Digital Signal Processing (EC 335)

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Lecture 6

Lecture Targets

- Discrete time Fourier transform (DTFT)

- Properties of DTFT

DTFT

- Periodicity in time domain results in sampling in Frequency domain
- Sampling in time-domain results in periodicity in Frequency domain
- Sampling is used to reflect the Discrete nature of the signal here

Complex exponential DT signal

$$e^{j\omega_o n}$$

$$\omega_o \rightarrow \omega_o + 2\pi$$

$$e^{j(\omega_o + 2\pi)n} = e^{j\omega_o n} \times \underbrace{e^{j2\pi n}}_{1}$$

$$e^{j(\omega_o + 2\pi)n} = e^{j\omega_o n}$$

All Complex exponential DT signals are periodic in frequency with a period of 2π. DT complex exponential signals will identical at ω, $\omega \pm 2\pi$, $\omega \pm 4\pi$,...

DTFT

$$C.T.F.T \Rightarrow X(j\omega) = \int_{-\infty}^{\infty} x(t).e^{-j\omega t}\,dt$$

$$D.T.F.T \Rightarrow X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n].e^{-j\omega n} \text{(Analysis Equation)}$$

$$I.C.T.F.T \Rightarrow x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega).e^{j\omega t}\,d\omega$$

$$I.D.T.F.T \Rightarrow X[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}).e^{j\omega n}\,d\omega \text{(Synthesis Equation)}$$

DTFT

Important Formulas for Summation

$$(i).\ \sum_{n=0}^{\infty}(a)^n = \frac{1}{1-a} \because \mathrm{GP}$$

$$(ii).\ \sum_{n=0}^{N}(a)^n = \frac{1-a^{N+1}}{1-a}$$

$$(iii).\ \sum_{n=0}^{N}(1)^n = N+1,\ \sum_{n=a}^{b}(1)^n = (b-a)+1,$$

$$(iv).\ \sum_{n=0}^{N} n = \frac{N(N+1)}{2}$$

$$(v).\ \sum_{n=0}^{N} n^2 = \frac{N(N+1)(2N+1)}{6}$$

$$(vi).\ \sum_{n=0}^{N} n^3 = \left(\frac{N(N+1)}{2}\right)^2$$

DTFT

$$x[n] = a^n u[n], 0 < a < 1$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n].e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n .e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^n$$

$$\therefore \sum_{n=0}^{\infty} (a)^n = \frac{1}{1-a}$$

$$X(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}$$

$$x[n] = \delta[n]$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n].e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n].e^{-j\omega n}$$

$$X(e^{j\omega}) = e^{-j\omega n}\Big|_{n=0}$$

$$X(e^{j\omega}) = 1$$

$$\boxed{x[n] = a^n u[n] + a^{-n} u[-n-1]}$$

$$x[n] = a^{-n} u[-n], 0 < a < 1$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n].e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{0} a^{-n}.e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{0} \left(ae^{j\omega}\right)^{-n}$$

Let

$$m = -n, n \to -\infty \Rightarrow m \to \infty$$

$$n \to 0 \Rightarrow m \to 0$$

$$X(e^{j\omega}) = \sum_{m=0}^{\infty} \left(ae^{j\omega}\right)^m$$

$$\therefore \sum_{n=0}^{\infty} (a)^n = \frac{1}{1-a}$$

$$X(e^{j\omega}) = \frac{1}{1-ae^{j\omega}}$$

DTFT

Two important results

$e^{-j\pi} = -1$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$x(n)(-1)^n; \omega = \pi$

The value of DTFT at $\omega = 0$

$$X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n]$$

$x[n] = \{1,2,3,4,7\}$, origin at 3

The value of DTFT at $(\omega = 0) = ???$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

at $n = 0$

$$x[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) d\omega$$

$$\Rightarrow \int_{-\pi}^{\pi} X(e^{j\omega}) d\omega = 2\pi . x[0]$$

DTFT

If $x[n]=\{-4, 3, -2, 3, -4\}$ origin at -2, find

(i). $x\left[e^{j0}\right]$

(ii). $x\left[e^{j\pi}\right]$

(iii). $\int_{-\pi}^{\pi} x\left[e^{j\omega}\right] d\omega$ -4π

Properties of DTFT

Linearity

$x_1[n] \to X_1(e^{j\omega})$

$x_2[n] \to X_2(e^{j\omega})$

$\alpha x_1[n] + \beta x_2[n] \to \alpha X_1(e^{j\omega}) + \beta X_2(e^{j\omega})$

Time – shifting

$x[n] \to X(e^{j\omega})$

$x[n \pm n_o) \to \left(e^{\pm j\omega n_o}\right) X(e^{j\omega})$

$\delta[n] \to 1$

$\delta[n-1] \to e^{-j\omega} \times 1 = e^{-j\omega}$

$\delta[n+1] \to e^{j\omega} \times 1 = e^{j\omega}$

Time – Reversal

$x[n] \to X(e^{j\omega})$

$x[-n] \to X(e^{-j\omega})$

$a^n u[n] \to \dfrac{1}{1 - ae^{-j\omega}}$

$a^{-n} u[-n] \to \dfrac{1}{1 - ae^{j\omega}}$

Differencing Propoerty

$x[n] \to X(e^{j\omega})$

$x[n] - x[n - n_o] \to \left[1 - e^{-j\omega n_o}\right] X(e^{j\omega})$

First difference

$x[n] - x[n-1] \to \left[1 - e^{-j\omega}\right] X(e^{j\omega})$

Accumulation Propoerty

$x[n] \to X(e^{j\omega})$

$\sum_{k=-\infty}^{n} x[n] \to \left[\dfrac{1}{1 - e^{-j\omega}}\right] X(e^{j\omega})$

$x_1[n] \to X_1(e^{j\omega}), x_2[n] \to X_2(e^{j\omega})$

$x_1[n] * x_2[n] \to X_1(e^{j\omega}) \times X_2(e^{j\omega})$

$x_1[n] = \left(\dfrac{1}{4}\right)^n u[n], x_2[n] = \left(\dfrac{1}{3}\right)^n u[n]$

Find DTFTof $x[n] = x_1[n] * x_2[n]$

$\therefore a^n u[n] = \dfrac{1}{1 - ae^{jw}}$

Expansion Property

$x[n] \to X(e^{j\omega})$

$x\left[\dfrac{n}{k}\right] \to X\left(e^{jk\omega}\right)$

Transform will be

periodic with period $\dfrac{2\pi}{\mathrm{k}}$

$x[n] = \{1, 2, 3\}$

$x[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2]$

$X(e^{j\omega}) = 1 + 2e^{-j\omega} + 3e^{-j2\omega}$..........(1)

$x_1[n] = x\left[\dfrac{n}{2}\right] = \{1, 0, 2, 0, 3\}$

$x_1[n] = \delta[n] + 2\delta[n-2] + 3\delta[n-4]$

$X_1(e^{j\omega}) = 1 + 2e^{-j2\omega} + 3e^{-j4\omega}$

Using Expansion Property

Replace $\omega \to \mathrm{k}\omega \therefore k = 2 \Rightarrow \omega = 2\omega$

in Eq(1)

$X_1(e^{j\omega}) = 1 + 2e^{-j2\omega} + 3e^{-j4\omega}$

Discrete Fourier Transform (DFT)

- **DTFT is used to evaluate the frequency response of DT signal.**

$$x[n] \rightarrow X(e^{j\omega})$$

- **The signal in Frequency domain (applying DTFT) will always be periodic. Why????**

- **DFT is also used to evaluate the frequency response of DT signal.**

- **What is the difference between DTFT and DFT then???**

- **The transform from DTFT is of <u>continuous nature.</u> We cannot use continuous signals in digital signal processors. So, we might use DFT instead of DTFT if the application is DSP.**