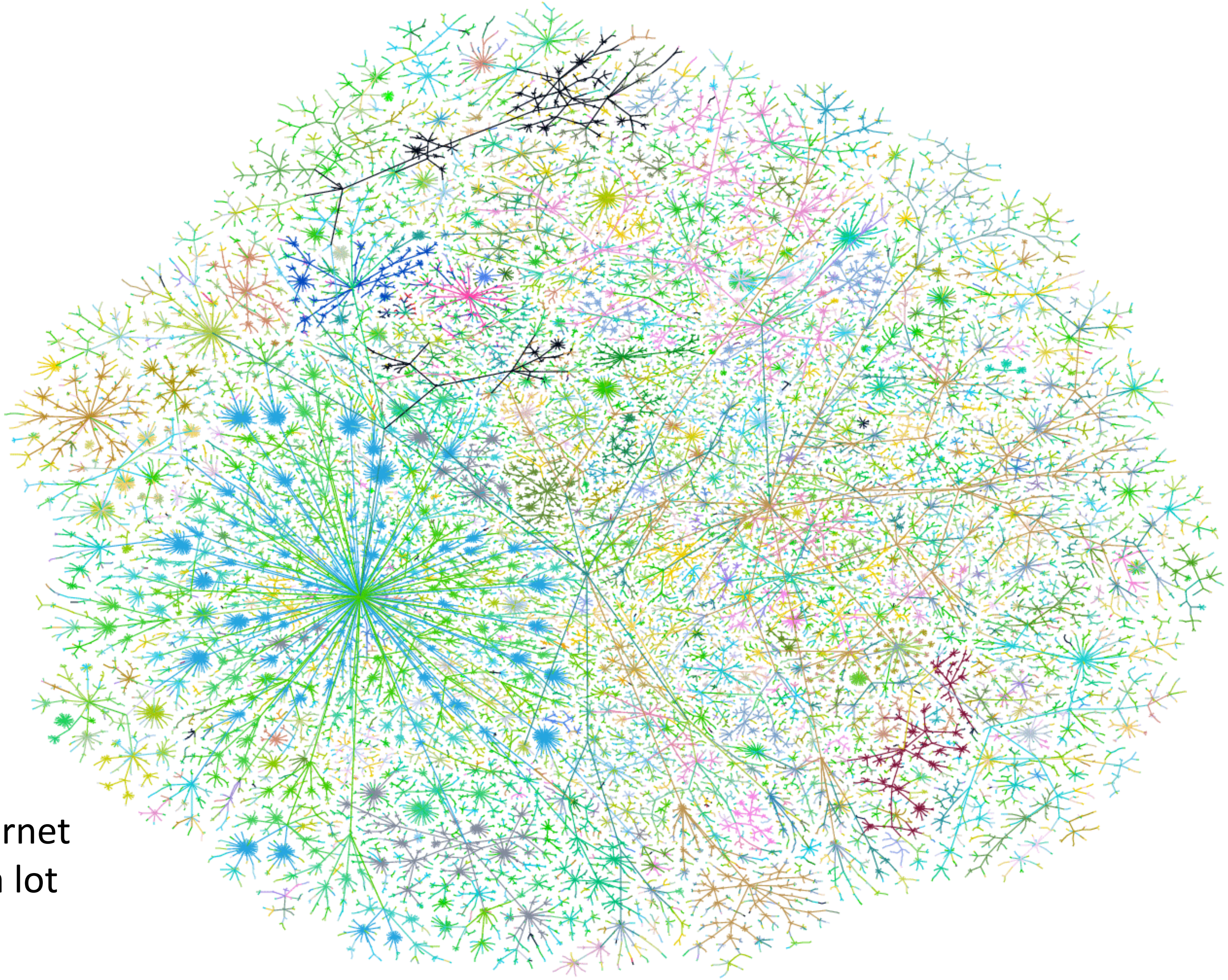
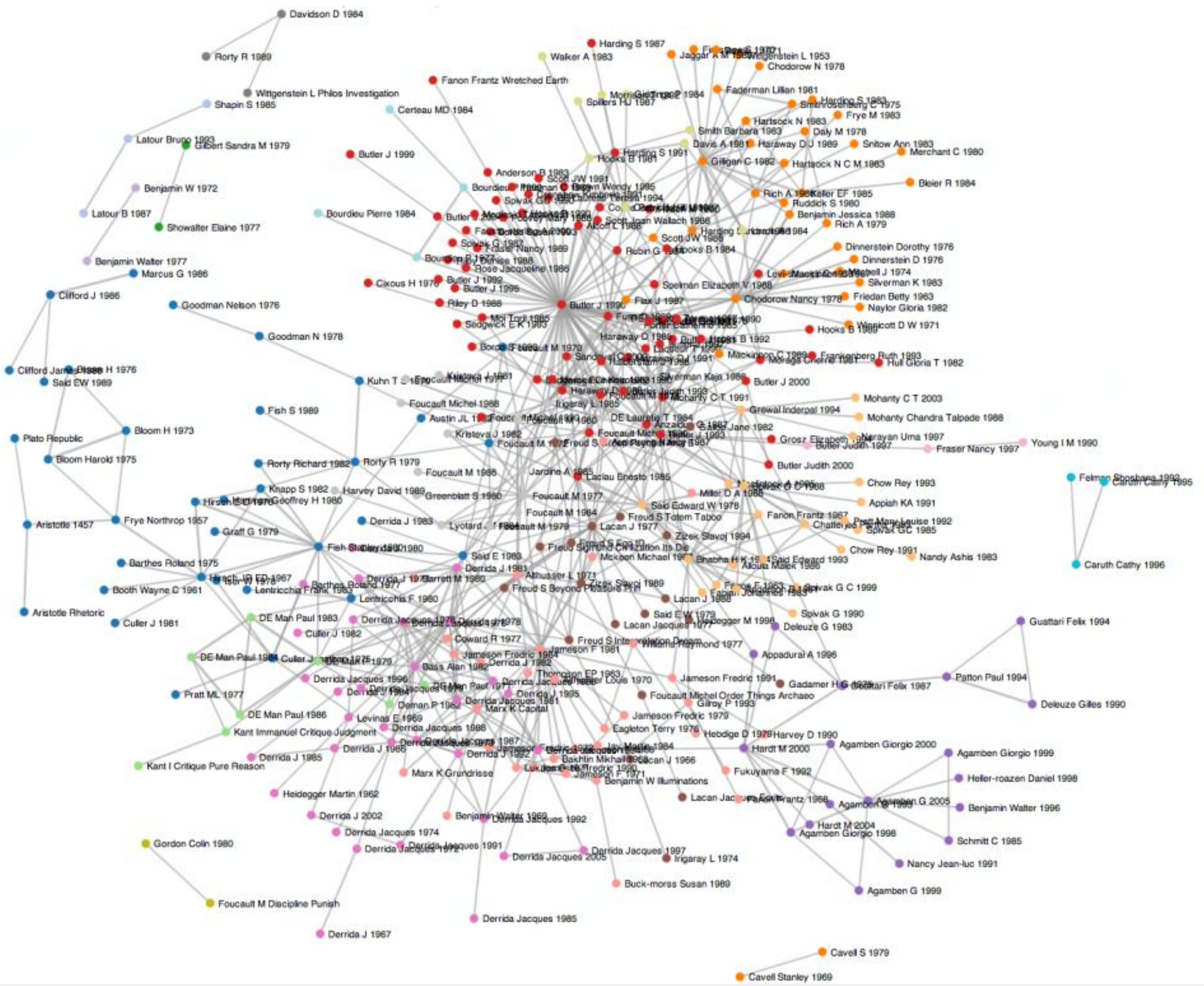
Graphs

Graphs



Graph of the internet
(circa 1999…it's a lot
bigger now…)

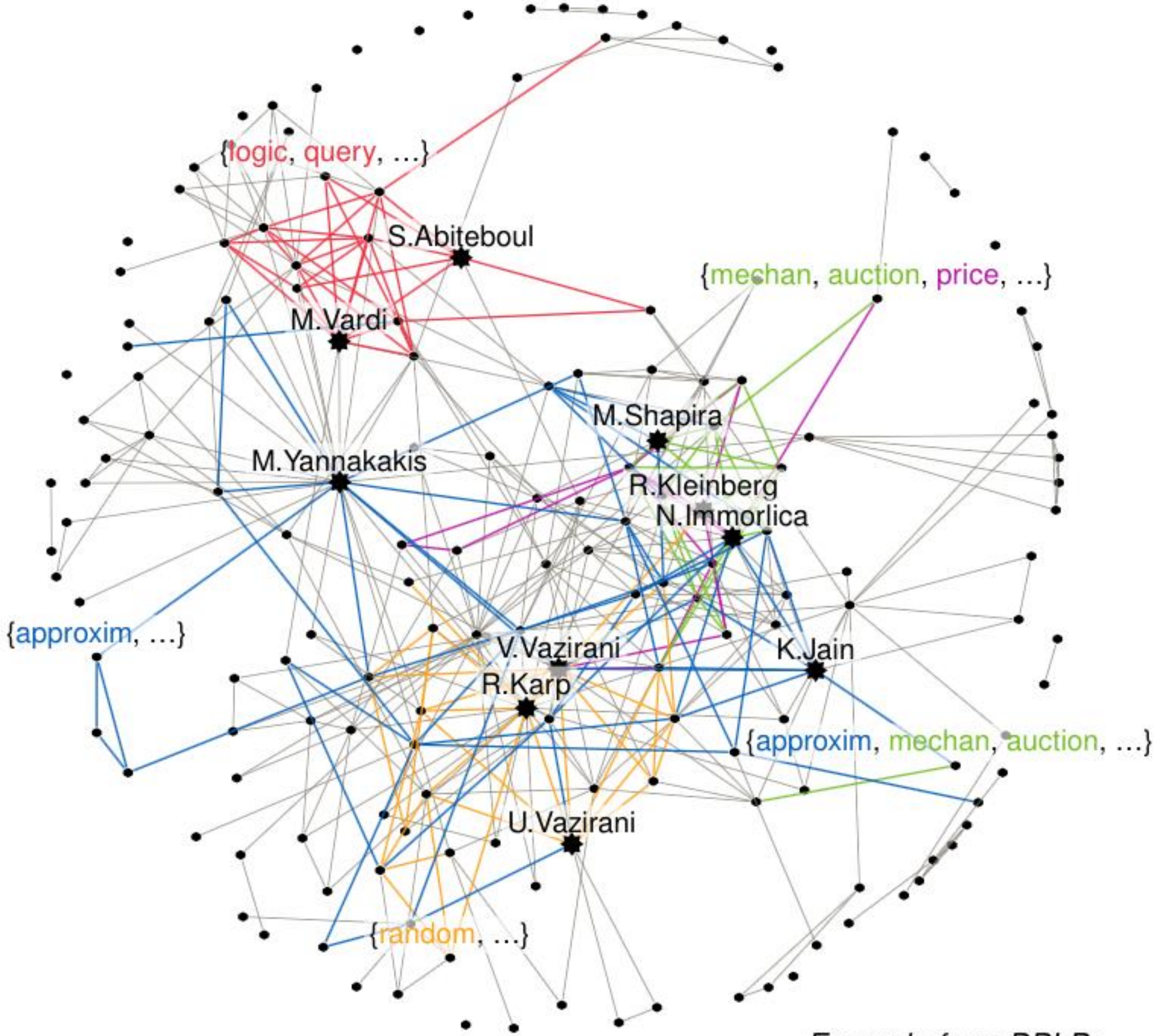
Graphs



Citation graph of literary theory academic papers

Graphs

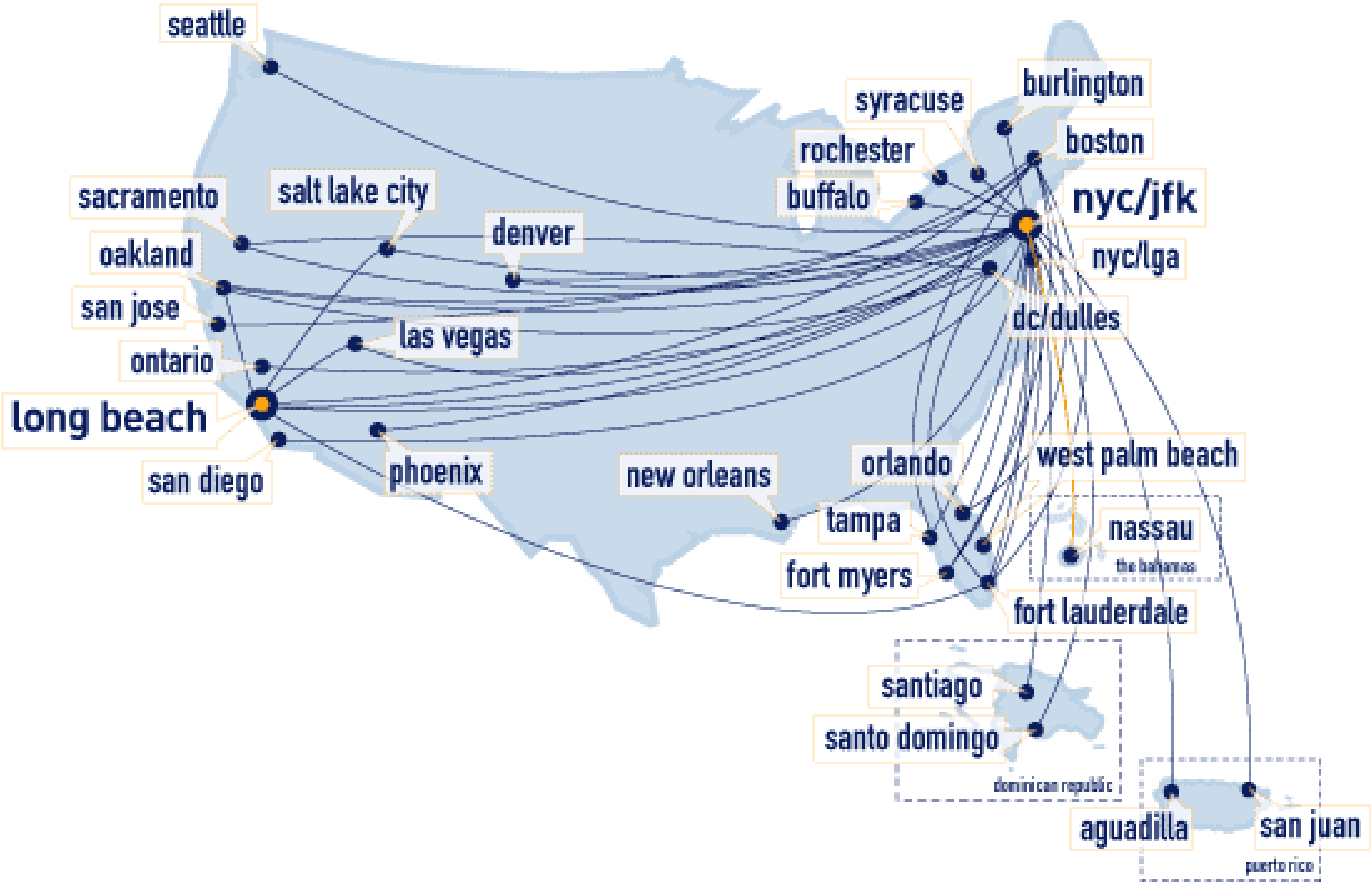
Theoretical Computer Science academic communities



Example from DBLP:
Communities within the co-authors of Christos H. Papadimitriou

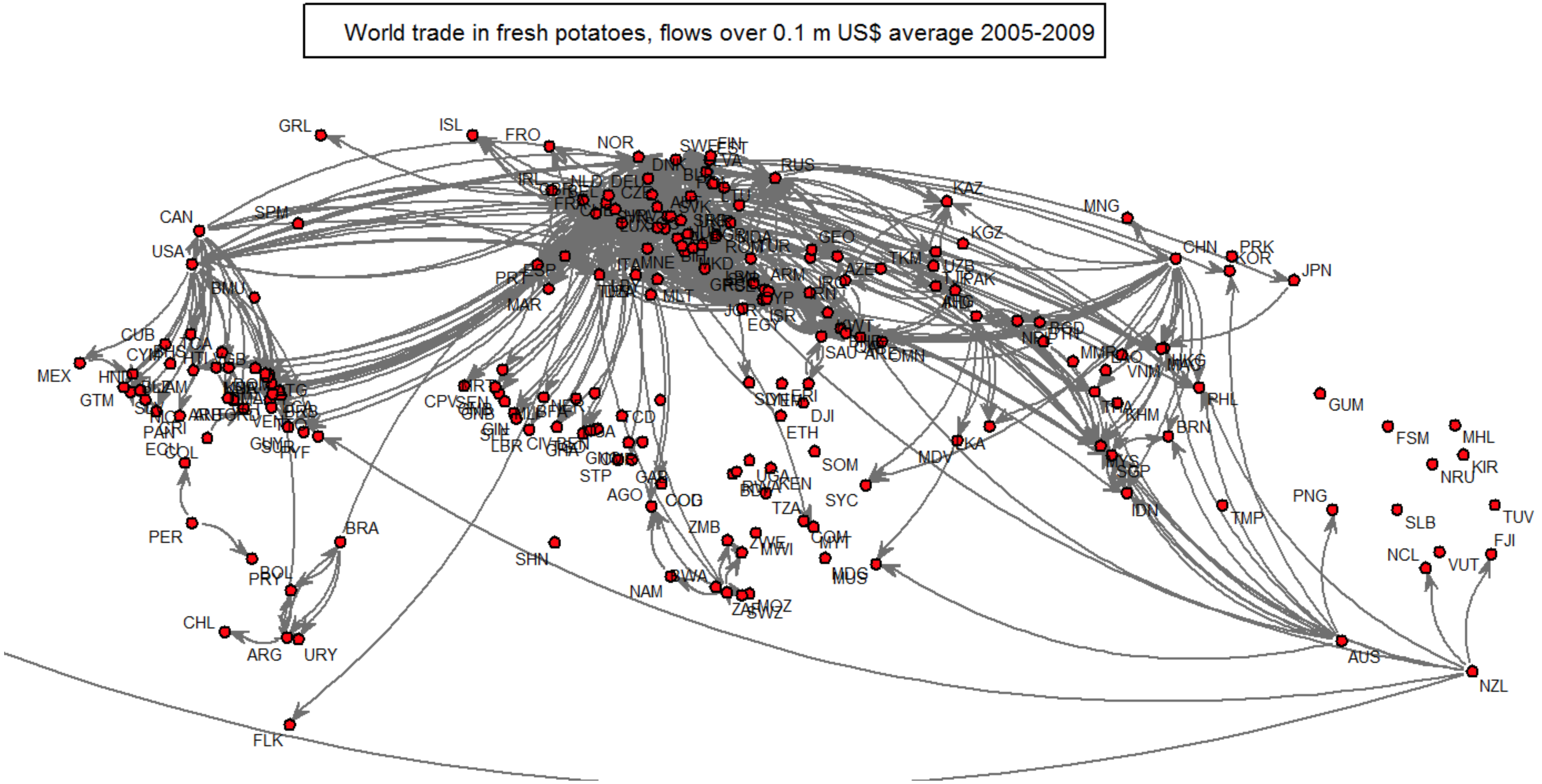
Graphs

jetblue flights



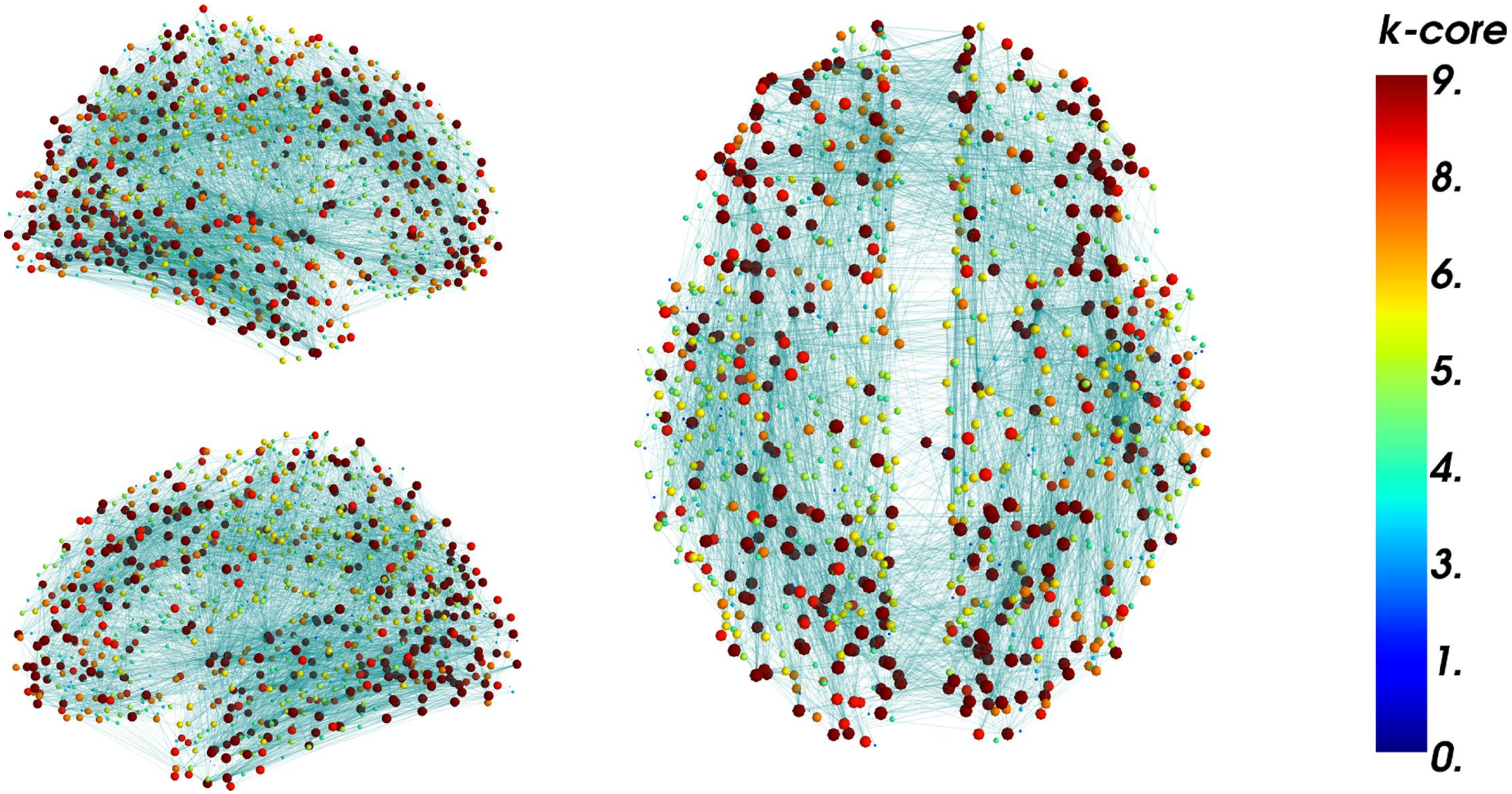
Graphs

Potato trade



Graphs

Neural connections
in the brain



Graphs

- **There are a lot of graphs.**

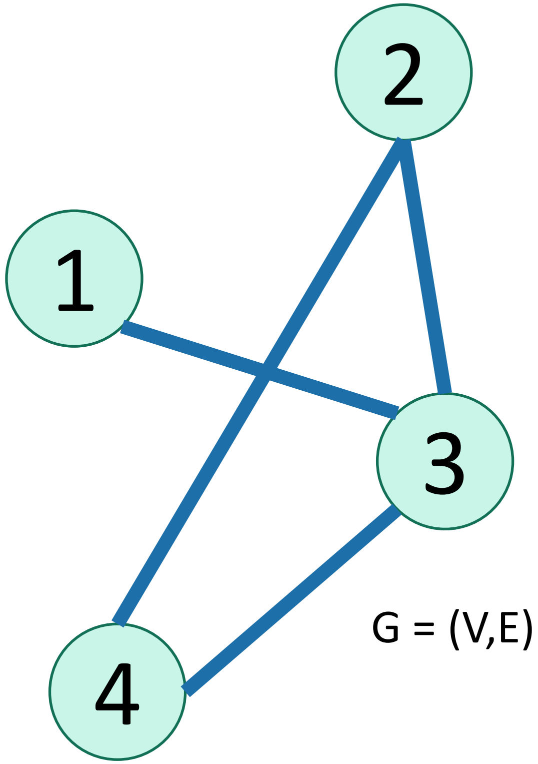
- We want to answer questions about them.
  - Efficient routing?
  - Community detection/clustering?
  - An ordering that respects dependencies?

- This is what we'll do for the next several lectures.

Undirected Graphs

- Has vertices and edges
  - V is the set of vertices
  - E is the set of edges
  - Formally, a graph is G = (V,E)

- Example
  - V = {1,2,3,4}
  - E = { {1,3}, {2,4}, {3,4}, {2,3} }

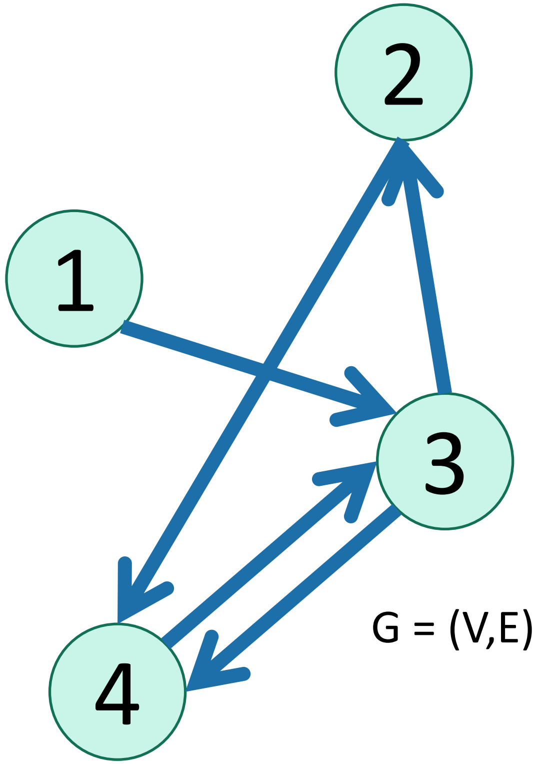


- The degree of vertex 4 is 2.
  - There are 2 edges coming out.
- Vertex 4's neighbors are 2 and 3

Directed Graphs

- Has vertices and edges
  - V is the set of vertices
  - E is the set of **DIRECTED** edges
  - Formally, a graph is G = (V,E)

- Example
  - V = {1,2,3,4}
  - E = { (1,3), (2,4), (3,4), (4,3), (3,2) }



- The in-degree of vertex 4 is 2.
- The out-degree of vertex 4 is 1.
- Vertex 4's incoming neighbors are 2
- Vertex 4's outgoing neighbor is 3.

Graph Representation

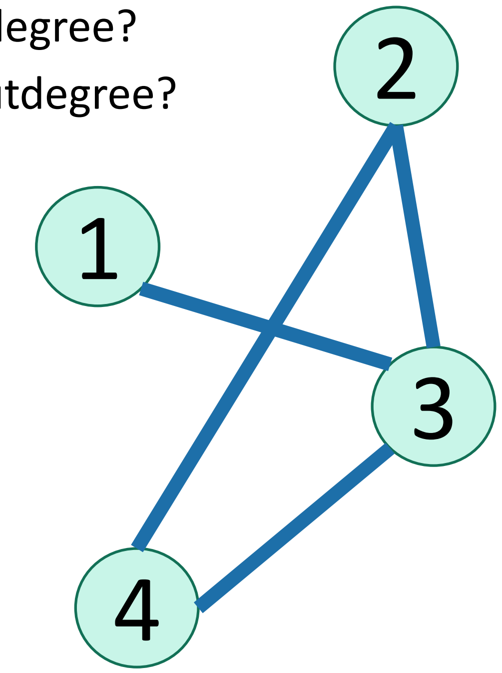
How do we represent graphs?

- Option 1: adjacency matrix

Indegree?

Outdegree?

$$
\begin{array}{c c}
 & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}
\end{array}
$$



How do we represent graphs?

- Option 1: adjacency matrix

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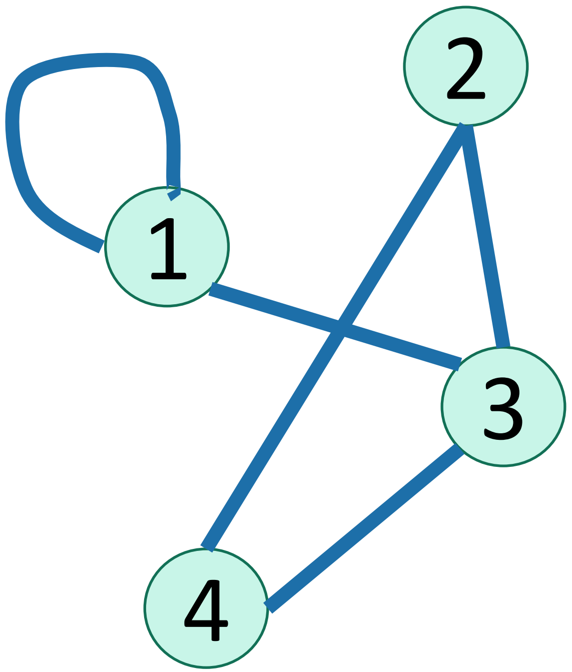
$$



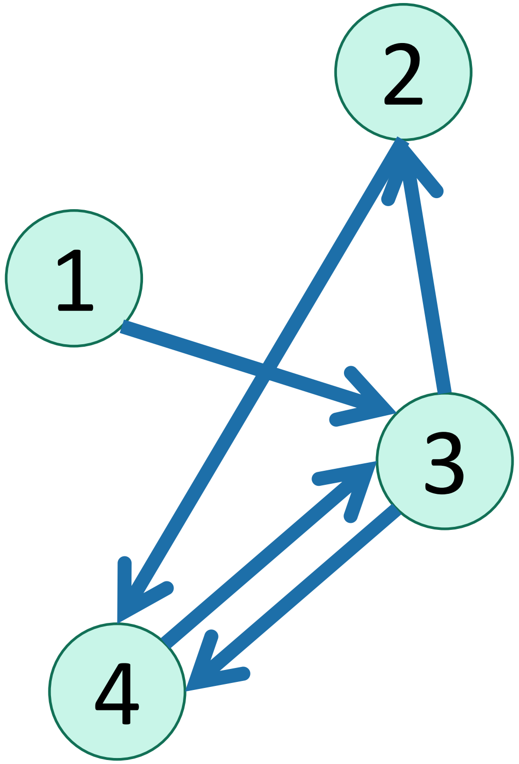
How do we represent graphs?

- Option 1: adjacency matrix

Destination

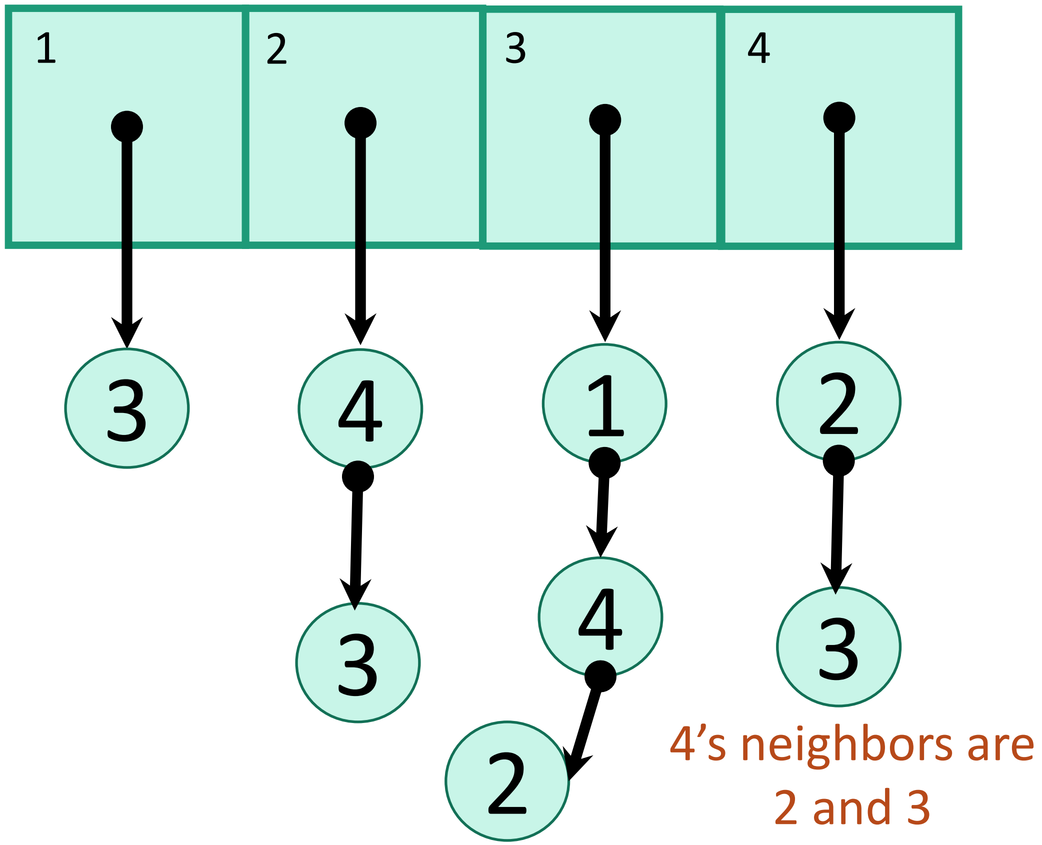
Source

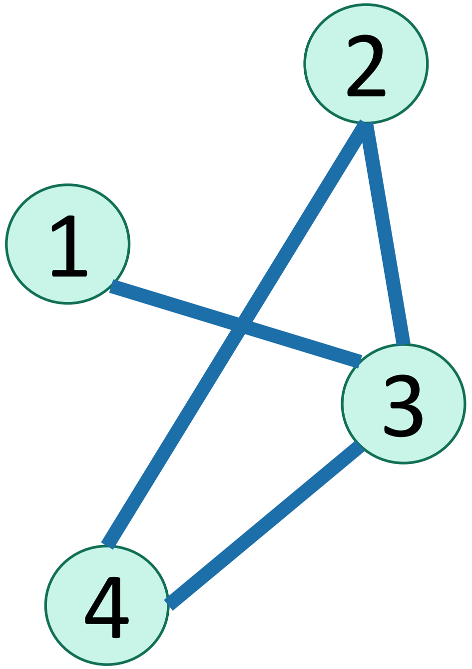
$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$



How do we represent graphs?

- Option 2: linked lists.





How would you modify this for directed graphs?

In either case

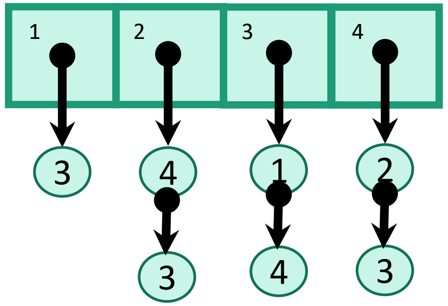
- May think of vertices storing other information
  - Attributes (name, IP address, …)
  - helper info for algorithms that we will perform on the graph
- We will want to be able to do the following ops:
  - Edge Membership: Is edge e in E?
  - Neighbor Query: What are the neighbors of vertex v?

Trade-offs

Say there are n vertices and m edges.

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Generally better for **sparse** graphs



Edge membership Is e = {v,w} in E?	$O(1)$	$O(\deg(v))$ or $O(\deg(w))$
Neighbor query Give me v's neighbors.	$O(n)$	$O(\deg(v))$
Space requirements	$O(n^2)$	$O(n + m)$

We'll assume this representation for the rest of the class

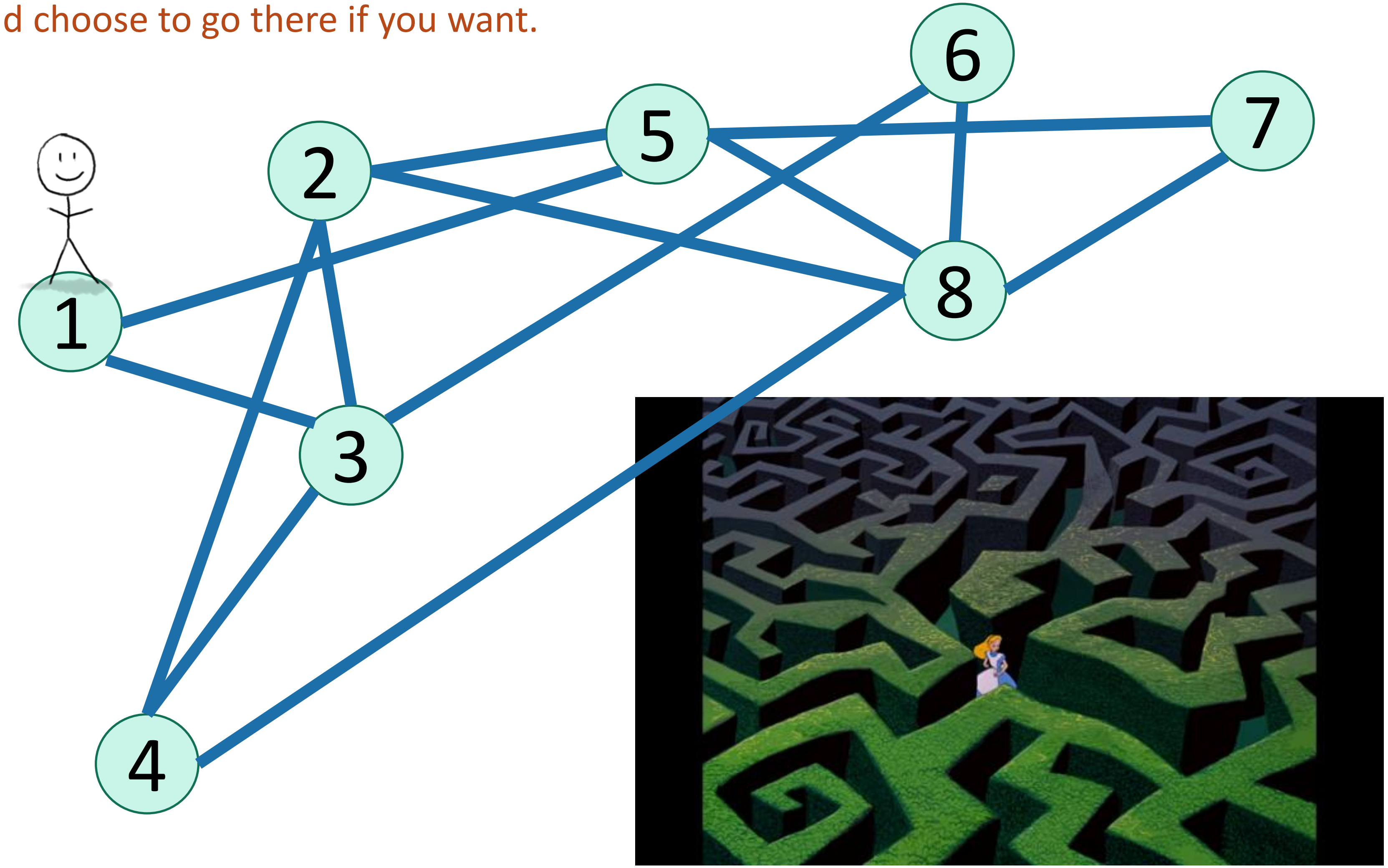
Assignment Question 1

- Design a Graph Class with following functions
  - EdgeMemberShip
  - NeighbourQuery
  - Add Edge
  - Constructor(List of Vertices)

Depth-first search

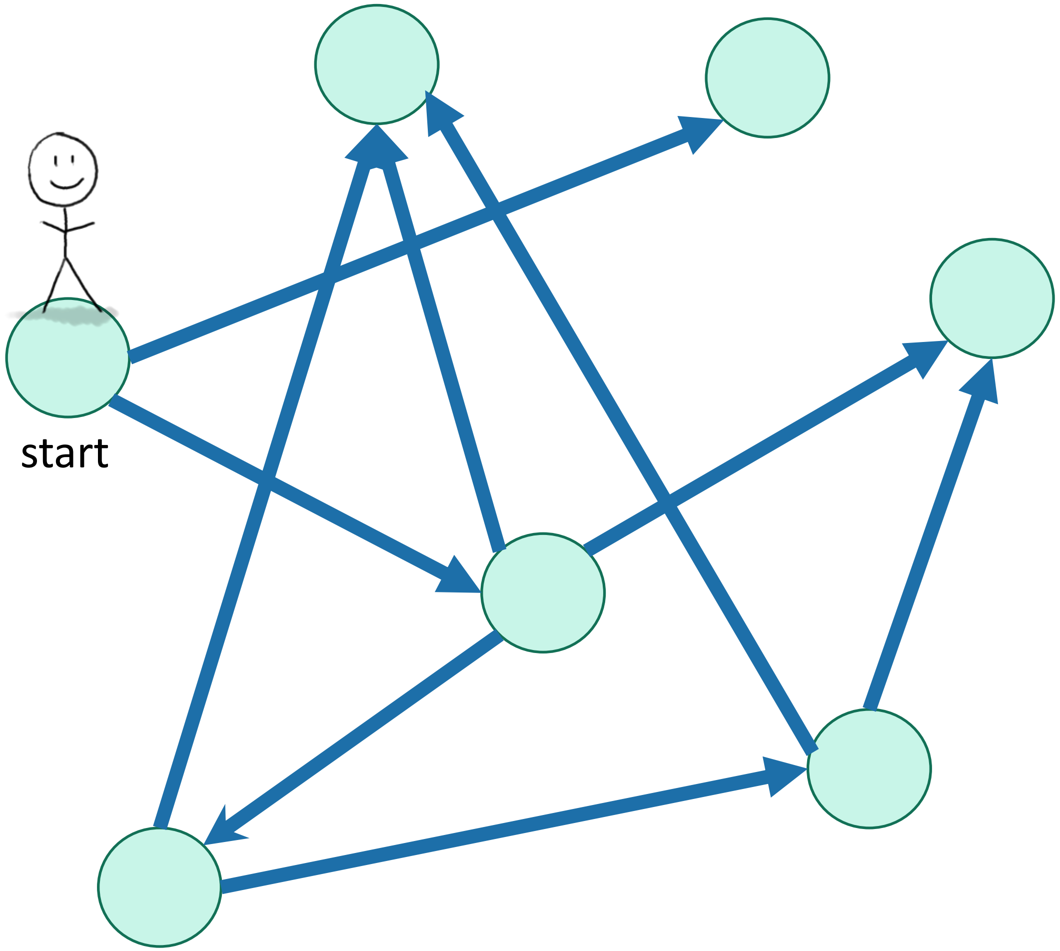
How do we explore a graph?

At each node, you can get a list of neighbors,
and choose to go there if you want.



Depth First Search

Exploring a labyrinth with chalk and a piece of string



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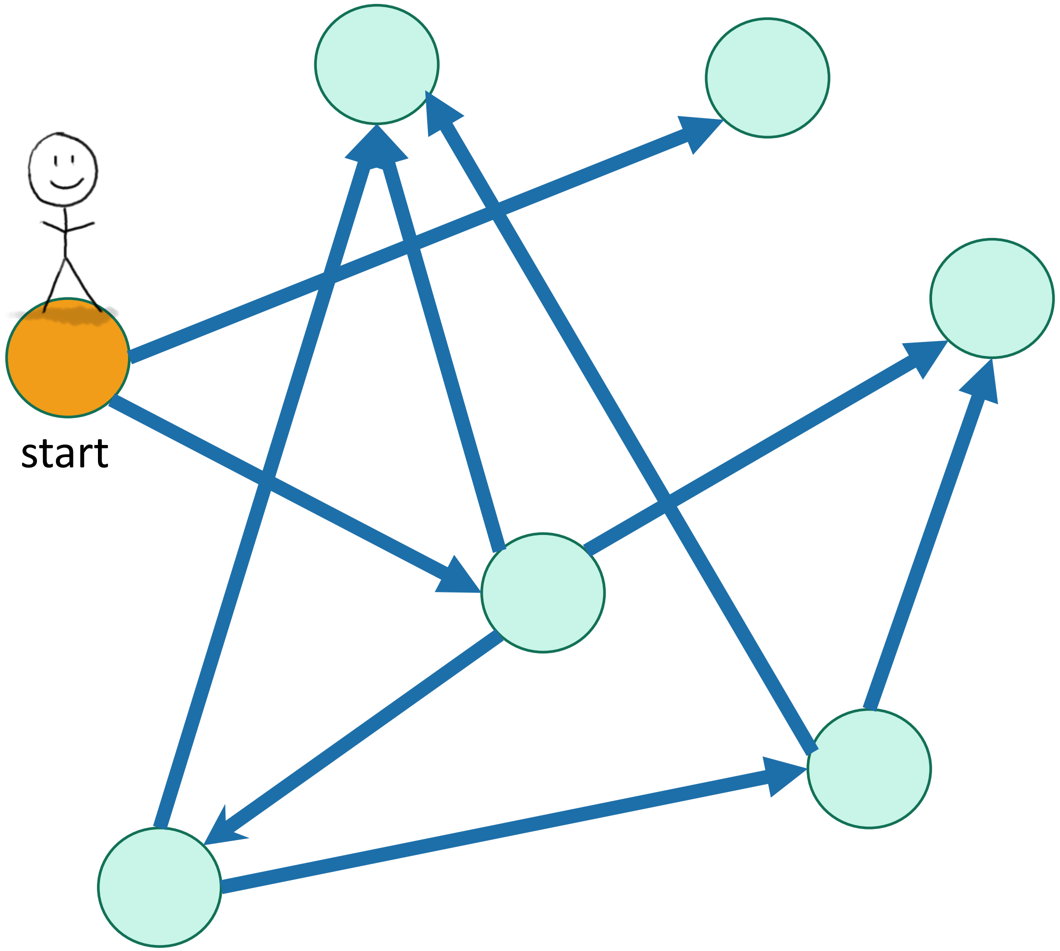

Not been there yet

Been there, haven't explored all the paths out.

Been there, have explored all the paths out.

Depth First Search

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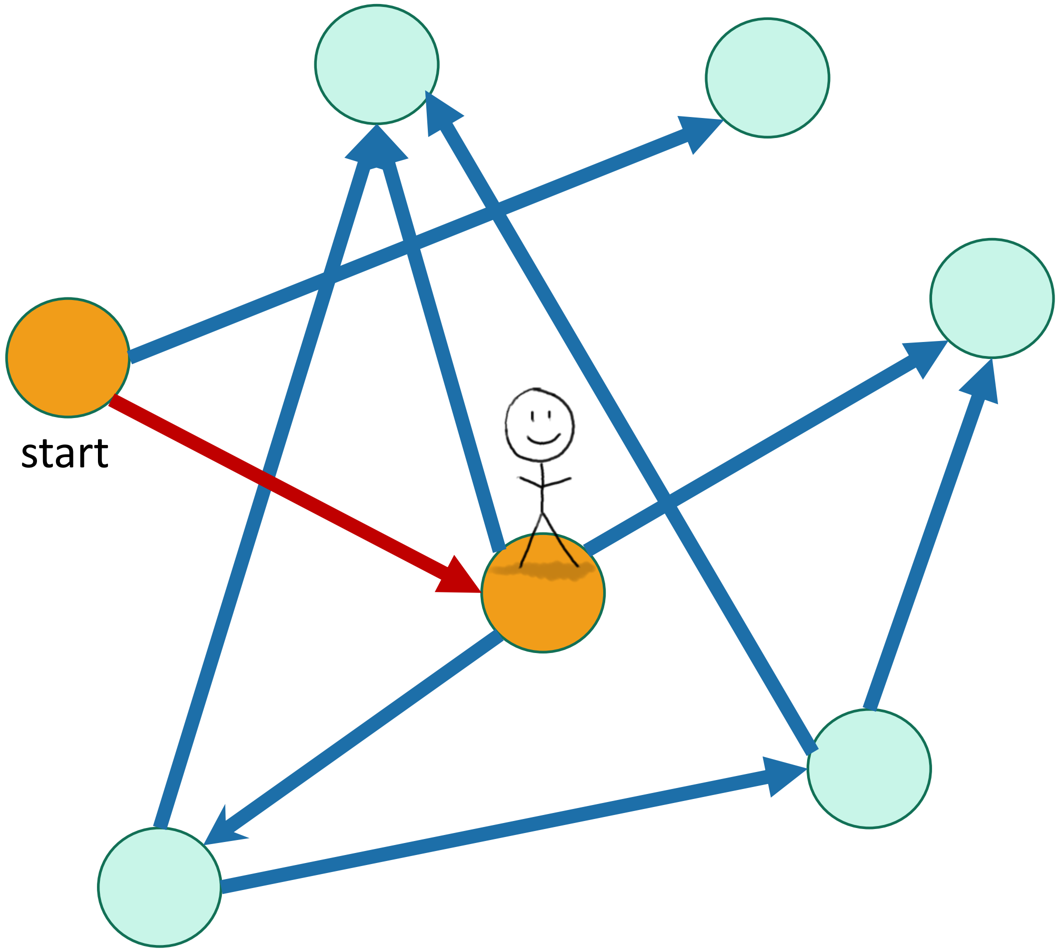

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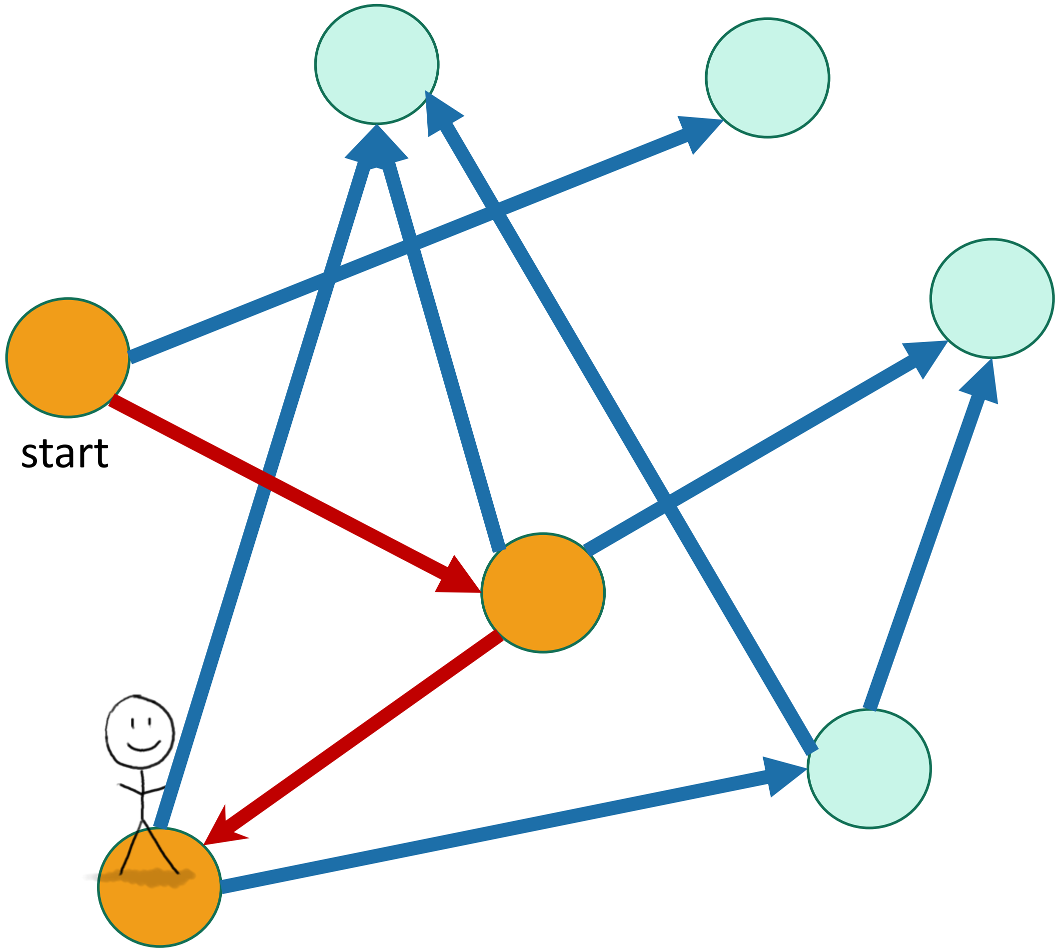

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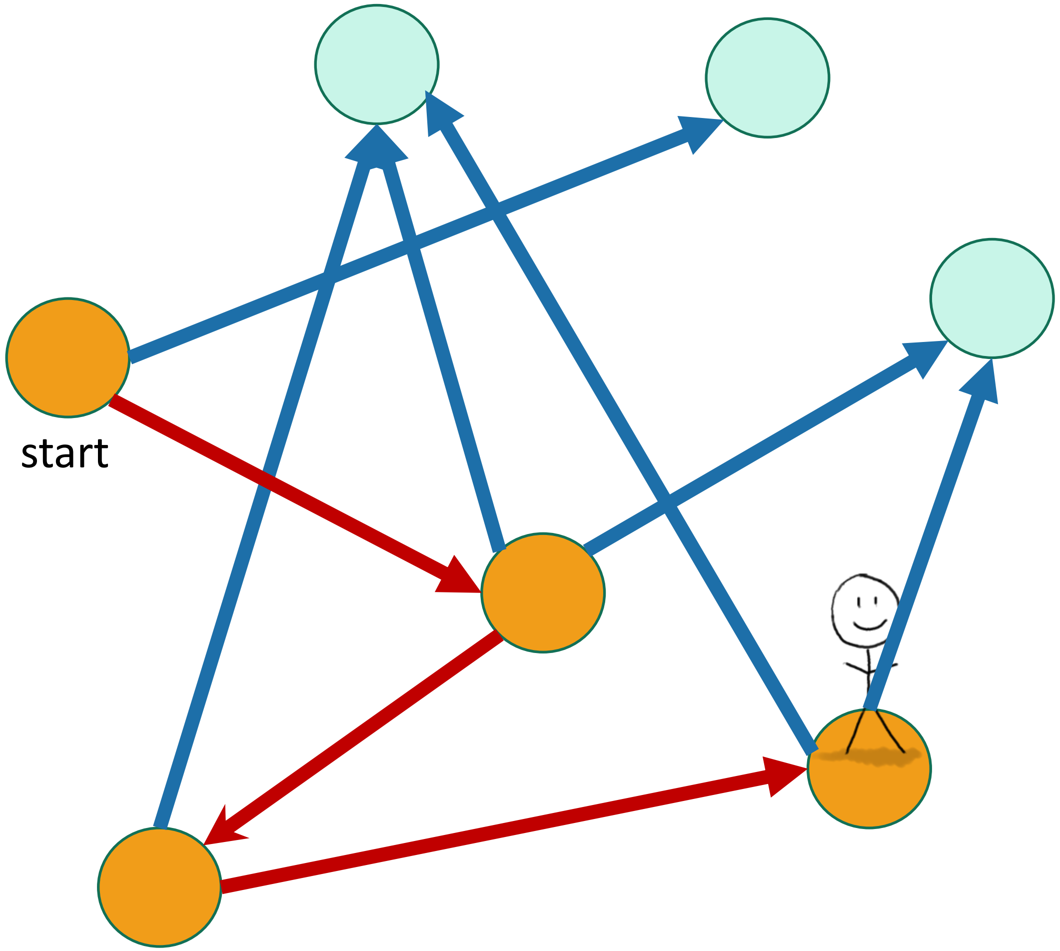

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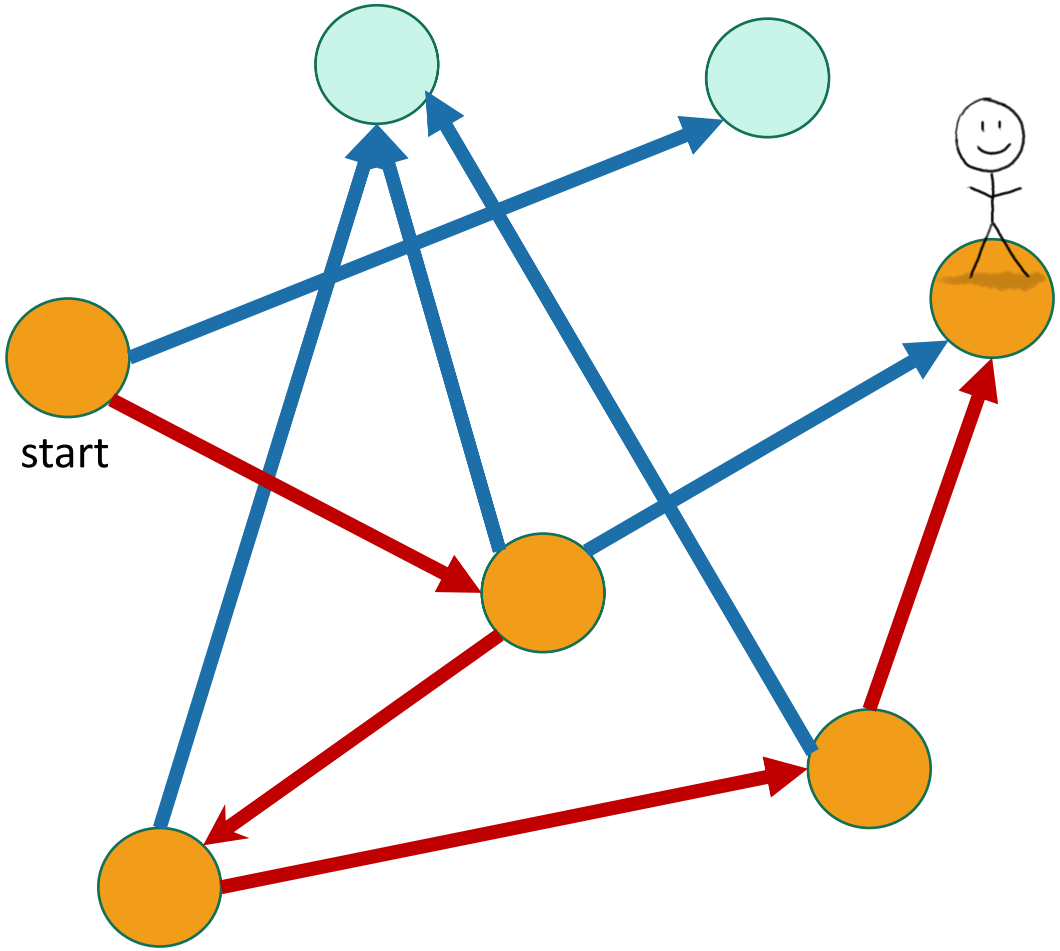

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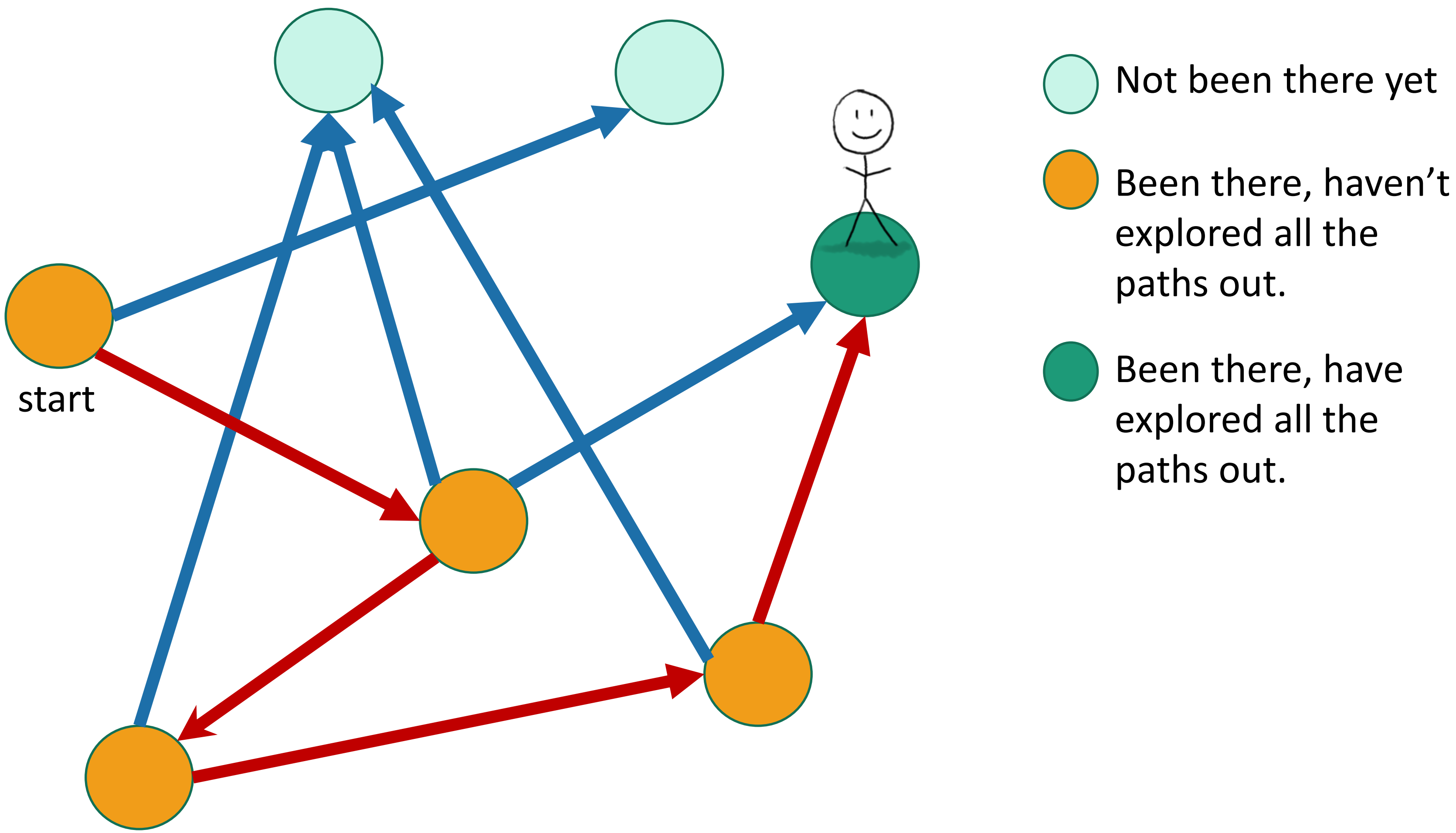

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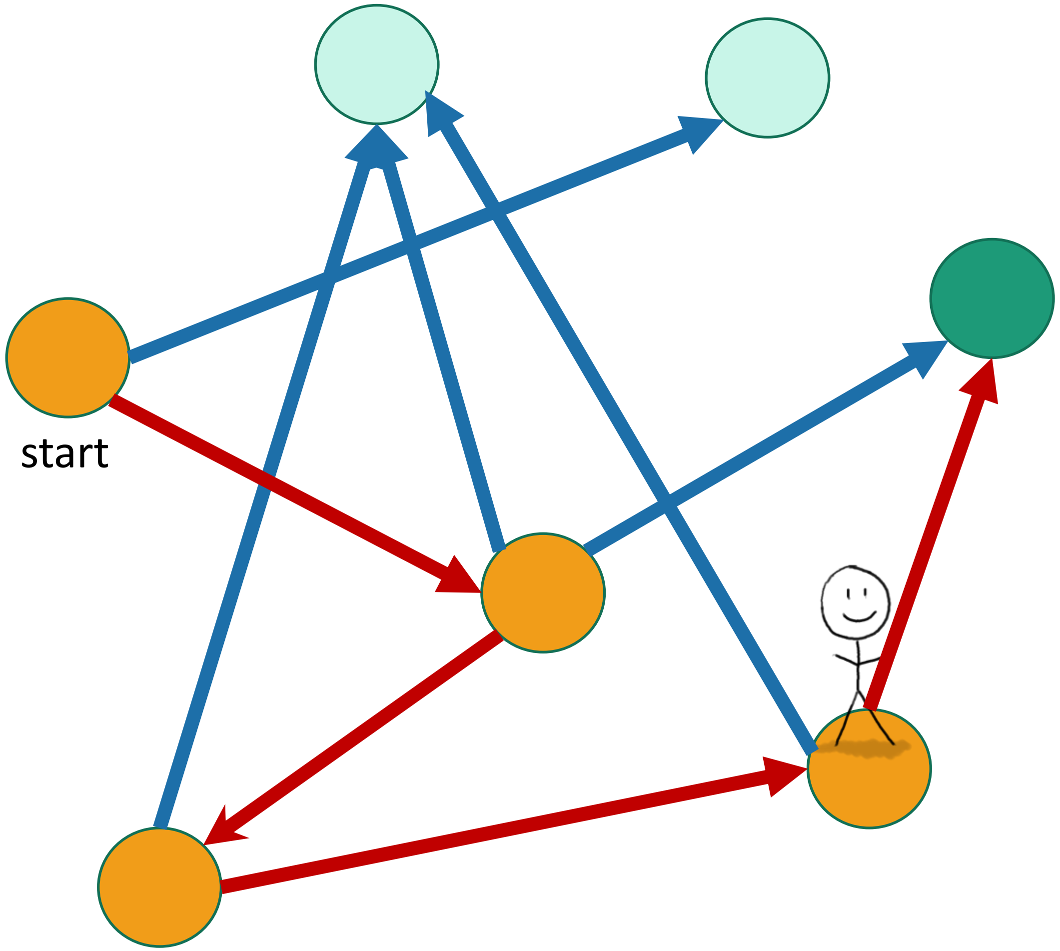

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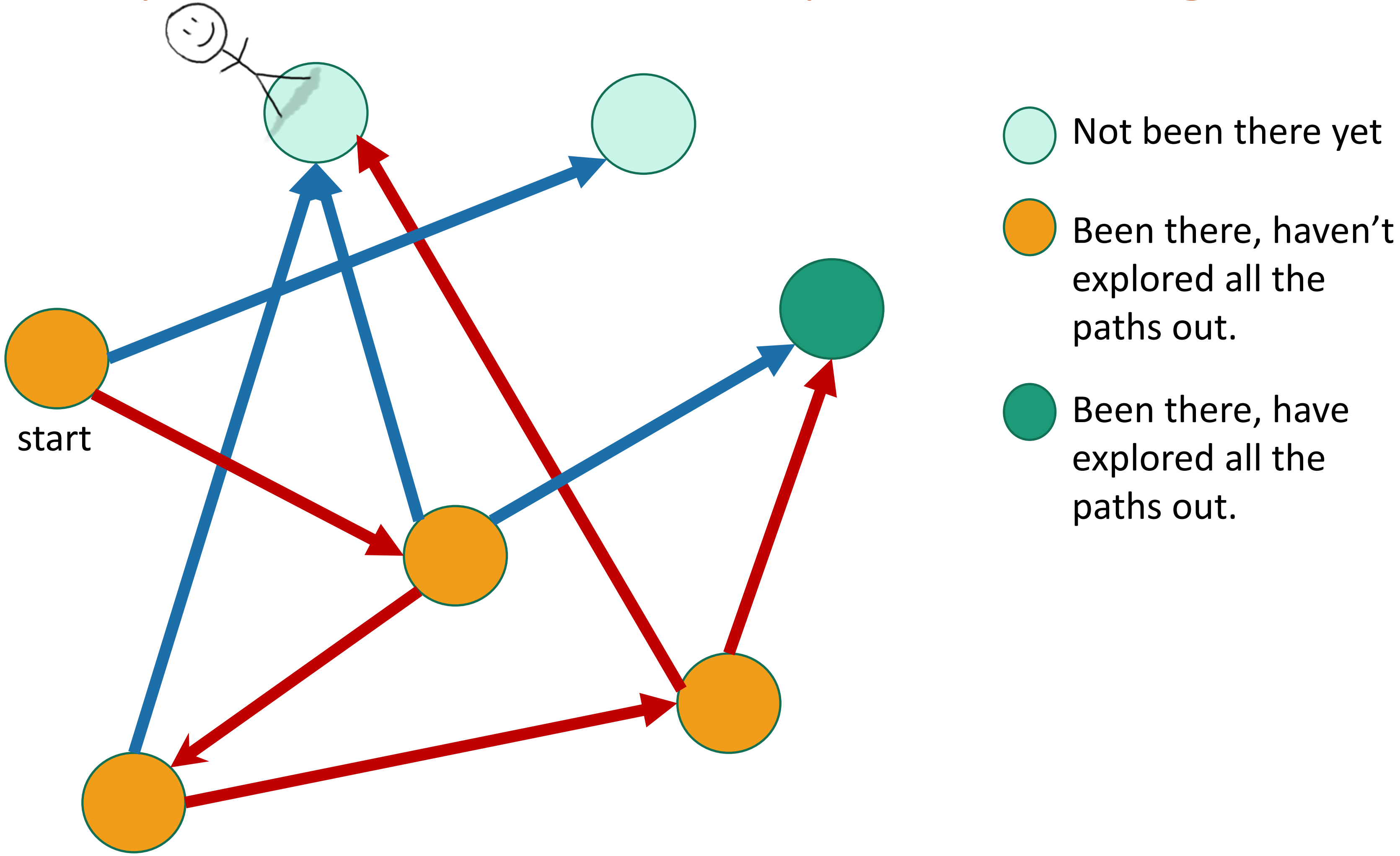

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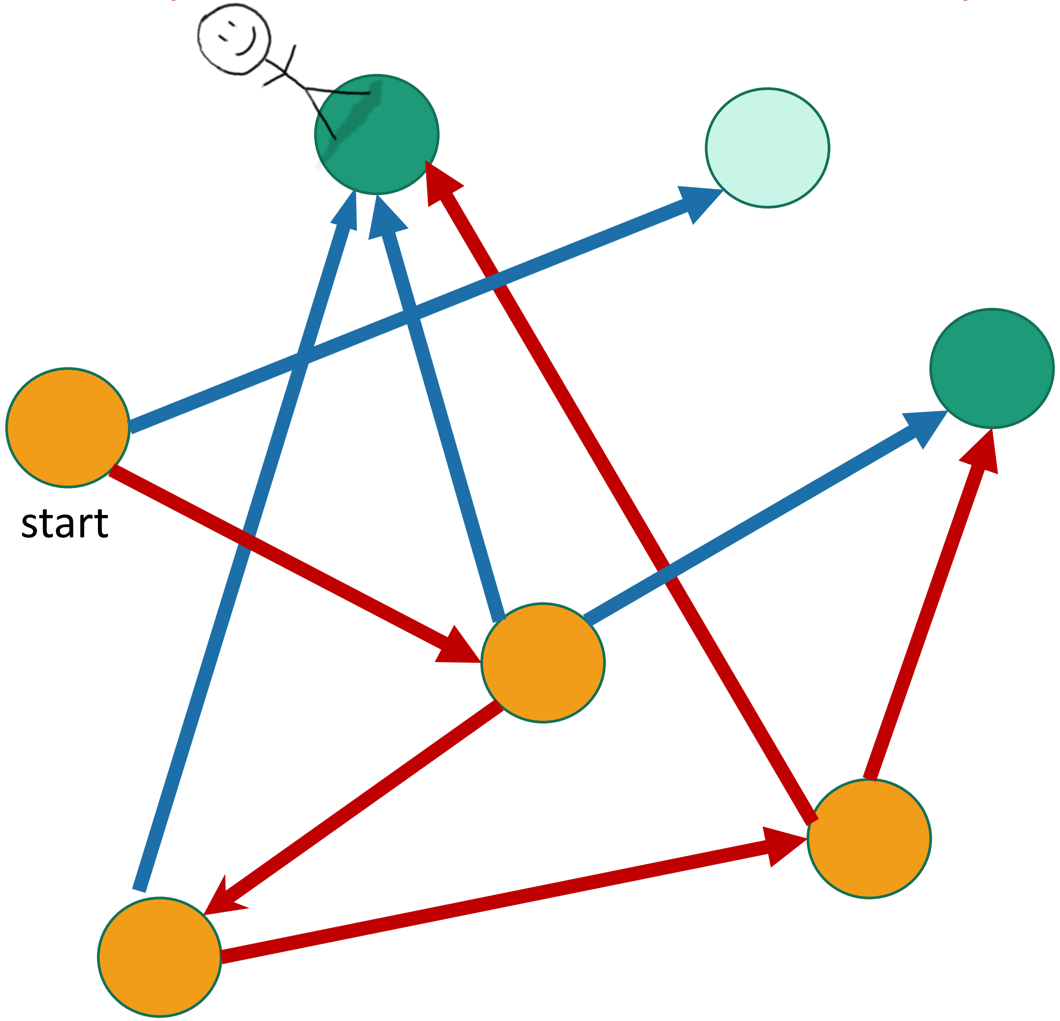

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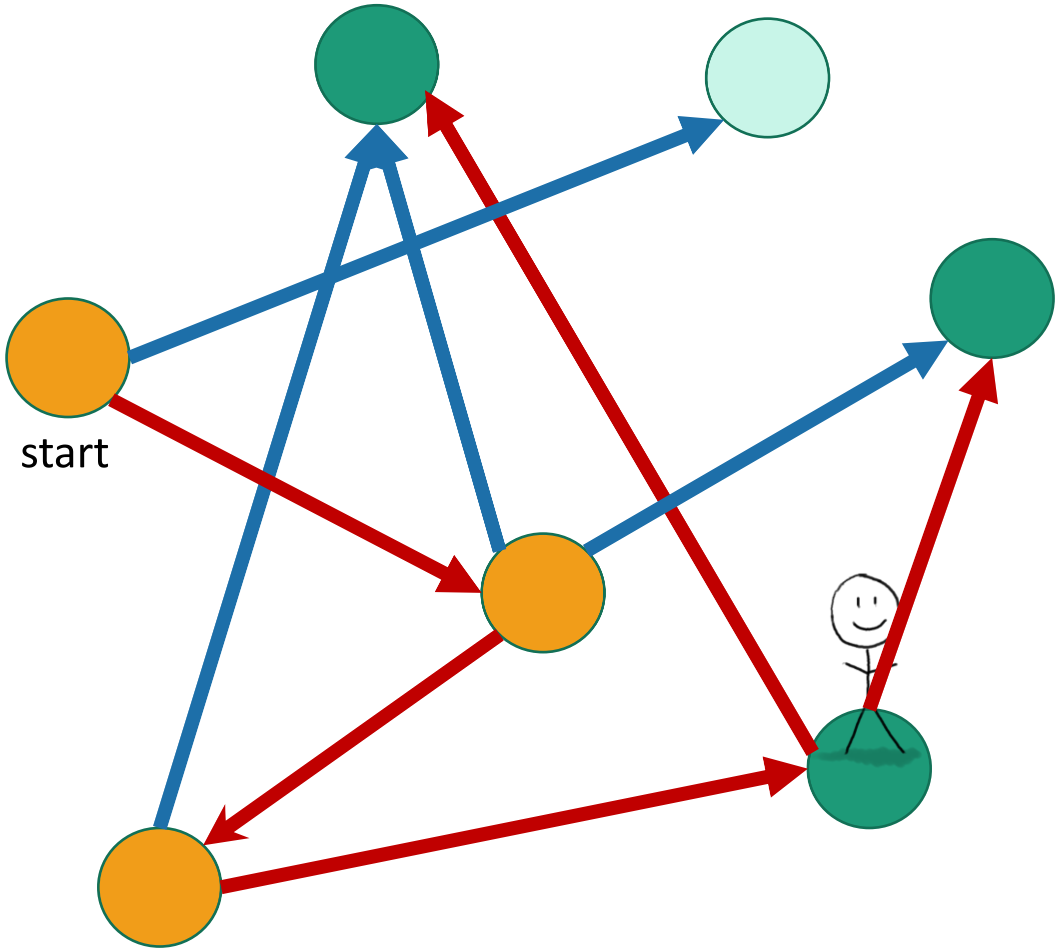

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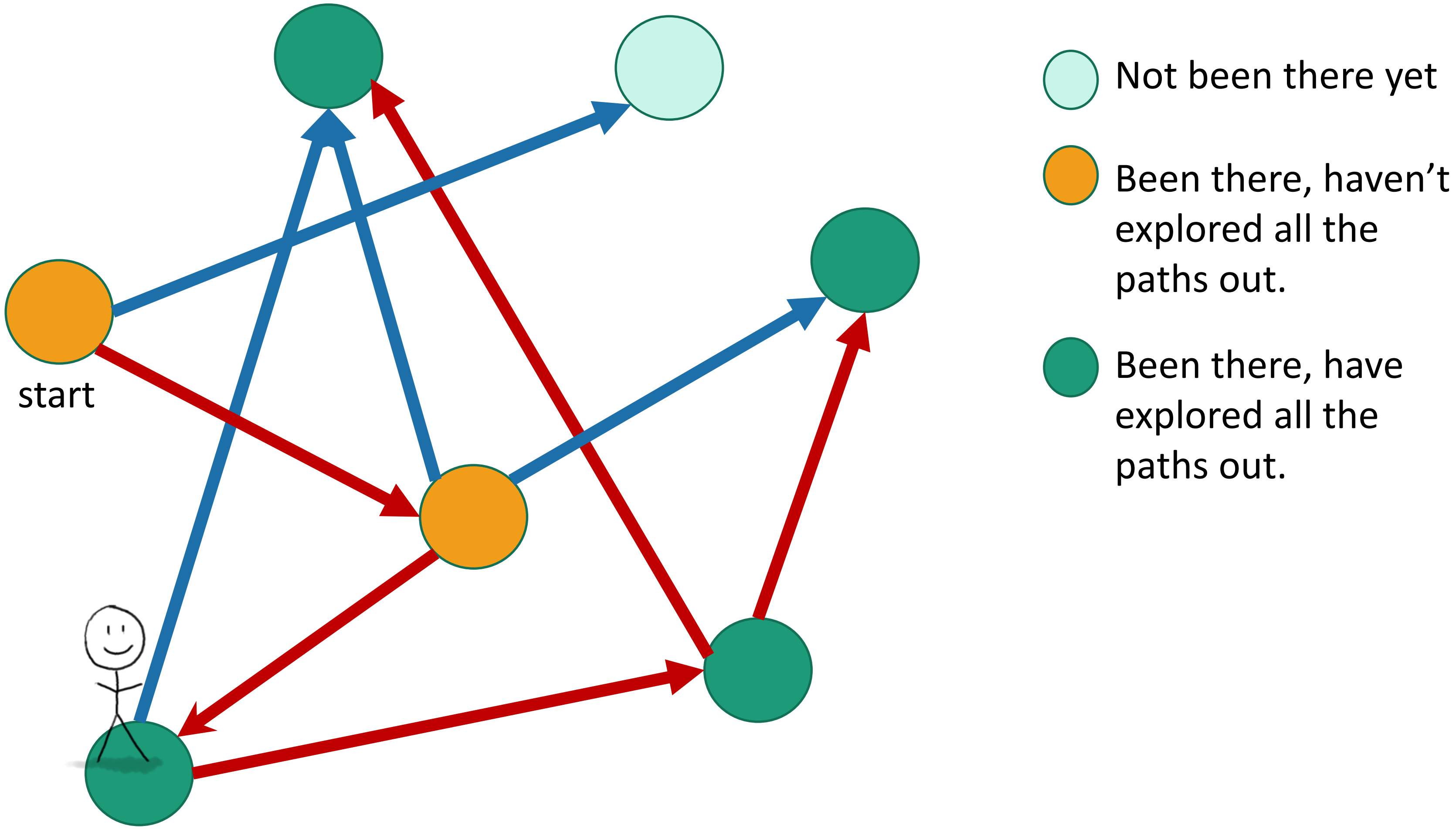

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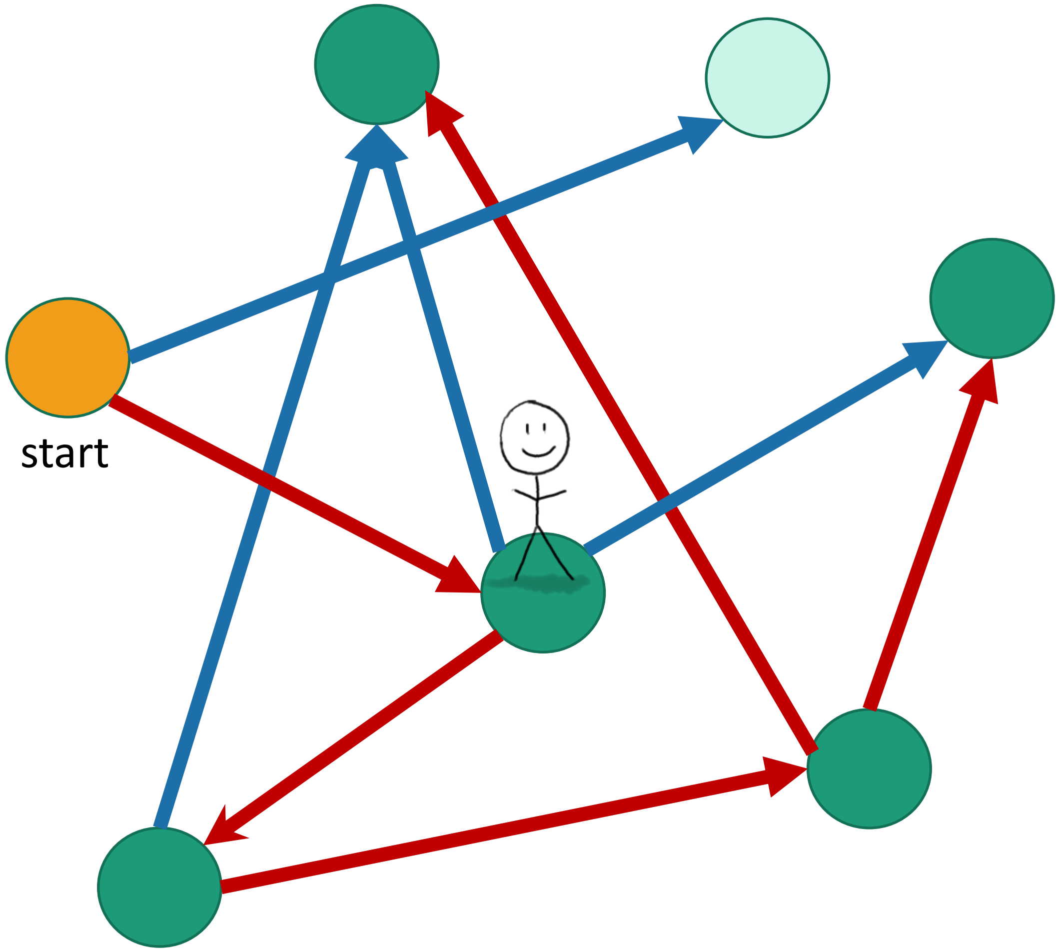

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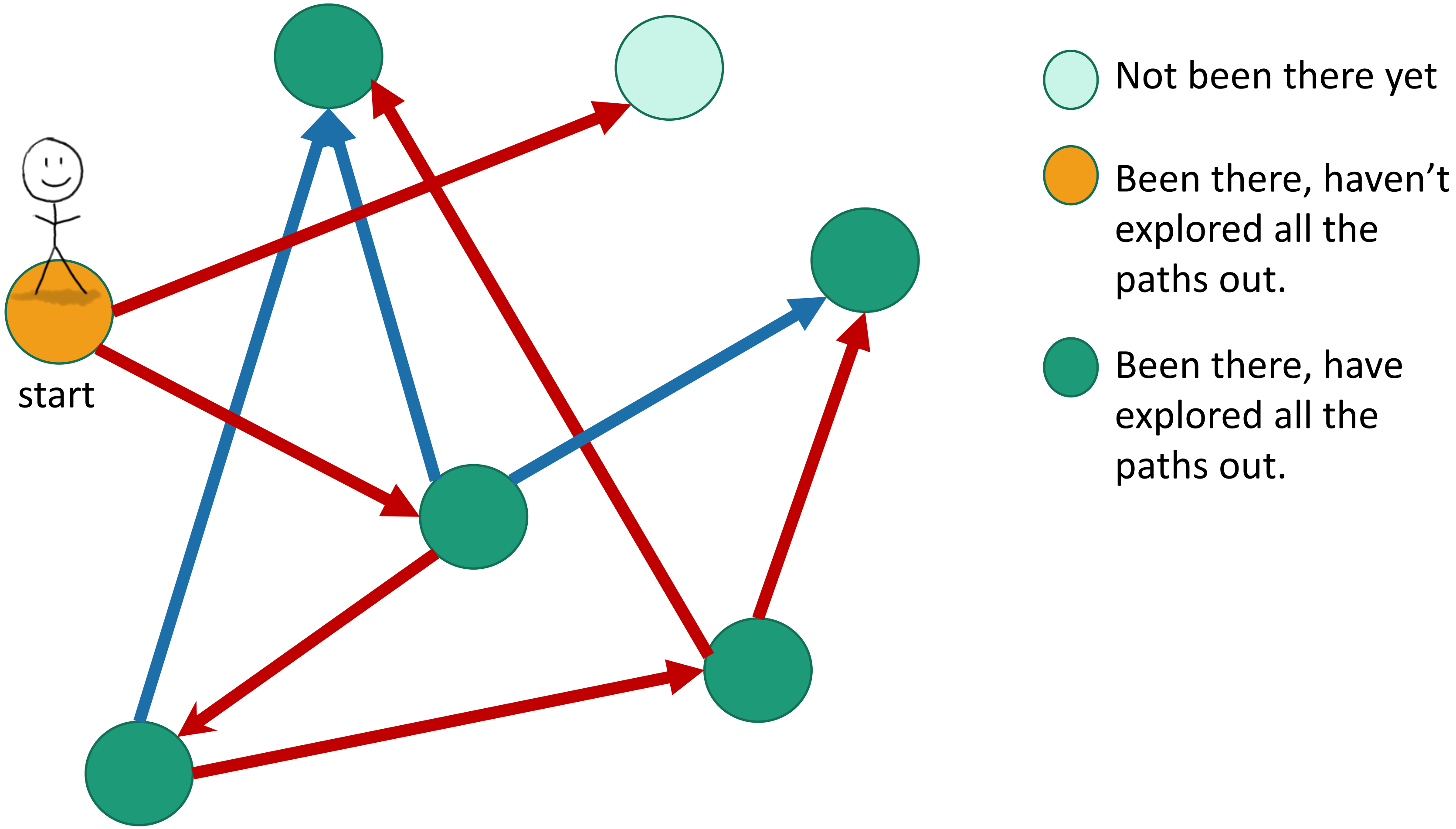

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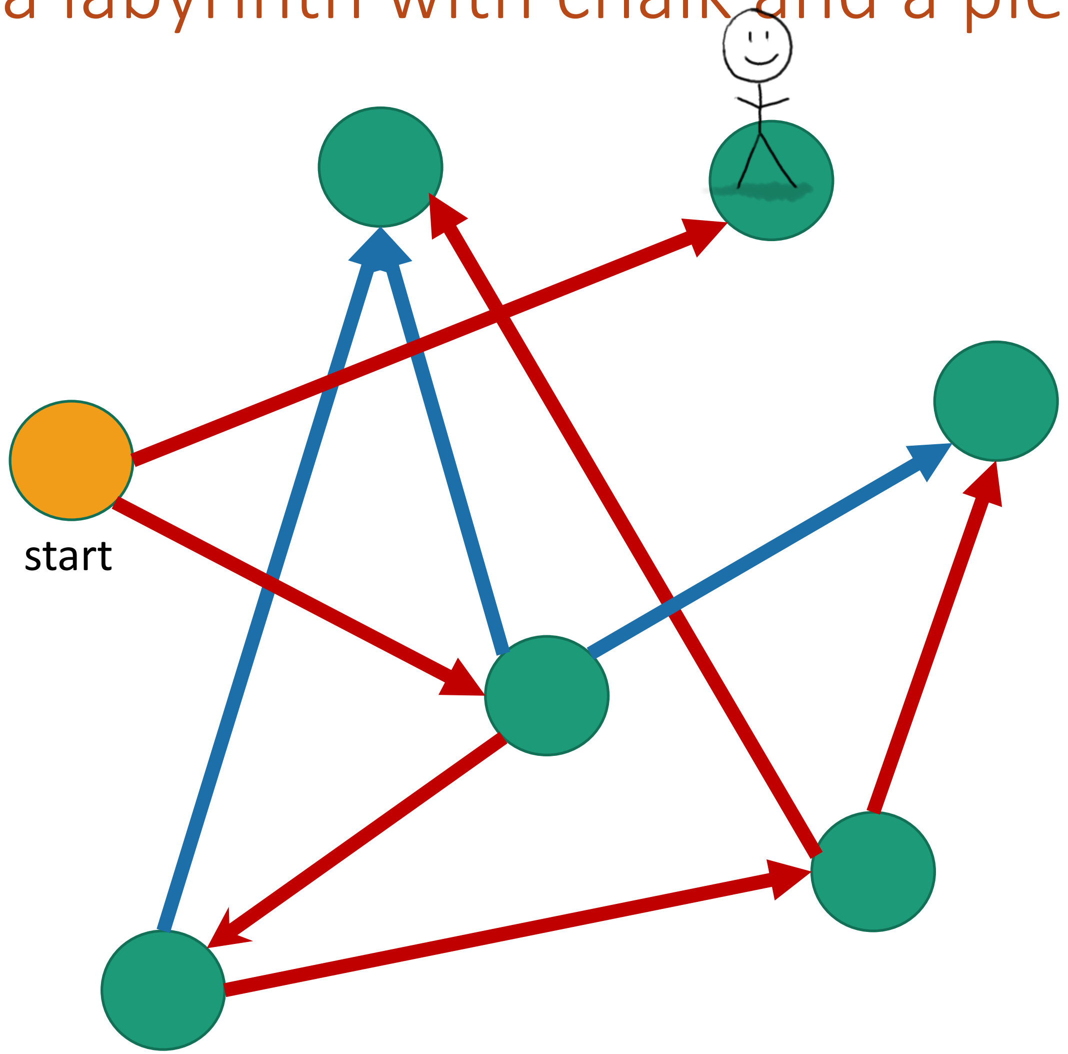

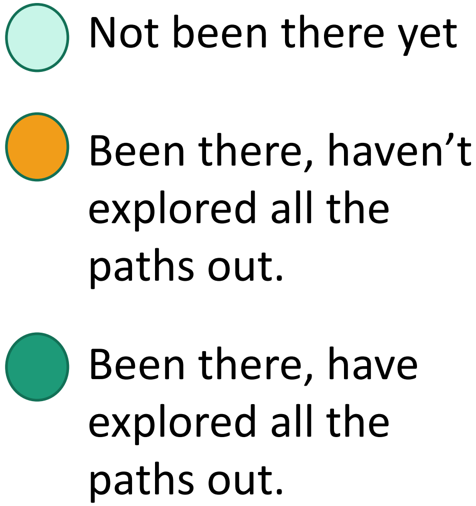

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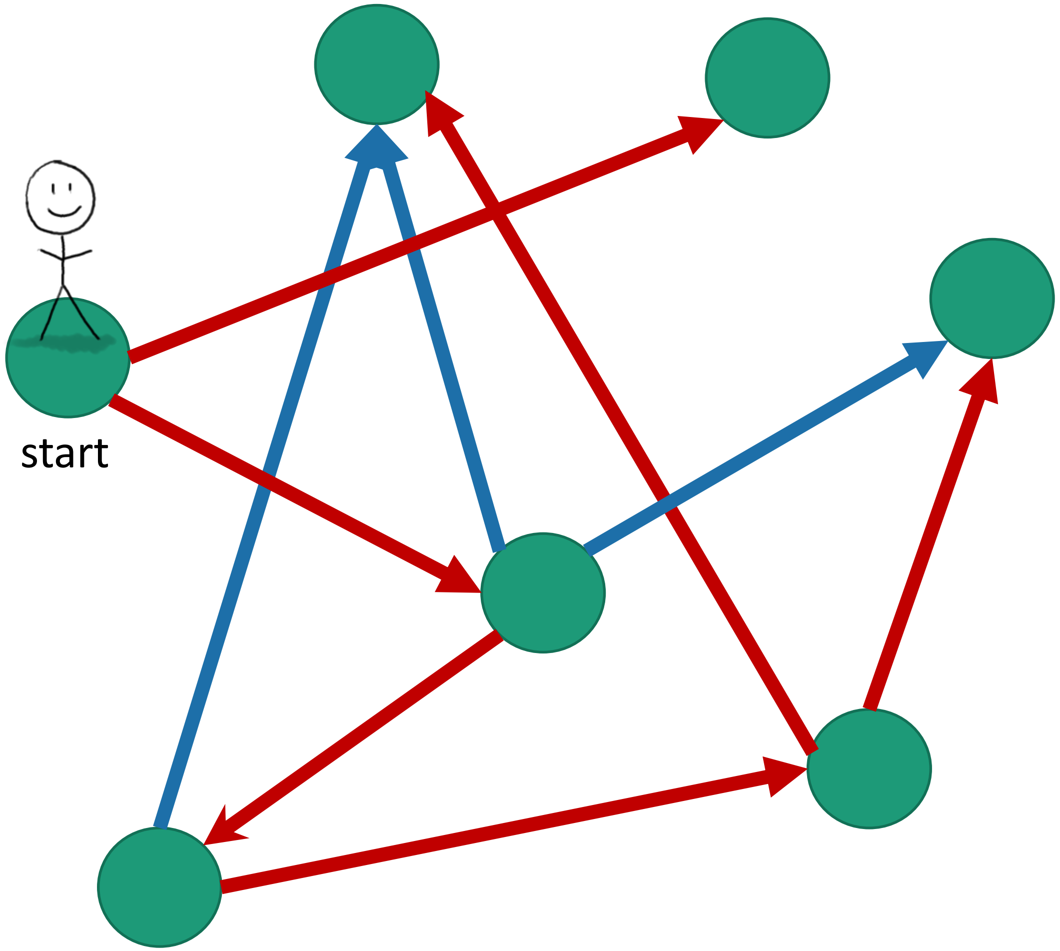

Not been there yet

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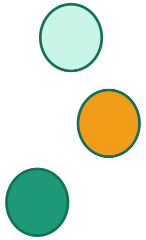
Been there, have explored all the paths out.

Labyrinth:
explored!

Depth First Search

Exploring a labyrinth with pseudocode

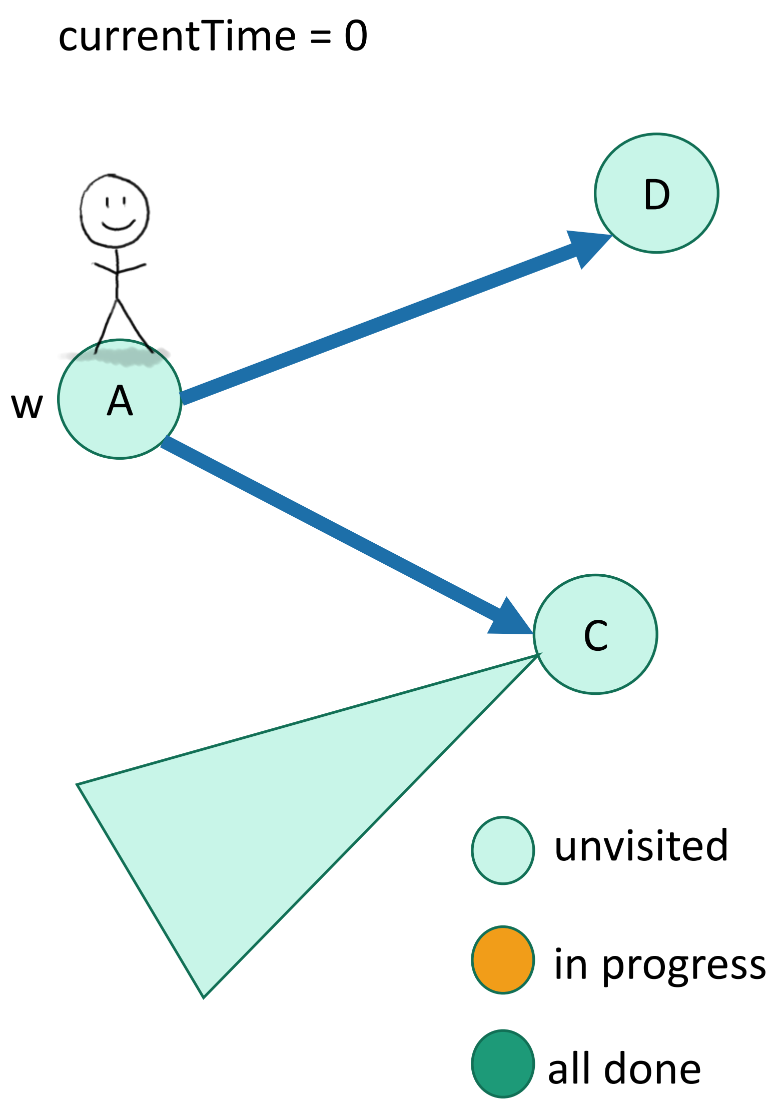
- Each vertex keeps track of whether it is:
  - Unvisited
  - In progress
  - All done



- Each vertex will also keep track of:
  - The time we **first enter it**.
  - The time we finish with it and mark it **all done**.

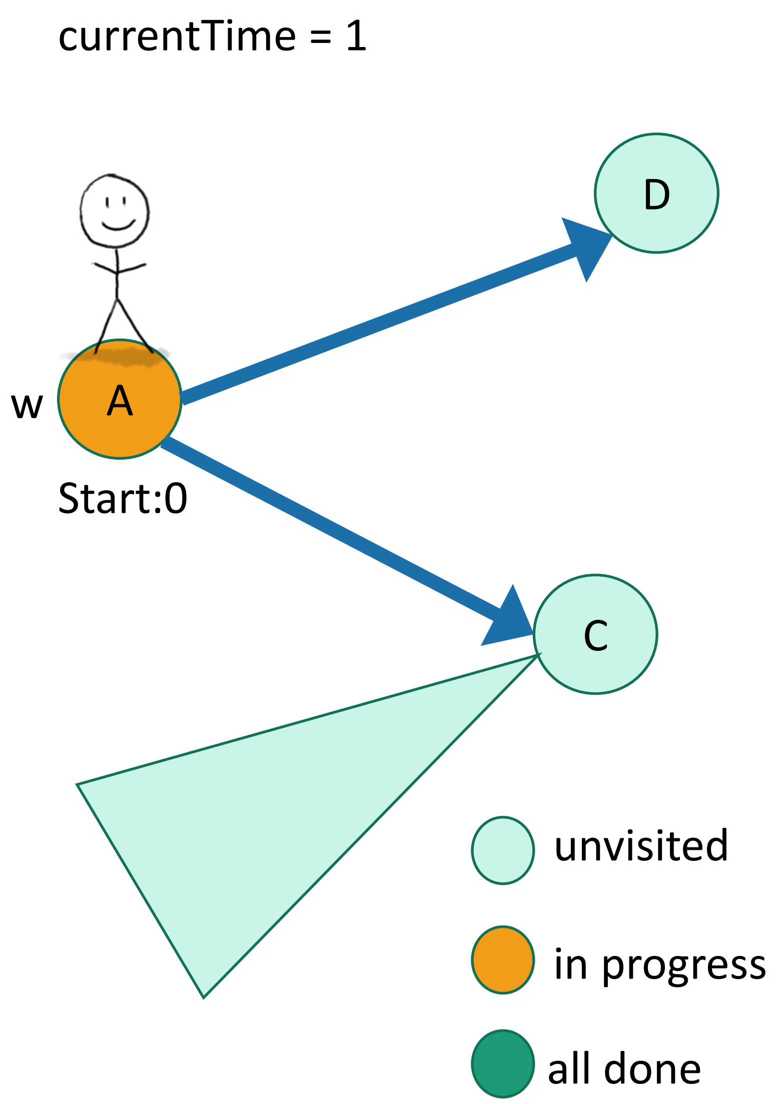
You might have seen other ways to implement DFS than what we are about to go through. This way has more bookkeeping, but more intuition – also, the bookkeeping will be useful later!

Depth First Search



- **DFS**(w, currentTime):
  - w.entryTime = currentTime
  - currentTime ++
  - Mark w as in progress.
  - **for** v in w.neighbors:
    - **if** v is unvisited:
      - currentTime
        = **DFS**(v, currentTime)
      - currentTime ++
  - w.finishTime = currentTime
  - Mark w as **all done**
  - **return** currentTime

Depth First Search



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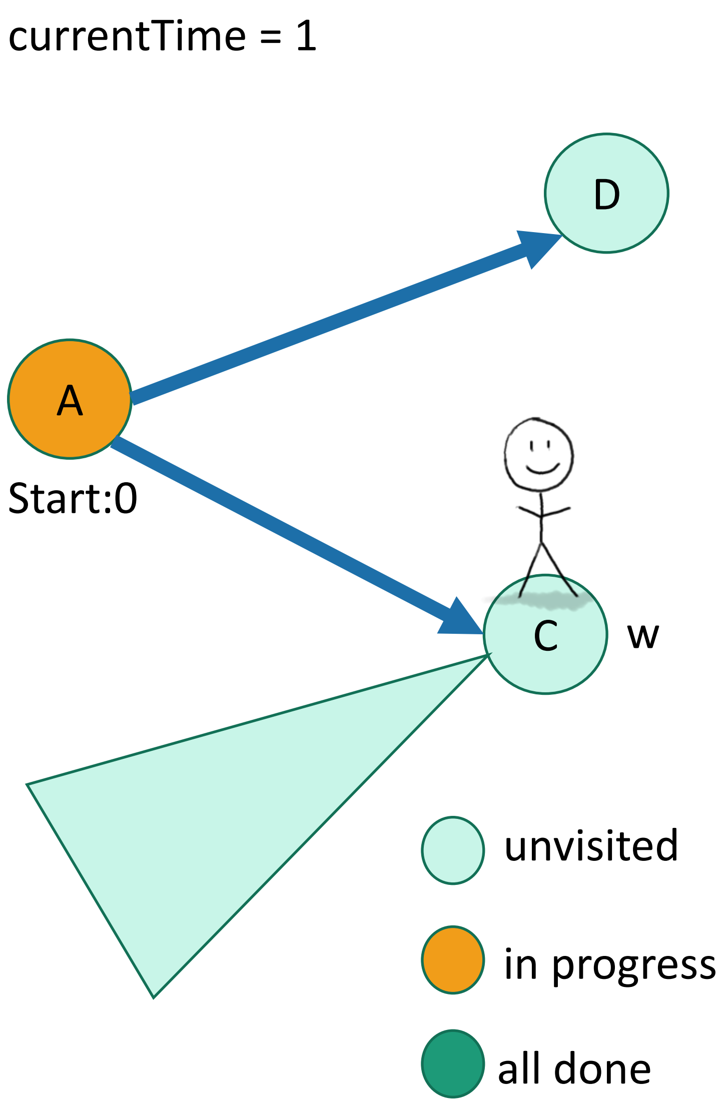

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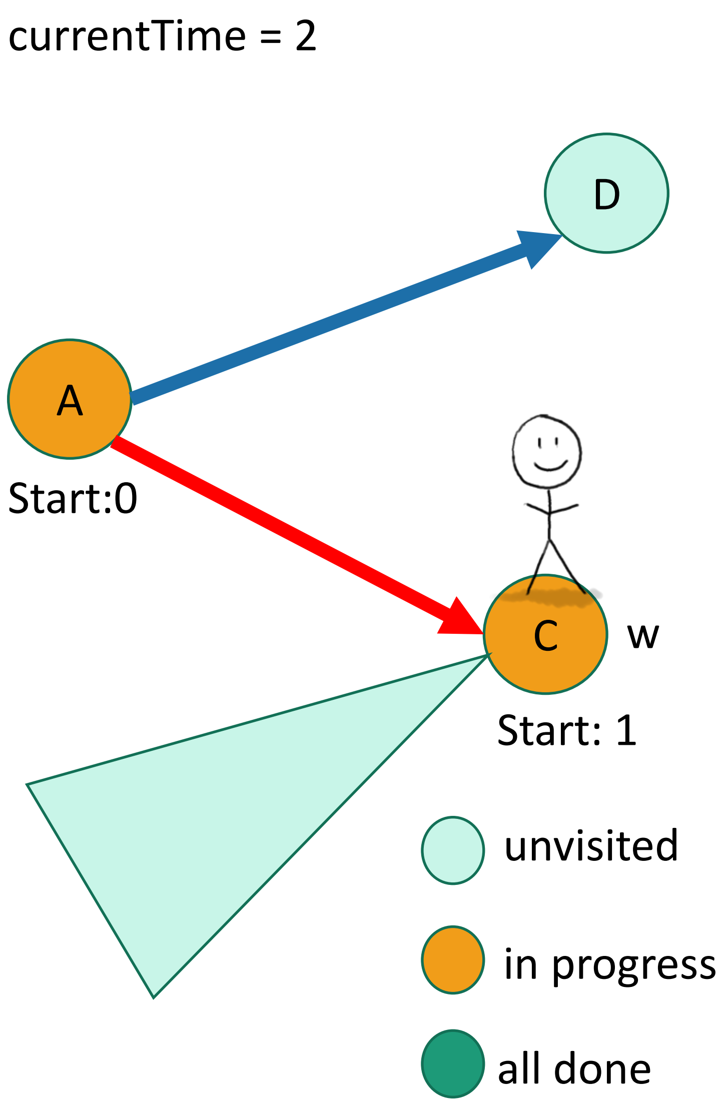

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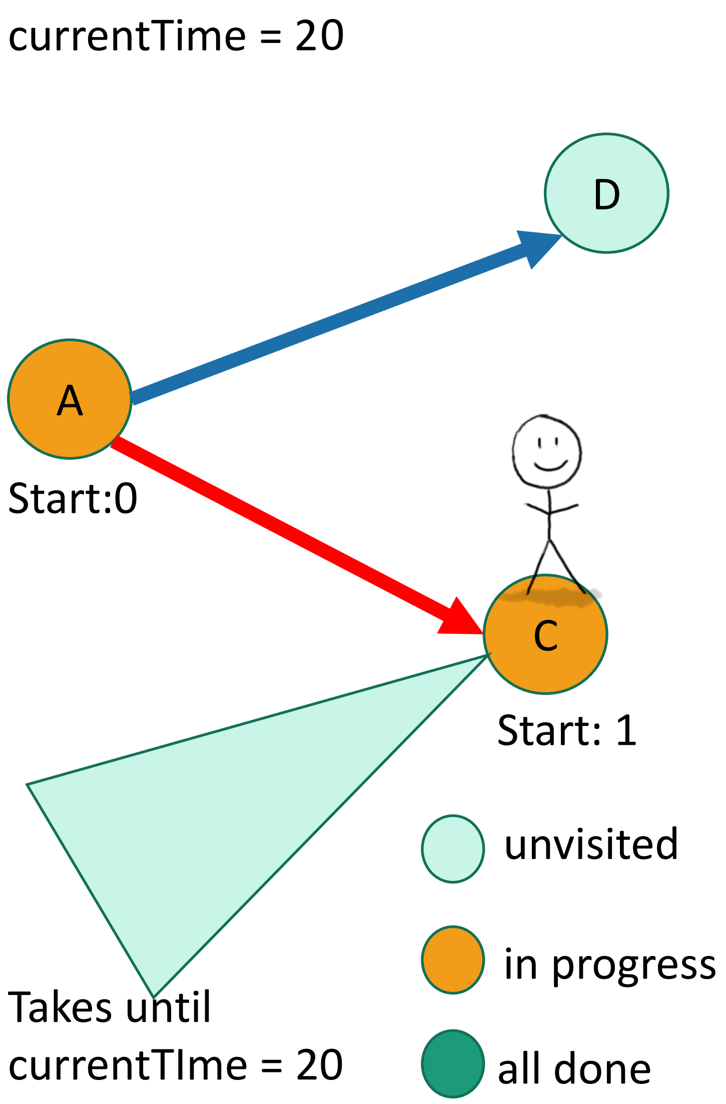

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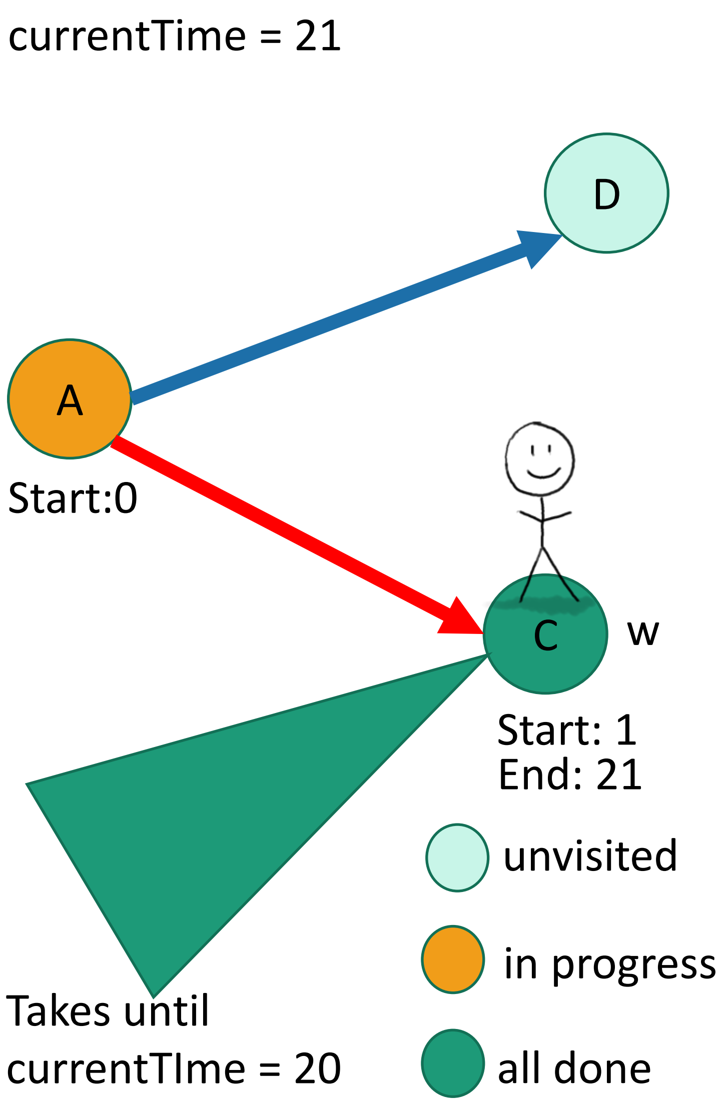
Depth First Search



- **DFS**(w, currentTime):
  - w.startTime = currentTime
  - currentTime ++
  - Mark w as **in progress**.
  - **for** v in w.neighbors:
    - **if** v is **unvisited**:
      - currentTime
        = **DFS**(v, currentTime)
      - currentTime ++
  - w.finishTime = currentTime
  - Mark w as **all done**
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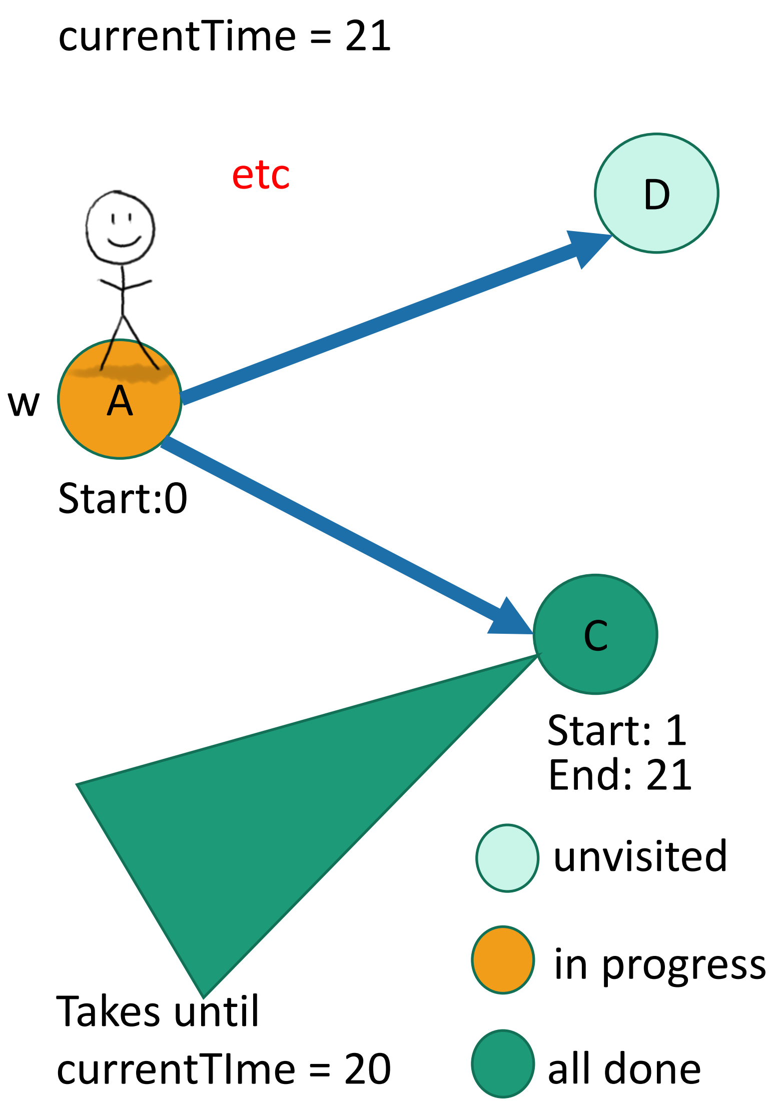
Depth First Search



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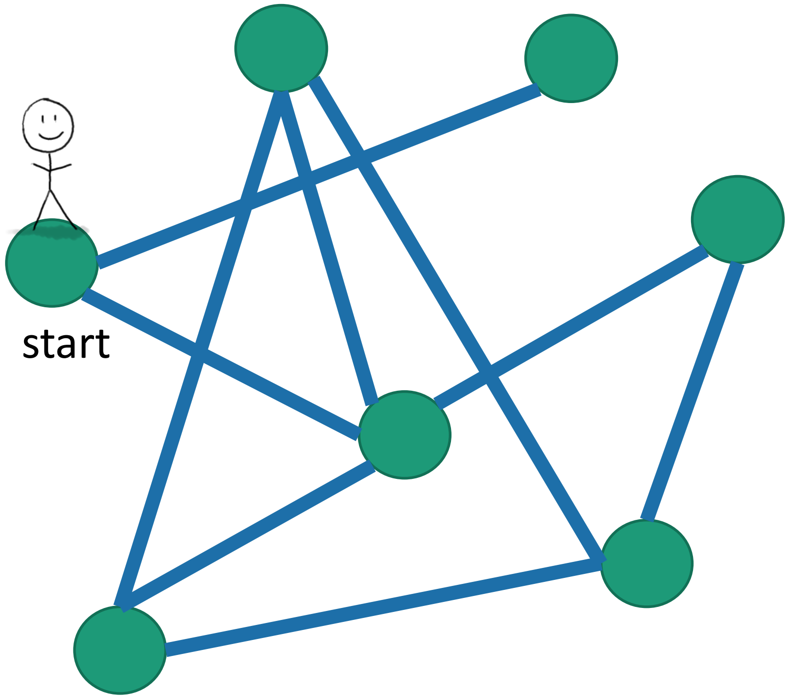
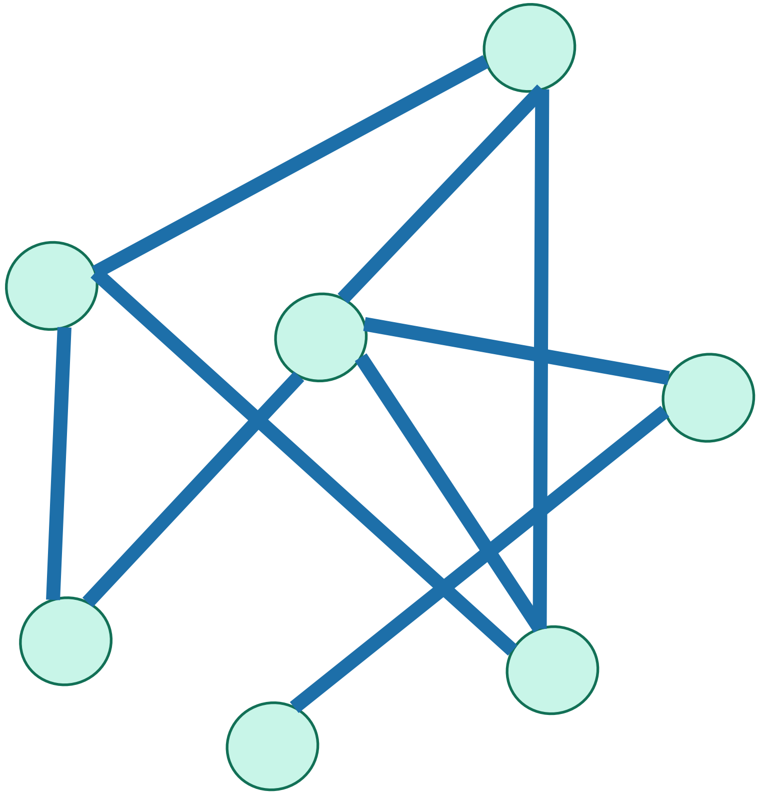
DFS finds all the nodes reachable from the starting point



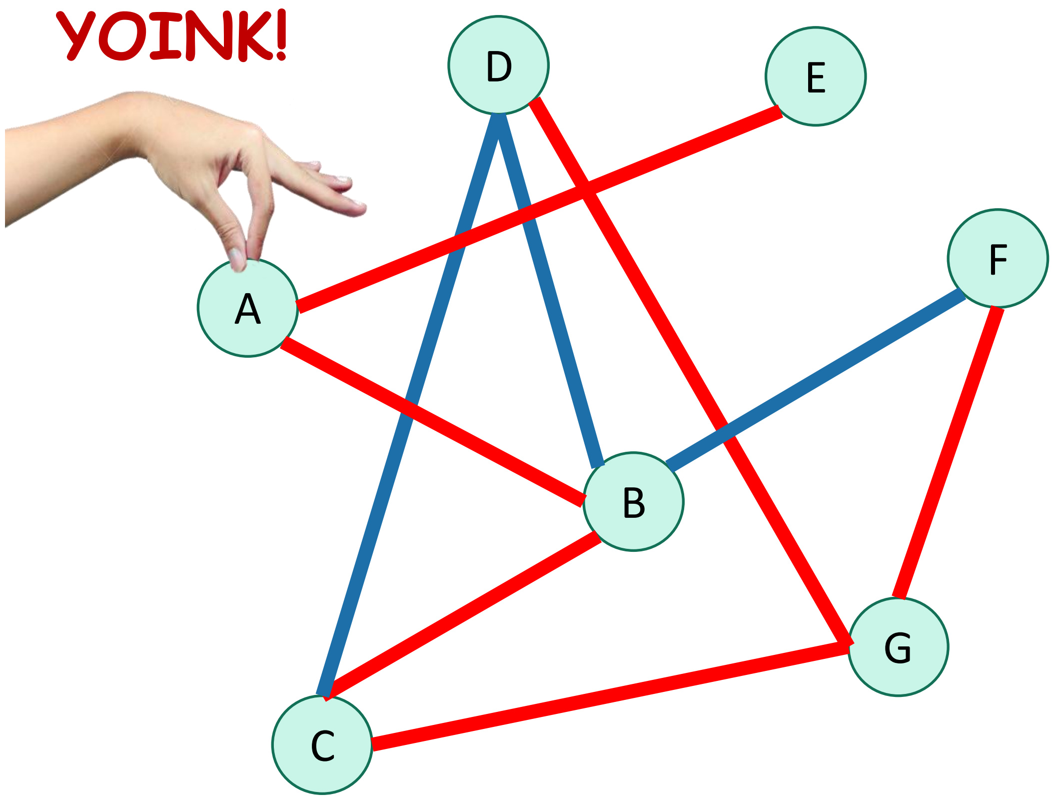
In an undirected graph, this is called a **connected component.**

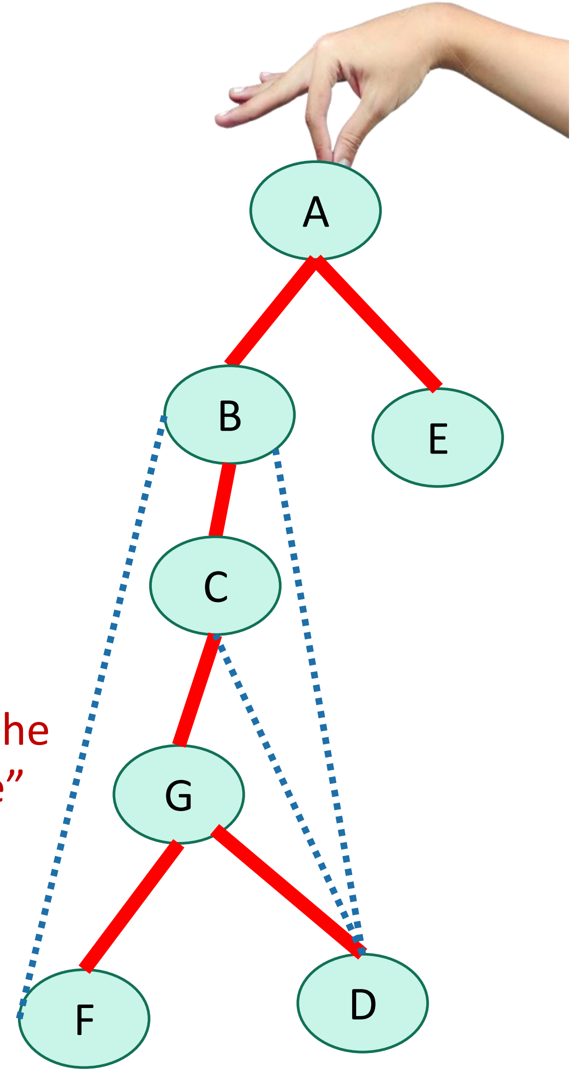


One application: finding connected components.

Why is it called depth-first?

- We are implicitly building a tree:





Call this the "DFS tree"

- And first we go as deep as we can.

Running time

To explore just the connected component we started in

- We look at each edge only once.
- And basically don't do anything else.
- So…

$$O(m)$$

- (Assuming we are using the linked-list representation)

Verify this formally!

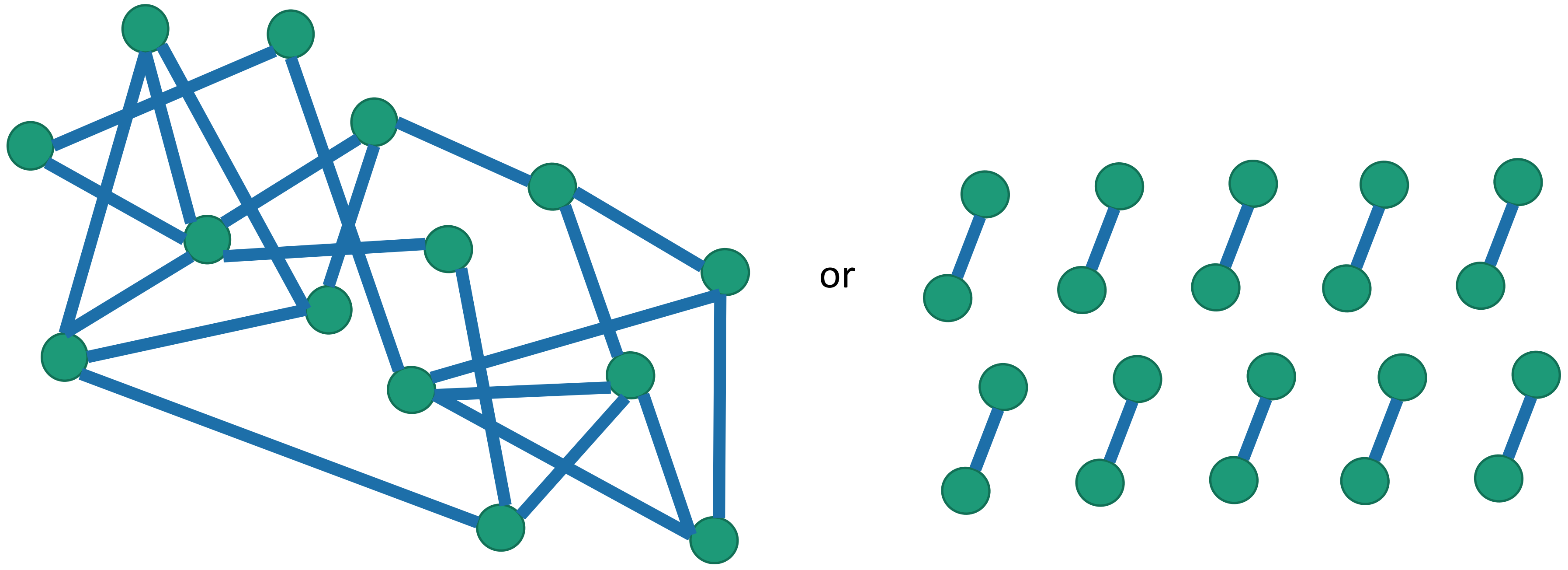
Ollie the over-achieving ostrich

Running time

To explore the whole thing

- Explore the connected components one-by-one.
- This takes time

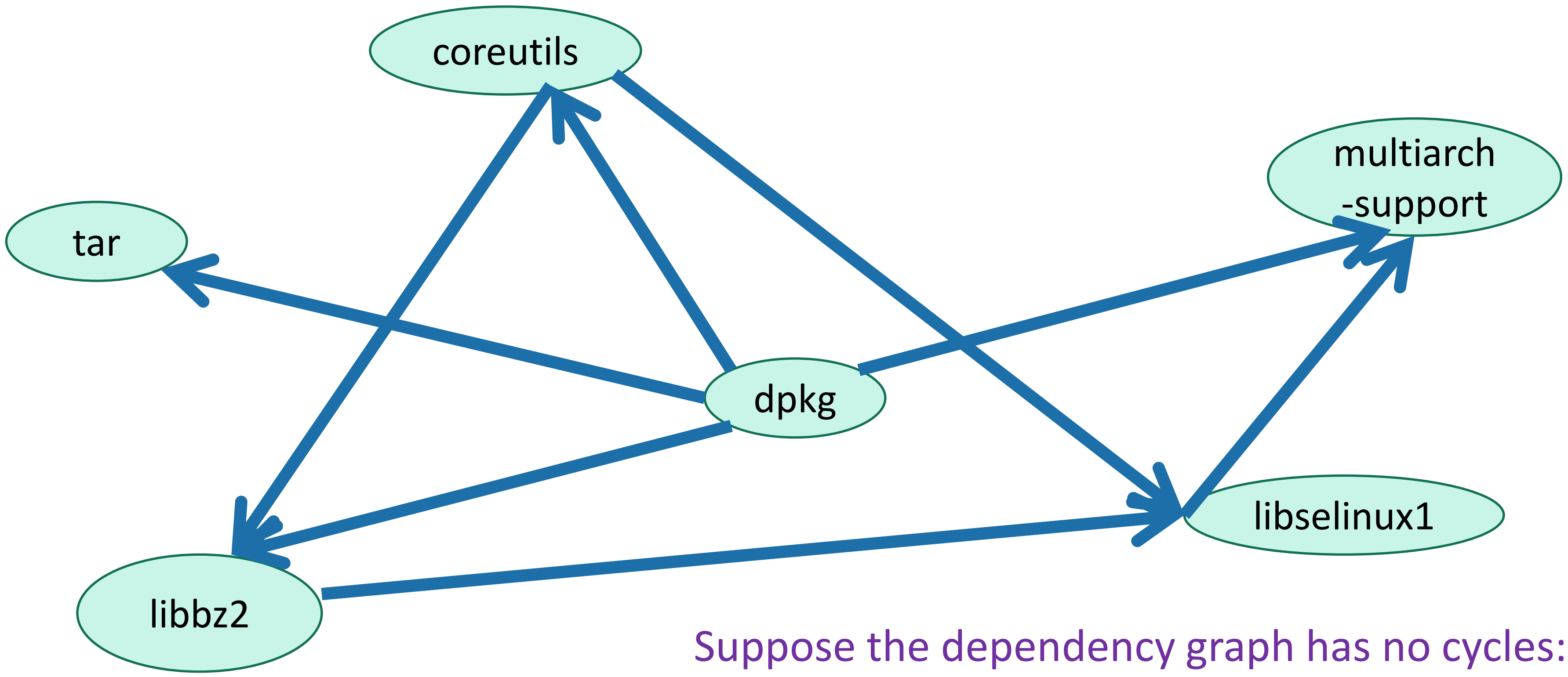
$$O(n + m)$$



Applications of DFS

topological sorting: Application of DFS

- Example: package dependency graph
- Question: in what order should I install packages?

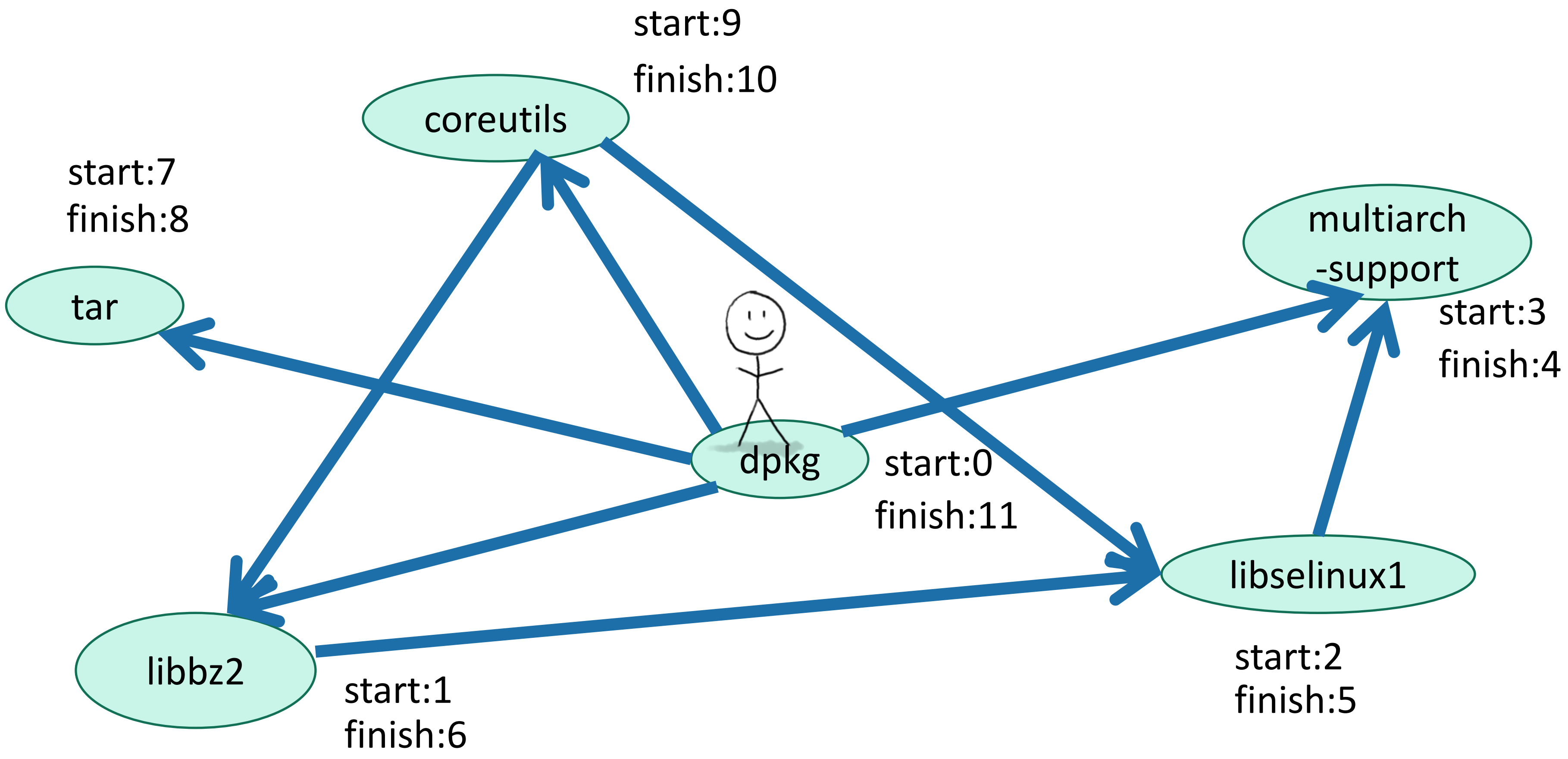


Suppose the dependency graph has no cycles:
it is a **Directed Acyclic Graph (DAG)**

Let’s do DFS

Observations:

- The start times don’t seem that useful.
- But the packages we should include earlier have larger **finish times.**



Suppose the underlying graph has no cycles

This is not an accident

Claim: In general, we'll always have:

finish: [larger]

finish: [smaller]

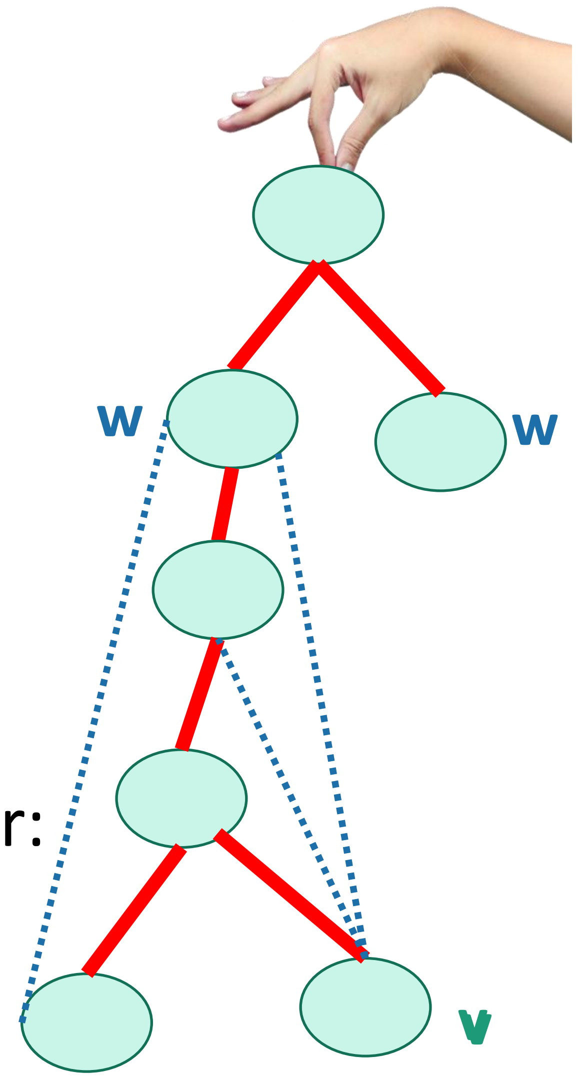
To understand why, let's go back to that DFS tree.

A more general statement

(this holds even if there are cycles)

This is called the “parentheses theorem” in CLRS

(check this statement carefully!)



- If v is a descendent of w in this tree:

- If w is a descendent of v in this tree:

- If neither are descendents of each other:

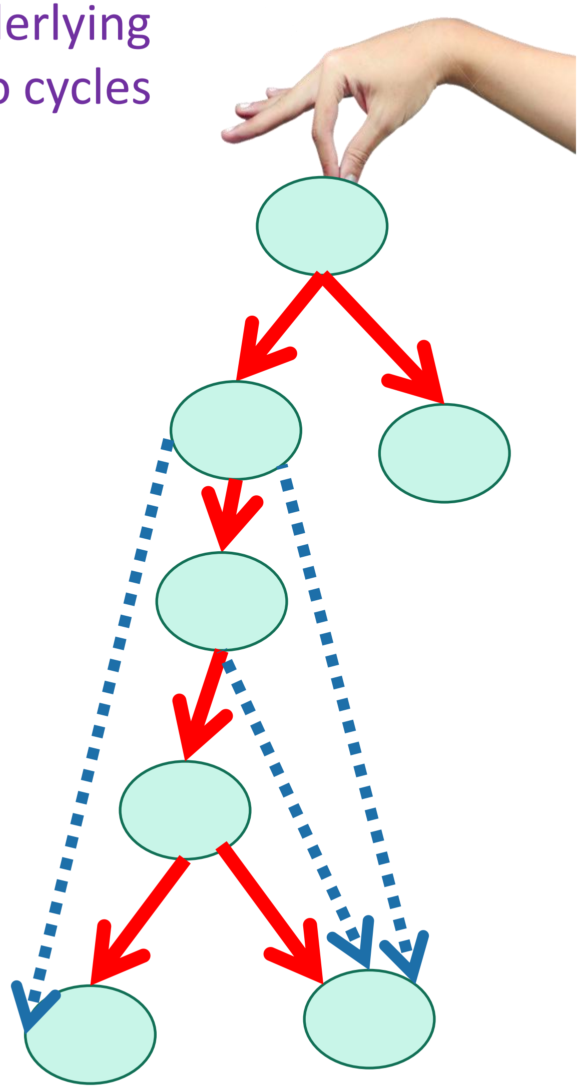
(or the other way around)

So to prove this ->

If A → B

Then B.finishTime < A.finishTime

Suppose the underlying graph has no cycles



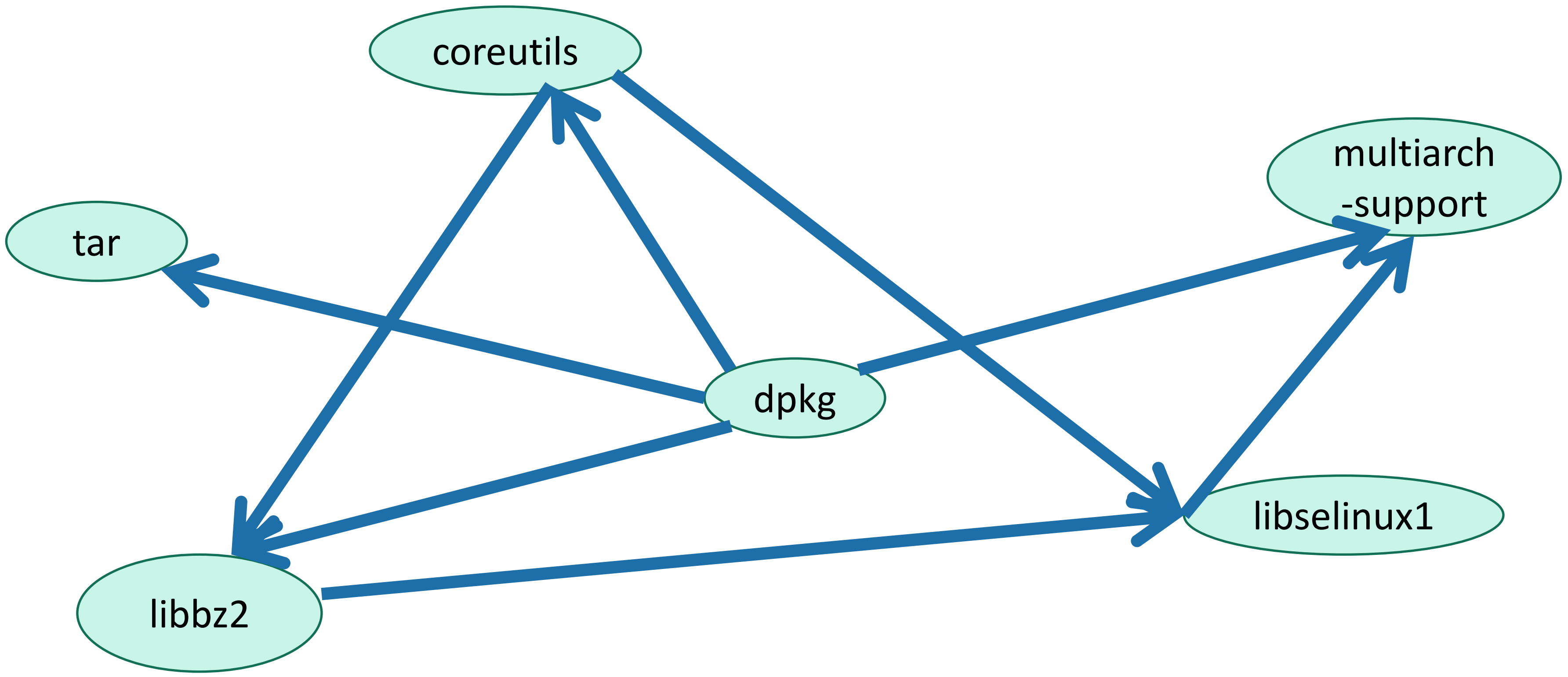
- Since the graph has no cycles, B must be a descendent of A in that tree.
  - All edges go down the tree.

- Then

- aka, B.finishTime < A.finishTime.

Back to this problem

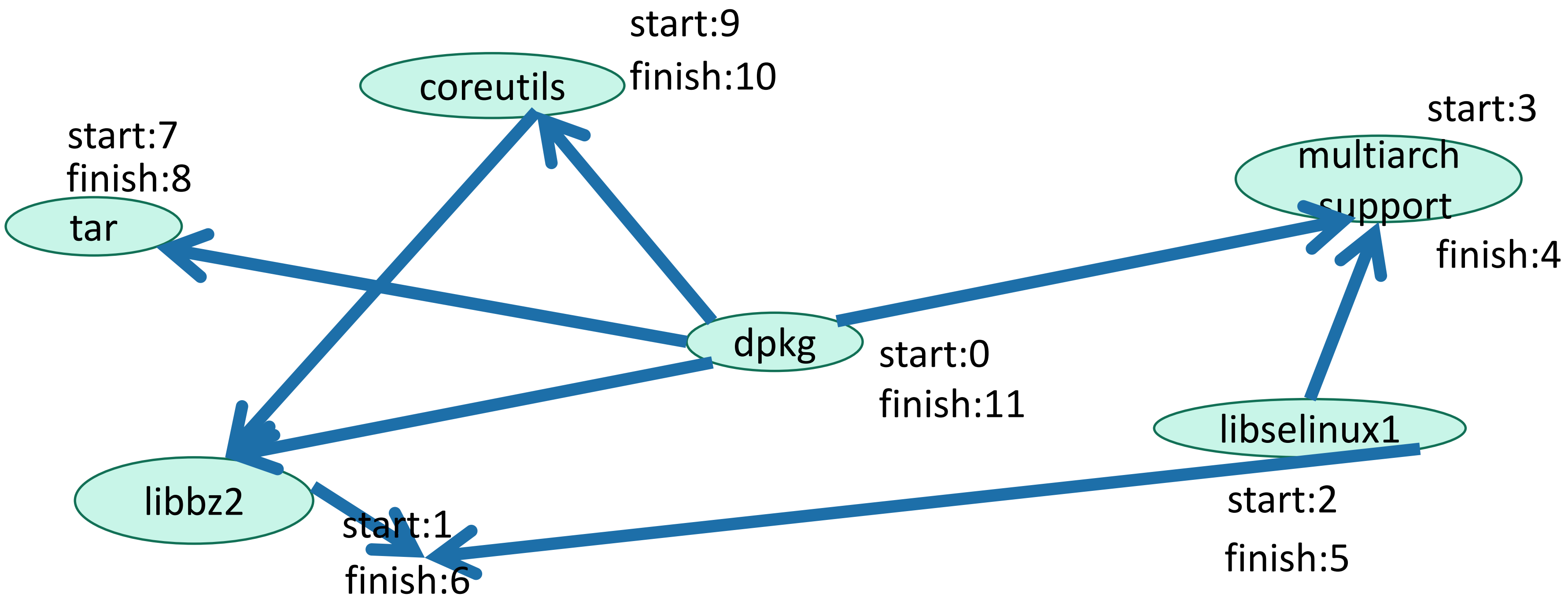
- Example: package dependency graph
- Question: in what order should I install packages?



In reverse order of finishing time

- Do DFS
- Maintain a list of packages, in the order you want to install them.
- When you mark a vertex as **all done**, put it at the **beginning** of the list.

- `dpkg`
- `coreutils`
- `tar`
- `libbz2`
- `libselinux1`
- `multiarch_support`

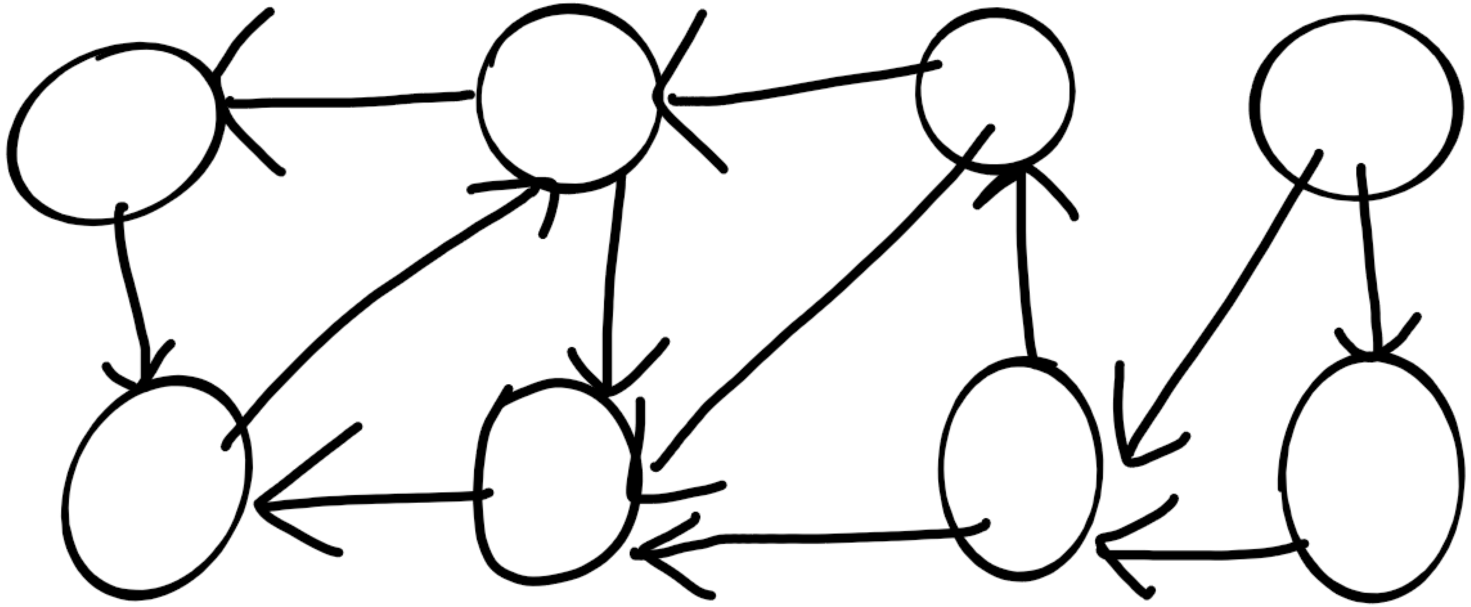


What did we just learn?

- DFS can help you solve the **TOPOLOGICAL SORTING PROBLEM**
  - That's the fancy name for the problem of finding an ordering that respects all the dependencies
- Thinking about the DFS tree is helpful.

Classification of Edges

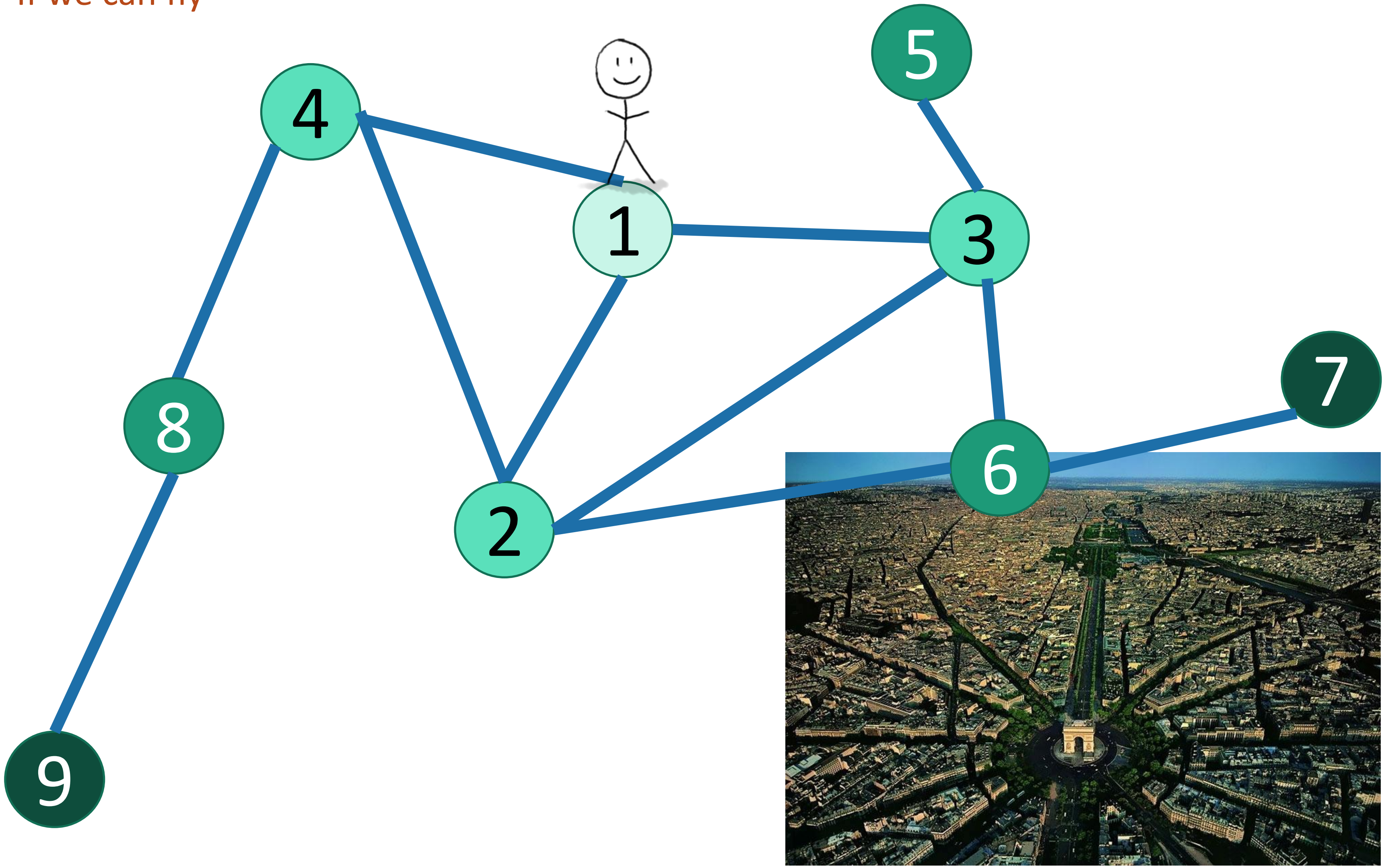
- Tree Edge
- Back Edge
- Forward Edge
- Cross Edge



Breadth-First Search

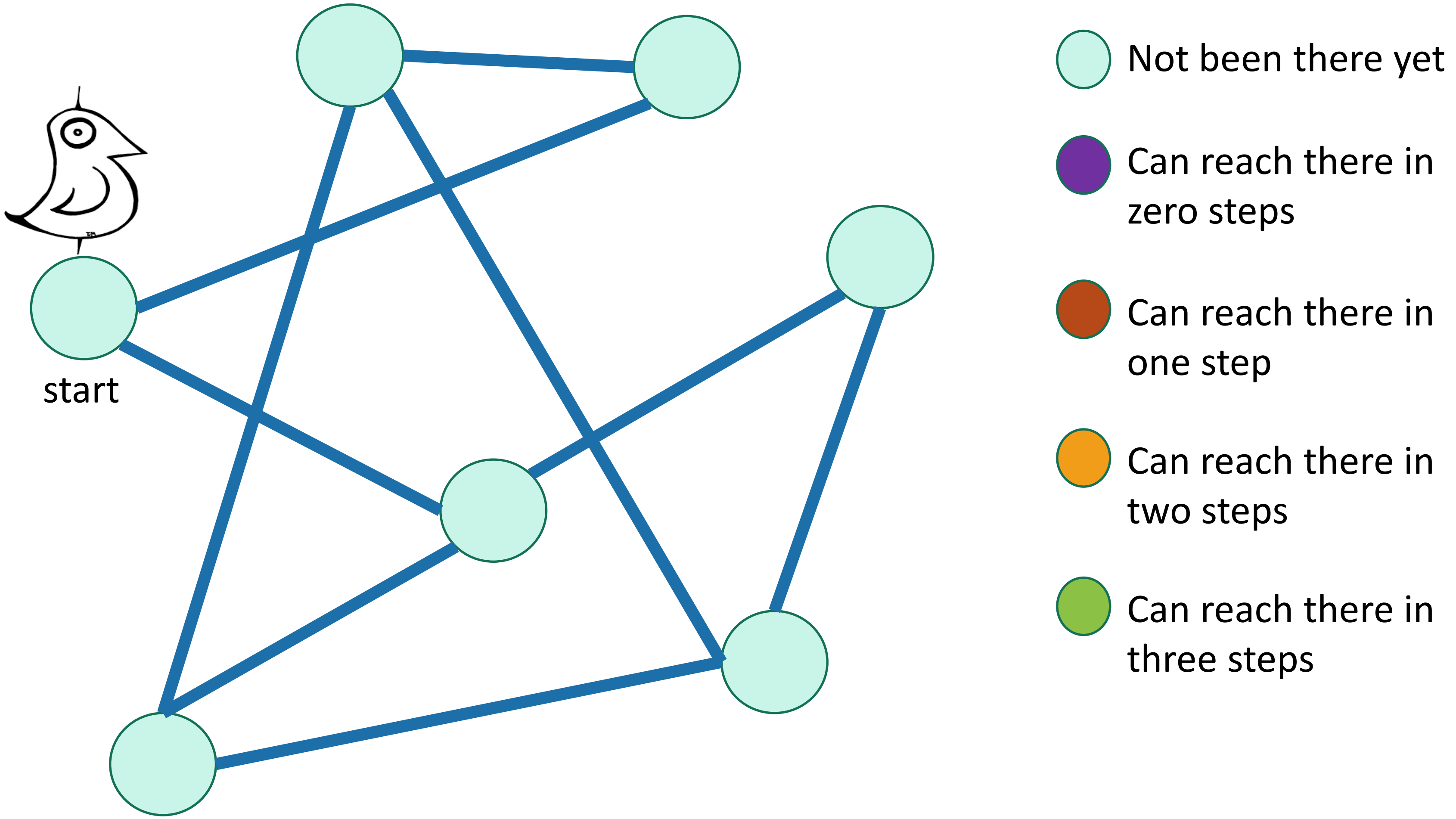
How do we explore a graph?

If we can fly



Breadth-First Search

Exploring the world with a bird's-eye view



Breadth-First Search

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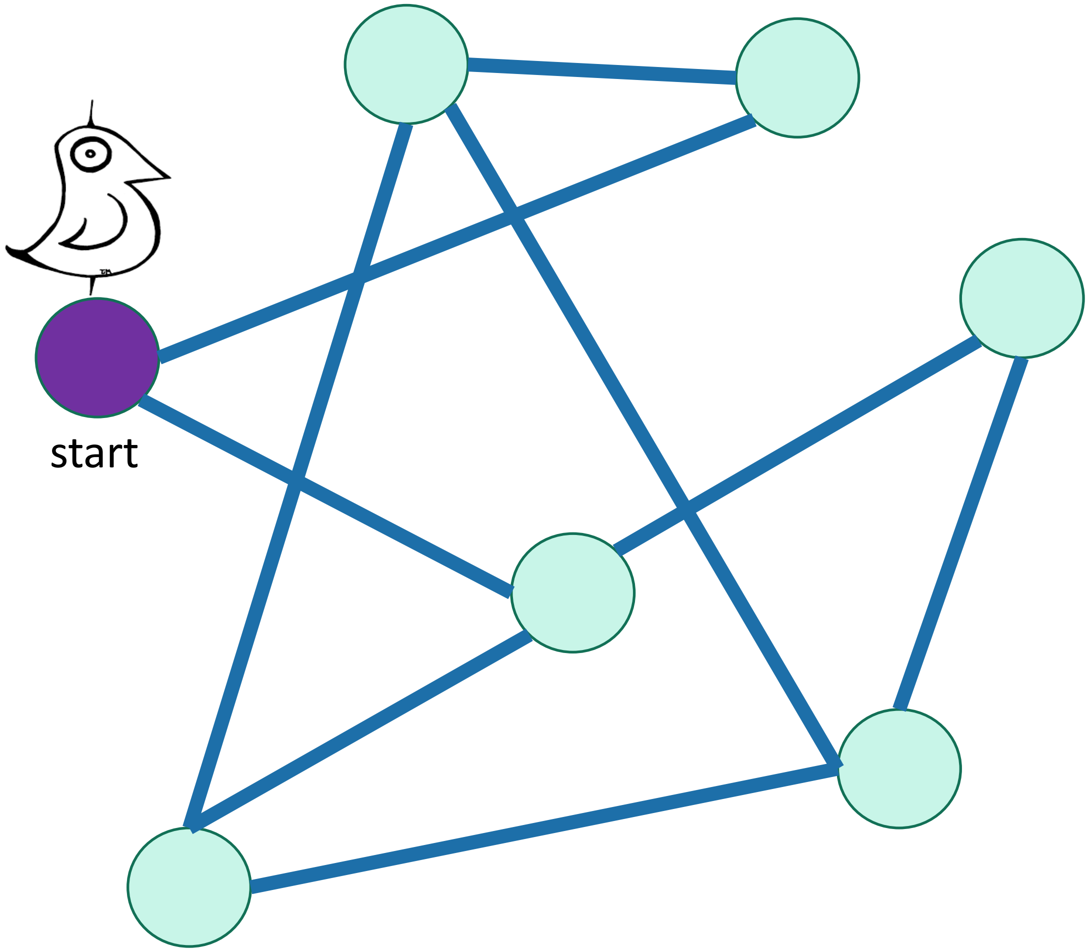


Not been there yet

Can reach there in zero steps

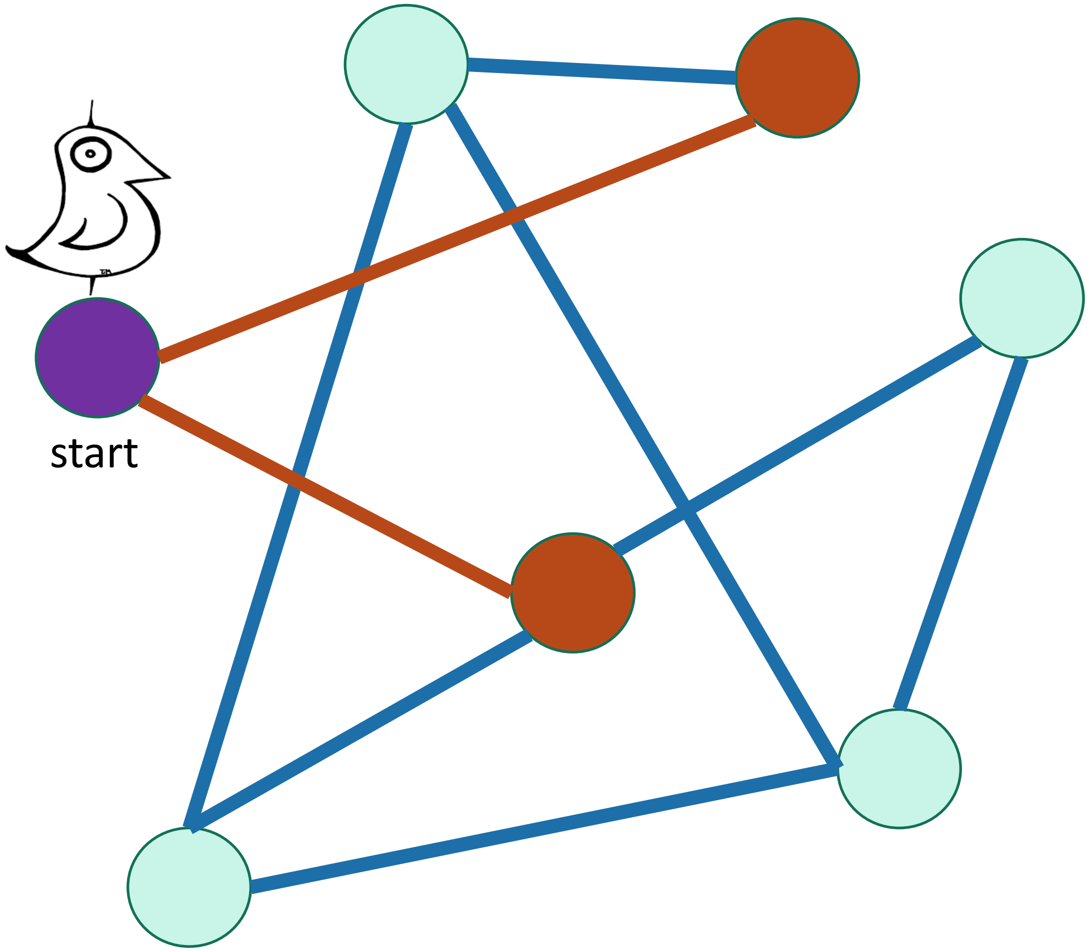
Can reach there in one step

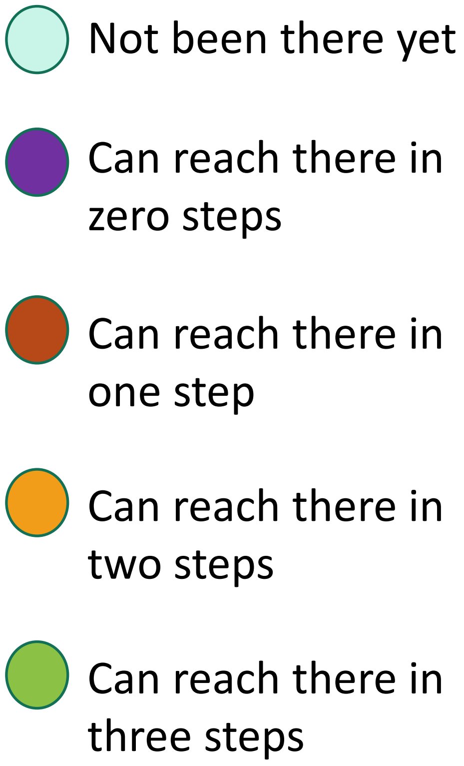
Can reach there in two steps

Can reach there in three steps

Breadth-First Search

Exploring the world with a bird's-eye view





Breadth-First Search

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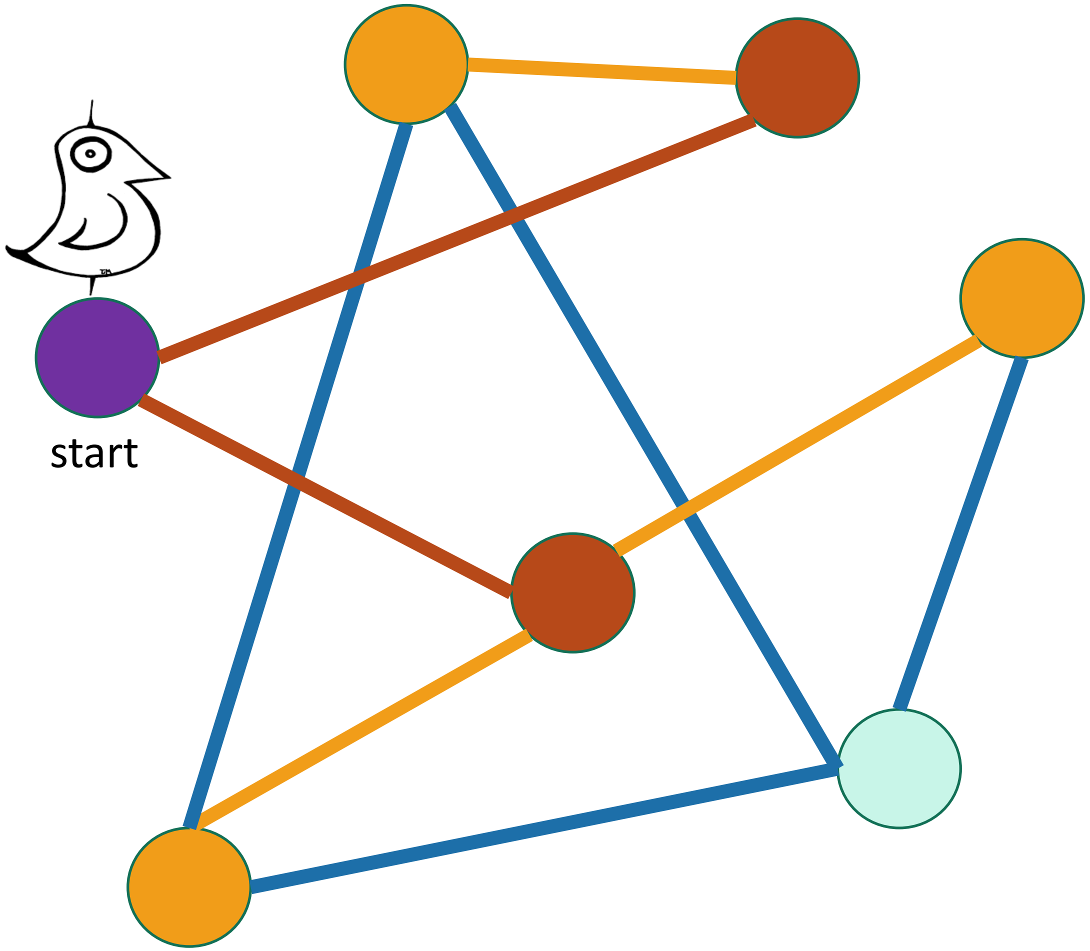

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Breadth-First Search

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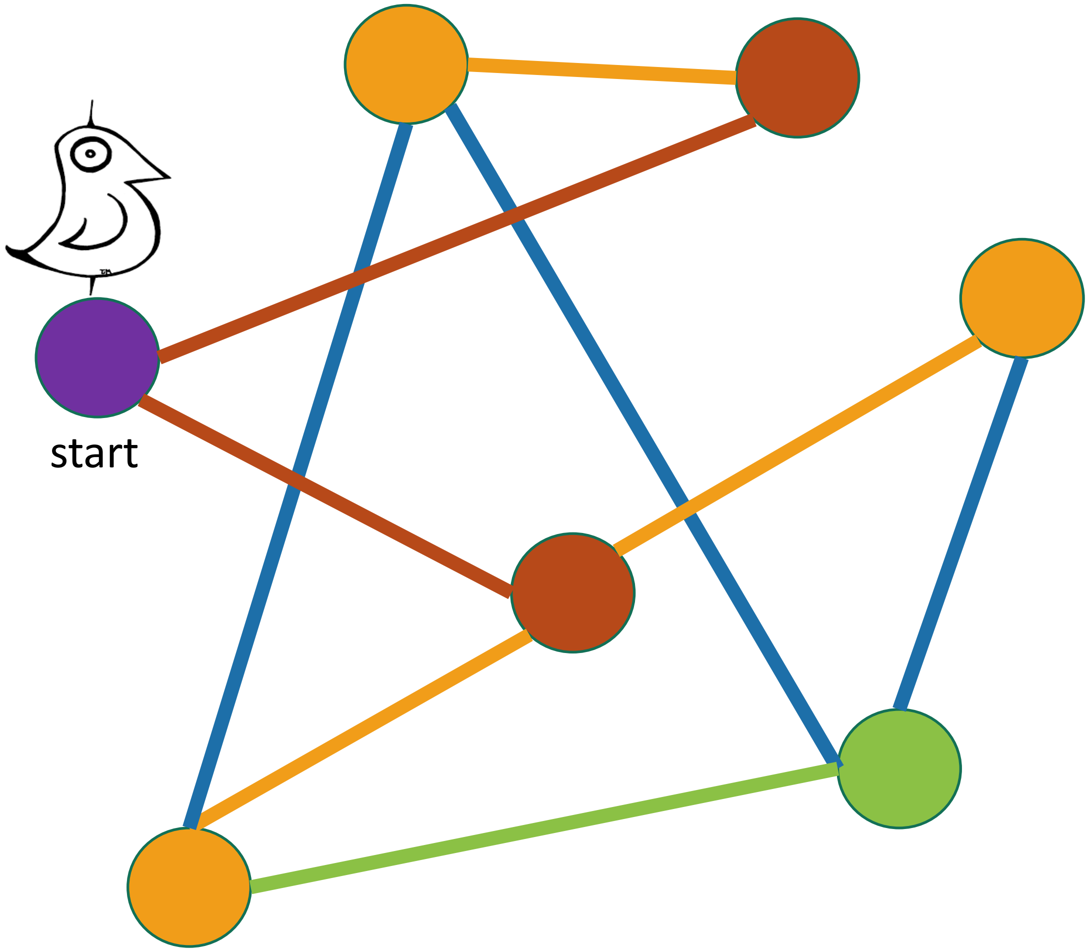
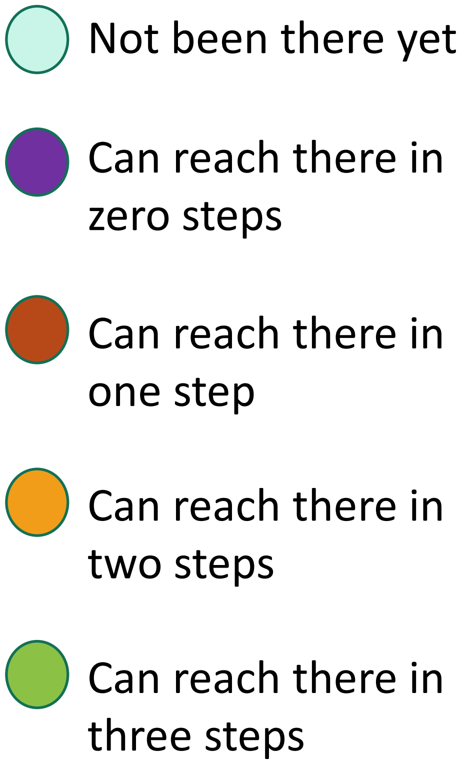




World:

explored!

Same disclaimer as for DFS: you may have seen other ways to implement this, this will be convenient for us.

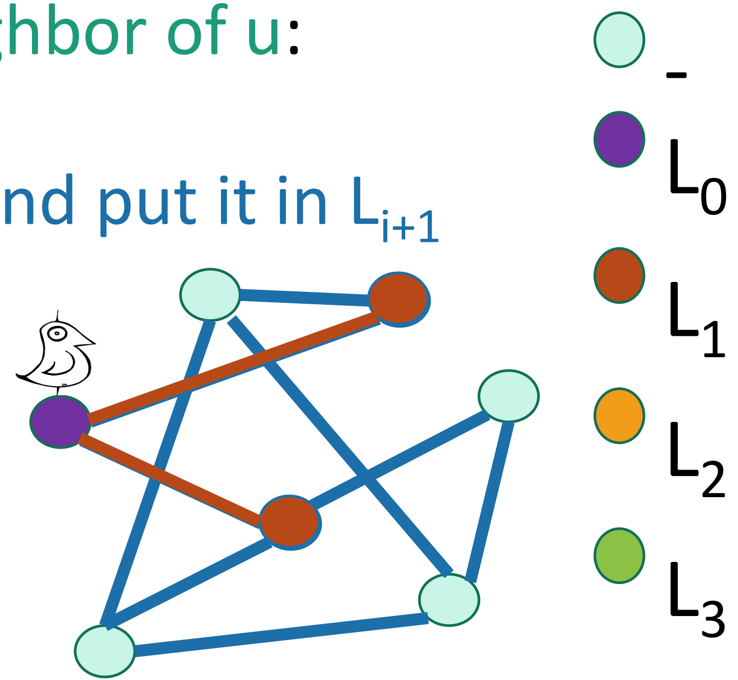
Breadth-First Search

Exploring the world with pseudocode

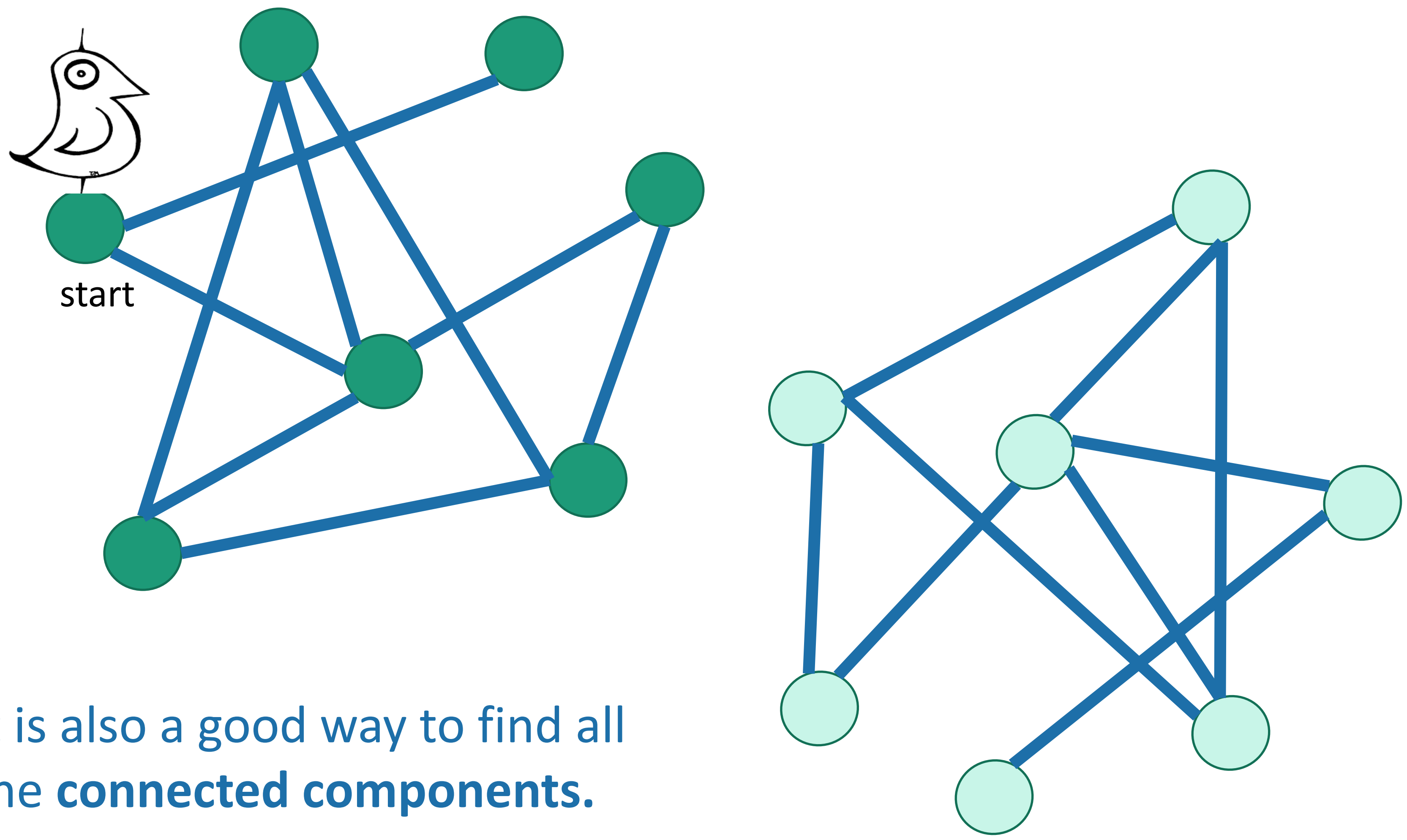
- Set L_i = {} for i=1,…,n
- L_0 = {w}, where w is the start node
- **For** i = 0, …, n-1:
  - **For** u in L_i:
    - **For** each v which is a neighbor of u:
      - **If** v isn't yet visited:
        - mark v as visited, and put it in L_{i+1}

L_i is the set of nodes we can reach in i steps from w

Go through all the nodes in L_i and add their unvisited neighbors to L_{i+1}



BFS also finds all the nodes reachable from the starting point



It is also a good way to find all the **connected components.**

Running time

To explore the whole thing

- Explore the connected components one-by-one.
- Same argument as DFS: running time is

$$O(n + m)$$

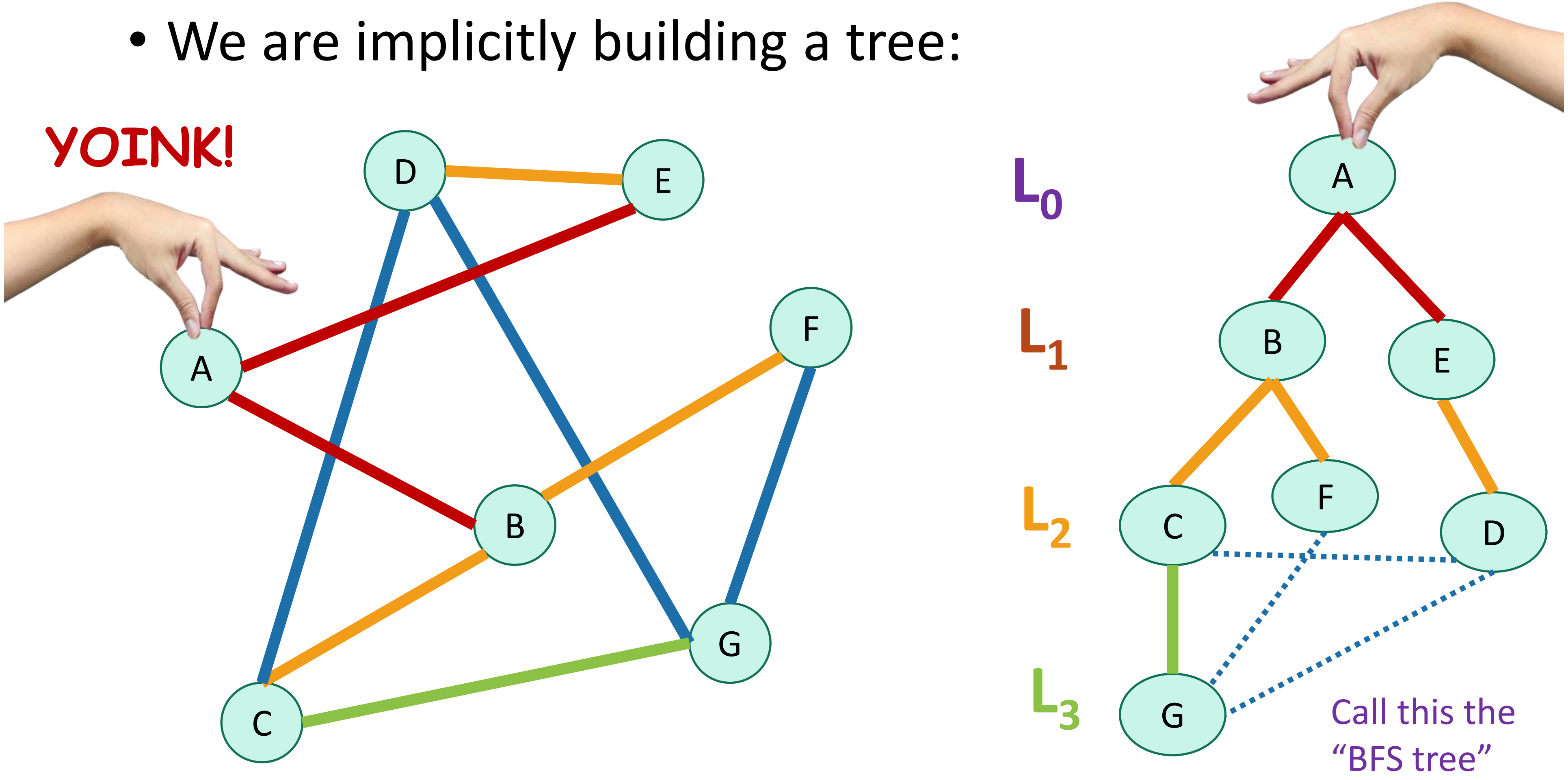
Verify these!

Ollie the over-achieving ostrich

- Like DFS, BFS also works fine on directed graphs.

Why is it called breadth-first?

- We are implicitly building a tree:

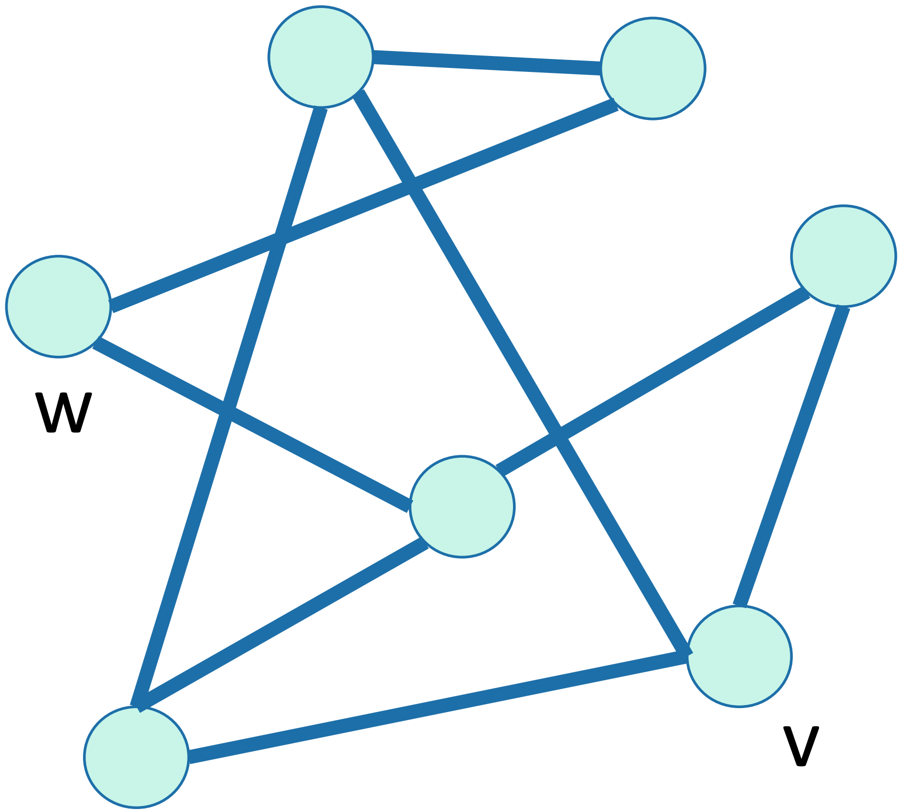


- And first we go as broadly as we can.

Applications of BFS

Application: shortest path

- How long is the shortest path between w and v?



Application: shortest path

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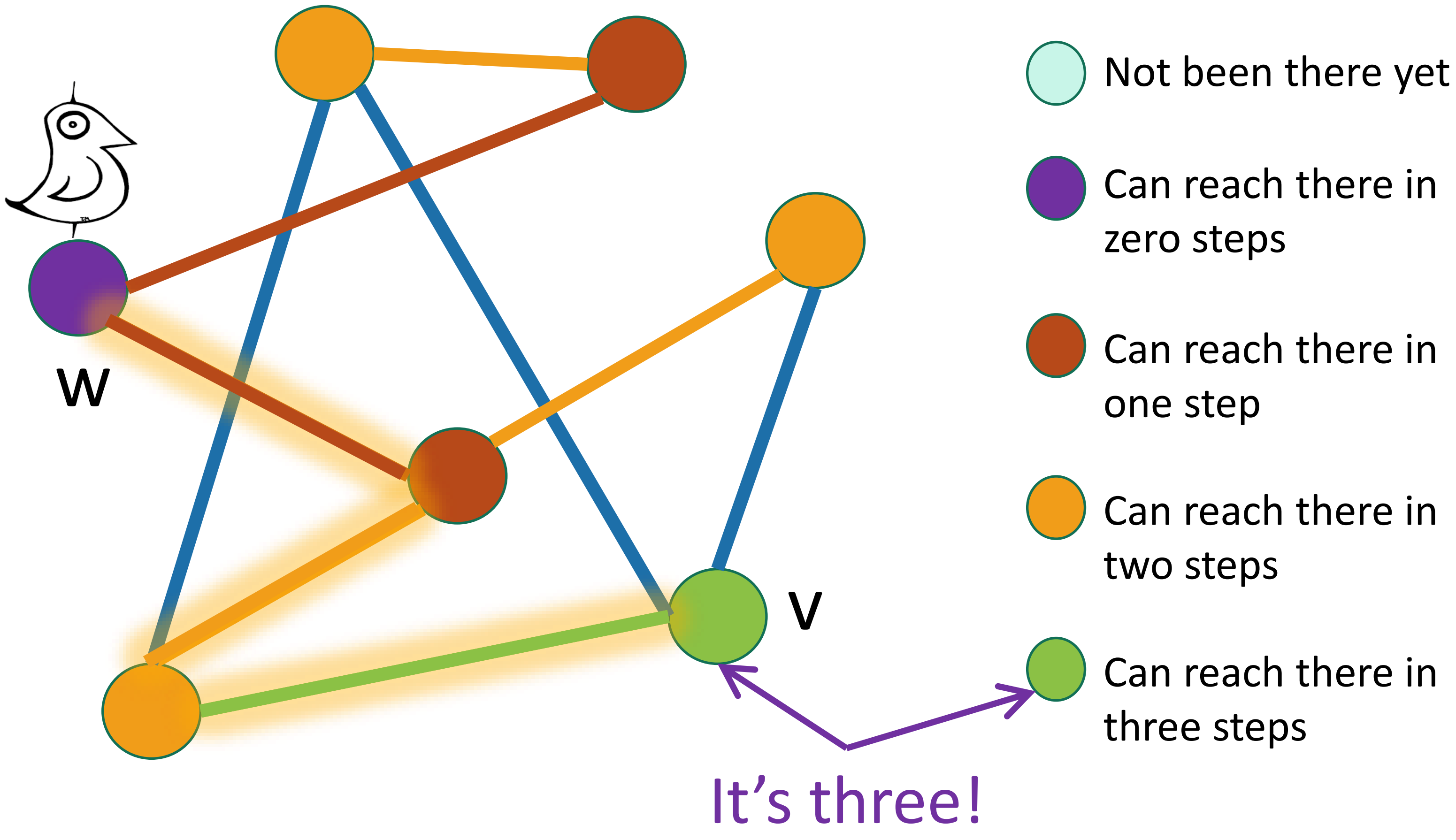


To find the distance between w and all other vertices v

The distance between two vertices is the length of the shortest path between them.

- Do a BFS starting at w
- For all v in L_i (the i'th level of the BFS tree)
  - The shortest path between w and v has length i
  - A shortest path between w and v is given by the path in the BFS tree.
- If we never found v, the distance is infinite.

What did we just learn?

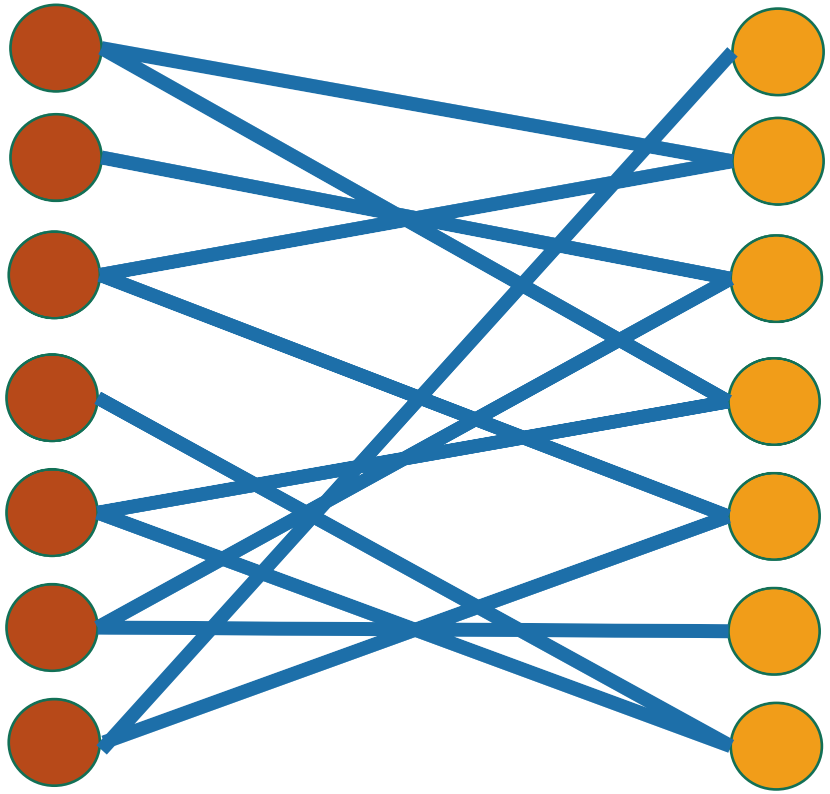
- The BFS tree is useful for computing distances between pairs of vertices.
- We can find the shortest path between u and v in time O(m).

The BSF tree is also helpful for:

- Testing if a graph is bipartite or not.

Application: testing if a graph is bipartite

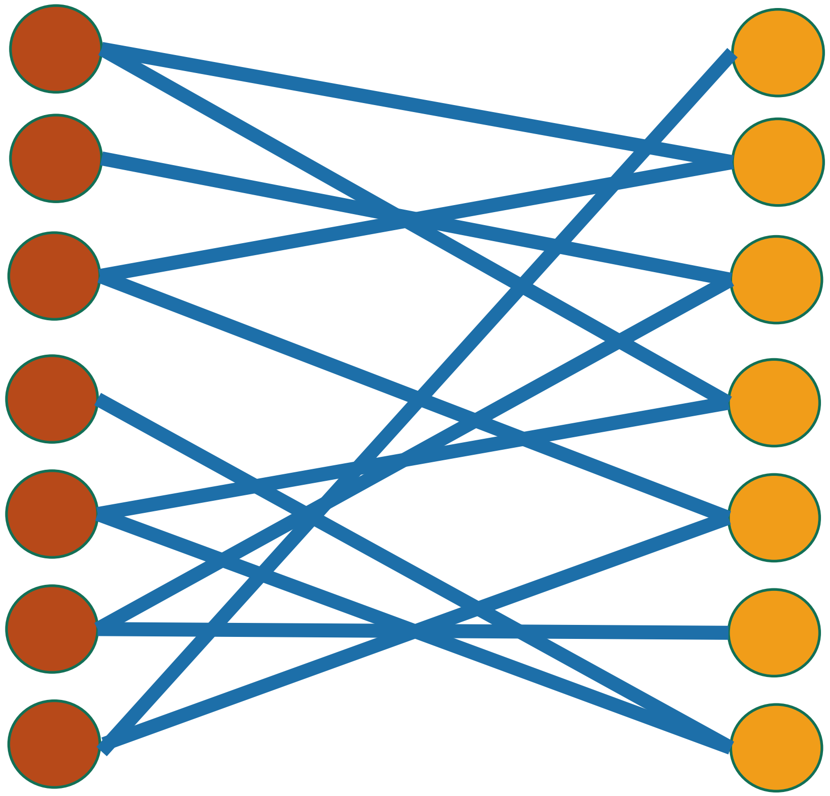
- Bipartite means it looks like this:



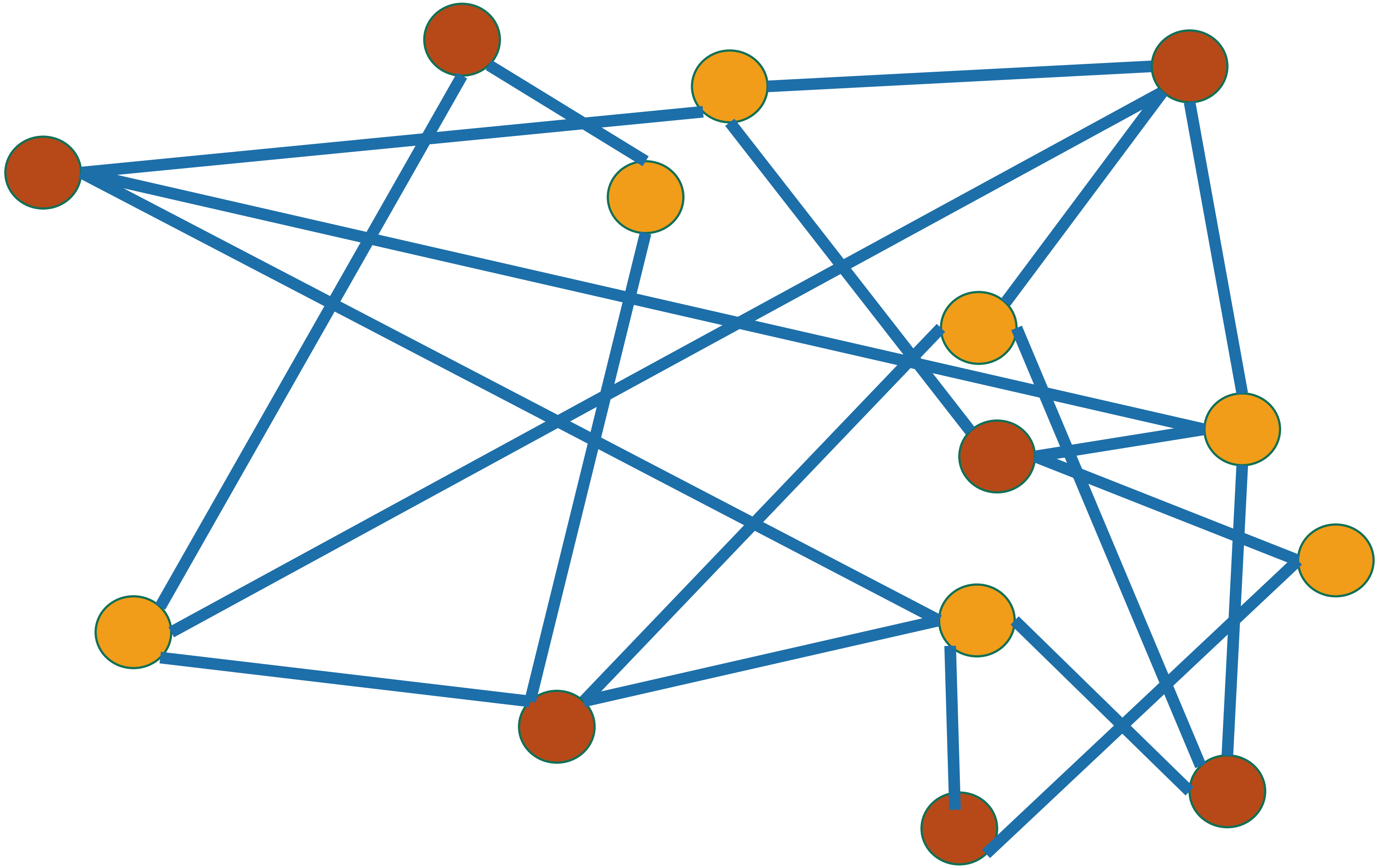
Can color the vertices red and orange so that there are no edges between any same-colored vertices

Example:
are students
are classes
if the student is enrolled in the class

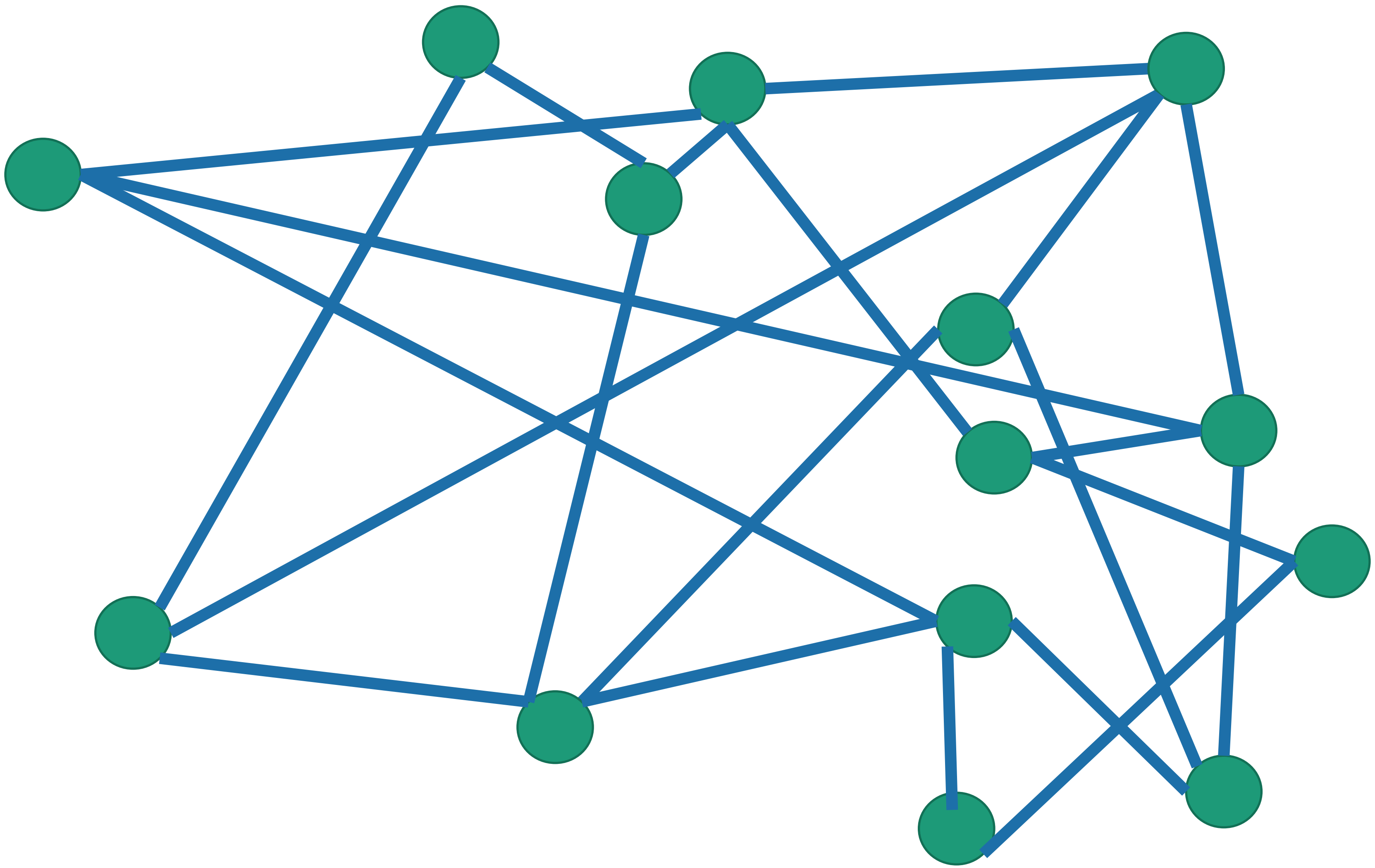
Is this graph bipartite?



How about this one?

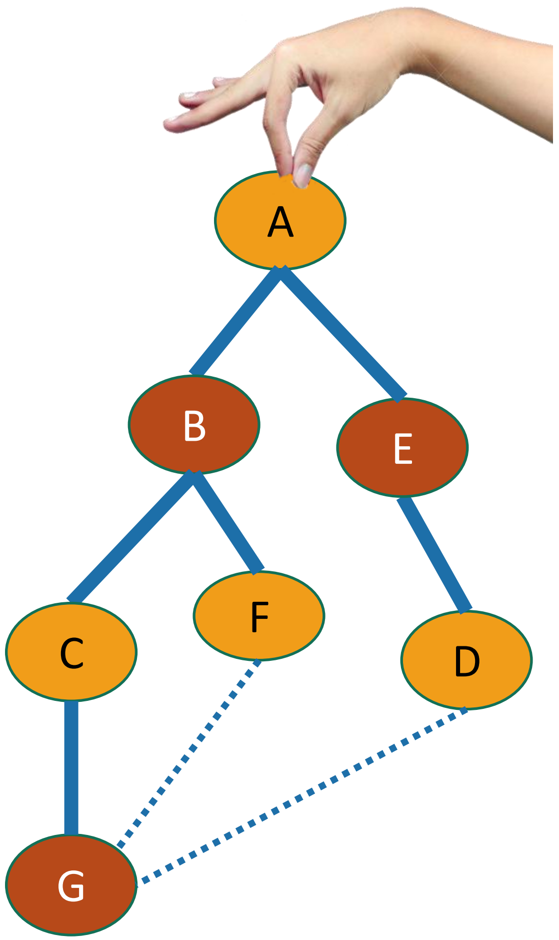


How about this one?



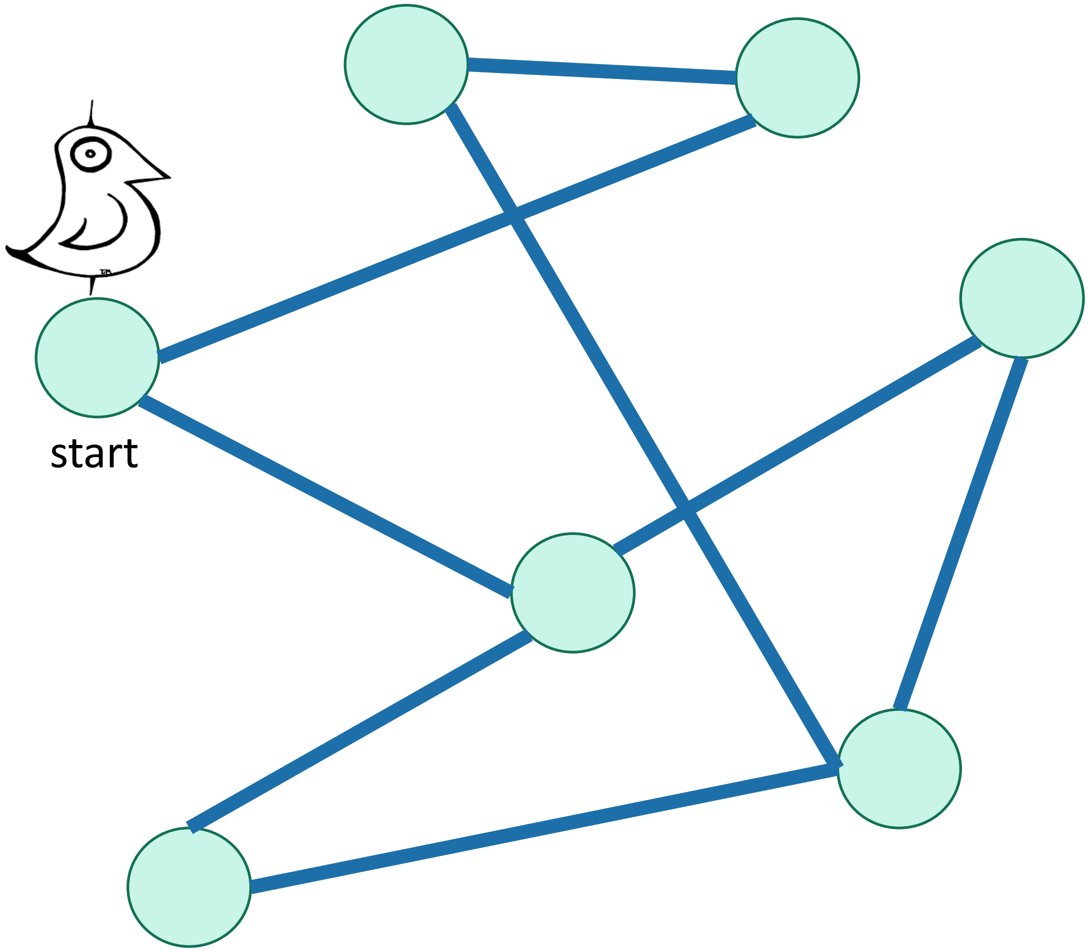
Solution using BFS

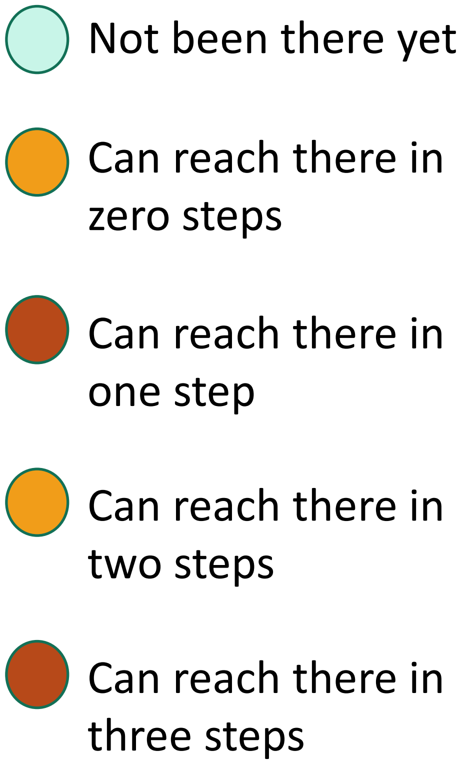
- Color the levels of the BFS tree in alternating colors.
- If you ever color a node so that you never color two connected nodes the same, then it is bipartite.
- Otherwise, it's not.



Breadth-First Search

For testing bipartite-ness





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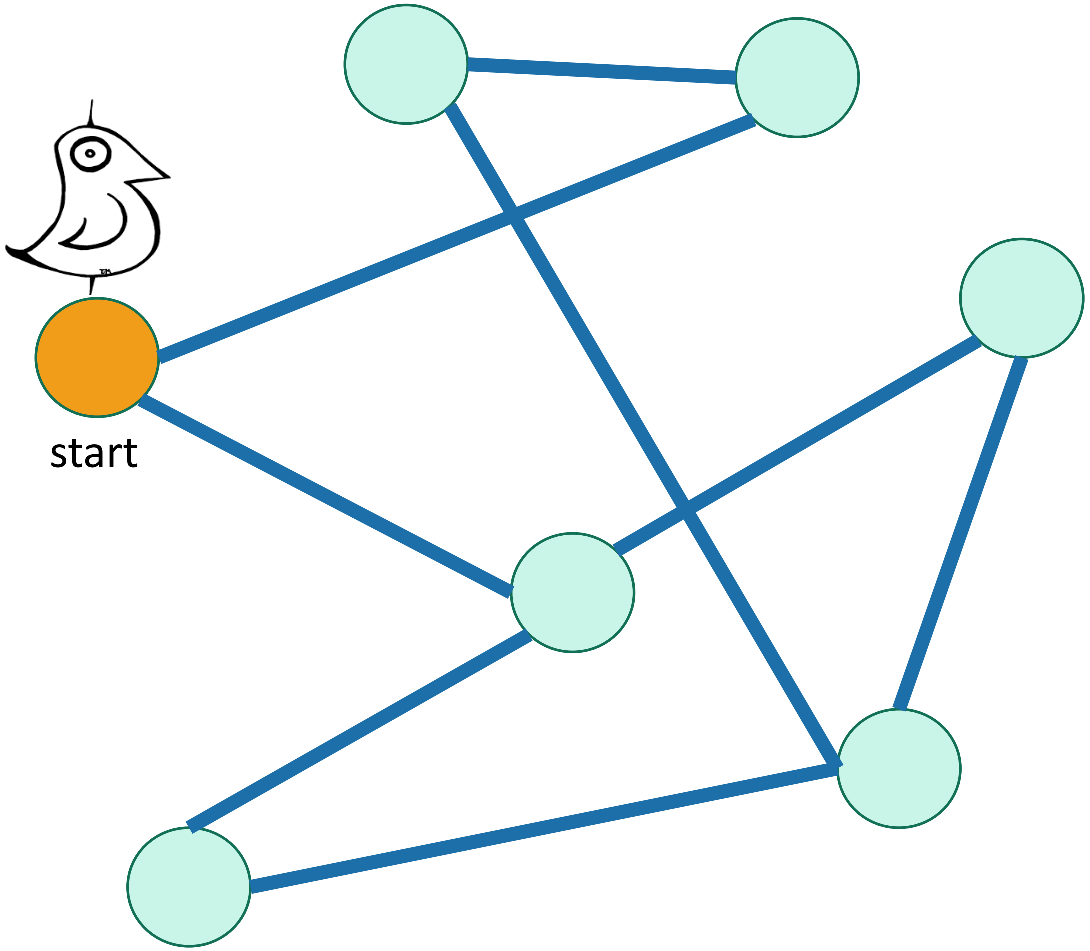

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Breadth-First Search

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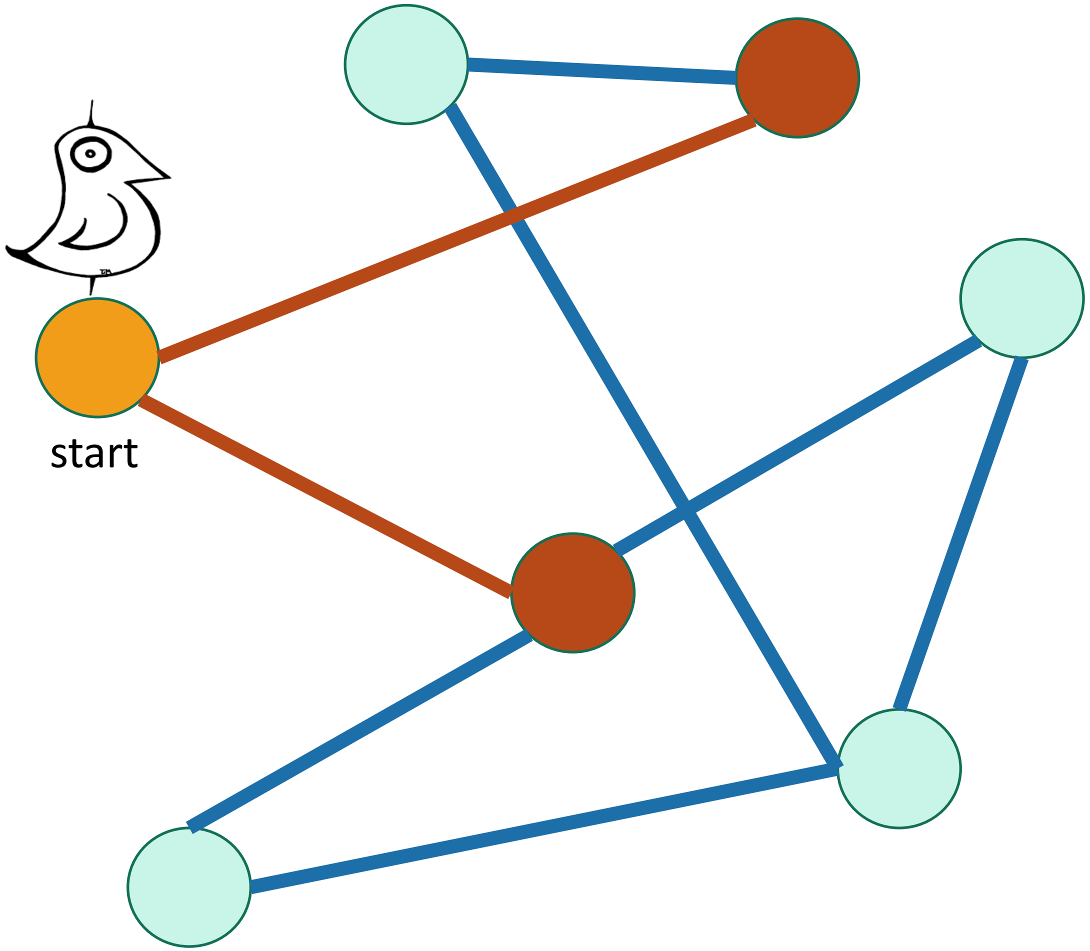

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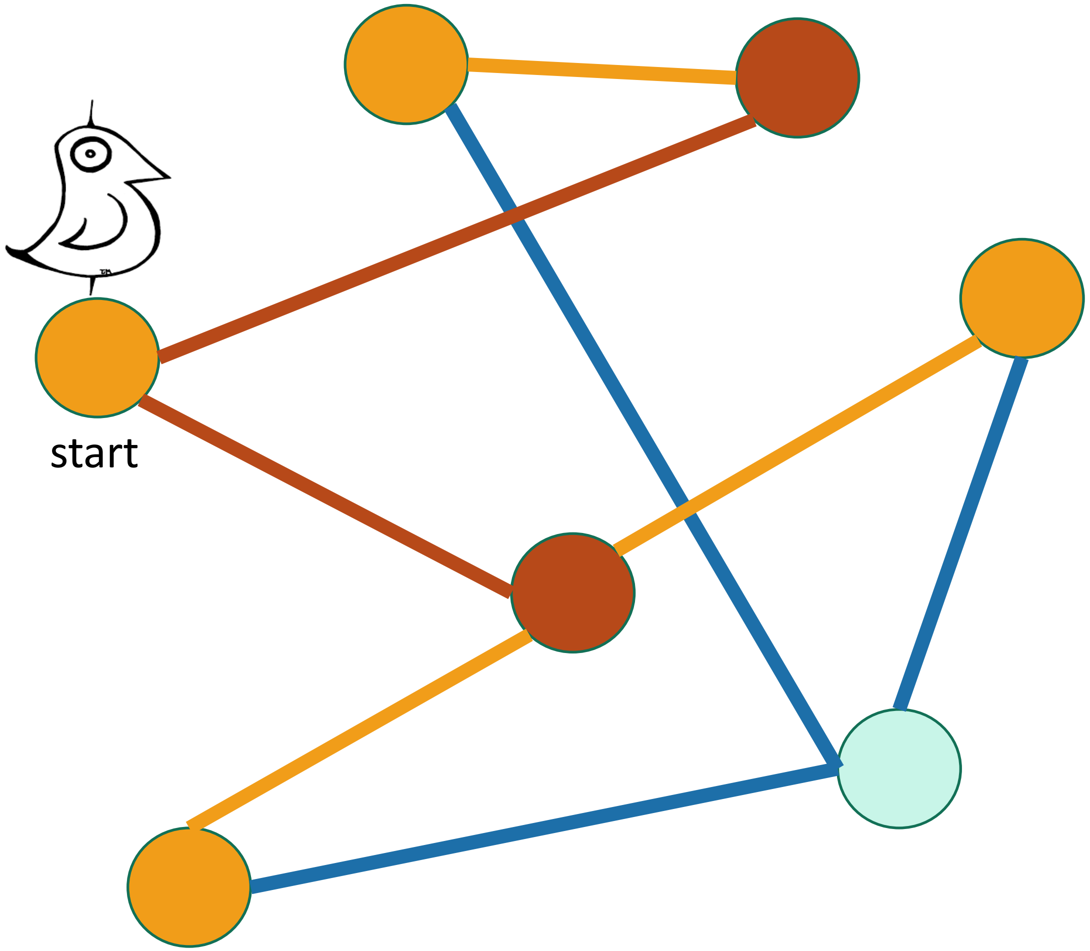

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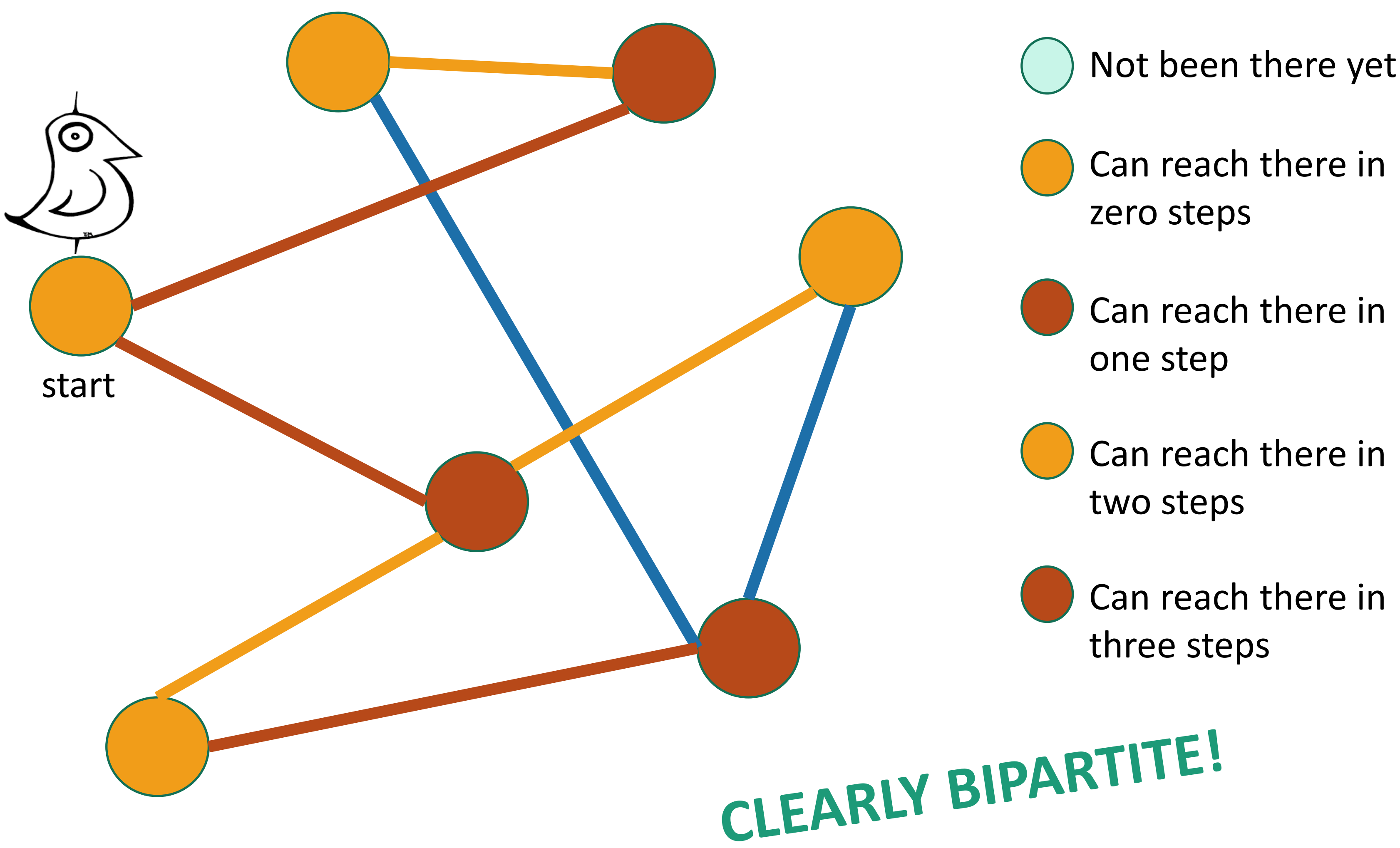

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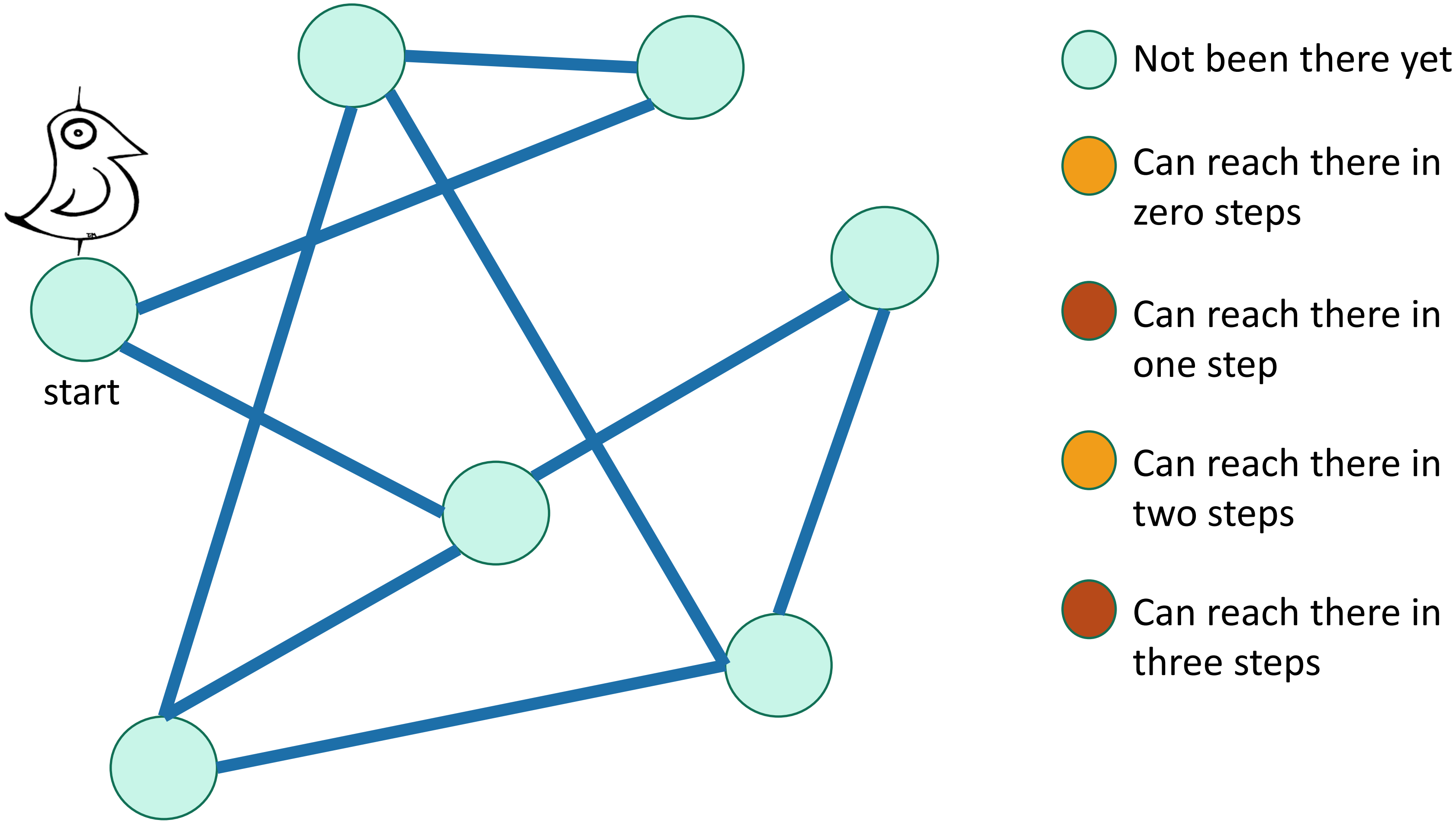

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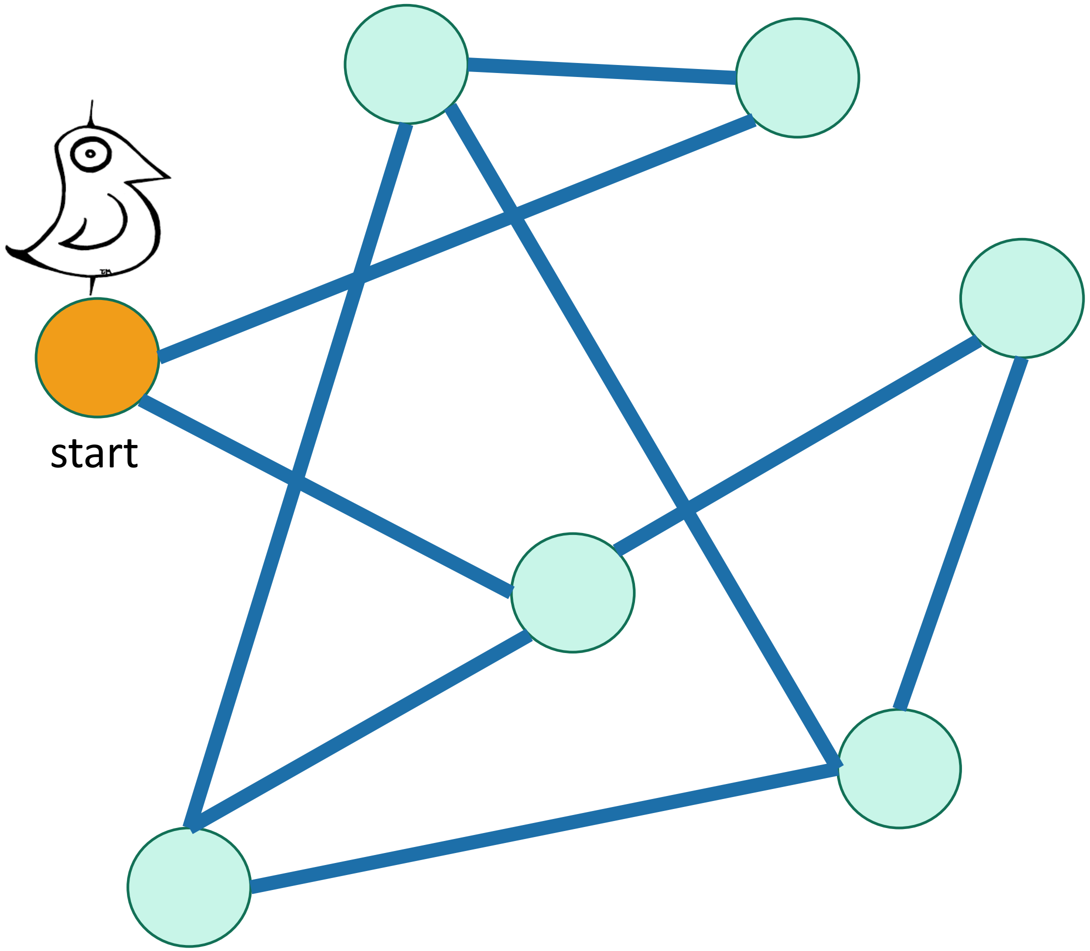

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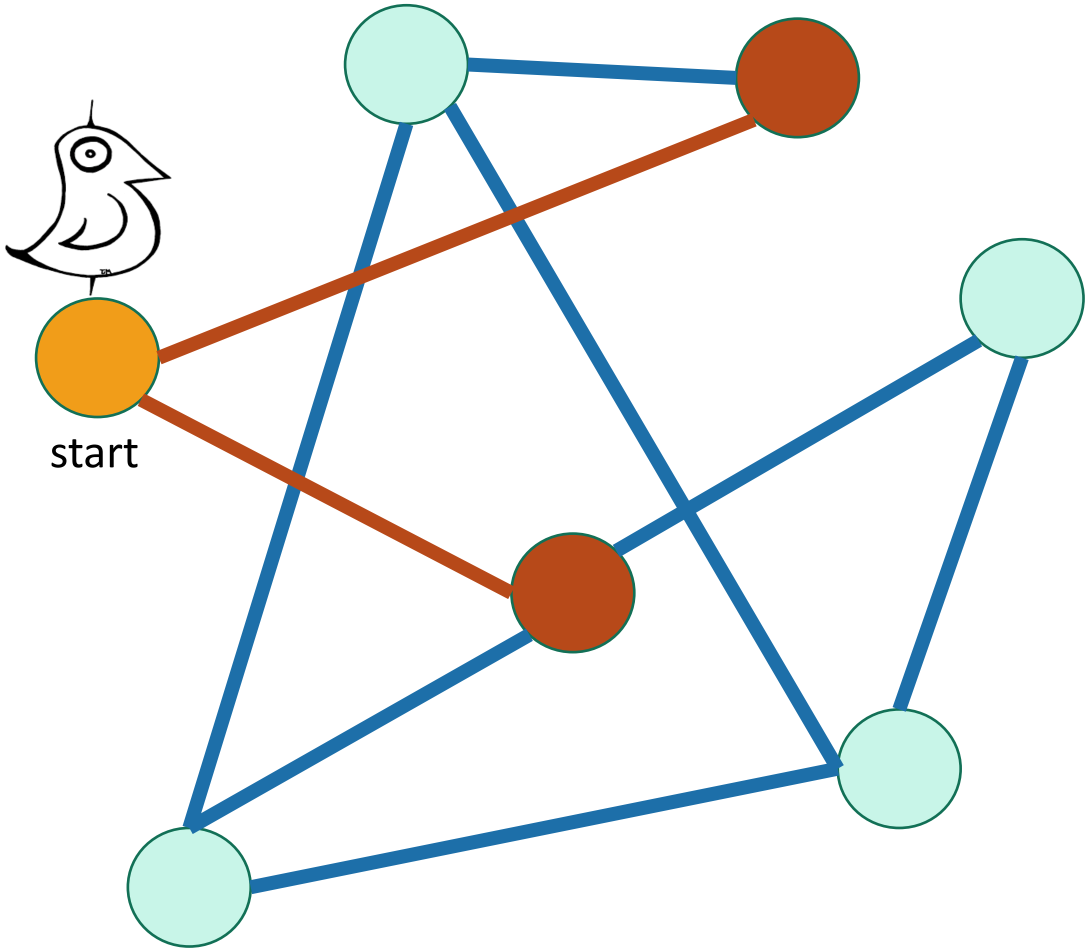

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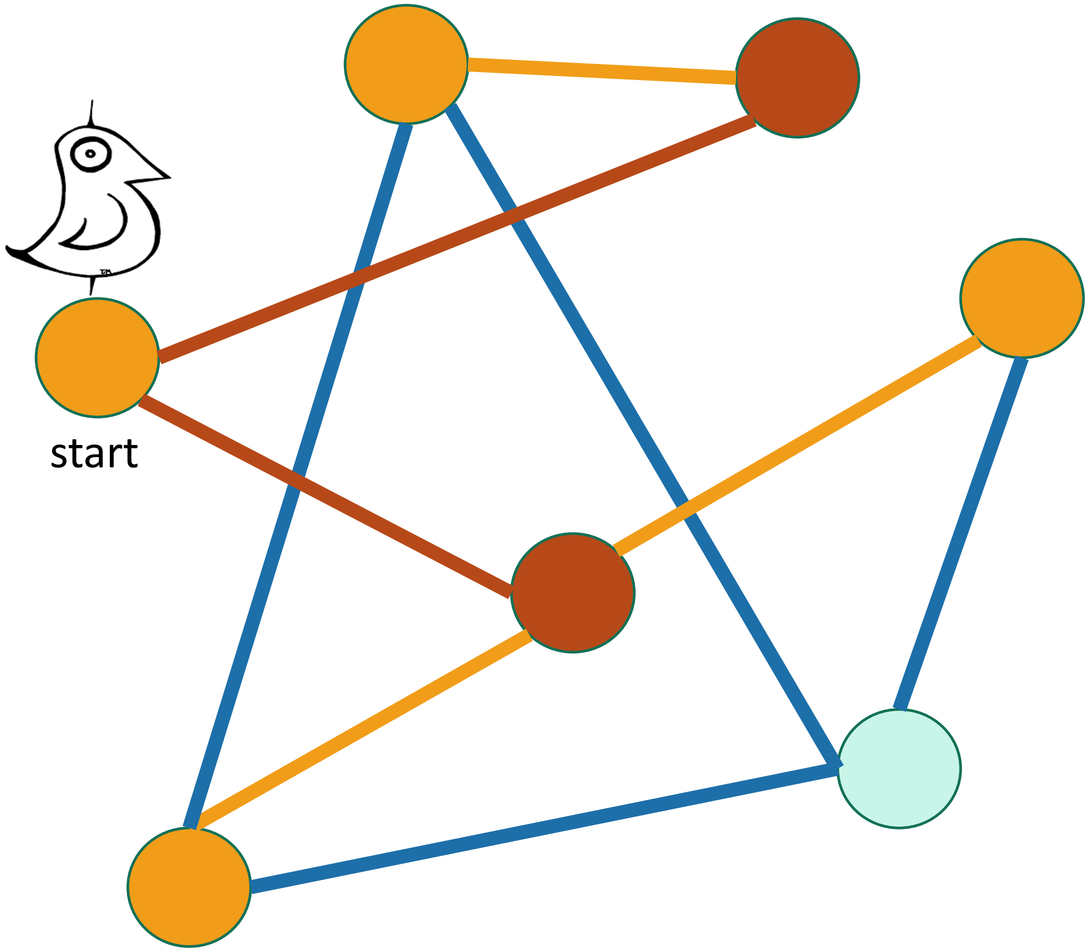

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Breadth-First Search

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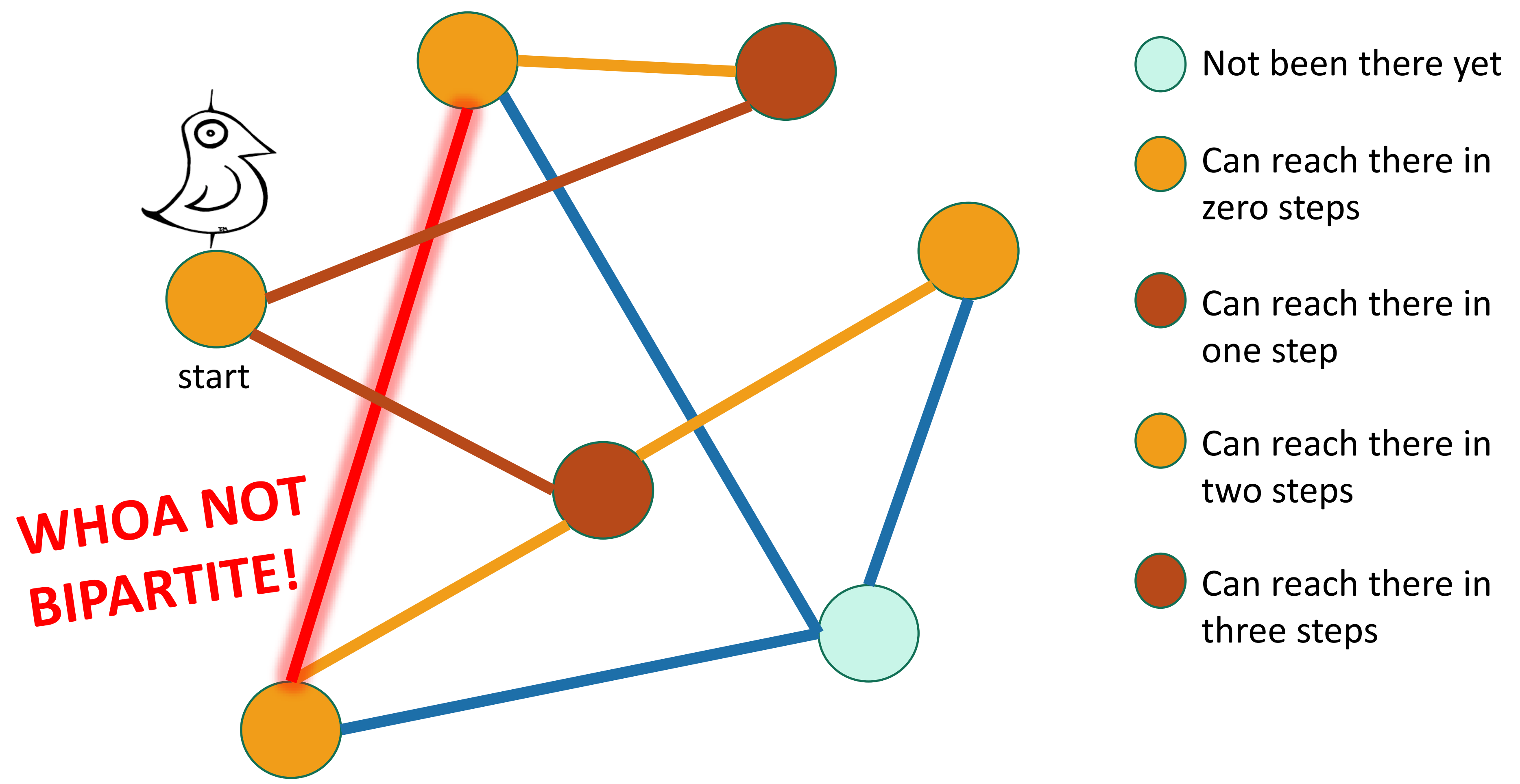

What did we just learn?

BFS can be used to detect bipartite-ness in time $O(n + m)$.

- Consider a hash table of size m = 1000 and a corresponding hash function
  $h(k) = \lfloor m\ (kA \bmod 1) \rfloor$ for . Compute the locations to which the keys 61, 62, 63, 64, and 65 are mapped. A is equal to 0.618033