Solution for the ET-ALA reexam (Q3-2015)

PROBLEM 1.

Let the matrix A and the vector $\mathbf{b}$ be given by

$$A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 3 & 7 & 8 & -3 \\ -1 & -3 & -3 & 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}.$$

1. Calculate the solutions to the matrix equation $A\mathbf{x} = \mathbf{b}$.

PROBLEM 1. Solution

The augmented matrix is constructed and row reduced

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & -1 & 1 \\ 3 & 7 & 8 & -3 & 3 \\ -1 & -3 & -3 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & -2 & -2 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

From the reduced matrix it is evident that x_4 is a free variable. In parametric form the full solution becomes

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 2 \\ -1 \\ 1 \end{bmatrix}$$

PROBLEM 2.
Let the following matrix and vector be given

$$A = \begin{bmatrix} 4 & -14 & 8 \\ 1 & -5 & 4 \\ 1 & -7 & 6 \end{bmatrix}, \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

1. Show that $\mathbf{v}_1$ is an eigenvector of A and find the corresponding eigenvalue.
2. Find the other eigenvectors and eigenvalues of A.
3. Can any vector in $\mathbb{R}^3$ be written as a linear combination of the eigenvectors?

PROBLEM 2. Solution
To show that $\mathbf{v}_1$ is an eigenvector of A we insert it into the eigenvalue equation, $A\mathbf{x} = \lambda\mathbf{x}$. This gives

$$A\mathbf{v}_1 = \begin{bmatrix} 4 & -14 & 8 \\ 1 & -5 & 4 \\ 1 & -7 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 10 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

Thus the vector fullfils the eigenvalue equation and the corresponding eigenvalue is $\lambda_1 = 2$.

The other eigenvalues and eigenvectors can be found with matlab

```
>> A=[4 -14 8;1 -5 4;1 -7 6]

A =
     4   -14     8
     1    -5     4
     1    -7     6

>> [P,D]=eig(A)

P =
    0.9045   -0.8165   -0.7493
    0.3015   -0.4082    0.2446
    0.3015   -0.4082    0.6154

D =
    2.0000         0         0
         0    1.0000         0
         0         0    2.0000
```

It is seen that $\lambda_1 = \lambda_2 = 2$ is a repeated eigenvalue. The two corresponding eigenvectors does not resemble $\mathbf{v}_1$. This is not an error, however, as $\mathbf{v}_1$ can be formed as a linear combination of the two $\lambda = 2$ eigenvectors.

The answer to the final question is yes. The matrix has one 2-dimensional and one 1-dimensional eigenspace. Together the eigenvectors thus span $\mathbb{R}^3$ as can also be seen by $P \sim I$.

PROBLEM 3.

If A is a diagonalizable and positive semidefinite $n \times n$ matrix, then the square root of A can be defined as $A^{1/2} = PD^{1/2}P^{-1}$ where

$$D = \begin{bmatrix} \sqrt{\lambda_1} & 0 & \dots & \dots & 0 \\ 0 & \sqrt{\lambda_2} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \sqrt{\lambda_n} \end{bmatrix} \quad \text{and } P = [\mathbf{v}_1 \dots \mathbf{v}_n]$$

where λ_i and $\mathbf{v}_i$ are corresponding eigenvalues and eigenvectors of A.

1. Show that $(A^{1/2})^2 = A$

Let a 3×3 matrix be given as

$$A = \begin{bmatrix} 4 & -3 & 3 \\ 0 & 1 & 8 \\ 0 & 0 & 9 \end{bmatrix}$$

2. Calculate $A^{1/2}$.

PROBLEM 3. Solution

This is done by inserting the definition

$$\begin{aligned}(A^{1/2})^2 &= (PD^{1/2}P^{-1})(PD^{1/2}P^{-1}) \\ &= PD^{1/2}(P^{-1}P)D^{1/2}P^{-1} \\ &= PD^{1/2}D^{1/2}P^{-1} \\ &= PDP^{-1} \\ &= A\end{aligned}$$

To calculate $A^{1/2}$ the eigenvalues and eigenvectors of A are needed. Since the matrix is upper triangular the eigenvalues can directly read or matlab can be used

```
>> A=[4 -3 3; 0 1 8; 0 0 9]

A =
     4    -3     3
     0     1     8
     0     0     9

>> [P,D]=eig(A)

P =
    1.0000    0.7071         0
         0    0.7071    0.7071
         0         0    0.7071


D =
     4     0     0
     0     1     0
     0     0     9
```

The matrix $D^{1/2}$ then becomes

$$D^{1/2} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

And the square root of A is then

$$A^{1/2} = PD^{1/2}P^{-1} = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

PROBLEM 4.
For the statements given below, state whether they are true or false and justify your answer for each statement.

1. For a $m \times n$ matrix with $m < n$ the set of row vectors form a basis for the row space.
2. The polynomial space $\mathbb{P}_2$ is a subspace of $\mathbb{P}_3$.
3. The vectors $f(t) = e^{-2t}$ and $g(t) = e^{-3t}$ are linearly independent.

PROBLEM 4. Solution
Statement 1 is **false**. A matrix with $m < n$ could have pivots in all rows and the row vectors would then form a basis for the row space. However, it is not stated the problem whether this is the case. An example of matrix with $m < n$ and where the statement is false is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Statement 2 is **false**. Polynomials in $\mathbb{P}_2$ have the form $p(t) = a_0 + a_1 t + a_2 t^2$ and are described by the three coefficients a_0, a_1 and a_2. Polynomials in $\mathbb{P}_3$ have the form $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ and are described by the four coefficients a_0, a_1, a_2 and a_3. One could argue though, that a 'pseudo-connection' exist if only polynomials in $\mathbb{P}_3$ with $a_3 = 0$ are considered.

Statement 3 is **true**. We check for linear independence as usual by looking for non-trivial solution to $c_1 f(t) + c_2 g(t) = 0$. This gives

$$c_1 e^{-2t} + c_2 e^{-3t} = 0$$

The only solution of c_1 and c_2 that makes this equation true for all values of t is $c_1 = c_2 = 0$ and the two vectors are thus linearly independent.

PROBLEM 5.
Assume that corresponding values of time t and output y have been measured for a system as shown in this table

t	y
0	1
1	3
2	6
3	6
4	8
5	11

It is assumed that the system can be fitted to a linear model of the form $y_1(t) = \beta_0 + \beta_1 t^2$.

1. Write up the observation vector and the design matrix for the problem.
2. Determine the two model parameters.
3. Comment on whether the assumed model is appropriate.

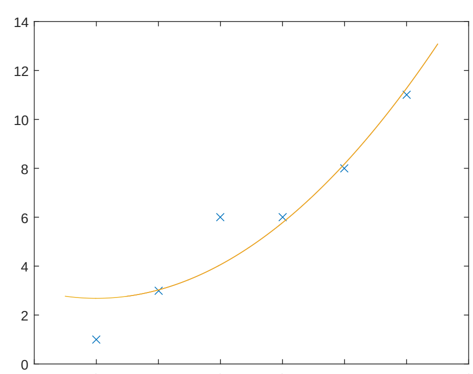
PROBLEM 5. Solution
The design matrix X and the observation vector $\mathbf{y}$ are

$$X = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 4 \\ 1 & 9 \\ 1 & 16 \\ 1 & 25 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 3 \\ 6 \\ 6 \\ 8 \\ 11 \end{bmatrix}$$

The model parameters are calculated as $\boldsymbol{\beta} = (X^T X)^{-1} X^T \mathbf{y}$. Using Matlab the parameters are

$$\boldsymbol{\beta} = \begin{bmatrix} 2.6834 \\ 0.3436 \end{bmatrix}$$

The number $||\mathbf{y} - \hat{\mathbf{y}}||$ is a measure of ther misfit. The number by itself is however not a clear indication. The data and the model can be plotted to get a different view. As seen below the model gives a reasonable fit but e.g. a straight line could probably give an equally good fit.



PROBLEM 6.
This problem is based on case 5: "Error-Detecting and Error-Correcting Codes". All calculations must therefore be done using $\mathbb{Z}_2$ arithmetics . Let the matrix A be given by

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

1. Determine a basis for the column space of A, a basis for the null space of A and the rank of A.

PROBLEM 6. Solution
The matrix is row reduced using $\mathbb{Z}_2$ arithmetics. This gives

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The matrix has pivots in the first three pivots. The $\underline{\underline{\text{rank of the matrix is hence three}}}$ and a basis for col A is given by the first three columns of A

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The basis for the null space of A is found by solving $A\mathbf{x} = \mathbf{0}$.

$$[A|\mathbf{0}] = \left[\begin{array}{cccc|c} 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

The matrix contains one free variable and the solution becomes

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

The basis vector for the null space is thus

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$