**PROBLEM 1.**

Consider the following matrix and vector

$$A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 4 & 1 & 7 \\ 3 & 2 & 1 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix}.$$

1. Is $\mathbf{x} = \begin{bmatrix} 1 & 2 & 1 & 1 \end{bmatrix}^T$ a solution of $A\mathbf{x} = \mathbf{b}$?
2. Determine the general solution of $A\mathbf{x} = \mathbf{b}$.
3. How many vectors are there in the solution set?

**PROBLEM 1. Solution**

The first problem is solved by direct calculation

$$A\mathbf{x} = \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 4 & 1 & 7 \\ 3 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 15 \\ 9 \end{bmatrix} \neq \mathbf{b}$$

Thus $\mathbf{x}$ is NOT a solution of $A\mathbf{x} = \mathbf{b}$.

The second problem is solved by row reduction of the augmented matrix

$$\left[\, A \mid \mathbf{b} \,\right] = \left[\begin{array}{cccc|c} 2 & 1 & 0 & 3 & 2 \\ -1 & 4 & 1 & 7 & 7 \\ 3 & 2 & 1 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & -5 & -1 \end{array}\right]$$

From this we see that the system is consistent and contains one free variable, $x_4$. The general solution is written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -3 \\ 5 \\ 1 \end{bmatrix}$$

Due to the free variable there is an infinite amount of vectors in the solution set.

**PROBLEM 2.**

Let a matrix be given by

$$A = \begin{bmatrix} 4 & -3 \\ 1 & 0 \end{bmatrix}.$$

1. Find the characteristic equation.
2. Calculate, *by hand*, the eigenvalues and eigenvectors of $A$.
3. Show that the vector $\mathbf{y} = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$ can be written as a linear combination of $A$'s eigenvectors.

**PROBLEM 2. Solution**

The characteristic equation is calculated as $\det(A - \lambda I) =$ and becomes

$$\begin{vmatrix} 4-\lambda & -3 \\ 1 & -\lambda \end{vmatrix} = 0 \iff \lambda^2 - 4\lambda + 3 = 0.$$

The eigenvalues are found by solving the characteristic equation

$$\lambda = \frac{4 \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 3}}{2} = \frac{4 \pm \sqrt{4}}{2} = \begin{cases} 3 \\ 1 \end{cases}$$

The eigenvectors are the non-trivial solutions of $(A - \lambda I)\mathbf{x} = \mathbf{0}$ for each eigenvalue

$$\lambda_1 = 3, \qquad [A - \lambda_1 I | 0] = \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 1 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 1, \qquad [A - \lambda_2 I | 0] = \left[\begin{array}{cc|c} 3 & -3 & 0 \\ 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

As we have two linear independent eigenvectors the set of these forms a basis for $\mathbb{R}^2$ and the vector $\mathbf{y}$ can be written as a linear combination of these i.e. $\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. The weights $c_1$ and $c_2$ are found as

$$\left[\begin{array}{cc|c} 3 & 1 & 9 \\ 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -3 \end{array}\right].$$

Thus $\mathbf{y} = 4\mathbf{v}_1 - 3\mathbf{v}_2$.

**PROBLEM 3.**

For the statements given below, state whether they are true or false and justify your answer for each statement.

1. The determinant of $\begin{bmatrix} 2a & 2b \\ c & a \end{bmatrix}$ is twice as big as the determinant of $\begin{bmatrix} a & b \\ c & a \end{bmatrix}$.
2. If $\mathbf{x}(t) = \mathbf{v}_1 e^{\lambda_1 t}$ solves $\mathbf{x}' = A\mathbf{x}$ with $\mathbf{v}_1$ and $\lambda_1$ being an eigenvector and corresponding eigenvalue of $A$, then $\mathbf{x}(t) = 2\mathbf{v}_1 e^{\lambda_1 t}$ also solves $\mathbf{x}' = A\mathbf{x}$.
3. If one or more of the singular values of an $n \times n$ matrix $A$ is equal to zero, then $A\mathbf{x} = \mathbf{b}$ will be inconsistent for all $\mathbf{b}$.

**PROBLEM 3. Solution**

The first statement is **true**. This is seen from Theorem 3c in chapter 3 or by direct computation of the determinants.

The second statement is **true**. If $\mathbf{x}(t)$ solves the differential equation, then any linear combination of $\mathbf{x}(t)$ also solves the differential equation.

The third statement is **false**. If $\mathbf{b}$ is in the column space of $A$ the equation $A\mathbf{x} = \mathbf{b}$ will be consistent, but this can not be determined from knowing that one or more singular values are equal to zero.

**PROBLEM 4.**

The following data points have been measured in an experiment

| $x$ | $y$ |
|---|---|
| 1 | 7 |
| 2 | 7 |
| 3 | 8 |
| 4 | 8 |
| 5 | 9 |

It is assumed that the points lie on a straight line.

1. Write up the necessary vectors, matrices and equations to fit the linear model $y = \beta_0 + \beta_1 x$ to data.
2. Calculate the model parameters for the linear model by solving the equation using the least squares method.

Assume that it is later realized that the last two data points might contain errors and should only be weighted half as much as the first three data points.

3. Calculate a new linear model using these assumptions.

**PROBLEM 4. Solution**

Following the procedure from chapter 6.6 the observation vector and the design matrix becomes

$$\mathbf{y} = \begin{bmatrix} 7 \\ 7 \\ 8 \\ 8 \\ 9 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix}$$

Which is solved using

$$\beta = (X^T X)^{-1} X^T \mathbf{y} = \begin{bmatrix} 6.3 \\ 0.5 \end{bmatrix}$$

Hence $y = 6.3 + 0.5x$ is the linear model of the data

The data points are weighted by following the procedure from chapter 6.8 and introducing the matrix

$$W = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

and computing

$$\beta = ((WX)^T WX)^{-1} (WX)^T W \mathbf{y} = \begin{bmatrix} 6.33 \\ 0.49 \end{bmatrix}$$

and the new linear model is then $y = 6.33 + 0.49x$.

**PROBLEM 5.**

Consider a plane in $\mathbb{R}^3$ given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + s \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \quad t, s \in \mathbb{R}$$

1. Does the point $\mathbf{x} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ lie on the plane?
2. Is the plane a subspace of $\mathbb{R}^3$?

**PROBLEM 5. Solution**

The first problem is solved by inserting $\mathbf{x} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ in the equation and simplifying the expression

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + s \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \iff \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + s \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

This equation is then solved using the standard procedure of row reducing the augmented matrix

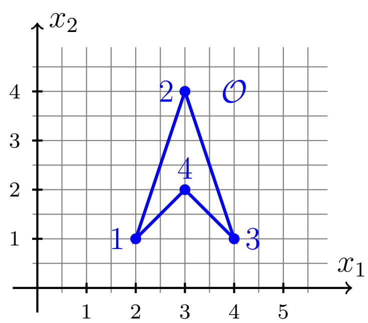
$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 1 & -1 & 1 \\ 3 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

From the bottom row we see that the equations are inconsistent and the point $\mathbf{x}$ does not lie on the plane.

For the plane to be a subspace of $\mathbb{R}^3$ it has to be closed under addition, closed under multiplication and contain the zero vector. Due to the offset $\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T$ the plane will not pass through the origin unless this vector happens to be a linear combination of the two vectors $\begin{bmatrix} 2 & 1 & 3 \end{bmatrix}^T$ and $\begin{bmatrix} -1 & -1 & 2 \end{bmatrix}^T$. This is easily seen not to be the case and the plane is therefore not a subspace of $\mathbb{R}^3$.

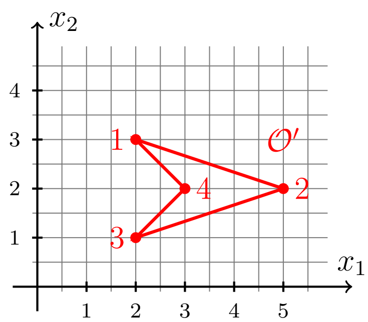
**PROBLEM 6.**

In the case *Computer Graphics in Automotive Design*, homogeneous coordinates were introduced. In this problem, homogeneous coordinates in $\mathbb{R}^2$ are used. Consider the following arrow-like object, denoted $\mathcal{O}$:




1. Determine the data matrix and the adjacency matrix for $\mathcal{O}$.

In the figure below $\mathcal{O}$ has transformed into the new object $\mathcal{O}'$ plotted in red.




2. Determine the transformation matrix that transforms $\mathcal{O}$ into $\mathcal{O}'$.

**PROBLEM 6. Solution**

The data and adjacency matrices are given by

$$D = \begin{bmatrix} 2 & 3 & 4 & 3 \\ 1 & 4 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

From the figure it is evident that the object is rotated 90° clockwise ($\varphi = -90°$) around point 4. This rotation is done by translating the figure so point 4 is at the origin, followed by the 90° rotation and a translation back to the original position. The three matrices that achieves this are:

$$T_1 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} \cos(-90°) & -\sin(-90°) & 0 \\ \sin(-90°) & \cos(-90°) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

The composite transformation is given by $T_2RT_1$ and evaluates to

$$T_2RT_1 = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix}.$$