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Interpolation:- Interpolation is one of the fundamental operation in mathematics. It has been the foundation of classical numerical analysis. Finite difference play an important role in numerical techniques, where tabulated values of the unknown function are available.

Finite Difference Operator:

Forward Difference Operator: For a given table of values (x_k, y_k), where $k = 0, 1, 2, \ldots, n$ of a function $y = f(x)$, with equally spaced abscissas. we define the Forward difference operator Δ (Delta) as follows.

$$\Delta y_i = y_{i+1} - y_i \quad , \quad i = 0, 1, 2, 3, \ldots, (n-1)$$

i.e, we have

$$\Delta y_0 = y_1 - y_0$$

$$\Delta y_1 = y_2 - y_1$$

$$\ldots\ldots\ldots$$

$$\Delta y_{n-1} = y_n - y_{n-1}$$

where Δ is called forward difference operator. also known as first forward difference operator.

(1a) Similarly, the difference of first forward operator are called second forward difference operator and are denoted by

$$\Delta^2 y_0, \Delta^2 y_1, \dots \Delta^2 y_{n-1}$$, Thus in general

$$\Delta^p y_r = \Delta^{p-1} y_{r+1} - \Delta^{p-1} y_r$$

Table

x	y	Δ	Δ^2	Δ^3
x_0	y_0			
		$\Delta y_0 = y_1 - y_0$		
x_1	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$	
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$
x_2	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$	
		$\Delta y_2 = y_3 - y_1$		
x_3	y_3			

$y_0 \Rightarrow$ Leading Term

$\Delta y_0, \Delta^2 y_0, \Delta^3 y_0 \Rightarrow$ Leading differences.

As we know

$$y = f(x)$$

$$\Delta f(x) = f(x+h) - f(x)$$

② Backward Difference Operator.

∇ (del)

$$\nabla y_1 = y_i - y_{i-1} \text{ where } i = n, (n-1), \ldots, 1$$

$$\nabla y_1 = y_1 - y_0$$

$$\nabla y_2 = y_2 - y_1$$

$$- - - - -$$

$$\nabla y_n = y_n - y_{n-1}$$

Second Backward difference

$$\nabla^2 y_2, \nabla^2 y_3, \ldots, \nabla^2 y_n$$

i-e, we have

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$$

$$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$$

$$- - -$$

$$\nabla^2 y_n = \nabla y_n - \nabla y_{n-1}$$

In general,

$$\nabla^k y_i = \nabla^{k-1} y_i - \nabla^{k-1} y_{i-1}, \quad i = n, (n-1), \ldots, k$$

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(2a) Table

x					
x_0	y_0				
x_1	y_1	∇y_1	$\nabla^2 y_2$		
x_2	y_2	∇y_2		$\nabla^3 y_3$	
x_3	y_3	∇y_3	$\nabla^2 y_3$		$\nabla^4 y_4$
x_4	y_4	∇y_4	$\nabla^2 y_4$	$\nabla^3 y_4$	

$\nabla f(x) = f(x) - f(x-h)$

Central difference operator:

δ (sigma) $\quad \cancel{\delta y_{1/2} = y_1 - y_0}$ $\quad \delta y_{1/2} = y_1 - y_0$

$$\delta f(x) = f(x + h/2) - f(x - h/2)$$

Shift operator:

E

$$E f(x) = f(x+h)$$

$$E^2 f(x) = f(x+2h)$$

$$E^{-1} f(x) = f(x-h)$$

$$E^{-2} f(x) = f(x-2h)$$

$$E^{\frac{1}{2}} f(x) = f(x + h/2)$$

(3) Relation b/w Difference operators.

1) $\Delta = E - 1$

2) $\nabla = 1 - E^{-1}$

1) Proof

$$\Delta = E - 1$$

$$\Delta f(x) = f(x+h) - f(x)$$

$$= E f(x) - f(x)$$

$$\Delta f(x) = (E-1) f(x)$$

$$\boxed{\Delta = E - 1}$$

2)

$$\nabla f(x) = f(x) - f(x-h)$$

$$= f(x) - E^{-1} f(x)$$

$$\nabla f(x) = f(x)\left(1 - E^{-1}\right)$$

$$\boxed{\nabla = 1 - E^{-1}}$$

(3a) **Newton's Forward difference formula**

For Equally spaced data

Let $y_x = f(x)$ be a function which takes the value $y_0, y_1, \ldots, y_n$ corresponding to the values of $x_0, x_1, \ldots x_n$. The values of x be at equal distance intervals.

Let $\quad x_i = x_0 + ih, \quad i = 0, 1, 2, \ldots n$

Let $y(x)$ be the polynomial of nth degree in x.

$$y(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)\cancel{(x - x_1)}(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + \cdots + a_n(x - x_0)(x - x_1) \cdots (x - x_{n-1}) \quad (1)$$

here $(n+1) \Rightarrow$ constants

Now when $x = x_0$

$$y(x_0) = a_0$$

$$\boxed{y_0 = a_0}$$

when $x = x_1$.

$$y_1 = a_0 + a_1(x_1 - x_0)$$

$$y_1 = a_0 + a_1 h$$

$$y_1 = y_0 + a_1 h$$

$$y_1 - y_0 = a_1 h$$

$$a_1 = \frac{y_1 - y_0}{h}$$

$$\boxed{a_1 = \frac{\Delta y_0}{h}}$$

when $x = x_2$

$$y_2 = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

$$= a_0 + a_1(2h) + a_2(2h)(h)$$

$$y_2 = a_0 + 2a_1 h + 2h^2 a_2$$

$$\cancel{y_2 - y_0}$$

$$y_2 - y_1 = a_0 + 2a_1 h + 2h^2 a_2 - y_1$$

$$= \cancel{a_0} + 2a_1 h + 2h^2 a_2 - \cancel{a_0} - a_1 h$$

$$\Delta y_1 = a_1 h + 2h^2 a_2$$

$$\Delta y_1 = \frac{\Delta y_0}{\cancel{h}} \cdot \cancel{h} + 2h^2 a_2$$

$$\Delta y_1 - \Delta y_0 = 2h^2 a_2$$

$$a_2 = \frac{\Delta^2 y_0}{2! h^2}$$

Similarly,

$$a_3 = \frac{\Delta^3 y_0}{3! h^3} \quad \text{and so on.}$$

(4a) By substituting $a_0, a_1, a_2, \ldots$ in (1), we get

$$y(x) = y_0 + \left(\frac{\Delta y_0}{h}\right)(x-x_0) + \frac{\Delta^2 y_0}{2!h^2}(x-x_0)(x-x_1)$$
$$+ \frac{\Delta^3 y_0}{3!h^3}(x-x_0)(x-x_1)(x-x_2) + \cdots + \frac{\Delta^n y_0}{n!h^n}(x-x_0)(x-x_1)\cdots(x-x_{n-1}) \quad (2)$$

Let us put

$$x - x_0 = ph$$
$$x = ph + x_0$$

$$\boxed{p = \frac{x-x_0}{h}}$$

Now

$$x - x_1 = (x - x_0) + x_0 - x_1$$
$$= ph + (-h)$$
$$= (p-1)h$$

Also

$$x - x_2 = x - x_0 + x_0 - x_2$$
$$= ph - 2h$$
$$= (p-2)h \quad \text{and so on.}$$

By putting in (2)

$$y(x_0+ph) = y_0 + \left(\frac{\Delta y_0}{\cancel{h}}\right)(p\cancel{h}) + \frac{\Delta^2 y_0}{2!\cancel{h^2}}[p\cancel{h}(p-1)\cancel{h}]$$
$$+ \frac{\Delta^3 y_0}{3!\cancel{h^3}}[p\cancel{h}(p-1)\cancel{h}(p-2)\cancel{h}] + \cdots$$

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$
$$+ \frac{p(p-1)(p-2)\cdots[p-(n-1)]}{n!}\Delta^n y_0$$

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Note ⑤ i) First two terms of this series give the results for the linear interpolation.

ii) First three ⟹ parabolic interpolation.

Example: Estimate the value of $\sin\theta$ at $\theta = 35^\circ$, using Newton Forward difference formula using given table.

θ	10	20	30	40	50
$\sin\theta$	0.1736	0.3420	0.5000	0.6428	0.7660

Formation of Forward Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0					
		$\Delta y_0 = y_1 - y_0$				
x_1	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$			
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 =$		
x_2	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$		$\Delta^4 y_0 =$	
		$\Delta y_2 = y_3 - y_2$		$\Delta^3 y_1 =$		$\Delta^5 y_0$
x_3	y_3		$\Delta^2 y_2 = \Delta y_3 - \Delta y_1$		$\Delta^4 y_1 =$	
		$\Delta y_3 = y_4 - y_3$		$\Delta^3 y_2 =$		
x_4	y_4		$\Delta^2 y_3 = \Delta y_4 - \Delta y_1$			
		$\Delta y_4 = y_5 - y_4$				
x_5	y_5					

Q# Find the Cubic Polinomial which Takes following values.

x	0	1	2	3
$f(x)$	1	2	1	10

also find $f(4)$ and $f'(4)$

Sol Forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
$x_0 = 0$	$y_0 = 1$			
		$\Delta y_0 = 1$		
1	2		$\Delta^2 y_0 = -2$	
		$1-2 = -1$		$\Delta^3 y_0 = 12$
2	1		$\Delta^2 y_1 = 10$	
		$10-1=9$		
3	10			

here $x_0 = 0$, $y_0 = 1$ $h = 1$ $\Rightarrow$ $P = \frac{x - x_0}{h} = \frac{x-0}{1}$

$P = x$

Now by Newton Forward Difference interpolation formula.

$$f(x) = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 \ldots$$

$$\Rightarrow f(x) = 1 + x \cdot 1 + \frac{x(x-1)}{2 \cdot 1}(-2) + \frac{x(x-1)(x-2)}{3 \cdot 2 \cdot 1}(12)$$

$$= 1 + x + \left(-\frac{\cancel{2}(x^2-x)}{\cancel{2}}\right) + \frac{\cancel{12}^{2}(x^3-2x^2-x^2+2x)}{\cancel{6}_{1}}$$

$$= 1 + x - x^2 + x + 2(x^3 - 3x^2 + 2x)$$

$$= 1 + \underline{2x} - \underline{x^2} + 2x^3 - \underline{6x^2} + \underline{4x}$$

$$f(x) = 2x^3 - 7x^2 + 6x + 1$$

$$f'(x) = 6x^2 - 14x + 6$$

$$f(4) = 2(4)^3 - 7(4)^2 + 6(4) + 1$$

$$= 128 - 112 + 24 + 1$$

$$f(4) = 41$$

$$f'(4) = 6(4)^2 - 14(4) + 6$$

$$= 96 - 56 + 6$$

$$= 46$$

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