In the grading of the exercises special attention will be paid to check that answers are substantiated and that the procedure of calculations is well documented. When results are achieved using a calculator or a pc it should be noted in the paper. All 6 problems are weighted equally in the grading. Your solution to the problem set can be written in Danish or English as you prefer.

PROBLEM 1.

1. Find the general solution to the system of equations below. Give the solution in vector-form.

$$\begin{aligned} -x_1 + 2x_3 + 6x_4 - 14x_5 &= 7 \\ 3x_1 - x_2 - 4x_3 - 18x_4 + 26x_5 &= 5 \\ -2x_1 + x_2 + 4x_3 + 14x_4 - 24x_5 &= 2 \end{aligned}$$

2. Let A, B and C all be $n \times n$ and invertible. Reduce the expressions below as much as possible. Account for the rules used in each step.

a) $A^{-1}(B^T A^T)^T C I^T C^{-1} B$
b) $(A^T I C^T A + B^T (I - (B^T)^{-1}))^T + I$
c) $(A^{-1})^2 A (B^T C)^T$

PROBLEM 2.

Consider the matrix A and the stacked matrices $B = \begin{bmatrix} A \\ A \end{bmatrix}$ and $C = \begin{bmatrix} A & A \\ A & A \end{bmatrix}$.
Three important subspaces of any matrix are: the *null space*, *column space* and *row space*.

1. Which of the three subspaces mentioned above are the same for A and B? Justify your answer.
2. Which of the three subspaces mentioned above are the same for B and C? Justify your answer.

PROBLEM 3.

Let $H = \begin{bmatrix} 0 & -9 & 5 \\ -1 & 3 & -7 \\ 1 & -12 & 7 \end{bmatrix}$.

1. Using row operations, find the determinant of H.
2. Let A and B be $n \times n$. Label each of the following statements *true* or *false*. Justify each answer by a short proof or counterexample.

(a) If $\det(A) = 0$ then rank $A < n$.
(b) $\det(-A) = (-1)^n \det(A)$.
(c) If B is invertible, then $\det(B^{-1}) = \frac{1}{\det(B)}$.

PROBLEM 4.

This problem is concerned with computer graphics and homogeneous coordinates as introduced in case 3. For simplicity, only 2D homogeneous coordinates $(x, y, 1)$ are used.

An object $\mathcal{O}$ has the nodes $n_1, n_2, \ldots, n_5$ with coordinates

$$\mathcal{C}_n = \{(1,1),(1,3),(3,3),(3,1),(2,2)\}$$

and the adjacency matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

1. Sketch the shape of the object, $\mathcal{O}$.
2. Using homogeneous coordinates, find a matrix T, which translates the $\mathcal{O}$, such that it is centered in $(0, 0)$. The translated object is called $\mathcal{O}_T$. Sketch $\mathcal{O}_T$.
3. Find a matrix R, which rotates $\mathcal{O}_T$ by 3° counterclockwise around (0,0). Sketch the rotated $\mathcal{O}_T$.
4. Using T and R from above, write a matrix expression for a matrix M which rotates the original shape $\mathcal{O}$ by 30° counterclockwise around its center, while keeping the center at its original coordinates. Compute the entries of M and sketch the rotated object $\mathcal{O}_M$.

PROBLEM 5.

Consider the $A = \begin{bmatrix} 61 & 36 & -42 \\ -60 & -35 & 42 \\ 30 & 18 & -20 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$.

1. Verify that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are eigenvectors of A and find the corresponding eigenvalues, $\{\lambda_1, \lambda_2, \lambda_3\}$.
2. Find matrices P and D such that $A = PDP^{-1}$ where D is diagonal.
3. Assuming that A is diagonalizable, write a matrix expression for A^{-1} in terms of P and D. Explain how we can easily find the eigenvalues of A^{-1} if we know the eigenvalues of A.

PROBLEM 6.

A scientist has measured a dataseries using an apparatus:

t	1.0	2.5	4.0	5.5	7.0
y	1.9	-2.7	-2.1	0.6	-1.8

She knows that the apparatus is linear, and that the data is contaminated by noise. Thus, the measurements can be represented in the form of a general linear model:

$$\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon} \tag{1}$$

where:

$\mathbf{y} = \begin{bmatrix} y_1, & y_2, & \ldots, & y_5 \end{bmatrix}^T$ is the observation vector.
$\boldsymbol{\beta} = \begin{bmatrix} \beta_1, \beta_2, \beta_3 \end{bmatrix}^T$ is the parameter vector.
X is the design matrix.
$\boldsymbol{\epsilon}$ is the residual vector representing the noise.

Her model of the apparatus (which is linear in terms of the variables $\{\beta_1, \beta_2, \beta_3\}$) is as follows:

$$y = \beta_1 \cdot \sin(1.5t) + \beta_2 \cdot t^3 + \beta_3 \cdot \log_{10}(t) \tag{2}$$

1. Construct the design matrix X, such that it corresponds to the model equation in Eq. (2).
2. Compute the least-squares solution, $\hat{\boldsymbol{\beta}}$, to Eq. (1).
3. There exists a matrix P such that $P\mathbf{y} = \hat{\boldsymbol{\beta}}$. Give a matrix expression for P.