National University of Modern Languages, Islamabad
Faculty of Engineering & Computer Science
Department of Computer Science
BSCS-V- Numerical Computing
Lecturer: Dr. Muhammad Mohsin

Interpolation

Interpolation means to insert or add something and in numerical computing, it means to find new data points, using the given or known data points.
It is actually a powerful method of construction or estimating a function $f(x)$ from the known distinct data points such that the function passes through them.
We define

Function = Polynomial + Error,

$$f(x) = P(x) + E,$$

$$f(x) \sim P(x).$$

Through two distinct points, we can construct a unique polynomial of degree one, that is, a linear polynomial (or a straight line). In general, through n distinct points, we can construct a unique polynomial of degree $n-1$.

Lagrange Interpolation:

Suppose that we are given the following two distinct data points.

x_0	x_1
$f(x_0)$	$f(x_1)$

So Lagrange linear polynomial (of degree one) is defined as

$$P_1(x) = L_0(x)f(x_0) + L_1(x)f(x_1),$$

where $L_0(x)$ and $L_1(x)$ are called Lagrange's Fundamental Polynomials and are defined as

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \quad \text{and} \quad L_1(x) = \frac{x - x_0}{x_1 - x_0}.$$

Similarly, for three distinct data points,

x_0	x_1	x_2
$f(x_0)$	$f(x_1)$	$f(x_2)$

Lagrange quadratic polynomial (of degree two) is defined as

$$P_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2),$$

where

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}, \quad L_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}$$

and $$L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}.$$

Task: For the following four data points, write a cubic Lagrange polynomial (of degree three).

x_0	x_1	x_2	x_3
$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$

Example: Find the Lagrange polynomial for the following data.

$x_0 = 0$	$x_1 = 1$	$x_2 = 3$
$f(x_0) = 1$	$f(x_1) = 3$	$f(x_2) = 55$

In addition, interpolate or find $f(2)$.

Solution: Since we are given three data points, so Lagrange quadratic polynomial of degree two is given by

$$P_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2),$$

where

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-1)(x-3)}{(-1)(-3)} = \frac{1}{3}(x^2-4x+3),$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0)(x-3)}{(1)(-2)} = \frac{1}{2}(3x-x^2),$$

and $L_2(x) = \dfrac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \dfrac{(x)(x-1)}{(3)(2)} = \dfrac{1}{6}(x^2-x).$

Therefore,

$$P_2(x) = \frac{1}{3}(x^2-4x+3)\,(1) + \frac{1}{2}(3x-x^2)\,(3) + \frac{1}{6}(x^2-x)(55).$$

We next take L.C.M. to write

$$\begin{aligned} P_2(x) &= \frac{1}{6}(2x^2-8x+6+27x-9x^2+55x^2-55x) \\ &= \frac{1}{6}(48x^2-36x+6) \\ &= 8x^2-6x+1 \\ &= f(x). \end{aligned}$$

So our required function is

$$f(x) = 8x^2-6x+1.$$

Now,

$$f(2) = 8(2)^2-6(2)+1 = 21.$$

Task: Find the Lagrange polynomial for the following four data points.

x	-1	2	4	6
$f(x)$	-9	0	56	208

In addition, interpolate or find $f(3)$.

Numerical Differentiation

The general method for deriving the numerical differentiation formula is to differentiate the interpolation polynomial. The Lagrange's quadratic interpolation polynomial is given by

$$P(x) = L_0(x)f_0 + L_1(x)f_1 + L_2(x)f_2,$$

$$P(x) = L_0(x)y_0 + L_1(x)y_1 + L_2(x)y_2.$$

Differentiating both sides with respect to x, we obtain

$$P'(x) = L_0'(x)y_0 + L_1'(x)y_1 + L_2'(x)y_2,$$

$$P''(x) = L_0''(x)y_0 + L_1''(x)y_1 + L_2''(x)y_2$$

and so on.

Example: Perform the numerical differentiation to find the first derivative, at $x = 2$, of the following data, using the Lagrange's interpolation formula.

$$y(0) = 1, y(1) = 3, y(3) = 55.$$

Solution: The Lagrange's quadratic polynomial, for the given data, is given by

$P_2(x) = P(x) = 8x^2 - 6x + 1.$ (That has been derived before.)

Differentiating both sides with respect to x, we obtain
$P'(x) = 16x - 6.$

Therefore, we obtain

$P'(2) = 16(2) - 6 = 26.$
