In the grading of the exercises special attention will be paid to check that answers are substantiated and that the procedure of calculations is well documented. When results are achieved using a calculator or a pc it should be noted in the paper. All 6 problems are weighted equally in the grading. Your solution to the problem set can be written in Danish or English as you prefer.

PROBLEM 1.
Let the following four vectors be given

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 5 \\ 1 \\ 1 \\ 6 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 7 \\ -15 \\ -2 \end{bmatrix}$$

1. Determine whether the set of vectors $\{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3\}$ is linearly dependent.
2. Solve the vector equation $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{b}$.

PROBLEM 2.
Assume it is requested to find the solution to the homogenous matrix equation $A\mathbf{x} = \mathbf{0}$ for some unknown 4×4 matrix A. The augmented matrix has been row reduced and the result is

$$[A|\mathbf{0}] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

1. Find the solution of $A\mathbf{x} = \mathbf{0}$.
2. Determine the rank of A.
3. Discuss whether the equation $A\mathbf{x} = \mathbf{b}$ can be solved if $\mathbf{b} = \begin{bmatrix} 3 \\ 4 \\ 1 \\ 2 \end{bmatrix}$

PROBLEM 3.
Consider the system $\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k$ with matrices

$$A = \begin{bmatrix} 0 & 2 & 3 \\ 1 & -1 & 2 \\ 1 & 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

and let $\mathbf{x}_0 = \mathbf{0}$.

1. Find the controllability matrix for the system and show that the system

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k,$$

   is controllable.

2. Find control vectors $\mathbf{u}_0,\ \mathbf{u}_1,\ \mathbf{u}_2$ that will force the system to $\mathbf{y} = \begin{bmatrix} 24 \\ 61 \\ 71 \end{bmatrix}$.

PROBLEM 4.

For the statements given below, state whether they are true or false and justify your answer for each statement.

1. If the matrix equation $A\mathbf{x} = \mathbf{0}$ has the solution $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ then $\mathbf{x} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ is also a solution.
2. Every $m \times n$ matrix has exactly n pivots.
3. For a $n \times n$ matrix the eigenvalues and the singular values are identical.

PROBLEM 5.

A 2D-vector with elements x_1 and x_2 can be rotated an angle ψ around the origin using a matrix multiplication.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} \cos(\psi) & -\sin(\psi) \\ \sin(\psi) & \cos(\psi) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Assume that a vector is rotated through first an angle ψ and then through a second angle θ for a total rotation of $\theta + \psi$.

1. Use the above matrix transformation to derive expressions for $\cos(\theta + \psi)$ and $\sin(\theta + \psi)$.

Consider the matrix

$$C = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$$

2. Show how the transformation $\mathbf{x} \mapsto C\mathbf{x}$ can be written as a scaling and rotation of $\mathbf{x}$ and find the values for the scaling and rotation.

PROBLEM 6.

Consider the following three vectors in $\mathbb{R}^3$.

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and } \mathbf{y} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

Let W be the subspace spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$.

1. Determine an orthogonal basis for W.
2. Find the orthogonal projection of $\mathbf{y}$ onto W.