Imaging Science

Exercise 1

An Trieu (3523966)
study programme: M.Sc. INFOTECH

Valentina Roldan (3519666)
study programme: M.Sc. INFOTECH

Khanh Quynh Nguyen (3517671)
study programme: M.Sc. INFOTECH

Abdul Rehman (3440146)
study programme: M.Sc. INFOTECH

Abdul Rehman (3440146)
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1 Discrete Convolution

Consider the discrete signal f:

$$f_i = \begin{cases} 1/2 & i = 0 \\ 1/2 & i = 1 \\ 0 & else \end{cases}$$

Let f_i be the following signal:

Position(i)	-3	-2	-1	0	1	2	3
value (g_i)	0	0	0	1/2	1/2	0	0

Let w_i be the following convolution kernel (mirrored weights):

Position(i)	-3	-2	-1	0	1	2	3
value (g_i)	0	0	0	1/2	1/2	0	0

a) First Convolution

The result $(g * w)_i$ at location i = -2:

Position(m)	-3	-2	-1	0	1	2	3
value (g_m)	0	0	0	1/2	1/2	0	0
value (w_{i-m})	1/2	1/2	0	0	0	0	0
Result $(g * w)_i$	0	0	-	-	-	-	-

The result $(g * w)_i$ at location i = -1:

Position(m)	-3	-2	-1	0	1	2	3
value (g_m)	0	0	0	1/2	1/2	0	0
value (w_{i-m})	0	1/2	1/2	0	0	0	0
Result $(g * w)_i$	0	0	0	-	-	-	-

The result $(g * w)_i$ at location i = 0:

Position(m)	-3	-2	-1	0	1	2	3
value (g_m)	0	0	0	1/2	1/2	0	0
value (w_{i-m})	0	0	1/2	1/2	0	0	0
Result $(g * w)_i$	0	0	0	1/4	-	-	-

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The result $(g * w)_i$ at location i = 1:

Position(m)	-3	-2	-1	0	1	2	3
value (g_m)	0	0	0	1/2	1/2	0	0
value (w_{i-m})	0	0	0	1/2	1/2	0	0
Result $(g * w)_i$	0	0	0	1/4	1/2	-	-

The result $(g * w)_i$ at location i = 2:

Position(m)	-3	-2	-1	0	1	2	3
value (g_m)	0	0	0	1/2	1/2	0	0
value (w_{i-m})	0	0	0	0	1/2	1/2	0
Result $(g * w)_i$	0	0	0	1/4	1/2	1/4	-

The result $(g * w)_i$ at location i = 3:

Position(m)	-3	-2	-1	0	1	2	3
value (g_m)	0	0	0	1/2	1/2	0	0
value (w_{i-m})	0	0	0	0	0	1/2	1/2
Result $(g * w)_i$	0	0	0	1/4	1/2	1/4	0

b) 2^{nd} Convolution
Use result a new discrete signal f_i:

Position(i)	-3	-2	-1	0	1	2	3
value (g_i)	0	0	0	1/4	1/2	1/4	0

Let w_i be the following convolution kernel (mirrored weights):

Position(i)	-3	-2	-1	0	1	2	3
value (g_i)	0	0	0	1/2	1/2	0	0

The result $(g * w)_i$ at location i = 0:

Position(m)	-3	-2	-1	0	1	2	3
value (g_m)	0	0	0	1/4	1/2	1/4	0
value (w_{i-m})	0	0	1/2	1/2	0	0	0
Result $(g * w)_i$	0	0	0	1/8	-	-	-

The result $(g * w)_i$ at location i = 1:

Position(m)	-3	-2	-1	0	1	2	3
value (g_m)	0	0	0	1/4	1/2	1/4	0
value (w_{i-m})	0	0	0	1/2	1/2	0	0
Result $(g * w)_i$	0	0	0	1/8	3/8	-	-

The result $(g * w)_i$ at location i = 2:

Position(m)	-3	-2	-1	0	1	2	3
value (g_m)	0	0	0	1/4	1/2	1/4	0
value (w_{i-m})	0	0	0	0	1/2	1/2	0
Result $(g * w)_i$	0	0	0	1/8	3/8	3/8	-

The result $(g * w)_i$ at location i = 3:

Position(m)	-3	-2	-1	0	1	2	3
value (g_m)	0	0	0	1/4	1/2	1/4	0
value (w_{i-m})	0	0	0	0	0	1/2	1/2
Result $(g * w)_i$	0	0	0	1/8	3/8	3/8	1/8

c) 3^{rd} Convolution
Use result a new discrete signal f_i:

Position(i)	-3	-2	-1	0	1	2	3
value (g_i)	0	0	0	1/8	3/8	3/8	1/8

Let w_i be the following convolution kernel (mirrored weights):

Position(i)	-3	-2	-1	0	1	2	3	4
value (g_i)	0	0	0	1/2	1/2	0	0	0

The result $(g * w)_i$ at location i = 0:

Position(m)	-3	-2	-1	0	1	2	3	4
value (g_m)	0	0	0	1/8	3/8	3/8	1/8	0
value (w_{i-m})	0	0	1/2	1/2	0	0	0	0
Result $(g * w)_i$	0	0	0	1/16	-	-	-	-

The result $(g * w)_i$ at location i = 1:

Position(m)	-3	-2	-1	0	1	2	3	4
value (g_m)	0	0	0	1/8	3/8	3/8	1/8	0
value (w_{i-m})	0	0	0	1/2	1/2	0	0	0
Result $(g * w)_i$	0	0	0	1/16	1/4	-	-	-

The result $(g * w)_i$ at location i = 2:

Position(m)	-3	-2	-1	0	1	2	3	4
value (g_m)	0	0	0	1/8	3/8	3/8	1/8	0
value (w_{i-m})	0	0	0	0	1/2	1/2	0	0
Result $(g * w)_i$	0	0	0	1/16	1/4	3/8	-	-

The result $(g * w)_i$ at location i = 3:

Position(m)	-3	-2	-1	0	1	2	3	4
value (g$_m$)	0	0	0	1/8	3/8	3/8	1/8	0
value (w$_{i-m}$)	0	0	0	0	0	1/2	1/2	0
Result (g * w)$_i$	0	0	0	1/16	1/4	3/8	1/4	-

The result $(g * w)_i$ at location i = 4:

Position(m)	-3	-2	-1	0	1	2	3	4
value (g$_m$)	0	0	0	1/8	3/8	3/8	1/8	0
value (w$_{i-m}$)	0	0	0	0	0	0	1/2	1/2
Result (g * w)$_i$	0	0	0	1/16	1/4	3/8	1/4	1/16

2 Properties of the Convolution

With the convolution formula:

$$(f * w)_i = \sum_{k=-\infty}^{\infty} f_{i-k} w_k$$

and three infinite discrete signal f, g, w, proof the following properties:

Linearity: $(\alpha * f + \beta * g) * w = \alpha(f * w) + \beta(g * w)$
Proof: We have:

$$(\alpha * f + \beta * g) * w = \sum_{k=-\infty}^{\infty} (\alpha f_{i-k} + \beta g_{i-k}) w_k$$

$$= (\sum_{k=-\infty}^{\infty} \alpha f_{i-k} w_k + \beta g_{i-k} w_k)$$

$$= \alpha \sum_{k=-\infty}^{\infty} f_{i-k} w_k + \beta \sum_{k=-\infty}^{\infty} g_{i-k} w_k$$

$$= \alpha f * w + \beta g * w$$

Therefore, the convolution possesses the linearity property

Commutativity: $f * w = w * f$
Proof: We have:

$$f * w = \sum_{k=-\infty}^{\infty} f_{i-k} w_k$$

Let $m = i - k => k = i - m$. When $k = \infty => m = -\infty$ and $k = -\infty => m = \infty$. With the assumption that the signals fulill all necessary conditions such that a reordering of an infinite series is allowed, the formular f*w equal to:

$$f * w = \sum_{m=\infty}^{-\infty} f_m w_{i-m} = \sum_{m=-\infty}^{\infty} f_m w_{i-m} = w * f$$

Therefore, the convolution possesses the Commutativity property
Identity: For which signal e does $f * e = f$ hold?
The above equation is equivalent to

$$\sum_{k=-\infty}^{\infty} f_{i-k} e_k = f_i$$

We try the following signal e:

$$e_k = \begin{cases} 1 & k = 0 \\ 0 & otherwise \end{cases}$$

With this signal, $f * e$ is equal to:

$$\sum_{k=-\infty}^{\infty} f_{i-k} e_k = ... + f_{i-0} e_0 + f_{i-1} e_1 + ...$$

Only $e_0 = 1$, hence:

$$\sum_{k=-\infty}^{\infty} f_{i-k} e_k = ... + f_{i-0} 1 + f_{i-1} 0 + ... \ = f_i$$

=> e_i satisfy the condition of Identity and the convolution has at least one identity element => Convolution has identity property

3 Quantisation, Noise Models, Error Measures

a) Images

Abbildung 1: Top left: orignal
Top right: q = 3 with no noise
Bottom left: q = 3 and a = 0.25 Gaussian noise
Bottom right: q = 3 and a = 0.25 uniform noise