PROBLEM 1.

Consider the following matrix and vector

$$A = \begin{bmatrix} 1 & -2 & 1 & -1 & 4 \\ 2 & -2 & 5 & 3 & 6 \\ 1 & 3 & 1 & 4 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 7 \\ 1 \\ -3 \end{bmatrix}.$$

1. Show that $\mathbf{x} = \begin{bmatrix} 2 & -1 & -4 & 1 & 2 \end{bmatrix}^T$ is a solution of $A\mathbf{x} = \mathbf{b}$.
2. Compute the general solution of $A\mathbf{x} = \mathbf{b}$ and write the solution in parametric form.

PROBLEM 1. Solution

The first problem is solved by direct calculation

$$A\mathbf{x} = \begin{bmatrix} 1 & -2 & 1 & -1 & 4 \\ 2 & -2 & 5 & 3 & 6 \\ 1 & 3 & 1 & 4 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ -4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ -3 \end{bmatrix} = \mathbf{b}.$$

Thus $\mathbf{x}$ is a solution of $A\mathbf{x} = \mathbf{b}$.

The second problem is solved by row reduction of the augmented matrix

$$\left[\, A \mid \mathbf{b} \,\right] = \left[\begin{array}{ccccc|c} 1 & -2 & 1 & -1 & 4 & 7 \\ 2 & -2 & 5 & 3 & 6 & 1 \\ 1 & 3 & 1 & 4 & -1 & -3 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 2 & 6 \\ 0 & 1 & 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 & 0 & -3 \end{array}\right].$$

As expected from the first problem we see that the system is consistent and contains two free variables, x_4 and x_5. The general solution is written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ -3 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

PROBLEM 2.

Consider the matrix A and three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ given by

$$A = \begin{bmatrix} 5 & -1 & -3 \\ 3 & 1 & -3 \\ 1 & -1 & 1 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

1. Show that the three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis for $\mathbb{R}^3$.
2. Show that the three vectors are all eigenvectors for A and determine the eigenvalue corresponding to each eigenvector.

Let a vector be given by

$$\mathbf{y} = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}.$$

3. Find the coordinates of $\mathbf{y}$ in the eigenvector basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

PROBLEM 2. Solution

The three vectors are a basis for $\mathbb{R}^3$ if they are linearly independent and span $\mathbb{R}^3$. This can be checked by assembling the vectors in a matrix and row reducing.

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

As the matrix is row equivalent to the identity matrix we can conclude that the vectors are indeed linearly independent and span $\mathbb{R}^3$ and hence forms a basis for $\mathbb{R}^3$.

The second problem can be solved by direct computation

$$A\mathbf{v}_1 = \begin{bmatrix} 5 & -1 & -3 \\ 3 & 1 & -3 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

From the above it is seen that $A\mathbf{v}_1 = 4\mathbf{v}_1$ and hence $\lambda_1 = 4$. Similar computations for $\mathbf{v}_2$ and $\mathbf{v}_3$ reveals that $\lambda_2 = 2$ and $\lambda_3 = 1$.

The coordinates of $\mathbf{y}$ in the eigenvector basis are the c_i's

$$\begin{bmatrix} \mathbf{y} \end{bmatrix}_{ev} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}, \quad \text{where } \mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

This gives

$$\mathbf{y} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_2 \end{bmatrix} \iff \begin{bmatrix} c_1 \\ c_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}^{-1} \mathbf{y} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \mathbf{y} \end{bmatrix}_{ev}.$$

PROBLEM 3.

For the statements given below, state whether they are true or false and justify your answer for each statement.

1. if A is an $n \times n$ matrix and $\mathbf{b}$ is an $n \times 1$ vector and $A\mathbf{x} = \mathbf{b}$ is inconsistent then $A\mathbf{x} = \mathbf{0}$ have both trivial and nontrivial solutions.
2. If A is an $n \times n$ matrix and all singular values of A are greater than zero, then $A\mathbf{x} = \mathbf{b}$ is consistent for all values of $\mathbf{b}$.
3. If $\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda_1 t}$ and $\mathbf{x}_2(t) = \mathbf{v}_2 e^{\lambda_2 t}$ are both solutions of the system of differential equations $\mathbf{x}' = A\mathbf{x}$, then the sum $\mathbf{x}_1(t) + \mathbf{x}_2(t) = \mathbf{v}_1 e^{\lambda_1 t} + \mathbf{v}_2 e^{\lambda_2 t}$ is also a solution of $\mathbf{x}' = A\mathbf{x}$.

PROBLEM 3. Solution

The first statement is **true**. As property g. in the Invertible Matrix Theorem "The equation $A\mathbf{x} = \mathbf{b}$ has one solution for each $\mathbf{b}$ in $\mathbb{R}^n$" is false in this case, the matrix A is not invertible. Therefore property d. "The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution" is also false i.e. the equation $A\mathbf{x} = \mathbf{0}$ has both trivial and nontrivial solutions.

The second statement is **true**. In this case, property x. in the Invertible Matrix Theorem "A has n nonzero singular values" is true. Therefore A is an invertible matrix and property g. "The equation $A\mathbf{x} = \mathbf{b}$ has one solution for each $\mathbf{b}$ in $\mathbb{R}^n$" is true.

The third statement is **true**. If $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are both solutions of $\mathbf{x}' = A\mathbf{x}$ then any linear combination of the solutions is also a solution due to the linearity of differentiation and matrix multiplication.

PROBLEM 4.

Let a quadratic form be given as

$$Q(\mathbf{x}) = 4x_1^2 - 6x_1x_2 - 10x_1x_3 - 10x_1x_4 - 6x_2x_3 - 6x_2x_4 - 2x_3x_4.$$

As usual, a quadratic form can also be written as $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$.

1. Find the matrix A of the quadratic form.
2. Determine whether A is positive definite, negative definite or indefinite.
3. Find the maximum and minimum values of $Q(\mathbf{x})$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$.

PROBLEM 4. Solution

The symmetric matrix A is given by

$$A = \begin{bmatrix} 4 & -3 & -5 & -5 \\ -3 & 0 & -3 & -3 \\ -5 & -3 & 0 & -1 \\ -5 & -3 & -1 & 0 \end{bmatrix}.$$

To classify the matrix, the eigenvalues are calculated with MATLAB and found to be $\{-9,\ 1,\ 3,\ 9\}$. As A has both positive and negative eigenvalues the correct classification is **indefinite**.

According to Theorem 6 in chapter 7.3 the maximum and minimum values of $Q(\mathbf{x})$ subject to $\mathbf{x}^T\mathbf{x} = 1$ are the maximum and eigenvalues of A, which in this case gives

$$\max\{\mathbf{x}^T A\mathbf{x} \mid \|\mathbf{x}\| = 1\} = 9, \qquad \min\{\mathbf{x}^T A\mathbf{x} \mid \|\mathbf{x}\| = 1\} = -9$$

PROBLEM 5.

Let the vector space $\mathbb{P}_2$ have the inner product defined by evaluation at -2, -1, 1 and 2. Let $p_0(t) = 1$, $p_1(t) = t$ and $p_2(t) = t^2$.

1. Compute the distance between p_0 and p_2.
2. Compute the orthogonal projection of p_2 onto the subspace spanned by p_0 and p_1.

Another polynomium is given by $p_a(t) = 1 + c$, where c is a real number.

3. Determine c so that p_2 and p_a are orthogonal.

PROBLEM 5. Solution

The distance between vectors p_0 and p_2 is given by

$$\operatorname{dist}(p_0, p_2) = \|p_0 - p_2\| = \sqrt{\langle p_0 - p_2, p_0 - p_2\rangle}.$$

Computing $p_0 - p_2 = 1 - t^2$ and using the standard trick of evaluating at the given points to turn the inner product computation into a dot product gives

$$p_0 - p_2 = 1 - t^2 \mapsto \begin{bmatrix} -3 \\ 0 \\ 0 \\ -3 \end{bmatrix}, \qquad \operatorname{dist}(p_0, p_2) = \sqrt{\begin{bmatrix} -3 \\ 0 \\ 0 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 0 \\ 0 \\ -3 \end{bmatrix}} = \sqrt{18} \approx 4.24$$

The orthogonal projection is given by

$$\hat{p}_2 = \frac{\langle p_2, p_0\rangle}{\langle p_0, p_0\rangle} p_0 + \frac{\langle p_2, p_1\rangle}{\langle p_1, p_1\rangle} p_1$$

To ease the computation the following mapping is used

$$p_0 \mapsto \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad p_1 \mapsto \begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \quad p_2 \mapsto \begin{bmatrix} 4 \\ 1 \\ 1 \\ 4 \end{bmatrix}.$$

And the inner products becomes

$$\langle p_2, p_0\rangle = 10, \quad \langle p_0, p_0\rangle = 4, \quad \langle p_2, p_1\rangle = 0, \quad \langle p_1, p_1\rangle = 10.$$

Hence

$$\hat{p}_2 = \frac{\langle p_2, p_0\rangle}{\langle p_0, p_0\rangle} p_0 + \frac{\langle p_2, p_1\rangle}{\langle p_1, p_1\rangle} p_1 = \frac{10}{4} 1 + \frac{0}{10} t = \frac{5}{2}$$

Vectors p_2 and p_a to be orthogonal if $\langle p_2, p_a\rangle = 0$. Evaluating p_a at the given points gives

$$p_a \mapsto \begin{bmatrix} 1+c \\ 1+c \\ 1+c \\ 1+c \end{bmatrix}.$$

The inner product equation to be solved is then

$$\langle p_2, p_a\rangle = 0 \iff \begin{bmatrix} 4 \\ 1 \\ 1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1+c \\ 1+c \\ 1+c \\ 1+c \end{bmatrix} = 0 \iff 10(1+c) = 0 \iff c = -1.$$

PROBLEM 6.

In the case *Computer Graphics in Automotive Design*, homogeneous coordinates were introduced. In this problem, homogeneous coordinates in $\mathbb{R}^2$ are used. Consider an arrow-like object $\mathcal{O}$ with nodes $n_1, n_2, \ldots, n_5$ with coordinates

$$C = \{(2,0), (4,2), (4,3), (3,3), (1,1)\}$$

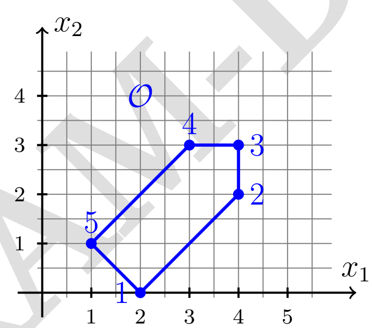
and adjacency matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

1. Sketch the object.
2. Find a translation matrix that moves the object so the arrowhead is at the center of the coordinate system.
3. Find a transformation matrix that will keep the arrowhead of the object at its original position, but make the arrow point in the opposite direction.

PROBLEM 6. Solution

Plotting the object gives



The arrowhead is at (4,3). To move this point into the (0,0) position the following translation matrix is used.

$$T = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}.$$

The required transformation is obtained by translating the arrowhead to the (0,0) position, rotating the arrow 180° (or flipping i.e. $x_1 \mapsto -x_1$ and $x_2 \mapsto -x_2$) followed by a translation back to the (4,3) position. In total the operation becomes

$$T_{flip} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 8 \\ 0 & -1 & 6 \\ 0 & 0 & 1 \end{bmatrix}.$$

Applying the transformation to the data gives the following plot.