Digital Signal Processing (EC 335)

Dr Zaki Uddin

MTS, CEME, NUST.

Lecture 7

Lecture Targets

- Discrete Fourier transform (DFT)

- Properties of DFT

Discrete Fourier Transform (DFT)

- **DTFT is used to evaluate the frequency response of DT signal.**

$$x[n] \rightarrow X(e^{j\omega})$$

- **The signal in Frequency domain (applying DTFT) will always be periodic. Why????**

- **DFT is also used to evaluate the frequency response of DT signal.**

- **What is the difference between DTFT and DFT then???**

- **The transform from DTFT is of <u>continuous nature.</u> We cannot use continuous signals in digital signal processors. So, we might use DFT instead of DTFT if the application is DSP.**

Discrete Fourier Transform (DFT)

As we know that we evaluate the DTFT at unit circle. Now for N-point DFT, we will equally sampled the DTFT in frequency domain.

$$\omega = \left(\frac{2\pi}{N}\right)k$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n].e^{-j\omega n}$$

$$X(k) = \sum_{n=0}^{N-1} x[n].e^{-j\frac{2\pi kn}{N}} \qquad ,\mathrm{k} = 0, 1, 2......, \mathrm{N}-1$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X(k).e^{j\frac{2\pi kn}{N}} ,\mathrm{n} = 0, 1, 2......, \mathrm{N}-1$$

Discrete Fourier Transform (DFT)

$$X(0) = \sum_{n=0}^{N-1} x[n] = x[0] + x[1] + x[2] + x[3] + \ldots\ldots\ldots\ldots\ldots$$

$$X\left(\frac{N}{2}\right) = \sum_{n=0}^{N-1} x[n].(-1)^n = x[0] - x[1] + x[2] - x[3] + \ldots\ldots\ldots\ldots\ldots$$

$$e^{-j\frac{2\pi kn}{N}} \text{ at k = N/2} = e^{-j\frac{2\pi Nn}{N.2}} = (e^{-j\pi})^n = (-1)^n$$

$$X(0) + X\left(\frac{N}{2}\right) = 2\left[x[0] + x[2] + x[4] + \ldots\ldots\ldots\ldots\ldots\ldots\right]$$

$$X(0) - X\left(\frac{N}{2}\right) = 2\left[x[1] + x[3] + x[5] + \ldots\ldots\ldots\ldots\ldots\ldots\right]$$

[1 2 3 7 9 10]

x[0] 15

x[5] 7

Discrete Fourier Transform (DFT) as Linear filter

- **Twiddle factor is used for the efficient computation of the DFT and FFT, which are widely used in many applications such as signal processing, audio and image processing, data compression, and digital communication systems. The use of twiddle factors has enabled these applications to process and analyze signals more quickly and efficiently, which has had a significant impact on the development of modern technologies.**

$$W_N = e^{-j\frac{2\pi}{N}} = \text{Twidle Factor}$$

$$X(k) = \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ . \\ . \\ . \\ X(N-1) \end{bmatrix} = X_N \qquad x(n) = \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ . \\ x(N-1) \end{bmatrix} = x_N$$

Discrete Fourier Transform (DFT) as a Linear filter

$$e^{-j\frac{2\pi}{N}kn} = W_N^{kn} = [W_N]$$

$\cos 2\pi - j\sin 2\pi$
$(+1)$

$$[W_N] = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots\cdots & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \cdots\cdots & W_N^{N-1} \\ W_N^0 & W_N^2 & W_N^4 & \cdots\cdots & W_N^{2(N-1)} \\ \cdot & & & & \\ \cdot & & & & \\ W_N^0 & W_N^{N-1} & W_N^{2(N-1)} & \cdots\cdots & W_N^{(N-1)^2} \end{bmatrix}$$

1 2 3 - —

2 4 6 8 - - .

3 6 9 - - · · -

Discrete Fourier Transform (DFT) as a Linear filter

$$X_N = [W_N] x_N \quad \text{(Analysis Equation)}$$

$$x_n = [W_N]^* X_N \quad \text{(Synthesis Equation)}$$

$$X(k) = \sum_{n=0}^{N-1} x[n].W_N^{kn}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X(k).W_N^{-kn}$$

Discrete Fourier Transform (DFT) as a Linear filter

$$X_N = [W_N] x_N \quad \text{(Analysis Equation)}$$

$$x_n = [W_N]^* X_N \quad \text{(Synthesis Equation)}$$

$$X(k) = \sum_{n=0}^{N-1} x[n].W_N^{kn}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X(k).W_N^{-kn}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X(k).\left(W_N^*\right)^{kn}$$

For any one value of k, number of complex multiplication= N

For any one value of k, number of complex addition= N-1

Total number of multiplications= N.N=N^2

Total number of additions= (N-1).N=N^2-N

Any problem in such a large calculation???

Normally, the calculations are performed by computers. Performing the aforesaid calculations will make any difference???

Complexity does not matter. Processing time is a crucial factor.

$$X(k) = \sum_{n=0}^{N-1} x[n].W_N^{kn}$$

$$X(k) = x[0].W_N^0 + x[1].W_N^k + \ldots\ldots\ldots + x[N-1].W_N^{k(N-1)}$$

Properties of DFT

Properties of Twiddle factor

$$W_N = e^{-j\frac{2\pi}{N}} = \text{Twidle Factor}$$

1). $W_N^{K+N} = W_N^K$

$$W_N^K = e^{-j\frac{2\pi}{N}K}$$

$$W_N^{K+N} = e^{-j\frac{2\pi}{N}(K+N)} = e^{-j\frac{2\pi}{N}K}$$

2). $W_N^{K+\frac{N}{2}} = -W_N^K$

3). $W_N^2 = W_{N/2}$

(i) Periodicity

If $x[n+N] = x[n]$

Then $x[K+N] = x[K]$

(ii)Linearity

$x_1[n] \to X_1(K)$

$x_2[n] \to X_2(K)$

$\alpha x_1[n] + \beta x_2[n] \to \alpha X_1(K) + \beta X_2(K)$

(iii) Time shifting

$x[n] \to x(K)$

$$x((n-n_0))_N \to e^{-j\frac{2\pi k}{N}.n_0} x(K)$$

(iv) Frequency shifting

$x[n] \to x(K)$

$$e^{j\frac{2\pi l}{N}n} x[n] \to X((k-l))_N$$

(v)Expansion in time

$x[n]$ DFT is $X(K)$

$x[n/k]$ DFT will repeat itself in $2\pi/k$

(vi) Circular Convolution

$x_1[n] \to X_1(K), x_2[n] \to X_2(K)$

$w[n] = x_1[n] \otimes x_2[n] \to X_1(K) \times X_2(K)$

$$\boxed{x_1[n] \otimes x_2[n] = \sum_{m=0}^{N-1} x_1(m) x_2((n-m))_N}$$

(vii) Multiplication

$$x_1[n] \times x_2[n] \to \frac{1}{N}[X_1(K) \otimes X_2(K)]$$

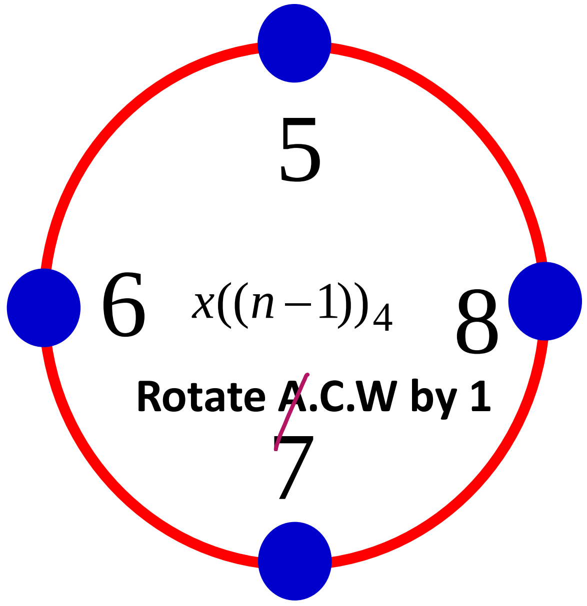
Representation on Circle

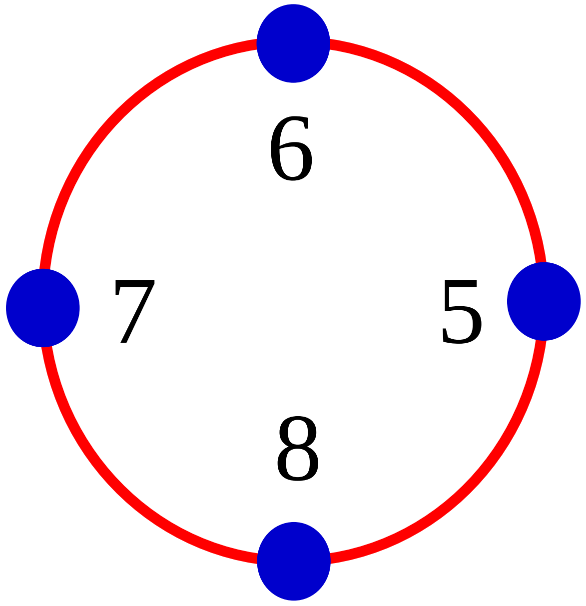
$x[n] = \{5, 6, 7, 8\}$

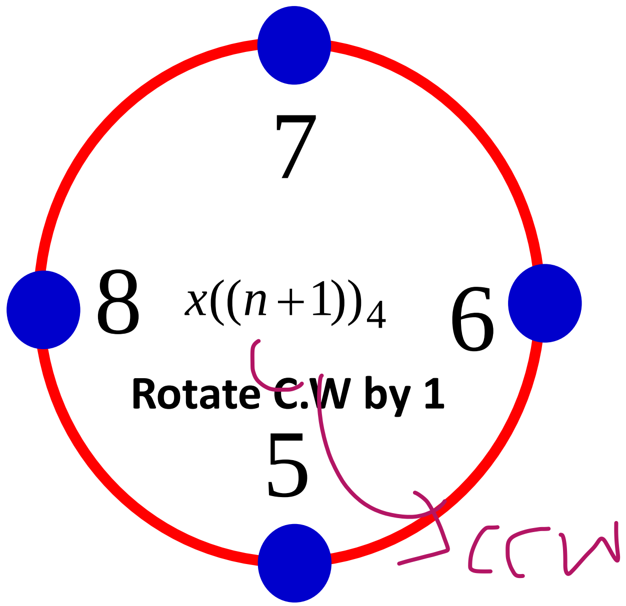
Draw

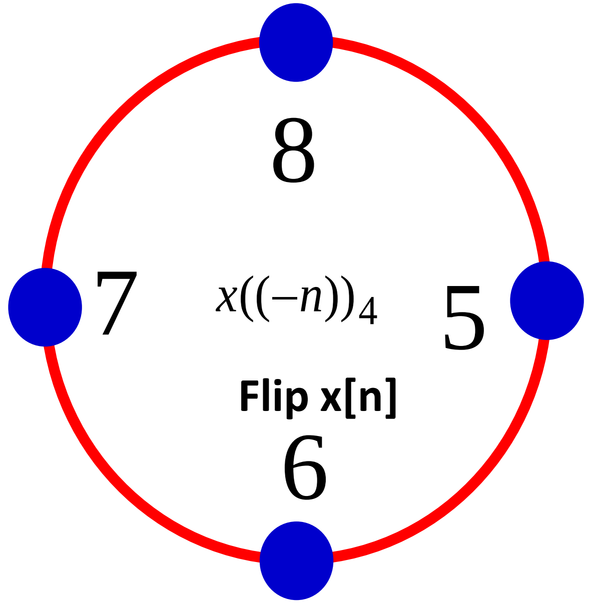
$x((n))_4, x((n-1))_4$

$x((n+1))_4, x((-n))_4$









Circular Convolution

$$x_1[n] = \{2, 3, 4, 5\}$$

$$x_2[n] = \{5, 6, 2, 1\}$$

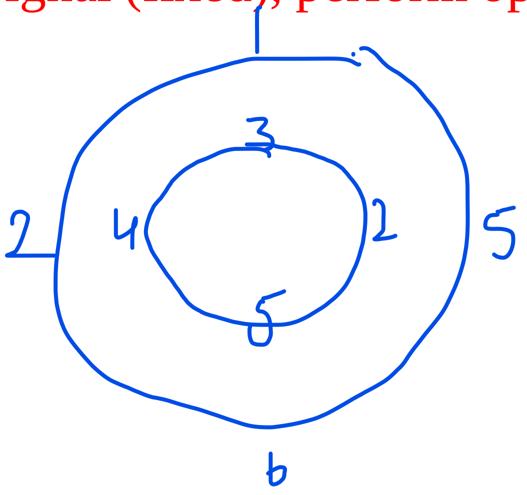
$$y[n] = x_1[n] \otimes x_2[n]???$$

$$x_1[n] \otimes x_2[n] = \sum_{m=0}^{N-1} x_1(m) x_2((n-m))_N$$

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \end{bmatrix} = \begin{bmatrix} 2 & 5 & 4 & 3 \\ 3 & 2 & 5 & 4 \\ 4 & 3 & 2 & 5 \\ 5 & 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 2 \\ 1 \end{bmatrix}$$

By definition method:

Draw first signal (fixed), perform operation on second signal



Example

The N - point DFT of $x[n] = a^n \; 0 \le n \le N$ -1 is $X(K)$

Find $X(K)$ at $K = 2, a = 0.5$ and $N = 4$

$$X(K) = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}}$$

$$X(K) = \sum_{n=0}^{N-1} a^n e^{-j\frac{2\pi kn}{N}} = \sum_{n=0}^{N-1} a^n \left(e^{-j\frac{2\pi k}{N}} \right)^n$$

$$X(K) = \sum_{n=0}^{N-1} \left(a e^{-j\frac{2\pi k}{N}} \right)^n$$

$$\therefore \sum_{n=0}^{N} (a)^n = \frac{1-a^{N+1}}{1-a}$$

$$X(K) = \frac{1-\left(a e^{-j\frac{2\pi k}{N}} \right)^N}{1-\left(a e^{-j\frac{2\pi k}{N}} \right)} = \frac{1-a^N \overbrace{\left(e^{-j\frac{2\pi k}{N}} \right)^N}^{1}}{1-a e^{-j\frac{2\pi k}{N}}}$$

$$X(2) = \frac{1-(0.5)^4}{1-0.5(-1)} = ???$$

Example

Two seq; $[a, b, c]$ and $[A, B, C]$ are related as

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^{-1} & W_3^{-2} \\ 1 & W_3^{-2} & W_3^{-4} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

X[K] **x[n]**

$$where, W_3 = e^{j\frac{2\pi}{3}}$$

If another seq; $[p, q, r]$ is derived as

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^{1} & W_3^{2} \\ 1 & W_3^{2} & W_3^{4} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & W_3^{2} & 0 \\ 0 & 0 & W_3^{4} \end{bmatrix} \begin{bmatrix} A/3 \\ B/3 \\ C/3 \end{bmatrix}$$

Find the relation b/w $[p, q, r]$ and $[a, b, c]$

Slide 13, basic definition of Twiddle factor

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = 1/3 \begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^{1} & W_3^{2} \\ 1 & W_3^{2} & W_3^{4} \end{bmatrix} \begin{bmatrix} A \\ W_3^{2} B \\ W_3^{4} C \end{bmatrix}$$

X'[n] **X'[K]**

Twiddle factor is defined with +ve sign, what will happen???

$$x[n] = [a, b, c] \rightarrow X(K) = [A, B, C]$$

$$x'[n] = [p, q, r] \rightarrow X'(K) = [A, W_3^{2}.B, W_3^{4}.C]$$

$$X'(K) = W_3^{2K} \times X(K)$$

Example

$$X'(K) = W_3^{2K} \times X(K)$$

$$X'(K) = e^{j\frac{2\pi}{3}\times 2K} \times X(K)$$

Time shifting

$$x[n] \rightarrow X(K)$$

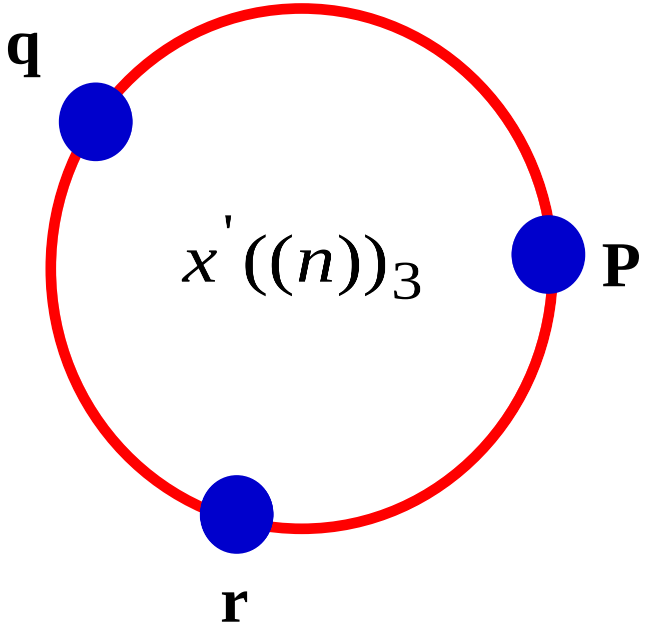
$$x((n+n_0))_N \rightarrow e^{j\frac{2\pi k}{N}.n_0} X(K)$$

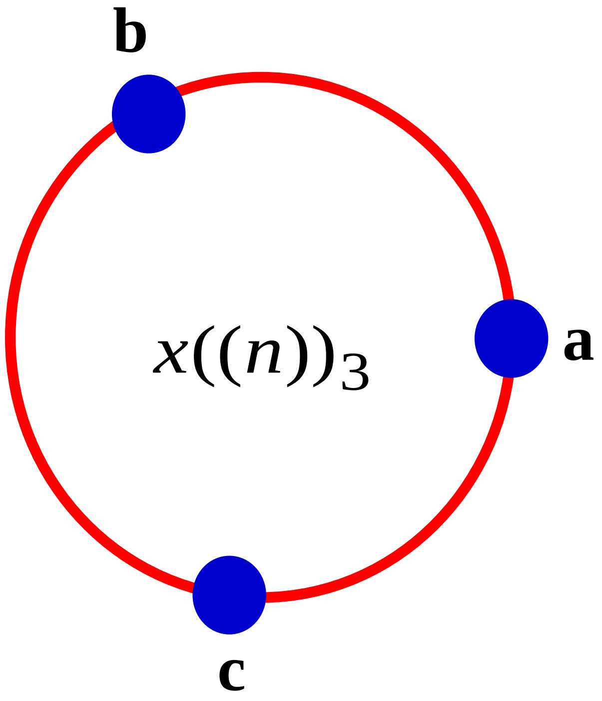
$$X'(K) = e^{j\frac{2\pi K}{3}2} \times X(K)$$

$$x[n] \rightarrow X(K)$$

$$x'[n] = x((n+2))_3 \rightarrow e^{j\frac{2\pi K}{3}\times 2} \times X(K) = X'(K)$$

$$x'[n] = x((n+2))_3$$

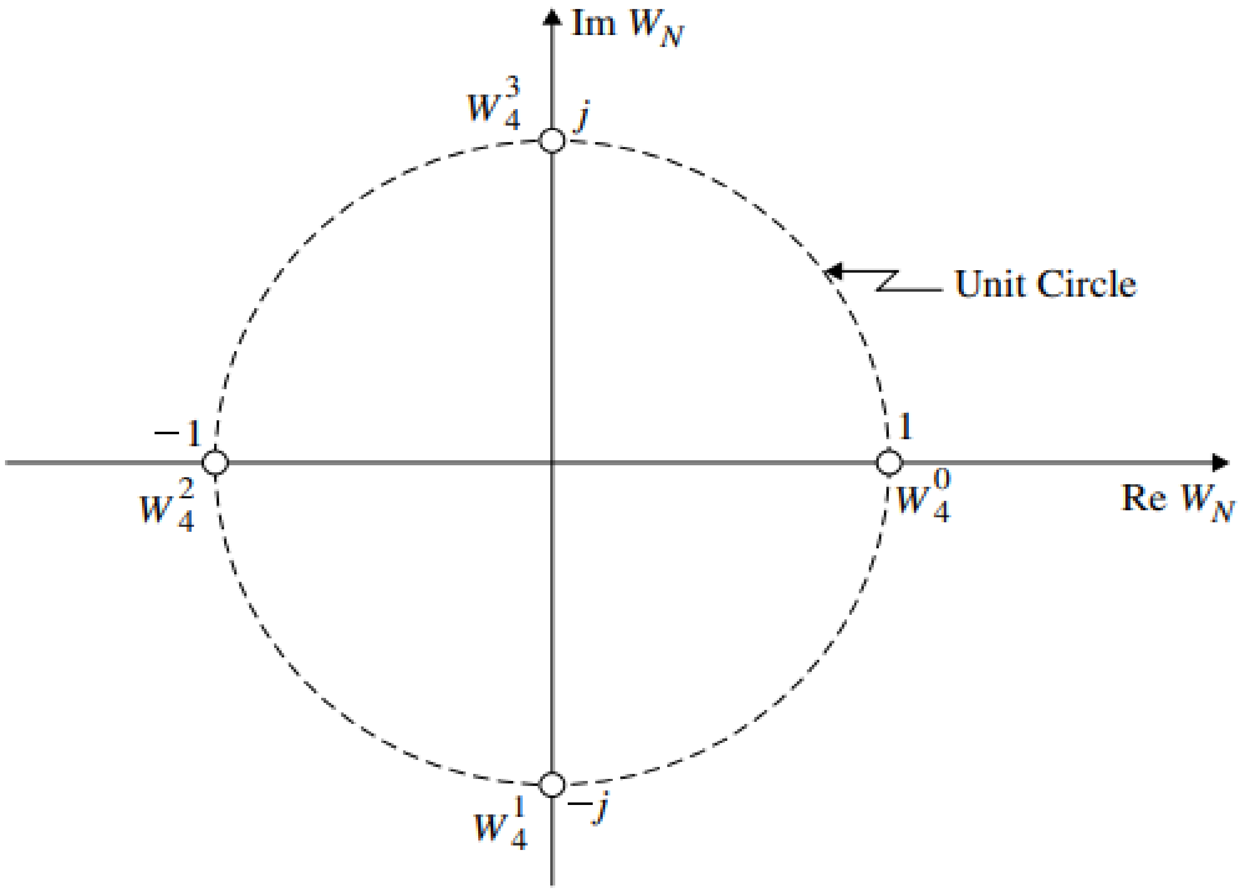




$$x((n+2))_3 = ?????$$

Four points, Six points and 8 points Twiddle factor

$$
W_N = \begin{array}{c c} & \begin{array}{cccc} n=0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} k=0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \end{array}
$$



$$
W_N = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \quad \begin{bmatrix} 1 \\ j \\ -1 \\ -j \end{bmatrix}
$$

Six points and 8 points Twiddle factor

Example

Find 4 - point DFT of

$$x[n] = \{1, j, -1, -j\}$$

Find the IDFT of the following function with N = 4.

$$X(k) = \{1,0,1,0\}$$

Practice question

Find 4 - point DFT of

$x[n] = \{1,1,1,1\}$

$x[n] = \{1,1,0,0\}$

$x[n] = \cos(pi.n)$, where $n = 0,1,2,3,....$

$x[n] = \sin \dfrac{n\pi}{2}$, where $n = 0,1,2,3,....$

$x[n] = \delta[n]$

Find IDFT of the following signals with $N = 4$

$X[K] = \{6, (-2 + j2), -2, (-2 - j2)\}$