



Digital Signal Processing (EC 335)

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Lecture 6

Lecture Targets

- Discrete time Fourier transform (DTFT)
- Properties of DTFT

DTFT

- ❑ Periodicity in time domain results in sampling in Frequency domain
- ❑ Sampling in time-domain results in periodicity in Frequency domain
- ❑ Sampling is used to reflect the Discrete nature of the signal here

Complex exponential DT signal

$$e^{j\omega_o n}$$

$$\omega_o \rightarrow \omega_o + 2\pi$$

$$e^{j(\omega_o + 2\pi)n} = e^{j\omega_o n} \times \underbrace{e^{j2\pi n}}_1$$

$$e^{j(\omega_o + 2\pi)n} = e^{j\omega_o n}$$

All Complex exponential DT signals are periodic in frequency with a period of 2π . DT complex exponential signals will be identical at ω , $\omega \pm 2\pi$, $\omega \pm 4\pi$,...

DTFT

$$C.T.F.T \Rightarrow X(j\omega) = \int_{-\infty}^{\infty} x(t).e^{-j\omega t} dt$$

$$D.T.F.T \Rightarrow X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n].e^{-j\omega n} \text{ (Analysis Equation)}$$

$$I.C.T.F.T \Rightarrow x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega).e^{j\omega t} d\omega$$

$$I.D.T.F.T \Rightarrow X[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}).e^{j\omega n} d\omega \text{ (Synthesis Equation)}$$

DTFT

Important Formulas for Summation

$$(i). \sum_{n=0}^{\infty} (a)^n = \frac{1}{1-a} \therefore \text{GP}$$

$$(ii). \sum_{n=0}^N (a)^n = \frac{1-a^{N+1}}{1-a}$$

$$(iii). \sum_{n=0}^N (1)^n = N+1, \sum_{n=a}^b (1)^n = (b-a)+1,$$

$$(iv). \sum_{n=0}^N n = \frac{N(N+1)}{2}$$

$$(v). \sum_{n=0}^N n^2 = \frac{N(N+1)(2N+1)}{6}$$

$$(vi). \sum_{n=0}^N n^3 = \left(\frac{N(N+1)}{2} \right)^2$$

DTFT

$$x[n] = a^n u[n], 0 < a < 1$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n \cdot e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} \left(a e^{-j\omega} \right)^n$$

$$\therefore \sum_{n=0}^{\infty} (a)^n = \frac{1}{1-a}$$

$$X(e^{j\omega}) = \frac{1}{1 - a e^{-j\omega}}$$

$$x[n] = \delta[n]$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n] \cdot e^{-j\omega n}$$

$$X(e^{j\omega}) = e^{-j\omega n} \Big|_{n=0}$$

$$X(e^{j\omega}) = 1$$

$$x[n] = a^n u[n] + a^{-n} u[-n-1]$$

$$x[n] = a^{-n} u[-n], 0 < a < 1$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^0 a^{-n} \cdot e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^0 \left(a e^{j\omega} \right)^{-n}$$

Let

$$m = -n, n \rightarrow -\infty \Rightarrow m \rightarrow \infty$$

$$n \rightarrow 0 \Rightarrow m \rightarrow 0$$

$$X(e^{j\omega}) = \sum_{m=0}^{\infty} \left(a e^{j\omega} \right)^m$$

$$\therefore \sum_{n=0}^{\infty} (a)^n = \frac{1}{1-a}$$

$$X(e^{j\omega}) = \frac{1}{1 - a e^{j\omega}}$$

DTFT

Two important results

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Handwritten notes: $e^{-j\pi} = -1$ (green), $x[n](-1)^n$; $\omega = \pi$ (blue)

The value of DTFT at $\omega = 0$

$$X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n]$$

$x[n] = \{1, 2, 3, 4, 7\}$, origin at 3

The value of DTFT at $(\omega = 0) = ???$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega$$

at $n = 0$

$$x[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) d\omega$$

$$\Rightarrow \int_{-\pi}^{\pi} X(e^{j\omega}) d\omega = 2\pi \cdot x[0]$$

DTFT

$x[0] \times 2$

If $x[n] = \{-4, 3, -2, 3, -4\}$ origin at -2, find



(i). $x[e^{j0}]$

(ii). $x[e^{j\pi}]$

(iii). $\int_{-\pi}^{\pi} x[e^{j\omega}] d\omega$ -4π

Properties of DTFT

Linearity

$$x_1[n] \rightarrow X_1(e^{j\omega})$$

$$x_2[n] \rightarrow X_2(e^{j\omega})$$

$$\alpha x_1[n] + \beta x_2[n] \rightarrow \alpha X_1(e^{j\omega}) + \beta X_2(e^{j\omega})$$

Time – shifting

$$x[n] \rightarrow X(e^{j\omega})$$

$$x[n \pm n_o] \rightarrow (e^{\pm j\omega n_o}) X(e^{j\omega})$$

$$\delta[n] \rightarrow 1$$

$$\delta[n-1] \rightarrow e^{-j\omega} \times 1 = e^{-j\omega}$$

$$\delta[n+1] \rightarrow e^{j\omega} \times 1 = e^{j\omega}$$

Time – Reversal

$$x[n] \rightarrow X(e^{j\omega})$$

$$x[-n] \rightarrow X(e^{-j\omega})$$

$$a^n u[n] \rightarrow \frac{1}{1 - ae^{-j\omega}}$$

$$a^{-n} u[-n] \rightarrow \frac{1}{1 - ae^{j\omega}}$$

Differencing Propoerty

$$x[n] \rightarrow X(e^{j\omega})$$

$$x[n] - x[n - n_o] \rightarrow [1 - e^{-j\omega n_o}] X(e^{j\omega})$$

First difference

$$x[n] - x[n-1] \rightarrow [1 - e^{-j\omega}] X(e^{j\omega})$$

Accumulation Propoerty

$$x[n] \rightarrow X(e^{j\omega})$$

$$\sum_{k=-\infty}^n x[k] \rightarrow \left[\frac{1}{1 - e^{-j\omega}} \right] X(e^{j\omega})$$

$$x_1[n] \rightarrow X_1(e^{j\omega}), x_2[n] \rightarrow X_2(e^{j\omega})$$

$$x_1[n] * x_2[n] \rightarrow X_1(e^{j\omega}) \times X_2(e^{j\omega})$$

$$x_1[n] = \left(\frac{1}{4} \right)^n u[n], x_2[n] = \left(\frac{1}{3} \right)^n u[n]$$

Find DTFT of $x[n] = x_1[n] * x_2[n]$

$$\therefore a^n u[n] = \frac{1}{1 - ae^{j\omega}}$$

Expansion Property

$$x[n] \rightarrow X(e^{j\omega})$$

$$x\left[\frac{n}{k}\right] \rightarrow X(e^{jk\omega})$$

Transform will be

periodic with period $\frac{2\pi}{k}$

$$x[n] = \{1, 2, 3\}$$

$$x[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2]$$

$$X(e^{j\omega}) = 1 + 2e^{-j\omega} + 3e^{-j2\omega} \dots\dots\dots(1)$$

$$x_1[n] = x\left[\frac{n}{2}\right] = \{1, 0, 2, 0, 3\}$$

$$x_1[n] = \delta[n] + 2\delta[n-2] + 3\delta[n-4]$$

$$X_1(e^{j\omega}) = 1 + 2e^{-j2\omega} + 3e^{-j4\omega}$$

Using Expansion Property

Replace $\omega \rightarrow k\omega \therefore k = 2 \Rightarrow \omega = 2\omega$

in Eq(1)

$$X_1(e^{j\omega}) = 1 + 2e^{-j2\omega} + 3e^{-j4\omega}$$

Discrete Fourier Transform (DFT)

□ DTFT is used to evaluate the frequency response of DT signal.

$$x[n] \rightarrow X(e^{j\omega})$$

□ The signal in Frequency domain (applying DTFT) will always be periodic. Why????

□ DFT is also used to evaluate the frequency response of DT signal.

□ What is the difference between DTFT and DFT then???

□ The transform from DTFT is of continuous nature. We cannot use continuous signals in digital signal processors. So, we might use DFT instead of DTFT if the application is DSP.