Solution for the ET-ALA exam (Q3-2015)

PROBLEM 1.
Let the following four vectors be given

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 2 \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \qquad \mathbf{u}_3 = \begin{bmatrix} 5 \\ 1 \\ 1 \\ 6 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 7 \\ -15 \\ -2 \end{bmatrix}$$

1. Determine whether the set of vectors $\{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3\}$ is linearly dependent.
2. Solve the vector equation $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{b}$.

PROBLEM 1. Solution
We check for linear (in)dependence by solving the equation $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{0}$ and look for non-trivial solution. The augmented matrix $[\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3|\mathbf{0}]$ is written down and row reduced

$$\left[\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ -1 & 1 & 1 & 0 \\ 3 & -1 & 1 & 0 \\ 2 & 2 & 6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From the reduced matrix it is evident that $\mathbf{u}_3 = \mathbf{u}_1 + 2\mathbf{u}_2$, thus non-trivial solutions exist and the vectors are therefore linearly dependent.

The vector equation is solved by writing up the augmented matrix $[\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3|\mathbf{b}]$ and row reducing.

$$\left[\begin{array}{ccc|c} 1 & 2 & 5 & 2 \\ -1 & 1 & 1 & 7 \\ 3 & -1 & 1 & -15 \\ 2 & 2 & 6 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The set of equations is consistent and c_3 is a free variable. The solution can be written in parametric form as

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

PROBLEM 2.

Assume it is requested to find the solution to the homogenous matrix equation $A\mathbf{x} = \mathbf{0}$ for some unknown 4×4 matrix A. The augmented matrix has been row reduced and the result is

$$[A|\mathbf{0}] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

1. Find the solution of $A\mathbf{x} = \mathbf{0}$.
2. Determine the rank of A.
3. Discuss whether the equation $A\mathbf{x} = \mathbf{b}$ can be solved if $\mathbf{b} = [3\ 4\ 1\ 2]^T$.

PROBLEM 2. Solution

From the matrix it is seen that x_2, x_3 and x_4 are free variables and $x_1 = 0$. The solution can thus be written in parametric form as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The $\underline{\underline{\text{rank of } A \text{ is one}}}$ as there is only one pivot in the matrix.

Based on the information given it can't be decided whether $A\mathbf{x} = \mathbf{b}$ can be solved. The system can be consistent or inconsistent. Knowledge of the numbers in the matrix is needed to determine which situation we have.

PROBLEM 3.
Consider the system $\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k$ with matrices

$$A = \begin{bmatrix} 0 & 2 & 3 \\ 1 & -1 & 2 \\ 1 & 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

and let $\mathbf{x}_0 = \mathbf{0}$.

1. Find the controllability matrix for the system and show that the system

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k,$$

   is controllable.

2. Find control vectors $\mathbf{u}_0$, $\mathbf{u}_1$, $\mathbf{u}_2$ that will force the system to $\mathbf{y} = \begin{bmatrix} 24 \\ 61 \\ 71 \end{bmatrix}$.

PROBLEM 3. Solution
The controllability matrix M is given by

$$M = [B\ AB\ A^2B] = \begin{bmatrix} 2 & -5 & -10 \\ -1 & 1 & -14 \\ -1 & -4 & -19 \end{bmatrix}.$$

By row reducing it is easily checked that M is row equivalent with the identity matrix and the system is therefore controllable.

The control vectors are found by solving the following system of equations.

$$[B\ AB\ A^2B \mid \mathbf{y}] = \left[\begin{array}{ccc|c} 2 & -5 & -10 & 24 \\ -1 & 1 & -14 & 61 \\ -1 & -4 & -19 & 71 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -4 \end{array}\right].$$

The solution is therefore $\mathbf{u}_2 = -3$, $\mathbf{u}_1 = 2$ and $\mathbf{u}_0 = -4$.

PROBLEM 4.
For the statements given below, state whether they are true or false and justify your answer for each statement.

1. If the matrix equation $A\mathbf{x} = \mathbf{0}$ has the solution $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ then $\mathbf{x} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ is also a solution.
2. Every $m \times n$ matrix has exactly n pivots.
3. For a $n \times n$ matrix the eigenvalues and the singular values are identical.

PROBLEM 4. Solution
Statement 1 is **true**. If $\mathbf{u}$ is a solution of $A\mathbf{x} = \mathbf{0}$ i.e. $A\mathbf{u} = \mathbf{0}$, then $c\mathbf{u}$ is also a solution as $A(c\mathbf{u}) = cA\mathbf{u} = c\mathbf{0} = \mathbf{0}$.

Statement 2 is **false**. The maximum number of pivots in an $m \times n$ matrix is the smaller of the numbers m and n. However, an arbitrary $m \times n$ matrix can have any number of pivots between zero and this maximum.

An example of a $m \times n$ matrix that does not have n pivots is the following 3×2 matrix with only 1 pivot

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Statement 3 is **false**. The eigenvalues are calculated on the matrix A and the singular values are the square roots of the eigenvalues of $A^T A$. These numbers will normally be different. A good example is the matrix below where the eigenvalues are complex, but the singular values are real (as always).

```
>> A=[3 7;-1 3]

A =
     3     7
    -1     3

>> eig(A)

ans =
   3.0000 + 2.6458i
   3.0000 - 2.6458i

>> svd(A)

ans =
    8.0000
    2.0000
```

PROBLEM 5.
A 2D-vector with elements x_1 and x_2 can be rotated an angle ψ around the origin using a matrix multiplication.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} \cos(\psi) & -\sin(\psi) \\ \sin(\psi) & \cos(\psi) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Assume that a vector is rotated through first an angle ψ and then through a second angle θ for a total rotation of $\theta + \psi$.

1. Use the above matrix transformation to derive expressions for $\cos(\theta + \psi)$ and $\sin(\theta + \psi)$.

Consider the matrix

$$C = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$$

2. Show how the transformation $\mathbf{x} \mapsto C\mathbf{x}$ can be written as a scaling and rotation of $\mathbf{x}$ and find the values for the scaling and rotation.

PROBLEM 5. Solution
Rotating through first an angle ψ and then through a second angle θ for a total rotation of $\psi + \theta$ amounts to multiplying with two rotation matrices i.e.

$$R_{\theta+\psi} = R_\theta R_\psi$$

Using the above defined rotation matrix gives

$$\begin{bmatrix} \cos(\theta+\psi) & -\sin(\theta+\psi) \\ \sin(\theta+\psi) & \cos(\theta+\psi) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\psi) & -\sin(\psi) \\ \sin(\psi) & \cos(\psi) \end{bmatrix}$$

By doing the multiplication on the right hand side of the above equation and comparing left hand and right hand side it is easily seen that

$$\cos(\theta + \psi) = \cos(\theta)\cos(\psi) - \sin(\theta)\sin(\psi)$$

and

$$\sin(\theta + \psi) = \sin(\theta)\cos(\psi) + \cos(\theta)\sin(\psi)$$

The second part of the problem is solved by following the procedure from chapter 5.5. We recognize the form

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos(\psi) & -\sin(\psi) \\ \sin(\psi) & \cos(\psi) \end{bmatrix}$$

with $a = 3$ and $b = 4$. The scaling and rotation parameters are

$$r = \sqrt{3^2 + 4^2} = 5, \qquad \psi = \tan^{-1}\left(\frac{4}{3}\right) = 53.13^\circ$$

PROBLEM 6.
Consider the following three vectors in $\mathbb{R}^3$.

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and } \mathbf{y} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

Let W be the subspace spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$.

1. Determine an orthogonal basis for W.
2. Find the orthogonal projection of $\mathbf{y}$ onto W.

PROBLEM 6. Solution
The Gram-Schmidt procedure is used to find an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$

$$\mathbf{v}_1 = \mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 = \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

The orthogonal projection $\hat{\mathbf{y}}$ is given by

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{y} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$