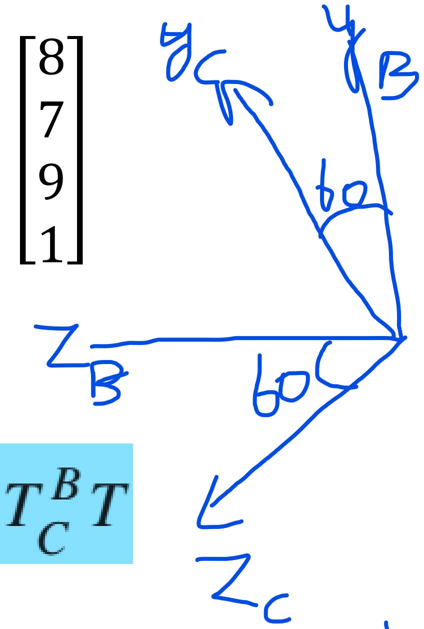
Frame {B} is rotated relative to frame {A} about z-axis by 30 degree and translated 4 units in x-axis and 3 units in y axis. Frame {C} is rotated relative to frame {B} about x-axis by 60 degrees and translated 6 units in x-axis and 5 units in z-axis. Find the position of P relative to frame {A} if ${}^cP = [8\ 7\ 9]^T$

$${}^A_BT = \begin{bmatrix}0.866 & -0.5 & 0 & 4\\ 0.5 & 0.866 & 0 & 3\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \qquad {}^B_CT = \begin{bmatrix}1 & 0 & 0 & 6\\ 0 & 0.5 & -0.866 & 0\\ 0 & 0.866 & 0.5 & 5\\ 0 & 0 & 0 & 1\end{bmatrix} \qquad {}^cP = \begin{bmatrix}8\\ 7\\ 9\\ 1\end{bmatrix}$$



$${}^A_CT = \begin{bmatrix}0.866 & -0.25 & 0.433 & 9.196\\ 0.5 & 0.433 & -0.75 & 6\\ 0 & 0.866 & 0.5 & 5\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\boxed{{}^AP = {}^A_BT\,{}^B_CT\,{}^CP} \qquad \boxed{{}^A_CT = {}^A_BT\,{}^B_CT}$$

$${}^AP = \begin{bmatrix}0.866 & -0.25 & 0.433 & 9.196\\ 0.5 & 0.433 & -0.75 & 6\\ 0 & 0.866 & 0.5 & 5\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}8\\ 7\\ 9\\ 1\end{bmatrix}$$

$$\begin{vmatrix} X_BX_C & X_BY_C & X_BZ_C\\ Y_BX_C & Y_BY_C & Y_BZ_C\\ Z_BX_C & Z_B\cdot Y_C & Z_BZ_C\end{vmatrix}$$

$${}^AP = \begin{bmatrix}18.24\\ 6.28\\ 15.56\\ 1\end{bmatrix}$$

$$\begin{vmatrix} 1 & 0 & 0\\ 0 & \cos 60 & -\sin 60\\ 0 & \sin 60 & \cos 60\end{vmatrix}$$

Transformation Arithmetic

$$
{}^B_AT = \left[\begin{array}{c|c} {}^A_BR^T & -{}^A_BR^T\,{}^AP_{BORG} \\ \hline 0\quad 0\quad 0 & 1 \end{array}\right]
$$

Inverting Transform (Proof)

To find ${}^B_AT$, we must compute ${}^B_AR$ and ${}^BP_{AORG}$ from ${}^A_BR$ and ${}^AP_{BORG}$. First, recall from our discussion of rotation matrices that

$$
{}^B_AR = {}^A_BR^T. \tag{2.42}
$$

Next, we change the description of ${}^AP_{BORG}$ into $\{B\}$ by using (2.13):

$$
{}^B({}^AP_{BORG}) = {}^B_AR\,{}^AP_{BORG} + {}^BP_{AORG}. \tag{2.43}
$$

The left-hand side of (2.43) must be zero, so we have

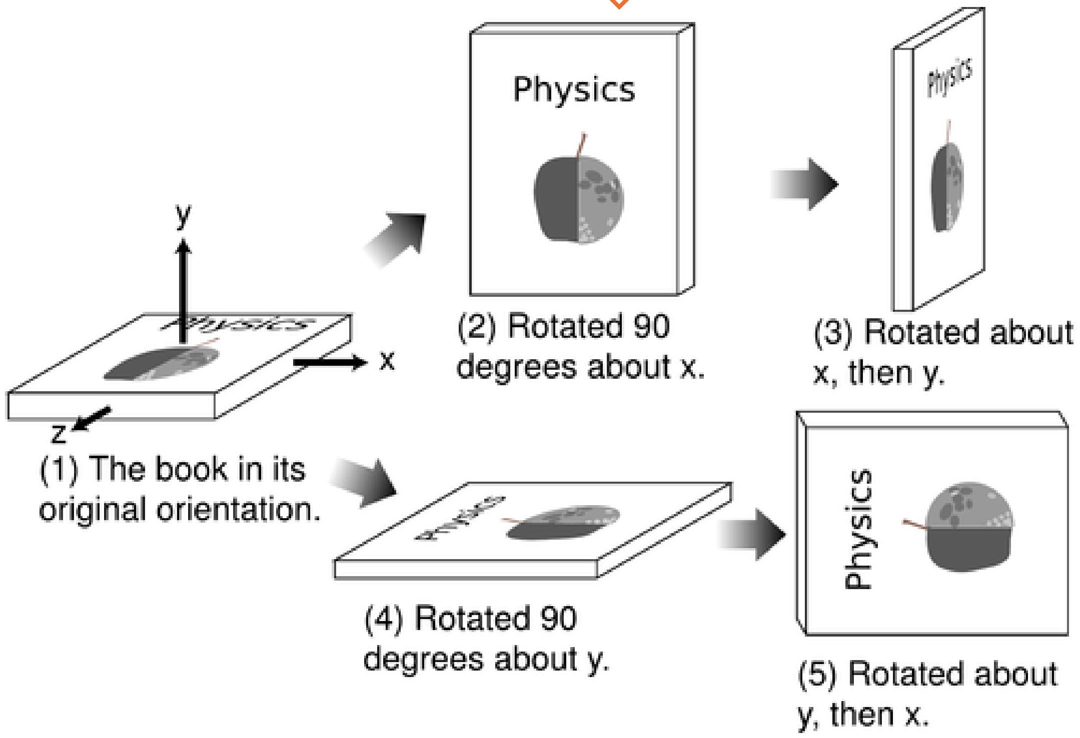
$$
{}^BP_{AORG} = -{}^B_AR\,{}^AP_{BORG} = -{}^A_BR^T\,{}^AP_{BORG}. \tag{2.44}
$$

Using (2.42) and (2.44), we can write the form of ${}^B_AT$ as

$$
{}^B_AT = \left[\begin{array}{c|c} {}^A_BR^T & -{}^A_BR^T\,{}^AP_{BORG} \\ \hline 0\quad 0\quad 0 & 1 \end{array}\right]. \tag{2.45}
$$

Transformation Arithmetic

Transformation matrices multiplication is not commutative



EXAMPLE 2.7

Consider two rotations, one about $\hat{Z}$ by 30 degrees and one about $\hat{X}$ by 30 degrees:

$$R_z(30) = \begin{bmatrix} 0.866 & -0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} \tag{2.60}$$

$$R_x(30) = \begin{bmatrix} 1.000 & 0.000 & 0.000 \\ 0.000 & 0.866 & -0.500 \\ 0.000 & 0.500 & 0.866 \end{bmatrix} \tag{2.61}$$

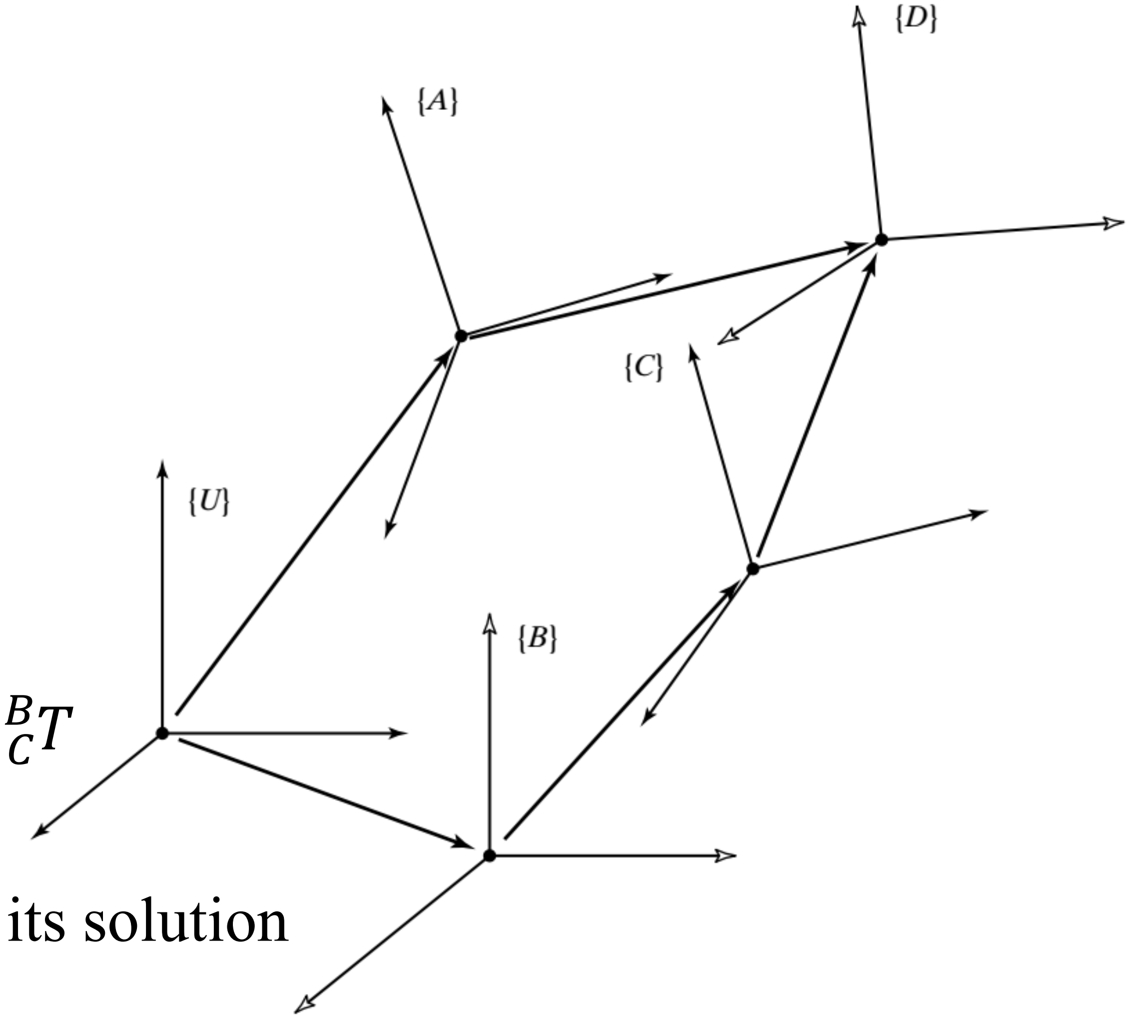
$$\begin{aligned} R_z(30)R_x(30) &= \begin{bmatrix} 0.87 & -0.43 & 0.25 \\ 0.50 & 0.75 & -0.43 \\ 0.00 & 0.50 & 0.87 \end{bmatrix} \\ \neq R_x(30)R_z(30) &= \begin{bmatrix} 0.87 & -0.50 & 0.00 \\ 0.43 & 0.75 & -0.50 \\ 0.25 & 0.43 & 0.87 \end{bmatrix} \end{aligned} \tag{2.62}$$

Transform Equations

$$ {}^U_D T = {}^U_A T \, {}^A_D T; $$

$$ {}^U_D T = {}^U_B T \, {}^B_C T \, {}^C_D T. $$

$$ {}^U_A T \, {}^A_D T = {}^U_B T \, {}^B_C T \, {}^C_D T. $$



Consider that all transforms are known except ${}^B_C T$

Here, we have one transform equation and

one unknown transform; hence, we easily find its solution

to be

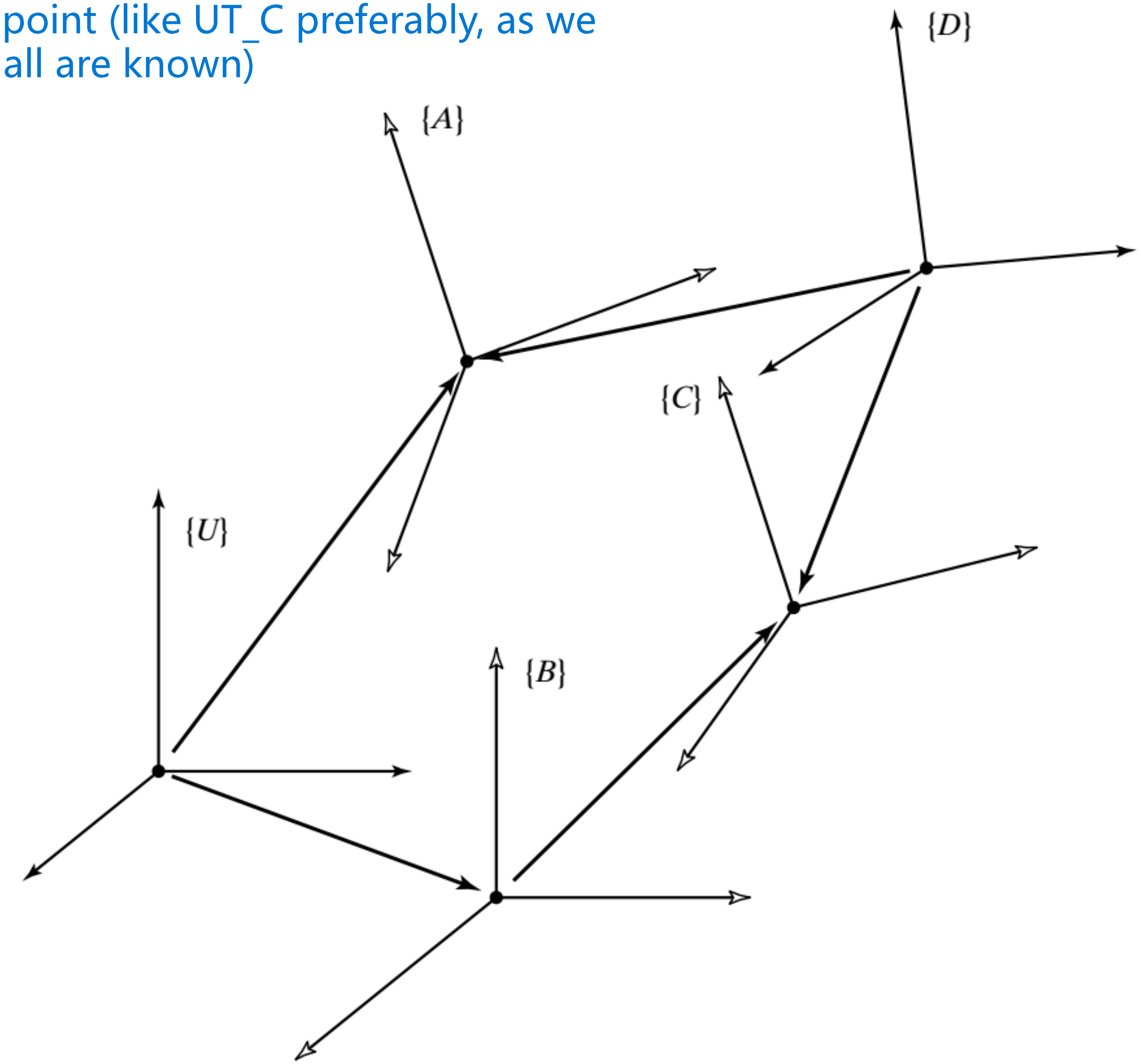
$$ {}^B_C T = {}^U_B T^{-1} \, {}^U_A T \, {}^A_D T \, {}^C_D T^{-1} $$

Transform Equations

if we want to calculate UT_A we choose 2 paths to an unknown point (like UT_C preferably, as we don't know its position so it could be cancelled out or known if all are known)
and apply equations of the path

$$\boxed{\begin{gathered} {}^{U}_{C}T = {}^{U}_{A}T \; {}^{D}_{A}T^{-1} \; {}^{D}_{C}T \\[2em] {}^{U}_{C}T = {}^{U}_{B}T \; {}^{B}_{C}T. \end{gathered}}$$

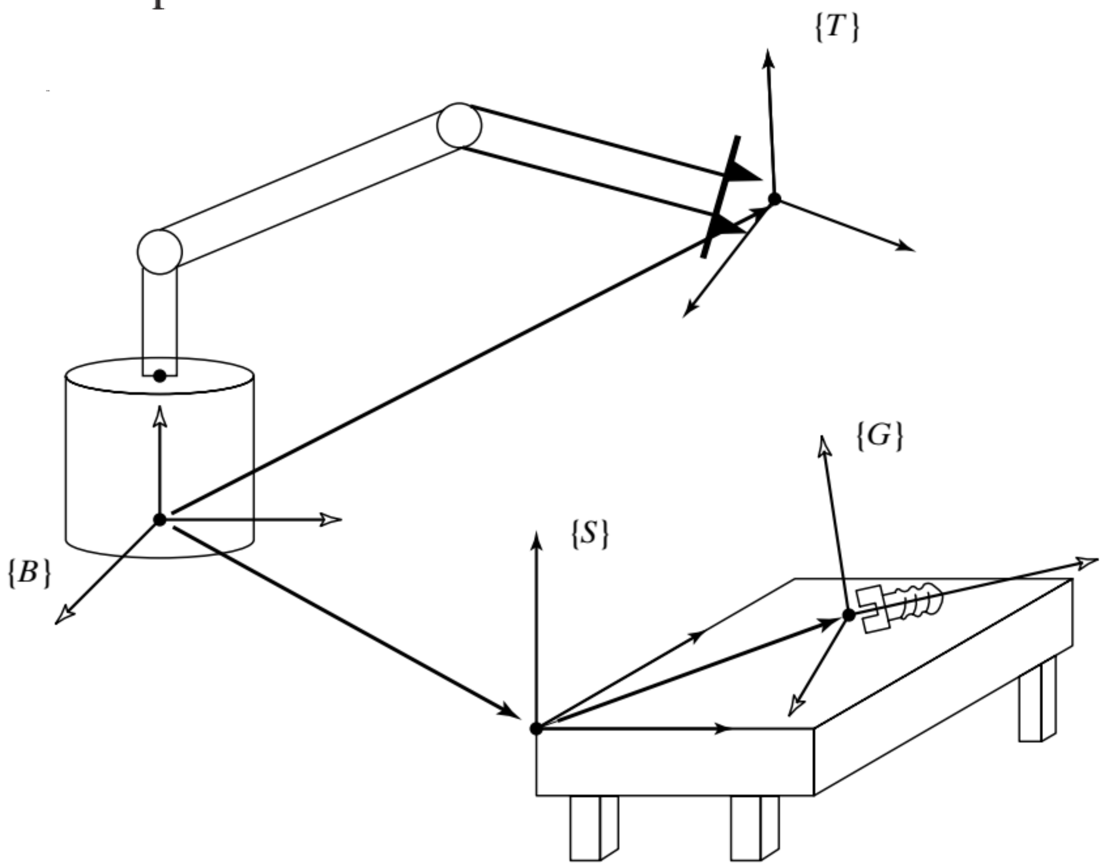
$$\boxed{{}^{U}_{A}T = {}^{U}_{B}T \; {}^{B}_{C}T \; {}^{D}_{C}T^{-1} \; {}^{D}_{A}T.}$$



Example

Assume that we know the transform ${}^{B}_{T}T$ in Fig. 2.16, which describes the frame at the manipulator's fingertips $\{T\}$ relative to the base of the manipulator, $\{B\}$, that we know where the tabletop is located in space relative to the manipulator's base (because we have a description of the frame $\{S\}$ that is attached to the table as shown, ${}^{B}_{S}T$), and that we know the location of the frame attached to the bolt lying on the table relative to the table frame—that is, ${}^{S}_{G}T$. Calculate the position and orientation of the bolt relative to the manipulator's hand, ${}^{T}_{G}T$.

$${}^{B}_{G}T = {}^{B}_{S}T \; {}^{S}_{G}T \qquad\qquad {}^{B}_{G}T = {}^{B}_{T}T \; {}^{T}_{G}T$$



Equating above two to get the bolt frame
Relative to the tool frame

$${}^{T}_{G}T = {}^{B}_{T}T^{-1} \; {}^{B}_{S}T \; {}^{S}_{G}T.$$

More on Representation of Orientation

- Orientation is by giving a 3 × 3 rotation matrix.
- Rotation matrices are special in that all columns are mutually orthogonal and have unit magnitude
- Determinant of Rotation matrices is always equal to +1 (Proper Orthonormal)

Cayley's formula for orthonormal matrices

For any proper orthonormal matrix R, there exists a skew-symmetric matrix *S* such that

$$R = (I_3 - S)^{-1}(I_3 + S)$$

$$S = \begin{bmatrix} 0 & -s_z & s_y \\ s_z & 0 & -s_x \\ -s_y & s_x & 0 \end{bmatrix}.$$

More on Representation of Orientation

- Therefore any 3 × 3 rotation matrix can be specified by just three parameters

- This means there are six constraints on the nine elements of a rotation matrix

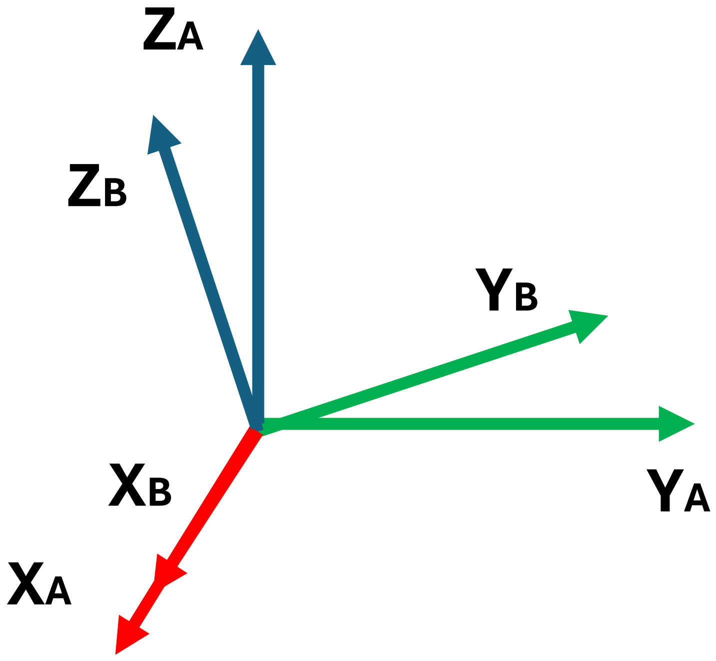
- A human operator at a computer terminal who wishes to type in the specification of the desired orientation of a robot's hand would have a hard time inputting a nine-element matrix with orthonormal columns

- A representation that requires only three numbers would be simpler

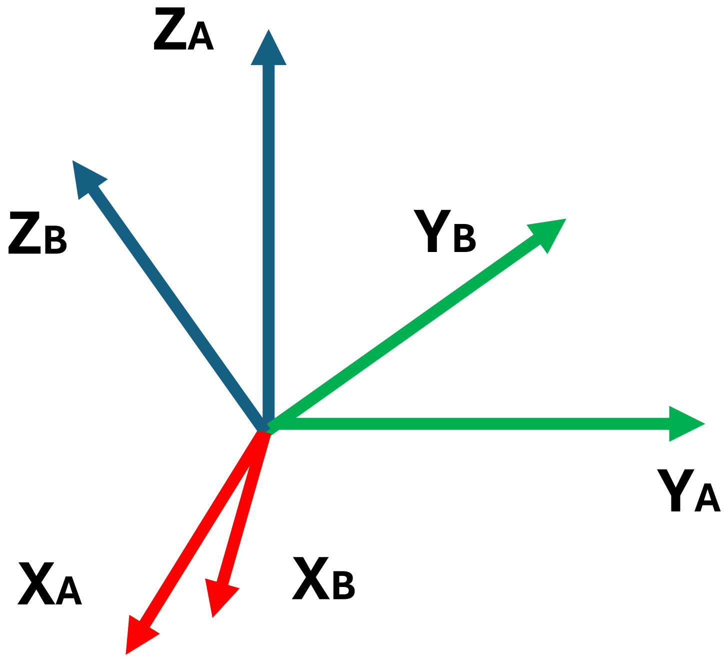
$$
\begin{aligned}
|\hat{X}| &= 1, \\
|\hat{Y}| &= 1, \\
|\hat{Z}| &= 1, \\
\hat{X} \cdot \hat{Y} &= 0, \\
\hat{X} \cdot \hat{Z} &= 0, \\
\hat{Y} \cdot \hat{Z} &= 0.
\end{aligned}
$$

X-Y-Z Fixed Angles

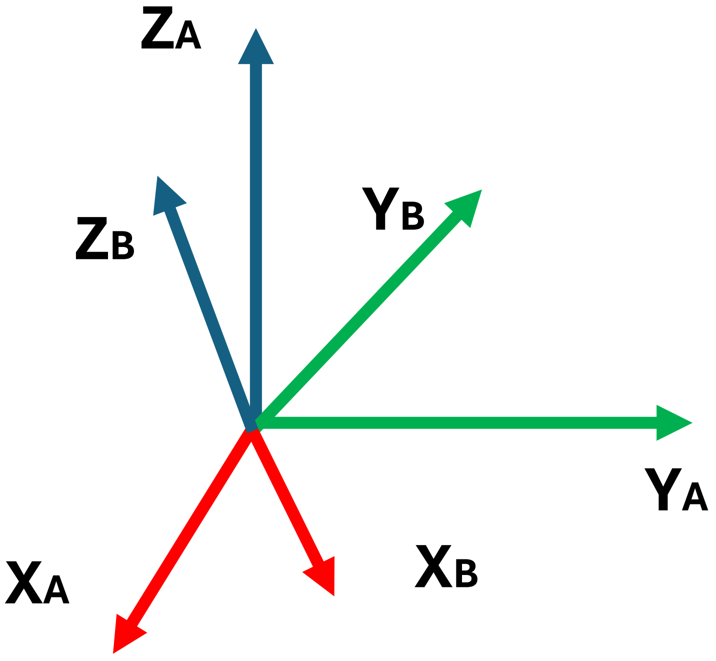
Start with the frame coincident with a known reference frame $\{A\}$. Rotate $\{B\}$ first about $\hat{X}_A$ by an angle γ, then about $\hat{Y}_A$ by an angle β, and, finally, about $\hat{Z}_A$ by an angle α.



Rotation about X_A-axis



Rotation about Y_A-axis



Rotation about Z_A-axis

X-Y-Z Fixed Angles

Start with the frame coincident with a known reference frame $\{A\}$. Rotate $\{B\}$ first about $\hat{X}_A$ by an angle γ, then about $\hat{Y}_A$ by an angle β, and, finally, about $\hat{Z}_A$ by an angle α.

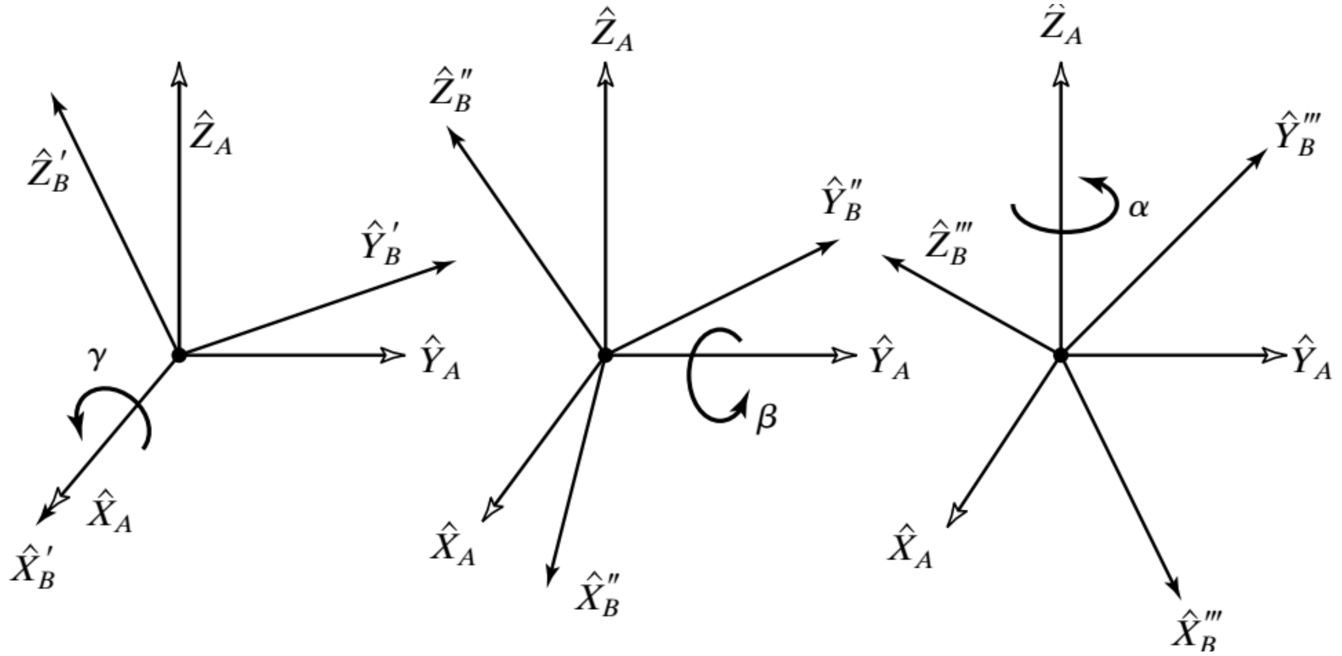


FIGURE 2.17: X–Y–Z fixed angles. Rotations are performed in the order $R_X(\gamma)$, $R_Y(\beta)$, $R_Z(\alpha)$.

X-Y-Z Fixed Angles

2nd 1st

$$ {}^{A}_{B}R_{XYZ}(\gamma, \beta, \alpha) = R_Z(\alpha)R_Y(\beta)R_X(\gamma) $$

$$ = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} $$

$$ {}^{A}_{B}R_{XYZ}(\gamma, \beta, \alpha) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix} $$

The inverse problem, that of extracting equivalent X–Y–Z fixed angles from a rotation matrix, is often of interest

X-Y-Z Fixed Angles

$$ {}^A_B R_{XYZ}(\gamma, \beta, \alpha) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

$$ {}^A_B R_{XYZ}(\gamma, \beta, \alpha) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix} $$

$$ \beta = \text{Atan2}(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}), $$

$$ \alpha = \text{Atan2}(r_{21}/c\beta, r_{11}/c\beta), $$

$$ \gamma = \text{Atan2}(r_{32}/c\beta, r_{33}/c\beta), $$

X-Y-Z Fixed Angles

Although a second solution exists, by using the positive square root in the formula for β, we always compute the single solution for which $-90.0° \leq \beta \leq 90.0°$.

If $\beta = 90.0°$, then a solution can be calculated to be

$$\beta = 90.0°,$$

$$\alpha = 0.0,$$

$$\gamma = \mathrm{Atan2}(r_{12}, r_{22}).$$

If $\beta = -90.0°$, then a solution can be calculated to be

$$\beta = -90.0°,$$

$$\alpha = 0.0,$$

$$\gamma = -\mathrm{Atan2}(r_{12}, r_{22}).$$

[3] Atan2(y, x) computes $\tan^{-1}(\frac{y}{x})$ but uses the signs of both x and y to identify the quadrant in which the resulting angle lies. For example, Atan $2(-2.0, -2.0) = -135°$, whereas Atan $2(2.0, 2.0) = 45°$, a distinction which would be lost with a single-argument arc tangent function. We are frequently computing angles that can range over a full 360°, so we will make use of the Atan2 function regularly. Note that Atan2 becomes undefined when both arguments are zero. It is sometimes called a "4-quadrant arc tangent," and some programming-language libraries have it predefined.

Frame {B} was initially coincident with {A}. We then rotated {B} about Z_A-axis by 60. Then we rotated about Y_A-axis by 45 and finally we rotated it about X_A-axis by 30. Calculate the resultant rotation matrix.

$$R_X(30) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C30 & -S30 \\ 0 & S30 & C30 \end{bmatrix}$$

$$R_Y(45) = \begin{bmatrix} C45 & 0 & S45 \\ 0 & 1 & 0 \\ -S45 & 0 & C45 \end{bmatrix}$$

$$R_Z(60) = \begin{bmatrix} C60 & -S60 & 0 \\ S60 & C60 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^{A}_{B}R_{ZYX}(60,45,30) = R_X(30) \,.\, R_Y(45) \,.\, R_Z(60) =$$

$$\begin{bmatrix} 0.35 & -0.61 & 0.71 \\ 0.93 & 0.13 & -0.35 \\ 0.13 & 0.78 & 0.61 \end{bmatrix}$$

Find the Fixed angles of rotation (γ, β, α) for the following XYZ rotation matrix.

$$
{}^{A}_{B}R = \begin{bmatrix} 0.9077 & -0.2946 & 0.2989 \\ 0.3304 & 0.9408 & -0.0760 \\ -0.2588 & 0.1677 & 0.9513 \end{bmatrix}
$$

$$
\beta = \text{Atan2}(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}),
$$

$$
\alpha = \text{Atan2}(r_{21}/c\beta, r_{11}/c\beta),
$$

$$
\gamma = \text{Atan2}(r_{32}/c\beta, r_{33}/c\beta),
$$

β = 15°

α = 20°

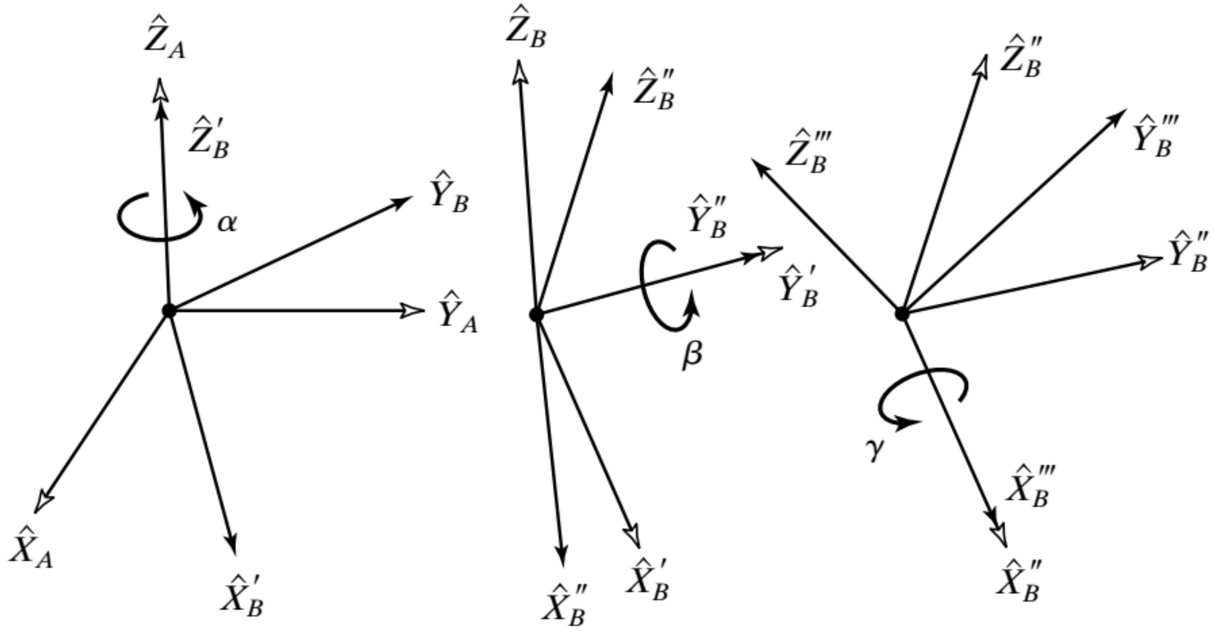
γ = 10°

Z-Y-X Euler Angles

Start with the frame coincident with a known frame $\{A\}$. Rotate $\{B\}$ first about $\hat{Z}_B$ by an angle α, then about $\hat{Y}_B$ by an angle β, and, finally, about $\hat{X}_B$ by an angle γ.

- In this representation, each rotation i performed about an axis of the moving system {B} rather than one of the fixed reference {A}

- Such sets of three rotations are calle **Euler Angles**



Z-Y-X Euler Angles

$$
{}^{A}_{B}R_{Z'Y'X'} = R_Z(\alpha)R_Y(\beta)R_X(\gamma)
$$

$$
= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}
$$

$$
{}^{A}_{B}R_{Z'Y'X'}(\alpha, \beta, \gamma) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}
$$

Relations for finding *Z-Y-X* Euler angles from a given rotation matrix are exactly same as the *X-Y-Z* Fixed Angles (as the rotation matrix is same)

Z–Y–Z Euler angles

Start with the frame coincident with a known frame $\{A\}$. Rotate $\{B\}$ first about $\hat{Z}_B$ by an angle α, then about $\hat{Y}_B$ by an angle β, and, finally, about Z_b by an angle γ.

$$
{}^{A}_{B}R_{Z'Y'Z'}(\alpha, \beta, \gamma) = \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}
$$

if $\sin\beta \neq 0$, it follows that

$$
\beta = \mathrm{Atan2}\left(\sqrt{r_{31}^2 + r_{32}^2}, r_{33}\right),
$$

$$
\alpha = \mathrm{Atan2}(r_{23}/s\beta, r_{13}/s\beta),
$$

$$
\gamma = \mathrm{Atan2}(r_{32}/s\beta, -r_{31}/s\beta).
$$

Z–Y–Z Euler angles

If $\beta = 0.0$, then a solution can be calculated to be

$$\beta = 0.0,$$
$$\alpha = 0.0,$$
$$\gamma = \text{Atan2}(-r_{12}, r_{11}).$$

If $\beta = 180.0°$, then a solution can be calculated to be

$$\beta = 180.0°,$$
$$\alpha = 0.0,$$
$$\gamma = \text{Atan2}(r_{12}, -r_{11}).$$