# Digital Signal Processing (EC 335)

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Lecture 3
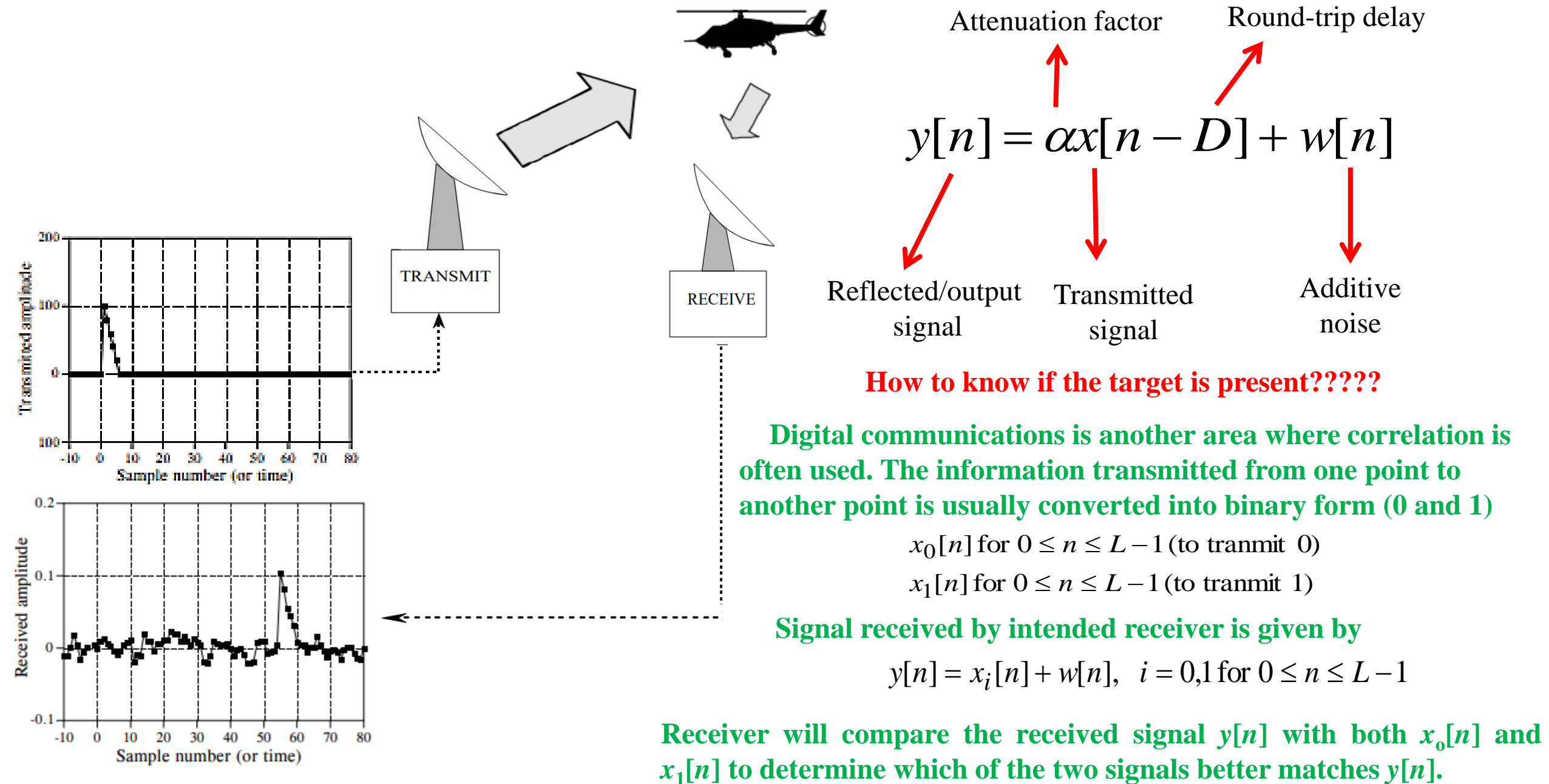
# Lecture Targets

❑ Correlation

❑ Cross Correlation

❑ Auto Correlation

❑ Properties of Correlation

# Correlation

➢ A signal operation similar to signal convolution, but with completely different physical meaning, is signal correlation.

➢ The signal correlation operation can be performed either with one signal (autocorrelation) or between two different signals (cross correlation).

➢ Physically, signal autocorrelation indicates how the signal energy (power) is distributed within the signal, and as such is used to measure the signal power.

➢ Typical applications of signal autocorrelation are in radar, sonar, satellite, and wireless communications systems.

➢ Devices that measure signal power using signal correlation are known as signal correlators.

➢ There are also many applications of signal cross correlation in signal processing systems, especially when the signal is corrupted by another undesirable signal (noise) so that the signal estimation (detection) from a noisy signal has to be performed.

➢ Signal cross correlation can be also considered as a measure of similarity of two signals.

# Correlation

Attenuation factor    Round-trip delay

$$y[n] = \alpha x[n - D] + w[n]$$

Reflected/output signal    Transmitted signal    Additive noise

**How to know if the target is present?????**

**Digital communications is another area where correlation is often used. The information transmitted from one point to another point is usually converted into binary form (0 and 1)**

$$x_0[n] \text{ for } 0 \leq n \leq L-1 \text{ (to tranmit 0)}$$
$$x_1[n] \text{ for } 0 \leq n \leq L-1 \text{ (to tranmit 1)}$$

**Signal received by intended receiver is given by**

$$y[n] = x_i[n] + w[n], \quad i = 0,1 \text{ for } 0 \leq n \leq L-1$$

**Receiver will compare the received signal $y[n]$ with both $x_0[n]$ and $x_1[n]$ to determine which of the two signals better matches $y[n]$.**



TRANSMIT

RECEIVE

# Correlation

**Cross Correlation of Energy signals**: Let $x(t)$ and $y(t)$ be the two Energy signals, then

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t).y^*(t-\tau)dt = \int_{-\infty}^{\infty} x(t+\tau).y^*(t)dt$$

If $x(t)$ and $y(t)$ are real

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t).y(t-\tau)dt = \int_{-\infty}^{\infty} x(t+\tau).y(t)dt \qquad \text{Eq (1)}$$

$$R_{yx}(\tau) = \int_{-\infty}^{\infty} y(t).x^*(t-\tau)dt = \int_{-\infty}^{\infty} y(t+\tau).x^*(t)dt$$

If $x(t)$ and $y(t)$ are real

$$R_{yx}(\tau) = \int_{-\infty}^{\infty} y(t).x(t-\tau)dt = \int_{-\infty}^{\infty} y(t+\tau).x(t)dt \qquad \text{Eq (2)}$$

# Correlation

**Put $\tau = -\tau$ in Equation (1), we have**

$$R_{xy}(-\tau) = \int_{-\infty}^{\infty} x(t).y(t+\tau)dt \qquad R_{yx}(\tau) = \int_{-\infty}^{\infty} y(t).x(t-\tau)dt = \int_{-\infty}^{\infty} y(t+\tau).x(t)dt$$

$$\boxed{R_{xy}(-\tau) = R_{yx}(\tau)}$$

Let $x(t)$ and $y(t)$ be the two Power signals, then

$$R_{xy}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t).y^*(t-\tau)dt = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t+\tau).y^*(t)dt$$

*If the signal is power signal and periodic, then we can eliminate this term*

# Correlation

$$R_{yx}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} y(t).x^*(t-\tau)dt = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} y(t+\tau).x^*(t)dt$$

$$\boxed{R_{xy}(-\tau) = R_{yx}(\tau)}$$

**Auto Correlation for Energy Signal**

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t).x^*(t-\tau)dt = \int_{-\infty}^{\infty} x(t+\tau).x^*(t)dt$$

If $x(t)$ is real

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t).x(t-\tau)dt = \int_{-\infty}^{\infty} x(t+\tau).x(t)dt$$

$$\boxed{R_{xx}(-\tau) = R_{xx}(\tau)}$$

# Correlation

## Auto Correlation for Power Signal

$$R_{xx}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t).x^*(t-\tau)dt = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t+\tau).x^*(t)dt$$

If $x(t)$ is real

$$R_{xx}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t).x(t-\tau)dt = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t+\tau).x(t)dt$$

# Relation b/w Correlation and Convolution

$$R_{xy}(\tau) = \int\limits_{-\infty}^{\infty} x(t).y(t-\tau)dt$$

$$x(t) * y(t) = \int\limits_{-\infty}^{\infty} x(\tau).y(t-\tau)d\tau$$

$$x(\tau) * y(-\tau) = \int\limits_{-\infty}^{\infty} x(t).y(t-\tau)dt$$

$$R_{xy}(\tau) = x(\tau) * y(-\tau)$$

$$R_{yx}(\tau) = y(\tau) * x(-\tau)$$

# Properties of Correlation

1) $\quad R_{xy}(-\tau) = R_{yx}(\tau) \qquad R_{xx}(-\tau) = R_{xx}(\tau)$

## 2) For Energy Signal

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t).x^*(t-\tau)dt$$

$$R_{xx}(0) = \int_{-\infty}^{\infty} x(t).x^*(t)dt$$

$$R_{xx}(0) = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$R_{xx}(0) = E_x$$

## 3) For Power Signal

$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t).x^*(t-\tau)dt$$

$$R_{xx}(0) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t).x^*(t)dt$$

$$R_{xx}(0) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt$$

$$R_{xx}(0) = P_x$$

# Properties of Correlation

$$\left| R_{xy}(\tau) \right| \le \sqrt{R_{xx}(0).R_{yy}(0)} = \sqrt{E_x.E_y}$$

$$\left| R_{xx}(\tau) \right| \le \sqrt{R_{xx}(0).R_{xx}(0)} = \sqrt{E_x.E_x} = E_x$$

If $x(t)$ is a periodic with a period T, then its ACF will also be periodic with same period T

$$x(t+T) = x(t)$$

$$R_{xx}(\tau + T) = R_{xx}(\tau)$$

# DT Energy Signal

*Cross − Correlation*

$$R_{xy}[m] = \sum_{n=-\infty}^{\infty} x[n].y^*[n-m] = \sum_{n=-\infty}^{\infty} x[n+m].y^*[n]$$

$$R_{yx}[m] = \sum_{n=-\infty}^{\infty} y[n].x^*[n-m] = \sum_{n=-\infty}^{\infty} y[n+m].x^*[n]$$

$$R_{xy}[-m] = R_{yx}[m]$$

*Auto − Correlation*

$$R_{xy}[m] = \sum_{n=-\infty}^{\infty} x[n].x^*[n-m] = \sum_{n=-\infty}^{\infty} x[n+m].x^*[n]$$

# DT Power Signal

*Cross – Correlation*

$$R_{xy}[m] = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x[n].y^*[n-m] = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x[n+m].y^*[n]$$

$$R_{yx}[m] = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} y[n].x^*[n-m] = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} y[n+m].x^*[n]$$

$$R_{xy}[-m] = R_{yx}[m]$$

*Auto – Correlation*

$$R_{xx}[m] = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x[n].x^*[n-m]$$

# Properties of Correlation

$$R_{xy}[m] = x[m] * y[-m]$$

$$R_{yx}[m] = y[m] * x[-m]$$

$$R_{xx}[m] = x[m] * x[-m]$$

**For Power Signal**

$$R_{xx}[0] = P_x$$

$$\left| R_{xy}[m] \right| \leq \sqrt{R_{xx}[0].R_{yy}[0]} = \sqrt{E_x.E_y}$$

$$R_{xy}[-m] = R_{yx}[-m]$$

$$\left| R_{xx}[m] \right| \leq \sqrt{R_{xx}[0].R_{xx}[0]} = \sqrt{E_x.E_x} = E_x$$

$$R_{xx}[-m] = R_{xx}[m] (Even)$$

**For Energy Signal**

$$R_{xx}[0] = E_x$$

# Example

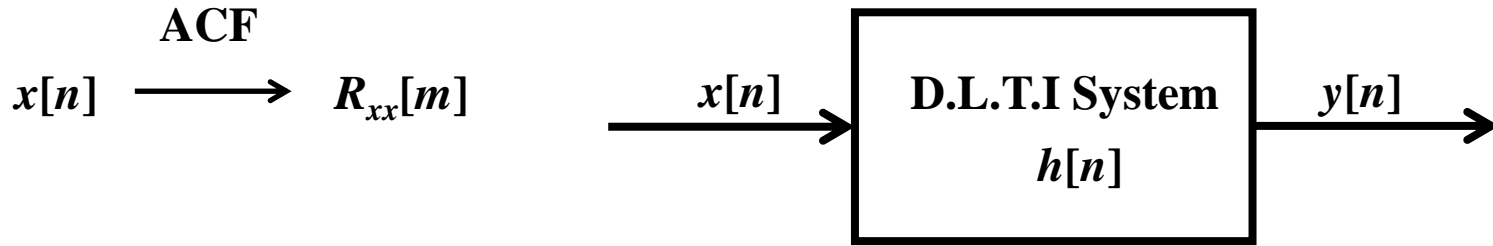Use graphical method to find cross correlation of the following

$x_1[n]=\{1, 2, 3, 4\}$

$x_2[n]=\{0, 1, 2, 3\}$

Compute the autocorrelation of

$x[n] = a^n u[n], 0 < a < 1$

# Correlation Examples

**ACF**

$x[n] \longrightarrow R_{xx}[m]$

$x[n] \longrightarrow$ | D.L.T.I System $h[n]$ | $\longrightarrow y[n]$

**Find**

$R_{yx}[m] \quad R_{xy}[m] \quad R_{yy}[m]$

$R_{yx}[m] = y[m] * x[-m]$

$\therefore \; y[m] = x[m] * h[m]$

$R_{yx}[m] = \{x[m] * h[m]\} * x[-m]$

$R_{yx}[m] = \left\{\underbrace{x[m] * x[-m]}_{R_{xx}[m]}\right\} * h[m]$

$R_{yx}[m] = R_{xx}[m] * h[m]$

$R_{xy}[m] = x[m] * y[-m]$

$\therefore \; y[-m] = x[-m] * h[-m]$

$R_{yx}[m] = x[m] * \{x[-m] * h[-m]\}$

$R_{yx}[m] = \left\{\underbrace{x[m] * x[-m]}_{R_{xx}[m]}\right\} * h[-m]$

$R_{yx}[m] = R_{xx}[m] * h[-m]$

$R_{yy}[m] = y[m] * y[-m]$

$R_{yx}[m] = \{x[m] * h[m]\} * \{x[-m] * h[-m]\}$

$R_{yx}[m] = \{x[m] * x[-m]\} * \{h[m] * h[-m]\}$

$R_{yx}[m] = R_{xx}[m] * R_{hh}[m]$

# Wiener-Khintchine Theorem: Correlation Application

**Wiener-Khintchine Theorem:** The power spectral density (PSD) is equal to the Fourier transform of the autocorrelation function.

**Power spectral density (PSD):** Power of the signal per unit bandwidth is known as PSD.

$$S_x(e^{j\omega}) = \sum_{k=-\infty}^{\infty} R_{xx}[k] e^{-j\omega k}$$