Formal Methods in Software Engineering

LECTURE # 6,7

Degree of a Predicate

- A one-place predicate forms a statement when combined with a single subject.

- The following are examples
  - **Jay** is clever
  - Kay is a Sophomore
  - Chris sleeps soundly
  - Max is very unhappy

Degree of a Predicate

- On the other hand, a two-place predicate takes two grammatical subjects, which is to say that it forms a statement when combined with two names.

  - Jones is taller than Smith
  - New York is south of Boston
  - Jay admires Kay
  - Kay respects Jay
  - Jay is a cousin of Kay

SINGULAR TERMS

- A singular term refers to a single individual – a person, place, thing, event, etc.,
- In order to decide whether a noun phrase qualifies as a singular term, the simplest thing to do is to check whether the noun phrase can be used properly with the singular verb form ‘is’. If the noun phrase requires the plural form ‘are’, then it is not a singular term, but is rather a plural term

SINGULAR TERMS

- Or we can say that
- In predicate logic, every subject is a singular term.

COMPOUND FORMULAS

- if Jay is a Freshman, then Kay is a Freshman
  - $Fj \rightarrow Fk$
- Kay is not a Freshman
  - $\sim Fk$
- neither Jay nor Kay is a Freshman
  - $\sim Fj \ \& \sim Fk$
- Jay respects Kay, but Kay does not respect Jay
  - $Rjk \ \& \sim Rkj$

Formula with Quantification

- Any formula can be prefixed by either a universal quantifier or an existential quantifier
- Px it is perfect
- Then let us quantify it both universally and existentially, as

Formula with Quantification

- $\forall xPx$ everything is such that it is perfect
- $\exists xPx$ at least one thing is such that it is perfect

Propositional Functions

- In logic, predicates can be obtained by removing some or all of the nouns from a statement. For instance, let P stand for "is a student at Bedford College" and let Q stand for "is a student at." Then both P and Q are *predicate symbols*.

Predicates and Quantified Statements

Predicate variables:

$P(x)$ = "x is a student at Bedford College"
$Q(x, y)$ = "x is a student at y"

- When concrete values are substituted in place of predicate variables, a statement results.

Propositional Functions

- To write in Predicate Logic 'x is greater than 3'
  - We introduce a functional symbol for the **predicate** and
  - Put the subject as an **argument** (to the functional symbol): $P(x)$
- Terminology
  - $P(x)$ is a statement
  - P is a predicate or propositional function
  - x as an argument

Propositional Functions

- Examples:
  - Father(x): unary predicate
  - Brother(x.y): binary predicate
  - Sum(x,y,z): ternary predicate
  - P(x,y,z,t): n-ary predicate

Propositional Functions

- **Definition:** A statement of the form $P(x_1,x_2,\ldots, x_n)$ is the value of the propositional symbol P.
- Here: $(x_1,x_2,\ldots, x_n)$ is an n-tuple and P is a predicate
- We can think of a propositional function as a function that
  - Evaluates to true or false
  - Takes one or more arguments
  - Expresses a predicate involving the argument(s)
  - Becomes a proposition when values are assigned to the arguments

Propositional Functions: Example

- Let $Q(x,y,z)$ denote the statement '$x^2+y^2=z^2$'
  - What is the truth value of $Q(3,4,5)$?

    $Q(3,4,5)$ is true

  - What is the truth value of $Q(2,2,3)$?

    $Q(2,3,3)$ is false

How many values of (x,y,z) make the predicate true?

There are infinitely many values that make the proposition true

Domain of a Predicate

- The **domain** of a predicate variable is the set of all values that may be substituted in place of the variable.

- Domain" may also be known as "domain of discourse", "universe of discourse", "universal set", or simply "universe". The last three terms are usually used in set theory.

Domain of a Predicate

- When an element in the domain of the variable of a one-variable predicate is substituted for the variable, the resulting statement is either true or false. The set of all such elements that make the predicate true is called the truth set of the predicate.

Domain of a Predicate

- If $P(x)$ is a predicate and x has domain D, the **truth set** is the set of all elements of D that make $P(x)$ true when they are substituted for x.

- The truth set of $P(x)$ is denoted $\{x \in D \mid P(x)\}$.

- *In set theory, the symbol | is used to mean "such that".*

Domain of a Predicate

- Let $Q(n)$ be the predicate "n is a factor of 8."
- Find the truth set of $Q(n)$ if

{1, 2, 4, 8} because these are exactly the positive integers that divide 8 evenly.

  - the domain of n is the set **Z** of all integers.

{1, 2, 4, 8, -1, -2, -4, -8} because the negative integers -1, -2, -4 and -8 also divide into 8 evenly.

Domain of a Predicate

- Consider the statement '*x*>3', does it make sense to assign to *x* the value 'blue'?
- Intuitively, the **universe of discourse** is the set of all things we wish to talk about; that is the set of all objects that we can sensibly assign to a variable in a propositional function.
- What would be the universe of discourse for the propositional function below be:

EnrolledCSE235(x)='x is enrolled in CSE235'

Free and bound variables

- Consider two predicates
- n>5…..Eq. 1

$\exists n : N \bullet n > 5$.....Eq.2

- In equation 1, n is free as its value may not be determined or restricted
- In equation 2, the variable n is bound, it is existentially quantified variable.

Free and bound variables

- Example

$$(\forall y : People \bullet y \text{ likes robin}) \vee x \text{ likes richard}$$

- In this statement, y and x are two variables. y is universally quantified and it is bound, while x is free variable.

Substitution

- Consider statement,

$$\exists n : N \bullet n > 5$$

- we may replace the name n with any term t, This process is called substitution: the expression p[t/n] denotes the fact that the term t is being substituted for the variable name n in the predicate p.

$$n > 5[3/n]$$

- Example, we may denote the substitution of 3 for n in the predicate n>5 by

Substitution

- Also, we may denote the substitution of 3+4 for n in n>5 by

  n>5[3+4/n]

Substitution

- More than one substitution in case of more than one free variable
- Consider predicate,

$$x \text{ is happy} \wedge y \text{ is sad}$$

- Here x and y are both free variables. We may substitute nigel for x as follows,

$$(x \text{ is happy} \wedge y \text{ is sad})[\text{nigel}/x] \Leftrightarrow$$

- Now substituting ken for y, $\text{nigel is happy} \wedge y \text{ is sad}$

$$(\text{nigel is happy} \wedge y \text{ is sad})[\text{ken}/y]$$

$$\text{nigel is happy} \wedge \text{ken is sad}$$

Substitution

- Similarly, in one step, overall substitution may be denoted as,

$$(x \text{ is happy} \wedge y \text{ is sad})[nigel/x][ken/y]$$

- These two substitutions take place sequentially, first nigel is substituted for x, then ken is substituted for y. However, for simultaneous substitutions,

$$(x \text{ is happy} \wedge y \text{ is sad})[nigel/x, ken/y]$$

Substitution

- To illustrate the difference between sequential and simultaneous substitution, consider the example,

$$(x > 3 \wedge y > 7)[y / x, 8 / y]$$

$$\Leftrightarrow \; y > 3 \wedge 8 > 7$$

$$(x > 3 \wedge y > 7)[y / x][8 / y]$$

$$\Leftrightarrow \; y > 3 \wedge y > 7[8 / y]$$

$$\Leftrightarrow 8 > 3 \wedge 8 > 7$$

$$\Leftrightarrow \mathit{true}$$

- In former there is only one occurrence of y and in latter there are two occurrences of y

Restriction

- Filters role in predicate logic
- Considering an example, all natural numbers which are prime numbers and are greater than 2 are odd, may be written as,

$$\forall n : N \bullet (Prime(n) \land n > 2) \Rightarrow Odd(n)$$

- Using a filter, statement may be written as,

$$\forall \mathrm{n} : \mathrm{N} \mid \mathrm{Prime(n)} \land \mathrm{n} > 2 \bullet \mathrm{Odd(n)}$$

- Form of restricted predicates as,

$$\text{quantifier quantification} \mid \text{restriction} \bullet \text{predicate}$$

Exercise

- Represent the following statement in the form,

quantifier quantification | restriction • predicate

- Everyone who likes cheese likes pizza
- There is a person who likes cheese that likes pizza
- Everyone who likes cheese also likes pizza and likes bananas

Solution

$$\forall p : People \mid likes(p, cheese) \bullet likes(p, pizza)$$

$$\exists p : People \mid likes(p, cheese) \bullet likes(p, pizza)$$

$$\forall p : People \mid likes(p, cheese) \bullet likes(p, pizza) \wedge likes(p, bananas)$$

Uniqueness

- Existential quantifier, allows us to represent statements such as “there is at least one x, such that…”.
- In order to be more specific, consider, “there is exactly one x, such that…”
- Consider an example statement,

$$\exists x : W \bullet x + 1 = 1$$

it is the case that the predicate x+1=1 is true for exactly one number:0

Uniqueness

- The operator which allow us to state that " there is exactly one natural number, x, is denoted as

$$\exists_1$$

$$\exists_1 x : N \bullet x + 1 = 1$$

- As an example,

$$\exists_1 \mathrm{c} : \mathrm{Country} \bullet \mathrm{c\ is\ a\ richest}$$

Uniqueness

- Assuming the statement

$$\exists_1 x : X \bullet p$$

- The μ operator allows us to do exactly this, the statement (μx:X | p) is read as " the unique x from set X such that p holds of x"
- Example, (μx:N | x+1=1) where μ is associated with value 0

Exercise

- Write the following in terms of μ expressions
  - The tallest mountain in the world
  - The height of the tallest mountain
  - The oldest person in the world
  - The nationality of the oldest person in the world

Solution

$$(\mu m : \text{Mountain} \mid \text{tallestin world}(m))$$

$$(\mu m : \text{Mountain} \mid \text{tallest in world}(m) \bullet \text{height}(m))$$

$$(\mu p : \text{People} \mid \text{oldest in world}(p)$$

$$(\mu p : \text{People} \mid \text{oldest in world}(p) \bullet \text{nationality}(p))$$