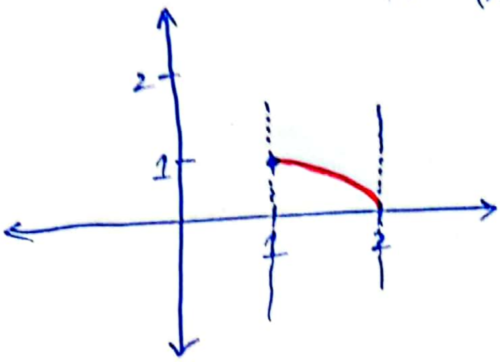
Q1 :

Solution :

$$y = \sqrt{2x - x^2}$$

$$\Rightarrow y^2 = 2x - x^2$$

$$\Rightarrow (x-1)^2 + y^2 = 1$$



The right-half of ~~semi circle~~ quarter circle of radii 1 centered at (1,0)

$$\int_1^2 \int_0^{\sqrt{2x-x^2}} \frac{1}{(x^2+y^2)^2}\, dy\,dx = \int_0^{\frac{\pi}{4}} \int_{\frac{1}{\cos\theta}}^{2\cos\theta} \frac{r}{r^4}\, dr\,d\theta \longrightarrow 5$$

$$= \int_0^{\frac{\pi}{4}} \int_{\frac{1}{\cos\theta}}^{2\cos\theta} \frac{1}{r^3}\, dr\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left[\frac{-1}{2r^2} \Big|_{1/\cos\theta}^{2\cos\theta} \right] d\theta$$

$$= \frac{\pi}{16} \longrightarrow 5$$

Q2:

Solution:

(a) Rectangular coordinates:—

$$z = \sqrt{x^2+y^2}\ ,\quad x^2+y^2=1 \Rightarrow y=\sqrt{1-x^2}$$

Since we are dealing in the Ist octant, therefore

$$x = y = z = 0$$

$$0 \le z \le \sqrt{x^2+y^2}$$

$$0 \le y \le \sqrt{1-x^2}$$

and $\quad 1-x^2 \geq 0$

$$\Rightarrow -1 \le x \le 1$$

so $\quad 0 \le x \le 1$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{x^2+y^2}} (6+4y)\, dz\, dy\, dx \longrightarrow 5$$

(b) Cylindrical Coordinates:

$$z = \sqrt{x^2+y^2} \quad ; \quad x^2+y^2=1$$

$$\Rightarrow z = r \qquad \Rightarrow r^2 = 1$$

$$\Rightarrow 0 \le z \le r \qquad \Rightarrow r = 1$$

$$\Rightarrow 0 \le r \le 1$$

$$f(x,y,z) = 6+4y$$

$$\Rightarrow f(r,\theta,z) = 6+4r\sin\theta$$

$$\int_0^{\frac{\pi}{2}}\int_0^{1}\int_0^{r} (6+4r\sin\theta)\, r\, dz\,dr\,d\theta \longrightarrow$$ 5

(c) Spherical coordinates:-

$$z = \sqrt{x^2+y^2}$$

$$\Rightarrow \rho\cos\phi = \sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta}$$

$$\Rightarrow \rho\cos\phi = \rho\sin\phi$$

$$\Rightarrow \tan\phi = 1$$

$$\Rightarrow \phi = \frac{\pi}{4}$$

$$\Rightarrow \frac{\pi}{4} \le \phi \le \frac{\pi}{2}$$

$$0 \le \rho \le \cosec\phi$$

and $0 \le \theta \le 2\pi$

$$\int_0^{\pi/2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\cosec\phi} \rho^2\sin(\phi)\left[6+4\rho\sin\phi\sin\theta\right] d\rho\,d\phi\,d\theta$$ 5

d) Let's start evaluating the integral using cylindrical coordinates:

$$\int_0^{\frac{\pi}{2}}\int_0^1\int_0^r (6r + 4r^2\sin\theta)\,dz\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 \left((6r + 4r^2\sin\theta)z \Big|_0^r \right) dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 (6r^2 + 4r^3\sin\theta)\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(2r^3 + r^4\sin\theta \Big|_0^1 \right] d\theta$$

$$= \int_0^{\frac{\pi}{2}} (2 + \sin\theta)\,d\theta$$

$$= \left[2\theta - \cos\theta\right]_0^{\frac{\pi}{2}}$$

$$= \left[\pi - \cos\frac{\pi}{2}\right] - \left[0 - \cos(0)\right]$$

$$= \pi + 1 \longrightarrow 5$$

3. (a). Define a vector field (F), flow integral, circulation around a curve (C) and flux of a vector field (F) across C [4].

Each definition contain one mark.

Vector field:

A vector field, denoted by F, assigns a vector to each point in space. This means that for any point P in space, $F(P)$ gives a vector representing some physical quantity like force, velocity, or electromagnetic field strength, at that point. Mathematically, vector field is denoted as,

$$F = M(x,y,z)i + N(x,y,z)j + P(x,y,z)k$$

Flow and Circulation integral:

DEFINITIONS If $\mathbf{r}(t)$ parametrizes a smooth curve C in the domain of a continuous velocity field $\mathbf{F}$, the **flow** along the curve from $A = \mathbf{r}(a)$ to $B = \mathbf{r}(b)$ is

$$\text{Flow} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds. = \text{F. dr/dt}$$

The integral is called a **flow integral**. If the curve starts and ends at the same point, so that $A = B$, the flow is called the **circulation** around the curve.

Flux:

DEFINITION If C is a smooth simple closed curve in the domain of a continuous vector field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ in the plane, and if $\mathbf{n}$ is the outward-pointing unit normal vector on C, the **flux** of $\mathbf{F}$ across C is

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_C \left(M \frac{dy}{ds} - N \frac{dx}{ds} \right) ds = \oint_C M\, dy - N\, dx.$$

Definition contain three marks and three are of two properties.

(b) Define a conservative field and write its equivalent statements [6].

DEFINITIONS Let $\mathbf{F}$ be a vector field defined on an open region D in space, and suppose that for any two points A and B in D the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ along a path C from A to B in D is the same over all paths from A to B. Then the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **path independent in D** and the field $\mathbf{F}$ is **conservative on D**.

Properties:

(i)

a field $\mathbf{F}$ is conservative if and only if it is the gradient field of a scalar function f—that is, if and only if $\mathbf{F} = \nabla f$ for some f.

(ii)

THEOREM 3—Loop Property of Conservative Fields The following statements are equivalent.

1. $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ around every loop (that is, closed curve C) in D.

2. The field $\mathbf{F}$ is conservative on D.

The following test can also be used as a definition

Component Test for Conservative Fields

Let $\mathbf{F} = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ be a field on an open simply connected domain whose component functions have continuous first partial derivatives. Then, $\mathbf{F}$ is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

(c) $F = (e^x\cos y + yz)\hat{i} + (xz - e^x\sin y)\hat{j} + (xy+z)\hat{k}$

$M = e^x\cos y + yz$, $N = xz - e^x\sin y$ and $P = xy + z$

$$\frac{\partial P}{\partial y} = x = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = y = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = -e^x\sin y + z = \frac{\partial M}{\partial y}$$

$\Rightarrow$ F is conservative. $\longrightarrow$ 3

$\longrightarrow$ Potential Function:

$$\nabla f = F$$

$$\Rightarrow \frac{\partial f}{\partial x} = e^x\cos y + yz, \quad \frac{\partial f}{\partial y} = xz - e^x\sin y, \quad \frac{\partial f}{\partial z} = xy + z$$

Consider $\frac{\partial f}{\partial x} = e^x\cos y + yz$

$$\Rightarrow \int \partial f = \int (e^x\cos y + yz)\, \partial x$$

$$\Rightarrow f = e^x\cos y + xyz + g(y,z)$$

$$\Rightarrow \frac{\partial f}{\partial y} = -e^x\sin y + xz + \frac{\partial g}{\partial y}$$

$$\Rightarrow \cancel{xz - e^x\sin y} = \frac{\partial g}{\partial y} + \cancel{xz - e^x\sin y}$$

$$\Rightarrow \quad \frac{\partial g}{\partial y} = 0$$

$$\Rightarrow \quad g = h(z)$$

$$\Rightarrow \quad f = e^{x}\cos y + xyz + h(z)$$

$$\Rightarrow \quad \frac{\partial f}{\partial z} = xy + \frac{dh}{dz}$$

$$\Rightarrow \quad xy + z = xy + \frac{dh}{dz}$$

$$\Rightarrow \quad \frac{dh}{dz} = z$$

$$\Rightarrow \quad h = \frac{z^2}{2} + c$$

$$\Rightarrow \quad f = e^{x}\cos y + xyz + \frac{z^2}{2} + c \longrightarrow 5$$

d

$F = x^2 i - yj$ ~~along~~ $x = y^2$ from $(4,2)$ to $(1,-1)$.

$$r = xi + yj$$

$$\Rightarrow r = y^2 i + yj \Rightarrow dr = (2yi + j)dy$$

Also

$$F = x^2 i - yj$$

$$\Rightarrow F = y^4 i - yj$$

$$\int_C F \cdot dr = \int_C (y^4 i - yj) \cdot (2yi + j)dy$$

$$= \int_2^{-1} (2y^5 - y)dy$$

$$= -\frac{39}{2}$$

7