In the grading of the exercises special attention will be paid to check that answers are substantiated and that the procedure of calculations is well documented. When results are achieved using a calculator or a pc it should be noted in the paper. All 6 problems are weighted equally in the grading. Your solution to the problem set can be written in Danish or English as you prefer.

PROBLEM 1.

Consider the following matrix and vector

$$A = \begin{bmatrix} 2 & 1 & 4 \\ -1 & 0 & -3 \\ 3 & 2 & 5 \\ 1 & -2 & 7 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ -4 \\ 4 \\ 12 \end{bmatrix}.$$

1. Is $\mathbf{x} = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}^T$ a solution of $A\mathbf{x} = \mathbf{b}$?
2. Determine the general solution of $A\mathbf{x} = \mathbf{b}$.
3. How many vectors are there in the solution set?

PROBLEM 2.

In the following two equations A, B, C, I and X are all $n \times n$ matrices.

$$(i) \quad A(X + I) = B, \qquad (ii) \quad XA = XB + C.$$

1. Solve equations (i) and (ii) for X and account for any assumptions made.

Next, consider an invertible $n \times n$ matrix A with the following property

$$A^2 = 5A + 2I.$$

2. Show that $A^3 = 27A + 10I$ and $A^{-1} = \frac{1}{2}(A - 5I)$.

PROBLEM 3.

For the statements given below, state whether they are true or false and justify your answer for each statement.

1. For an 2×3 matrix A with rank 2 and a 2×1 vector $\mathbf{b}$ the equation $A\mathbf{x} = \mathbf{b}$ will always have a solution.
2. The rotation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ where θ is a real scalar is an orthogonal matrix.
3. Eigenvalues must be nonzero scalars.

PROBLEM 4.

Consider the 3×3 matrix A given as

$$A = \begin{bmatrix} 1 & 6 & 2 \\ 1 & 6 & 4 \\ 1 & 3 & 2 \end{bmatrix}.$$

1. Show that the column vectors of A are linearly independent and span $\mathbb{R}^3$.
2. Show that no pair of column vectors from A is orthogonal.
3. Calculate an orthogonal basis for $\mathbb{R}^3$ using the column vectors from A and the Gram-Schmidt procedure.

PROBLEM 5.

This problem is based on the case "Error-Detecting and Error-Correcting Codes". All calculations must therefore be done using $\mathbb{Z}_2$ arithmetics, i.e. with binary numbers. Let the matrix A and the vector $\mathbf{x}$ be given by

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

1. Calculate $A\mathbf{x}$.
2. Determine the rank of A and calculate a basis for the null space of A.

PROBLEM 6.

Consider the set of all solutions to the differential equation

$$y'(x) + y(x) = 0,$$

where the prime denotes the derivative, i.e. $y'(x) = \frac{dy}{dx}$. In this problem it will be shown that the set of solutions to the differential equation fulfil the necessary properties to form a vector space.

1. Show that the zero function $y_0(x) = 0$ is in the set.
2. Show that if a function $y_1(x)$ is in the set, then $cy_1(x)$ is also in the set, where c is a scalar.
3. Show that if two functions $y_1(x)$ and $y_2(x)$ each are in the set, then $y_1(x) + y_2(x)$ is also in the set.