**Cayley-Hamilton theorem (without proof):**

The Cayley-Hamilton theorem states that every square matrix satisfies its own characteristic equation, meaning that if you substitute the matrix into its characteristic polynomial, the result is the zero matrix.

Here's a more detailed explanation:

* **Characteristic Polynomial:**

For a square matrix A, the characteristic polynomial p(x) is defined as p(x) = det(xI - A), where I is the identity matrix and x is a variable.

* **Cayley-Hamilton Theorem:**

The theorem states that if you replace the variable x in the characteristic polynomial p(x) with the matrix A, the resulting matrix expression p(A) will equal the zero matrix.

* **Example:**

Let's say A is a 2x2 matrix: A = [[a, b], [c, d]].

* + The characteristic polynomial p(x) would be: p(x) = det(xI - A) = det([[x, 0], [0, x]] - [[a, b], [c, d]]) = det([[x-a, -b], [-c, x-d]]) = (x-a)(x-d) - (-b)(-c) = x^2 - (a+d)x + (ad-bc).
  + According to the Cayley-Hamilton theorem, if we substitute A into p(x), we get: p(A) = A^2 - (a+d)A + (ad-bc)I = 0.

**Video tutorial:** <https://youtu.be/PCO6y793jz4?si=_xFVZz_B--4o0HgV>

**Finding inverse and power of a matrix by Cayley-Hamilton theorem :**

The Cayley-Hamilton theorem states that every square matrix satisfies its own characteristic equation, which can be used to find the inverse and powers of a matrix.

Here's a breakdown of how to use the Cayley-Hamilton theorem:

1. Find the Characteristic Polynomial:

* For a matrix A, the characteristic polynomial p(λ) is defined as det(λI - A), where I is the identity matrix and λ is a scalar.
* For a 2x2 matrix A = [[a, b], [c, d]], the characteristic polynomial is p(λ) = λ² - (a+d)λ + (ad-bc).
* For a 3x3 matrix, the characteristic polynomial is p(λ) = λ³ - Tλ² + T₁λ - |A|, where T is the trace of A (sum of diagonal elements), T₁ is the sum of the minors of the main diagonal elements, and |A| is the determinant of A.

2. Apply the Cayley-Hamilton Theorem:

* The theorem states that if p(λ) is the characteristic polynomial, then p(A) = 0 (the zero matrix).
* This means you can substitute the matrix A for λ in the characteristic polynomial and replace any constant terms with the corresponding multiple of the identity matrix I.

3. Finding the Inverse:

* If the characteristic polynomial is p(λ) = λ³ + aλ² + bλ + c, then A³ + aA² + bA + cI = 0.
* To find the inverse, multiply both sides of the equation by A⁻¹: A⁻¹(A³ + aA² + bA + cI) = A⁻¹(0).
* This simplifies to A² + aA + bI + cA⁻¹ = 0.
* Solve for A⁻¹: A⁻¹ = -(A² + aA + bI) / c.

4. Finding Powers of a Matrix:

* The Cayley-Hamilton theorem allows you to express higher powers of a matrix as a linear combination of lower powers.
* For example, if you have the characteristic polynomial p(λ) = λ³ + aλ² + bλ + c, you can write any power of A (e.g., A⁴) as a linear combination of A², A, and I.
* To find A⁴, you can use the relationship A³ + aA² + bA + cI = 0 to express A³ in terms of A², A, and I, and then multiply by A to find A⁴.

Example (Finding the Inverse):

Let's say you have a matrix A and its characteristic polynomial is p(λ) = λ² - 5λ + 6.

1. **Apply the Cayley-Hamilton theorem:** A² - 5A + 6I = 0.
2. **Multiply by A⁻¹:** A⁻¹(A² - 5A + 6I) = A⁻¹(0).
3. **Simplify:** A - 5I + 6A⁻¹ = 0.
4. **Solve for A⁻¹:** 6A⁻¹ = -A + 5I => A⁻¹ = (-A + 5I) / 6.

**Video tutorial:** [**https://youtu.be/8BEv1mQpJNU?si=OafPUNZdxXdAdG2-**](https://youtu.be/8BEv1mQpJNU?si=OafPUNZdxXdAdG2-)

**Diagonalization of a matrix**:

Diagonalization of a matrix involves transforming it into a diagonal matrix (where non-diagonal elements are zero) by finding an invertible matrix and its inverse, such that A = PDP⁻¹, where D is the diagonal matrix, P is the matrix of eigenvectors, and P⁻¹ is the inverse of P.

Here's a more detailed explanation:

What is Diagonalization?

* **Diagonal Matrix:**

A square matrix where all elements except those on the main diagonal are zero.

* **Diagonalizable Matrix:**

A square matrix that can be transformed into a diagonal matrix through a similarity transformation.

* **Similarity Transformation:**

A transformation of a matrix A to a similar matrix B, where B = P⁻¹AP, and P is an invertible matrix.

* **Invertible Matrix (P):**

A matrix that has an inverse.

* **Diagonal Matrix (D):**

The diagonal matrix obtained after diagonalization, with eigenvalues of the original matrix along the diagonal.

How to Diagonalize a Matrix:

1. **1. Find Eigenvalues and Eigenvectors:**
   * Calculate the eigenvalues (λ) of the matrix A by solving the characteristic equation: det(A - λI) = 0, where I is the identity matrix.
   * For each eigenvalue, find the corresponding eigenvectors by solving (A - λI)v = 0, where v is the eigenvector.
2. **2. Construct the Matrix P:**
   * Form a matrix P whose columns are the linearly independent eigenvectors of A.
3. **3. Construct the Matrix D:**
   * Create a diagonal matrix D where the diagonal elements are the eigenvalues of A.
4. **4. Verify Diagonalization:**
   * Check if A = PDP⁻¹ holds true.

When is Diagonalization Possible?

* A matrix is diagonalizable if and only if it has a complete set of linearly independent eigenvectors.
* A matrix is diagonalizable if and only if the geometric multiplicity of each eigenvalue equals its algebraic multiplicity.

Why is Diagonalization Useful?

* **Simplifies Matrix Powers:** Calculating Aⁿ becomes easier with diagonalization: Aⁿ = PDⁿP⁻¹.
* **Factorization and Decoupling:** Diagonalization helps in understanding and manipulating complex systems.
* **Analyzing Eigenvalues and Eigenvectors:** It provides insights into the structure and behavior of linear transformations.
* **Understanding Fundamental Linear Transformations:** Diagonalization helps in understanding how linear transformations affect vectors and spaces.

Video tutorial: <https://youtu.be/05tJyiBy0eY?si=iHHzs2BkMU__-mj0>

**Quadratic forms and nature of the quadratic forms:**

A quadratic form is a polynomial where all terms are of degree two, and its nature, or classification, can be determined by its behavior and the signs of its eigenvalues, such as positive definite, negative definite, or indefinite.

Here's a more detailed explanation:

What is a Quadratic Form?

* A quadratic form is a polynomial expression in which all terms are of degree two, meaning they involve either the square of a variable or the product of two different variables.
* A general quadratic form in n variables can be represented as: Q(x) = x^T A x, where x is a vector of variables, and A is a symmetric matrix.
* **Example:** f(x, y) = ax^2 + 2bxy + cy^2 is a quadratic form in two variables, x and y.

Nature (Classification) of Quadratic Forms:

The nature or classification of a quadratic form depends on the signs of its eigenvalues or, equivalently, the signs of the coefficients of the squared terms in the canonical form.

* **Positive Definite:**

A quadratic form is positive definite if it is always positive for all non-zero values of its variables.

* + **Example:** Q(x, y) = x^2 + y^2.
  + All eigenvalues are positive.
* **Negative Definite:**

A quadratic form is negative definite if it is always negative for all non-zero values of its variables.

* + **Example:** Q(x, y) = -x^2 - y^2.
  + All eigenvalues are negative.
* **Indefinite:**

A quadratic form is indefinite if it can be both positive and negative for different values of its variables.

* + **Example:** Q(x, y) = x^2 - y^2.
  + It has both positive and negative eigenvalues.
* **Positive Semi-definite:**

A quadratic form is positive semi-definite if it is always non-negative, but can be zero for some values of its variables.

* + All eigenvalues are non-negative.
* **Negative Semi-definite:**

A quadratic form is negative semi-definite if it is always non-positive, but can be zero for some values of its variables.

* + All eigenvalues are non-positive.

Determining the Nature:

* **Eigenvalues:**

The nature of a quadratic form can be determined by analyzing the eigenvalues of the corresponding symmetric matrix A.

* **Canonical Form:**

A quadratic form can be transformed into a canonical form (also known as diagonal form) where the cross-product terms (like 2bxy) are eliminated.

* **Hessian Matrix:**

In multivariable calculus, the Hessian matrix, which is the matrix of second partial derivatives, is used to determine the nature of a critical point of a function.

Applications:

* **Optimization:**

Quadratic forms play a crucial role in optimization problems, especially in determining the nature of critical points (minima, maxima, or saddle points) of functions.

* **Linear Algebra:**

Quadratic forms are fundamental in linear algebra, particularly in the study of symmetric matrices and their eigenvalues.

* **Statistics:**

Quadratic forms are used in statistics, for example, in analyzing the variance of a dataset.

Video tutorial: <https://youtu.be/1GFIy2_lH1Y?si=1Wlzlto3T6pHPXe3>

**Reduction of quadratic form to canonical form by orthogonal transformation:**

To reduce a quadratic form to its canonical form using an orthogonal transformation, you first represent the quadratic form as a matrix, find the eigenvalues and eigenvectors of the matrix, form an orthogonal matrix from the eigenvectors, and then use this matrix to transform the original quadratic form into its canonical form, which is a sum of squares.

Here's a more detailed explanation:

1. Represent the Quadratic Form as a Matrix:

* Let the quadratic form be represented as Q(x) = x^T A x, where x is a column vector of variables and A is a symmetric matrix formed from the coefficients of the quadratic form.
* For example, if the quadratic form is ax^2 + by^2 + cxy, then A = [[a, c/2], [c/2, b]].

2. Find Eigenvalues and Eigenvectors:

* Find the eigenvalues (λ) of the matrix A by solving the characteristic equation |A - λI| = 0, where I is the identity matrix.
* For each eigenvalue, find the corresponding eigenvectors by solving the equation (A - λI)v = 0, where v is the eigenvector.

3. Form an Orthogonal Matrix:

* Normalize the eigenvectors to obtain unit eigenvectors (vectors of length 1).
* Form an orthogonal matrix P whose columns are the normalized eigenvectors.

4. Transform the Quadratic Form:

* Let x = Py, where y is the new coordinate system.
* Substitute x = Py into the quadratic form Q(x) = x^T A x.
* This results in Q(Py) = (Py)^T A (Py) = y^T P^T A P y.
* Since P is orthogonal, P^T P = I, so Q(Py) = y^T P^T A P y = y^T (P^T A P) y.
* The matrix P^T A P is a diagonal matrix D whose diagonal elements are the eigenvalues of A.
* Therefore, the canonical form of the quadratic form is Q(y) = y^T D y = λ1y1^2 + λ2y2^2 + ... + λny\_n^2, where λ1, λ2, ..., λn are the eigenvalues of A.

Example:

Let's say the quadratic form is Q(x, y) = 2x^2 + 5y^2 + 3z^2 + 4xy.

1. **Matrix Representation:** A = [[2, 2], [2, 5]].
2. **Eigenvalues and Eigenvectors:** (Solve for eigenvalues and eigenvectors of A).
3. **Orthogonal Matrix:** (Form P using normalized eigenvectors).
4. **Transformation:** (Use x = Py and substitute to get the canonical form).

Video tutorial: <https://youtu.be/GCtDvZLsG4Q?si=6-eDtueaIharMm0H>