In the grading of the exercises special attention will be paid to check that answers are substantiated and that the procedure of calculations is well documented. When results are achieved using a calculator or a pc it should be noted in the paper. All 6 problems are weighted equally in the grading. Your solution to the problem set can be written in Danish or English as you prefer.

PROBLEM 1.

1. Find the general solution to the system of equations below. Give the solution in vector-form.
$$\begin{aligned} -x_1 + 2x_3 + 6x_4 - 14x_5 &= 7 \\ 3x_1 - x_2 - 4x_3 - 18x_4 + 26x_5 &= 5 \\ -2x_1 + x_2 + 4x_3 + 14x_4 - 24x_5 &= 2 \end{aligned}$$
2. Let A, B and C all be $n \times n$ and invertible. Reduce the expressions below as much as possible. Account for the rules used in each step.
   a) $A^{-1}(B^T A^T)^T C I^T C^{-1} B$
   b) $(A^T I C^T A + B^T(I - (B^T)^{-1}))^T + I$
   c) $(A^{-1})^2 A (B^T C)^T$

PROBLEM 1. Solution

1. In matrix form we have:
$$[A|b] = \left[\begin{array}{ccccc|c} -1 & 0 & 2 & 6 & -14 & 7 \\ 3 & -1 & -4 & 18 & 26 & 5 \\ -2 & 1 & 4 & 14 & -24 & 2 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -4 & 2 & 7 \\ 0 & 1 & 0 & 2 & 4 & -12 \\ 0 & 0 & 1 & 1 & -6 & 7 \end{array}\right]$$
   With x_4 and x_5 free, the solution is:
$$\mathbf{x} = x_4 \begin{bmatrix} 4 \\ -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ -4 \\ 6 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 7 \\ -12 \\ 7 \\ 0 \\ 0 \end{bmatrix}$$
2. The solutions are:
   a) $A^{-1}(B^T A^T)^T C I^T C^{-1} B$
   $= A^{-1}(B^T A^T)^T B$ — Rules: $XI = X$ and $X^{-1}X = XX^{-1} = I$
   $= A^{-1}ABB$ — Rules: $(XY)^T = Y^T X^T$ and $(X^T)^T = X$
   $= \underline{\underline{B^2}}$ — Rules: $X^{-1}X = I$, $IX = X$ and $XX = X^2$
   b) $(A^T I C^T A + B^T(I - (B^T)^{-1}))^T + I$
   $= (A^T C^T A + B^T - I)^T + I$ — Rules: $XI = X$ and $X^{-1}X = XX^{-1} = I$
   $= (A^T C^T A)^T + (B^T)^T - I + I$ — Rules: $(X+Y)^T = X^T + Y^T$ and $I^T = I$
   $= \underline{\underline{A^T C A + B}}$ — Rules: $(XY)^T = Y^T X^T$ and $(X^T)^T = X$
   c) $(A^{-1})^2 A (B^T C)^T$
   $= A^{-1}(B^T C)^T$ — Rules: $X^2 = XX$, $X^{-1}X = I$ and $XI = X$
   $= \underline{\underline{A^{-1} C^T B}}$ — Rules: $(XY)^T = Y^T X^T$ and $(X^T)^T = X$

PROBLEM 2.

Consider the matrix A and the stacked matrices $B = \begin{bmatrix} A \\ A \end{bmatrix}$ and $C = \begin{bmatrix} A & A \\ A & A \end{bmatrix}$.

Three important subspaces of any matrix are: the *null space*, *column space* and *row space*.

1. Which of the three subspaces mentioned above are the same for A and B? Justify your answer.

2. Which of the three subspaces mentioned above are the same for B and C? Justify your answer.

PROBLEM 2. Solution

First, we make a table of the vector spaces, where each subspace lives (assuming that A is $m \times n$):

In $\mathbb{R}^?$:	A	B	C
Nul	n	n	$2n$
Col	m	$2m$	$2m$
Row	n	n	$2n$

1. We look at each subspace for A and B:

   Null space: It is easily seen that $B \sim \begin{bmatrix} A \\ 0 \end{bmatrix}$. The bottom half of B then has no effect on the null space, and therefore $\underline{\underline{\mathrm{Nul}(A) = \mathrm{Nul}(B)}}$.

   Column space: $\underline{\underline{\mathrm{Col}(A) \neq \mathrm{Col}(B)}}$ (see table).

   Row space: While the null space of a matrix is the subspace that *is not* spanned by the rows of a matrix, the row space is the subspace that *is* spanned by the rows, i.e. the row space is orthogonal to the null space. Therefore, because $\mathrm{Nul}(A) = \mathrm{Nul}(B)$ they also have the same row space.

2. We look at each subspace for B and C:

   Null space: $\underline{\underline{\mathrm{Nul}(B) \neq \mathrm{Nul}(C)}}$ (see table).

   Column space: The columns of C are obviously linearly dependent, so we can remove half and still have the same column space. Thus, $\underline{\underline{\mathrm{Col}(B) = \mathrm{Col}(C)}}$.

   Row space: $\underline{\underline{\mathrm{Nul}(B) \neq \mathrm{Nul}(C)}}$ (see table).

PROBLEM 3.

Let $H = \begin{bmatrix} 0 & -9 & 5 \\ -1 & 3 & -7 \\ 1 & -12 & 7 \end{bmatrix}$.

1. Using row operations, find the determinant of H.
2. Let A and B be $n \times n$. Label each of the following statements *true* or *false*. Justify each answer by a short proof or counterexample.
   (a) If $\det(A) = 0$ then $\operatorname{rank} A < n$.
   (b) $\det(-A) = (-1)^n \det(A)$.
   (c) If B is invertible, then $\det(B^{-1}) = \frac{1}{\det(B)}$.

PROBLEM 3. Solution

1. Using row operations, we reduce H to a triangular matrix. The determinant of a triangular matrix is the product of the diagonal entries:

$$\det(H) = \begin{vmatrix} 0 & -9 & 5 \\ -1 & 3 & -7 \\ 1 & -12 & 7 \end{vmatrix} = \underbrace{(-1)\begin{vmatrix} 1 & -12 & 7 \\ -1 & 3 & -7 \\ 0 & -9 & 5 \end{vmatrix}}_{\text{Swap } r_1 \text{ and } r_3} = \underbrace{(-1)\begin{vmatrix} 1 & -12 & 7 \\ 0 & -9 & 0 \\ 0 & -9 & 5 \end{vmatrix}}_{r_2 = r_2 + r_1} \tag{1}$$

$$= \underbrace{(-1)\begin{vmatrix} 1 & -12 & 7 \\ 0 & -9 & 0 \\ 0 & 0 & 5 \end{vmatrix}}_{r_3 = r_3 - r_2} = (-1) \cdot 1 \cdot (-9) \cdot 5 = \underline{\underline{45}} \tag{2}$$

2. a) True. If $\operatorname{rank} A < n$ then A is not invertible and therefore $\det(A) = 0$.
   b) True. $-A$ can be constructed by scaling each row by (-1). Each time we scale a row by a constant, the determinant is scaled by the same constant. Therefore, when scaling all n rows, we get $\det(-A) = (-1)^n \det(A)$.
   c) True. We have: $\det(B^{-1})\det(B) = \det(B^{-1}B) = \det(I) = 1$ and thus $\det(B^{-1}) = \frac{1}{\det(B)}$.

PROBLEM 4.

This problem is concerned with computer graphics and homogeneous coordinates as introduced in case 3. For simplicity, only 2D homogeneous coordinates $(x, y, 1)$ are used.

An object $\mathcal{O}$ has the nodes $n_1, n_2, \ldots, n_5$ with coordinates

$$\mathcal{C}_n = \{(1,1), (1,3), (3,3), (3,1), (2,2)\}$$

and the adjacency matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

1. Sketch the shape of the object, $\mathcal{O}$.
2. Using homogeneous coordinates, find a matrix T, which translates the $\mathcal{O}$, such that it is centered in $(0, 0)$. The translated object is called $\mathcal{O}_T$. Sketch $\mathcal{O}_T$.
3. Find a matrix R, which rotates $\mathcal{O}_T$ by 3° counterclockwise around (0,0). Sketch the rotated $\mathcal{O}_T$.
4. Using T and R from above, write a matrix expression for a matrix M which rotates the original shape $\mathcal{O}$ by 30° counterclockwise around its center, while keeping the center at its original coordinates. Compute the entries of M and sketch the rotated object $\mathcal{O}_M$.

PROBLEM 4. Solution

1. See sketches below.
2. $\mathcal{O}$ is centered in $(2, 2)$, so we translate it by $\Delta x = \Delta y = -2$:

$$T = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

   We write $\mathcal{C}_n$ in homogeneous coordinates and perform the transformation:

$$C_{n,h} = \begin{bmatrix} 1 & 1 & 3 & 3 & 2 \\ 1 & 3 & 3 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad , \qquad TC_{n,h} = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & 1 & 1 & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

3. 3° in radians: $\theta = 3° \cdot \pi/180° \simeq 0.0524$.

$$R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \simeq \begin{bmatrix} 0.9986 & -0.0523 & 0 \\ 0.0523 & 0.9986 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Performing first the translation and then the rotation, we get:

$$RTC_{n,h} \simeq \begin{bmatrix} -0.9463 & -1.0510 & 0.9463 & 1.0510 & 0 \\ -1.0510 & 0.9463 & 1.0510 & -0.9463 & 0 \\ 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \end{bmatrix}$$

4. To perform the operation we must shift the center of $\mathcal{O}$ to $(0,0)$ using T, rotate it 10 times with R and shift it back using T^{-1}:

$$M = T^{-1}R^{10}T \simeq \begin{bmatrix} 0.8660 & -0.5000 & 1.2679 \\ 0.5000 & 0.8660 & -0.7321 \\ 0.0000 & 0.0000 & 1.0000 \end{bmatrix}.$$

Applying M to $C_{n,h}$, we get:

$$MC_{n,h} \simeq \begin{bmatrix} 1.6340 & 0.6340 & 2.3660 & 3.3660 & 2.0000 \\ 0.6340 & 2.3660 & 3.3660 & 1.6340 & 2.0000 \\ 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \end{bmatrix}$$

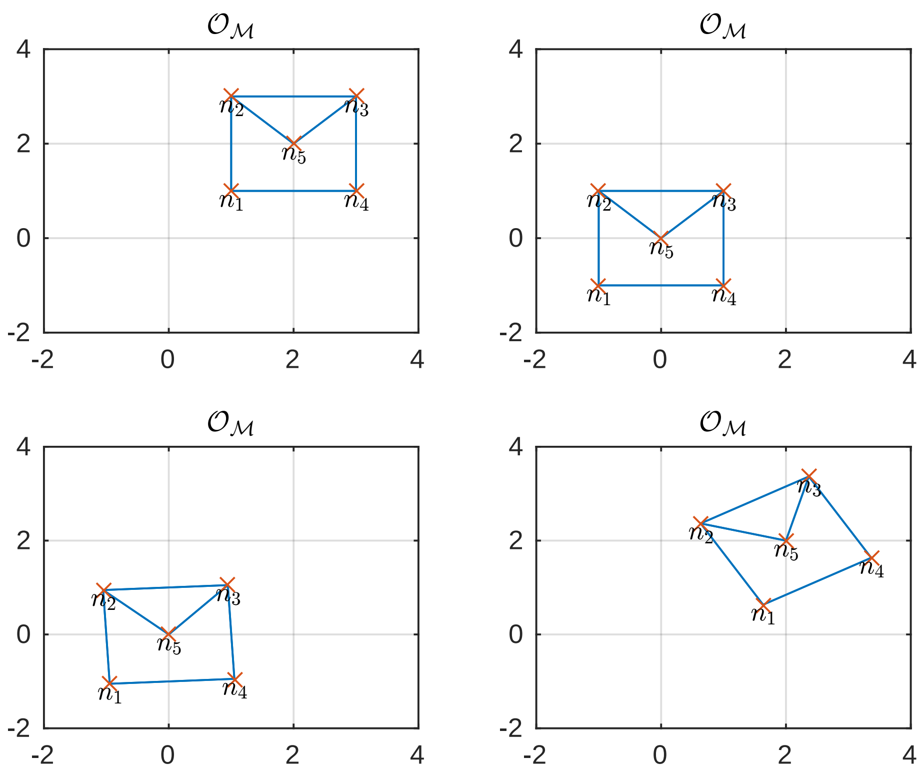

Figure 1: Sketches for problem 4.

PROBLEM 5.

Consider the $A = \begin{bmatrix} 61 & 36 & -42 \\ -60 & -35 & 42 \\ 30 & 18 & -20 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$.

1. Verify that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are eigenvectors of A and find the corresponding eigenvalues, $\{\lambda_1, \lambda_2, \lambda_3\}$.
2. Find matrices P and D such that $A = PDP^{-1}$ where D is diagonal.
3. Assuming that A is diagonalizable, write a matrix expression for A^{-1} in terms of P and D. Explain how we can easily find the eigenvalues of A^{-1} if we know the eigenvalues of A.

PROBLEM 5. Solution

1. Observe that $A\mathbf{v}_1 = 4\mathbf{v}1$, $A\mathbf{v}_2 = \mathbf{v}_2$ and $A\mathbf{v}_3 = \mathbf{v}_3$. Thus, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are eigenvectors corresponding to the eigenvalues $\{4, 2, 2\}$.
2. We let $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$ and $D = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ and verify that $PDP^{-1} = \begin{bmatrix} 61 & 36 & -42 \\ -60 & -35 & 42 \\ 30 & 18 & -20 \end{bmatrix} = A$.
3. We know that $(PDP^{-1})^k = PD^kP^{-1}$, so $A^{-1} = (PDP^{-1})^{-1} = PD^{-1}P^{-1}$. D^{-1} is diagonal with entries $d_{ii}^{-1} = \frac{1}{d_{ii}}$, so the eigenvalues of A^{-1} are simply the reciprocal of the eigenvalues of A.

PROBLEM 6.

A scientist has measured a dataseries using an apparatus:

t	1.0	2.5	4.0	5.5	7.0
y	1.9	-2.7	-2.1	0.6	-1.8

She knows that the apparatus is linear, and that the data is contaminated by noise. Thus, the measurements can be represented in the form of a general linear model:

$$\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon} \tag{1}$$

where:

$\mathbf{y} = \begin{bmatrix} y_1, & y_2, & \ldots, & y_5 \end{bmatrix}^T$ is the observation vector.
$\boldsymbol{\beta} = \begin{bmatrix} \beta_1, \beta_2, \beta_3 \end{bmatrix}^T$ is the parameter vector.
X is the design matrix.
$\boldsymbol{\epsilon}$ is the residual vector representing the noise.

Her model of the apparatus (which is linear in terms of the variables $\{\beta_1, \beta_2, \beta_3\}$) is as follows:

$$y = \beta_1 \cdot \sin(1.5t) + \beta_2 \cdot t^3 + \beta_3 \cdot \log_{10}(t) \tag{2}$$

1. Construct the design matrix X, such that it corresponds to the model equation in Eq. (2).
2. Compute the least-squares solution, $\hat{\boldsymbol{\beta}}$, to Eq. (1).
3. There exists a matrix P such that $P\mathbf{y} = \hat{\boldsymbol{\beta}}$. Give a matrix expression for P.

PROBLEM 6. Solution

1. The matrix is:

$$X = \begin{bmatrix} \sin(1.5t_1) & t_1^3 & \log_{10}(t_1) \\ \sin(1.5t_2) & t_2^3 & \log_{10}(t_2) \\ \sin(1.5t_3) & t_3^3 & \log_{10}(t_3) \\ \sin(1.5t_4) & t_4^3 & \log_{10}(t_4) \\ \sin(1.5t_5) & t_5^3 & \log_{10}(t_5) \end{bmatrix} = \begin{bmatrix} 0.9975 & 1 & 0 \\ -0.5716 & 15.6250 & 0.3979 \\ -0.2794 & 64.0000 & 0.6021 \\ 0.9226 & 166.3750 & 0.7404 \\ -0.8797 & 343.0000 & 0.8451 \end{bmatrix}$$

2. $\hat{\boldsymbol{\beta}}$ is given by: $\hat{\boldsymbol{\beta}} = (X^T X)^{-1} X^T \mathbf{y} = X^+ \mathbf{y}$.

$$\hat{\boldsymbol{\beta}} = (X^T X)^{-1} X^T \mathbf{y} \tag{3}$$

$$= \begin{bmatrix} 0.3560 & -0.2220 & -0.1019 & 0.3811 & -0.1568 \\ 0.0007 & -0.0031 & -0.0030 & -0.0001 & 0.0036 \\ -0.0790 & 0.9587 & 1.0814 & 0.5285 & -0.5016 \end{bmatrix} \begin{bmatrix} 1.9 \\ -2.7 \\ -2.1 \\ 0.6 \\ -1.8 \end{bmatrix} \tag{4}$$

$$= \begin{bmatrix} 2.0006 \\ 0.0093 \\ -3.7895 \end{bmatrix} \tag{5}$$

3. P is the so-called Moore-Penrose pseudo-inverse (or just pseudo-inverse) of X, often denoted by X^+ or $X^\dagger$:

$$P = X^+ = (X^T X)^{-1} X^T$$

   P could also have been constructed using some matrix factorization (e.g. SVD or QR).