Solution for the ET-ALA exam (Q1-2014)

PROBLEM 1.

1. Determine the solution(s) of the following vector equation

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ 11 \end{bmatrix}$$

Let B be an invertible $n \times n$ matrix.

2. Reduce the expression $B^3B(B^{-1})^TB^{-1}BB^T(BB)B^{-1}(B^{-1})^2$ as much as possible and account for the rules used in each step of the reduction.

PROBLEM 1. Solution

The problem is solved by writing the augmented matrix and row reducing it.

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -3 \\ 2 & 1 & -1 & -1 \\ 1 & 0 & 3 & 11 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

There are no free variables and the solution $x_1 = 2$, $x_2 = -2$ and $x_3 = 3$ is therefore unique.

The expression is reduced as follows

$$\begin{aligned}
&B^3B(B^{-1})^TB^{-1}BB^T(BB)B^{-1}(B^{-1})^2 = \\
&B^3B(B^{-1})^TIB^T(BB)B^{-1}(B^{-1})^2 = \\
&B^3B(B^{-1})^TB^T(BB)B^{-1}(B^{-1})^2 = \\
&B^3B(B^T)^{-1}B^T(BB)B^{-1}(B^{-1})^2 = \\
&B^3BI(BB)B^{-1}(B^{-1})^2 = \\
&B^3B(BB)B^{-1}(B^{-1})^2 = \\
&B^6(B^{-1})^3 = \\
&B^3
\end{aligned}$$

Where the rules $BB^{-1} = I$, $(B^{-1})^T = (B^T)^{-1}$, $(B^{-1})^2 = B^{-1}B^{-1}$ was used together with the fact that an I can be inserted and removed at will.

PROBLEM 2.
Consider the following 2×2 matrix

$$A = \begin{bmatrix} 1 & q \\ 4 & 2 \end{bmatrix}$$

where q is a parameter that can be adjusted.

1. Calculate the eigenvalues of A as a function of q.
2. Calculate q so that $\lambda = 5$ is an eigenvalue of A.

PROBLEM 2. Solution
The eigenvalues are calculated in the usual fashion by solving $\det(A - \lambda I) = 0$.

$$\begin{vmatrix} 1-\lambda & q \\ 4 & 2-\lambda \end{vmatrix} = 0 \iff (1-\lambda)(2-\lambda) - 4q = 0 \iff \lambda^2 - 3\lambda + 2 - 4q = 0$$

This is a standard quadratic equation and the solution is therefore given by

$$\lambda = \frac{3 \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot (2-4q)}}{2 \cdot 1} = \underline{\underline{\frac{3 \pm \sqrt{9 - 8(1-2q)}}{2}}}$$

This result can also be given in several other ways e.g.

$$\lambda = \frac{3}{2} \pm \frac{1}{2}\sqrt{1 + 16q}$$

The value of q giving $\lambda = 5$ is solved for as follows

$$\begin{aligned}
&\frac{3 \pm \sqrt{9 - 8(1-2q)}}{2} = 5 \iff \\
&3 \pm \sqrt{9 - 8(1-2q)} = 10 \iff \\
&\pm \sqrt{9 - 8(1-2q)} = 7 \iff \\
&9 - 8(1-2q) = 49 \iff \\
&-8(1-2q) = 40 \iff \\
&1 - 2q = -5 \iff \\
&-2q = -6 \iff \\
&\underline{\underline{q = 3}}
\end{aligned}$$

A faster way is to just insert $\lambda = 5$ in $\lambda^2 - 3\lambda + 2 - 4q = 0$ from which $q = 3$ is also found.

PROBLEM 3.

For the statements given below, state whether they are true or false and justify your answer for each statement.

1. If $A\mathbf{x} = \mathbf{b}$ has the solution $\mathbf{x} = \mathbf{c}$, then $A\mathbf{x} = 2\mathbf{b}$ has the solution $\mathbf{x} = \frac{1}{2}\mathbf{c}$.
2. The matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ is positive definite.
3. Two vectors, $\mathbf{a}$ and $\mathbf{b}$ are linearly dependent if $\mathbf{a}^T\mathbf{b} \neq 0$.

PROBLEM 3. Solution

The first statement is false. Since $\mathbf{x} = \mathbf{c}$ is a solution we know that $A\mathbf{c} = \mathbf{b}$. Inserting $\mathbf{x} = \frac{1}{2}\mathbf{c}$ in $A\mathbf{x} = 2\mathbf{b}$ gives

$$A\frac{1}{2}\mathbf{c} = 2\mathbf{b} \iff \frac{1}{2}A\mathbf{c} = 2\mathbf{b}$$

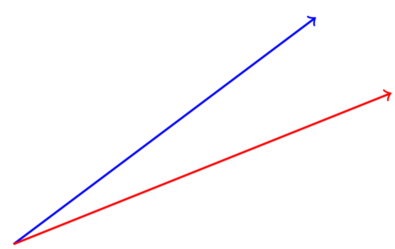
but with $A\mathbf{c} = \mathbf{b}$ this becomes

$$\frac{1}{2}\mathbf{b} = 2\mathbf{b}$$

which is clearly false except for $\mathbf{b} = \mathbf{0}$.

The second statement is false. The eigenvalues of A are -0.2361 and 4.2361, however for the matrix to be positive definite, all eigenvalues must be positive.

The third statement is false. If $\mathbf{a}^T\mathbf{b} = 0$ the vectors are orthogonal and therefore also linear independent. However linearly independent vectors are not necessarily orthogonal. A counter example of two linearly independent, yet not orthogonal vectors are shown here



PROBLEM 4.

In case 1 cubic splines was introduced. In this problem we continue this line of thought and determine the parameters of an unknown function satisfying a number of constraints. Let the function $f(x)$ be given by

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

The function must pass through the following data points

x	$f(x)$
1	3
3	2

Further, the first and second derivatives of the function must satisfy

$$f'(1) = 3, \quad f''(2) = -1$$

1. Write down the equations needed to determine a_0, a_1, a_2, and a_3.
2. Solve the equations and write down the full expression for $f(x)$.

PROBLEM 4. Solution

To determine the 4 unknown parameters 4 linear equations are set up. Since the function has to pass through the two data points the first two equations becomes

$$\begin{aligned} a_0 + a_1 1 + a_2 1^2 + a_3 1^3 &= 3 \\ a_0 + a_1 3 + a_2 3^2 + a_3 3^3 &= 2 \end{aligned}$$

The derivatives of the function are

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 \quad \text{and} \quad f''(x) = 2a_2 + 6a_3 x$$

The derivative constraints therefore turns into

$$a_1 + 2a_2 + 3a_3 = 3$$

and

$$2a_2 + 12a_3 = -1$$

These four equations can be written as an augmented matrix and solved with the standard methods.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 1 & 3 & 9 & 27 & 2 \\ 0 & 1 & 2 & 3 & 3 \\ 0 & 0 & 2 & 12 & -1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -5.5 \\ 0 & 1 & 0 & 0 & 15.25 \\ 0 & 0 & 1 & 0 & -8 \\ 0 & 0 & 0 & 1 & 1.25 \end{array}\right]$$

and hence

$$f(x) = -5.5 + 15.25x - 8x^2 + 1.25x^3$$

PROBLEM 5.
Consider the following matrix and vector

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 1 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

1. Make a QR decomposition of A with the method from the textbook.
2. Solve the equation $A\mathbf{x} = \mathbf{b}$ using the QR decomposition, i.e. substitute $A = QR$ to obtain a simplified problem.

PROBLEM 5. Solution
The matrix Q contains a orthonormal basis for A. The columns of A are clearly not orthogonal. Therefor the Gram-Schmidt procedure is used to find an orthogonal basis.

$$\mathbf{a}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \qquad \mathbf{a}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

This gives orthogonal vectors

$$\mathbf{q}_1 = \mathbf{a}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \qquad \mathbf{q}_2 = \mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{q}_1}{\mathbf{q}_1 \cdot \mathbf{q}_1}\mathbf{q}_1 = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

These vectors are normalized and forms Q

$$Q = \begin{bmatrix} 2/3 & 1/3 \\ 2/3 & -2/3 \\ 1/3 & 2/3 \end{bmatrix}$$

Next the R matrix is found from $A = QR \Rightarrow R = Q^T A$ since $Q^T Q = I$.

$$R = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$$

The equation $A\mathbf{x} = \mathbf{b}$ is solved with the QR decomposition as

$$A\mathbf{x} = \mathbf{b} \iff QR\mathbf{x} = \mathbf{b} \iff Q^T QR\mathbf{x} = Q^T\mathbf{b} \iff R\mathbf{x} = Q^T\mathbf{b}$$

Since R is an upper-triangular matrix, the needed row operations are simple. With $Q^T\mathbf{b} = [0 \;\; 3]^T$ the augmented matrix $[R|Q^T\mathbf{b}]$ is

$$\left[\begin{array}{cc|c} 3 & 1 & 0 \\ 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 3 \end{array}\right]$$

Thus $\mathbf{x} = [-1 \;\; 3]^T$ is the desired solution of $A\mathbf{x} = \mathbf{b}$.

PROBLEM 6.
Let a weighted inner product between two vectors in $\mathbb{R}^3$ be given by

$$< \mathbf{x}, \mathbf{y} >= 3x_1y_1 + 2x_2y_2 + x_3y_3,$$

and let two vectors, $\mathbf{x}$ and $\mathbf{y}$ be given by

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

1. Calculate $< \mathbf{x}, \mathbf{y} >$, $||\mathbf{x}||$ and the distance between $\mathbf{x}$ and $\mathbf{y}$, $\mathrm{dist}(\mathbf{x}, \mathbf{y})$ using the weighted inner product.
2. Show that the triangle inequality holds for the weighted inner product and the two vectors $\mathbf{x}$ and $\mathbf{y}$.

PROBLEM 6. Solution
The inner product is calculated as

$$< \mathbf{x}, \mathbf{y} >= 3 \cdot 1 \cdot 2 + 2 \cdot 1 \cdot (-1) + (-1) \cdot 3 = 1$$

The norm of $\mathbf{x}$ is given by $||\mathbf{x}|| = \sqrt{< \mathbf{x}, \mathbf{x} >}$ and is

$$||\mathbf{x}|| = \sqrt{3 \cdot 1 \cdot 1 + 2 \cdot 1 \cdot 1 + (-1) \cdot (-1)} = \sqrt{6} \approx 2.4495$$

The distance between $\mathbf{x}$ and $\mathbf{y}$ is given by $||\mathbf{x} - \mathbf{y}||$. The vector $\mathbf{x} - \mathbf{y}$ is $[-1 \;\; 2 \;\; -4]^T$ and the distance is therefore

$$\mathrm{dist}(\mathbf{x}, \mathbf{y}) = \sqrt{3 \cdot (-1) \cdot (-1) + 2 \cdot 2 \cdot 2 + (-4) \cdot (-4)} = \sqrt{27} \approx 5.1962$$

The triangle inequality states $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$. To show that the inequality holds we calculate

$$||\mathbf{y}|| = \sqrt{3 \cdot 2 \cdot 2 + 2 \cdot (-1) \cdot (-1) + 3 \cdot 3} = \sqrt{23} \approx 4.7958$$

and $\mathbf{x} + \mathbf{y} = [3 \;\; 0 \;\; 2]^T$ giving

$$||\mathbf{x} + \mathbf{y}|| = \sqrt{3 \cdot 3 \cdot 3 + 2 \cdot 0 \cdot 0 + 2 \cdot 2} = \sqrt{31} \approx 5.5678$$

since $\sqrt{31} \approx 5.5678$ and $\sqrt{23} + \sqrt{6} \approx 7.2453$ the inequality holds as expected.