National University of Computer and Emerging Sciences, Lahore Campus

Course Name:	Differential Equations	Course Code:	MT-1006
Degree Program:	BCS, BDS, BSR	Semester:	Spring 2023
Exam Duration:	3 Hours	Total Marks:	70
Paper Date:	09-06-2023	Weight	50%
Section:	All	Page(s):	2
Exam Type:	Final		

Student : Name: M. Section: 2A

Instruction/Notes: Attemp ... allowed. Exchange of stati

$\frac{a_{n+1}}{a_n}$ $L>1$ diverg $L<1$ converge $L=1$ inconclu

Question no. 1: (CLO-01) (5+5 marks)

a) Use the ratio test to find whether the given series converges or diverges. Also, find the radius and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x+2)^n}{n\,2^n}$ and check the convergence or divergence at endpoints.

b) Show that the p-series,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

(p *is* a real constant) converges if $p > 1$, and diverges if $p \le 1$.

Question no. 2: (CLO-02) (10 marks)

The sugar-making process contains a step called "inversion" in which cane sugar is dissolved in water. The rate at which the quantity of unconverted sugar is changing is proportional to the amount present. If 550 kg are present initially and 400 kg are present after 10 hours, how much is left after 15 hours? At what rate is the unconverted sugar changing after 15 hours?

Question no. 3: (CLO-03) (10 marks)

Determine whether the equation $(2xy + x)dy + (2y^2 + 2y + 4x^2)dx = 0$ is exact. If not, make it exact and solve the differential equation.

$\frac{M_y - N_x}{N}$ $e^{-\int p(x)dx}$

Question no. 4: (CLO-04) (10 marks)

Determine the solution of the given differential equation by using undetermined coefficients approach.

$$y'' - 3y' = 8e^{3x} + 4\sin x.$$

Ae^{mx}

$A \sin \alpha x + B \cos \alpha x$

Question no. 5: (CLO-04) (1+3+6 marks)

Identify the given differential equation and find the general solution of given differential equation by using appropriate technique.

$$x^2y'' + xy' - y = lnx.$$

$y = ct^{x}$
$y = ct^{x}(\ln x)$
$y = e^{\alpha}(c_1 \cos (\beta \ln x))$

Question no. 6: (CLO-05) (10 marks)

Find the Fourier series for the function

$$f(x) = \begin{cases} x, & -\pi < x < 0, \\ \pi - x, & 0 < x < \pi. \end{cases}$$

Question no. 7: (CLO-05) (5+2+3 marks)

a) Check that the given set of functions is orthogonal on the indicated interval.

$\left\{1, sin\frac{m\pi}{p}x, cos\frac{n\pi}{p}x\right\}$; $n = 1, 2, 3, \ldots$, $m = 1, 2, 3, \ldots$; $[-p, p]$.

b) Classify the given partial differential equation as hyperbolic, parabolic or elliptic.

$$2\frac{\partial^2 u}{\partial x \partial y} - 3\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial x} = 0.$$

c) Verify that the indicated function $u(x,t) = \ln(x^2 + t^2)$ is a solution of given partial differential equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = 0.$$

(OR)

Question no. 7: (CLO-05) (10 marks)

Use separation of variables technique to find the product solutions for the given partial differential equation.

$$c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}.$$

Q#1 a) Use the ratio test to find whether the given series converges or diverges. Also, find the radius and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x+2)^n}{n 2^n}$ and check the convergence or divergence at endpoints.

Sol:- $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x+2)^n}{n2^n}$ — (1)

$$\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n2^n}{(n+2)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x+2)\,n}{(n+1)\,2}\right|$$

$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)\left|\frac{x+2}{2}\right|$$

$$= \lim_{n\to\infty}\left(\frac{1}{1+1/n}\right)\left|\frac{x+2}{2}\right|$$

$$= 1\left|\frac{x+2}{2}\right|$$

$$= \frac{x+2}{2}$$

The series converges if

$$\left|\frac{x+2}{2}\right| < 1$$

$$|x+2| < 2$$

$$-2 < x+2 < 2$$

$$\boxed{-4 < x < 0}$$

when $x = -4$ in eq. (1)

we have $\sum_{n=1}^{\infty} -\frac{1}{n}$, is a divergent series.

when $x = 0$ in (1)

We have $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$, the

alternately Harmonic series which converges conditionally.
The interval of convergence of is,

$$-4 < x \leq 0$$

and Radius is 2.
The series converges conditionally at $x=0$.

Q#1 b). Show that the p-series,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

(p is a real constant) converges if $p>1$, and diverges if $p \le 1$.

Sol:- The given series is called p-series.
If $p=1$, we have Harmonic Series which we know diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$ is

Harmonic series.

P- Series Test:-

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

① If $p>1$, then the series converges
② If $p \le 1$, then the series diverges.

Case 1 If $p>1$, $a_n = \frac{1}{n^p}$

$$a_n = f(x)$$

$$f(x) = \frac{1}{x^p}$$

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b x^{-p}\,dx$$

$$= \lim_{b\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^b$$

$$= \lim_{b\to\infty} \left[\frac{(b)^{-p+1}}{-p+1} - \frac{(1)^{-p+1}}{-p+1}\right]$$

$$= \frac{1 \lim_{b\to\infty}}{1-p} \left[\frac{1}{(b)^{p-1}} - 1\right]$$

$(b)^{p-1} \to \infty$ as $b \to \infty$ bcz $p-1>0$, $p>1$

$$= \frac{1}{1-p}\left[0-1\right]$$

$$= \frac{1}{p-1} \Rightarrow \int a_n \text{ converges.}$$

The series converges by integral test for $p>1$

Case 2 If $p \le 1$

$$\int_1^\infty a_n \, dx = \int_1^\infty \frac{1}{x^p} \, dx = \lim_{b \to \infty} \int_1^b x^{-p} \, dx$$

$$= \lim_{b \to \infty} \left. \frac{x^{-p+1}}{-p+1} \right|_1^b$$

$$= \lim_{b \to \infty} \left[\frac{(b)^{-p+1}}{-p+1} - \frac{(1)^{-p+1}}{-p+1} \right]$$

$$= \frac{(+\infty)^{1-p}}{1-p} - \frac{1}{1-p} \qquad \begin{array}{l} p<1 \\ \Rightarrow 1-p>0 \end{array}$$

$$= \frac{(\infty)^{+ve}}{1-p} = \infty \quad \text{(Diverges)}$$

Since $\int a_n \, dx$ diverges. Hence the series

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

Q# 2 Sol:- Let $A(t)$ be the amount of unconverted sugar

$$\frac{dA}{dt} = k A(t) \quad \text{--- (1)}$$

Initial condition :-

$$A(0) = 550 \text{ kg}$$

$$A(10) = 400 \text{ kg}.$$

$$\int \frac{dA}{A(t)} = \int k \, dt$$

$$\ln|A| = kt + c \quad \text{--- (2)}$$

Taking Anti log on b/s

$$A(t) = e^{kt+c}$$

$$A(t) = e^{kt} \cdot e^{c} \qquad \because e^{c} = c$$

$$\boxed{A(t) = ce^{kt}} \quad \text{--- (3)}$$

using $A(0) = 550$ kg

put $t=0$ in eq. (3)

$$A(0) = e^{k(0)} c$$

$$\boxed{550 = c}$$

Put in (3)

$$\boxed{A(t) = 550 e^{kt}} \quad \text{--- (4)}$$

$$A(10) = 550 e^{k(10)}$$

$$400 = 550 e^{10k}$$

$$e^{10k} = \frac{400}{550} = \frac{40}{55}$$

$$\ln(e^{10k}) = \ln\left(\frac{40}{55}\right)$$

$$10k = \ln\left(\frac{8}{11}\right)$$

$$\boxed{k = \frac{1}{10} \ln\left(\frac{8}{11}\right)}$$

Put value of k in eq (4)

$$A(t) = 550\, e^{\frac{1}{10}\ln\left(\frac{8}{11}\right)t}$$

$$A(t) = 550\left(e^{\ln\left(\frac{8}{11}\right)}\right)^{t/10} \quad \text{--- (5)}$$

$$A(t) = 550\left(\frac{8}{11}\right)^{t/10}$$

⇒ How much is left after 15 hours.

when $\boxed{t=15}$ in eq. (5)

$$A(15) = 550\left(\frac{8}{11}\right)^{15/10}$$

$$= \boxed{341.12 \text{ kg}}$$

At what rate is the unconverted sugar changing after 15 hours.

$$\left.\frac{dA}{dt}\right|_{t=15} = k\,A(t)$$

$$\frac{dA}{dt} = \frac{1}{10}\ln\left(\frac{8}{11}\right) \cdot 341.12$$

$$\boxed{\frac{dA}{dt} = 10.863 \text{ kg/hr}}$$

Q#3. Determine whether the given equation is exact or not? If not, make it exact and solve

$$(2y^2+2y+4x^2)dx + (2xy+x)dy = 0 \quad \text{--- (1)}$$

Sol:- $M = 2y^2+2y+4x^2$

$N = 2xy+x$

$$\frac{\partial M}{\partial y} = 4y+2 \quad , \quad \frac{\partial N}{\partial x} = 2y+1$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$4y+2 \neq 2y+1$ (Not exact)

Now we find I.F

$$\frac{M_y - N_x}{N} = \frac{4y+2-(2y+1)}{2xy+x}$$

$$= \frac{\cancel{2y+1}}{x(\cancel{2y+1})} = \frac{1}{x}$$

Since it is a function of x so

$$I.F = e^{\int 1/x\,dx} = \cancel{e}^{\cancel{\ln} x} = x$$

Multiply I.F with eq. (1)

$$x(2y^2+2y+4x^2)dx + x(2xy+x)dy = 0$$

$$(2xy^2+2xy+4x^3)dx + (2x^2y+x^2)dy = 0$$

$M = 2xy^2+2xy+4x^3 \quad , \quad N = 2x^2y+x^2$

$$\Rightarrow \frac{\partial M}{\partial y} = 4xy+2x \quad , \quad \frac{\partial N}{\partial x} = 4xy+2x$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad , \quad \text{(Exact eq.)}$$

Now we solve D. Eq.

$M = \frac{\partial f}{\partial x}$, $N = \frac{\partial f}{\partial y}$

$\frac{\partial f}{\partial x} = 2xy^2 + 2xy + 4x^3$ — (2) ; $\frac{\partial f}{\partial y} = 2x^2y + x^2$ — (3)

Integrating (3) w.r.t y.

$f(x,y) = \cancel{2}x^2\frac{y^2}{\cancel{2}} + x^2y + h(x)$

$f(x,y) = x^2y^2 + x^2y + h(x)$ — (4)

$\frac{\partial f}{\partial x} = 2xy^2 + 2xy + h'(x)$ — (5)

Equating (2) and (5)

$\cancel{2xy^2} + \cancel{2xy} + 4x^3 = \cancel{2xy^2} + \cancel{2xy} + h'(x)$

$\Rightarrow h'(x) = 4x^3$

Integrating this, we get $h(x) = x^4 + c$

Putting in (4)

$f(x,y) = x^2y^2 + x^2y + x^4 + c$

Q# 4 Determine the solution of the given differential equation by using undetermined coefficients approach.

$$y'' - 3y' = 8e^{3x} + 4\sin x$$

Sol:- The auxiliary equation for the associat. Homogenous eq.

$$y'' - 3y' = 0$$

$$m^2 - 3m = 0$$

$$m(m-3) = 0$$

$$m_1 = 0 \quad , \quad m_2 = 3$$

$$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$y_c = c_1 e^{0(x)} + c_2 e^{3x}$$

$$y_c = c_1 + c_2 e^{3x}$$

The particular solution

$$y_p = Ae^{3x} + B\cos x + C\sin x$$

Hence our assumption is clearly present in complementary solution. This means e^{3x} is a solution of D.Eq. and a constant multiple Ae^{3x} substituted in D.E. gives no result.

So we consider a particular solution by multiplying the factor x with term Ae^{3x}.

$$y_p = Axe^{3x} + B\cos x + C\sin x.$$

$$y_p' = 3Axe^{3x} + Ae^{3x} - B\sin x + C\cos x$$

$$y_p'' = 3Ae^{3x} + 9Axe^{3x} + 3Ae^{3x} - B\cos x - C\sin x$$

Substituting the values in given D.Eq.

$$y_p'' - 3y_p' = 8e^{3x} + 4\sin x$$

$$3Ae^{3x} + 9Axe^{3x} + 3Ae^{3x} - B\cos x - C\sin x - 3\left[3Axe^{3x} + Ae^{3x} - B\sin x + C\cos x\right] = 8e^{3x} + 4\sin x.$$

$$\Rightarrow 6Ae^{3x} + 9Axe^{3x} - B\cos x - C\sin x - 9Axe^{3x} - 3Ae^{3x} + 3B\sin x - 3C\cos x = 8e^{3x} + 4\sin x.$$

$$\Rightarrow 3Ae^{3x} + 3B\sin x - B\cos x - C\sin x - 3C\cos x = 8e^{3x} + 4\sin x$$

$$\Rightarrow 3Ae^{3x} + \cos x(-B-3C) + \sin x(3B - C) = 8e^{3x} + 4\sin x$$

Comparing the coefficients

e^{3x} :- $3A = 8 \Rightarrow \boxed{A = 8/3}$

$\sin x$:- $3B - C = 4$ — (1)

$\cos x$:- $-B - 3C = 0$ — (2)

Solving (1) and (2)

$-10C = 4 \Rightarrow \boxed{C = -2/5}$

Put in (2)

$-B - 3\left(-\frac{2}{5}\right) = 0 \Rightarrow \boxed{B = \frac{6}{5}}$

Hence the particular solution.

$$y_p = \frac{8}{3}e^{3x} + \frac{6}{5}\cos x - \frac{2}{5}\sin x$$

The general solution is

$$y = y_c + y_p$$

$$\Rightarrow \boxed{y = c_1 + c_2 e^{3x} + \frac{8}{3}e^{3x} + \frac{6}{5}\cos x - \frac{2}{5}\sin x}$$

Q# 5: Identify the given differential equation and find the general solution of given differential equation by using appropriate technique.

$$x^2y'' + xy' - y = \ln x$$

Sol:- Assume $y = x^m$ is a solution of D.E.

$y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$

Firstly we find complementary solution corresponding to homogenous eq.

$$x^2y'' + xy' - y = 0$$

$$x^2 m(m-1)x^{m-2} + xmx^{m-1} - x^m = 0$$

$$x^m m(m-1) + mx^m - x^m = 0$$

$$x^m(m(m-1) + m - 1) = 0$$

$$\Rightarrow m(m-1) + m - 1 = 0$$

$$m^2 - m + m - 1 = 0$$

$$m^2 - 1 = 0$$

$$(m-1)(m+1) = 0$$

$$m_1 = 1 \quad , \quad m_2 = -1$$

$$y_c = c_1y_1 + c_2y_2$$

$$y_c = c_1x^{m_1} + c_2x^{m_2}$$

$$y_c = c_1x + c_2x^{-1}$$

where $y_1 = x$ and $y_2 = x^{-1}$.

Now by using variation of parameter, we find a particular solution.

$$y_p = u_1y_1 + u_2y_2$$

Now we put given D.Eq. into standard form by dividing x^2 on both sides.

$$y'' + \frac{y'}{x} - \frac{y}{x^2} = \frac{\ln x}{x^2}$$

Here, $f(x) = \dfrac{\ln x}{x^2}$

We find Wronskian

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W = \begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix} = -x^{-1} - x^{-1} = -2x^{-1} \neq 0$$

We find U_1 and U_2

$$U_1 = -\int \frac{y_2 f(x)}{W} dx, \quad U_2 = \int \frac{y_1 f(x)}{W} dx$$

$$\Rightarrow U_1 = \cancel{-}\int \frac{\cancel{x^{-1}} \cdot \ln x / x^2}{\cancel{-}2\cancel{x^{-1}}} dx = \frac{1}{2}\int \frac{\ln x}{x^2} dx$$

Now integrating by parts,

$$U_1 = \frac{1}{2}\int \frac{\ln x}{x^2} dx$$

$$= \frac{1}{2}\left[\frac{-1}{x}\ln x + \int \frac{1}{x}\frac{d\ln x}{dx} \cdot dx\right]$$

$$= \frac{1}{2}\left[-\frac{1}{x}\ln x + \int \frac{1}{x}\cdot\frac{1}{x} dx\right]$$

$$= \frac{1}{2}\left[-\frac{1}{x}\ln x + \int \frac{1}{x^2} dx\right]$$

$$= \frac{1}{2}\left[-\frac{1}{x}\ln x + \frac{x^{-1}}{-1}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{x}\ln x - \frac{1}{x}\right] \Rightarrow \boxed{U_1 = -\frac{1}{2}x^{-1}\ln x - \frac{1}{2}x^{-1}}$$

$$\Rightarrow U_2 = -1\int \frac{y_1 f(x)}{W} dx$$

$$U_2 = \int \frac{\cancel{x}\ \ln x / x^2}{-2x^{-1}} dx$$

$$= -\int \frac{\ln x}{2\cancel{x}\,\cancel{x^{-1}}} dx$$

$$U_2 = -\int \frac{\ln x}{2}\,dx$$

$$U_2 = -\frac{1}{2}\int 1.\ln x \,dx$$

$$= -\frac{1}{2}\left[x.\ln x - \int x \frac{d}{dx}\ln x\right] dx$$

$$= -\frac{1}{2}\left[x\ln x - \int x\frac{1}{x}\,dx\right]$$

$$= -\frac{1}{2}\left[x\ln x - \int 1\,dx\right]$$

$$\boxed{U_2 = -\frac{1}{2}x\ln x + \frac{1}{2}x}$$

$$y_p = \left(-\frac{1}{2}x^{-1}\ln x - \frac{1}{2}x^{-1}\right)x + \left(-\frac{1}{2}x\ln x + \frac{1}{2}x\right)x^{-1}$$

General solution is :-

$$y = y_c + y_p$$

$$y = c_1 x + c_2 x^{-1} + \left(-\frac{1}{2}x^{-1}\ln x - \frac{1}{2}x^{-1}\right)x + \left(-\frac{1}{2}x\ln x + \frac{1}{2}x\right)x^{-1}$$

$$y = c_1 x + c_2 x^{-1} - \frac{1}{2}\ln x \cancel{-\frac{1}{2}} - \frac{1}{2}\ln x \cancel{+\frac{1}{2}}$$

$$\boxed{y = c_1 x + c_2 x^{-1} - \ln x}$$ — Ans

Q#6 Sol:- $f(x) = \begin{cases} x & ; -\pi < x < 0 \\ \pi - x & ; 0 < x < \pi \end{cases}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{P}x + b_n \sin\frac{n\pi}{P}x\right) \quad (1)$$

$$a_0 = \frac{1}{P}\int_{-P}^{P} f(x)\,dx$$

$$a_n = \frac{1}{P}\int_{-P}^{P} f(x)\cos\frac{n\pi}{P}x\,dx$$

$$b_n = \frac{1}{P}\int_{-P}^{P} f(x)\sin\frac{n\pi}{P}x\,dx$$

$$a_0 = \frac{1}{\pi}\left\{\int_{-\pi}^{0} x\,dx + \int_{0}^{\pi}(\pi - x)\,dx\right\}$$

$$a_0 = \frac{1}{\pi}\left\{\frac{x^2}{2}\Big|_{-\pi}^{0} + \left|\pi x - \frac{x^2}{2}\right|_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{0 - \frac{\pi^2}{2} + \left(\pi(\pi) - \frac{\pi^2}{2}\right)\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{\pi^2}{2} + \pi^2 - \frac{\pi^2}{2}\right\}$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\frac{n\pi}{\pi}x\,dx$$

$$a_n = \frac{1}{\pi}\left\{\int_{-\pi}^{0} x\cos nx + \int_{0}^{\pi}(\pi - x)\cos nx\,dx\right\}$$

$$I_1 \quad + \quad II_2$$

$\Rightarrow I_1$:- $$a_n = \frac{1}{\pi}\left\{x\frac{\sin nx}{n}\Big|_{-\pi}^{0} - \int_{-\pi}^{0}\frac{\sin nx}{n}\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\pi\frac{\cancel{\sin n\pi}^{\,0}}{n} + \frac{1}{n}\frac{\cos nx}{n}\Big|_{-\pi}^{0}\right\}$$

$$= \frac{1}{\pi}\left\{ \pi \frac{\cancel{\sin n\pi}^{0}}{n} + \frac{\cos nx}{n^2} \right|_{-\pi}^{0}$$

$$= \frac{1}{\pi}\left\{ \pi \frac{\cancel{\sin n\pi}^{0}}{n} + \left(\frac{\cos n(0)}{n^2} - \frac{\cos n\pi}{n^2}\right)\right\}$$

$$= \frac{1}{\pi}\left(\frac{1}{n^2} - \frac{\cos n\pi}{n^2}\right) \qquad \boxed{\because \cos nx = (-1)^n}$$

$$a_n = \frac{1}{n^2\pi}\left(1 - \frac{\cos n\pi}{n^2}\right)$$

$$\boxed{a_n = \frac{1}{n^2\pi}\left(1 - (-1)^n\right)} \quad \text{--- (i)}$$

I_2 :-

$$= \frac{1}{\pi}\left\{ (\pi - x)\frac{\sin nx}{n}\Big|_0^{\pi} - \int_0^{\pi} \frac{\sin nx}{n}(-1)\,dx \right\}$$

$$= \frac{1}{\pi}\left\{ 0 - \frac{\pi \cancel{\sin n(0)}^{0}}{n} + \frac{1}{n}\int_0^{\pi} \sin nx \, dx \right\}$$

$$= \frac{1}{\pi}\left\{ -\frac{1}{n}\left(\frac{\cos nx}{n}\right)\Big|_0^{\pi} \right\}$$

$$= \frac{-1}{n^2\pi}\left\{\cos n\pi - \cos(0)\right\}$$

$$= \frac{-1}{n^2\pi}\left\{(-1)^n - 1\right\}$$

$$= \frac{1}{n^2\pi}\left\{1 - (-1)^n\right\} \quad \text{--- (ii)}$$

Add (i) and (ii)

$$= \frac{1}{n^2\pi}\left\{1 - (-1)^n\right\} + \frac{1}{n^2\pi^2}\left\{1 - (-1)^n\right\}$$

$$\boxed{a_n = \frac{2}{n^2\pi}\left(1 - (-1)^n\right)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin \frac{\pi x}{\pi}\, dx \qquad \boxed{p = \pi}$$

$$b_n = \frac{1}{\pi}\left\{ \int_{-\pi}^{0} x \sin nx\, dx + \int_0^{\pi} (\pi - x)\sin nx\, dx \right.$$

$$b_n = I_1 + I_2$$

$$I_1 = \frac{1}{\pi}\int_{-\pi}^{0} x \sin nx \, dx$$

$$= \frac{1}{\pi}\left\{ -x\frac{\cos nx}{n}\Big|_{-\pi}^{0} + \int_{-\pi}^{0} \frac{\cos nx}{n}\, dx \right\}$$

$$= \frac{1}{\pi}\left\{ 0 - (\pi)\frac{\cos n\pi}{n} + \frac{1}{n}\frac{\sin nx}{n}\Big|_{\pi}^{0} \right\}$$

$$= \frac{1}{\pi}\left\{ -\pi\frac{\cos n\pi}{n} + 0 \right\}$$

$$I_1 = -\frac{\cos n\pi}{n}$$

$$I_1 = \frac{-(-1)^n}{n} \quad \text{— (i)}$$

$$I_2 = \frac{1}{\pi}\int_{0}^{\pi} (\pi - x)\sin nx \, dx$$

$$= \frac{1}{\pi}\left\{ -(\pi - x)\frac{\cos nx}{n}\Big|_{0}^{\pi} + \int_{0}^{\pi} \frac{\cos nx}{n}(-1)\, dx \right\}$$

$$= \frac{1}{\pi}\left\{ 0 + \frac{\pi \cos n(0)}{n} + \frac{1}{n}\cancel{\sin}\frac{nx}{n}\Big|_{0}^{\pi} \right\}$$

$$= \frac{1}{\pi}\left\{ \frac{\pi\cos(0)}{n} \right\}$$

$$= \frac{1}{n} \quad \text{— (ii)}$$

Add (i) and (ii)

$$= -\frac{(-1)^n}{n} + \frac{1}{n}$$

$$\boxed{b_n = -\frac{1}{n}(-1)^n + \frac{1}{n}}$$

put values of coefficient in fourier series

$$f(x) = 0 + \sum_{n=1}^{\infty}\left(\frac{2}{n^2\pi}\left(1 - (-1)^n\right)\cos nx \right.$$
$$\left. + \left(\frac{1}{n} - \frac{1}{n}(-1)^n\right)\sin nx \right)$$

$$f(x) = \sum_{n=1}^{\infty} \left\{ \frac{2}{n^2\pi}\left(1-(-1)^n\right)\cos nx + \left(\frac{1}{n} - \frac{1}{n}(-1)^n\right)\sin nx \right.$$

Sol.

Q#7 The set of functions

a) $\left\{1, \cos\frac{n\pi x}{p}, \sin\frac{m\pi x}{p}\right\}$ $n = 1, 2, 3,$ $m = 1, 2, 3;$

is orthogonal? $[-p, p]$.

Sol:- $$\int_{-p}^{p} 1 \cdot \cos\frac{n\pi x}{p}\,dx = \int_{-p}^{p} \cos\frac{n\pi x}{p}\,dx$$

$$= \frac{p}{n\pi}\left(\sin\frac{n\pi x}{p}\right)\Big|_{-p}^{p}$$

$$= \frac{p}{n\pi}\left[\sin\frac{n\pi p}{p} - \sin\frac{n\pi(-p)}{p}\right]$$

$$= \frac{p}{n\pi}\left[\cancel{\sin n\pi}^{\,0} + \cancel{\sin n\pi}^{\,0}\right]$$

$$\int_{-p}^{p} 1 \cdot \cos\frac{n\pi x}{p}\,dx = 0$$

Similarly $$\int_{-p}^{p} 1 \cdot \sin\frac{m\pi x}{p}\,dx = \int_{-p}^{p} \sin\frac{m\pi x}{p}\,dx$$

$$= -\frac{p}{m\pi}\left(\cos\frac{m\pi x}{p}\right)\Big|_{-p}^{p}$$

$$= \frac{-p}{m\pi}\left(\cos\frac{m\pi p}{p} - \cos\frac{m\pi(-p)}{p}\right)$$

$$= \frac{-p}{m\pi}\left(\cancel{\cos m\pi} - \cancel{\cos m\pi}\right)$$

$$\int_{-p}^{p} \sin\frac{m\pi x}{p}\,dx = 0$$

$$\Rightarrow \int_{-p}^{p} \sin\left(\frac{m\pi x}{p}\right)\cos\left(\frac{n\pi x}{p}\right)dx$$

$$= \frac{1}{2}\int_{-p}^{p}\left\{\sin\left(\frac{m\pi x - n\pi x}{p}\right) + \sin\left(\frac{m\pi x + n\pi x}{p}\right)\right\}dx$$

$$= \frac{1}{2}\int_{-p}^{p} \left\{ \sin\left(\frac{m-n}{p}\right)\pi x + \sin\frac{(m+n)}{p}\pi x \right\} dx$$

$$= \frac{1}{2}\left[\frac{-p}{(m-n)\pi}\cos\left(\frac{m-n}{p}\right)\pi x + \left(\frac{-p}{(m+n)\pi}\right)\cos\left(\frac{m+n}{p}\right)\pi x\right]_{-p}^{p}$$

$$= \frac{1}{2}\left[\frac{-p}{(m-n)\pi}\left(\cos\frac{(m-n)}{\cancel{p}}\pi \cancel{p} - \cos\frac{(m-n)}{\cancel{p}}\pi(-\cancel{p})\right) + \frac{-p}{(m+n)\pi}\left(\cos\frac{(m+n)}{\cancel{p}}\pi \cancel{p} - \cos\frac{(m+n)}{\cancel{p}}\pi(-\cancel{p})\right)\right]$$

$$= 0$$

Hence the given set is orthogonal set.

Q#7 b) Classify the given partial differential equation as hyperbolic, parabolic or elliptic.

$$2\frac{\partial^2 u}{\partial x \partial y} - 3\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial x} = 0$$

Sol:- Here $B = 2$, $C = -3$ and $A = 0$

So $B^2 - 4AC = (2)^2 - 4(0)(-3)$

$= 4 - 0 = 4 > 0$

So the given equation is hyperbolic.

Q#7 c) Verify that the indicated function $u(x,t) = \ln(x^2 + t^2)$ is a solution of given partial differential equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = 0$$

Sol:- D.E :- $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial t^2} = 0$ —— (A)

solution :- $u(x,t) = \ln(x^2 + t^2)$

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2+t^2} \quad ; \quad \frac{\partial u}{\partial t} = \frac{2t}{x^2+t^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{2(x^2+t^2)-2x(2x)}{(x^2+t^2)^2} = \frac{2x^2+2t^2-4x^2}{x+}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{2t^2-2x^2}{(x^2+t^2)^2} \quad \text{— (1)}$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = \frac{2(x^2+t^2)-2t(2t)}{(x^2+t^2)^2} = \frac{2x^2+2t^2-4t^2}{(x^2+t^2)^2}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{-2t^2+2x^2}{(x^2+t^2)^2} \quad \text{— (2)}$$

Verifying eq. (1) and (2) for (A)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = 0$$

$$\Rightarrow \frac{2t^2-2x^2}{(x^2+t^2)^2} + \left(\frac{-2t^2+2x^2}{(x^2+t^2)^2}\right) = 0$$

$$\Rightarrow \frac{\cancel{2t^2}-\cancel{2x^2}-\cancel{2t^2}+\cancel{2x^2}}{(x^2+t^2)^2} = 0$$

$$\Rightarrow 0 = 0.$$

verified.

Q:8 Use seperation of variables technique to find the product solutions for the given partial differential equation.

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} \quad \text{--- } (1)$$

Sol:- The solution of equation of the form

$$u(x,y) = X(x)\,Y(y) \quad \text{--- } (2)$$

$$\frac{\partial u}{\partial x} = X'Y \quad , \quad \frac{\partial u}{\partial y} = XY'$$

$$\frac{\partial^2 u}{\partial x} = X''Y \quad , \quad \frac{\partial^2 u}{\partial y^2} = XY''$$

Put values in eq. (1)

$$c^2 X''Y = XY''$$

$$\frac{X''}{X} = \frac{Y''}{c^2 Y} = -\lambda$$

It reduces to two ODEs.

$$\frac{X''}{X} = -\lambda \quad , \quad \frac{Y''}{c^2 Y} = -\lambda$$

$$X'' + \lambda X = 0 \quad (3) \quad , \quad Y'' + c^2 \lambda Y = 0 \quad (4)$$

There are three cases.

Case 1:- $\lambda = 0$

Put $\lambda = 0$ in eq (3) and (4)

$$X'' = 0 \quad , \quad Y'' = 0$$

$$X''(x) = 0 \quad , \quad Y''(y) = 0$$

$$X = c_1 x + c_2 \qquad Y(y) = c_3 y + c_4$$

$$\Rightarrow \boxed{u(x,y) = (c_1 x + c_2)(c_3 y + c_4)}$$

Case 2:- if $\lambda < 0 \Rightarrow \lambda = -\alpha^2$

Put values in eq. (3) and eq. (4)

$X'' - \alpha^2 X = 0$, $Y'' - c^2\alpha^2 Y = 0$

Auxiliary eq:-

$m^2 - \alpha^2 = 0$, $m^2 - c^2\alpha^2 = 0$

$m = \pm\alpha$, $m = \pm c\alpha$

$X(x) = c_5 \cosh\alpha x + c_6 \sinh\alpha x$

and $Y(y) = c_7 \cosh\alpha cy + c_8 \sinh\alpha cy$

The product solution

$\Rightarrow u(x,y) = X(x)\,Y(y)$

$$\Rightarrow \boxed{u(x,y) = (c_5 \cosh\alpha x + c_6 \sinh\alpha x)(c_7 \cosh\alpha cy + c_8 \sinh\alpha cy)}$$

Case 3: If $\lambda > 0 \Rightarrow \lambda = \alpha^2$

Put values in eq. (3) and eq. (4)

$X'' + \alpha^2 X = 0$, $Y'' + \alpha^2 c^2 Y = 0$

Auxiliary eq:-

$m^2 + \alpha^2 = 0$, $m^2 + \alpha^2 c^2 = 0$

$m = \pm\alpha i$, $m = \pm\alpha c i$

$X(x) = c_9 \cos\alpha x + c_{10} \sin\alpha x$

$Y(y) = c_{11} \cos\alpha cy + c_{12} \sin\alpha cy$

The product solution

$u(x,y) = X(x)\,Y(y)$

$$\Rightarrow \boxed{u(x,y) = (c_9 \cos\alpha x + c_{10} \sin\alpha x)(c_{11} \cos\alpha cy + c_{12} \sin\alpha cy)}$$