**<u>Notice</u>: In the grading of the exercises special attention will be paid to check that the answers are substantiated and that the procedure of calculations is well documented. When results are achieved using a calculator or a pc it should be noted in the solution. All six problems are weighted equally in the grading. Your solution to the problem set can be written in Danish or English as you prefer.**

## Problem 1

Let the matrix $A$ and vector $\boldsymbol{b}$ be given by:

$$A = \begin{bmatrix} 2 & -1 & 2 & 1 \\ 4 & -3 & 0 & 7 \\ -1 & 0 & -3 & 2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ 6 \\ \alpha \end{bmatrix}$$

a) Determine $\alpha$ so the system of equations $A\boldsymbol{x} = \boldsymbol{b}$ are consistent.

b) Determine all solutions to the consistent system of equations $A\boldsymbol{x} = \boldsymbol{b}$.

## Problem 2

The following four vectors are given by:

$$\boldsymbol{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ 9 \\ 13 \\ 4 \end{bmatrix}$$

a) Determine whether the set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly independent.

b) Write – if possible – $\boldsymbol{b}$ as a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_3$.

## Problem 3

For the statements given below, state whether they are true or false and justify your answer for each statement:

a) If $A$ and $B$ are two symmetric $n \times n$ matrices, then $C = A - 2B$ is also symmetric.

b) The determinant of $\begin{bmatrix} 3a & 3b & 3c \\ 3c & 3a & 3b \\ 3b & 3c & 3a \end{bmatrix}$ is nine times the determinant of $\begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$.

c) If $A$ is an real $n \times n$ matrix with $n$ odd (3, 5, 7, …) and $A$ has a complex eigenvalue, then $A$ will also have a real eigenvalue.

## Problem 4

In the following two equations $A, B, C$ and $I$ are all $n \times n$ matrices.

(i) $A(X + 2I)B = BA$ (ii) $XA = C + XB$

a) Solve equation (i) and (ii) for $X$ and account for any assumptions made.

Next, consider an invertible $n \times n$ matrix $A$ with the following property:

$$A^2 = A - I$$

b) Show that $A^3 = -I$ and $A^{-1} = -A^2$.

## Problem 5

Let the matrix $A$ and vector $\boldsymbol{x}$ be given by:

$$A = \begin{bmatrix} -1 & -1 & -2 & 3 \\ 2 & 6 & 4 & -14 \\ 1 & 5 & 2 & -11 \\ 4 & 1 & 8 & -6 \end{bmatrix} \text{ and } \boldsymbol{x} = \begin{bmatrix} 2 \\ -3 \\ 2 \\ 1 \end{bmatrix}$$

a) Determine $rank\ A$ and $dim\ Nul\ A$.

b) Find a orthonormal basis for $Nul\ A$.

c) Is $\boldsymbol{x}$ in $Nul\ A$ ?

## Problem 6

A quadric form is given by: $Q(\boldsymbol{x}) = 3x_1^2 + 4x_2^2 - 2x_3^2 - 6x_1x_2 + 2x_1x_3 - 8x_2x_3$

a) Determine the matrix $A$, so $Q(\boldsymbol{x}) = \boldsymbol{x}^T A \boldsymbol{x}$

b) Find the principal axes and rewrite the quadratic form in the coordinate system of the principal axes.

c) Find the maximum and minimum values of $Q(\boldsymbol{x})$ subject to the constraint $||\boldsymbol{x}|| = 1$.