Methods of Proof

To understand written mathematics, one must understand what makes up a correct mathematical argument, that is, a proof. This requires an understanding of the techniques used to build proofs. The methods we will study for building proofs are also used throughout computer science, such as the rules computers used to reason, the techniques used to verify that programs are correct, etc. Many theorems in mathematics are implications, p $\rightarrow$q. The techniques of proving implications give rise to different methods of proofs.

Basic Terminologies before Methods of Proof:

a. **Logic:** *Logic is the study of the principles and methods that distinguishes between a valid and an invalid argument*

b. **Statement/Proposition:** A statement is a declarative sentence that is either true or false but not both. A statement is also referred to as a proposition, Example: 2+2 = 4, It is Sunday today

c. **Conditional Statement:** A **conditional statement**, symbolized by p q, is an if-then **statement** in which p is a hypothesis and q is a conclusion. The logical connector in a **conditional statement** is denoted by the symbol. The **conditional** is defined to be true unless a true hypothesis leads to a false conclusion.
For example: Consider the statement:
"If you earn an A in Math, then I'll buy you a computer."
p: "You earn an A in Math," and
q: "I will buy you a computer."
The original statement is then saying:
if p is true, then q is true, or, more simply, if p, then q.
We can also phrase this as p implies q, and we write p $\rightarrow$ q.

INVERSE OF A CONDITIONAL STATEMENT:

The inverse of the conditional statement **p $\rightarrow$ q** is **~p $\rightarrow$ ~q**

CONVERSE OF A CONDITIONAL STATEMENT:

The converse of the conditional statement **p $\rightarrow$ q** is **q $\rightarrow$p**

CONTRAPOSITIVE OF A CONDITIONAL STATEMENT:

The contrapositive of the conditional statement **p $\rightarrow$ q** is**~ q $\rightarrow$ ~ p**

ARGUMENT:

An **argument** is a list of statements called **premises** (or **assumptions** or **hypotheses**) followed by a statement called the **conclusion.**

P1	Premise
P2 Premise	
P3	Premise
. . . .	
Pn	Premise

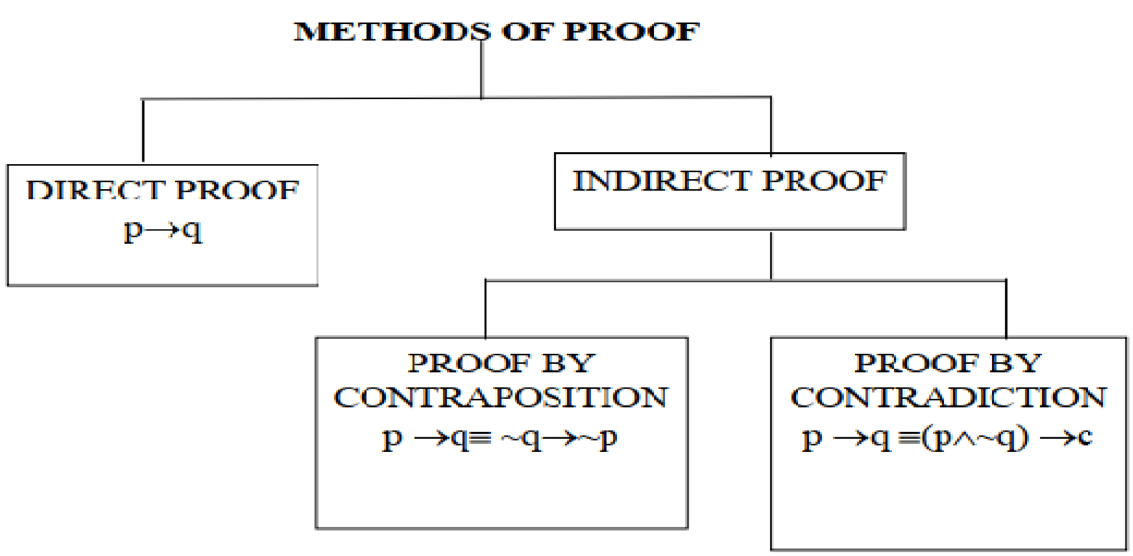
$\therefore$C Conclusion

What is a Proof?

a) A proof is a valid argument that establishes the truth of a theorem (as the conclusion)
b) Statements in a proof can include the axioms (something assumed to be true), the premises, and previously proved theorems
c) Rules of inference, and definitions of terms, are used to draw intermediate conclusions from the other statements, tying the steps of a proof
d) Final step is usually the conclusion of theorem

Related Terms:

a) Lemma: a theorem that is not very important – We sometimes prove a theorem by a series of lemmas
b) Corollary : a theorem that can be easily established from a theorem that has been proved
c) Conjecture : a statement proposed to be a true statement, usually based on partial evidence, or intuition of an expert



Some basic regarding odds and even numbers:

An integer n is even if, and only if, $n = 2k$ for some integer k.

An integer n is odd if, and only if, $n = 2k + 1$ for some integer k

1. **Direct Method:**
   $p \rightarrow q$

Sample Solution-1:

EXERCISE:

Prove that the sum of two odd integers is even.

SOLUTION:

Let m and n be two odd integers. Then by definition of odd numbers

$m = 2k + 1 \quad$ for some $k \in Z$

$n = 2l + 1 \quad$ for some $l \in Z$

$$
\begin{aligned}
\text{Now} \quad m + n &= (2k + 1) + (2l + 1) \\
&= 2k + 2l + 2 \\
&= 2(k + l + 1) \\
&= 2r \qquad \text{where } r = (k + l + 1) \in Z
\end{aligned}
$$

Hence $m + n$ is even.

Sample Solution-II:

EXERCISE:

Prove that the square of an even integer is even.

SOLUTION:

Suppose n is an even integer. Then $n = 2k$

Now

$$
\begin{aligned}
\text{square of } n = n^2 &= (2 \cdot k)^2 \\
&= 4k^2 \\
&= 2 \cdot (2k^2) \\
&= 2 \cdot p \quad \text{where } p = 2k^2 \in Z
\end{aligned}
$$

Hence, n^2 is even. (proved)

Sample Solution-1II and IV:

EXERCISE:

Prove that if n is an odd integer, then $n^3 + n$ is even.

SOLUTION:

Let n be an odd integer, then $n = 2k + 1$ for some $k \in Z$

Now $n^3 + n = n(n^2 + 1)$

$= (2k + 1)((2k+1)^2 + 1)$

$= (2k + 1)(4k^2 + 4k + 1 + 1)$

$= (2k + 1)(4k^2 + 4k + 2)$

$= (2k + 1)\, 2.\, (2k^2 + 2k + 1)$

$= 2\cdot(2k + 1)(2k^2 + 2k + 1) \qquad k \in Z$

= an even integer

EXERCISE:

Prove that, if the sum of any two integers is even, then so is their difference.

SOLUTION:

Suppose m and n are integers so that m + n is even. Then by definition of even numbers

$m + n = 2k \qquad$ for some integer k

$\Rightarrow \quad m = 2k - n \;\ldots\ldots\ldots\ldots\ldots\ldots(1)$

Now $m - n = (2k - n) - n \qquad$ using (1)

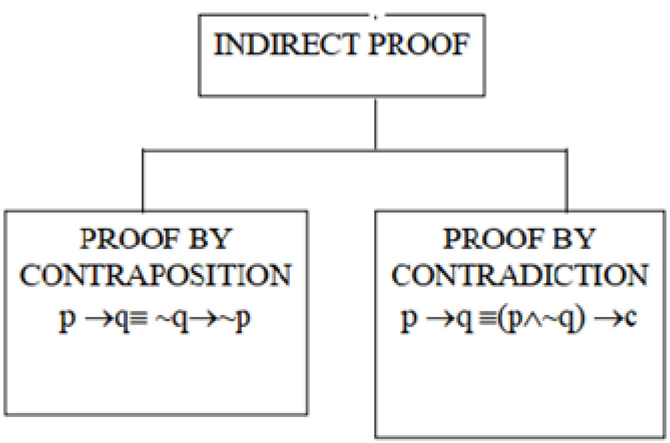
$= 2k - 2n$

$= 2(k - n) = 2r \qquad$ where $r = k - n$ is an integer

Hence m - n is even.

2. Indirect Method of Proof:

There are two subcategories, as shown below



i. Proof by Contraposition: i.e $p \to q \cong \sim q \to \sim p$

A proof by contraposition is based on the logical equivalence between a statement and its contrapositive. Therefore, the implication $p \to q$ can be proved by showing that its contrapositive $\sim q \to \sim p$ is true. The contrapositive is usually proved directly.
The method of proof by contrapositive may be summarized as:

a) Express the statement in the form if p then q
b) Rewrite this statement in the contrapositive form if not q then not p.
c) Prove the contrapositive by a direct proof.

Sample Solution-1:

EXERCISE:

Prove that for all integers n, if n^2 is even then n is even.

PROOF:

The contrapositive of the given statement is:
"if n is not even (odd) then n^2 is not even (odd)"
We prove this contrapositive statement directly.
Suppose n is odd. Then $n = 2k + 1$ for some $k \in Z$
Now $n^2 = (2k+1)^2 = 4k^2 + 4k + 1$
$= 2\cdot(2k^2 + 2k) + 1$
$= 2\cdot r + 1$ where $r = 2k^2 + 2k \in Z$
Hence n^2 is odd. Thus the contrapositive statement is true and so the given statement is true.

Sample Solution-II:

EXERCISE:

Prove that if $3n + 2$ is odd, then n is odd.

PROOF:

The contrapositive of the given conditional statement is

" if n is even then $3n + 2$ is even"
Suppose n is even, then
$n = 2k$ for some $k \in Z$
Now $3n + 2 = 3(2k) + 2$
$= 2. (3k + 1)$
$= 2.r$ where $r = (3k + 1) \in Z$
Hence $3n + 2$ is even. We conclude that the given statement is true since its contrapositive is true.

Proof by Contradiction

A proof by contradiction is based on the fact that either a statement is true or it is false but not both. Hence the supposition, that the statement to be proved is false, leads logically to a contradiction, impossibility or absurdity, then the supposition must be false. Accordingly, the given statement must be true.

To use contradiction, firs of all we have to convert, conditional statement to the following format.

$p \rightarrow q \cong p \wedge \sim q$

Many theorems in mathematics are conditional statements ($p \rightarrow q$). Now the negation of he implication $p \rightarrow q$ is

$\sim(p \rightarrow q) \equiv \sim(\sim p \vee q)$

Sample Solution-I:

EXERCISE:

Give a proof by contradiction for the statement:
"If n^2 is an even integer then n is an even integer."

PROOF:

Suppose n^2 is an even integer and n is not even, so that n is odd.
Hence $n = 2k + 1$ for some integer k.

$$
\begin{aligned}
\text{Now} \quad n^2 &= (2k + 1)^2 \\
&= 4k^2 + 4k + 1 \\
&= 2 \cdot (2k^2 + 2k) + 1
\end{aligned}
$$

Sample Solution-II:

EXERCISE:

Prove by contradiction method, the statement: If n and m are odd integers, then n + m is an even integer.

SOLUTION:

Suppose n and m are odd and n + m is not even (odd i.e by taking contradiction).

Now $n = 2p + 1$ for some integer p
and $m = 2q + 1$ for some integer q

$$
\begin{aligned}
\text{Hence} \quad n + m &= (2p + 1) + (2q + 1) \\
&= 2p + 2q + 2 = 2 \cdot (p + q + 1)
\end{aligned}
$$

which is even, contradicting the assumption that n + m is odd.