2) < 0, \quad f(3) > 0$

$4x = x^3 - 9$	$x^3 = 4x + 9$
$x = \frac{x^3 - 9}{4}$	$x = (4x+9)^{1/3}$
$x' = \frac{3x^2}{4}$	$x' = \frac{4}{3}(4x+9)^{-2/3}$
$\lvert F'(x) \rvert < 1$	$\lvert F'(x) \rvert < 1$

So,. $F(x) = (4x+9)^{1/3}$

take $x_0 = 2.7$

$x_1 = F(x_0)$

$x_1 = 2.70534$

$x_2 = F(x_1)$

$x_2 = 2.70631$

$x_3 = F(x_2)$

$x_3 = 2.70648$

Q2:

$$\begin{bmatrix} -1 & 3 & 1 \\ 2 & 2 & 5 \\ 4 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 5 \end{bmatrix}$$

Sol

Gauss Jacobi's Method

Step 1:

Firstly, we make the system diagonally dominant.

* $4x_1 + x_2 - x_3 = 5$ —— ①

$-x_1 + 3x_2 + x_3 = -4$ —— ②

$2x_1 + 2x_2 + 5x_3 = 1$ —— ③

Step 2: Taking $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 0$

Now, we separate x_1, x_2 & x_3 from Eq ①, ② & ③

Iteration 1: ← $x_1^{(1)} = 5/4$

$x_2^{(1)} = -4/3$

$x_3^{(1)} = 1/5$

~~Iteration 2:~~ *

$$x_1 = \frac{5 - x_2 + x_3}{4}$$

$$x_2 = \frac{-4 + x_1 - x_3}{3}$$

$$x_3 = \frac{1 - 2x_1 - 2x_2}{5}$$

Iteration 2:

$x_1^{(2)} = 1.6333$

$x_2^{(2)} = -0.9833$

$x_3^{(2)} = 0.2333$

By using Gauss-Seidel method.

$x_1^{(2)} = 1.6333$

$x_2^{(2)} = -0.9889$

$x_3^{(2)} = 0.2333$

$$x_1^{(3)} = \frac{5+(0.9889)+0.2333}{4}$$

$$\boxed{x_1^{(3)} = 1.5556}$$

$$x_2^{(3)} = \frac{-4+1.5556-0.2333}{3}$$

$$\boxed{x_2^{(3)} = -0.89257}$$

$$x_3^{(3)} = \frac{1-2(1.5556)+2(0.89257)}{5}$$

$$\boxed{x_3^{(3)} = -0.065212}$$

Q3 (A)

Solve the system given in Q2 using Doolittle Method.

Sol

$$\begin{bmatrix} -1 & 3 & 1 \\ 2 & 2 & 5 \\ 4 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 5 \end{bmatrix}$$

$$A \quad X = B$$

Let

$$A = LU \text{ s.t}$$

$$\begin{bmatrix} -1 & 3 & 1 \\ 2 & 2 & 5 \\ 4 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\begin{bmatrix} -1 & 3 & 1 \\ 2 & 2 & 5 \\ 4 & 1 & -1 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix}$$

$\boxed{u_{11}=-1}$, $\boxed{u_{12}=3}$, $\boxed{u_{13}=1}$

$l_{21}u_{11}=2$, $l_{21}u_{12}+u_{22}=2$

$\boxed{l_{21}=-2}$ $\quad -6+u_{22}=2$

$\boxed{u_{22}=8}$

$l_{21}u_{13}+u_{23}=5$

$-2+u_{23}=5$, $l_{31}u_{12}+l_{32}u_{22}=1$

$\boxed{u_{23}=7}$ $\quad -12+8l_{32}=1$

$\boxed{l_{32}=\frac{13}{8}}$

$l_{31}u_{11}=4$

$\boxed{l_{31}=-4}$

$l_{31}u_{13}+l_{32}u_{23}+u_{33}=-1$

$-4+\frac{13}{8}\times 7+u_{33}=-1$

$u_{33}=3-\frac{91}{8}$

$\boxed{u_{33}=\frac{-67}{8}}$

$A=LU$

$$\begin{bmatrix} -1 & 3 & 1 \\ 2 & 2 & 5 \\ 4 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -4 & 13/8 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 1 \\ 0 & 8 & 7 \\ 0 & 0 & -67/8 \end{bmatrix}$$

So,

$$AX = B$$

$$LUX = B$$

let $UX = Z$ ——— *)

$$LZ = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -4 & 13/8 & 1 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 5 \end{bmatrix}$$

$$\boxed{Z_1 = -4}$$

$$-2Z_1 + Z_2 = 1$$

$$8 + Z_2 = 1$$

$$\boxed{Z_2 = -7}$$

$$-4Z_1 + \frac{13}{8} Z_2 + Z_3 = 5$$

$$16 - \frac{91}{8} + Z_3 = 5$$

$$\boxed{Z_3 = 3/8}$$

put the value of Z_1, Z_2 & Z_3 in Eq *)

$$\begin{bmatrix} -1 & 3 & 1 \\ 0 & 8 & 7 \\ 0 & 0 & -67/8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \\ 3/8 \end{bmatrix}$$

$$-\frac{67}{8}x_3 = \frac{3}{8}$$

$$\boxed{x_3 = \frac{-3}{67}}$$

$$8x_2 = -7 - 7x_3$$
$$= \frac{-448}{67}$$

$$\boxed{x_2 = \frac{-56}{67}}$$

$$-x_1 + 3x_2 + x_3 = -4$$
$$-x_1 = -4 - 3x_2 + x_3$$
$$-x_1 = \frac{-97}{67}$$

$$\boxed{x_1 = \frac{97}{67}}$$

(B)

Solve the following system using Gauss Elimination with partial pivoting method

$$\begin{bmatrix} 4 & 10 & 8 \\ 10 & 26 & 26 \\ 8 & 26 & 61 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 44 \\ 128 \\ 214 \end{bmatrix}$$

Sol.

$$\left[\begin{array}{ccc:c} 4 & 10 & 8 & 44 \\ 10 & 26 & 26 & 128 \\ 8 & 26 & 61 & 214 \end{array}\right]$$

Partial Pivoting

$$\approx \left[\begin{array}{ccc:c} 10 & 26 & 26 & 128 \\ 4 & 10 & 8 & 44 \\ 8 & 26 & 61 & 214 \end{array}\right]$$

$$\approx \left[\begin{array}{ccc:c} 10 & 26 & 26 & 128 \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & b_2^{(2)} \\ 0 & a_{32}^{(2)} & a_{33}^{(2)} & b_3^{(2)} \end{array}\right]$$

$$a_{22}^{(2)} = a_{22}^{(1)} - \left[\frac{a_{21}^{(1)}}{a_{11}^{(1)}}\right] a_{12}^{(1)}$$

$$= 10 - \left(\frac{4}{10}\right)(26)$$

$$\boxed{a_{22}^{(2)} = -2/5}, \quad \boxed{a_{23}^{(2)} = \frac{-12}{5}}, \quad \boxed{a_{32}^{(2)} = \frac{26}{5}}$$

$$b_2^{(2)} = b_2^{(1)} - \left[\frac{4}{10}\right](128)$$

$$\boxed{b_2^{2} = -36/5}, \quad \boxed{a_{33}^{(2)} = 201/5}, \quad \boxed{b_3^{(2)} = \frac{558}{5}}$$

$$\approx \left[\begin{array}{ccc:c} 10 & 26 & 26 & 128 \\ 0 & -2/5 & -12/5 & -36/5 \\ 0 & 26/5 & 201/5 & 558/5 \end{array}\right]$$

Partial Pivoting:

$$\approx \left[\begin{array}{ccc:c} 10 & 26 & 26 & 128 \\ 0 & 26/5 & 201/5 & 558/5 \\ 0 & -2/5 & -12/5 & -36/5 \end{array}\right]$$

$$\approx \left[\begin{array}{ccc:c} 10 & 26 & 26 & 128 \\ 0 & 26/5 & 201/5 & 558/5 \\ 0 & 0 & a_{33}^{(3)} & b_3^{(3)} \end{array}\right]$$

$$\boxed{a_{33}^{(3)} = 9/13}, \quad \boxed{b_3^{(3)} = 18/13}$$

$$\approx \left[\begin{array}{ccc:c} 10 & 26 & 26 & 128 \\ 0 & 26/5 & 201/5 & 558/5 \\ 0 & 0 & 9/13 & 18/13 \end{array}\right]$$

$$\frac{9}{13} x_3 = \frac{18}{13} \Rightarrow \boxed{x_3 = 2}$$

$$\frac{26}{5} x_2 + \frac{201}{5} \times 2 = \frac{558}{5} \Rightarrow \boxed{x_2 = 6}$$

$$10 x_1 + 26(6) + 26(2) = 128 \Rightarrow \boxed{x_1 = -8}$$