Mathematical Induction

Mathematical Induction is a mathematical technique which is used to prove a statement, a formula or a theorem is true for every natural number.

In order to use Mathematical Induction, consider the give statement as:

Let P(n) be a propositional function defined for all positive integers n. P(n) is true for every positive integer n if

1. Basis Step:
The proposition P(1) is true.

2. Inductive Step:
If P(k) is true then P(k + 1) is true for all integers k ≥ 1.

i.e. $\forall k \quad p(k) \to P(k+1)$

So, to prove a statement or formula, we have to use two main steps as mentioned above. i.e

Basis Step and Inductive Step: Consider the following examples.

EXAMPLE:

Use Mathematical Induction to prove that

$$1+2+3+\cdots+n=\frac{n(n+1)}{2} \quad \text{for all integers } n \geq 1$$

SOLUTION:

Let $$P(n): 1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

1.Basis Step:

P(1) is true.

For n = 1, left hand side of P(1) is the sum of all the successive integers starting at 1 and ending at 1, so LHS = 1 and RHS is

$$R.H.S=\frac{1(1+1)}{2}=\frac{2}{2}=1$$

so the proposition is true for n = 1.

2. Inductive Step: Suppose P(k) is true for, some integers $k \geq 1$.

(1) $$1+2+3+\cdots+k=\frac{k(k+1)}{2}$$

To prove P(k + 1) is true. That is,

(2) $$1+2+3+\cdots+(k+1)=\frac{(k+1)(k+2)}{2}$$

Consider L.H.S. of (2)

$$\begin{aligned}1+2+3+\cdots+(k+1) &= 1+2+3+\cdots+k+(k+1)\\ &= \frac{k(k+1)}{2}+(k+1) \qquad \text{using (1)}\\ &= (k+1)\left[\frac{k}{2}+1\right]\\ &= (k+1)\left[\frac{k+2}{2}\right]\\ &= \frac{(k+1)(k+2)}{2} = \text{RHS of (2)}\end{aligned}$$

Hence by principle of Mathematical Induction the given result true for all integers greater or equal to 1.

EXERCISE:

Use mathematical induction to prove that
$1+3+5+\ldots+(2n-1) = n^2$ for all integers $n \geq 1$.

SOLUTION:

Let P(n) be the equation $1+3+5+\ldots+(2n-1) = n^2$

1. **Basis Step**:

   P(1) is true
   For n = 1, L.H.S of P(1) = 1and
   R.H.S =2(1)-1 = 1
   Hence the equation is true for n = 1

2. **Inductive Step**:

Suppose P(k) is true for some integer $k \geq 1$. That is,

$$1 + 3 + 5 + \ldots + (2k - 1) = k^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

To prove P(k+1) is true; i.e.,

$$1 + 3 + 5 + \ldots + [2(k+1)-1] = (k+1)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Consider L.H.S. of (2)

$$\begin{aligned} 1+3+5+\cdots+[2(k+1)-1] &= 1+3+5+\cdots+(2k+1) \\ &= 1+3+5+\cdots+(2k-1)+(2k+1) \\ &= k^2+(2k+1) \qquad \text{using (1)} \\ &= (k+1)^2 \\ &= \text{R.H.S. of (2)} \end{aligned}$$

Thus P(k+1) is also true. Hence by mathematical induction, the given equation is true for all integers $n \geq 1$.

EXERCISE:

Use mathematical induction to prove that

$1+2+2^2 + \ldots + 2^n = 2^{n+1} - 1$ for all integers $n \geq 0$

SOLUTION:

Let P(n): $1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$

1. **Basis Step:**

P(0) is true.

For n = 0

L.H.S of P(0) = 1

R.H.S of P(0) = $2^{0+1} - 1 = 2 - 1 = 1$

Hence P(0) is true.

2. **Inductive Step:**

Suppose P(k) is true for some integer $k \geq 0$; i.e.,

$1+2+2^2+\ldots+2^k = 2^{k+1} - 1$……………………(1)

To prove P(k+1) is true, i.e.,

$1+2+2^2+\ldots+2^{k+1} = 2^{k+1+1} - 1$…………………(2)

Consider LHS of equation (2)

$1+2+2^2+\ldots+2^{k+1} = (1+2+2^2+\ldots+2^k) + 2^{k+1}$

$= (2^{k+1} - 1) + 2^{k+1}$

$= 2 \cdot 2^{k+1} - 1$

$= 2^{k+1+1} - 1 =$ R.H.S of (2)

Hence P(k+1) is true and consequently by mathematical induction the given propositional function is true for all integers $n \geq 0$.

EXERCISE:

Prove by mathematical induction

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$ for all integers $n \geq 1$

SOLUTION:

Let P(n) be the given equation.

1.Basis Step:

P(1) is true

For n = 1

L.H.S of P(1) = $\frac{1}{1\cdot 2}=\frac{1}{1\times 2}=\frac{1}{2}$

R.H.S of P(1) = $\frac{1}{1+1}=\frac{1}{2}$

Hence P(1) is true

2.Inductive Step:

Suppose P(k) is true, for some integer k≥1. That is

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k(k+1)}=\frac{k}{k+1}$$

To prove P(k+1) is true. That is

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(k+1)(k+1+1)}=\frac{k+1}{(k+1)+1} \quad \text{........(2)}$$

Now we will consider the L.H.S of the equation (2) and will try to get the R.H.S by using equation (1) and some simple computation.

Consider LHS of (2)

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(k+1)(k+2)}$$

$$=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k(k+2)+1}{(k+1)(k+2)}$$

$$=\frac{k^2+2k+1}{(k+1)(k+2)}$$

$$=\frac{(k+1)^2}{(k+1)(k+2)}$$

$$=\frac{k+1}{(k+2)}$$

$=$ RHS of (2)

Hence P(k+1) is also true and so by Mathematical induction the given equation is true for all integers n ≥1.