PROBLEM 1.

Consider the following matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & -1 \\ 0 & 1 & 3 \\ 1 & 1 & 3 \end{bmatrix}.$$

1. Determine the solution to the homogeneous matrix equation $A\mathbf{x} = \mathbf{0}$.
2. Can $A\mathbf{x} = \mathbf{b}$ be solved for any $\mathbf{b}$?

PROBLEM 1. Solution

The augmented matrix is written up and reduced

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 1 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

There are no free variables and we only get the trivial solution $\mathbf{x} = \mathbf{0}$.

The equation $A\mathbf{x} = \mathbf{b}$ can't be solved for any $\mathbf{b}$. A is a 4×3 matrix and the three columns do not span $\mathbb{R}^4$. Hence $A\mathbf{x} = \mathbf{b}$ can only be solved for $\mathbf{b} \in \mathrm{col}(A)$.

PROBLEM 2.

Consider the following matrix and vector

$$A_1 = \begin{bmatrix} 4 & -3 \\ 3 & -3 \end{bmatrix}, \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

1. Use a graphical method to decide if $\mathbf{v}_1$ is an eigenvector of A_1 and if so, determine the corresponding eigenvalue.

Another matrix is given by

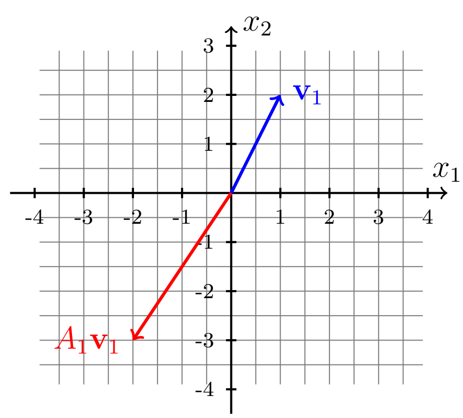
$$A_2 = \begin{bmatrix} 3 & 2 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & -2 & 0 & -1 \\ 0 & 2 & 1 & 2 \end{bmatrix}.$$

2. Determine the eigenspaces and the dimensions of these.

3. Explain whether A_2 is a diagonalizable matrix.

PROBLEM 2. Solution

The vectors $\mathbf{v}_1$ and $A_1\mathbf{v}_1$ are plotted



From the plot it is evident that the two vectors $\mathbf{v}_1$ and $A_1\mathbf{v}_1$ doesn't lie on a straight line, i.e. $A_1\mathbf{v}_1 \neq \lambda\mathbf{v}_1$ and thus $\mathbf{v}_1$ is not an eigenvector of A_1.

The eigenvalues and eigenvectors are found with MATLAB

```
A2 =
     3     2     1    -1
     0     1    -1     1
     0    -2     0    -1
     0     2     1     2

>> [P,D]=eig(A2)

P =
    1.0000   -0.5000    0.1552    0.5000
         0    0.5000    0.5704   -0.5000
         0    0.5000   -0.5704    0.5000
         0   -0.5000    0.5704    0.5000

D =
    3.0000         0         0         0
         0   -1.0000         0         0
         0         0    3.0000         0
         0         0         0    1.0000
```

From the above we see that the eigenvalues are -1, 1 and 3 with multiplicities of 1, 1 and 2. The dimension of each eigenspaces is equal to the multiplicity of the eigenvalue. Bases for the eigenspaces are thus

$$\lambda = -1 : \left\{ \begin{bmatrix} -0.5000 \\ 0.5000 \\ 0.5000 \\ -0.5000 \end{bmatrix} \right\}, \quad \lambda = 1 : \left\{ \begin{bmatrix} 0.5000 \\ -0.5000 \\ 0.5000 \\ 0.5000 \end{bmatrix} \right\}, \quad \lambda = 3 : \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.1552 \\ 0.5704 \\ -0.5704 \\ 0.5704 \end{bmatrix} \right\}.$$

As seen from the above, A_2 has a complete set of linearly independent eigenvectors and is hence diagonalizable according to theorem 5.5.

PROBLEM 3.

For the statements given below, state whether they are true or false and justify your answer for each statement.

1. The columns of a 7×3 matrix with rank 3 are linearly independent.
2. A 2×2 matrix A with columns $\mathbf{a}_1$ and $\mathbf{a}_2$ is invertible if $\mathbf{a}_1 \neq \mathbf{a}_2$.
3. If A and B are $n \times n$ matrices and A is invertible then $A^{-1}BA = B$.

PROBLEM 3. Solution

The first statement is **true**. In our usual picture the matrix consist of 3 $\mathbb{R}^7$ column vectors. Since the rank is 3, we have 3 pivots, i.e. one in each column and the vectors are therefore linearly independent.

The second statement is **false**. Even if $\mathbf{a}_1 \neq \mathbf{a}_2$ they can still be linearly dependent. To ensure invertibility we must have linearly independent columns i.e. $\mathbf{a}_1 \neq c\mathbf{a}_2$.

The third statement is **false**. If the expression is multiplied with A from the left we get $BA = AB$ but matrix multiplication is generally not commutative and $BA \neq AB$.

PROBLEM 4.

Consider the first four Laguerre polynomials shown here.

$$L_0(x) = 1, \qquad L_1(x) = -x + 1$$
$$L_2(x) = \frac{1}{2}x^2 - 2x + 1, \qquad L_3(x) = -\frac{1}{6}x^3 + \frac{3}{2}x^2 - 3x + 1.$$

1. Show that the four Laguerre polynomials form a basis for $\mathbb{P}_3$.

Let a vector in $\mathbb{P}_3$ be given as $z(x) = x^3$.

2. Write $z(x)$ as a linear combination of the four Laguerre polynomials.

PROBLEM 4. Solution

To form a basis the set of vectors must be linearly independent and span $\mathbb{P}_3$. Exploiting the isomorphism between $\mathbb{R}^4$ and $\mathbb{P}_3$ and rewriting the polynomials as column vectors we get

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 & -\frac{1}{6} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

From the above we see that the polynomials are linearly independent and spans $\mathbb{P}_3$, hence they form a basis.

To find $z(x) = x^3$ as a linear combination of the four Laguerre polynomials the isomorphism is again used

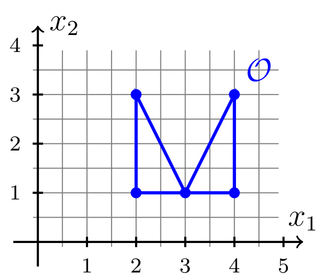
$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & -3 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{6} & 1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & -18 \\ 0 & 0 & 1 & 0 & 18 \\ 0 & 0 & 0 & 1 & -6 \end{array}\right].$$

Hence

$$\begin{aligned} x^3 &= 6L_0(x) - 18L_1(x) + 18L_2(x) - 6L_3(x) \\ &= 6(1) - 18(-x+1) + 18(\frac{1}{2}x^2 - 2x + 1) - 6(-\frac{1}{6}x^3 + \frac{3}{2}x^2 - 3x + 1). \end{aligned}$$

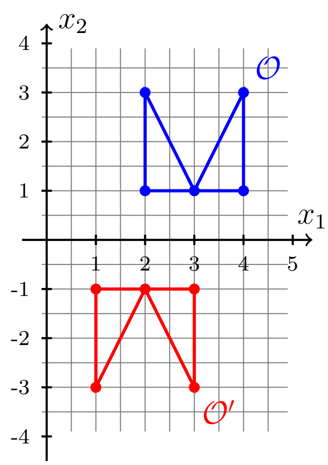
PROBLEM 5.

In the case *Computer Graphics in Automotive Design*, homogeneous coordinates were introduced. In this problem, homogeneous coordinates in $\mathbb{R}^2$ are used. Consider the following object, denoted $\mathcal{O}$:



1. Determine the data matrix and the adjacency matrix for $\mathcal{O}$.

In the figure below $\mathcal{O}$ has transformed into the new object $\mathcal{O}'$ plotted in red.



2. Determine the transformation matrix that transforms $\mathcal{O}$ into $\mathcal{O}'$.

PROBLEM 5. Solution

Starting at the top left point and going clockwise around the figure the data and adjacency matrices are

```
D =                                          A =
       2     3     4     4     2                    0     1     0     0     1
       3     1     3     1     1                    1     0     1     1     1
       1     1     1     1     1                    0     1     0     1     0
                                                    0     1     1     0     0
                                                    1     1     0     0     0
```

The transformation is composed of a translation one step to the left on the x-axis and a reflection across the x-axis. The two movements can also be carried in the reverse order. The matrices are

$$T_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad T_1T_2 = T_2T_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

PROBLEM 6.

Let V be the vector space of all $n \times n$ matrices and define the inner product between two matrices in V as

$$< A, B >= \mathrm{Tr}(B^T A).$$

Where Tr is the usual trace function, i.e. the sum of the diagonal elements.
Consider matrices

$$A = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & b \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix},$$

where b is a scalar.

1. Calculate the norm of A.
2. Determine b so that A and B are orthogonal.

The matrices A and C are orthogonal.

3. Find a nonzero matrix D that is orthogonal to both A and C.

PROBLEM 6. Solution

The norm of A is calculated as

$$||A|| = \sqrt{< A, A >} = \sqrt{Tr(A^T A)}.$$

Since

$$A^T A = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 4 \\ 4 & 2 \end{bmatrix}$$

we have Tr(A)=12 and $||A|| = \sqrt{12}$.

The matrices are orthogonal if their inner product is zero.

$$\mathrm{Tr}(B^T A) = \mathrm{Tr}\left(\begin{bmatrix} 1 & 2 \\ 2 & b \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} \right) = \mathrm{Tr}\left(\begin{bmatrix} 7 & 3 \\ 3b+2 & b+2 \end{bmatrix} \right) = b + 9$$

From the above is it seen that the matrices are orthogonal for $\underline{\underline{\text{b=-9}}}$.

Since A and C are orthogonal, we can find the matrix D by selecting a random matrix D_R and subtract the orthogonal projections onto A and D.

$$D = D_R - \frac{< D_R, A >}{< A, A >} A - -\frac{< D_R, C >}{< C, C >} C.$$

Note, not any random matrix can be used, D_R can't be a linear combination of A and C. Choosing $Dr = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and doing the calculations in MATLAB gives

```
>> Dr=ones(2,2)

Dr =
     1     1
     1     1
```

```
>> D=Dr-trace(Dr'*A)/trace(A'*A)*A-trace(Dr'*C)/trace(C'*C)*C

D =
    0.3667    0.7000
   -0.4333    0.2333
```

The calculations can easily be verified

```
>> trace(D'*A)

ans =

     0

>> trace(D'*C)

ans =

   4.4409e-16
```

Where the last non-zero value is caused by round-off effects.