Digital Signal Processing (EC 335)

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Lecture 2

Lecture Targets

- Time Domain Analysis
- Time invariant System
- Linear System
- Linear Time Invariant (LTI) System

System

What is system????

Any operation on the signal is system, for example

$$y[n]=2x[n]$$

Some operation on the signal is natural, which are undesired. Example is communication between cellphone and BTS. How?

What to do next to process the undesired Signal???

Linear and Non-Linear Systems

Linear System: Systems following the **Principle of Superposition.**

Superposition: 1) Homogeneity $x[n] \to y[n] \Rightarrow \alpha\, x[n] \to \alpha\, y[n]$

2) Additive if $x_1[n] \to y_1[n]$ **and** $x_2[n] \to y_2[n]$

$$x_1[n] + x_2[n] \to y_1[n] + y_2[n]$$

If

$$\boxed{\alpha x_1[n] + \beta x_2[n] \to \alpha y_1[n] + \beta y_2[n]}$$

Linear System

Why are we interested in Linear Systems?

99% of DSP operations are for linear systems

What, if the system is non-linear? Like parabola etc.

We will approximate the system is linear for small interval of time. However, we will use adaptive signal processing schemes. In this course, the target will be linear systems only.

Linear and Non-linear Systems

Check the following system: Linear or Non-linear

A. Squaring System: $y[n]=x^2[n]$

1. **Check using general principle of superposition**
2. **Check by using the following signals**

   $x_1[n]=\{1, -1\}$ **and** $x_2[n]=\{-1, 1\}$

B. System that add previous sample to the current sample

Check by using the following signals

$x_1[n]=\{1, -1\}$ **and** $x_2[n]=\{-1, 1\}$

Time-invariant and Time-variant Systems

Time Invariant System: If the **time shift** in the **input of the system** results an **identical time shift** in **output of the system** without changing the nature of the output.

To check if the system is time-invariant??

1. **Find the delayed response of the system …….. $y\,[n-n_0]$**
2. **Find $\Gamma[x[n-n_0]]=y[n,n_0]$….Response of the system for delayed input**

IF

$y[n,n_0]=y\,[n-n_0]$, Then the System is Time-invariant

Time-invariant and Time-variant Systems

check if $y[n] = x^2[n]$ is time - invariant?

Step 1: Find the dealyed response of the system.

Hint: Replace n by $n\text{-}n_0$

$$y[n\text{-}n_0] = \left[x^2[n\text{-}n_0]\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Step 2: Find the response of the system for delayed input.

Hint: Write the original function, and then give one time shift

$$y[n, n_0] = x^2[n - n_0] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$y[n\text{-}n_0] = y[n, n_0] \Rightarrow \text{Time invariant system}$$

check if $y[n] = x[n] + x[-n]$ is time - invariant?

Time-invariant and Time-variant Systems

To check if $y[n] = x[3n]$ is time - invariant?

$$y[n-n_0] = x[3(n-n_0)]\text{..........................}(1)$$

$$y[n,n_0] = x[3n-n_0]\text{..............................}(2)$$

$y[n-n_0]$ is not equal to $y[n,n_0]$

$\Rightarrow$ Time variant system

Discrete time Convolution

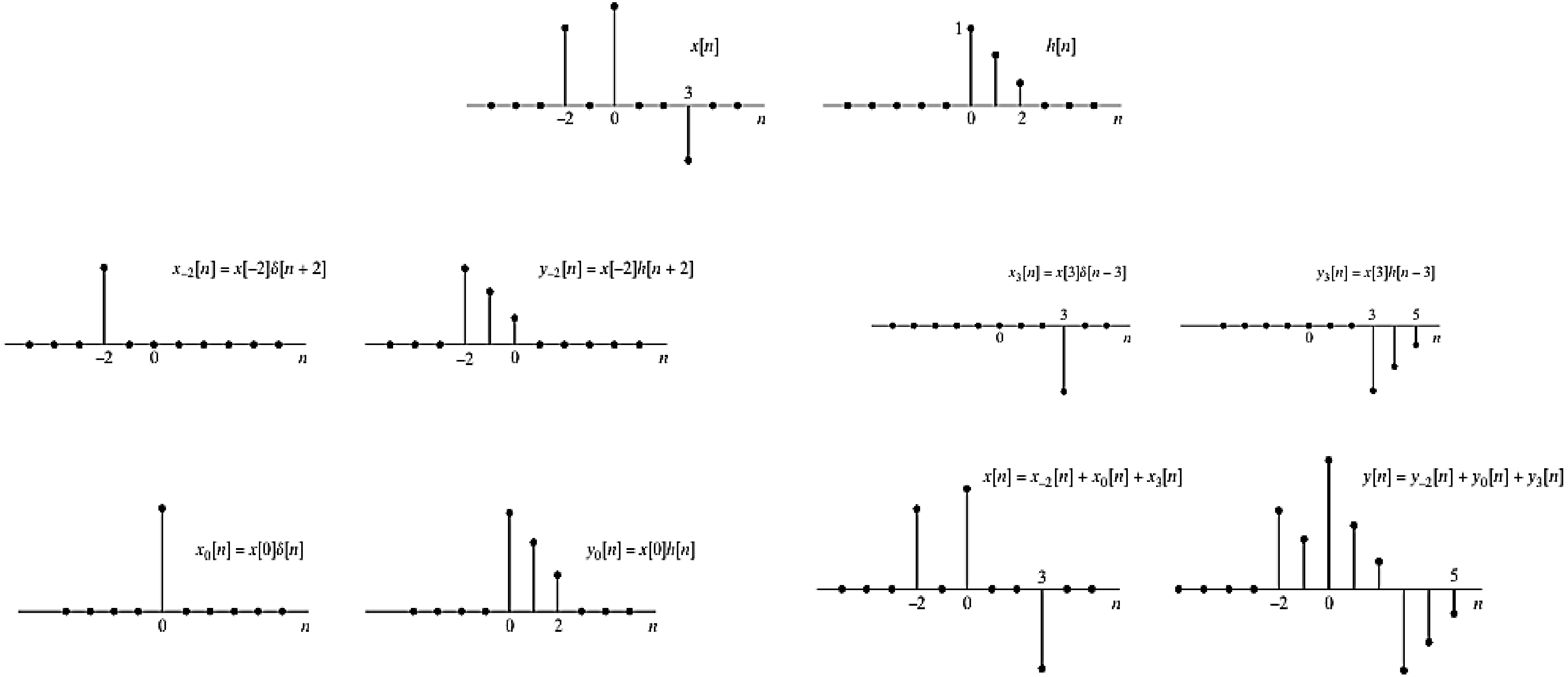
Three types of problems in DSP

1. **Identification/Estimation problem:** Where we identify the impulse response of the system. For example Channel estimation.
2. **Design problem:** Where we design impulse response for the system, for example, we have noise in the system or multipath fading problem, and we wish to design an impulse response for such system.
3. **Implementation problem:** If we know impulse response, then how you compute the output. Using convolution for example.

$$y[n] = x[n] * h[n]$$

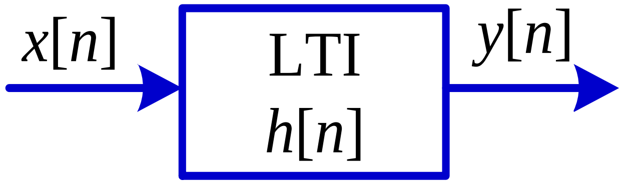
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Discrete time Convolution



Discrete time Convolution

Convolution of DT signals: Convolution Summation



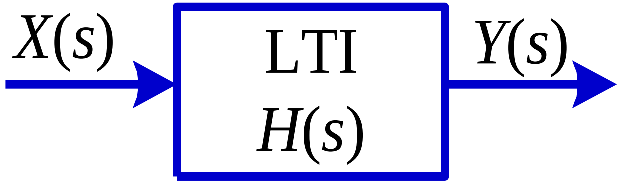
$$y[n] = x[n] * h[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

All properties of CT convolution holds true for DT convolution

Laplace Transform

Convolution is used in Time domain



$$Y(s) = X(s)H(s)$$

Inverse Laplace Transform
Can find $y(t)$

Laplace transform is used in Frequency domain

Discrete time Convolution

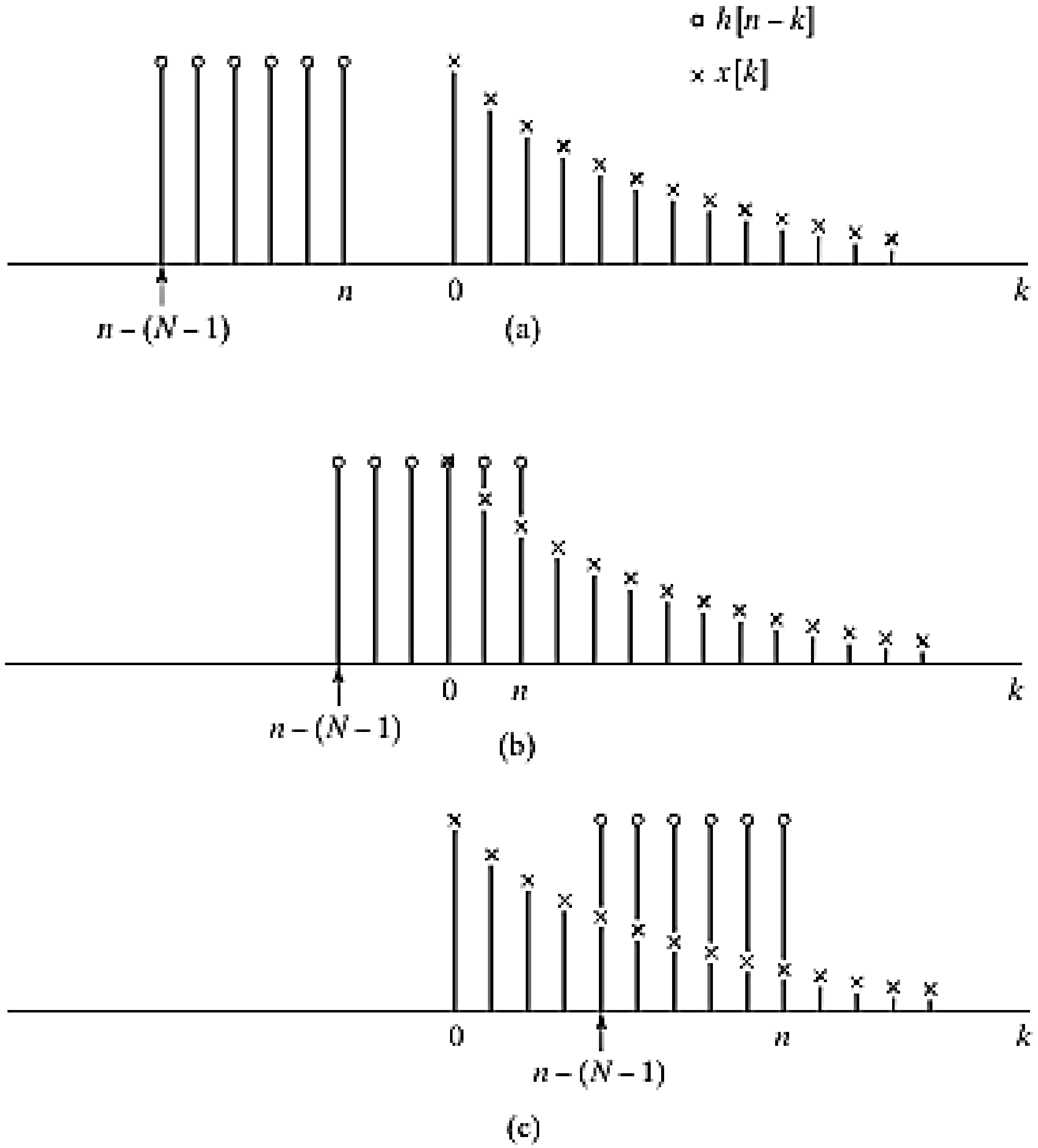
$$h[n]=\begin{cases}1, & 0\le n\le N-1\\ 0, & \text{otherwise}\end{cases}$$

$$x[n]=\begin{cases}a^n, & n\ge 0\\ 0, & \text{otherwise}\end{cases}$$

or

$$x[n]=a^n u[n]$$

$$y[n]=0, n<0$$



$$y[n]=\sum_{k=0}^{n}a^k \text{ for } 0\le n\le N-1$$

$$y[n]=\frac{1-a^{n+1}}{1-a}$$

$$y[n]=\sum_{k=n-N+1}^{n}a^k \text{ for } N-1<n$$

$$y[n]=\frac{a^{n-N+1}-a^{n+1}}{1-a}$$

$$y[n]=a^{n-N+1}\left(\frac{1-a^N}{1-a}\right)$$

$$\therefore \sum_{k=N_1}^{N_2}a^k=\frac{a^{N_1}-a^{N_2+1}}{1-a}$$

DT Convolution

$x[n]=u[n], x[n]=u[n]$

Find the output $y[n]$?

As we know that

$$y[n]=x[n]*h[n]$$

$$y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]$$

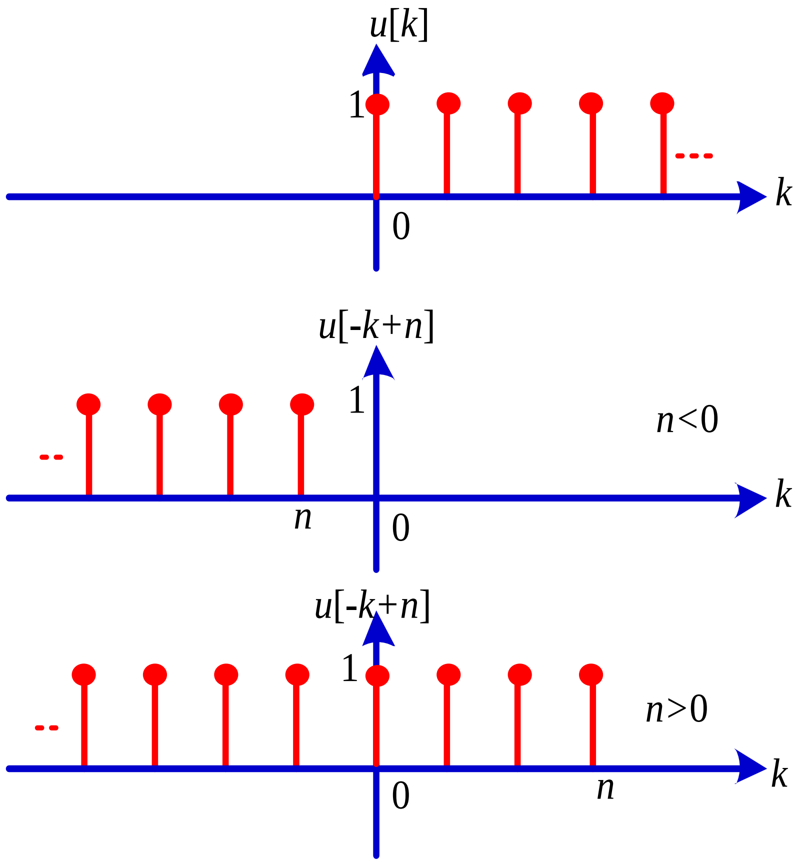
Given

$$x[n]=h[n]\Rightarrow x[k]=u[k]$$

$$h[n]=u[k]\Rightarrow h[n-k]=u[-k+n]$$

$$y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]$$

Refer to Graph



$$\boxed{\begin{aligned}u[n]*u[n]&=(n+1)u[n]\\&=r(n+1):\text{Draw } r(n+1)\end{aligned}}$$

$$y[k]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]$$

$$y[k]=0, k<0$$

$$\left(\begin{array}{l}\text{As there is no common area}\\ \text{between } u[k] \text{ and } u[n-k]\end{array}\right)$$

for $k\geq 0$

$$y[k]=\sum_{k=0}^{n}\underbrace{u[k]}_{1}\underbrace{u[n-k]}_{1}$$

$$y[k]=\sum_{k=0}^{n}1=n+1$$

$$y[k]=n+1, k\geq 0$$

$$y[k]=\begin{cases}0, & k<0\\ n+1, & k\geq 0\end{cases}$$

Discrete time Convolution

$$x[n] * \delta[n] = x[n]$$

$$x[n-1] * \delta[n+3] = x[n+2]$$

$$\underbrace{\{u[n-1] - u[n-2]\}}_{\delta[n-1]} * \underbrace{\{u[n-1] - u[n-2]\}}_{\delta[n-1]}$$

$$\Rightarrow \delta[n-1] * \delta[n-1] = \delta[n-2]$$

$$x[n] = (0.5)^n u[n]$$

$$h[n] = \delta[n+2] + 0.5\delta[n+1]$$

$$y[n] = x[n] * h[n]$$

$$y[n] = x[n] * \left[\delta[n+2] + 0.5\delta[n+1]\right]$$

$$y[n] = x[n] * \delta[n+2] + x[n] * 0.5\delta[n+1]$$

$$y[n] = x[n+2] + 0.5x[n+1]$$

$$y[n] = (0.5)^{n+2} u[n+2] + 0.5.(0.5)^{n+1} u[n+1]$$

$$y[n] = (0.5)^{n+2} \{u[n+2] + u[n+1]\}$$

Discrete time Convolution

$$x[n]=\left(\frac{1}{2}\right)^n u[n]$$

$$h[n]=\delta[n]-\frac{1}{2}\delta[n-1]$$

$$y[n]=x[n]*h[n]$$

$$y[n]=x[n]*\left\{\delta[n]-\frac{1}{2}\delta[n-1]\right\}$$

$$y[n]=x[n]*\delta[n]-x[n]*\frac{1}{2}\delta[n-1]$$

$$y[n]=x[n]-\frac{1}{2}x[n-1]$$

$$y[n]=\left(\frac{1}{2}\right)^n u[n]-\frac{1}{2}.\left(\frac{1}{2}\right)^{n-1} u[n-1]$$

$$y[n]=\left(\frac{1}{2}\right)^n \{u[n]-u[n-1]\}$$

$$y[n]=\left(\frac{1}{2}\right)^n \{\delta[n]\}$$

$$\therefore\ x[n].\delta[n]=x[0].\delta[n]$$

$$y[n]=\left(\frac{1}{2}\right)^0 \{\delta[n]\}$$

$$y[n]=\delta[n]$$

Let three LTI systems are connected in cascade with their impulse responses are given as

$$h_1[n]=\left(\frac{1}{2}\right)^n u[n],h_2[n]=u[n+3],h_3[n]=\delta[n]-\delta[n-1]$$

Find the overall Impulse response of the System?

$$h[n]=h_1[n]*h_2[n]*h_3[n]$$

$$h[n]=h_1[n]*\{h_2[n]*h_3[n]\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$h_2[n]*h_3[n]=u[n+3]*\{\delta[n]-\delta[n-1]\}$$

$$h_2[n]*h_3[n]=\{u[n+3]*\delta[n]\}-\{u[n+3]*\delta[n-1]\}$$

$$h_2[n]*h_3[n]=u[n+3]-u[n+2]=\delta[n+3]\ldots\ldots\ldots\ldots(2)$$

$$h[n]=\left(\frac{1}{2}\right)^n u[n]*\delta[n+3]$$

$$h[n]=\left(\frac{1}{2}\right)^{n+3} u[n+3]$$

Convolution of Sequences

$$x[n]=\{1,2,3\}$$

$$h[n]=\{1,2,3\}$$

$$x[n]=\delta[n]+2\delta[n-1]+3\delta[n-2]$$

$$h[n]=\delta[n]+2\delta[n-1]+3\delta[n-2]$$

$$y[n]=x[n]*h[n]$$

Method -1: Distributive property of Impulse

$$y[n]=\{\delta[n]*\delta[n]\}+\{\delta[n]*2\delta[n-1]\}+\{\delta[n]*3\delta[n-2]\}$$

$$+\{2\delta[n-1]*\delta[n]\}+\{2\delta[n-1]*2\delta[n-1]\}+\{2\delta[n-1]*3\delta[n-2]\}$$

$$+\{3\delta[n-2]*\delta[n]\}+\{3\delta[n-2]*2\delta[n-1]\}+\{3\delta[n-2]*3\delta[n-2]\}$$

$$y[n]=\delta[n]+2\delta[n-1]+3\delta[n-2]+2\delta[n-1]+4\delta[n-2]+6\delta[n-3]$$

$$+3\delta[n-2]+6\delta[n-3]+9\delta[n-4]$$

$$y[n]=\delta[n]+4\delta[n-1]+10\delta[n-2]+\ldots\ldots$$

Convolution of Sequences

$x[n]=\{1,2,3\}$

$h[n]=\{1,2,3\}$

Find $y[n]$?

Method – 2

$$y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]$$

$$y[0]=\sum_{k=-\infty}^{\infty}x[k]h[-k]$$

$x[k]$={0,0,1, 2, 3}

$\uparrow$

$x[-k]$={3,2,1, 0, 0}

$\uparrow$

$y[0]$=1

$$y[1]=\sum_{k=-\infty}^{\infty}x[k]h[1-k]$$

$h[1-k]=h[-(k-1)]$

Delay by 1

$x[k]$={0, 1, 2, 3,0}

$\uparrow$

$x[1-k]$={3,2,1, 0, 0}

$\uparrow$

$y[1]$=0+2+2+0=4

$$y[2]=\sum_{k=-\infty}^{\infty}x[k]h[2-k]$$

$x[k]$={1, 2, 3}

$\uparrow$

$x[2-k]$={3,2,1}

$\uparrow$

$y[1]$=3+4+3=10

$$y[-1]=\sum_{k=-\infty}^{\infty}x[k]h[-1-k]$$

$x[k]$={0,0,0,1, 2, 3}

$\uparrow$

$x[-1-k]$={3,2,1,0,0,0}

$\uparrow$

$y[-1]$=0

Convolution of Sequences

Tabular method

$h[n]$ \ $x[n]$	1	2	3
1	1	2	3
2	2	4	6
3	3	6	9

$$y[n] = \{1,4,10,12,9\}$$

Convolution of Sequences

Product method

$h[n]$ \ $x[n]$	1	2	3		
1	1	2	3	0	0
2	0	2	4	6	0
3	0	0	3	6	9
$y[n]$	1	4	10	12	9

$y[n]=\{1, 4, 10, 12, 9\}$

Convolution of Sequences

$x[n]=\{1, \underset{\uparrow}{2}, 3,4\}$

$h[n]=\{1,2,\underset{\uparrow}{2}, 1\}$

$y[n]=\{1, 4, 9, \underset{\uparrow}{15}, 16,11,4\}$

$h[n]$ \ $x[n]$	1	2	3	4			
1	1	2	3	4	0	0	0
2	0	2	4	6	8	0	0
2	0	0	2	4	6	8	0
1	0	0	0	1	2	3	4
$y[n]$	1	4	9	15	16	11	4

N= $y[n]$ sample position at origin

N_1= Number of samples at negative values of $x[n]$

N_2= Number of samples at negative values of $h[n]$

$N=N_1+N_2$

Practice Questions

$$x[n]=\{\underset{\uparrow}{1}, 2, 3, 4, 5\}$$

$$h[n]=\{\underset{\uparrow}{1}, 2, 3, 3, 2, 1\}$$

$$x[n]=\{\underset{\uparrow}{1}, 2, 3, 4, 5, 6\}$$

$$h[n]=\{\underset{\uparrow}{2}, -4, 6, -8\}$$

$$x[n]=\{-1/2, \underset{\uparrow}{2}, 1/3, 3/2\}$$

$$h[n]=\{1, -1/2, \underset{\uparrow}{2/3}\}$$

$$x[n]=u[n]-3u[n-2]+2u[n+4]$$

$$h[n]=u[n+1]-u[n-8]$$

Convolution of Sequences

Given $y[n] = x[n] * h[n]$

$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$

$x[n] \to$ Causal

If $y[0] = 1, y[1] = \frac{1}{2}$

Find $x[1]$??

Sol :

$$h[n] = \left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{16}, \ldots\right\}$$

$$x[n] = \{x[0], x[1], x[2], x[3], \ldots\}$$

Use any of the above method

to find $x[1]$

From the table

$x[0] = y[0] = 1$

$x[1] + \frac{1}{2}x[0] = y[1]$

Putting values, we have

$x[1] + \frac{1}{2}(1) = \frac{1}{2}$

$\Rightarrow x[1] = 0$

$h[n]$ \ $x[n]$	$x[0]$	$x[1]$	$x[2]$	-------	
1	$x[0]$	$x[1]$	$x[2]$		
1/2	0	$1/2x[0]$	$1/2x[1]$	$1/2x[2]$	
1/4	0	0	$1/4x[0]$	$1/4x[1]$	$1/4x[2]$
⋮	⋮	⋮			
$y[n]$	$y[0]$	$y[1]$			

Convolution of Sequences

Given that both $x[n]$ and $h[n]$ are non - zero

for n $= 0, 1, 2$ and is zero otherwise

$x[0] = 1, x[1] = 2, x[2] = 1$

$h[0] = 1$

$y[1] = 3, y[2] = 4$

Find $10y[3] + y[4]$??

Sol :

Use tabular method.

$h[n]$ \ $x[n]$	1	2	1
1	1	2	1
$h[1]$	$h[1]$	$2h[1]$	$h[1]$
$h[2]$	$h[2]$	$2h[2]$	$h[2]$

Given

$y[1] = 3, y[1] = 2 + h[1] \Rightarrow h[1] = 1$

$y[2] = 4, y[2] = h[2] + 2h[1] + 1$

$\Rightarrow h[2] = 1$

$y[3] = 2h[2] + h[1] \Rightarrow y[3] = 3$

$y[4] = h[2] \Rightarrow y[4] = 1$

Now

$10y[3] + y[4] = 31$

From the table, we have

$y[n] = \{1, 2 + h[1], h[2] + 2h[1] + 1, 2h[2] + h[1], h[2]\}$

DT Convolution

$$x[n]=\alpha^n u[n],h[n]=\left(-\frac{1}{2}\right)^n u[n-4]$$

$$y[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$y[n]=\sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$x[n-k]=\alpha^{n-k}u[n-k]$$

$$h[k]=\left(-\frac{1}{2}\right)^k u[k-4]$$

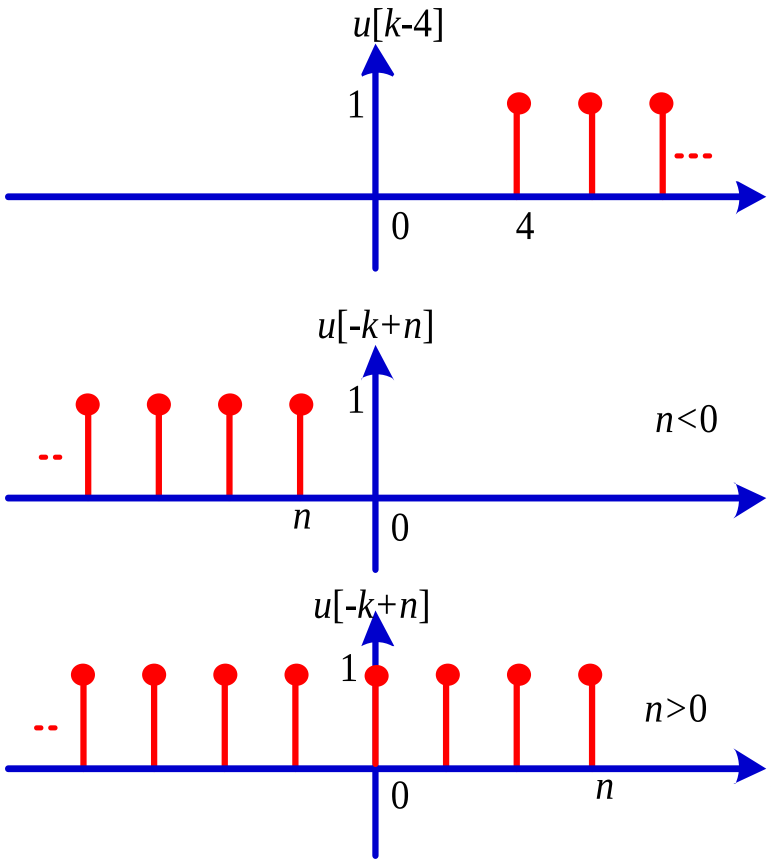
$$y[n]=\sum_{k=-\infty}^{\infty}\left(-\frac{1}{2}\right)^k \underbrace{u[k-4]}_{n\geq 4}.\alpha^{n-k}\underbrace{u[n-k]}_{-\infty<k<n}$$

$$y[n]=0,n<0$$

$$y[n]=\sum_{k=4}^{n}\left(-\frac{1}{2}\right)^k (\alpha)^n (\alpha)^{-k}$$

$$y[n]=(\alpha)^n\sum_{k=4}^{n}\left(-\frac{1}{2}\right)^k (\alpha)^{-k}$$

$$y[n]=(\alpha)^n\sum_{k=4}^{n}\left(-\frac{1}{2\alpha}\right)^k$$



$$y[n]=(\alpha)^n\sum_{k=4}^{n}\left(-\frac{1}{2\alpha}\right)^k$$

Let $k\text{-}4=m$

$$k=4\Rightarrow m=0$$

$$k=n\Rightarrow m=n-4$$

$$y[n]=(\alpha)^n\sum_{m=0}^{n-4}\left(-\frac{1}{2\alpha}\right)^{m+4}$$

$$y[n]=(\alpha)^n\left(-\frac{1}{2\alpha}\right)^4\sum_{m=0}^{n-4}\left(-\frac{1}{2\alpha}\right)^m$$

$$\therefore \sum_{n=0}^{N}(a)^n=\frac{1-a^{N+1}}{1-a}$$

$$y[n]=(\alpha)^{n-4}\left(-\frac{1}{2}\right)^4\left[\frac{1-\left(-\frac{1}{2\alpha}\right)^{n-4+1}}{1-\left(-\frac{1}{2\alpha}\right)}\right]$$

DT Convolution

If the response of the linear shift invariance system to a unit step (i.e. the step response) is

$$s[n] = n\left(\frac{1}{2}\right)^n u[n]$$

Find the unit sample (impulse response), *h*[*n*]?

As we know that

$\delta[n] = u[n] - u[n-1]$

So, $h[n]$ is related to $s[n]$

$$h[n] = s[n] - s[n-1]$$

$$h[n] = n\left(\frac{1}{2}\right)^n u[n] - (n-1)\left(\frac{1}{2}\right)^{n-1} u[n-1]$$

$$h[n] = n\left(\frac{1}{2}\right)^n u[n] - 2(n-1)\left(\frac{1}{2}\right)^n u[n-1]$$

DT Convolution

$$x[n] = e^{j2\pi n}$$

$$h[n] = \begin{cases} -2\sqrt{2} & \text{for } n = -1, +1 \\ 4\sqrt{2} & \text{for } n = -2, +2 \\ 0, \text{otherwise} \end{cases}$$

$$\text{Find } \; y[n] = x[n] * h[n]???$$

Hint: Plot *h*[*n*] first