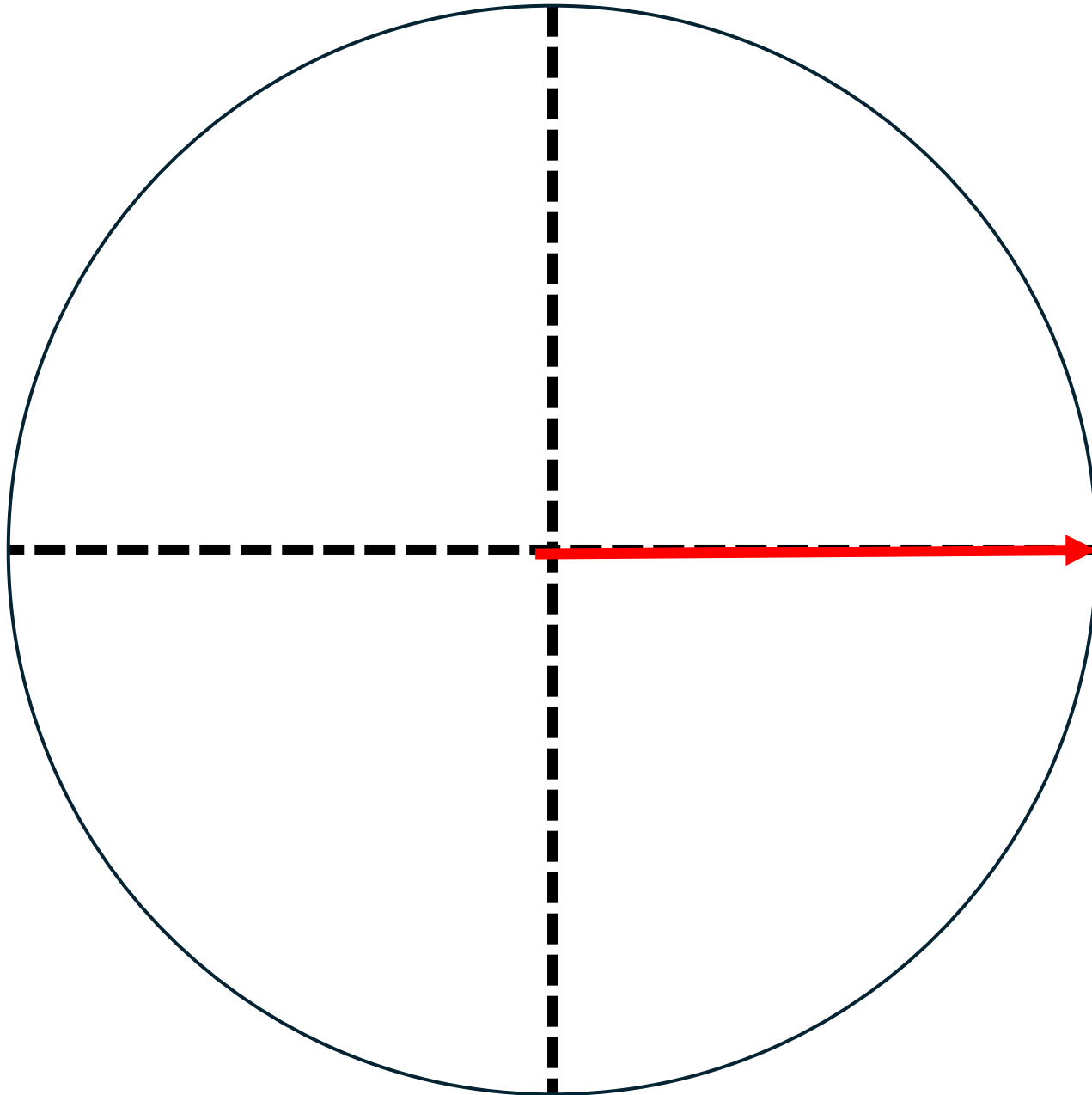


Why inverse of tangent gives better angle representation



$$\sin \beta = 0.5$$

$$\beta = 30^\circ$$

$$\beta = 150^\circ$$

To know the correct quadrant we need the value of $\cos \beta$ as well

$$\cos \beta = 0.866$$

$$\beta = \tan^{-1} \left(\frac{0.5}{0.866} \right) = 30^\circ$$

$$\cos \beta = -0.866$$

$$\beta = \tan^{-1} \left(\frac{0.5}{-0.866} \right) = 150^\circ$$

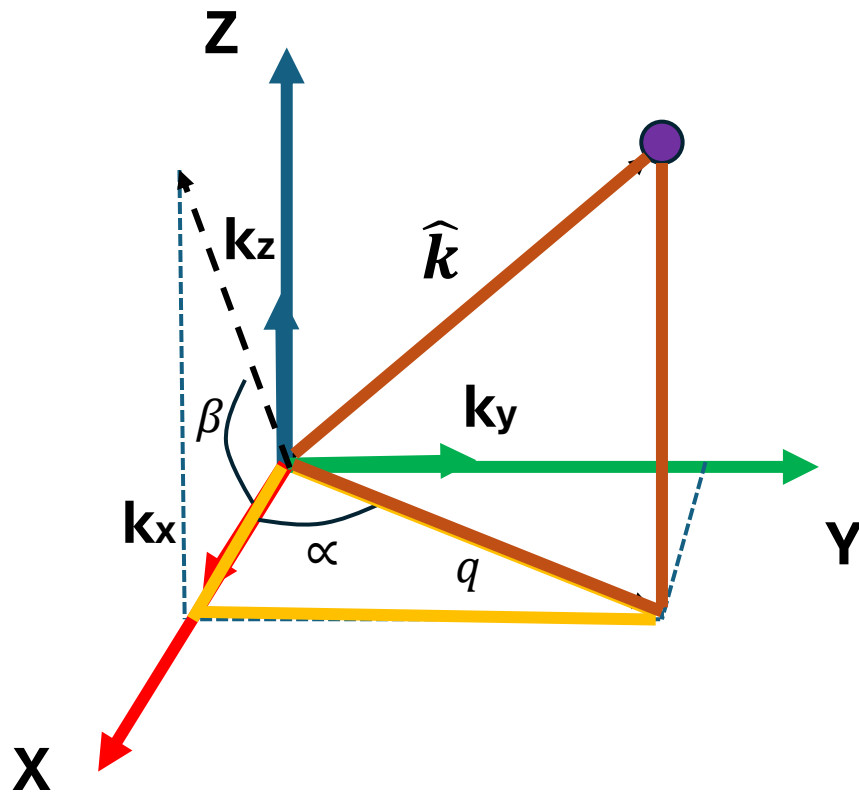
Equivalent angle-axis representations

Start with the frame coincident with a known frame $\{A\}$; then rotate $\{B\}$ about the vector ${}^A\hat{K}$ by an angle θ according to the right-hand rule.

- Vector \hat{K} is sometimes called the equivalent axis of a finite rotation.
- A general orientation of $\{B\}$ relative to $\{A\}$ may be written as ${}^A_B R(\hat{K}, \theta)$ or $R_K(\theta)$ and will be called the equivalent angle–axis representation.
- The specification of the vector ${}^A\hat{K}$ requires only two parameters, because its length is always taken to be one.
- The angle specifies a third parameter. Often, we will multiply the unit direction, \hat{K} with the amount of rotation, θ , to form a compact 3×1 vector description of orientation, denoted by K (no “hat”)

$$\hat{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} \quad \hat{q} = \begin{bmatrix} k_x \\ k_y \\ 0 \end{bmatrix}$$

β is equal to the angle between \hat{k} and \hat{q}

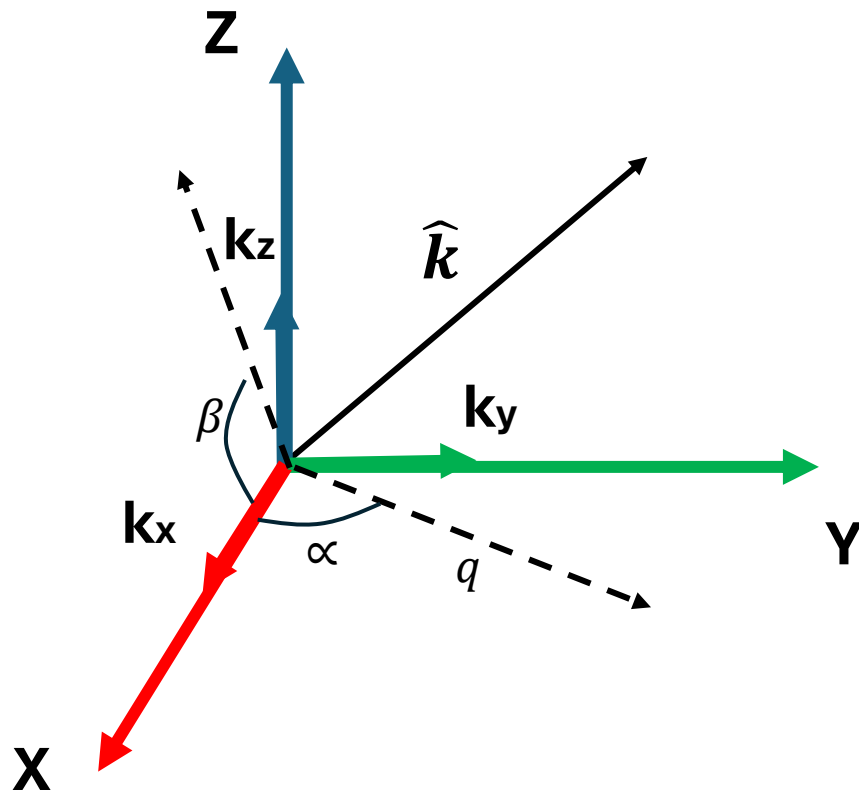


$$\cos(\alpha) = \frac{k_x}{q} \quad \sin(\alpha) = \frac{k_y}{q}$$

$$\cos(\beta) = \frac{q}{k} = q \quad \sin(\beta) = \frac{k_z}{k} = q$$

$$\hat{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} \quad \hat{q} = \begin{bmatrix} k_x \\ k_y \\ 0 \end{bmatrix}$$

β is equal to the angle between \hat{k} and \hat{q}



$$\cos(\alpha) = \frac{k_x}{q} \quad \sin(\alpha) = \frac{k_y}{q}$$

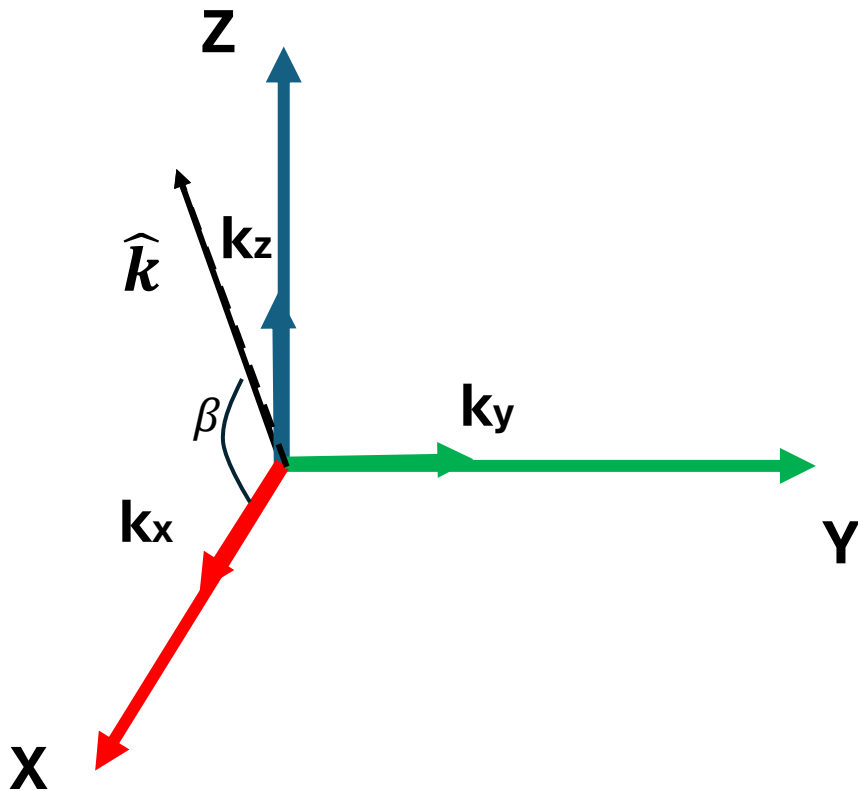
$$\cos(\beta) = \frac{q}{k} = q \quad \sin(\beta) = \frac{k_z}{k} = q$$

$$\hat{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} \quad \hat{q} = \begin{bmatrix} k_x \\ k_y \\ 0 \end{bmatrix}$$

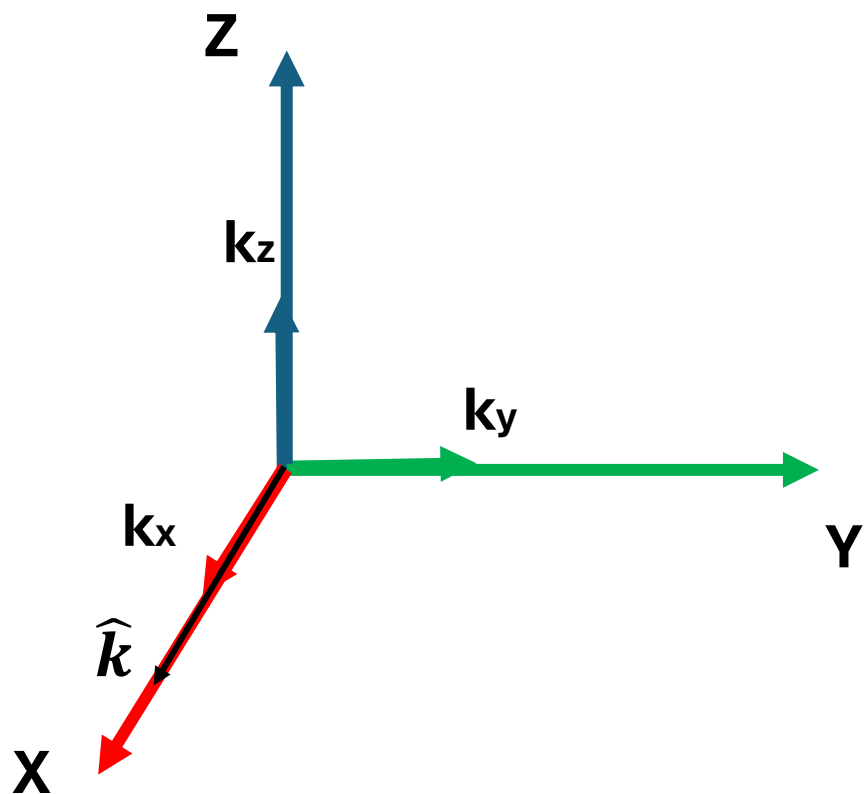
β is equal to the angle between \hat{k} and \hat{q}

$$\cos(\alpha) = \frac{k_x}{q} \quad \sin(\alpha) = \frac{k_y}{q}$$

$$\cos(\beta) = \frac{q}{k} = q \quad \sin(\beta) = \frac{k_z}{k} = q$$



$$R = R_z(-\alpha)$$



$$\hat{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} \quad \hat{q} = \begin{bmatrix} k_x \\ k_y \\ 0 \end{bmatrix}$$

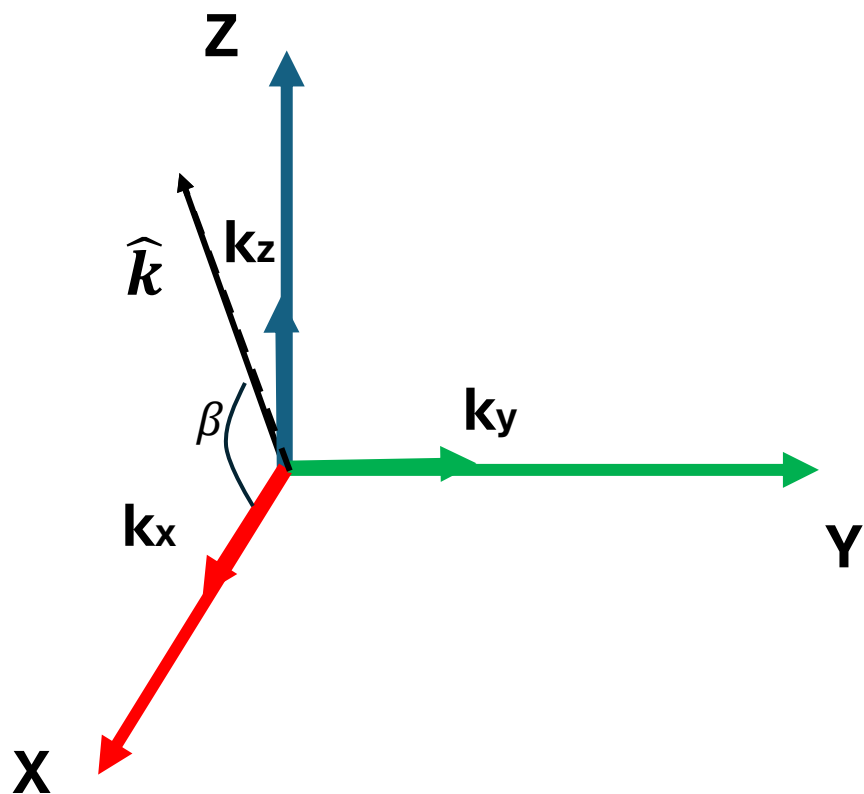
β is equal to the angle between \hat{k} and \hat{q}

$$\cos(\alpha) = \frac{k_x}{q} \quad \sin(\alpha) = \frac{k_y}{q}$$

$$\cos(\beta) = \frac{q}{k} = q \quad \sin(\beta) = \frac{k_z}{k} = q$$

$$R = R_y(\beta)R_z(-\alpha)$$

$$R = R_x(\theta)R_y(\beta)R_z(-\alpha)$$



$$\hat{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} \quad \hat{q} = \begin{bmatrix} k_x \\ k_y \\ 0 \end{bmatrix}$$

β is equal to the angle between \hat{k} and \hat{q}

$$\cos(\alpha) = \frac{k_x}{q} \quad \sin(\alpha) = \frac{k_y}{q}$$

$$\cos(\beta) = \frac{q}{k} = q \quad \sin(\beta) = \frac{k_z}{k} = q$$

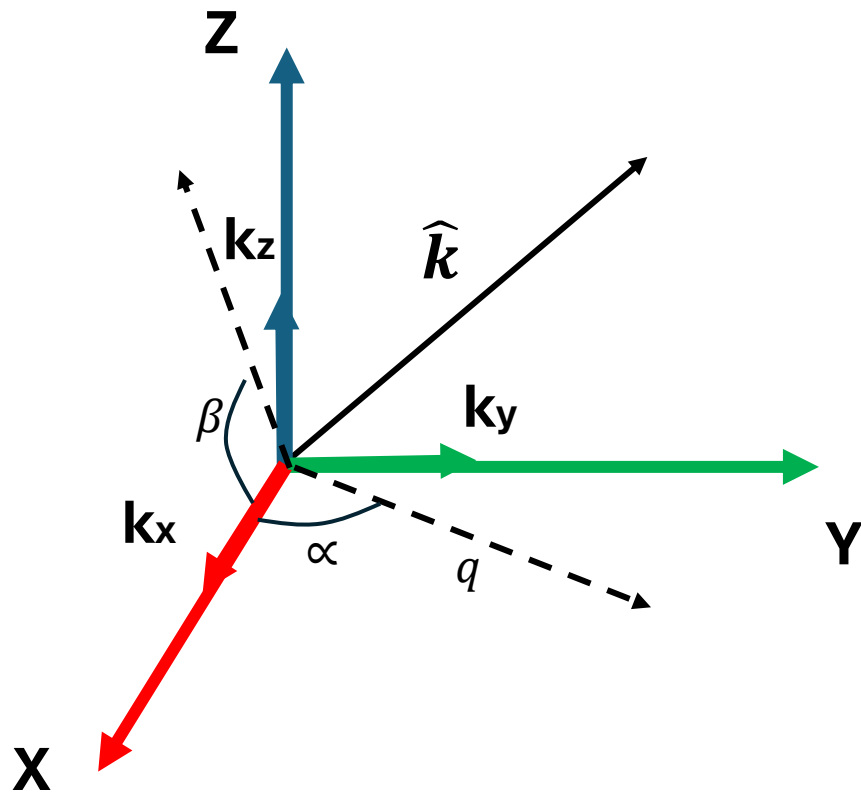
$$R = R_y(-\beta)R_x(\theta)R_y(\beta)R_z(-\alpha)$$

$$\hat{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} \quad \hat{q} = \begin{bmatrix} k_x \\ k_y \\ 0 \end{bmatrix}$$

β is equal to the angle between \hat{k} and \hat{q}

$$\cos(\alpha) = \frac{k_x}{q} \quad \sin(\alpha) = \frac{k_y}{q}$$

$$\cos(\beta) = \frac{q}{k} = q \quad \sin(\beta) = \frac{k_z}{k} = q$$



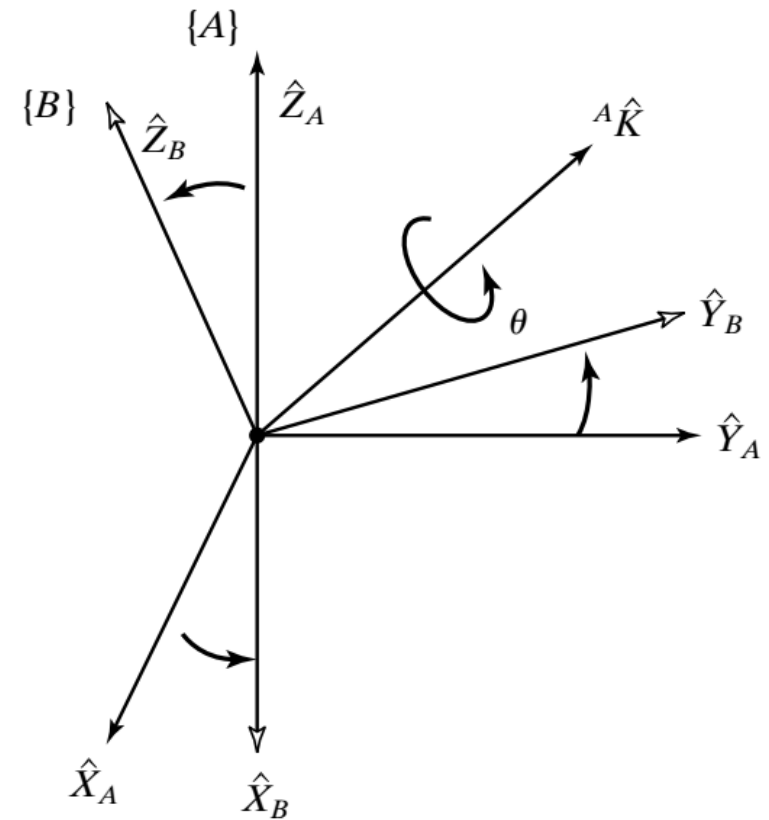
$$R = R_z(\alpha)R_y(-\beta)R_x(\theta)R_y(\beta)R_z(-\alpha)$$

Equivalent angle-axis representations

$$R_k(\theta) = R_z(\alpha)R_y(-\beta)R_x(\theta)R_y(\beta)R_z(-\alpha)$$

$$R_K(\theta) = \begin{bmatrix} k_x k_x v\theta + c\theta & k_x k_y v\theta - k_z s\theta & k_x k_z v\theta + k_y s\theta \\ k_x k_y v\theta + k_z s\theta & k_y k_y v\theta + c\theta & k_y k_z v\theta - k_x s\theta \\ k_x k_z v\theta - k_y s\theta & k_y k_z v\theta + k_x s\theta & k_z k_z v\theta + c\theta \end{bmatrix}$$

where $c\theta = \cos \theta$, $s\theta = \sin \theta$, $v\theta = 1 - \cos \theta$, and ${}^A\hat{K} = [k_x k_y k_z]^T$. The sign of θ is determined by the right-hand rule, with the thumb pointing along the positive sense of ${}^A\hat{K}$.



Equivalent angle-axis representations Inverse

$$R_K(\theta) = \begin{bmatrix} k_x k_x v\theta + c\theta & \underline{k_x k_y v\theta - k_z s\theta} & \underline{k_x k_z v\theta + k_y s\theta} \\ \underline{k_x k_y v\theta + k_z s\theta} & k_y k_y v\theta + c\theta & \underline{k_y k_z v\theta - k_x s\theta} \\ \underline{k_x k_z v\theta - k_y s\theta} & \underline{k_y k_z v\theta + k_x s\theta} & k_z k_z v\theta + c\theta \end{bmatrix}$$

To calculate inverse, we first need to find the trace of the matrix. Trace is the sum of the diagonals of the square matrix.

$$\text{Trace} = (k_x^2 v\theta + c\theta) + (k_y^2 v\theta + c\theta) + (k_z^2 v\theta + c\theta)$$

$$\text{Trace} = (k_x^2 + k_y^2 + k_z^2) v\theta + 3c\theta$$

Since \hat{k} is a unit vector, $k_x^2 + k_y^2 + k_z^2 = 1$.

$$\text{Trace} = 1 - c\theta + 3c\theta$$

$$\text{Trace} = 1 + 2c\theta$$

$$\theta = \text{Acos} \left(\frac{r_{11} + r_{22} + r_{33} - 1}{2} \right)$$

$$\hat{K} = \frac{1}{2 \sin \theta} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

Equivalent angle-axis representation

Example

$$\bullet \hat{K} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

K axis is along the standard x axis of the body. Find the equivalent transformation matrix using Equivalent angle-axis representation

$$R_K(\theta) = \begin{bmatrix} k_x k_x v\theta + c\theta & k_x k_y v\theta - k_z s\theta & k_x k_z v\theta + k_y s\theta \\ k_x k_y v\theta + k_z s\theta & k_y k_y v\theta + c\theta & k_y k_z v\theta - k_x s\theta \\ k_x k_z v\theta - k_y s\theta & k_y k_z v\theta + k_x s\theta & k_z k_z v\theta + c\theta \end{bmatrix}$$

$$R_K(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix}$$

Equivalent angle-axis representations

$${}^A_B R_K(\theta) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix},$$

$$\theta = \text{Acos} \left(\frac{r_{11} + r_{22} + r_{33} - 1}{2} \right) \qquad \hat{K} = \frac{1}{2 \sin \theta} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

Example

A frame $\{B\}$ is described as initially coincident with $\{A\}$. We then rotate $\{B\}$ about the vector ${}^A\hat{K} = [0.707 \ 0.707 \ 0.0]^T$ (passing through the point ${}^A P = [1.0 \ 2.0 \ 3.0]$) by an amount $\theta = 30$ degrees. Give the frame description of $\{B\}$.

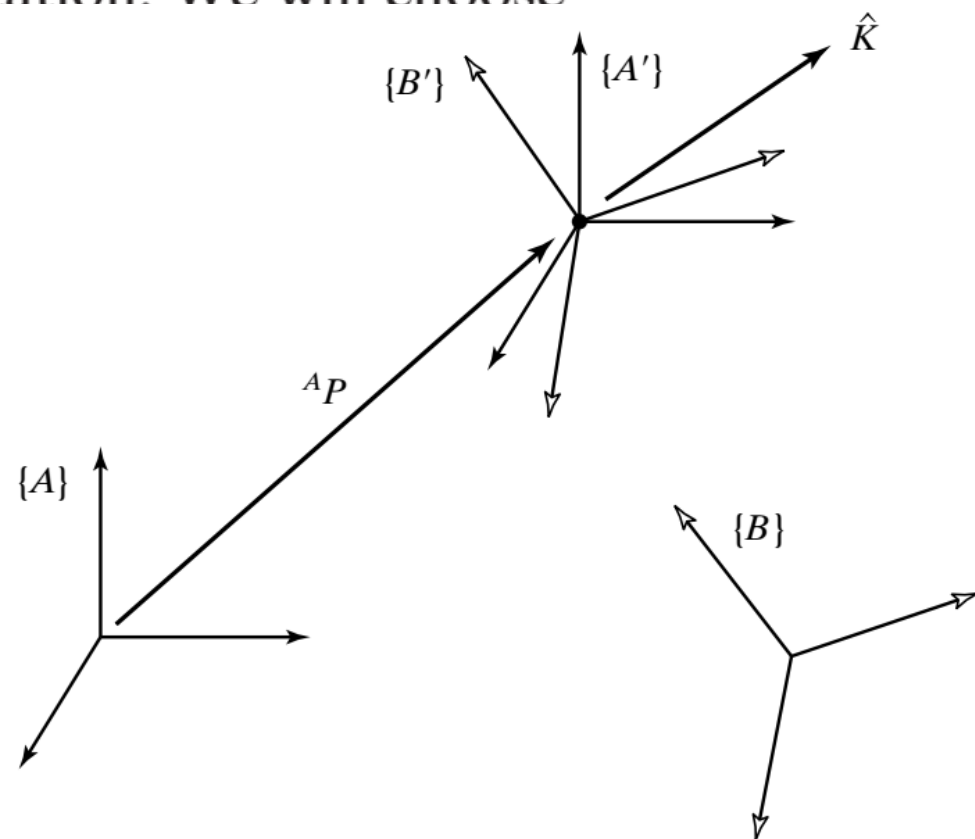
A frame $\{B\}$ is described as initially coincident with $\{A\}$. We then rotate $\{B\}$ about the vector ${}^A\hat{K} = [0.707 \ 0.707 \ 0.0]^T$ (passing through the point ${}^AP = [1.0 \ 2.0 \ 3.0]$) by an amount $\theta = 30$ degrees. Give the frame description of $\{B\}$.

Before the rotation, $\{A\}$ and $\{B\}$ are coincident. As is shown in Fig. 2.20, we define two new frames, $\{A'\}$ and $\{B'\}$, which are coincident with each other and have the same orientation as $\{A\}$ and $\{B\}$ respectively, but are translated relative to $\{A\}$ by an offset that places their origins on the axis of rotation. We will choose

$${}_{A'}^AT = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 1.0 \\ 0.0 & 1.0 & 0.0 & 2.0 \\ 0.0 & 0.0 & 1.0 & 3.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}$$

Similarly, the description of $\{B\}$ in terms of $\{B'\}$ is

$${}_{B'}^BT = \begin{bmatrix} 1.0 & 0.0 & 0.0 & -1.0 \\ 0.0 & 1.0 & 0.0 & -2.0 \\ 0.0 & 0.0 & 1.0 & -3.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}$$



Now, keeping other relationships fixed, we can rotate $\{B'\}$ relative to $\{A'\}$. This is a rotation about an axis that passes through the origin, so we can use (2.80) to compute $\{B'\}$ relative to $\{A'\}$. Substituting into (2.80) yields the rotation-matrix part of the frame description. There was no translation of the origin, so the position vector is $[0, 0, 0]^T$. Thus, we have

$${}^{A'}_{B'}T = \begin{bmatrix} 0.933 & 0.067 & 0.354 & 0.0 \\ 0.067 & 0.933 & -0.354 & 0.0 \\ -0.354 & 0.354 & 0.866 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}$$

$0.72(1 - \cos 30) + \cos 30$

Finally, we can write a transform equation to compute the desired frame,

$${}^A_B T = {}^A_{A'} T {}^{A'}_{B'} T {}^{B'}_B T,$$

which evaluates to

$${}^A_B T = \begin{bmatrix} 0.933 & 0.067 & 0.354 & -1.13 \\ 0.067 & 0.933 & -0.354 & 1.13 \\ -0.354 & 0.354 & 0.866 & 0.05 \\ 0.000 & 0.000 & 0.000 & 1.00 \end{bmatrix}.$$

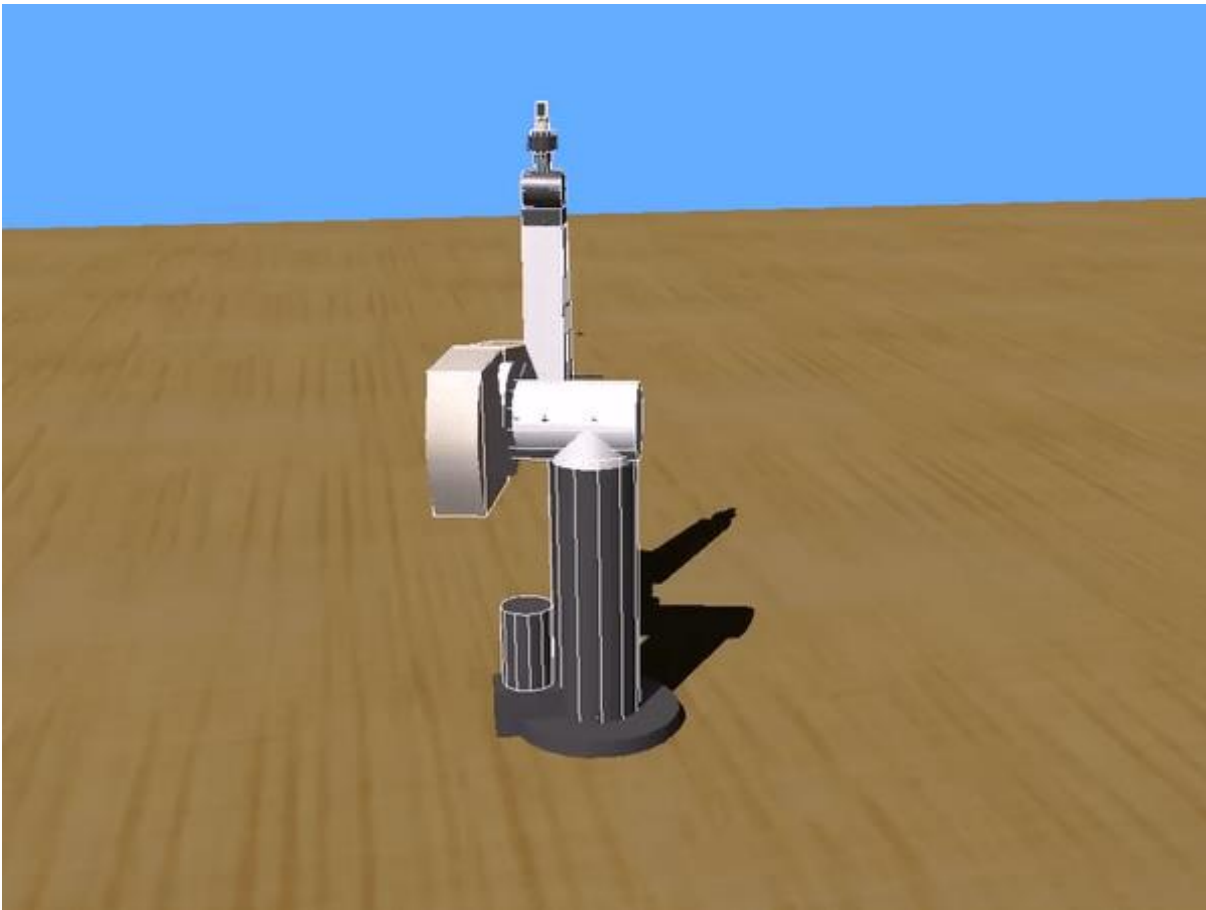
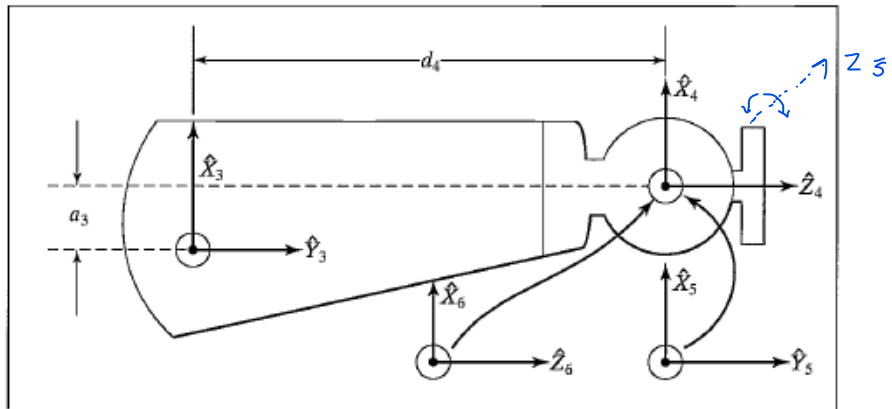
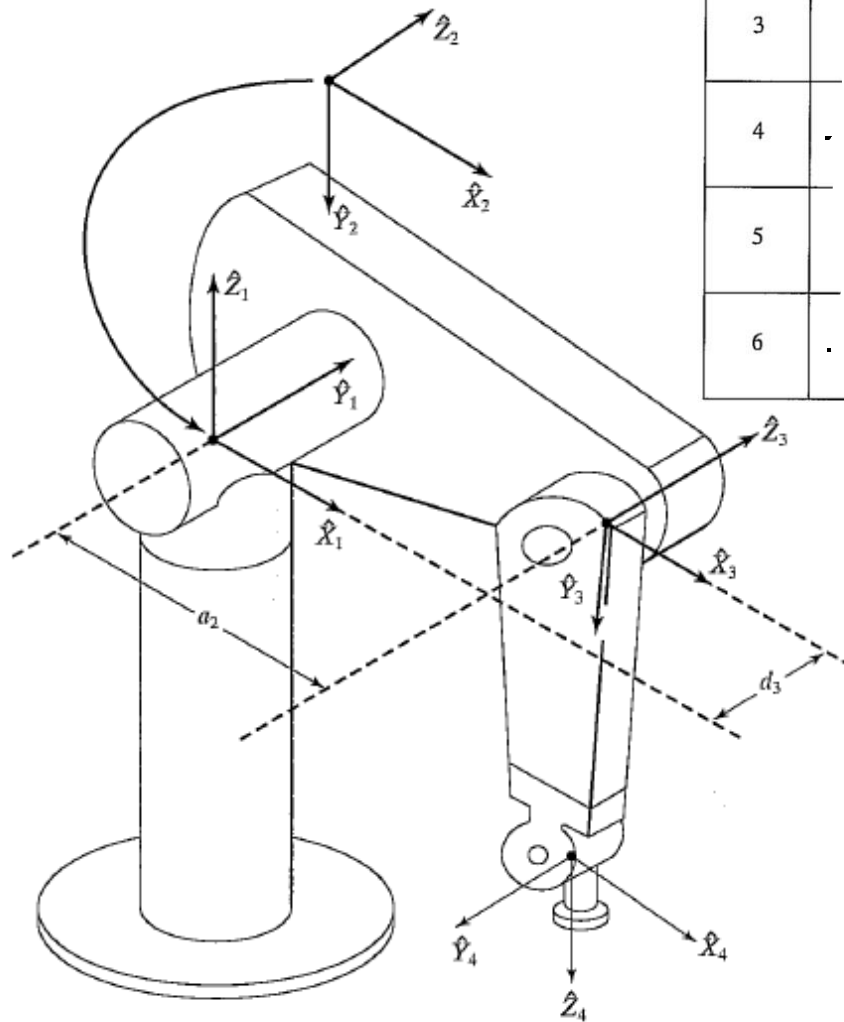
MANIPULATOR KINEMATICS

Forward kinematics

1. Identify joints and links of a robot.
2. Assign coordinate systems.
3. Find joint and link parameters (Denavit Hartenberg Table).
4. Derive Transformation matrices using the DH parameters.
5. Compute forward kinematics.

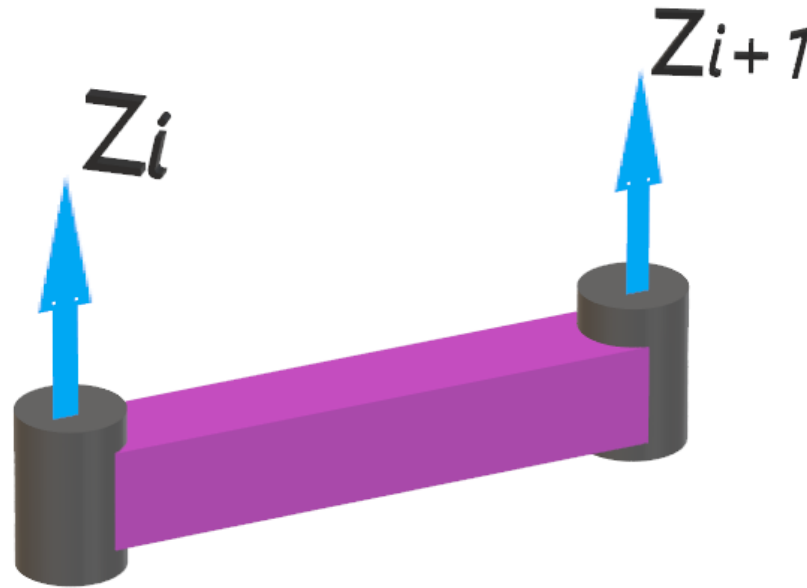
link twist l_length j_offset j_anl

i	$\alpha_i - 1$	$a_i - 1$	d_i	θ_i
1				
2				
3				
4				
5				
6				



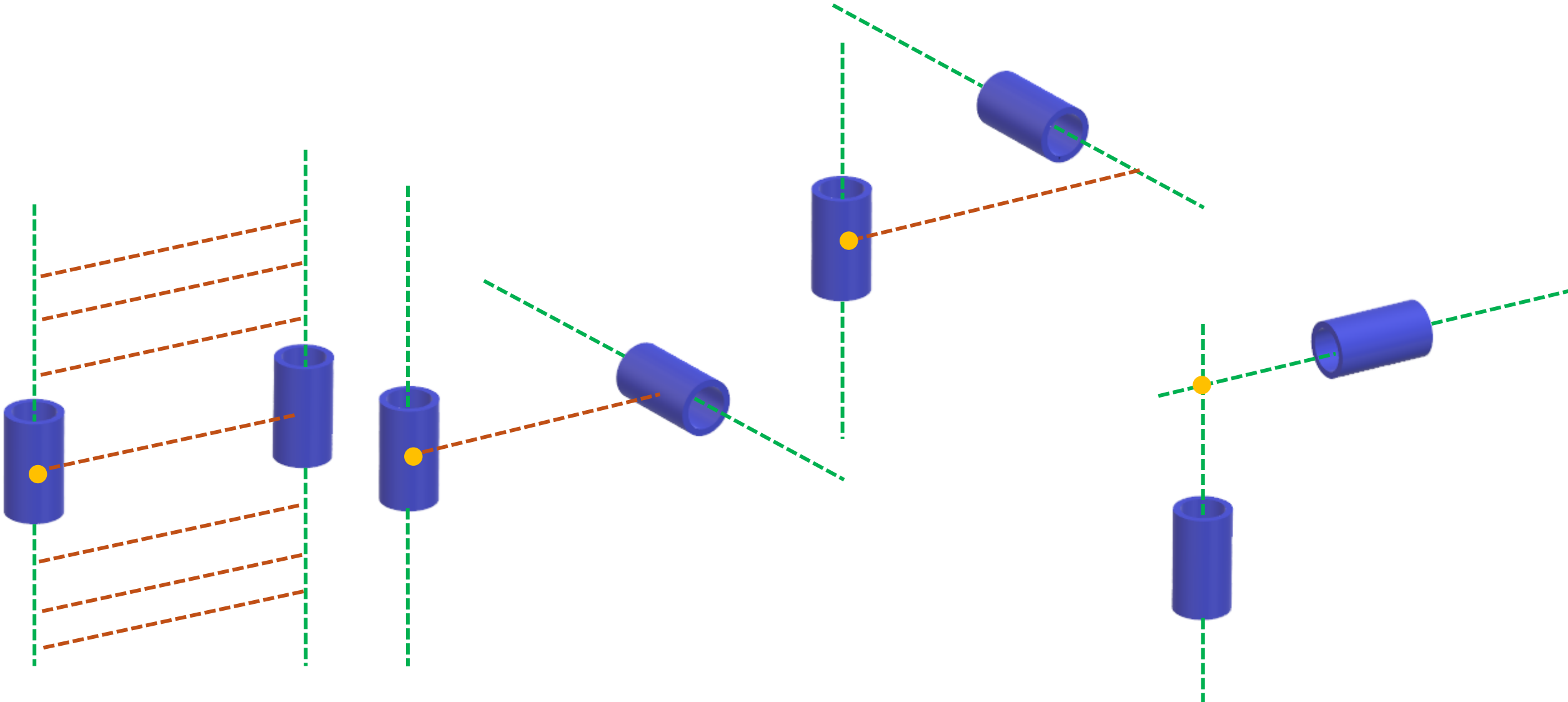
3.4 Convention for Affixing Frames to Links

Z axes: Assign Z-axes along the axis of rotation for revolute joints and axis of translation for prismatic joints.



3.4 Convention for Affixing Frames to Links

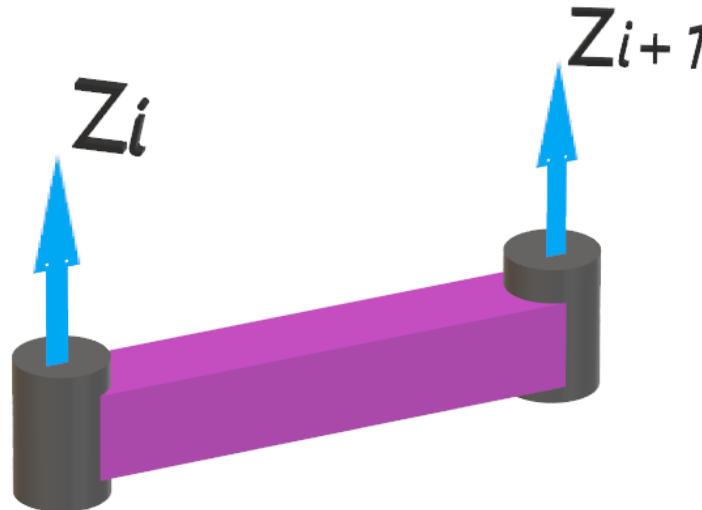
Origin of Frame: It is the common perpendicular or the point of intersection.



Assigning Frames to Robot Manipulator

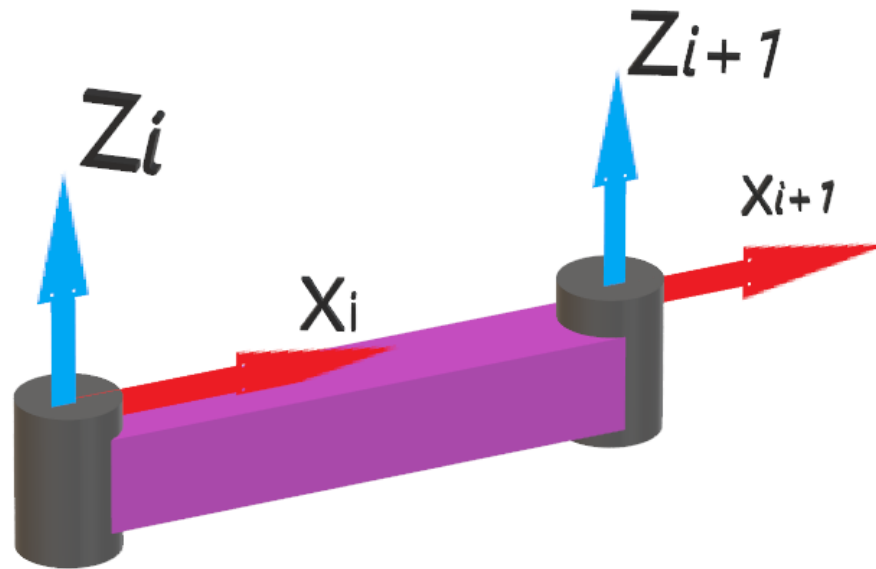
To set the origin of the i th frame.

Point of intersection between joint axes (i and $i+1$) or at the start of common normal between the joint (i and $i+1$)



Assigning Frames to Robot Manipulator

Assign X_i axis along the common normal and if the axes intersect, then X_i would be normal to the plane containing the two axes.

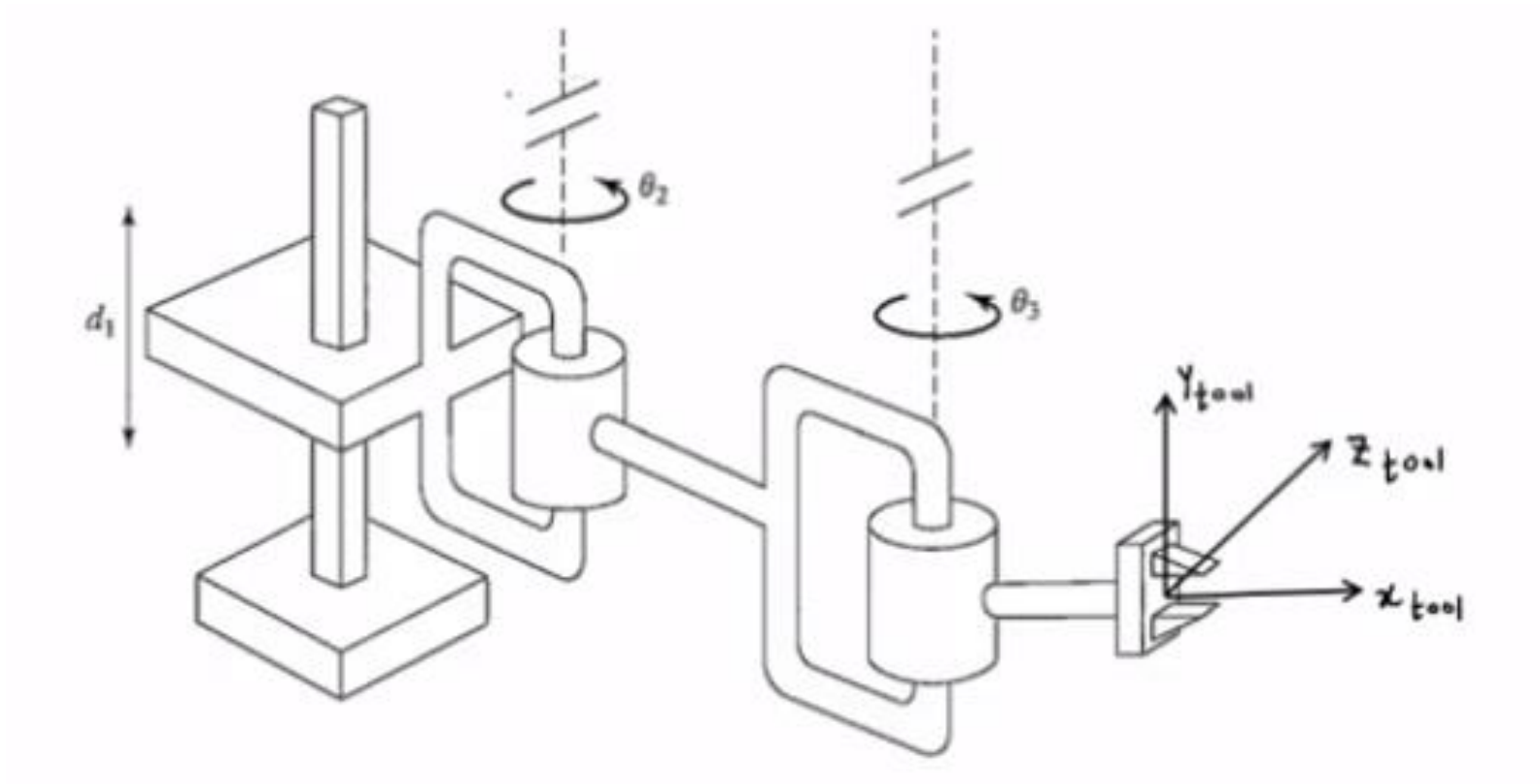


Assigning Frames to Robot Manipulator

Frame {0} and Frame {n}:

Frame {0} is generally attached to base of the robot and to match frame{1} such that most of the parameters can be made to zero.

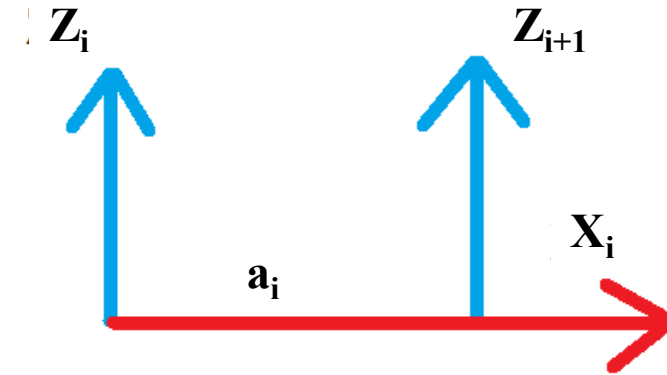
Frame {n} is attached to the end-effector of the robot.



DH Parameters

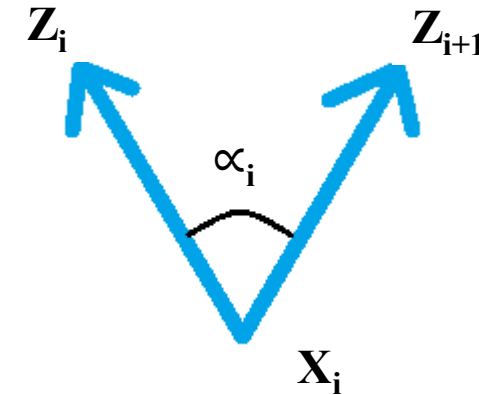
Link length:

a_i = Distance from Z_i to Z_{i+1} along X_i



Link twist:

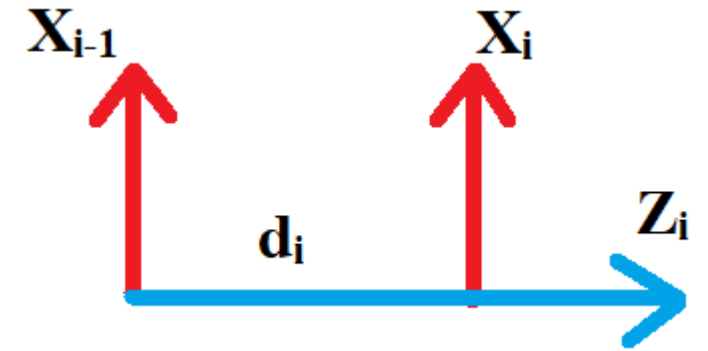
α_i = Angle from Z_i to Z_{i+1} about X_i



DH Parameters:

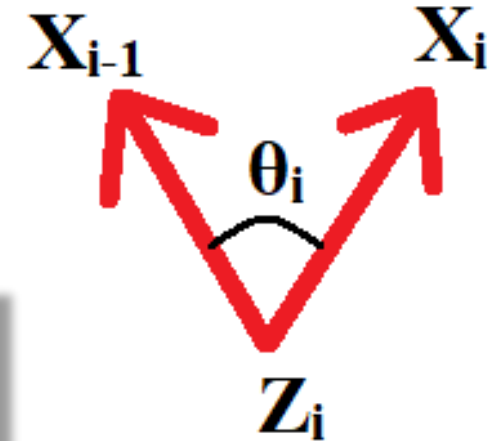
Joint offset:

d_i = Distance from X_{i-1} to X_i along Z_i



Joint angle:

θ_i = Angle from X_{i-1} to X_i about Z_i

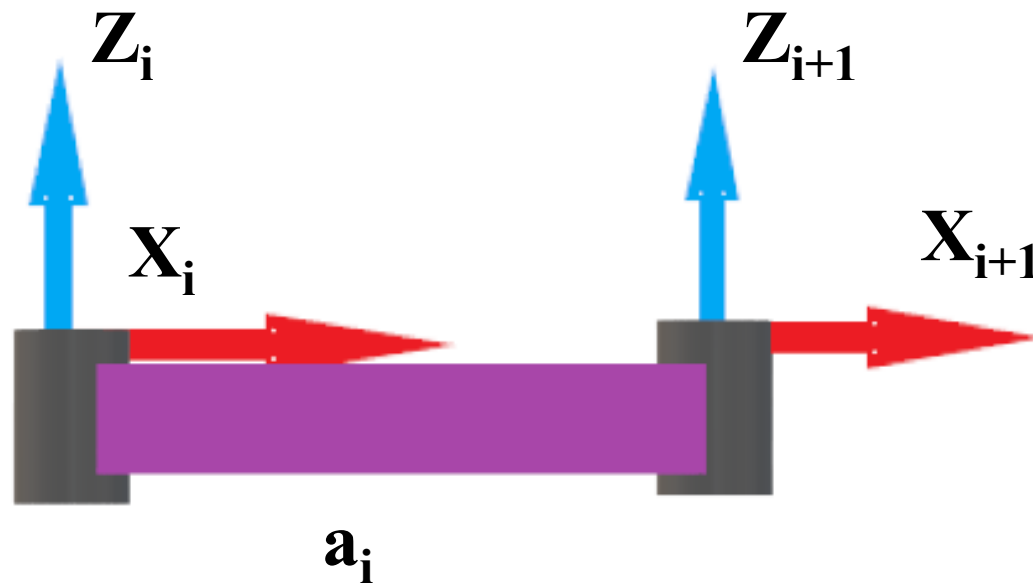


a_i = the distance from \hat{Z}_i to \hat{Z}_{i+1} measured along \hat{X}_i ;
 α_i = the angle from \hat{Z}_i to \hat{Z}_{i+1} measured about \hat{X}_i ;
 d_i = the distance from \hat{X}_{i-1} to \hat{X}_i measured along \hat{Z}_i ; and
 θ_i = the angle from \hat{X}_{i-1} to \hat{X}_i measured about \hat{Z}_i .

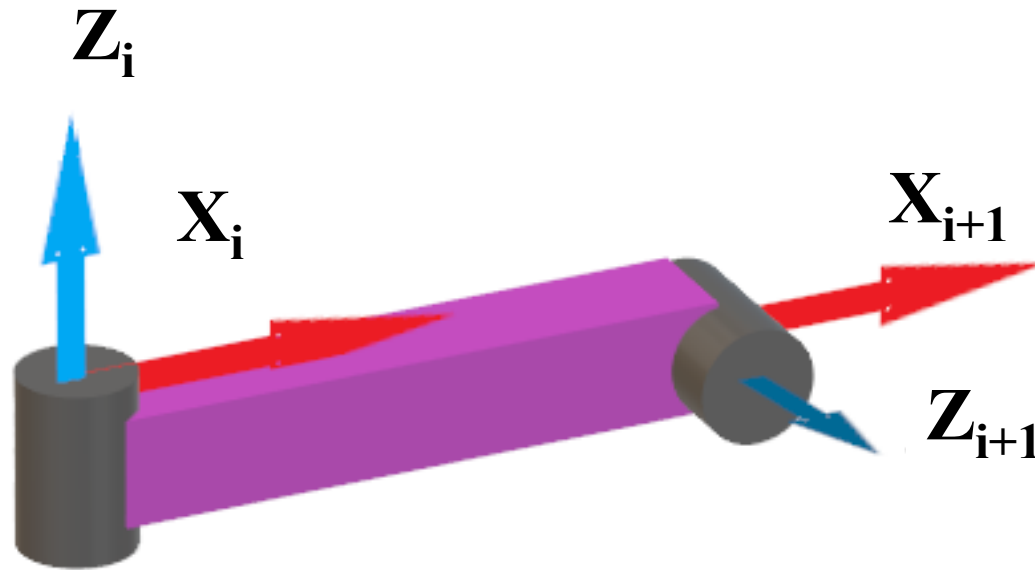
Link Description

Link length (a) and link twist (α) are the two parameters used to describe a link.

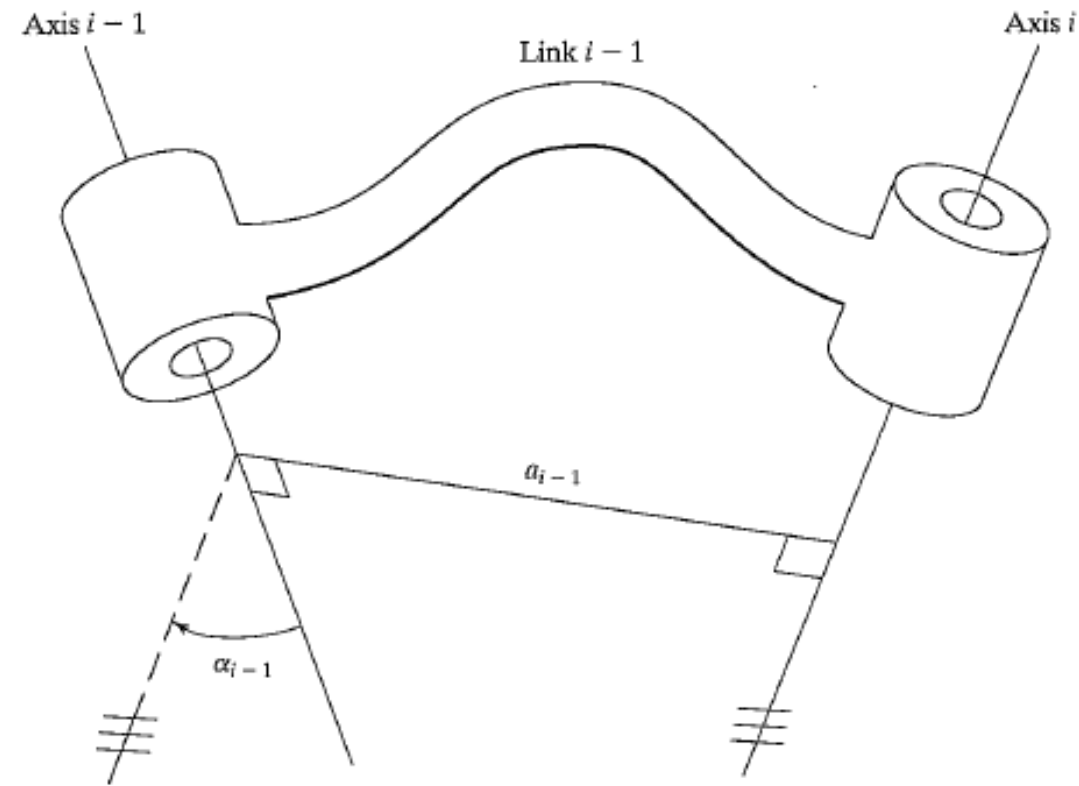
Link Length:



Twist Angle: The angle about the common perpendicular line between the joint axes

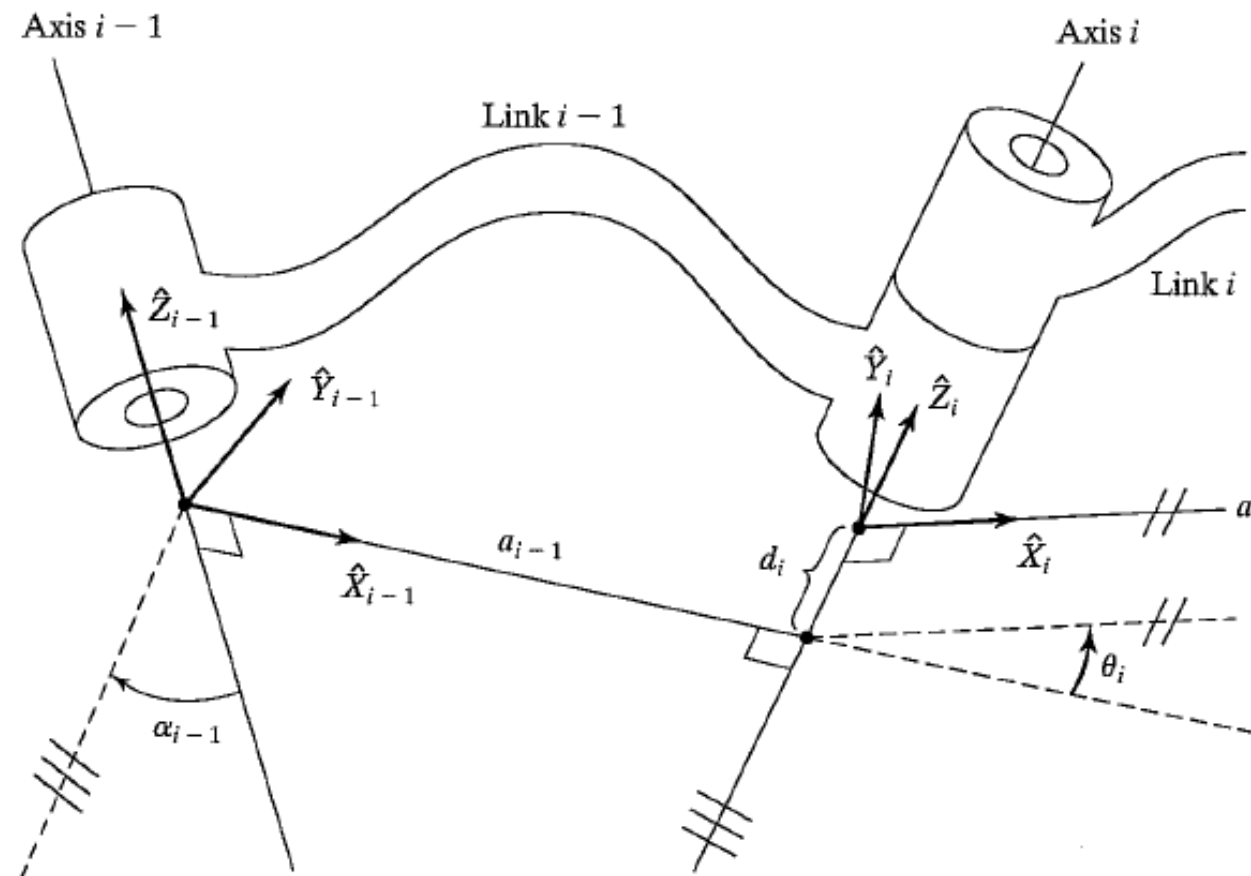


Link Description

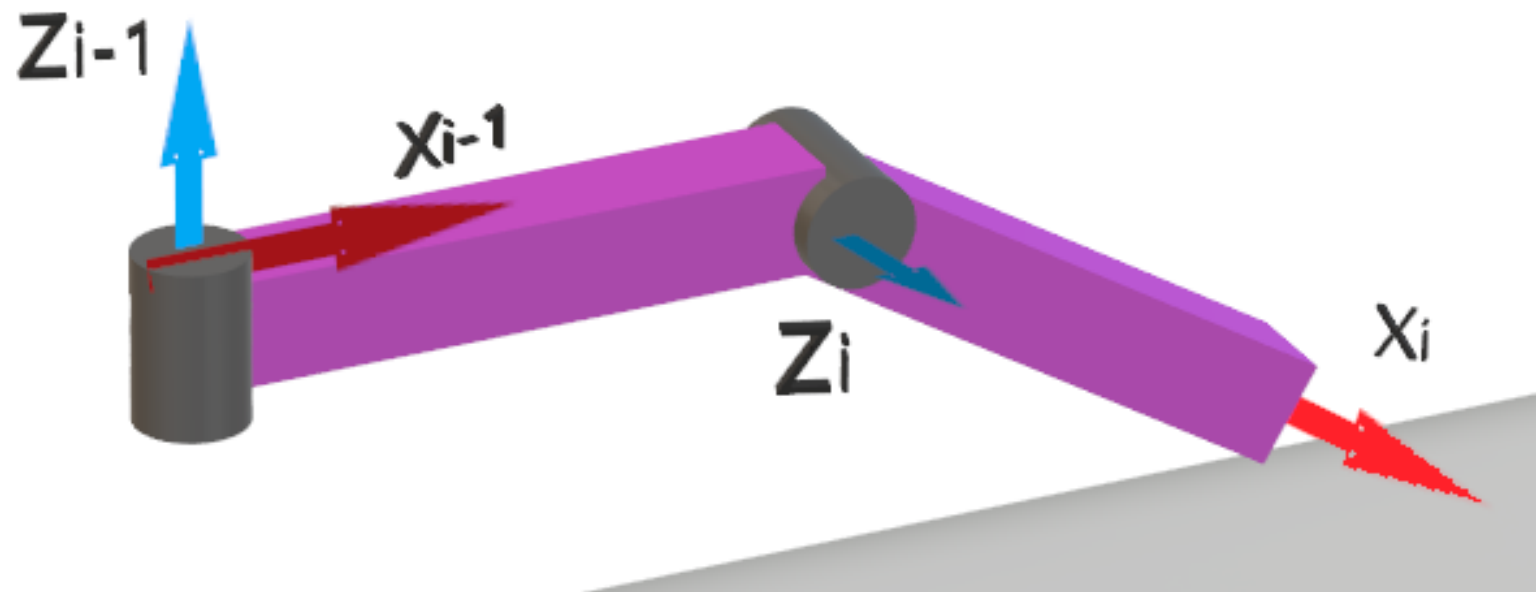


Joint Description: Two parameters are used to describe the joint. Joint offset (d_i) and joint angle (θ_i)

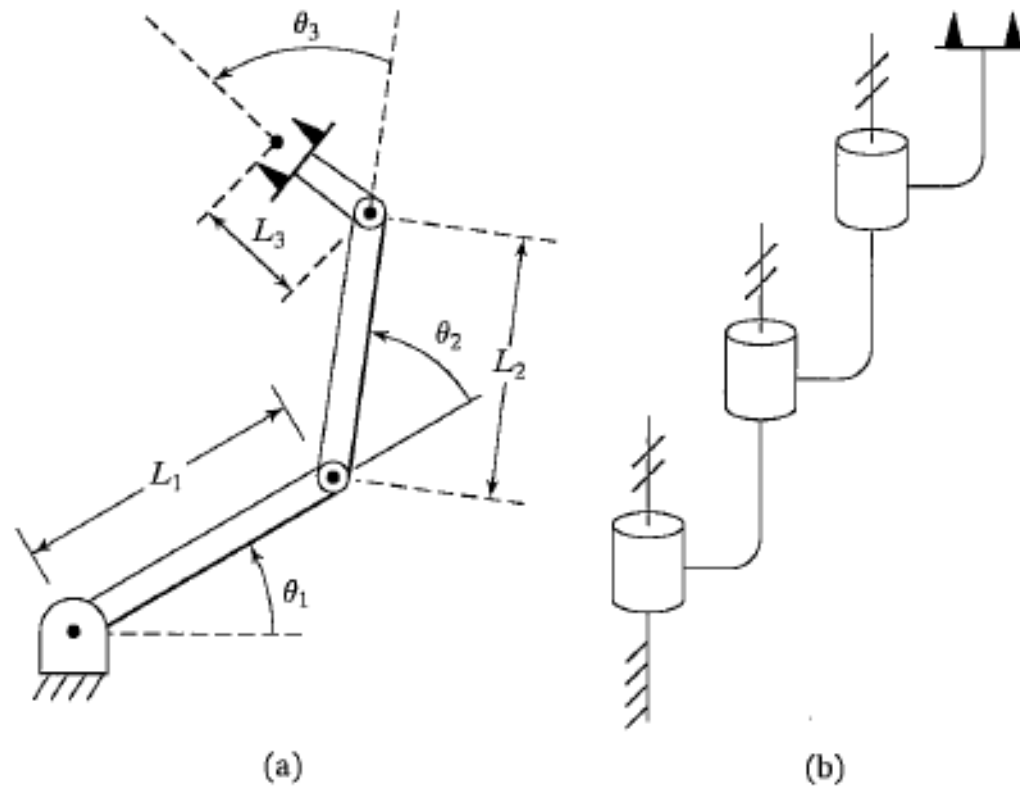
Joint Offset: The length along axis of motion between two common perpendiculars



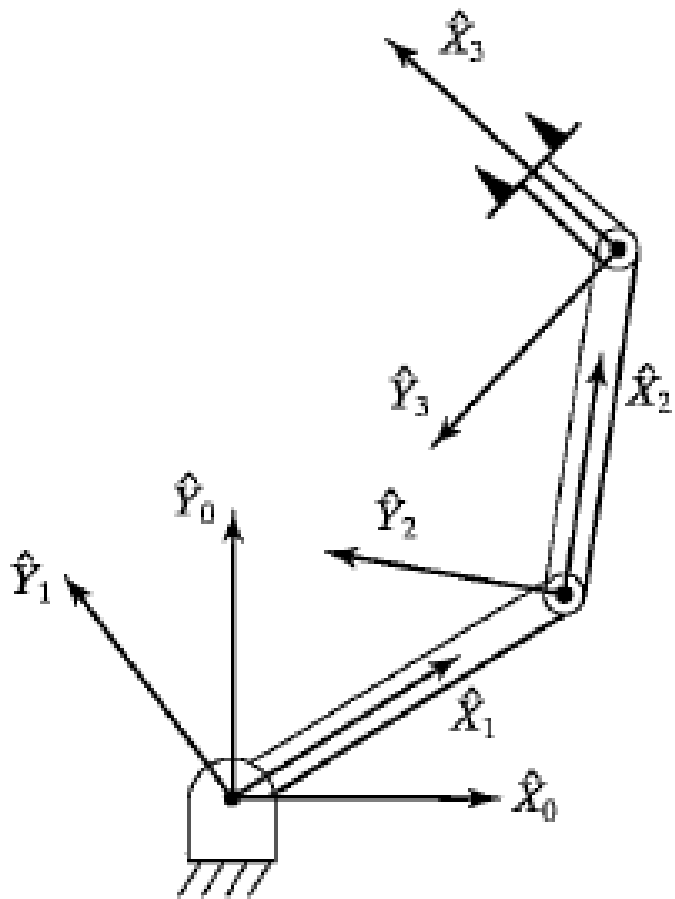
Joint Angle: The angle about the axis of motion between two common perpendiculars.



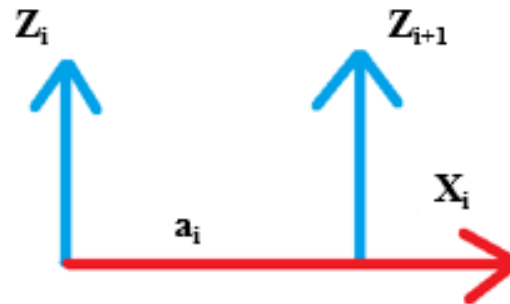
Example 3.3



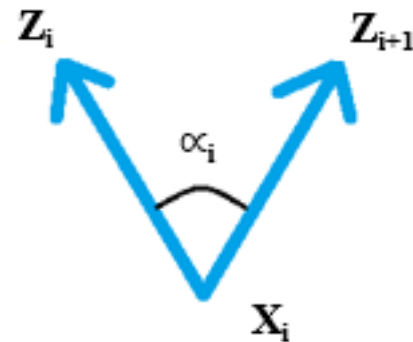
Example 3.3



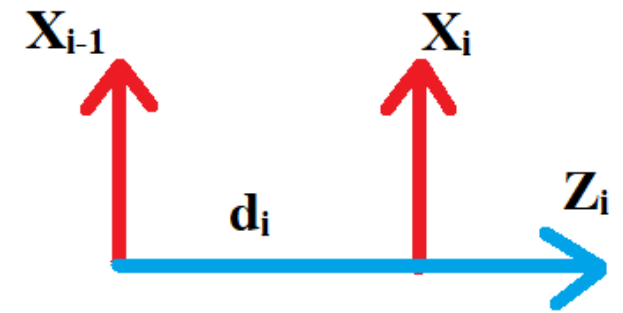
Link Length



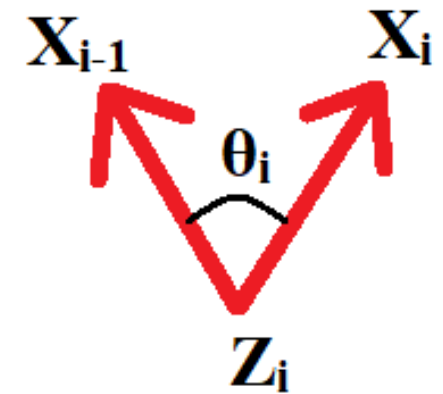
Link Twist



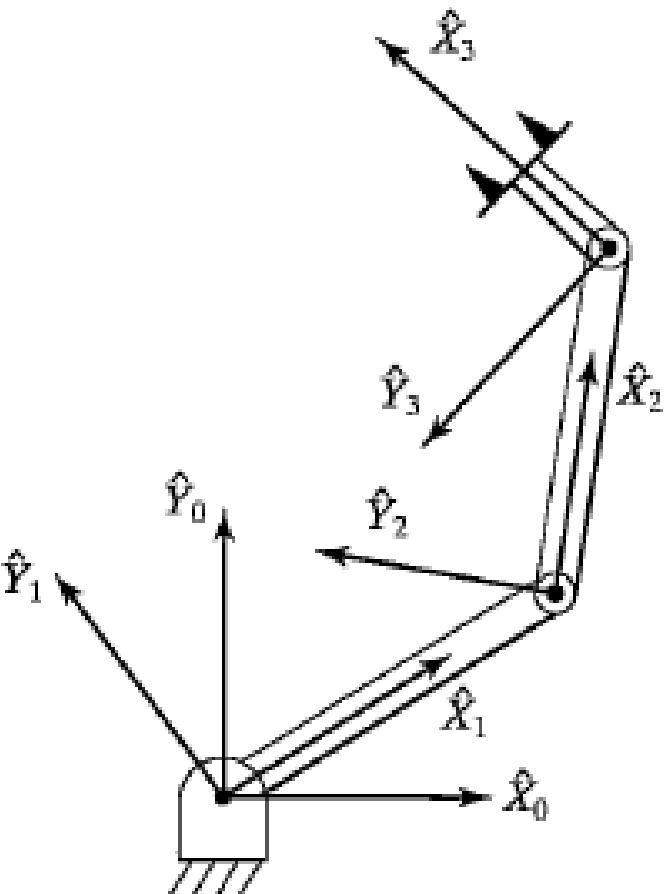
Joint Offset



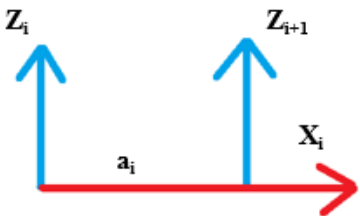
Joint Angle



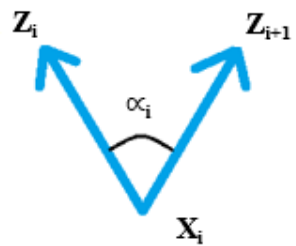
Example 3.3



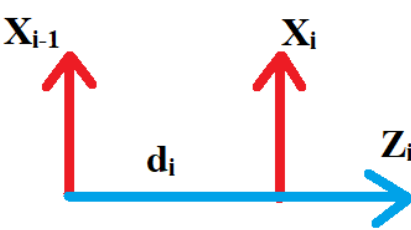
Link Length



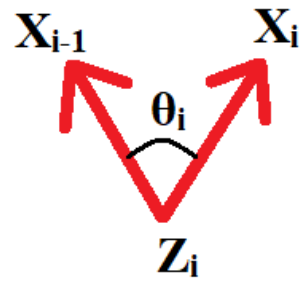
Link Twist



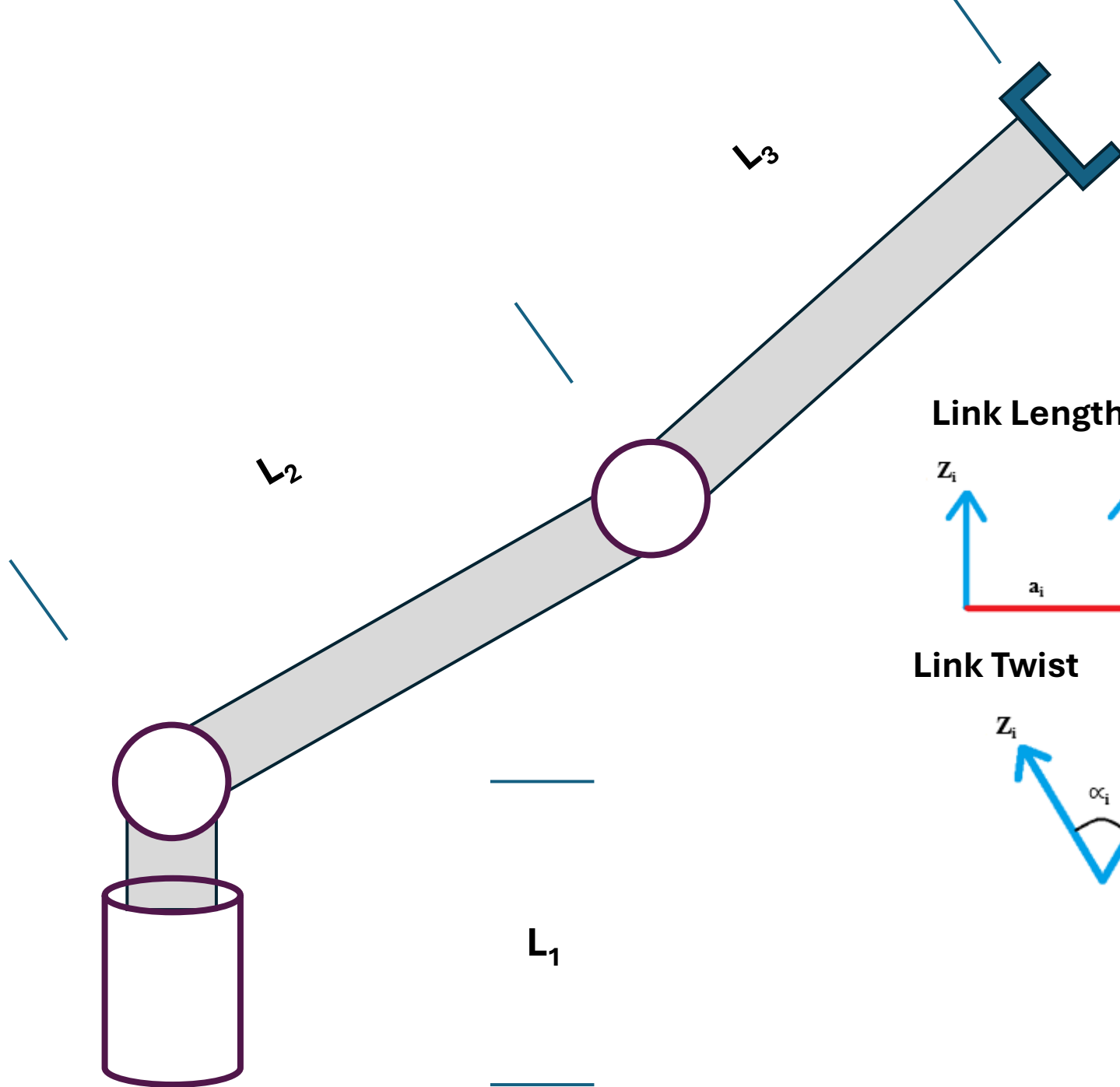
Joint Offset



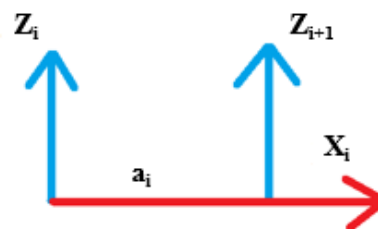
Joint Angle



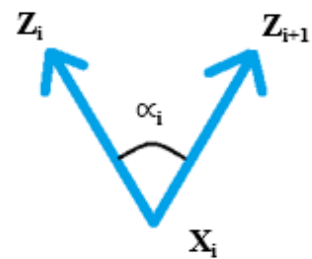
i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	L_1	0	θ_2
3	0	L_2	0	θ_3



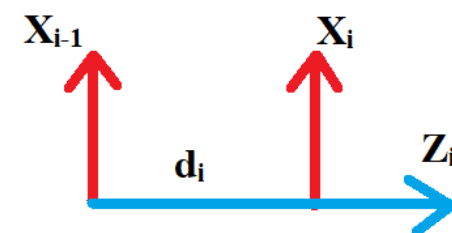
Link Length



Link Twist



Joint Offset



Joint Angle

