Formal Methods

Lecture # 5

Predicate Logic

(First Order Logic - FOL)

Syntax of FOL: Basic elements

- Constants KingJohn, 2,...
- Functions Sqrt, Likes...
- Variables x, y, a, b,...
- Connectives $\neg, \Rightarrow, \wedge, \vee, \Leftrightarrow$
- Equality =
- Quantifiers $\forall, \exists$

Predicates

- A predicate is a proposition whose truth depends on the value of one or more variables.
- For example, “n is a perfect square” is a predicate whose truth depends on the value of n.
- A function like notation is used to denote a predicate supplied with specific variable values. P(n)=“n is a perfect square”
- P(4) is true and P(5) is false.

Universal Quantification

- Universal Quantification denoted as $\forall$
- Universal Quantification allows us to capture statements of the form "for all" or "for every".
- For example, "every natural number is greater than or equal to zero" can be written formally as

$$\forall n : N \bullet n \geq 0$$

Universal Quantification

- A quantified statement consists of three parts: the quantifier, the quantification, and the predicate
- Every predicate logic statement can be considered as follows

$$\text{quantifier quantification} \bullet \text{predicate}$$

Universal Quantification

$$\forall p : People; d : Dog \bullet \text{p is owner} \wedge \text{p owns d}$$

Exercise

- Everybody likes Jaffa cakes
- All vegetarians don’t like Jaffa cakes
- Everybody either likes Jaffa cakes or is a vegetarian
- Either every body likes Jaffa cakes or everybody is a vegetarian

Solution

$\forall x : \text{People} \bullet x \text{ likes Jaffa cakes}$

$\forall x : \text{People} \bullet (x \text{ is vegetarian} \wedge \neg(x \text{ likes Jaffa cakes}))$

$\forall x : \text{People} \bullet (x \text{ likes Jaffa cakes} \vee x \text{ is vegetarian})$

$(\forall x : \text{People} \bullet x \text{ likes Jaffa cakes}) \vee (\forall x : \text{People} \bullet x \text{ is vegetarian})$

Universal Quantification

- Law 1

$$((\forall x : X \bullet p) \vee (\forall x : X \bullet q)) \Rightarrow (\forall x : X \bullet p \vee q)$$

- Example

$$((\forall \mathrm{p} : \mathrm{Person} \bullet \mathrm{p\ is\ wise}) \vee (\forall \mathrm{p} : \mathrm{Person} \bullet \mathrm{p\ is\ strong})) \Rightarrow$$
$$(\forall \mathrm{p} : \mathrm{Person} \bullet \mathrm{p\ is\ wise} \vee \mathrm{p\ is\ strong})$$

Universal Quantification

- Law 2

$$((\forall x : X \bullet p) \wedge (\forall x : X \bullet q)) \Rightarrow (\forall x : X \bullet p \wedge q)$$

- Example

$$((\forall \mathrm{p} : \mathrm{Person} \bullet \mathrm{p\ is\ wise}) \wedge (\forall \mathrm{p} : \mathrm{Person} \bullet \mathrm{p\ is\ strong}))$$

$$(\forall \mathrm{p} : \mathrm{Person} \bullet \mathrm{p\ is\ wise} \wedge \mathrm{p\ is\ strong})$$

Existential quantification

- Existential quantifier denoted by $\exists$
- Existential quantification is used to assert that a property holds of some (or at least one) elements of a set
- "Some natural numbers are divisible by 3" may be written as

$$\exists n : N \bullet n \bmod 3 = 0$$

Exercise

- Some people like Jaffa cakes
- Some vegetarians don’t like Jaffa cakes
- Some people either like Jaffa cakes or are vegetarian
- Either some people like Jaffa cakes or some people are vegetarian

Solution

$\exists x : People \bullet x \text{ likes Jaffa cakes}$

$\exists x : People \bullet x \text{ is vegetarian} \wedge \neg(x \text{ likes Jaffa cakes})$

$\exists x : People \bullet (x \text{ likes Jaffa cakes} \vee x \text{ is vegetarian})$

$(\exists x : People \bullet x \text{ likes Jaffa cakes}) \vee (\exists x : People \bullet x \text{ is vegetarian})$

Existential quantification

- Law 3

$$(\exists x : X \bullet p \vee q) -> ((\exists x : X \bullet p) \vee (\exists x : X \bullet q))$$

- Example

$$(\exists c : Car \bullet fast(c) \vee small(c)) ->$$
$$((\exists c : Car \bullet fast(c)) \vee (\exists c : Car \bullet small(c)))$$

Existential quantification

- Law 4

$$(\exists x : X \bullet p \wedge q) \Rightarrow ((\exists x : X \bullet p) \wedge (\exists x : X \bullet q))$$

- Example

$$\exists c : Car \bullet fast(c) \wedge small(c)) \Rightarrow$$

$$((\exists c : Car \bullet fast(c)) \wedge (\exists c : Car \bullet small(c)))$$

Satisfaction and validity

- The predicate n>3 can be considered neither true nor false unless we know the value associated with n

- A predicate p is valid if and only if it is true for all possible values of the appropriate type. That is, if a predicate p is associated with a variable x of type X, then p is valid if, and only if,

$$\forall x : X \bullet p \text{ is true}$$

- Example

The predicate $n \geq 0$ is valid as $\forall n : \mathrm{N} \bullet n \geq 0$ is equivalent to true

Satisfaction and validity

- A predicate p is satisfiable if and only if it is true for some values of the appropriate type. That is, if a predicate p is associated with a variable x of type X, then p is satisfiable if, and only if ,

$$\exists x : X \bullet p \text{ is true}$$

- Example

The predicate $n > 0$ is satisfiable as $\exists n : N \bullet n > 0$ is equivalent to true

Satisfaction and validity

- A predicate p is unsatisfiable if, and only if, it is false for all possible values of the appropriate type.

- If a predicate p is associated with a variable x of type X, then p is unsatisfiable if, and only if,

$$\forall x : X \bullet p \text{ is false}$$

Satisfaction and validity

- Valid predicates and tautologies are always true
- Satisfiable predicates and contingencies are sometimes true and sometimes false
- Unsatisfiable predicates and contradictions are never true

The negation of quantifiers

- The statement "some body like Brian" may be expressed via predicate logic as

$$\exists p : Person \bullet p \text{ likes Brian}$$

- To negate this expression, we may write as,

$$\neg\exists p : Person \bullet p \text{ likes Brian}$$

- which in natural language may be expressed as "nobody likes Brian"

The negation of quantifiers

- Logically saying “nobody likes Brian” is equivalent to saying “everybody does not like Brian”.
- The negation of quantifiers behaves exactly in this fashion, just as in natural language, “nobody likes Brian” and “everybody does not like Brian” are equivalent so in predicate logic
- And $\neg(\exists p : Person \bullet p \text{ likes Brian})$

$$\forall p : Person \bullet \neg(p \text{ likes Brian})$$

- are equivalent.

The negation of quantifiers

- Law 3

$$\neg(\exists x : X \bullet p) \Leftrightarrow \forall x : X \bullet \neg p$$
$$\neg(\forall x : X \bullet p) \Leftrightarrow \exists x : X \bullet \neg p$$

- When negation is applied to a quantified expression it flips quantifiers as it moves inwards(i.e negation turns all universal quantifiers to existential quantifiers and vice versa, and negates all predicates)