Set Theory

Set

- A set is a collection of well-defined objects that contains no duplicates
- The term “well defined” means that for a given value it is possible to determine whether or not it is a member of the set.
- There are many examples of sets such as
  - Set of natural numbers $\mathbb{N}$
  - The set of integer numbers $\mathbb{Z}$
  - The set of rational numbers $\mathbb{Q}$.

Venn Diagrams

- Venn diagrams may be used to represent sets pictorially
- They may be used to illustrate various set operations such as set union, intersection and set difference

Example

– The bank accounts in a bank;

– The set of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$;

– The integer numbers $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$;

– The non-negative integers $\mathbb{Z}+ = \{0, 1, 2, 3, \ldots\}$;

– The set of prime numbers $= \{2, 3, 5, 7, 11, 13, 17, \ldots\}$;

Finite Set

- A finite set may be defined by listing all of its elements.
- For example, the set A = {2, 4, 6, 8, 10} is the set of all even natural numbers less than or equal to 10.

Set

- The order in which the elements are listed is not relevant;
- The set {2, 4, 6, 8,10} is the same as the set {8, 4, 2, 10, 6}.

Sets defined by Predicate

- Sets may be defined by using a predicate to constrain set membership.

Forexample,

The set S = {n : $\mathbb{N}$: n 10 ^ n mod 2 = 0} also represents the set {2, 4,

- 6, 8, 10}.

The set of even natural numbers may be defined by a predicate over the set of natural numbers that restricts membership to the even numbers. It is defined by:

$$\text{Evens} = \{x \mid x \in \mathbb{N} \ \wedge \ even(x)\}.$$

Set Theory Common Notations

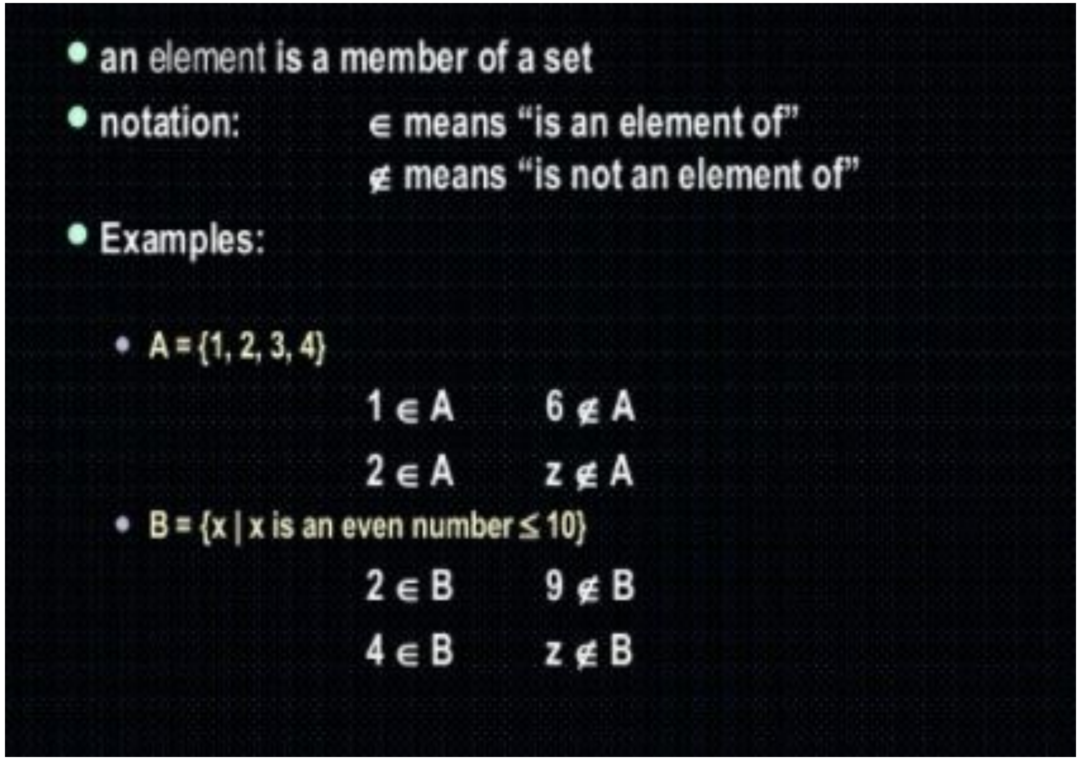
Symbol	Meaning
Upper case	designates set name
Lower case	designates set elements
{ }	enclose elements in set
$\in$ or	is (or is not) an element of
$\subseteq$	is a subset of (includes equal sets)
$\subset$	is a proper subset of
$\not\subset$	is not a subset of
$\supset$	is a superset of
| or :	such that (if a condition is true)
| |	the cardinality of a set

Elements of a Finite set

The elements of a finite set S are denoted by {x1, x2, … xn}.

- The expression $x \in S$ denotes that the element x is a member of the set S
- The expression $x \notin S$ indicates that x is not a member of the set S.

Exercise


an element is a member of a set
notation: ∈ means "is an element of"
∉ means "is not an element of"
Examples:
A = {1, 2, 3, 4}
1 ∈ A
6 ∉ A
2 ∈ A
z ∉ A
B = {x | x is an even number ≤ 10}
2 ∈ B
9 ∉ B
4 ∈ B
z ∉ B

Sub Set ,Super Set and Proper Sub set

A set S is a subset of a set T (denoted $S \subseteq T$) if whenever $s \in S$ then $s \in T$, and in this case the set T is said to be a superset of S (denoted $T \supseteq S$). Two sets S and T are said to be equal if they contain identical elements; that is, $S = T$ if and only if $S \subseteq T$ and $T \subseteq S$. A set S is a proper subset of a set T (denoted $S \subset T$) if $S \subseteq T$ and $S \neq T$. That is, every element of S is an element of T, and there is at least one element in T that is not an element of S. In this case, T is a proper superset of S (denoted $T \supset S$).

Example 4.2

(i) $\{1, 2\} \subseteq \{1, 2, 3\}$;
(ii) $\varnothing \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$;

Exercise

- $A = \{x \mid x \text{ is a positive integer} \leq 8\}$

  set A contains: 1, 2, 3, 4, 5, 6, 7, 8
- $B = \{x \mid x \text{ is a positive even integer} < 10\}$

  set B contains: 2, 4, 6, 8
- $C = \{2, 4, 6, 8, 10\}$

  set C contains: 2, 4, 6, 8, 10
- Subset Relationships

$A \subseteq A$	$A \not\subset B$	$A \not\subset C$
$B \subset A$	$B \subseteq B$	$B \subset C$
$C \not\subset A$	$C \not\subset B$	$C \subseteq C$

Empty Set

The empty set (denoted by $\varnothing$ or $\{\}$) represents the set that has no elements. Clearly, $\varnothing$ is a subset of every set. The singleton set containing just one element x is denoted by $\{x\}$, and clearly $x \in \{x\}$ and $x \neq \{x\}$. Clearly, $y \in \{x\}$ if and only if $x = y$.

Cardinality of Set

The cardinality (or size) of a finite set S defines the number of elements present in the set. It is denoted by $|S|$. The cardinality of an infinite[4] set S is written as $|S| = \infty$.

Example 4.3

(i) Given $A = \{2, 4, 5, 8, 10\}$, then $|A| = 5$.
(ii) Given $A = \{x \in \mathbb{Z} : x^2 = 9\}$, then $|A| = 2$.
(iii) Given $A = \{x \in \mathbb{Z} : x^2 = -9\}$, then $|A| = 0$.

Set Theoretical Operations

Cartesian Product

The Cartesian product allows a new set to be created from existing sets. The Cartesian[5] product of two sets S and T (denoted $S \times T$) is the set of ordered pairs $\{(s, t) \mid s \in S, t \in T\}$. Clearly, $S \times T \neq T \times S$ and so the Cartesian product of two sets is not commutative. Two ordered pairs (s_1, t_1) and (s_2, t_2) are considered equal if and only if $s_1 = s_2$ and $t_1 = t_2$.

Power Set

The power set of a set A (denoted $\mathbb{P}A$) denotes the set of subsets of A. For example, the power set of the set $A = \{1, 2, 3\}$ has eight elements and is given by:

$$\mathbb{P}A = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

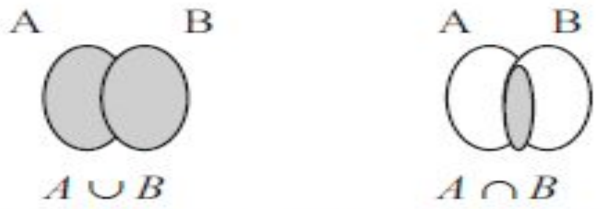
There are $2^3 = 8$ elements in the power set of $A = \{1, 2, 3\}$, and the cardinality of A is 3. In general, there are $2^{|A|}$ elements in the power set of A.

Set Theoretical Operations

<u>*Union and Intersection Operations*</u> The union of two sets A and B is denoted by $A \cup B$. It results in a set that contains all of the members of A and of B and is defined by:

$$A \cup B = \{r |\ r \in A \text{ or } r \in B\}.$$

For example, suppose $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$, then $A \cup B = \{1, 2, 3, 4\}$. Set union is a commutative operation; that is, $A \cup B = B \cup A$. Venn diagrams are used to illustrate these operations pictorially.



The intersection of two sets A and B is denoted by $A \cap B$. It results in a set containing the elements that A and B have in common and is defined by:

Set Theoretical Operations

Set Difference Operations

The set difference operation $A \backslash B$ yields the elements in A that are not in B. It is defined by

$$A \backslash B = \{a |\ a \in A \text{ and } a \notin B\}.$$

For A and B defined as A = {1, 2} and B = {2, 3}, we have $A \backslash B = \{1\}$ and $B \backslash A = \{3\}$. Clearly, set difference is not commutative; that is, $A \backslash B \neq B \backslash A$. Clearly, $A \backslash A = \varnothing$ and $A \backslash \varnothing = A$.

Properties of Set Theory Operations

Table 4.1 Properties of set operations

Property	Description
Commutative	Union and intersection operations are commutative; that is, $S \cup T = T \cup S$ $S \cup T = T \cap S$
Associative	Union and intersection operations are associative; that is, $R \cup (S \cup T) = (R \cup S) \cup T$ $R \cap (S \cap T) = (R \cap S) \cap T$
Identity	The identity under set union is the empty set $\varnothing$, and the identity under intersection is the universal set U $S \cup \emptyset = \emptyset \cup S = S$ $S \cap U = U \cap S = S$
Distributive	The union operator distributes over the intersection operator and vice versa $R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$ $R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$
De Morgan's[a] law	The complement of $S \cup T$ is given by: $(S \cup T)^c = S^c \cap T^c$ The complement of $S \cap T$ is given by: $(S \cap T)^c = S^c \cup T^c$

Relation

- **The relation** is the subset of the Cartesian product which contains only some of the ordered pair based on the relationships defined between the first and second elements. The relation is usually denoted by R.

Binary Relation

- A binary relation describes **a relationship between the elements of 2 sets**. If A and B are sets, then a binary relation R from A to B is a subset of the Cartesian product of A and B

Example

$A=\{1,2,9\}$ and $B=\{1,3,7\}$

Consider a relation where $a=b$ for the sets given above:

$R=\{(1,1)\}$

Consider a relation where $a<b$ for the sets given above:

$R=\{(1,3),(1,7),(2,3),(2,7)\}$

Inverse Relation

An inverse relation, as its name suggests, is the inverse of a relation. Let us recall what is a relation. A relation is the collection of ordered pairs. Let us consider two sets A and B. Then the set of all ordered pairs of the form (x, y) where $x \in A$ and $y \in B$ is called the cartesian product of A and B, which is denoted by A x B. Any subset of this cartesian product A x B is a relation.

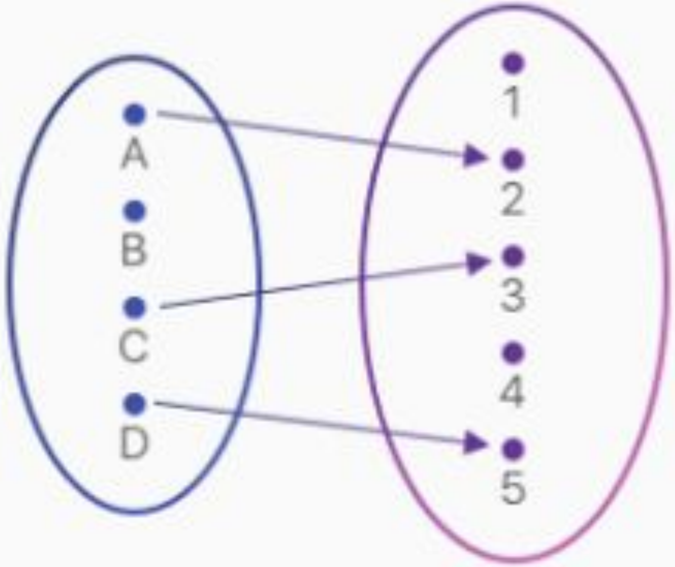
Inverse Relation

- An inverse relation (in set theory) is the inverse of a relation and is obtained by interchanging the elements of each ordered pair of the given relation. Let R be a relation from a set A to another set B. Then R is of the form $\{(x, y): x \in A \text{ and } y \in B\}$. The inverse relationship of R is denoted by R^{-1} and its formula is $R^{-1} = \{(y, x): y \in B \text{ and } x \in A\}$. i.e.,

Function

- A *function* in set theory world is simply a mapping of some (or all) elements from Set A to some (or all) elements in Set B

Function



Domain	{ A,B,C,D }
Arguments	{ A,C,D }
Codomain	{ 1,2,3,4,5 }
Image	{ 2,3,5 }

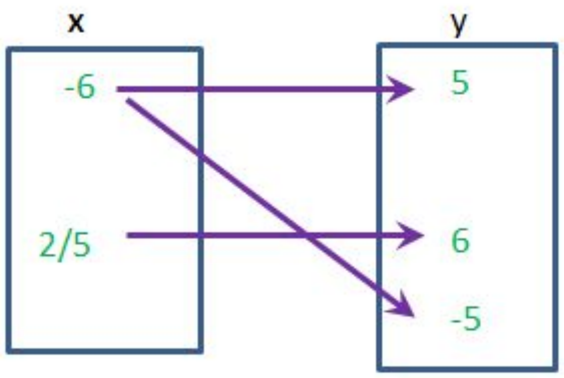
- In the example above, the collection of all the possible elements in A is known as the ***domain;*** while the elements in A that act as inputs are specially named ***arguments***. On the right, the collection of all possible outputs (also known as “range” in other branches), is referred to as the ***codomain***; while the collection of actual output elements in B mapped from A is known as the ***image.***

Function

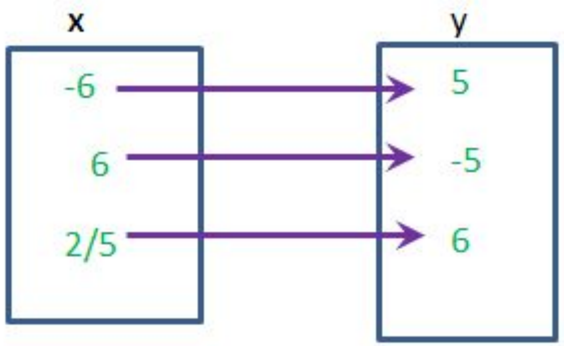
A function $f : X \to Y$ between sets X, Y assigns to each $x \in X$ a unique element $f(x) \in Y$. Functions are also called maps, mappings, or transformations. The set X on which f is defined is called the domain of f and the set Y in which it takes its values is called the codomain. We write f : x 7→ f(x) to indicate that f is the function that maps x to f(x).

Functions and Non Functions Example

The relation { (-6, 5), (-6, -5), (2/5, 6) }



d) The relation { (-6, 5), (6, -5), (2/5, 6) }



Exercise

- Check if the following ordered pairs are functions:
- W= {(1, 2), (2, 3), (3, 4), (4, 5)
- Y = {(1, 6), (2, 5), (1, 9), (4, 3)}