Digital Signal Processing (EC 335)

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Lecture 5

Lecture Targets

- Fourier series representation of DT periodic signals (DTFS)
- Properties of DTFS

CTFS

Synthesis Equation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Analysis Equation

$$a_k = \frac{1}{T}\int_0^T x(t).e^{-jk\omega_0 t}\,dt$$

DTFS

Synthesis Equation

$$x[n] = \sum_{k=0}^{N-1} c[k] e^{\frac{jk2\pi n}{N}}$$ — for sinosoidal (sin, cos)

The frequency domain signal *c*[*k*] of the DT signal *x*[*n*] will be also periodic.

Analysis Equation

$$c[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-\frac{jk2\pi n}{N}}$$

$$c[k+N] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-\frac{j2\pi(k+N)n}{N}}$$

$$\overset{1}{\cos 2\pi n} - j\,\overset{0}{\sin 2\pi n}$$

Periodicity property of *c*[*k*]

$$c[k+N] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-\frac{j2\pi kn}{N}} . \underbrace{e^{-\frac{j2\pi \cancel{N} n}{\cancel{N}}}}_{1}$$

$$c[k+N] = c[k]$$

DTFS

Significance of the term $\boxed{\dfrac{2\pi k}{N}}$ in DTFS equations

FS coefficient of DT signal will repeat itself after N sample, where N is the Period of the signal.

$\frac{2\pi}{N}$ rad/sample

What will be the gap (Frequency spacing) between the harmonics????

Can we find the location of each sample???

Properties of DTFS

Linearity

$$x_1[n] \to c_1(k)$$
$$x_2[n] \to c_2(k)$$
$$\alpha x_1[n] + \beta x_2[n] \to \alpha c_1(k) + \beta c_2(k)$$

Time-shifting

$$x[n] \to c(k)$$
$$x[n-n_o] \to e^{-\frac{j2\pi k n_o}{N}} \times c(k)$$
$$\therefore \frac{2\pi k}{N} = \omega_o$$

Time-scaling

$$x[n] \to c(k)$$
$$x[\alpha n] \to \alpha \times c(k)$$

Convolution property

$$x_1[n] * x_2[n] \to N \times c_1(k) \times c_2(k)$$

Periodic convolution

$$x_1[n] * x_2[n] = \sum_{m=0}^{N-1} x_1[m].x_2[n-m]$$

$$x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k].x_2[n-k]$$

What is the difference b/w periodic and Linear convolution?????

Properties of DTFS

Multiplication Property

$$x_1[n] \times x_2[n] \rightarrow c_1(k) * c_2(k)$$

$* \rightarrow$ Periodic or linear?

Time reversal

$$x[n] \rightarrow c(k)$$

$$x[-n] \rightarrow c(-k)$$

Frequency-shifting

$$x[n] \rightarrow c(k)$$

$$e^{j\frac{2\pi k_o n}{N}} \times x[n] \rightarrow c(k - k_o)$$

Example

Find the FS coefficient of

$$x[n]=\cos\frac{\pi}{3}n$$

Sol :

$$c[k]=\frac{1}{N}\sum_{n=0}^{N-1}x[n]e^{-\frac{jk2\pi n}{N}}$$

$$x[n]=\sum_{n=0}^{N-1}c[k]e^{\frac{jk2\pi n}{N}}$$

$$x[n]=\cos\frac{\pi}{3}n$$

$$\omega_o=\frac{\pi}{3}$$

$$N=\left(\frac{2\pi}{\omega_o}\right)m, \text{m = smallest integer}$$

$$N=\left(\frac{2\pi\times 3}{\pi}\right)\times m\Rightarrow N=6$$

$$x[n]=\sum_{k=-3}^{2}c[k]e^{\frac{j2\pi kn}{6}\rightarrow\frac{j\pi kn}{3}}$$

$$x[n]=c[-3]e^{j\frac{\pi}{3}n(-3)}+c[-2]e^{j\frac{\pi}{3}n(-2)}+c[-1]e^{j\frac{\pi}{3}n(-1)}$$

$$+c[0]+c[1]e^{j\frac{\pi}{3}n(1)}+c[2]e^{j\frac{\pi}{3}n(2)}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$x[n]=\cos\frac{\pi}{3}n=\frac{1}{2}e^{j\frac{\pi}{3}n}+\frac{1}{2}e^{-j\frac{\pi}{3}n}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Comparing (1) and (2), we have

$$c[-1]=\frac{1}{2},c[1]=\frac{1}{2}$$

$$c[-3]=c[-2]=c[0]=c[2]=0$$

$$\sin\theta=\frac{e^{j\theta}}{2j}-\frac{e^{-j\theta}}{2j}$$

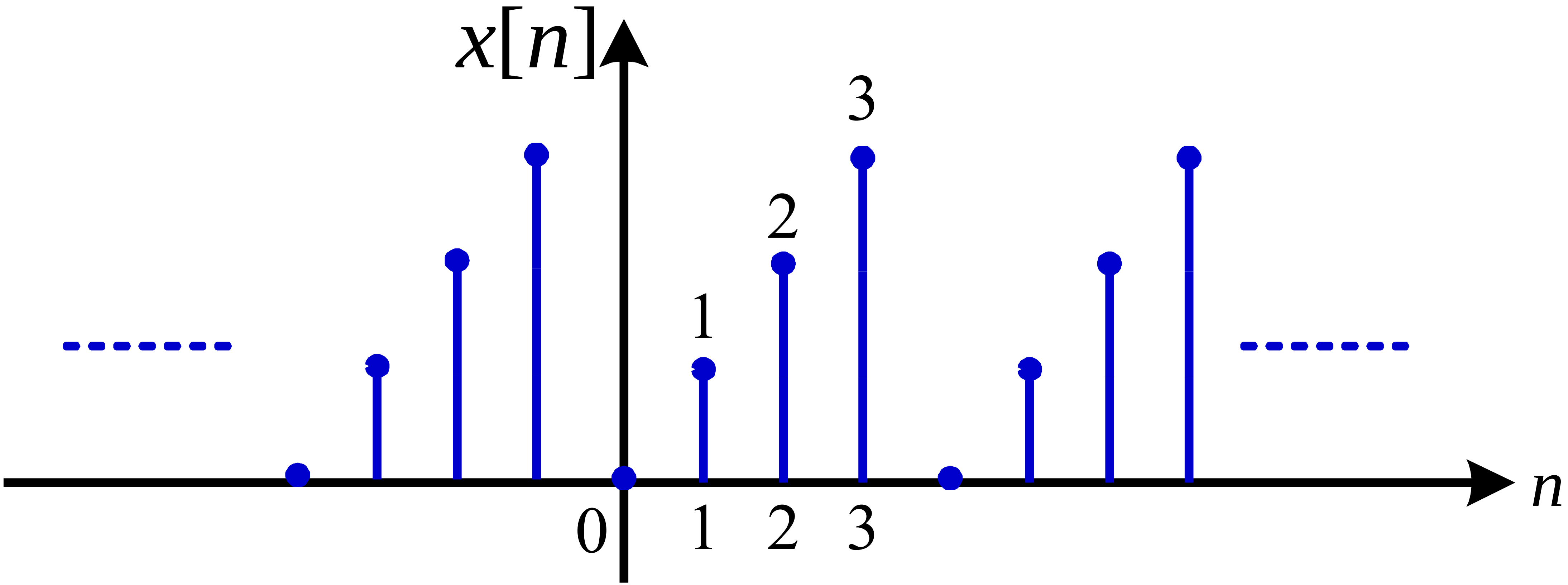
Plot $c[k]$?????

Find FS representation of

$$x[n]=\sin\left(\frac{2\pi}{7}\right)n$$

Example

Find the FS coefficient of the following signal



Example

$$C[k]=\frac{1}{N}\sum_{n=0}^{N-1}x[n]e^{-j\frac{k2\pi}{N}n}$$

$$N=4$$

$$C[k]=\frac{1}{4}\sum_{n=0}^{N-1}x[n]e^{-j\frac{k2\pi}{4}n}$$

$$C[k]=\frac{1}{4}\left\{\begin{array}{l}\underbrace{x[0]}_{0}e^{-j\frac{k\pi}{2}(0)}+\underbrace{x[1]}_{1}e^{-j\frac{k\pi}{2}(1)}\\ \underbrace{x[2]}_{2}e^{-j\frac{k\pi}{2}(2)}+\underbrace{x[3]}_{3}e^{-j\frac{k\pi}{2}(3)}\end{array}\right\}$$

$$C[k]=\frac{1}{4}\left\{\left(e^{-j\frac{\pi}{2}}\right)^k+2\left(e^{-j\pi}\right)^k+3\left(e^{-j\frac{3\pi}{2}}\right)^k\right\}$$

$$\left(e^{-j\frac{\pi}{2}}\right)^k=(-j)^k,\ \left(e^{-j\pi}\right)^k=(-1)^k,\ \left(e^{-j\frac{3\pi}{2}}\right)^k=(j)^k$$

$$C[k]=\frac{1}{4}\left\{(-j)^k+2(-1)^k+3(j)^k\right\}$$

Find C[3], C[7], C[-1]

Questions

Find the FS coefficient of the periodic impulse train

$$x[n] = \sum_{l=-\infty}^{\infty} \delta(n - Nl)$$

N should be strictly positive integer

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} \delta(n) e^{-j(2\pi/N)kn}.$$

From the sifting property of the *impulse*, we have

$$c_k = \frac{1}{N} e^{0}$$

$$c_k = \frac{1}{N} \quad \text{for all } k \in \mathbb{Z}.$$

Example

Find the FS coefficient of $$x[n]=1+\sin\left(\frac{2\pi}{N}n\right)+3\cos\left(\frac{2\pi}{N}n\right)+2\cos\left(\frac{4\pi}{N}n+\frac{\pi}{2}\right)$$

$$\omega_o=\frac{2\pi}{N}$$ m

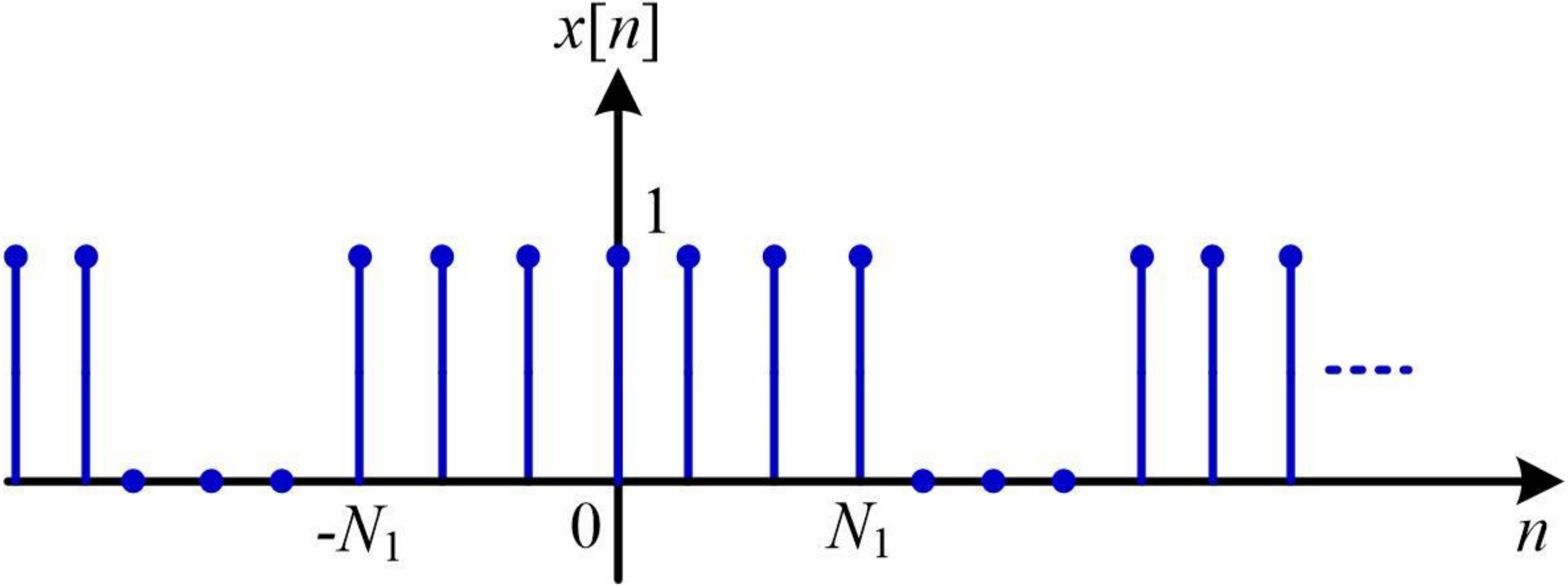
$$x[n]=1+\frac{1}{2j}e^{j\frac{2\pi}{N}n}-\frac{1}{2j}e^{-j\frac{2\pi}{N}n}+3.\frac{1}{2}e^{j\frac{2\pi}{N}n}+3.\frac{1}{2}e^{-j\frac{2\pi}{N}n}+2.\frac{1}{2}e^{j\left(\frac{4\pi}{N}n+\frac{\pi}{2}\right)}+2.\frac{1}{2}e^{-j\left(\frac{4\pi}{N}n+\frac{\pi}{2}\right)}$$

$$x[n]=1+e^{j\frac{2\pi}{N}n}\left[\frac{3}{2}+\frac{1}{2j}\right]+e^{-j\frac{2\pi}{N}n}\left[\frac{3}{2}-\frac{1}{2j}\right]+\underbrace{e^{j\frac{\pi}{2}}}_{j}.e^{j\frac{4\pi}{N}n}+\underbrace{e^{-j\frac{\pi}{2}}}_{-j}.e^{-j\frac{4\pi}{N}n}$$

$$c[0]=1,c[1]=\left[\frac{3}{2}+\frac{1}{2j}\right],c[-1]=\left[\frac{3}{2}-\frac{1}{2j}\right],c[2]=j,c[-2]=-j$$

Example

Find the FS coefficient of



Example

$n = -N_1 \rightarrow \boxed{m = n + N_1}$

$$c[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j\frac{2\pi k}{N}n}$$

$$c[k] = \frac{1}{N} \sum_{n=-N_1}^{N_1} \underbrace{x[n]}_{1} e^{-j\frac{2\pi k}{N}n}$$

$$c[k] = \frac{1}{N} \sum_{m=0}^{2N_1} e^{-j\frac{2\pi k(m-N_1)}{N}}$$

$$c[k] = \frac{1}{N} \sum_{m=0}^{2N_1} \underbrace{e^{-j\frac{2\pi k N_1}{N}}}_{\text{constant}} . e^{-j\frac{2\pi k m}{N}}$$

$$c[k] = \frac{e^{j\frac{2\pi k N_1}{N}}}{N} \sum_{m=0}^{2N_1} e^{-j\frac{2\pi k m}{N}}$$

$$c[k] = \frac{e^{j\frac{2\pi k N_1}{N}}}{N} \boxed{\sum_{m=0}^{2N_1} \left(e^{-j\frac{2\pi k}{N}} \right)^m}$$

$$\sum_{n=0}^{N} (a)^n = \frac{1-a^{N+1}}{1-a}$$

Example

Find the FS representation of the 5-periodic sequence given by

$$x[n] = \begin{cases} -\dfrac{1}{2}, n = -1 \\ 1, n = 0 \\ \dfrac{1}{2}, n = 1 \\ 0, n \in \{-2, 2\} \end{cases}$$

$$c_k = \frac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{-j(2\pi/N)kn}$$

$$c_k = \frac{1}{5} \sum_{n=-2}^{2} x(n) e^{-j(2\pi/5)kn}$$

$$= \frac{1}{5} \left(-\frac{1}{2} e^{-j(2\pi/5)(-1)k} + e^{-j(2\pi/5)(0)k} + \frac{1}{2} e^{-j(2\pi/5)(1)k} \right)$$

$$c_k = \frac{1}{5} \left(-\frac{1}{2} e^{j(2\pi/5)k} + 1 + \frac{1}{2} e^{-j(2\pi/5)k} \right)$$

$$= \frac{1}{5} \left(1 - \frac{1}{2} \left(e^{j(2\pi/5)k} - e^{-j(2\pi/5)k} \right) \right)$$

$$= \frac{1}{5} \left(1 - \frac{1}{2} \left[2j \sin\left(\frac{2\pi}{5} k \right) \right] \right)$$

$$= \frac{1}{5} \left[1 - j \sin\left(\frac{2\pi}{5} k \right) \right].$$

Practice Questions

Find the FS representation of the 8-periodic sequence given by

$$x[n] = \begin{cases} 1, n \in \{0..3\} \\ 0, n \in \{4..7\} \end{cases}$$

Find the FS representation of the 8-periodic sequence given by

$$x[n] = \begin{cases} n, n \in \{-2..2\} \\ 0, n = -3 \text{ or } n \in \{3..4\} \end{cases}$$

Convergence of DTFS

Since the analysis and synthesis equations for DTFS involve only finite sums (as opposed to infinite series), convergence is not a significant concern.

If an N-periodic sequence is bounded (i.e., is finite in value), its Fourier-series coefficient sequence will exist and be bounded and the Fourier series analysis and synthesis equations must converge.