PROBLEM 1.

Consider the following matrix and vector

$$A = \begin{bmatrix} 2 & 1 & 4 \\ -1 & 0 & -3 \\ 3 & 2 & 5 \\ 1 & -2 & 7 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ -4 \\ 4 \\ 12 \end{bmatrix}.$$

1. Is $\mathbf{x} = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}^T$ a solution of $A\mathbf{x} = \mathbf{b}$?
2. Determine the general solution of $A\mathbf{x} = \mathbf{b}$.
3. How many vectors are there in the solution set?

PROBLEM 1. Solution

To test if $\mathbf{x}$ is a solution $A\mathbf{x}$ is calculated

$$A\mathbf{x} = \begin{bmatrix} 2 & 1 & 4 \\ -1 & 0 & -3 \\ 3 & 2 & 5 \\ 1 & -2 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ 12 \\ 4 \end{bmatrix}$$

As the result is clearly different from $\mathbf{b}$, it is concluded that $\mathbf{x}$ is <u>not</u> a solution of $A\mathbf{x} = \mathbf{b}$.
The general solution of $A\mathbf{x} = \mathbf{b}$ is found by row reduction of the augmented matrix

$$\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right] = \left[\begin{array}{ccc|c} 2 & 1 & 4 & 4 \\ -1 & 0 & -3 & -4 \\ 3 & 2 & 5 & 4 \\ 1 & -2 & 7 & 12 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From this we see that the system is consistent and contains one free variable, x_3. The general solution is written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

Due to the free variable, there are an infinite number of solutions to $A\mathbf{x} = \mathbf{b}$ and the solution set contains an infinite amount of vectors.

PROBLEM 2.

In the following two equations A, B, C, I and X are all $n \times n$ matrices.

$$(i) \quad A(X+I) = B, \qquad (ii) \quad XA = XB + C.$$

1. Solve equations (i) and (ii) for X and account for any assumptions made.

Next, consider an invertible $n \times n$ matrix A with the following property

$$A^2 = 5A + 2I.$$

2. Show that $A^3 = 27A + 10I$ and $A^{-1} = \frac{1}{2}(A - 5I)$.

PROBLEM 2. Solution

The equations are solved as

$$A(X+I) = B \iff X + I = A^{-1}B \iff X = \underline{\underline{A^{-1}B - I}}$$

and

$$XA = XB + C \iff XA - XB = C \iff X(A-B) = C \iff X = \underline{\underline{C(A-B)^{-1}}}.$$

where it was assumed in the first equation that A was invertible and in the second equation that $A - B$ was invertible.

The second part of the problem is solved as

$$A^3 = AA^2 = A(5A+2I) = 5A^2 + 2A = 5(5A+2I) + 2A = 27A + 10I$$

and

$$A^2 = 5A+2I \iff A^{-1}A^2 = A^{-1}(5A+2I) \iff A = 5I+2A^{-1} \iff A-5I = 2A^{-1} \iff A^{-1} = \frac{1}{2}(A-5I).$$

PROBLEM 3.

For the statements given below, state whether they are true or false and justify your answer for each statement.

1. For an 2×3 matrix A with rank 2 and a 2×1 vector $\mathbf{b}$ the equation $A\mathbf{x} = \mathbf{b}$ will always have a solution.
2. The rotation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ where θ is a real scalar is an orthogonal matrix.
3. Eigenvalues must be nonzero scalars.

PROBLEM 3. Solution

The first statement is **true**. The situation corresponds to this augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & * & * \\ 0 & 1 & * & * \end{array}\right]$$

Where the $*$ are unknown numbers. It is seen that the system is consistent and it will therefore always have a solution. In fact, the system has an infinite amount of solutions as we also have a free variable.

The second statement is **true**. An orthogonal matrix have mutually orthogonal columns and each column have unit length. This can also be stated as $QQ^T = Q^TQ = I$. For the given rotation matrix, QQ^T becomes

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \cos\theta\sin\theta \\ \cos\theta\sin\theta - \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where $\cos^2\theta + \sin^2\theta = 1$ was used in the last step.

The third statement is **false**. Eigenvalues can be both zero and nonzero, only eigenvectors must be nonzero.

PROBLEM 4.

Consider the 3×3 matrix A given as

$$A = \begin{bmatrix} 1 & 6 & 2 \\ 1 & 6 & 4 \\ 1 & 3 & 2 \end{bmatrix}.$$

1. Show that the column vectors of A are linearly independent and span $\mathbb{R}^3$.
2. Show that no pair of column vectors from A is orthogonal.
3. Calculate an orthogonal basis for $\mathbb{R}^3$ using the column vectors from A and the Gram-Schmidt procedure.

PROBLEM 4. Solution

The column vectors $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ are linearly independent if the equation $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + c_3\mathbf{a}_3 = \mathbf{0}$ only has the trivial solution. To check this the augmented matrix $[A|\mathbf{0}]$ is row reduced to give

$$\left[\begin{array}{ccc|c} 1 & 6 & 2 & 0 \\ 1 & 6 & 4 & 0 \\ 1 & 3 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

From the above it is seen that the equation only has the trivial solution and the column vectors are therefore linearly independent. As there are 3 linearly independent $\mathbb{R}^3$ vectors these will span all of $\mathbb{R}^3$.

The non-orthogonality of the vectors are shown by direct calculation of inner products

$$\mathbf{a}_1^T\mathbf{a}_2 = 15, \quad \mathbf{a}_1^T\mathbf{a}_3 = 8, \quad \mathbf{a}_2^T\mathbf{a}_3 = 42.$$

As none of the inner products are zero, none of the pairs of vectors are orthogonal.

The orthogonal basis is calculated with the Gram-Schmidt procedure as follows

$$\mathbf{b}_1 = \mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{b}_2 = \mathbf{a}_2 - \frac{\mathbf{a}_2^T\mathbf{b}_1}{\mathbf{b}_1^T\mathbf{b}_1}\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$\mathbf{b}_3 = \mathbf{a}_3 - \frac{\mathbf{a}_3^T\mathbf{b}_1}{\mathbf{b}_1^T\mathbf{b}_1}\mathbf{b}_1 - \frac{\mathbf{a}_3^T\mathbf{b}_2}{\mathbf{b}_2^T\mathbf{b}_2}\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

PROBLEM 5.

This problem is based on the case "Error-Detecting and Error-Correcting Codes". All calculations must therefore be done using $\mathbb{Z}_2$ arithmetics, i.e. with binary numbers. Let the matrix A and the vector $\mathbf{x}$ be given by

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

1. Calculate $A\mathbf{x}$.
2. Determine the rank of A and calculate a basis for the null space of A.

PROBLEM 5. Solution

The product $A\mathbf{x}$ is calculated as follows

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 1 \cdot 0 + 1 \cdot 1 \\ 1 \cdot 1 + 1 \cdot 0 + 0 \cdot 1 \\ 1 \cdot 1 + 1 \cdot 0 + 0 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

To solve the second part of the problem the augmented matrix is row reduced

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

There are two pivots in the above matrix and thus rank $A = 2$. The basis for null A is found by reading of the solution to the homogeneous equation from the above. The solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

and the basis for null A thus consist of the single vector $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

PROBLEM 6.

Consider the set of all solutions to the differential equation

$$y'(x) + y(x) = 0,$$

where the prime denotes the derivative, i.e. $y'(x) = \frac{dy}{dx}$. In this problem it will be shown that the set of solutions to the differential equation fulfil the necessary properties to form a vector space.

1. Show that the zero function $y_0(x) = 0$ is in the set.
2. Show that if a function $y_1(x)$ is in the set, then $cy_1(x)$ is also in the set, where c is a scalar.
3. Show that if two functions $y_1(x)$ and $y_2(x)$ each are in the set, then $y_1(x) + y_2(x)$ is also in the set.

PROBLEM 6. Solution

The zero function is in the set of solutions if the function solves the differential equation. This is easily tested by inserting $y_0(x) = 0$ into the differential equation

$$y_0'(x) + y_0(x) = \iff 0' + 0 = 0 \iff 0 = 0.$$

As the last equation is obviously true, it can be concluded that the zero function is in the solution set.

Since $y_1(x)$ is in the solution set, it fulfils the differential equation, i.e.

$$y_1'(x) + y_1(x) = 0$$

To check if the $cy_1(x)$ is also in the solution set, the function is inserted into the differential equation

$$\Big(cy_1(x)\Big)' + cy_1(x) = 0 \iff cy_1'(x) + cy_1(x) = 0 \iff c\Big(y_1'(x) + y_1(x)\Big) = 0 \iff c \cdot 0 = 0 \iff 0 = 0$$

where the linearity of the differential operator was used in the first step and $y_1'(x) + y_1(x) = 0$ was used in the third step. As the last equation is obviously true, it can be concluded that if $y_1(x)$ is in the solution set, then $cy_1(x)$ is also is in the solution set. Note that the special case of $c = 0$ corresponds to the zero function i.e. $0 \cdot y_1(x) = y_0(x) = 0$.

To show the final step it is noted that $y_1(x)$ and $y_2(x)$ both solves the differential equation i.e.

$$y_1'(x) + y_1(x) = 0, \quad y_2'(x) + y_2(x) = 0$$

Therefore

$$\begin{aligned}
&\Big(y_1(x) + y_2(x)\Big)' + \Big(y_1(x) + y_2(x)\Big) = 0 \\
&\Updownarrow \\
&y_1'(x) + y_2'(x) + y_1(x) + y_2(x) = 0 \\
&\Updownarrow \\
&y_1'(x) + y_1(x) + y_2'(x) + y_2(x) = 0 \\
&\Updownarrow \\
&0 + 0 = 0
\end{aligned}$$

where the linearity of the differential operator was again used in the first step. As the last equation is obviously true, it can be concluded that if $y_1(x)$ and $y_2(x)$ are in the solution set, then $y_1(x) + y_2(x)$ is also is in the solution set.

As it has been shown that the solution set contains the zero function and is closed under multiplication and addition it can be concluded that the solution set forms a vector space.