National University of Computer and Emerging Sciences
Lahore Campus

Quantum Computing (CS4084)

Sessional-II Exam

Date: April 3nd 2024

Course Instructor(s)

Dr. Faisal Aslam

Total Time (Hrs):	**1**
Total Marks:	**26**
Total Questions:	**7**

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Roll No Section Student Signature

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Attempt all the questions.
Using calculator is not allowed
Cheat sheet(s) are not allowed

1. Write $5e^{\frac{5i\pi}{2}}$ in a) Cartesian form and b) in Modulus-argument form. [**2+2 Marks**]

   Solution:
   Note: Our complex number lies in the first quadrant. Thus, everything is very simple.

   a) Cartesian form: $5i$

   b) Modulus-argument form: $5(\cos\left(\frac{5\pi}{2}\right) + i\sin\left(\frac{5\pi}{2}\right))$.

2. Given a controlled Hadamard gate, what will be its output given its control bit is $|-\rangle$ and input bit is $|+\rangle$. [**5 Marks**]

   Solution:
   The controlled Hadamard gate, often denoted as C-H, applies a Hadamard gate to the target qubit only if the control qubit is in the state $|1\rangle$, otherwise it leaves the target qubit unchanged.

   We can first simplify our input

$$
\begin{aligned}
|+-\rangle &= \frac{|0\rangle + |1\rangle}{\sqrt{2}} \frac{|0\rangle - |1\rangle}{\sqrt{2}} \\
&= \frac{|00\rangle - |01\rangle + |10\rangle - |11\rangle}{2}
\end{aligned}
$$

   Now we apply Hadamard gate on the second qubit whenever the first qubit is 1.

$$
\frac{|00\rangle - |01\rangle + |1\rangle|H0\rangle - |1\rangle|H1\rangle}{2} = \frac{|00\rangle - |01\rangle + |1\rangle|+\rangle - |1\rangle|-\rangle}{2}
$$

3. Prove that the rows of a unitary matrix are orthonormal. Show all of your proof steps. [**3 Marks**]

   Solution:

   A matrix U is unitary if its conjugate transpose (also known as the adjoint) $U^\dagger$ is equal to its inverse, i.e., $U^\dagger = U^{-1}$.

   Now, to prove that the rows of U are orthonormal, we need to show two things:

   1. Each row vector has a magnitude of 1.
   2. Each pair of row vectors are orthogonal to each other.

   Lets rows of the unitary matrix U are r_1, r_2, ... r_n. Then we have

$$U.U^\dagger = I$$

$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_n \end{pmatrix} \begin{pmatrix} r_1^* & r_2^* & r_3^* & \cdots & r_n^* \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$\begin{pmatrix} r_1r_1^* & r_1r_2^* & r_1r_3^* & \cdots & r_1r_n^* \\ r_2r_1^* & r_2r_2^* & r_2r_3^* & \cdots & r_2r_n^* \\ r_3r_1^* & r_3r_2^* & r_3r_3^* & \cdots & r_3r_n^* \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r_nr_1^* & r_nr_2^* & r_nr_3^* & \cdots & r_nr_n^* \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

This shows that $\langle r_i|r_i\rangle = 1$ so the rows are normalized. Moreover $\langle r_i|r_j\rangle = 0, i \neq j$ hence the rows are orthogonal.

This completes the proof.

4. **Quantum Fourier transform:** Given your input $|1101\rangle$ is phase shifted twice [Hint: $1, \omega_4^2, \omega_4^4, \omega_4^6$] and then given to the inverse of the quantum Fourier transform matrix QFT^{-1}, what will be your output? Show your working clearly. [**5 Marks**]

   Solution:

   Our input is $\omega_{16}^{10}|13\rangle$. So the result will be 13 column of QFT^{-1} multiplied by the ω_{16}^{10}. That is

$$= \sum_{i=0}^{15} \omega_{16}^{-13\times i} \times \omega_{16}^{10}$$

5. **Simon's Algorithm**: Given a function $f : \{0,1\}^3 \to \{0,1\}^3, f(x) = f(x \oplus s)$, and $s = 110$. We construct a circuit for $U_f|x\rangle|000\rangle = |x\rangle|f(x)\rangle$. We give the circuit input the equal superposition of x and measure the second register. Given that our measurement of the second register is $f(101) = 111$, what is the content of the first register? [**2 Marks**]

   Solution:

   The contents of first registers will be all the inputs that can produce 111 in equal superposition:

$$= \frac{|101\rangle + |101 \bigoplus 110\rangle}{\sqrt{2}}$$
$$= \frac{|101\rangle + |011\rangle}{\sqrt{2}}$$

6. **Period Finding Algorithm**: Given a function $f : \{0,1\}^4 \to \{0,1\}^4, f(x) = x \mod 4$.

   (a) We construct a circuit for $U_f \left|x\right\rangle \left|0000\right\rangle = \left|x\right\rangle \left|f(x)\right\rangle$. We give the circuit input the equal superposition of x and measure the second register. Given that our measurement of the second register is $f(1011)$, what is the content of the first register? [**3 Marks**] **Solution:**

   $$\frac{\left|3\right\rangle + \left|7\right\rangle + \left|11\right\rangle + \left|15\right\rangle}{2}$$

   (b) Subsequently, we apply quantum Fourier transform on the content of first register. Then what is the period of the resultant function? [**2 Marks**] **Solution:**

   $$\frac{2^n}{r} = \frac{16}{4} = 4$$

7. **Deutsch-Jozsa Algorithm**: Discuss in a sentence or two why Deutsch-Jozsa will not work if the circuit initial state is $\left|0^n\right\rangle \left|0\right\rangle$ instead of $\left|0^n\right\rangle \left|1\right\rangle$. [**2 Marks**]

   Solution:
   The Deutsch-Jozsa algorithm relies on phase kickback which is made possible because the second qubit was $\left|1\right\rangle$. If the input is $\left|0^n 0\right\rangle$ then there will be no phase kickback and we will not be able to distinguish between even and odd functions.