In the grading of the exercises special attention will be paid to check that answers are substantiated and that the procedure of calculations is well documented. When results are achieved using a calculator or a pc it should be noted in the paper. All 6 problems are weighted equally in the grading. Your solution to the problem set can be written in Danish or English as you prefer.

PROBLEM 1.
The augmented matrix for a linear system is given by

$$\left[\begin{array}{cccc|c} 0 & 12 & h+3 & 24 & 26 \\ 4 & 8 & h+13 & 12 & 14 \\ 1 & 2 & 1 & 3 & 3 \end{array}\right]$$

1. Determine the general form solution of the system for the case where $h = -8$.
2. For what values of h is the system consistent?

PROBLEM 2.
Consider the matrix

$$A = \begin{bmatrix} 2 & 4 & p \\ 1 & q & 0 \\ 2 & 4 & 5 \end{bmatrix}$$

1. Using a cofactor expansion, find an expression for the determinant of A.
2. Mark each statement below True or False. Justify your answer. All matrices are $n \times n$.

   (a) If the determinant of a A is zero, then one column of A is a linear combination of the remaining columns.

   (b) If $\det(A) = k$ then $\det(A^3) = 3k$.

   (c) If $\det(A) = 1$ then $A = I$.

PROBLEM 3.

Let $A = \begin{bmatrix} -24 & -8 & -4 \\ 8 & -40 & -4 \\ -16 & 16 & -24 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix}$.

1. Justify that $\mathbf{v}_1$ is an eigenvector of A and find the corresponding eigenvalue λ_1.
2. Find the remaining eigenvalues, λ_2 and λ_3, of A.
3. Let $\mathbf{v}_2$ and $\mathbf{v}_3$ be eigenvectors of corresponding λ_2 and λ_3. Show that $\mathbf{w} = 3\mathbf{v}_2 - 2\mathbf{v}_3$ is also an eigenvector.

PROBLEM 4.

Consider the vectors $\mathbf{a} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} -4 \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{d} = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$.

1. Are the vectors linearly independent?
2. Pick out as few of the vectors as possible, such that they span $\mathbb{R}^3$.
3. Do $\mathbf{a}$ and $\mathbf{b}$ span $\mathbb{R}^2$?
4. Construct an orthonormal basis for $\mathbb{R}^3$ such that one of the basis vectors points in the same direction as $\mathbf{d}$.

PROBLEM 5.

Let $A = \begin{bmatrix} 1 & -1 & -3 & 2 & 4 & 2 \\ 2 & 1 & -3 & 3 & 10 & 3 \\ -3 & -1 & 5 & -6 & -16 & -6 \\ 2 & 0 & -4 & 4 & 10 & 4 \end{bmatrix}$.

1. Find a basis for $\operatorname{Nul} A$.
2. What is the rank of A and the dimension of $\operatorname{col} A$?
3. Let the linear transformation T be defined by $T(\mathbf{x}) = A\mathbf{x}$. Is T one-to-one? Justify your answer.

PROBLEM 6.

Let the singular value decomposition (SVD) of a 2×3 matrix A be given by

$$U = \begin{bmatrix} 0.4472 & -0.8944 \\ 0.8944 & 0.4472 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix} \text{ and } \quad V = \begin{bmatrix} 0.8729 & 0 & -0.4880 \\ 0.4364 & 0.4472 & 0.7807 \\ 0.2182 & -0.8944 & 0.3904 \end{bmatrix}.$$

where U and V contain the left and right singular vectors, respectively.

1. Using U, Σ and V, compute the pseudo-inverse of A.
2. Compute the least-squares solution to the system $A\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.
3. Explain the difference between a matrix inverse and a matrix pseudo-inverse.