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Problem 1

Let the matrix A and vector $\boldsymbol{b}$ be given by:

$$A = \begin{bmatrix} 2 & -1 & 2 & 1 \\ 4 & -3 & 0 & 7 \\ -1 & 0 & -3 & 2 \end{bmatrix} \text{ and } \boldsymbol{b} = \begin{bmatrix} 0 \\ 6 \\ \alpha \end{bmatrix}$$

a) Determine α so the system of equations $A\boldsymbol{x} = \boldsymbol{b}$ are consistent.

b) Determine all solutions to the consistent system of equations $A\boldsymbol{x} = \boldsymbol{b}$.

Solution:

a) Row reduction of the augmented matrix:

$$[A|\boldsymbol{b}] = \left[\begin{array}{cccc|c} 2 & -1 & 2 & 1 & 0 \\ 4 & -3 & 0 & 7 & 6 \\ -1 & 0 & -3 & 2 & \alpha \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 3 & -2 & -\alpha \\ 0 & 1 & 4 & -5 & -6 \\ 0 & 0 & 0 & 0 & \alpha - 3 \end{array}\right]$$

Third row: The system is consistent only for $\alpha = 3$.

b) For $\alpha = 3$:

$$[A|\boldsymbol{b}] = \left[\begin{array}{cccc|c} 2 & -1 & 2 & 1 & 0 \\ 4 & -3 & 0 & 7 & 6 \\ -1 & 0 & -3 & 2 & 3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 3 & -2 & -3 \\ 0 & 1 & 4 & -5 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \text{Two free variables: } x_3, x_4$$

From the row reduction it is seen, that the general solution to:

$$\begin{bmatrix} 2 & -1 & 2 & 1 \\ 4 & -3 & 0 & 7 \\ -1 & 0 & -3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix} \Rightarrow \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 \\ -6 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 5 \\ 0 \\ 1 \end{bmatrix}$$

Problem 2

The following four vectors are given by:

$$\boldsymbol{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ 9 \\ 13 \\ 4 \end{bmatrix}$$

a) Determine whether the set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly independent.

b) Write – if possible – $\boldsymbol{b}$ as a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_3$.

Solution:

a) Linear independence is checked by forming the matrix $V = [\boldsymbol{v}_1\ \boldsymbol{v}_2\ \boldsymbol{v}_3]$ and row reducing:

$$V = [\boldsymbol{v}_1\ \boldsymbol{v}_2\ \boldsymbol{v}_3] = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ -1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there are three pivot columns the three vectors are linearly independent.

b) Finding the possible linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_3$ to give the vector $\boldsymbol{b} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3$ is equivalent to row reduction of the augmented matrix:

$$[V|\boldsymbol{b}] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 1 & 1 & 2 & 9 \\ -1 & 1 & 2 & 13 \\ 0 & 1 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So: $\boldsymbol{b} = -2\boldsymbol{v}_1 - 3\boldsymbol{v}_2 + 7\boldsymbol{v}_3 = -2\begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + 7\begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4-3+7 \\ -2-3+14 \\ 2-3+14 \\ 0-3+7 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \\ 13 \\ 4 \end{bmatrix}$ o.k.

Problem 3

For the statements given below, state whether they are true or false and justify your answer for each statement:

a) If A and B are two symmetric $n \times n$ matrices, then $C = A - 2B$ is also symmetric.

b) The determinant of $\begin{bmatrix} 3a & 3b & 3c \\ 3c & 3a & 3b \\ 3b & 3c & 3a \end{bmatrix}$ is nine times the determinant of $\begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$.

c) If A is an real $n \times n$ matrix with n odd (3, 5, 7, …) and A has a complex eigenvalue, then A will also have a real eigenvalue.

Solution:

a) True, since $(A - 2B)^T = A^T - 2B^T = A - 2B$ (A and B symmetric $\Leftrightarrow A^T = A$ and $B^T = B$)

b) False. Multiplying a row with a constant, the determinant is multiplied with the same constant (Theorem 3.3c). So $\begin{vmatrix} 3a & 3b & 3c \\ 3c & 3a & 3b \\ 3b & 3c & 3a \end{vmatrix} = 3 \cdot 3 \cdot 3 \cdot \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = 27\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$.

c) True. Complex eigenvalues of real matrices allways comes in conjugate pairs $(\lambda, \bar{\lambda})$ (see the paragraph preceding Example 5 in Chap.5.5). So with n odd at least one eigenvalue of A must be real.

Problem 4

In the following two equations A, B, C and I are all $n \times n$ matrices.

(i) $A(X + 2I)B = BA$ (ii) $XA = C + XB$

a) Solve equation (i) and (ii) for X and account for any assumptions made.

Next, consider an invertible $n \times n$ matrix A with the following property:

$$A^2 = A - I$$

b) Show that $A^3 = -I$ and $A^{-1} = -A^2$.

Solution:

a) (i) $A(X + 2I)B = BA \Leftrightarrow X + 2I = A^{-1}BAB^{-1} \Leftrightarrow X = A^{-1}BAB^{-1} - 2I$
(ii) $XA = C + XB \Leftrightarrow X(A - B) = C \Leftrightarrow X = C(A - B)^{-1}$
Assuming that in (i) A and B and in (ii) $(A - B)$ are invertible.

b) $A^3 = AA^2 = A(A - I) = A^2 - A = A - I - A = -I$
$A = A^{-1}A^2 = A^{-1}(A - I) = A^{-1}A - A^{-1} = I - A^{-1} \Rightarrow A^{-1} = I - A = -A^2$

Problem 5

Let the matrix A and vector $\boldsymbol{x}$ be given by:

$$A = \begin{bmatrix} -1 & -1 & -2 & 3 \\ 2 & 6 & 4 & -14 \\ 1 & 5 & 2 & -11 \\ 4 & 1 & 8 & -6 \end{bmatrix} \text{ and } \boldsymbol{x} = \begin{bmatrix} 2 \\ -3 \\ 2 \\ 1 \end{bmatrix}$$

a) Determine $rank\ A$ and $dim\ Nul\ A$.

b) Find a orthonormal basis for $Nul\ A$.

c) Is $\boldsymbol{x}$ in $Nul\ A$?

Solution:

a) Row reduction of A: $\begin{bmatrix} -1 & -1 & -2 & 3 \\ 2 & 6 & 4 & -14 \\ 1 & 5 & 2 & -11 \\ 4 & 1 & 8 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, shows that A has two pivot columns and therefore $rank\ A = 2$.

From the Rank Theorem 4.14: $dim\ Nul\ A = Number\ of\ columns - \ rank\ A = 4 - 2 = 2$

b) From the row reduction of A it is seen that $Nul\ A$ is spanned by the two vectors:

$$\boldsymbol{v}_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

Since $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = -2$, they are not orthogonal, and we find an orthonormal set by using the Gram-Schmidt process with $\boldsymbol{v}_1$ as the first vector and the orthogonal complement of $\boldsymbol{v}_2$:

$$\widehat{\boldsymbol{v}}_2 = \boldsymbol{v}_2 - \frac{\boldsymbol{v}_1 \cdot \boldsymbol{v}_2}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1}\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} - \frac{-2}{5}\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 2 \\ 2/5 \\ 1 \end{bmatrix}$$

as the second vector.

Normalizing we get:

$$\boldsymbol{u}_1 = \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|} = \begin{bmatrix} -2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \\ 0 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{u}_2 = \frac{\widehat{\boldsymbol{v}}_2}{\|\widehat{\boldsymbol{v}}_2\|} = \sqrt{\frac{5}{26}}\begin{bmatrix} 1/5 \\ 2 \\ 2/5 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{130}}\begin{bmatrix} 1 \\ 10 \\ 2 \\ 5 \end{bmatrix}$$

as an orthonormal basis of $Nul\ A$.

c) Since $A \cdot \boldsymbol{x} = \begin{bmatrix} -1 & -1 & -2 & 3 \\ 2 & 6 & 4 & -14 \\ 1 & 5 & 2 & -11 \\ 4 & 1 & 8 & -6 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -20 \\ -20 \\ 15 \end{bmatrix} \neq \boldsymbol{0}$, $\boldsymbol{x}$ is NOT in $Nul\ A$.

Problem 6

A quadric form is given by: $Q(\boldsymbol{x}) = 3x_1^2 + 4x_2^2 - 2x_3^2 - 6x_1x_2 + 2x_1x_3 - 8x_2x_3$

a) Determine the matrix A, so $Q(\boldsymbol{x}) = \boldsymbol{x}^T A\boldsymbol{x}$

b) Find the principal axes and rewrite the quadratic form in the coordinate system of the principal axes.

c) Find the maximum and minimum values of $Q(\boldsymbol{x})$ subject to the constraint $\|\boldsymbol{x}\| = 1$.

Solution:

a) The symmetric matrix of the quadratic form is: $A = \begin{bmatrix} 3 & -3 & 1 \\ -3 & 4 & -4 \\ 1 & -4 & -2 \end{bmatrix}$.

b) The eigenvalues of A: $\lambda := \text{eigenvals}(A) = \begin{bmatrix} 7.976 \\ 1.057 \\ -4.033 \end{bmatrix}$

Diagonal matrix: $D = \begin{bmatrix} 7.976 & 0 & 0 \\ 0 & 1.057 & 0 \\ 0 & 0 & -4.033 \end{bmatrix}$

The corresponding eigenvectors: $P := \text{eigenvecs}(A) = \begin{bmatrix} 0.534 & 0.842 & 0.074 \\ -0.765 & 0.444 & 0.466 \\ 0.36 & -0.305 & 0.881 \end{bmatrix}$

Diagonalization: $P \cdot D \cdot P^{-1} = \begin{bmatrix} 3 & -3 & 1 \\ -3 & 4 & -4 \\ 1 & -4 & -2 \end{bmatrix} = A$ o.k.

The Principal axis (= eigenvectors): $\boldsymbol{e}_1 = \begin{bmatrix} 0.534 \\ -0.765 \\ 0.36 \end{bmatrix}$ $\boldsymbol{e}_2 = \begin{bmatrix} 0.842 \\ 0.444 \\ -0.305 \end{bmatrix}$ $\boldsymbol{e}_3 = \begin{bmatrix} 0.074 \\ 0.466 \\ 0.881 \end{bmatrix}$

Quadratic form: $Q(\boldsymbol{y}) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 = 7.976y_1^2 + 1.057y_2^2 - 4.033y_3^2$

c) Subject to constraint $\|\boldsymbol{x}\| = 1$: $Q(\boldsymbol{x})_{max} = \lambda_{max} = 7.976$

$Q(\boldsymbol{x})_{min} = \lambda_{min} = -4.033$