In the grading of the exercises special attention will be paid to check that answers are substantiated and that the procedure of calculations is well documented. When results are achieved using a calculator or a pc it should be noted in the paper. All 6 problems are weighted equally in the grading. Your solution to the problem set can be written in Danish or English as you prefer.

PROBLEM 1.
The augmented matrix for a linear system is given by

$$\left[\begin{array}{cccc|c} 0 & 12 & h+3 & 24 & 26 \\ 4 & 8 & h+13 & 12 & 14 \\ 1 & 2 & 1 & 3 & 3 \end{array}\right]$$

1. Determine the general form solution of the system for the case where $h = -8$.
2. For what values of h is the system consistent?

PROBLEM 1. Solution

1. The reduced echelon form of the matrix when $h = -8$ is:

$$\begin{bmatrix} 1 & 0 & 0 & -1 & -5 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & 2 \end{bmatrix}.$$

   Thus, the solution is:

$$x_1 = x_4 - 5 \quad , \quad x_2 = -2x_4 + 3 \quad , \quad x_3 = 2 \quad , \quad x_4 = \text{free}$$

   or in parametric vector form:

$$\mathbf{x} = x_4 \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -5 \\ 3 \\ 2 \\ 0 \end{bmatrix}.$$

2. The sequence of row operations $r_2 = r_2 - r_1$, $r_2 = r_2 - 4r_3$, and $r_1 = r_1 + r_2$ yields the matrix

$$\begin{bmatrix} 0 & 0 & h+9 & 0 & 2 \\ 0 & -12 & 6 & -24 & -24 \\ 1 & 2 & 1 & 3 & 3 \end{bmatrix}.$$

   Thus, the system is consistent for $h \neq -9$

PROBLEM 2.

Consider the matrix

$$A = \begin{bmatrix} 2 & 4 & p \\ 1 & q & 0 \\ 2 & 4 & 5 \end{bmatrix}$$

1. Using a cofactor expansion, find an expression for the determinant of A.
2. Mark each statement below True or False. Justify your answer. All matrices are $n \times n$.
   (a) If the determinant of a A is zero, then one column of A is a linear combination of the remaining columns.
   (b) If $\det(A) = k$ then $\det(A^3) = 3k$.
   (c) If $\det(A) = 1$ then $A = I$.

PROBLEM 2. Solution

1. By expansion along the second row the determinant is:

$$\begin{aligned} \det(A) &= (-1)^{2+1} \cdot 1 \cdot \begin{vmatrix} 4 & p \\ 4 & 5 \end{vmatrix} + (-1)^{2+2} \cdot q \cdot \begin{vmatrix} 2 & p \\ 2 & 5 \end{vmatrix} \\ &= -1(20 - 4p) + q(10 - 2p) \\ &= \underline{\underline{4p + 10q - 2pq - 20}}. \end{aligned}$$

2. a) True. When $\det(A) = 0$, A has less than n pivots, and therefore at least one column is a linear combination of the remaining columns.
   b) False. $\det(A^3) = \det(A)^3 = k^3$
   c) False. It is true that $\det(I) = 1$ but we can easily find another example: $\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1$.

PROBLEM 3.

Let $A = \begin{bmatrix} -24 & -8 & -4 \\ 8 & -40 & -4 \\ -16 & 16 & -24 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix}$.

1. Justify that $\mathbf{v}_1$ is an eigenvector of A and find the corresponding eigenvalue λ_1.
2. Find the remaining eigenvalues, λ_2 and λ_3, of A.
3. Let $\mathbf{v}_2$ and $\mathbf{v}_3$ be eigenvectors of corresponding λ_2 and λ_3. Show that $\mathbf{w} = 3\mathbf{v}_2 - 2\mathbf{v}_3$ is also an eigenvector.

PROBLEM 3. Solution

1. If $\mathbf{v}_1$ is an eigenvector, there exists a constant λ such that $A\mathbf{v}_1 = \lambda\mathbf{v}_1$.

   $$A\mathbf{v}_1 = \begin{bmatrix} -48 \\ -48 \\ 96 \end{bmatrix} = -24\mathbf{v}_1$$

   i.e. $\mathbf{v}_1$ is an eigenvector with eigenvalue -24.
2. Using matlab: $\lambda_2 = \lambda_3 = -32$.
3. We give two equally valid solutions.
   a) Because the eigenvalue $\lambda = -32$ has a multiplicity of 2, the corresponding eigenvectors span a two-dimensional eigenspace for $\lambda = -32$. Any vector in the eigenspace is an eigenvector, and thus any linear combination of eigenvectors corresponding to this eigenvalue is also an eigenvector.

   b) We wish to show that $A\mathbf{w} = \lambda\mathbf{w}$. We have:

   $$A(p\mathbf{v}_2 + q\mathbf{v}_3) = pA\mathbf{v}_2 + qA\mathbf{v}_3 = p\lambda\mathbf{v}_2 + q\lambda\mathbf{v}_3 = \lambda(p\mathbf{v}_2 + q\mathbf{v}_3)$$

   Substituting $\mathbf{w}$ for $(p\mathbf{v}_2 + q\mathbf{v}_3)$ above with $(p, q) = (3, -2)$, we see that $\mathbf{w}$ is indeed an eigenvector.

PROBLEM 4.

Consider the vectors $\mathbf{a} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} -4 \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{d} = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$.

1. Are the vectors linearly independent?
2. Pick out as few of the vectors as possible, such that they span $\mathbb{R}^3$.
3. Do $\mathbf{a}$ and $\mathbf{b}$ span $\mathbb{R}^2$?
4. Construct an orthonormal basis for $\mathbb{R}^3$ such that one of the basis vectors points in the same direction as $\mathbf{d}$.

PROBLEM 4. Solution

1. No. With 4 vectors in $\mathbb{R}^3$, the set must be linearly dependent.
2. E.g. $\mathbf{a}$, $\mathbf{b}$, $\mathbf{d}$. Verification: $\begin{bmatrix} \mathbf{a} & \mathbf{b} & \mathbf{d} \end{bmatrix} \sim I$.
3. No. Vectors in $\mathbb{R}^3$ cannot span $\mathbb{R}^2$. However, they do span a 2-dimensional subspace of $\mathbb{R}^3$.
4. We start with $\mathbf{d}$ and produce the remaining basis vectors by applying the Gram-Schmidt procedure to $\mathbf{a}$ and $\mathbf{b}$.

$$
\begin{aligned}
\mathbf{u}_1 &= \mathbf{d} \\
\mathbf{u}_2 &= \mathbf{a} - \frac{\mathbf{a}^T\mathbf{u}_1}{\mathbf{u}_1^T\mathbf{u}_1}\mathbf{u}_1 = \begin{bmatrix} 1.2941 \\ 0.9412 \\ 2.7059 \end{bmatrix} \\
\mathbf{u}_3 &= \mathbf{b} - \frac{\mathbf{b}^T\mathbf{u}_1}{\mathbf{u}_1^T\mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{b}^T\mathbf{u}_2}{\mathbf{u}_2^T\mathbf{u}_2}\mathbf{u}_2 = \begin{bmatrix} -2.7381 \\ 2.1905 \\ 0.5467 \end{bmatrix}
\end{aligned}
$$

We now have an orthogonal basis and all that is left is to normalize each vector, $\mathbf{n}_i = \frac{\mathbf{u}_i}{||\mathbf{u}_i||}$:

$$
N = \begin{bmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3 \end{bmatrix} = \begin{bmatrix} 0.4851 & 0.4117 & -0.7715 \\ 0.7276 & 0.2994 & 0.6172 \\ -0.4851 & 0.8608 & 0.1543 \end{bmatrix}
$$

We verify that $N^TN = I$.

PROBLEM 5.

Let $A = \begin{bmatrix} 1 & -1 & -3 & 2 & 4 & 2 \\ 2 & 1 & -3 & 3 & 10 & 3 \\ -3 & -1 & 5 & -6 & -16 & -6 \\ 2 & 0 & -4 & 4 & 10 & 4 \end{bmatrix}$.

1. Find a basis for $\operatorname{Nul} A$.
2. What is the rank of A and the dimension of $\operatorname{col} A$?
3. Let the linear transformation T be defined by $T(\mathbf{x}) = A\mathbf{x}$. Is T one-to-one? Justify your answer.

PROBLEM 5. Solution

1. We start by finding the reduced echelon form of the augmented matrix for $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} A & \mathbf{0} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The free variables are x_3, x_5 and x_6. The general form solution to $A\mathbf{x} = \mathbf{0}$ is:

$$\mathbf{x} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

The three vectors above form a basis for $\operatorname{Nul} A$.

2. We know that $\operatorname{rank} A + \operatorname{dimNul} A = n$ where n is the number of columns in A. From above we see that $\operatorname{dimNul} A = 3$ and thus we have:

$$\operatorname{rank} A = 6 - 3 = 3.$$

$\operatorname{dimCol} A = \operatorname{rank} A = 3$.

3. Since there are free variables (or equivalently, since $\operatorname{Nul} A \neq \{\mathbf{0}\}$), T is not one-to-one.

PROBLEM 6.

Let the singular value decomposition (SVD) of a 2×3 matrix A be given by

$$U = \begin{bmatrix} 0.4472 & -0.8944 \\ 0.8944 & 0.4472 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix} \text{ and } \quad V = \begin{bmatrix} 0.8729 & 0 & -0.4880 \\ 0.4364 & 0.4472 & 0.7807 \\ 0.2182 & -0.8944 & 0.3904 \end{bmatrix}.$$

where U and V contain the left and right singular vectors, respectively.

1. Using U, Σ and V, compute the pseudo-inverse of A.
2. Compute the least-squares solution to the system $A\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.
3. Explain the difference between a matrix inverse and a matrix pseudo-inverse.

PROBLEM 6. Solution

1. The pseudo-inverse is given by:

$$\begin{aligned} A^+ &= V_r \Sigma_r^{-1} U_r^T \text{ (where subscript } r \text{ denotes the first } r \text{ columns)} \\ &= \begin{bmatrix} 0.8729 & 0 \\ 0.4364 & 0.4472 \\ 0.2182 & -0.8944 \end{bmatrix} \begin{bmatrix} \frac{1}{12} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 0.4472 & 0.8944 \\ -0.8944 & 0.4472 \end{bmatrix} \\ &= \begin{bmatrix} 0.0325 & 0.0651 \\ -0.0837 & 0.0836 \\ 0.2081 & -0.0860 \end{bmatrix} \end{aligned}$$

2. The least-squares solution is found by applying the pseudo-inverse to $\mathbf{b}$:

$$\mathbf{x}_{\text{lsq}} = A^+\mathbf{b} = \begin{bmatrix} 0.0325 & 0.0651 \\ -0.0837 & 0.0836 \\ 0.2081 & -0.0860 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.1301 \\ -0.0838 \\ 0.3302 \end{bmatrix}$$

   Note: The last decimals may vary due to rounding.

3. Not all matrices have an inverse, but all have a pseudo-inverse. An inverse is applied to find the unique solution to a system $A\mathbf{x} = \mathbf{b}$ where A is $n \times n$ invertible, while a pseudo-inverse can be applied when A is $m \times n$ to find the least-squares solution.