Bernoulli's Equation

The differential equation

$$\frac{dy}{dx} + P(x)y = f(x)y^n \quad \text{—— (1)}$$

where n is any real number, is called Bernoulli's Eq. For $n=0$ & $n=1$ equation (1) is linear in y. Now for $y \neq 0$, (1) can be written as

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = f(x) \quad \text{—— (2)}$$

Let $\boxed{w = y^{1-n}}$, then

$$\frac{dw}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

With these substitutions (2) can be simplified to linear equation

$$\frac{dw}{dx} + (1-n)P(x)w = (1-n)f(x) \quad \text{—— (3)}$$

Solving (3) for w and using $y^{1-n} = w$ lead to a solution of (1).

Question 1

$$\frac{dy}{dx} + \frac{1}{x} y = xy^2 \quad \text{——— (1)}$$

$$P(x) = \frac{1}{x}, \quad f(x) = x, \quad n = 2$$

dividing eq (1) with y^2

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}\left(\frac{1}{y}\right) = x \quad \text{——— (2)}$$

Let $\frac{1}{y} = t$

$$-\frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$$

Using above substitution in eq (2)

$$-\frac{dt}{dx} + \frac{1}{x} t = x$$

$$\boxed{\frac{dt}{dx} - \frac{1}{x} t = -x} \quad \text{——— (3)}$$

Now eq (3) is linear in t & x.

$$I.F = e^{\int -1/x \, dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1} = 1/x$$

$$1/x \, dt/dx - 1/x^2 \, t = -1$$

$$\frac{d}{dx}\left[t \, 1/x\right] = -1 \quad \Rightarrow \quad t \cdot 1/x = -x + c$$

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$$t = -x^2 + cx$$

$$\frac{1}{y} = -x^2 + cx$$

$$y = \frac{1}{-x^2 + cx}$$ Solution.

2.4 EXACT DIFFERENTIAL EQUATION

A differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is said to be an exact differential equation if

$$M_y = N_x$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Method of Solution.

$$\int M\,dx + \int \text{Terms of } N \text{ free from } x \; dy = c$$

Q $(e^{2y} - y\cos xy)\,dx + (2xe^{2y} - x\cos xy + 2y)\,dy = 0$

$M \quad dx + \quad N \quad dy = 0$

$M = e^{2y} - y\cos xy$

$M_y = \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(e^{2y} - y\cos xy\right)$

$= 2e^{2y} - \left(y \, \partial/\partial y \cos xy + \cos xy \, \partial/\partial y \, y\right)$

$= 2e^{2y} - y(-x\sin xy) - \cos xy$

$= 2e^{2y} + xy\sin xy - \cos xy$

$N = 2xe^{2y} - x\cos xy + 2y$

$N_x = 2e^{2y} - \left[x\left[-y\sin xy\right] + \cos xy\right]$

$= 2e^{2y} + xy\sin xy - \cos xy.$

As $M_y = N_x$ => given diff eq is exact

For solution

$\int M\,dx + \int \text{Terms of } N \text{ free from } x \; dy = c$

$\int e^{2y} - y\cos xy \; dx + \int 2y \; dy = c$

$e^{2y}x - y\int \cos xy \; dx + y^2 = c$

$$e^{2y}x - y\frac{\sin xy}{y} + y^2 = c$$

$$xe^{2y} - \sin xy + y^2 = c$$

2 $$\left(\frac{1}{1+y^2} + \cos x - 2xy\right)\frac{dy}{dx} = y(y+\sin x), \quad y(0)=1$$

$$\left(\frac{1}{1+y^2} + \cos x - 2xy\right)dy = y(y+\sin x)dx$$

$$y(y+\sin x)\,dx + \left(-\frac{1}{1+y^2} - \cos x + 2xy\right)dy = 0$$

$$M \quad dx + \quad N \quad dy = 0$$

$$M = y(y+\sin x)$$

$$M_y = 2y + \sin x$$

$$N = -\frac{1}{1+y^2} - \cos x + 2xy$$

$$N_x = \sin x + 2y$$

The differential equation is exact. as $M_y = N_x$.

$\int M\,dx + \int$ Terms of N free from x $dy = c$

$$\int (y^2 + y\sin x)\,dx + \int -\frac{1}{1+y^2}\,dy = c$$

$$y^2x - y\cos x - \tan^{-1}y = c$$

A first order differential eq is a linear differential equation if it is or it can be written in the form

Using initial condition $y(0) = 1$

$$-1 - \tan^{-1} 1 = c$$

$$-1 - \pi/4 = c$$

NON EXACT DIFFERENTIAL EQUATION

A differential equation is said to be non-exact if

$$M_y \neq N_x$$

"We can reduce non exact diff eq to exact diff eq by multiplying non exact diff eq. with an Integrating factor."