LINEAR DIFFERENTIAL EQUATION

A first order differential eq is a linear differential equation if it is or it can be written in the form

$$\frac{dy}{dx} + P(x)\, y = f(x). \quad \text{——— (1)}$$

METHOD OF SOLUTION

(i) Write the differential eq in form (1).

(ii) From standard form (1), identify $P(x)$, then find the integrating factor $e^{\int P(x)\,dx}$.

(iii) Multiply the standard form of equation by the integrating factor. The left-hand side of the resulting equation is automatically the derivative of the integrating factor & y.

(iv) Integrate both sides to get solution.

Exercise 2.3

Find the general solution of the given differential eq. Give the largest interval over which the general solution is defined. Determine whether there are any transient terms in the general solution.

Q9 $\quad x\dfrac{dy}{dx} - y = x^2 \sin x$

$$\frac{dy}{dx} - \frac{1}{x}y = x \sin x$$

$$P(x) = -1/x$$

$$I.F = e^{\int -\frac{1}{x}\,dx}$$

$$= e^{-\ln x} = e^{\ln x^{-1}}$$

$$= \frac{1}{x}$$

multiplying I.F with standard linear diff eq.

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \sin x.$$

$$\frac{d}{dx}\left(\frac{1}{x}y\right) = \sin x.$$

$$d\left(\frac{1}{x}y\right) = \sin x\,dx$$

$$\int \cancel{d}\left(\frac{y}{x}\right) = \int \sin x\,dx.$$

$$\frac{y}{x} = -\cos x + c$$

$$y = -x\cos x + cx$$

No transient terms.

Q20

$$(x+2)^2 \frac{dy}{dx} = 5 - 8y - 4xy$$

$(x+2)^2 \frac{dy}{dx} = 5$ [redacted] $4(2y + xy)$

$$(x+2)^2 \frac{dy}{dx} + 4y(2+x) = 5$$

$$\frac{dy}{dx} + \frac{4(2+x)}{(x+2)^2} y = \frac{5}{(x+2)^2}$$

$$\frac{dy}{dx} + \frac{4}{(x+2)} y = \frac{5}{(x+2)^2}. \quad \text{——— } (1)$$

$$P(x) = 4/x+2$$

$$I.F = e^{\int \frac{4}{x+2} dx} = e^{4\int \frac{1}{x+2} dx} = e^{4 \ln(x+2)} = e^{\ln(x+2)^4}$$

$$= (x+2)^4.$$

multiplying the I.F with eq (1)

$$(x+2)^4 \frac{dy}{dx} + 4(x+2)^3 y = 5(x+2)^2$$

$$\int d\left(y \cdot (x+2)^4\right) = \int 5(x+2)^2 \, dx.$$

$$y\,(x+2)^4 = 5\,\frac{(x+2)^3}{3} + C$$

$$y = 5/3\,(x+2) + \frac{C}{(x+2)^4}$$

Q $\frac{dy}{dx} = \frac{y}{y-x}$ $y(5) = 2$

Above diff eq is homogeneous, it is also linear in x & y (x is dependent & y is independent)

$$\frac{dx}{dy} = \frac{y-x}{y}$$

$$\frac{dx}{dy} = 1 - \frac{x}{y}$$

$$\frac{dx}{dy} + \frac{1}{y} x = 1 \quad \text{——— (1)}$$

$$I.F = e^{\int 1/y \, dy} = e^{\ln y} = y$$

multiplying I.F with eq (1)

$$y \frac{dx}{dy} + x = y$$

$$\frac{d}{dy}(yx) = y$$

$$\int d(yx) = \int y \, dy$$

$$yx = \frac{y^2}{2} + c$$

$$x = \frac{y}{2} + \frac{c}{y}$$

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Using initial condition $y = 2,\ x = 5$

$$5 = \frac{2}{2} + \frac{c}{2}$$

$$5 - 1 = \frac{c}{2}$$

$$\boxed{C = 8}$$

Particular solution is

$$x = \frac{y}{2} + \frac{8}{y}$$

$$x = \frac{y^2 + 16}{2y}$$

I: $(-\infty, 0) \cup (0, \infty)$ for y

I: $(0, \infty)$ is the interval on which solution is defined.

Q Find a continuous solution satisfying each differential equation and the given initial condition.

$$\frac{dy}{dx} + 2y = f(x) \quad , \quad f(x) = \begin{cases} 1 & 0 \le x \le 3 \\ 0 & x > 3 \end{cases}$$

$$y(0) = 0$$

$$I.F = e^{\int 2\,dx} = e^{2x}$$

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = e^{2x}f(x)$$

$$\frac{d}{dx}\left(e^{2x}y\right) = e^{2x}f(x)$$

$$\int d\left(e^{2x}y\right) = \int e^{2x}f(x)\,dx$$

$$\begin{cases} y\,e^{2x} = \int 0\,dx \\ y\,e^{2x} = c_1 \implies y = c_1 e^{-2x} \end{cases} \quad \boxed{x > 3}$$

$$\begin{cases} y\,e^{2x} = \int e^{2x}\,dx \\ y\,e^{2x} = \frac{e^{2x}}{2} + c_2 \\ y = \frac{1}{2} + c_2 e^{-2x} \end{cases} \quad \boxed{0 \le x \le 3}$$

Using initial condition $y(0) = 0$

$$0 = \tfrac{1}{2} + c_2 e^0 \implies c_2 = -\tfrac{1}{2} \implies y = \tfrac{1}{2} - \tfrac{1}{2}e^{-2x}$$

For continuous solution, solution must be continuous at 3. ($x=3$)

$$\Rightarrow \quad c_1 e^{-2x} = \frac{1}{2} \; \frac{1}{2} e^{-2x}$$

$$c_1 e^{-2x} = \frac{1}{2}\left(1 - e^{-2x}\right)$$

$$c_1 = \frac{1}{2}\left(e^{2x} - 1\right)$$

$$c_1 = \frac{1}{2}\left(e^{6} - 1\right).$$