EE-379

LINEAR CONTROL SYSTEMS

Enrollment Codes for LMS:
Syn-A = 653729801
Syn-B = 025416389
Syn-C = 025791683

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Course Books

Textbook:

- **Design of Feedback Control Systems** (4^{th} Edition) by R.T. Stefani, C.J. Savant, B. Shahian, G.H. Hostetter

Reference Books:

- **Feedback Control Systems** (4^{th} Edition) by Charles L. Phillips and Royce D. Harbor
- **Control Systems Engineering** (7^{th} Edition) by Norman S. Nise
- **Modern Control Systems** (12^{th} Edition) by Richard C. Dorf and Robert H. Bishop
- **Modern Control Engineering** (5^{th} Edition) by Katsuhiko Ogata

- Students are encouraged to purchase the above-mentioned textbook and read it thoroughly.

Course Learning Outcomes and Evaluation

Learning Outcomes

CLO	Outcomes	Level	PLO
1.	The student will have the ability to analyse complex linear systems (single and multivariable, external and internal representation). This includes their stability, controller design and evaluation of closed loop response.	C4	**2**
2.	Apply mathematical/analytical skills, to analyse system designs using root-locus, frequency response, and state-space methods.	C4	**3**
3.	Ability to design controllers for linear discrete-time control systems so that their performance meets specified design criteria.	C5	**3**
4.	Knowledge and understanding to provide a basis or opportunity for originality in developing and applying control concepts in the context of research.	C2	**2**
5.	The student will be able to use modern analytical tools, test equipment and computer aided design to assemble different types of control systems and measure performance.	P4	**5**

Grading **Credit Hour: 3-1 (Theory-Lab)**

The grade of this course will be the weighted average of the following activities.

Theory 75%		**Lab 25%**	
Activities	**%**	**Activities**	**%**
Midterm Exam	30%	Midterm Exam	15%
Quiz	10%	Lab Tasks	45%
Assignment	10%	Lab Manual	05%
Project	10%	Lab Project	20%
Final Exam	40%	Lab Final Exam	15%

- **Project Submission Deadline = 14th Week**

Introduction to EE-379

1. Control system basics

- ✓ Open loop control, closed-loop control, input/output relation, process, plant, compensator, practical examples.
- ✓ Modeling of physical systems (electrical, mechanical, and electromechanical) in the form of a Transfer function.
- ✓ Control system representation in the form of signal flow graphs and block diagrams

2. Responses of the control systems

- ✓ Transient and steady-state response of the system
- ✓ Quantification of the stability of the system

3. Performance specifications of the control systems

- ✓ Steady state error, sensitivity, disturbances

4. Time and frequency domain analysis

- ✓ Analysis of control systems in time and frequency domains

5. Modern control systems

- ✓ State space representation, design, and analysis

Course Contents

Sr. no.	Description of the topics	Tentative Schedule
1	Basic concepts, Modelling, Transfer function	Week 1
2	Transfer functions, block diagrams and signal flow graphs	Week 2
3	Response of first and second order systems	Week 3
4	BIBO stability and Routh-Hurwitz Criterion	Week 4
5	Performance specifications of LTI control systems Part 1	Week 5
6	Performance specifications of LTI control systems Part 2	Week 6
7	Root Locus analysis	Week 7
8	Root Locus design	Week 8
9	**Midterm Exam**	**Week 9**
10	Frequency response analysis	Week 10
11	Frequency response design	Week 11
12	Nyquist Plots	Week 12
13	State Space analysis	Week 13
14	State space design	Week 14-16
15	**Final Exam**	**Week 17**

Classification of Control Systems Types

1. Based on Feedback

Open Loop Control Systems

- No feedback mechanism
- Output has no effect on the control action
- Simple and inexpensive, but less accurate
- Example: Electric kettle, washing machines

Closed Loop Control Systems

- Feedback is present
- Control action is dependent on the output
- More accurate but complex and costly
- Example: Air conditioning systems, cruise control in cars

2. Based on Time Response

Time-Invariant Control Systems

- The behavior of the system does not change with time
- Fixed characteristics
- Example: Electrical circuits with constant resistances, inductances, and capacitances

Time-Variant Control Systems

- The system parameters vary with time
- Changing characteristics
- Example: Moving vehicles, where speed, direction, etc., change over time.

Classification of Control Systems Types

3. Based on Nature of the Input/Output

Linear Control Systems

- System equations follow linearity principles (superposition and homogeneity)
- Easier to analyze and solve
- Example: Electrical circuits with linear components.

Non-Linear Control Systems

- The system does not follow linearity principles
- Harder to analyze but more realistic as many real-world systems are nonlinear
- Example: Chemical reactors, certain mechanical systems.

4. Based on the Control Strategy

Proportional Control (P)

- Control output is proportional to the error signal
- Example: Basic temperature control systems.

Proportional-Integral (PI) Control

- Combines proportional control and integral of the error
- Reduces steady-state error
- Example: Speed control in motors.

Proportional-Integral-Derivative (PID) Control

- Includes proportional, integral, and derivative control
- Fast, stable, and widely used in industrial systems
- Example: Robotic arm control

Classification of Control Systems Types

5. Based on the Signal Type

Continuous-Time Control Systems
- Signals are continuous over time
- Mathematical models are represented by differential equations
- Example: Analog circuits.

Discrete-Time Control Systems
- Signals are discrete (sampled at intervals).
- Mathematical models are represented by difference equations
- Example: Digital systems, sampled-data control systems.

Digital Control Systems
- Use digital signals for control
- Often employ microcontrollers or digital computers
- Example: Modern electronic devices and appliances.

6. Based on the Input-Output

SISO (Single Input Single Output) Control Systems
- Only one input and one output
- Example: A simple thermostat

MIMO (Multiple Input Multiple Output) Control Systems
- Multiple inputs and multiple outputs
- More complex but can handle larger systems
- Example: Aircraft flight control systems.

7. Adaptive Control Systems

- These systems adjust themselves automatically to changing conditions or parameters
- Example: Autonomous vehicles that adapt to road conditions.

Control Systems Theory

Classical Control Theory

- Classical control theory focuses primarily on the **analysis** and **design** of **linear time-invariant (LTI) systems**.
- It uses tools like **transfer functions** and **frequency response methods** to analyze system stability, transient behavior, and steady-state response

Key Characteristics

1. **Single-Input Single-Output (SISO) systems:** Primarily deals with systems with one input and one output.
2. **Time-domain analysis:** Uses techniques like step response, impulse response, and error analysis to study system performance.
3. **Frequency-domain analysis:** Uses tools like Bode plots, Nyquist plots, and Nichols charts to assess system stability and performance.
4. **Laplace transforms and transfer functions:** Widely used for modeling and analysis.
5. **Stability analysis:** Relies on criteria like the Routh-Hurwitz criterion, Nyquist criterion, and root locus techniques.

Classification in Classical Control Theory:

- **Open-Loop vs Closed-Loop:** Classical theory applies to both open-loop and closed-loop systems, though closed-loop (feedback) systems are more common
- **Compensator design:** Use of compensators (lead, lag, or lead-lag) to improve system performance
- **PID Controllers:** Proportional-Integral-Derivative (PID) control is the most widely used control approach in classical theory.

Limitations of Classical Control Theory:

1. Primarily restricted to **SISO systems** and linear models
2. Difficult to extend to complex, multi-variable (MIMO) systems
3. Does not easily handle non-linear systems or systems with time-varying parameters.

Control Systems Theory

Modern Control Theory

- Modern control theory expands the analysis and design techniques to include **multi-variable systems** and more complex models, including **non-linear** and **time-varying systems**.
- It primarily focuses on the **state-space representation** and matrix-based approaches for control system design.

Key Characteristics

1. **Multi-Input Multi-Output (MIMO) systems:** Capable of handling multiple inputs and outputs, unlike classical control.
2. **State-space representation:** A major shift from transfer functions to state-space models, where systems are described using vectors and matrices.
3. **Time-domain analysis:** More focus on time-domain methods using state-space models.
4. **Optimal control and robust control:** Modern control theory includes advanced techniques for optimizing system performance and **ensuring robustness under uncertainties**.
5. **Stability analysis:** **Lyapunov's stability criterion** is often used for analyzing system stability, especially for non-linear systems.
6. **Controllability and Observability:** Introduces the concepts of **controllability** (whether the system can be driven to a desired state) and **observability** (whether the system states can be inferred from outputs).

Control Systems Theory

Classification in Modern Control Theory

- **Linear vs Non-Linear Systems:**
  - **Linear systems** can be analyzed using matrix algebra and linear differential equations.
  - **Non-linear systems** require advanced methods, like linearization or Lyapunov functions, to analyze stability and control.
- **Time-Invariant vs Time-Variant Systems**
  - Time-invariant systems have fixed parameters, while time-variant systems have parameters that change over time.
- **Deterministic vs Stochastic Systems**
  - **Deterministic** systems are **predictable**, with no randomness.
  - **Stochastic** systems incorporate **randomness** and require probabilistic methods for analysis and control (e.g., Kalman filters)..
- **Adaptive and Robust Control Systems**
  - **Adaptive Control**: The control system adjusts its parameters in real time to adapt to changing system dynamics or uncertainties.
  - **Robust Control**: Ensures the system performs well even in the presence of model inaccuracies or external disturbances.

Techniques in Modern Control Theory

1. **Pole Placement:** A method used to design control systems by specifying the desired closed-loop pole locations using state feedback.
2. **Linear Quadratic Regulator (LQR):** An optimal control strategy that minimizes a cost function based on state and control effort.
3. **Kalman Filter:** A recursive algorithm used for estimating the state of a system in the presence of noise (widely used in modern control).

Extension of Modern Control Theory-Advance Control Systems:

1. H-Infinity ($H\infty$) Control
2. Sliding Mode Control (SMC)
3. Model Predictive Control (MPC)
4. Robust Control
5. Adaptive Control
6. Optimal Control (including Linear Quadratic Regulator - LQR)

Advantages of Modern Control Theory:

1. Applicable to MIMO systems and systems with complex interactions.
2. Can handle non-linear and time-variant systems.
3. Allows for optimal and robust control design to achieve desired performance under varying conditions.

Comparison: Classical vs. Modern Control Systems Theory

Feature	Classical Control Theory	Modern Control Theory
System Type	Primarily **SISO**, Linear	**MIMO**, Linear & Non-linear
Mathematical Tools	Transfer Functions, Bode/Nyquist Plots	State-Space Representation, Matrix Algebra
Time Domain vs Frequency Domain	Focuses on both time and frequency domains	Primarily time-domain (state-space approach)
Control Strategies	PID Control, Lead/Lag Compensators	Optimal Control (LQR), Robust Control, Adaptive Control
Application Complexity	Simple, mostly single variable systems	Complex, multi-variable systems, non-linear systems
Focus	Design for **stability** and **transient** response	Design for **optimality**, **robustness**, and **performance** under uncertainty

Concepts in Chapter 1

1. Introduction to the basic terminology of control systems followed by some real-world applications.
2. Describe the behavior of various systems including electrical, mechanical, rotational, electromechanical, etc.
3. Reduce differential equations representing system behavior into a suitable form using Laplace Transformation.
4. Generate a relationship between the input and output of each system block.
5. The block diagram can be reduced to just one input-output relationship, the system's overall transfer function.
6. Signal flow graphs development.

EE-379 Linear Control Systems

Week No. 1: Continuous Time System Description

1. Introduction to Control Systems
2. Basic Concepts and Terminologies
3. Modeling of Electrical, Mechanical, and Rotational systems
4. Analogies
5. System Description
6. Transfer Function
7. Relevant MATLAB Commands

Introduction to control system

- A control system is defined as a system of devices that regulates, manages, or commands the behavior of other devices to achieve the desired output.

Terminologies

System
An arrangement or combination of different physical components that are connected or related together to form an entire unit to achieve a certain objective. E.g., computer, classroom

Control
To regulate, direct and command a system so that the desired objective is achieved.

Plant/Process
Portion of the system to be controlled, It is fixed as far as the control system designer is concerned The designer's job is to ensure that the plant operates as required

Input
The applied or excitation signal applied to a control system to get a specific output

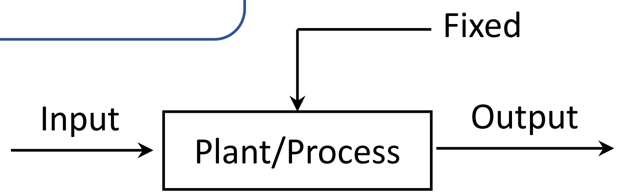
Output
The actual response obtained from a control system due to the application of the input.

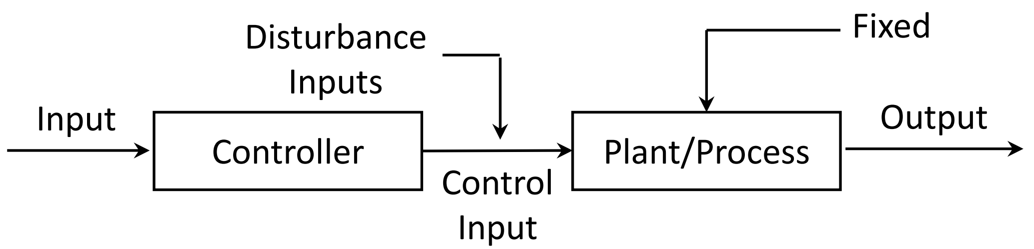
Controller
Internal or external element of the system used to control the plant or process The controller generates plant input signals designed to produce the desired outputs Some plant inputs are accessible to the designer and some are not available

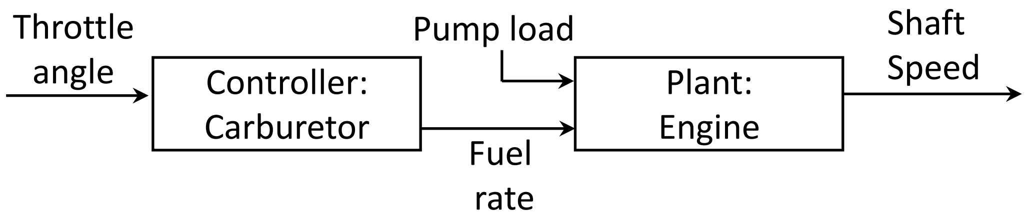
Disturbance
A disturbance is an uncontrollable input that has an undesired effect on the desired output of the system. It may be internal (produced within the system) or external

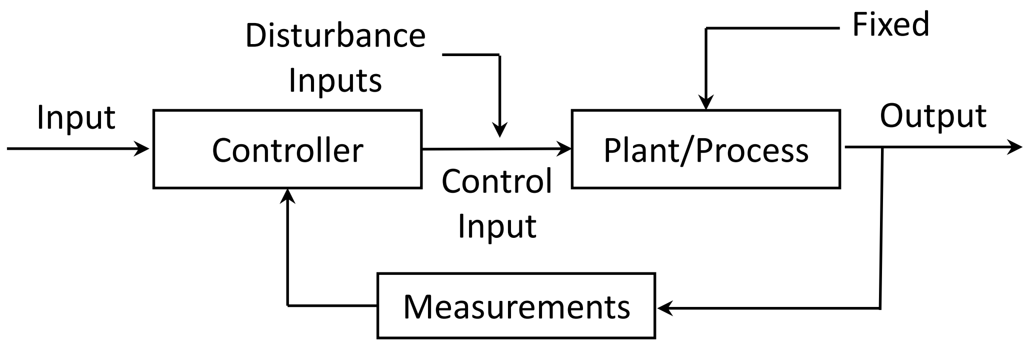
Open Loop System
A system in which the control inputs are not influenced by the plant outputs i.e. there is no feedback around the plant.

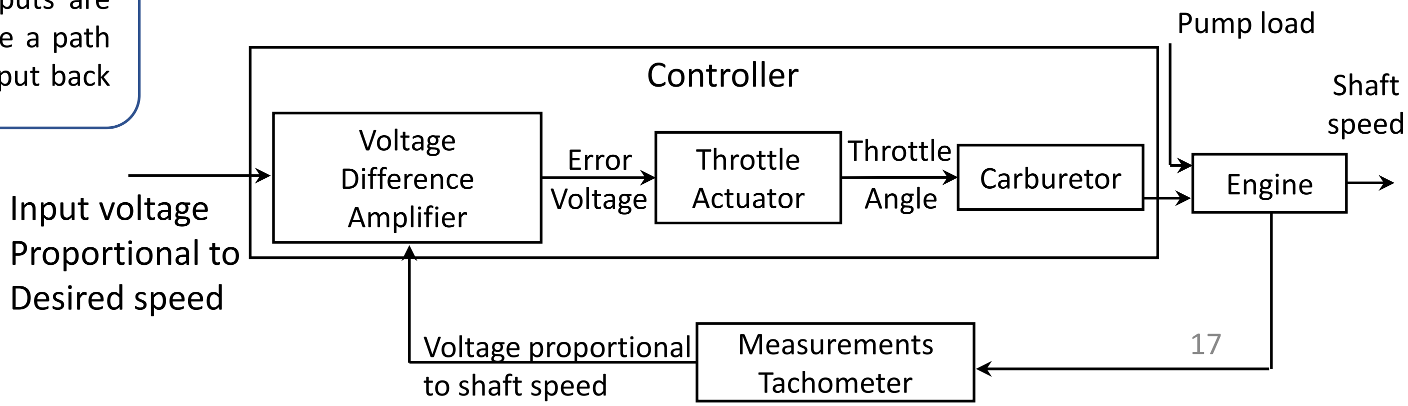
Closed Loop System
A system in which the control inputs are influenced by the plant outputs i e a path (or loop) is provided from the output back to the controller









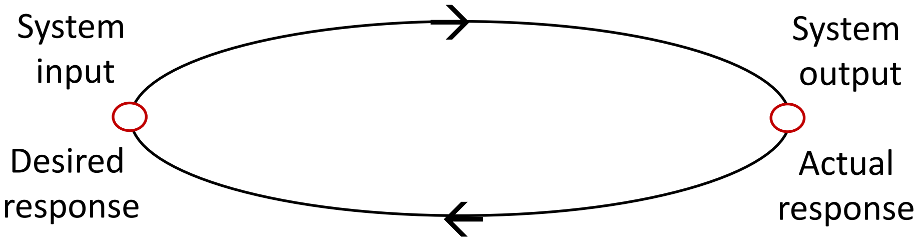


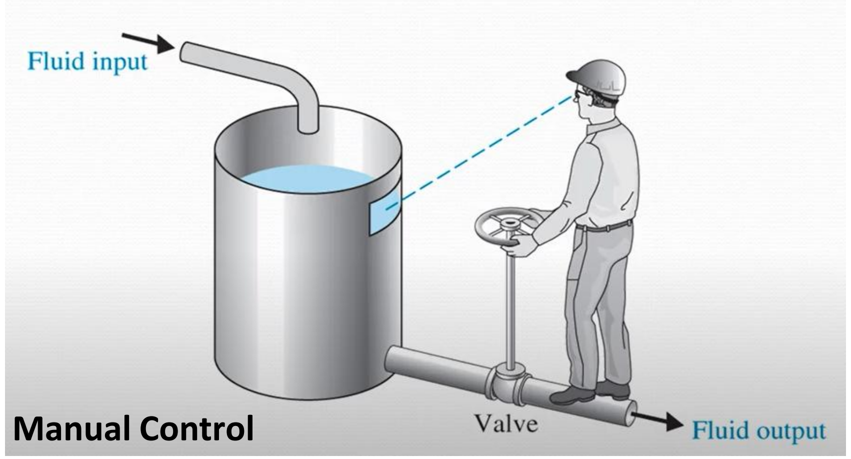
Practical Examples of Control Systems

Control System Components

Input → **Plant** → Output

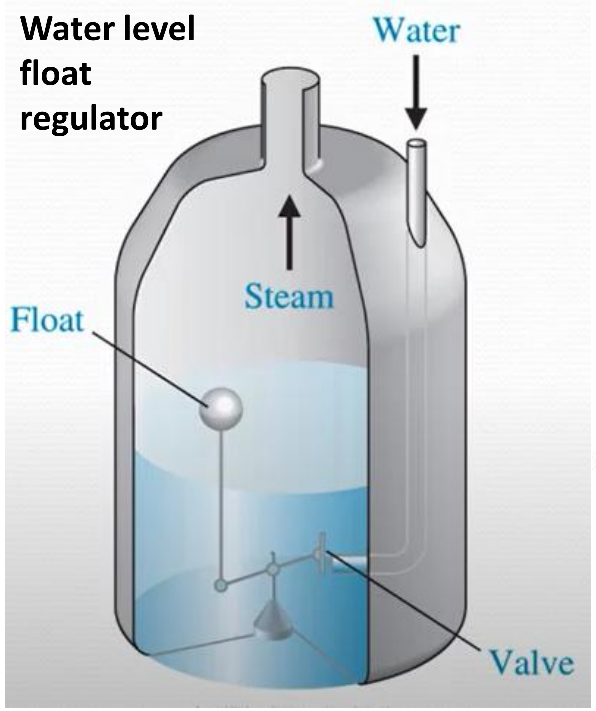
Variable → **Sensor** → Measurement Of variable

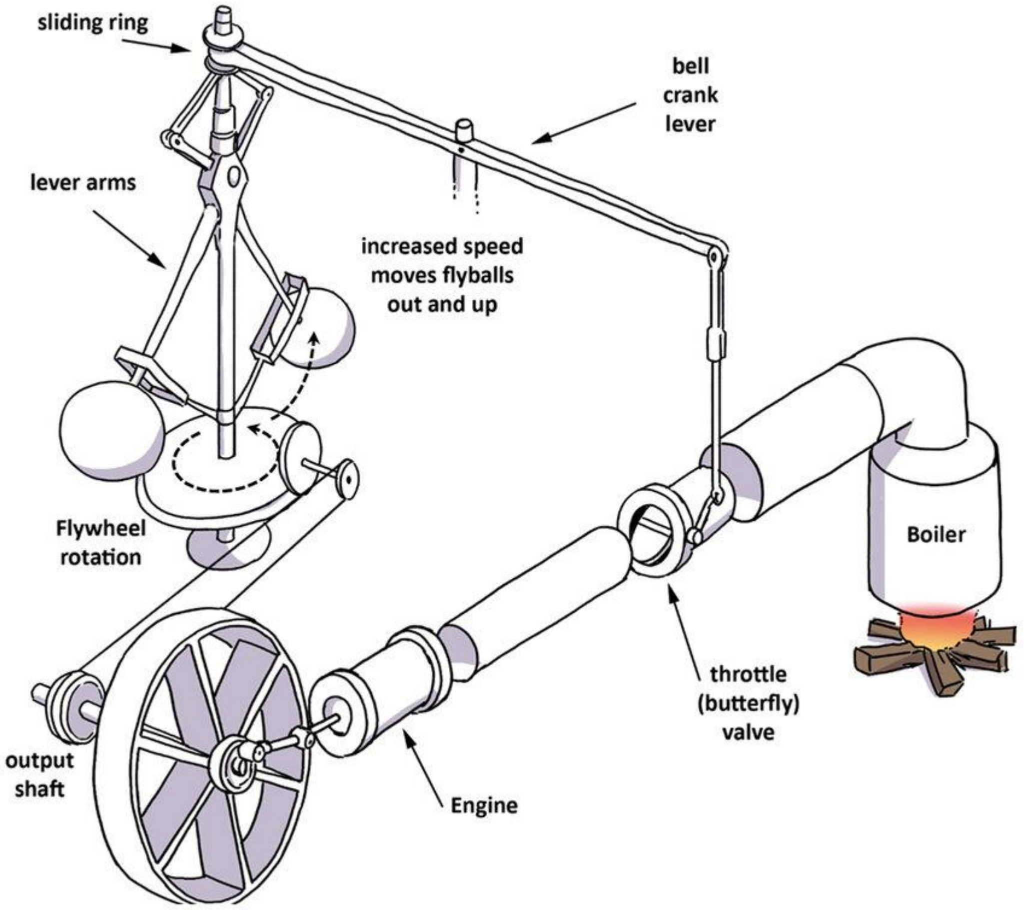




Manual Control

Automatic Control Water level float regulator





Watt's flywheel

Limitations of the manual control

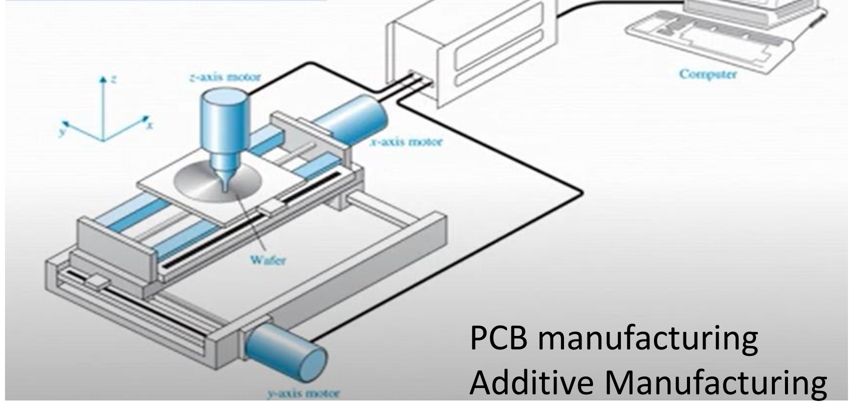
- High operating cost
- Reliability issues
- Inaccurate
- Bandwidth
- Safety-related concerns

Examples of naturally occurring automatic control systems

- Blood sugar regulation by the pancreas
- Adrenaline, heart rate, oxygen control
- Tracking of moving objects by our eyes
- Grasping of objects by hand
- Body temperature regulation

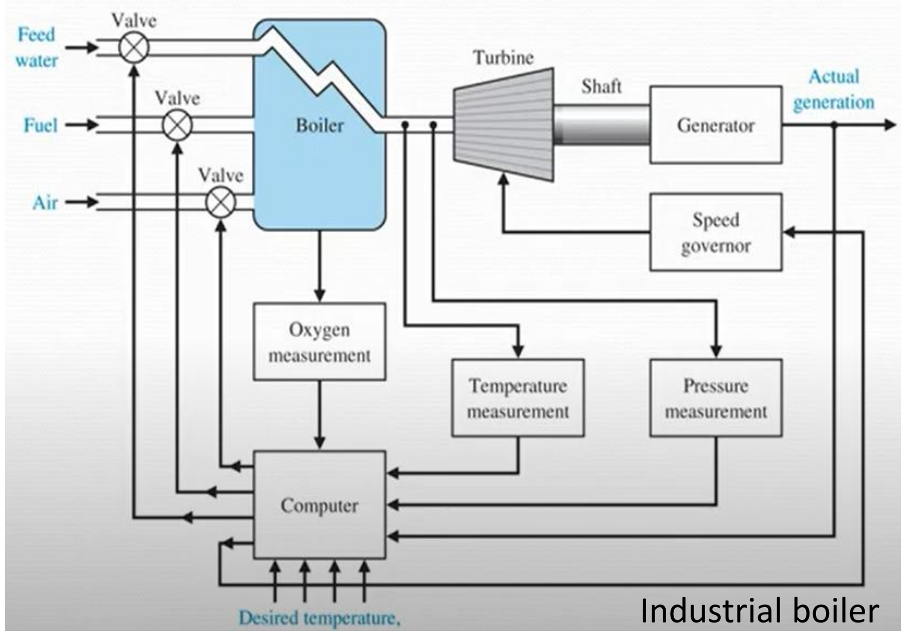
Examples of control systems

CNC machines

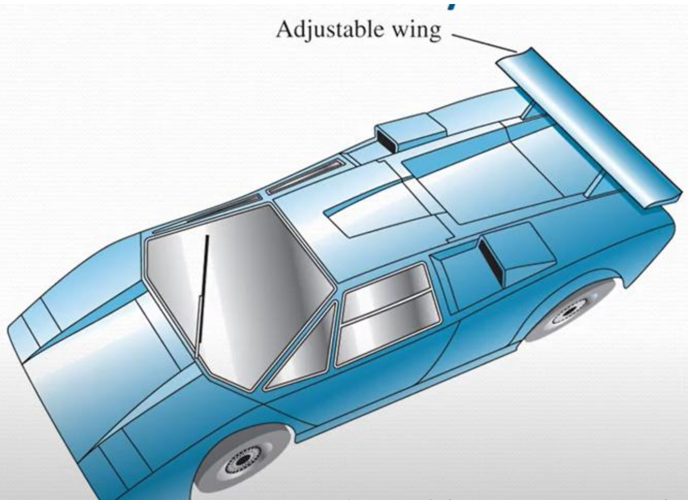


PCB manufacturing
Additive Manufacturing

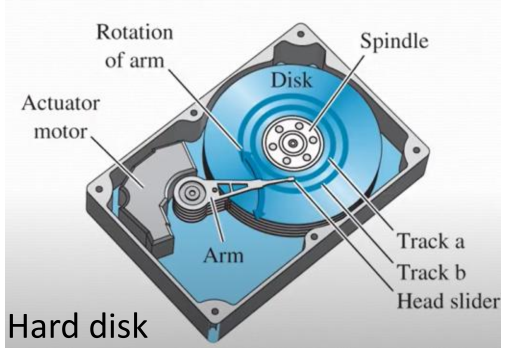
Unmanned aerial vehicle



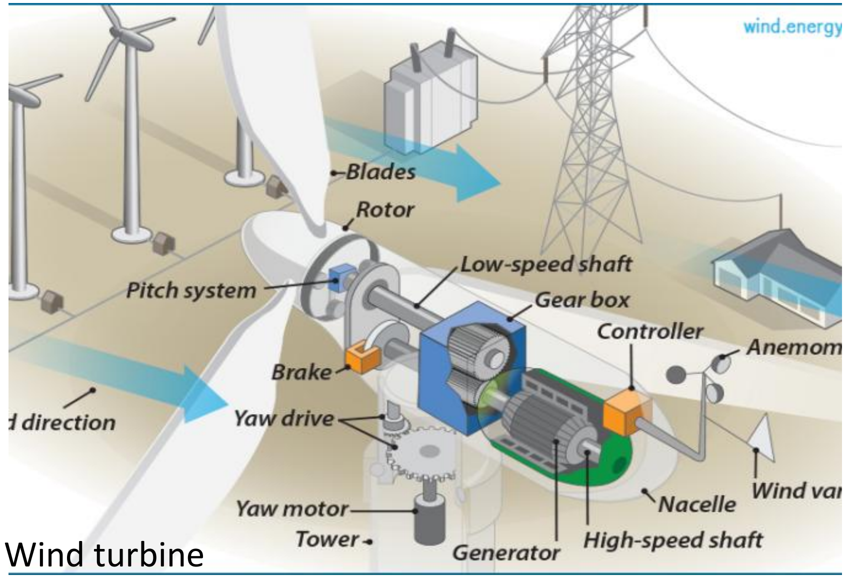
Industrial boiler



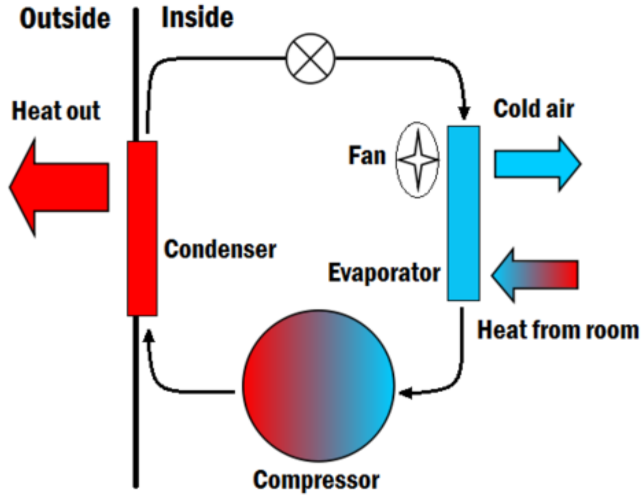
Adjustable wing control



Hard disk



Wind turbine

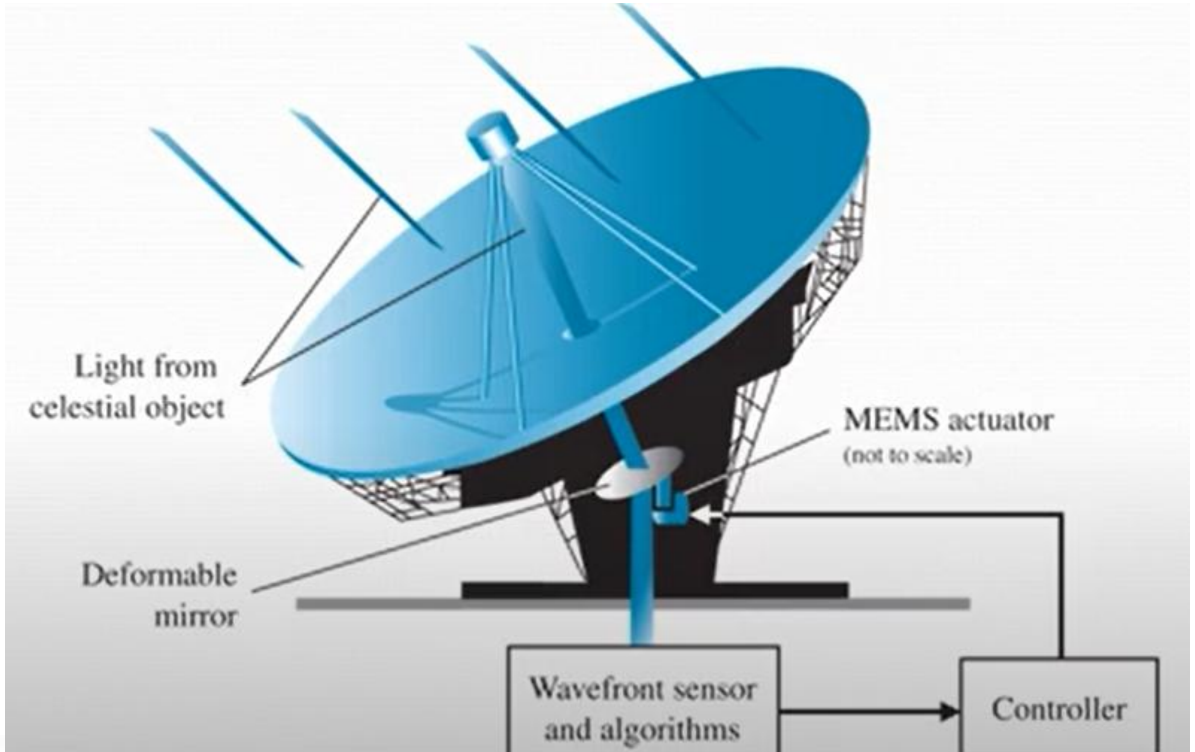


Air conditioning system


6113 RO

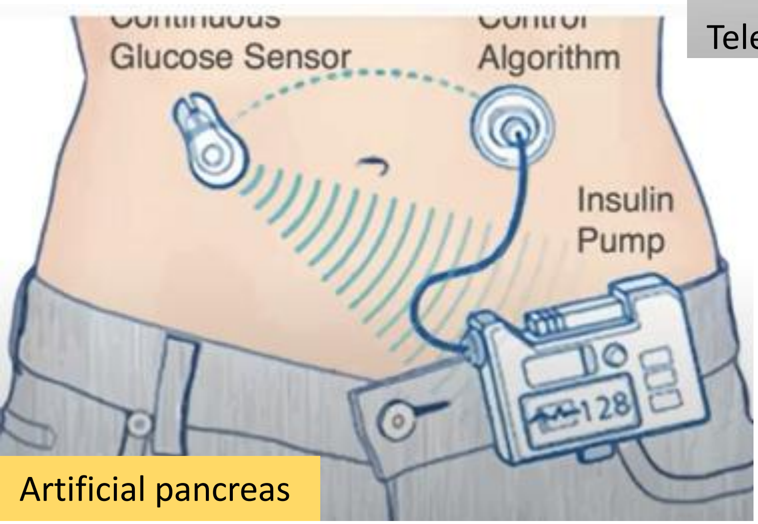
Examples of control systems

Surgical robot

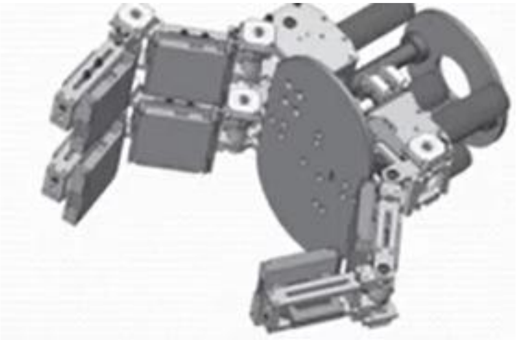


Telescope position control

Humanoid robot



Artificial pancreas



Robotic grippers

Assignment Project No. 1

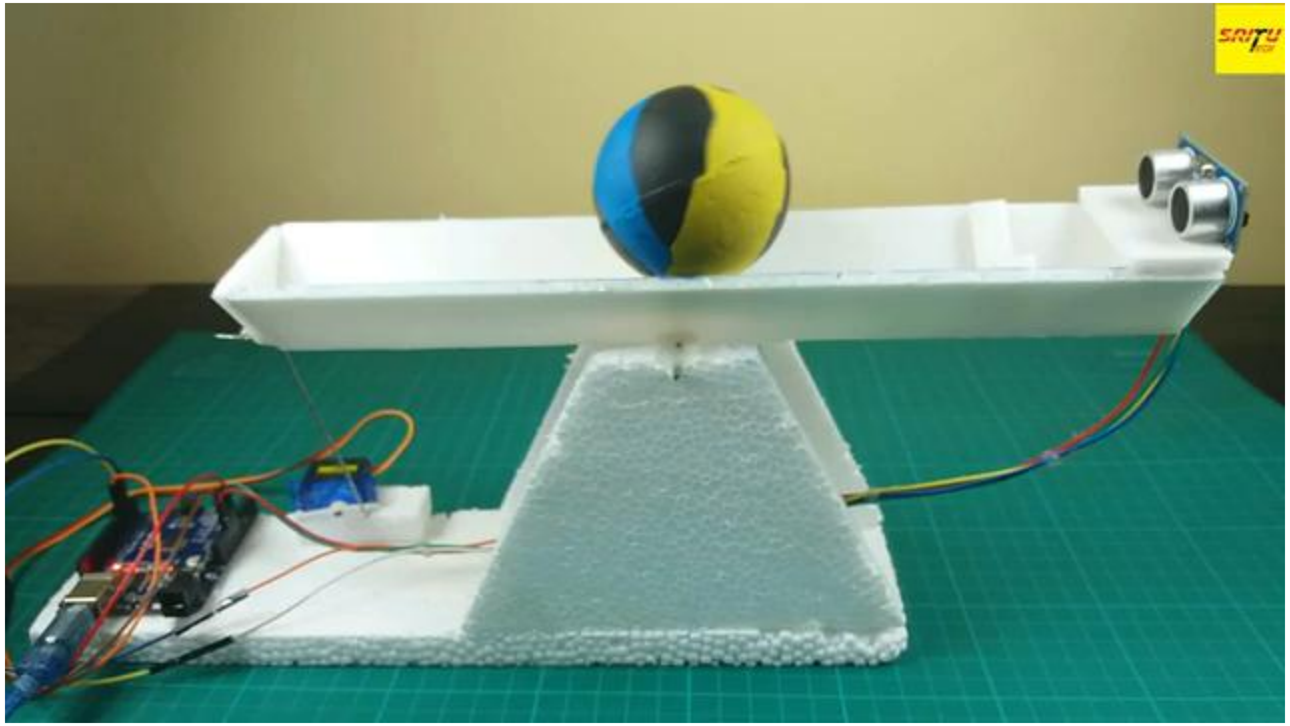
- Follow the tutorial shown in the video and implement the system using PID control.

Or

- You can also choose to make a very basic self balancing robot if you wish.

- The system shall be comprised of one sensor, one actuator, and one Microcontroller.

- Only 2 students/group are allowed. Practical demonstration and viva will be taken.



https://www.youtube.com/watch?v=YOPTksabdbM

Project Submission deadline: 29-Sep-25

Viva deadline: 29 Sep – 3 Oct 2025

Mathematical Modeling of Physical Systems

Modeling - Prerequisites

Control Engineers must be able to analyze and design systems of various kinds. E.g., for the "**speed control system design**", the control engineer must know:

- How **vacuum pressure affects** throttle setting? **(pneumatics)**
- How **temp. and pressure affect** the power out as the air-gas mixture explodes? **(Thermodynamics)**
- How will a car respond to the **power generated by the pistons** in the cylinder? **(mechanics)**
- How electrical devices may be used to measure and store important variables e.g., temp, vacc .press. **(electrical circuits)**

Modeling

Modeling - Characteristics

- It is necessary to build a **mathematical model** that behaves similarly to the **actual system** within a specific range (e.g., spring mass damper system may be used to simulate the motion of the vehicle within a certain range of the applied power).

- Linearization may be used to construct a model valid at some ranges.

- In order to model the system, the properties of the components must be known.

- Methods for analyzing the components of various systems are discussed.

Modeling

Linearization

- May be used to construct a model valid at some ranges.

  $$E.g., \quad y = f(x) = x^2$$

- Linear equation?
- **Taylor series expansion** can be used to get a linear approximation, that is valid for some operating conditions

  $$y \sim y_0 + f^1(x_0)(x - x_0)$$
  $$y \sim x_0^2 + 2x_0(x - x_0)$$

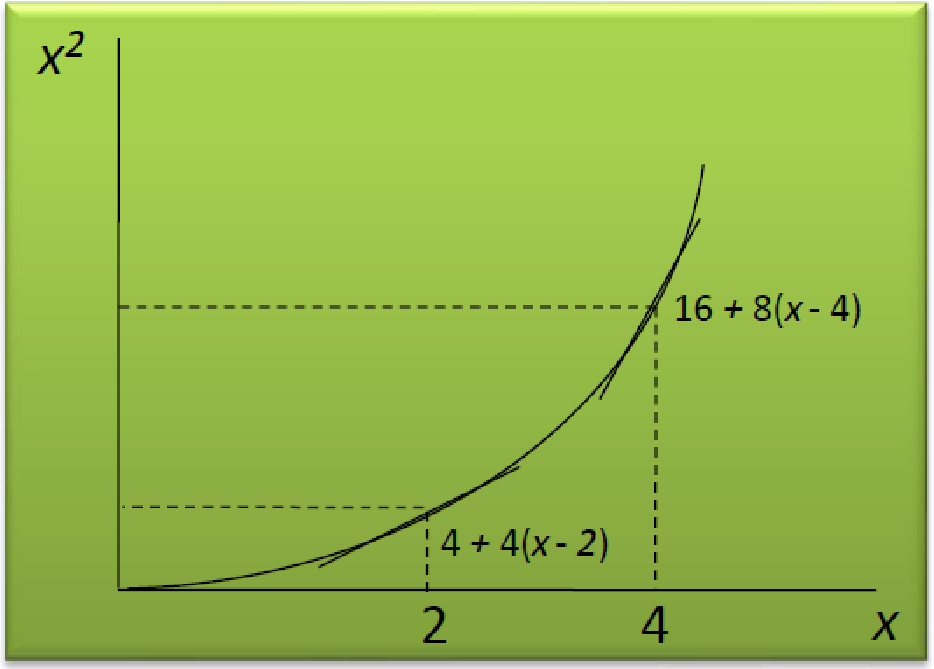
- If we choose $x_0 = 2$ then:

  $$y \sim 4 + 4(x - 2)$$

- If we choose $x_0 = 4$ then:

  $$y \sim 16 + 8(x - 4)$$

x	x^2	$4+4(x-2)$	$16+8(x-4)$
2.00	4.00	4.00	0.00
2.10	4.41	4.40	0.80
2.20	4.84	4.80	1.60
3.00	9.00	8.00	8.00
4.1	16.81	12.40	16.80



Modeling

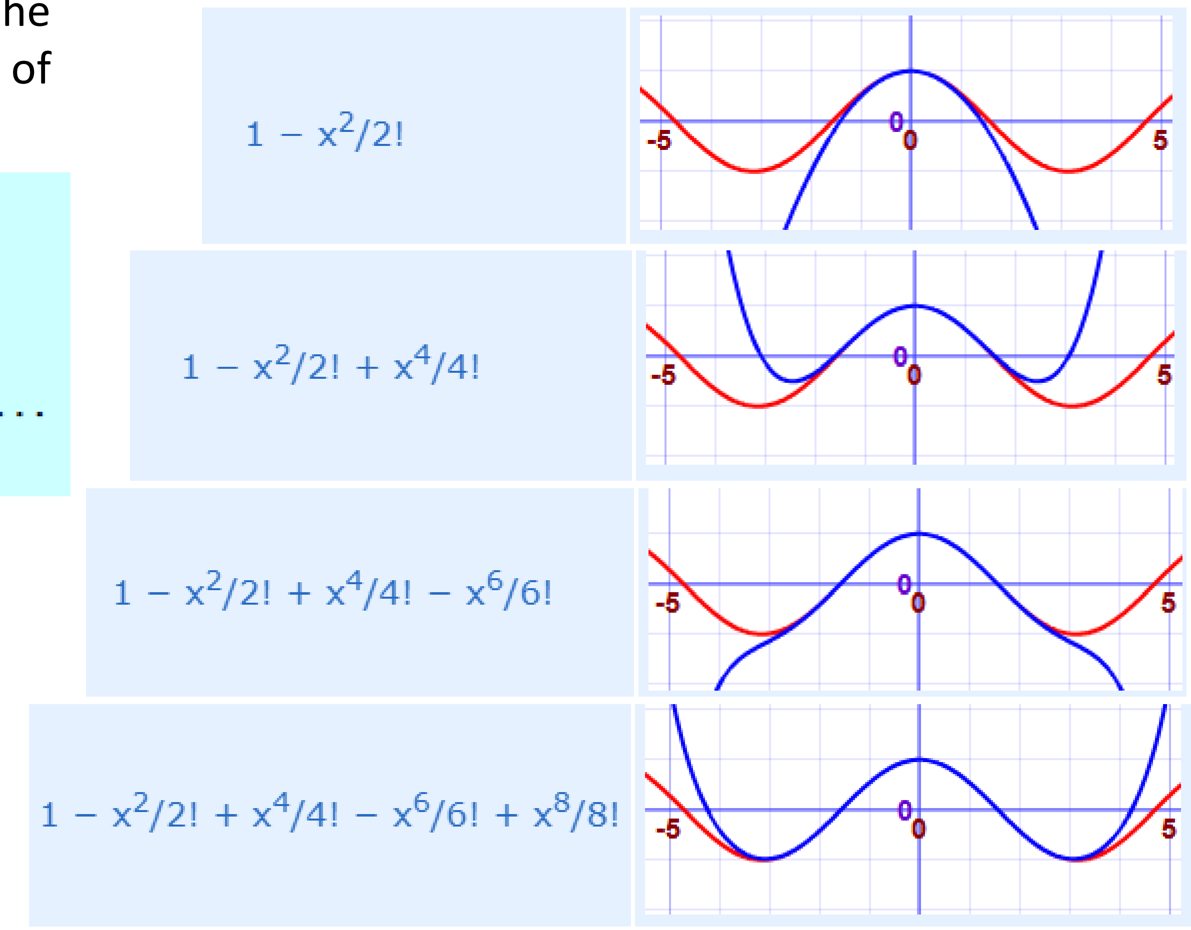
Basics of Taylor Series (Concepts Check)

- The Taylor series for a function is often useful in physical situations to approximate the value of the function near the expansion point x_o. It may be evaluated term-by-term in terms of the derivatives of the function.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$
$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

Taylor Series Expansion of Cos(x)

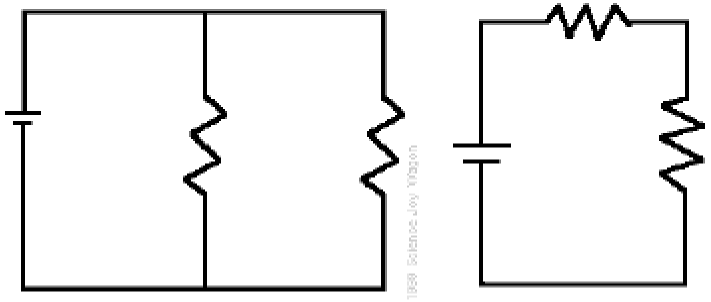


Modeling

Electrical Systems

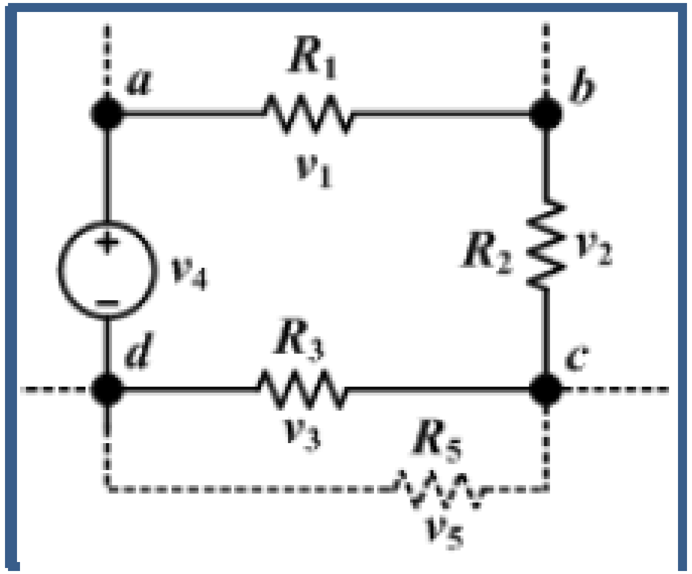
- Electrical networks are controlled by two **Kirchhoff's laws**:
  - The algebraic sum of voltages around a closed loop equals zero.

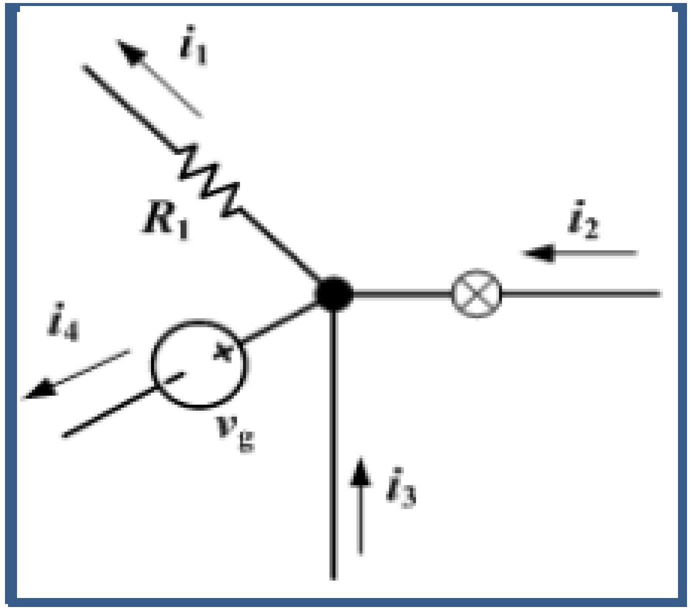
$$\sum_j v_j = 0$$



  - The algebraic sum of currents flowing into a circuit node equals zero

$$\sum_j i_j = 0$$

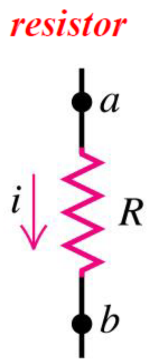


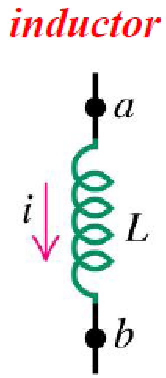


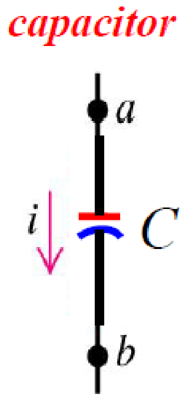
Modeling

Electrical Systems

- Voltage-Current Relations for Various Electronic Components.







Time Domain

$$V_R(t) = Ri(t)$$
$$\mathrm{i}(t) = \frac{1}{R} V_R(t)$$

$$V_L(t) = L\frac{di}{dt}$$
$$\mathrm{i}(t) = \frac{1}{L}\int_{-\infty}^{t} V_L(t)dt$$

$$V_C(t) = \frac{1}{C}\int_{-\infty}^{t} i(t)dt$$
$$\mathrm{i}(t) = C\frac{dV_C}{dt}$$

Freq. Domain

$$V_R(s) = RI(s)$$
$$\mathrm{I}(s) = \frac{1}{R} V_R(s)$$

$$V_L(s) = sLI(s)$$
$$\mathrm{I}(s) = \frac{1}{sL} V_L(s)$$

$$V_C(s) = \frac{1}{sC} I(s)$$
$$I(s) = sCV_C(s)$$

Modeling

Basics of Laplace Transform (Concepts Check)

- Laplace transform has **no physical significance** except that it transforms **the time domain signal** to a **complex frequency domain**. It is useful to **simplify mathematical computations** and it can be used for the **easy analysis of signals** and systems. The **stability of the system** is directly revealed when the transfer function of the system is known in the Laplace domain. LT is used for **solving differential equations**.

TABLE 2.1 Laplace transform table

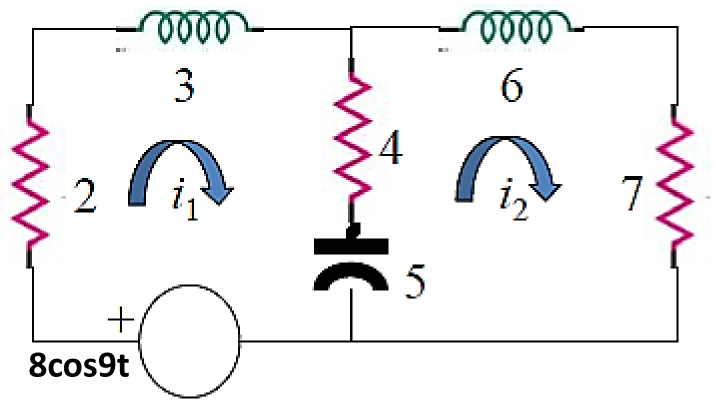
Item no.		$f(t)$	$F(s)$
1.	Impulse input	$\delta(t)$	1
2.	Step input	$u(t)$	$\frac{1}{s}$
3.	Ramp input	$tu(t)$	$\frac{1}{s^2}$
4.		$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5.		$e^{-at}u(t)$	$\frac{1}{s+a}$
6.		$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7.		$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

TABLE 2.2 Laplace transform theorems

Item no.	Theorem	Name
1.	$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t)+f_2(t)] = F_1(s)+F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)] = F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)] = e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - f'(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) = \lim_{s\to 0} sF(s)$	Final value theorem[1]
12.	$f(0+) = \lim_{s\to\infty} sF(s)$	Initial value theorem[2]

Modeling Example – Electrical Systems

Mesh Analysis



Equating algebraic sum of voltages around each mesh to zero gives

$$2i_1 + 3\frac{di_1}{dt} + 4(i_1 - i_2) + \frac{1}{5}\int_{-\infty}^{t}(i_1 - i_2)dt = 8\cos 9t$$

$$6\frac{di_2}{dt} + 7i_2 + 4(i_2 - i_1) + \frac{1}{5}\int_{-\infty}^{t}(i_2 - i_1)dt = 0$$

Collecting terms in terms of i_1 and i_2 gives

$$3\frac{di_1}{dt} + 6i_1 + \frac{1}{5}\int_{-\infty}^{t} i_1 dt - 4i_2 - \frac{1}{5}\int_{-\infty}^{t} i_2 dt = 8\cos 9t$$

$$-4i_1 - \frac{1}{5}\int_{-\infty}^{t} i_1 dt + 6\frac{di_2}{dt} + 11i_2 + \frac{1}{5}\int_{-\infty}^{t} i_2 dt = 0$$

Laplace transforming and assuming zero initial conditions

$$\left(3s + 6 + \frac{1}{5s}\right)I_1(s) + \left(-4 - \frac{1}{5s}\right)I_2(s) = \frac{8s}{s^2 + 9^2}$$

$$\left(-4 - \frac{1}{5s}\right)I_1(s) + \left(6s + 11 + \frac{1}{5s}\right)I_2(s) = 0$$

The above equations can be written in the standard form as

$$Z_{11}(s)I_1(s) + Z_{12}(s)I_2(s) = E_1(s)$$

$$Z_{21}(s)I_1(s) + Z_{22}(s)I_2(s) = E_2(s)$$

Where

Z_{11} (s)= sum of all impedances around the I_1 Mesh

Z_{12} (s)= Z_{11} (s)=sum of all impedances common toI_1 and I_2 Mesh

Z_{22} (s)= sum of all impedances around the I_2 Mesh

E_1 (s)= independent voltage source driving mesh I_1

E_2 (s)= independent voltage source driving mesh I_2

Solving these equations for I_2 using the Cramer's rule we get

$$I_2(s) = \frac{\begin{vmatrix} Z_{11} & E_1 \\ Z_{21} & E_2 \end{vmatrix}}{\begin{vmatrix} Z_{11} & Z_{12} \\ Z_{11} & Z_{22} \end{vmatrix}} = \frac{Z_{11}E_2 - Z_{21}E_1}{Z_{11}Z_{22} - Z_{12}Z_{21}} = \frac{\left[\frac{8s}{(s^2 + 9^2)}\right]\left[4 + \left(\frac{1}{5s}\right)\right]}{\left[3s + 6 + \left(\frac{1}{5s}\right)\right]\left[6s + 11 + \left(\frac{1}{5s}\right)\right] - \left[4 + \left(\frac{1}{5s}\right)\right]^2}$$

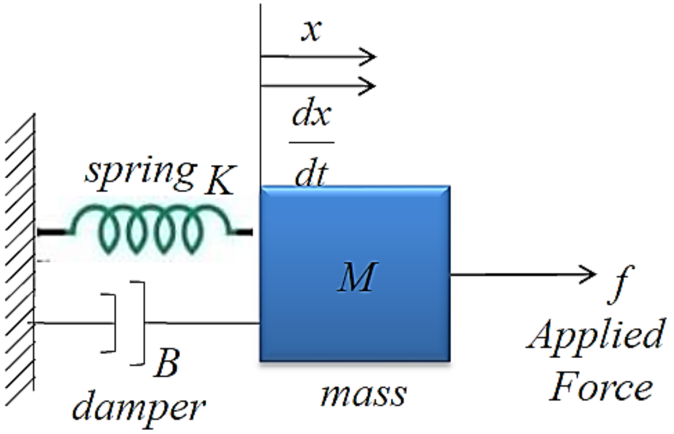
Modeling

Mechanical Systems

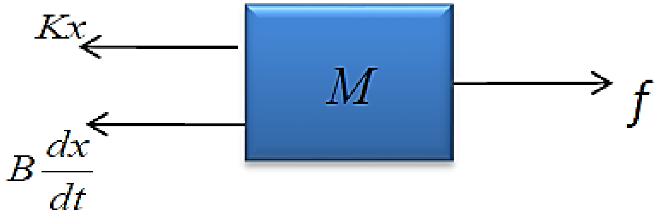
- Analysis of translational mechanical systems
  - Define **positions with directional senses for each mass** in the system
  - Draw a **free body diagram** for each mass (expressing forces in terms of mass position and velocity)
  - **Write an equation** for each mass equating the algebraic sum of forces acting in the same direction.

TABLE 2.4 Force-velocity, force-displacement, and impedance translational relationships for springs, viscous dampers, and mass

Component	Force-velocity	Force-displacement	Impedence $Z_M(s) = F(s)/X(s)$
Spring ($x(t)$, $f(t)$, K)	$f(t) = K\int_0^t v(\tau)d\tau$	$\boxed{f(t) = Kx(t)}$	K
Viscous damper ($x(t)$, $f(t)$, f_v)	$f(t) = f_v v(t)$	$\boxed{f(t) = f_v \frac{dx(t)}{dt}}$	$f_v s$
Mass ($x(t)$, M, $f(t)$)	$f(t) = M\frac{dv(t)}{dt}$	$\boxed{f(t) = M\frac{d^2x(t)}{dt^2}}$	Ms^2



Free Body Diagram



Newton's Law

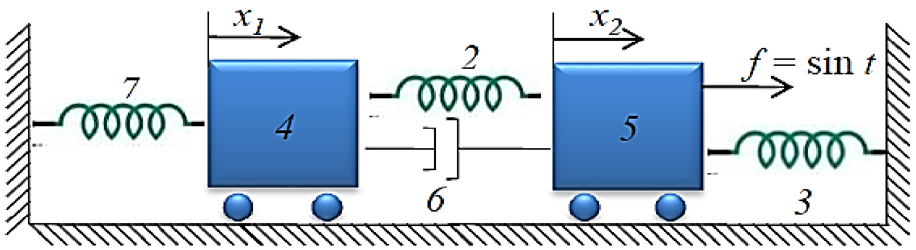
$$\sum F = ma$$

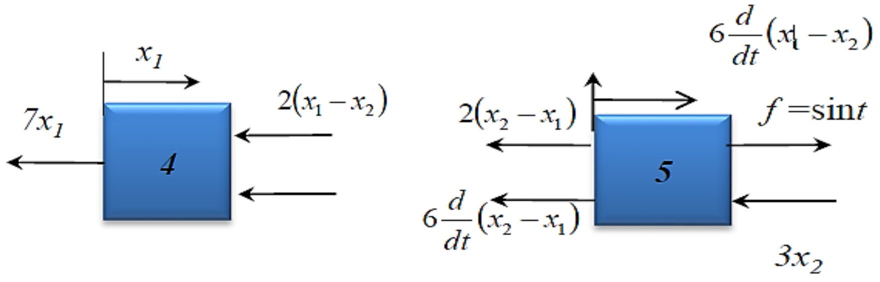
$$M\left(\frac{d^2x}{dt^2}\right) = -\mathrm{kx} - \mathrm{B}\frac{dx}{dt} + f$$

$$M\left(\frac{d^2x}{dt^2}\right) + \mathrm{B}\frac{dx}{dt} + \mathrm{kx} = f$$

$$s^2MX(s) + \mathrm{sBX(s)} + \mathrm{KX(s)} = F(s)$$

Modeling Example – Mechanical Systems





- Mechanical System Analysis

  Equating forces for the first mass gives

$$4\frac{d^2x_1}{dt^2} = -7x_1 - 2(x_1 - x_2) - 6\frac{d}{dt}(x_1 - x_2)$$

and similarly for the second mass

$$5\frac{d^2x_2}{dt^2} = f - 3x_2 - 2(x_2 - x_1) - 6\frac{d}{dt}(x_2 - x_1)$$

Collecting the terms we can write

$$4\frac{d^2x_2}{dt^2} + 6\frac{dx_1}{dt} + 9x_1 - 6\frac{dx_2}{dt} - 2x_2 = 0$$

$$-6\frac{dx_1}{dt} - 2x_1 + 5\frac{d^2x_2}{dt^2} + 6\frac{dx_2}{dt} + 5x_2 = \sin t$$

Laplace transforming and assuming zero initial conditions

$$(4s^2 + 6s + 9)X_1(s) + (-6s - 2)X_2(s) = 0$$

$$(-6s - 2)X_1(s) + (5s^2 + 6s + 5)X_2(s) = \frac{1}{s^2 + 1}$$

The above equations can be written in the standard form as

$$W_{11}X_1(s) + W_{12}X_2(s) = F_2(s)$$

$$W_{21}X_1(s) + W_{22}X_2(s) = F_2(s)$$

Where

$W_{11}(s)$= Inertial force and all forces attached to M_1

$W_{12}(s)$= $W_{21}(s)$=sum of forces connected to M_1 and M_2

$W_{22}(s)$= Inertial force and all forces attached to M_2

$F_1(s)$= independent driving force on M_1

$F_2(s)$= independent driving force on M_2

Solving these equations for X_1 using the Cramer's rule we get

$$X_1(s) = \frac{\begin{vmatrix} F_1(s) & W_{12} \\ F_2(s) & W_{22} \end{vmatrix}}{\begin{vmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{vmatrix}} = \frac{\left[\frac{1}{s^2+1}\right][6s+2]}{[(4s^2 + 6s + 9)(5s^2 + 6s + 5)][6s+2]^2}$$

Modeling

Rotational Systems

Analysis of rotational mechanical systems.

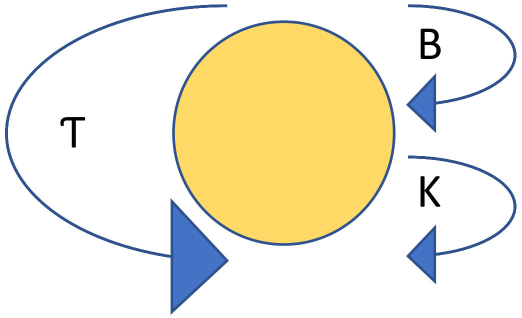
- Torque replaces force
- **Angular displacement** replaces translational displacement
- Mass is replaced by **inertia**

Steps

- Draw **angular positions** with directional senses for each rotational mass
- Draw a **free body diagram** for each rotational mass (expressing each torque in terms of angular positions of the masses)
- Write an **equation for each rotational mass** equating the algebraic sum of torques on it.

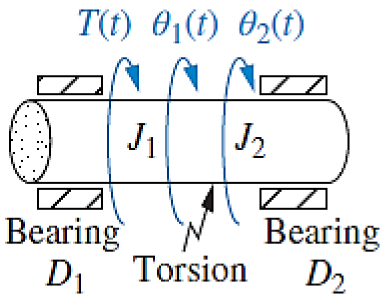
TABLE 2.5 Torque-angular velocity, torque-angular displacement, and impedance rotational relationships for springs, viscous dampers, and inertia

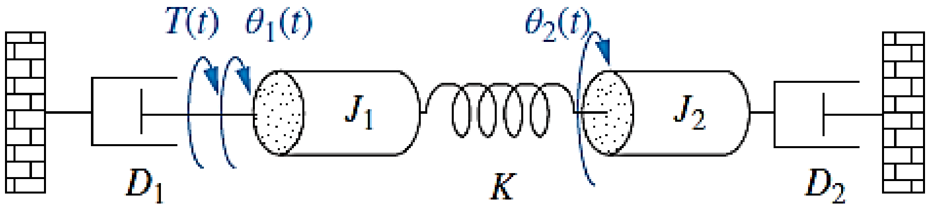
Component	Torque-angular velocity	Torque-angular displacement	Impedence $Z_M(s) = T(s)/\theta(s)$
Spring K, $T(t)$ $\theta(t)$	$T(t) = K\int_0^t \omega(\tau)d\tau$	$T(t) = K\theta(t)$	K
Viscous damper D, $T(t)$ $\theta(t)$	$T(t) = D\omega(t)$	$T(t) = D\frac{d\theta(t)}{dt}$	Ds
Inertia J, $T(t)$ $\theta(t)$	$T(t) = J\frac{d\omega(t)}{dt}$	$T(t) = J\frac{d^2\theta(t)}{dt^2}$	Js^2

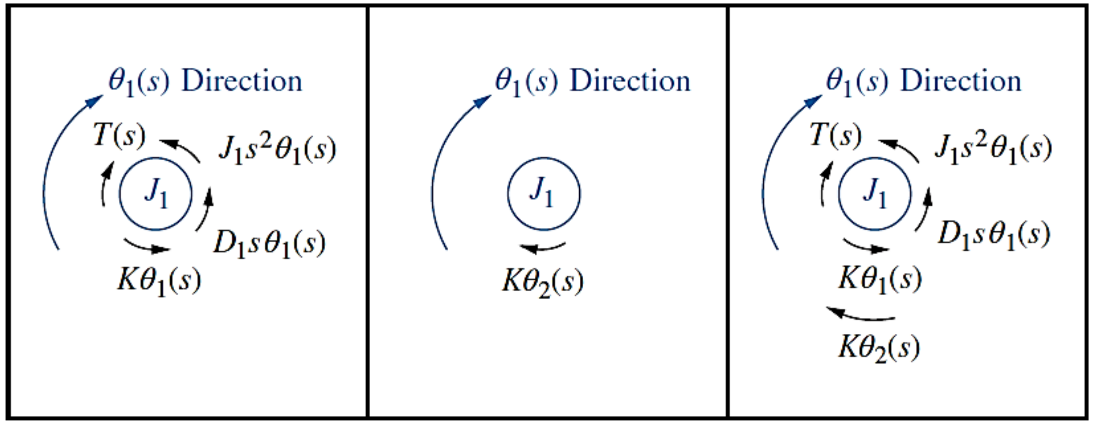


Free body diagram

Modeling Example – Rotational Systems



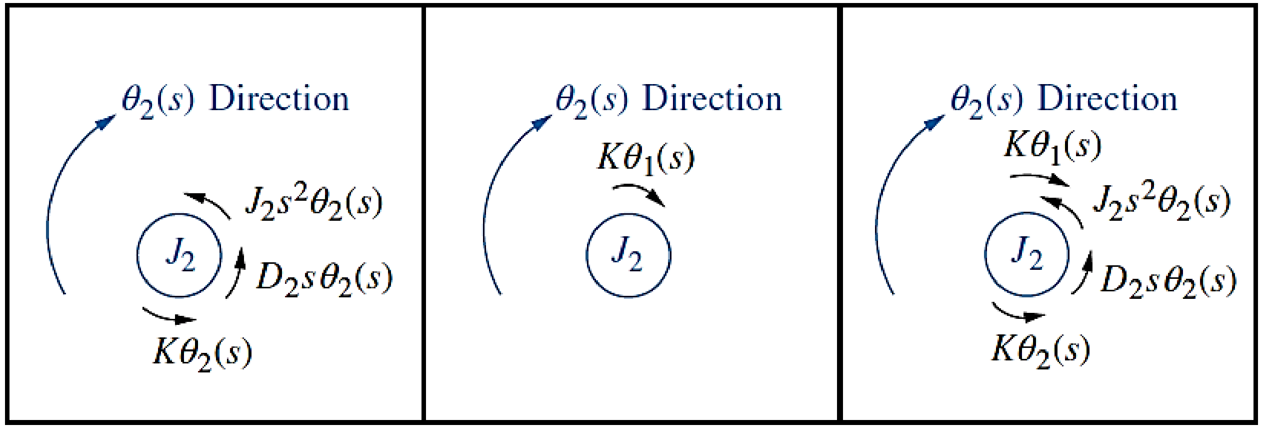




Torques on J_1 due only to the motion of J_1

Torques on J_1 due only to the motion of J_2

Final free body diagram for J_1



Torques on J_2 due only to the motion of J_2

Torques on J_2 due only to the motion of J_1

Final free body diagram for J_2

$$(J_1s^2 + D_1s + K)\theta_1(s) - K\theta_2(s) = T(s)$$

$$-K\theta_1(s) + (J_2s^2 + D_2s + K)\theta_2(s) = 0$$

$$\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } \theta_1 \end{bmatrix} \theta_1(s) - \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ \theta_1 \text{ and } \theta_2 \end{bmatrix} \theta_2(s) = \begin{bmatrix} \text{Sum of} \\ \text{applied torques} \\ \text{at } \theta_1 \end{bmatrix}$$

$$-\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ \theta_1 \text{ and } \theta_2 \end{bmatrix} \theta_1(s) + \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } \theta_2 \end{bmatrix} \theta_2(s) = \begin{bmatrix} \text{Sum of} \\ \text{applied torques} \\ \text{at } \theta_2 \end{bmatrix}$$

Analogies and Examples

What is an Analogy?

- **Different types of physical systems** that are modeled by the **same form of equations** are called analogous systems.
- Equations are derived similarly for all linear systems (electrical, mechanical, or rotational).
- An **electric circuit that is analogous** to a system from another discipline is called an **electric circuit analog**.

- Analogs can be obtained by comparing the describing equations, such as the **equations of motion** of a mechanical system, with **either electrical mesh or nodal equations**.

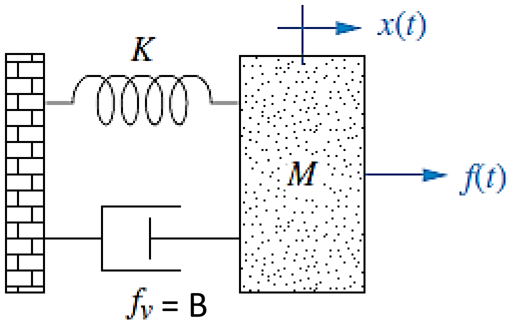
- Both mesh and nodal analogies can be constructed.
- When compared **with mesh equations**, the resulting electrical circuit is called a **series analog**.
- When compared **with nodal equations**, the resulting electrical circuit is called **a parallel analog**.

- **The procedure includes:**
  - **Write mechanical system equations.**
  - **Substitute electrical quantities** (using electrical network constants and variables).
  - **Interpret these equations** to yield the analog network.

Analogies and Examples

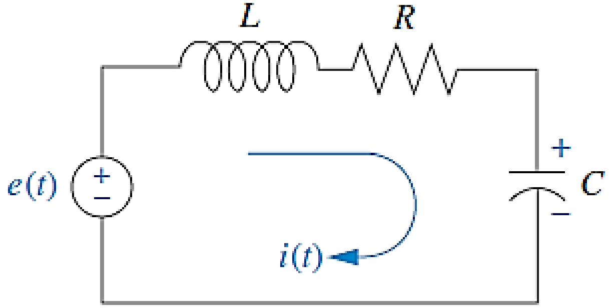
Series Analog

- Consider the translational mechanical system as shown:



$$(Ms^2 + Bs + K)X(s) = \mathrm{F}(s)$$

- Kirchhoff's mesh equation for the simple series RLC network as shown:



$$(Ls + R + \frac{1}{Cs})I(s) = \mathrm{E}(s)$$

The analogy does not exist?

$$(Ms^2 + Bs + K)X(s) = \mathrm{F}(s)$$

$$\frac{(Ms^2 + Bs + K)sX(s)}{s} = \left(Ms + B + \frac{K}{s}\right)V(s)$$

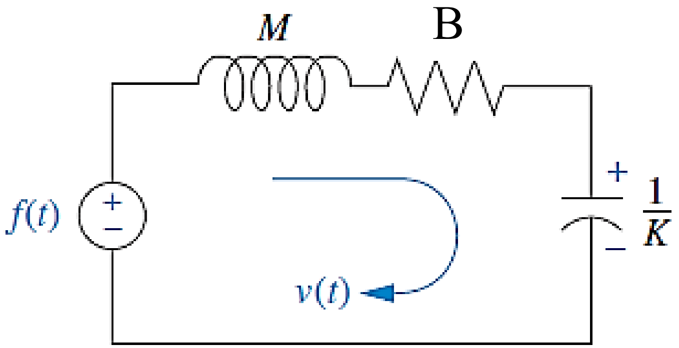
$$\left(sM_{ii} + B_{ii} + \frac{K_{ii}}{s}\right)V_i \ldots = F_i$$

$$\left(sJ_{ii} + B_{ii} + \frac{K_{ii}}{s}\right)\Omega_i \ldots = T_i$$

with

$$\left(sL_{ii} + R_{ii} + \frac{1}{sC_{ii}}\right)I_i \ldots = E_i$$

$$\left.\begin{matrix} M \\ J \end{matrix}\right\} \sim L,\ B \sim R,\ K \sim \frac{1}{C},\ \left.\begin{matrix} F \\ T \end{matrix}\right\} \sim E,\ \left.\begin{matrix} V \\ \Omega \end{matrix}\right\} \sim I$$



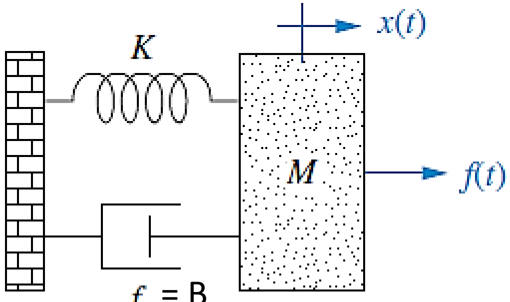
Series Analog

Component	Analogy
Mass M / Moment of Inertia J	Inductance L
Damping Constant B	Resistance R
Spring Constant K	Inverse of Capacitance 1/C
Force F/Torque T	Voltage E
Linear v/ angular Ω velocity	Current I

Analogies and Examples

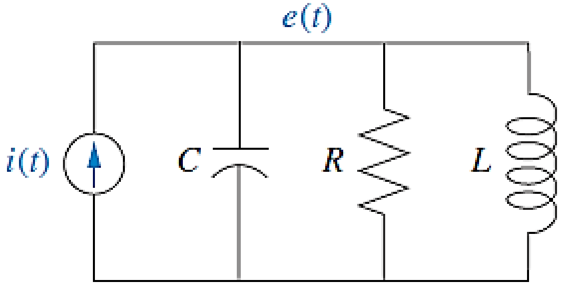
Parallel Analog

- Consider the translational mechanical system as shown:

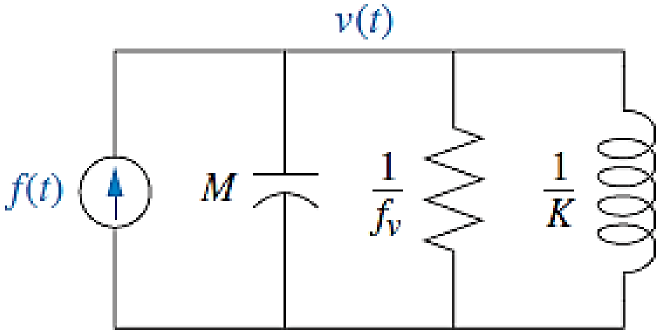


$$(Ms^2+Bs+K)X(s) = \mathrm{F}(s) = \left(Ms+B+\frac{K}{s}\right)V(s)$$

- Kirchhoff's nodal equation for the simple parallel RLC network as shown:



$$\left(Cs+\frac{1}{R}+\frac{1}{Ls}\right)E(s) = \mathrm{I}(s)$$



The parallel analog

$$\left(sM_{ii}+B_{ii}+\frac{K_{ii}}{s}\right)V_i\ldots=F_i$$

$$\left(sJ_{ii}+B_{ii}+\frac{K_{ii}}{s}\right)\Omega_i\ldots=T_i$$

with

$$\left(sC_{ii}+\frac{1}{R_{ii}}+\frac{1}{sL_{ii}}\right)E_i\ldots=I_i$$

Component	Analogy
Mass M / Moment of Inertia J	Capacitance C
Damping Constant B	Conductance 1/R
Spring Constant K	Inverse of Inductance 1/L
Force F/Torque T	Current I
Linear v/ angular Ω velocity	Voltage E

$$\left.\begin{matrix}M\\J\end{matrix}\right\}\sim C,\ B\sim\frac{1}{R},\ K\sim\frac{1}{L},\ \left.\begin{matrix}F\\T\end{matrix}\right\}\sim I,\ \left.\begin{matrix}V\\\Omega\end{matrix}\right\}\sim E$$

Relevant MATLAB Commands

MATLAB Commands

- The first command introduced is **syms s** this causes the command variable **s** to be used as a symbol.
- Next is **inv(matrix)** which takes the inverse of a matrix.
- Finally, here we introduce the command **simplify** which is used to combine symbolic terms and cancel where needed.

Example no.1

Recalling the equation for I_2 from Slide 21:

$$I_2(s) = \frac{\left[\frac{8s}{(s^2+9^2)}\right]\left[4+\left(\frac{1}{5s}\right)\right]}{\left[3s+6+\left(\frac{1}{5s}\right)\right]\left[6s+11+\left(\frac{1}{5s}\right)\right]-\left[4+\left(\frac{1}{5s}\right)\right]^2}$$

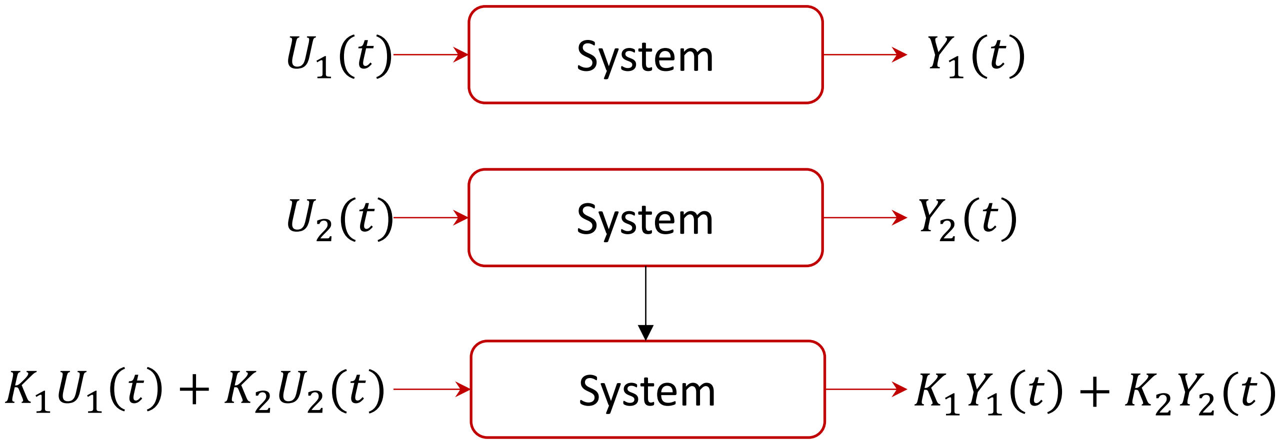
MATLAB Code:

```
syms s
t1=8*s/(s^2+81)
t2=4+1/5/s
t3=3*s+6+1/5/s
t4=6*s+11+1/5/s
t5=t2
t=t1*t2/(t3*t4-t5^2)
simplify t
```

System Description

Linear Time-Invariant Systems

- Definition of the linear and time-invariant system:
  - ✓ A system is linear if the **principle of superposition** applies.



Superposition

The **net response** produced by the simultaneous application of **two or more forcing functions** is the sum of the responses that would have been caused by **each stimulus individually**

- ✓ **For linear systems**, the response to several inputs can be calculated by treating **one input at a time and adding the results.**
- ✓ A system is time-invariant **if its parameters are stationary** with respect to time during system operation (a differential equation is linear if the coefficients are constants).
- ✓ Systems that are represented by differential equations whose **coefficients are functions of time** are called **linear time-varying systems**.

Examples of the time-invariant and time-variant systems?

System Description

Transfer Function

- The transfer function between a pair of input and output variables is the ratio of the Laplace transform of the **output to the input** (alternatively, the transfer function between an input variable and an output variable of a system is defined as the Laplace transform of the impulse response.

$$R(s) \longrightarrow \boxed{G(s)} \longrightarrow Y(s)$$

- The transfer function is defined only for **linear time-invariant LTI systems** (**not for nonlinear and time varying systems**).

- **All initial conditions** of the system are assumed to be **zero**.

- The transfer function of a continuous data system is expressed only as a **function of the complex variable 's'** , it is not a function of the real variable, time, or any other variable that is used as the independent variable.

System Description

Transfer Function - Example

- **For a single-input, single output system** with input $\boldsymbol{r(t)}$ and output $\boldsymbol{y(t)}$ the transfer function (or transmittance) relating output to the input is defined as follows:

$$T(s) = \left.\frac{Y(s)}{R(s)}\right|_{when\ all\ initial\ conditions\ are\ zero}$$

- Consider the system described by:

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = -\frac{dr}{dt} + 5r$$

- Laplace-transforming and collecting terms gives:

$$s^2Y(s) - sy(0) - y'(0) + 6[sY(s) - y(0)] + 8Y(s) = -[sR(s) - r(0)] + 5R(s)$$
$$Y(s)[s^2 + 6s + 8] = sy(0) + y'(0) + 6y(0) + r(0) + R(s)[-s + 5]$$

- Setting initial conditions to zero yields:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{-s + 5}{s^2 + 6s + 8}$$

System Description

Transfer Function

- In the following representation of the transfer function:

$$T(s) = \left.\frac{Y(s)}{R(s)}\right|_{\text{when all initial conditions are zero}}$$

- $\boldsymbol{Y(s)}$ and $\boldsymbol{R(s)}$ both are polynomials in **s** domain.

- Let $Y(s) = b_m s^m + b_{m-1}\, s^{m-1} + \cdots + b_0$

  $\mathrm{R}(s) = s^n + a_{n-1}\, s^{m-1} + \cdots + a_0$

$$T(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{s^n + a_{n-1} s^{m-1} + \cdots + a_0}$$

$$T(s) = \frac{K[(s-b_1)(s-b_2)\ldots(s-b_n)]}{[(s-a_1)(s-a_2)\ldots(s-a_n)]}$$

Numerator

Denominator

Where K is the gain factor

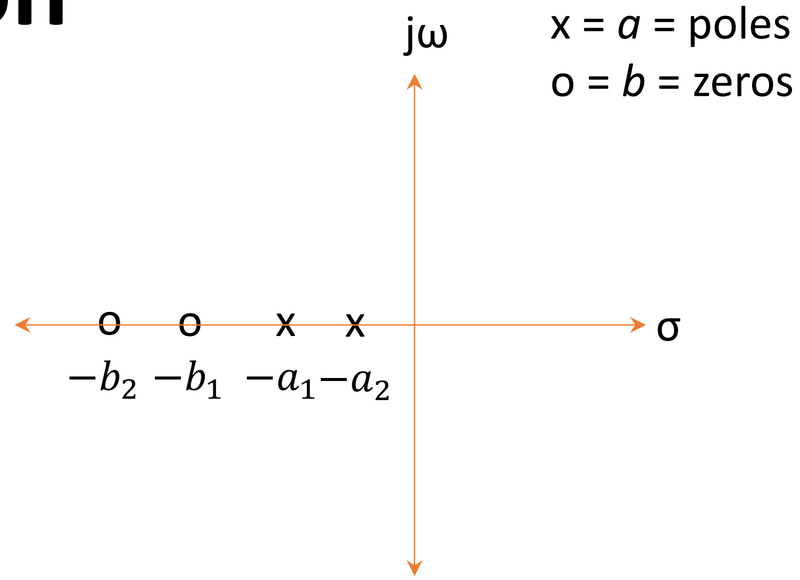
Zeros of a transfer function are defined as the values of s for which the magnitude of Transfer Function becomes zero.

Poles of a transfer function are defined as the values of s for which the magnitude of Transfer Function becomes infinity.

System Description

Transfer Function – Graphical Representation

- There are two axes of an s-plane.
- The first axis or σ-axis is the real axis and the values of σ are plotted along the real axis.
- The second or jω–axis is the imaginary axis and the jω values are plotted along the imaginary axis.



- The poles and zeros of the below-mentioned transfer function are illustrated as follows:

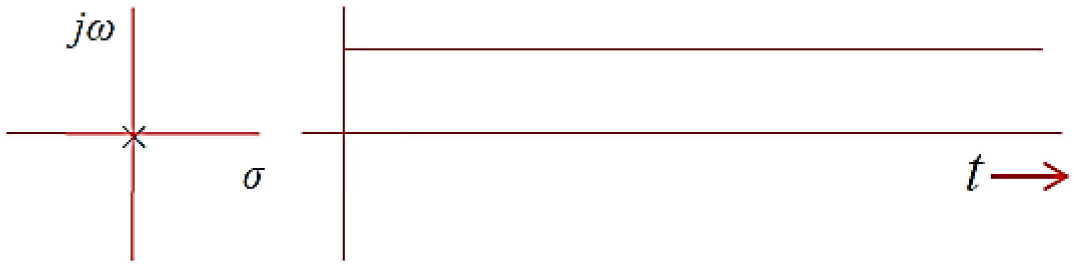
$$T(s) = \frac{K(s+b_1)(s+b_2)^2}{(s+a_1)(s+a_2)^3}$$

where K is the gain and
$0 < a_2 < a_1 < b_1 < b_2$

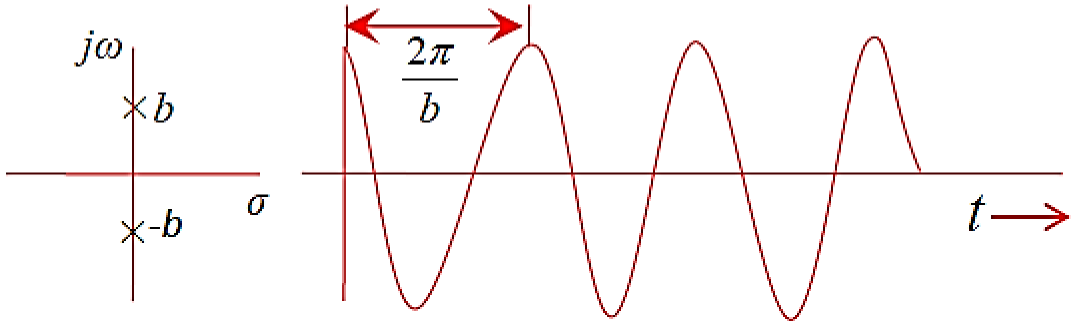
The denominator polynomial in terms of **s** of a TF is known as the **characteristic polynomial**. If this polynomial is set to zero, the **characteristic equation** is obtained which can be solved to obtain the **poles of the TF** denominator

System Description

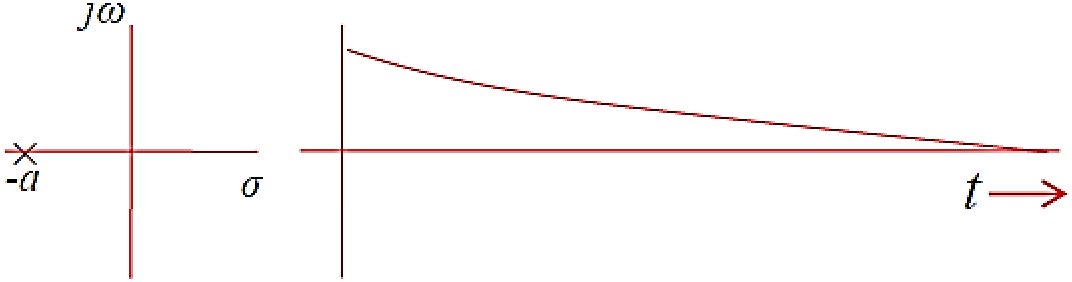
Representative time function for various pole locations on complex plane



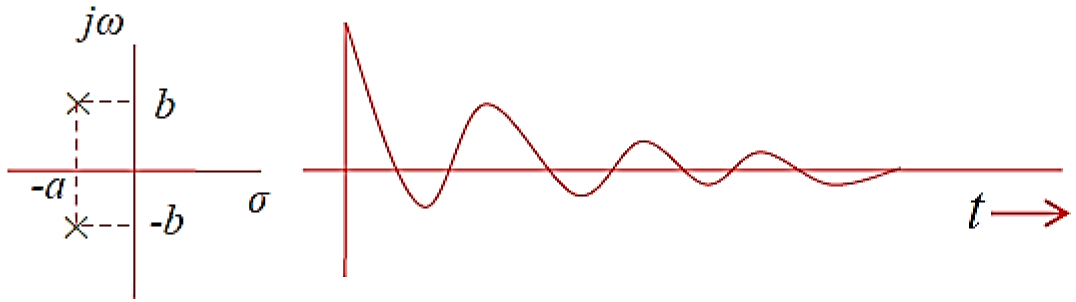
Constant K



Sinusoid with rad freq b, $A\cos(bt + \theta)$



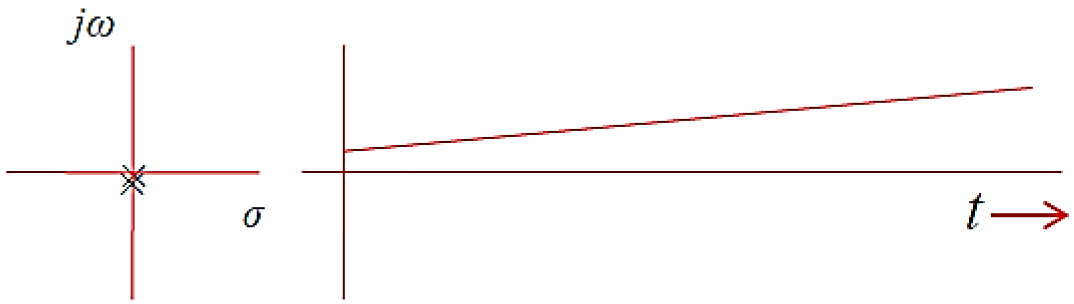
Decaying Exponential Ke^{-at}



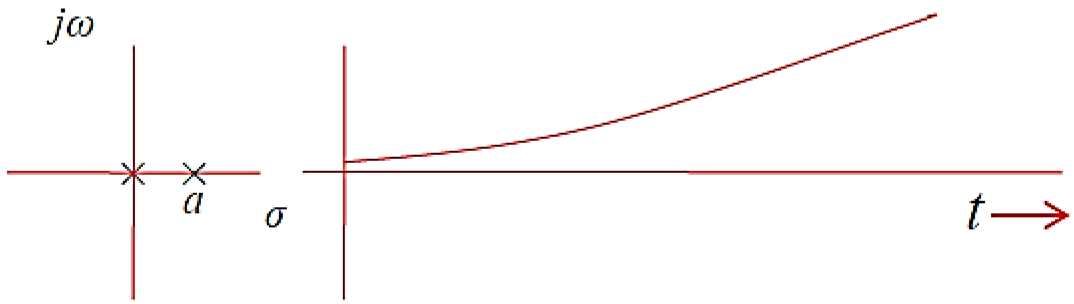
Decaying exponential times sinusoid, $Ae^{-at}\cos(bt + \theta)$. Exponential constant is $-a$ and sinusoidal radian frequency is b.

System Description

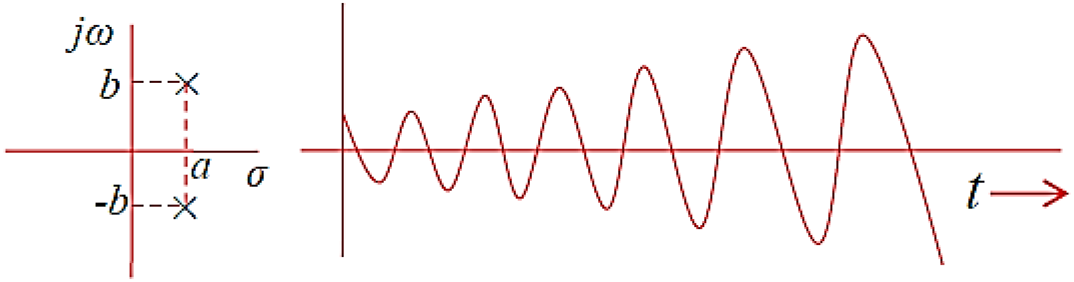
Representative time function for various pole locations on complex plane



Const K plus a const times t, $K_1 + K_2 t$



Expanding Exponential Ke^{at}



Expanding exponential times sinusoid,
$Ae^{at}\cos(\mathrm{bt} + \theta)$.

System Description

Transfer Function

- The type of **time function** corresponding to each **partial fraction expansion term** for a Laplace transformed signal depends upon:
  - The **location of roots in the complex plane**.
  - Whether the **roots are repeated**.

- **System Response**

In linear control systems (or more generally, linear time-invariant systems, LTI), the total response of a system can be split into two parts: $\mathbf{y}(\boldsymbol{t}) = \boldsymbol{y}_{\boldsymbol{zi}}(\boldsymbol{t}) + \boldsymbol{y}_{\boldsymbol{zs}}(\boldsymbol{t})$

Zero State Response

- The **system output** when the **initial conditions are all zero** is termed the **zero-state response** component.
- Its Laplace transform is simply the product of the transfer function and the input transform
- $T(s) = \frac{Y(s)}{R(s)} \longrightarrow T(s)R(s) = Y(s)$

Zero Input Part

- If the system **initial conditions are not zero**, there is an additional **output component** present called as the **zero-input part**.

System Description

Transfer Function

- If the following system

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = -\frac{dr}{dt} + 5r \quad \rightarrow \quad T(s) = \frac{Y(s)}{R(s)} = \frac{-s+5}{s^2+6s+8}$$

has zero initial conditions (zero-state) and input is:

$\mathrm{r(t)} = 7e^{-3t}$

$$\boxed{\begin{aligned}&s^2Y(s) - sy(0) - y'(0) + 6[sY(s) - y(0)] + 8Y(s) = -[sR(s) - r(0)] + 5R(s)\\&Y(s)[s^2 + 6s + 8] = sy(0) + y'(0) + 6y(0) + r(0) + R(s)[-s + 5]\end{aligned}}$$

- The system output is given by

$$Y_{zero\ state}(s) = T(s)R(s) = \frac{7(-s+5)}{(s^2+6s+8)(s+3)}$$

- **For zero input response**, lets assume $y(0) = 0,\ y'(0) = 1, and\ r(0^-) = 7.$

$$Y_{zero\ input}(s) = \frac{8}{(s^2+6s+8)}$$

- **Complete response is: $Y(s) = Y_{zero\ state}(s) + Y_{zero\ input}(s)$**

$$Y(s) = T(s)R(s) = \frac{7(-s+5)}{(s^2+6s+8)(s+3)} + \frac{8}{(s^2+6s+8)}$$

System Description

Transfer Function

- An alternative to the **zero input / zero state** approach is to separate the response into **natural** and **forced parts**
- The **natural component** consists of **all characteristic root terms** in the partial fraction expansion for the response. (response due to initial conditions)
- The **forced response component** is the **remainder of the response** and is composed of terms associated with the **input transform**. (**response due to force input**)
- For the previous system:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{-s+5}{s^2+6s+8}$$

Characteristics roots are -2, -4.

System Description

Transfer Function

- The response in partial fractions is:

$$Y(s) = \underbrace{\frac{7(-s+5)}{(s+2)(s+4)(s+3)}}_{\text{Zero state}} + \underbrace{\frac{8}{(s+2)(s+4)}}_{\text{Zero input}}$$

$$Y(s) = \underbrace{\frac{-56}{(s+3)}}_{\text{Forced component}} + \underbrace{\frac{28.5}{(s+2)} + \frac{27.5}{(s+4)}}_{\text{Natural component}}$$

Both zero-input and zero-state response contribute to the **natural response component**.

✓ The **natural response** consists of all the characteristics root terms of a response (**TF**) whereas the **forced response** is the remainder of the response and is composed of the terms associated with the input transform.

Relevant MATLAB Commands

MATLAB Commands related to Transfer Function

- num = [coefficients]
- den = [coefficients]
- roots (den)
- tf (num, den)
- zpk ([num roots], [den roots], gain k)
- ilaplace (Laplace function)
- laplace (time function)

time function: $1 + 4e^{-2t}\cos(3t)$

- To solve the transfer function:

$$T_s = \frac{5(s+6)}{s^2 + 6s + 25}$$

- num = [5 30]
- Den = [1 6 25]

MATLAB Code:

create transfer function

t = tf(num, den)

roots of the # polynomial

roots (den)

-3+j4, -3-j4

to get zero pole gain

t=zpk(-6, [-3+j*4 -3-j*4], 5)

---

syms s

Y = (5*(s+6))/(s^2+6*s+36)

y = ilaplace(Y)

---

syms t

ya= 1+4*exp(-2*t)*cos(3*t)

Ya= laplace(ya)