EE-379 Linear Control Systems

Week No. 6: Root Locus Analysis

- Background
- Pole Zero Plots
- General Method
- Construction Rules
- Examples

EE-379 Root Locus Analysis

Background

- **Chapter 1**: D.E can be written for various systems and can be solved, and the response could be divided into **forced** and **natural** components
- **Chapter 2**: Established definitions related to **natural response** and **types of stability** (stability is generally considered to be the property of the natural response)
- **Chapter 3**: The other response component (**steady state**) was considered that is the tendency of a device to follow (track) a command.
- **Chapter 4**: **Return to stability** and provide a very broad and useful measure as compared to the Routh Hurwitz criterion.

EE-379 Root Locus Analysis

Background

- **Logical approach to determine stability** would be to
  - ✓ Extract roots of the characteristic polynomial as the adjustable gain varies (computational packages i.e., MATLAB can readily perform this task)
  - ✓ Walter Evans (1940) developed a set of rules by which the path traced by roots of a closed loop characteristic equation can be sketched this plot is the root locus.
  - ✓ **There are two chapters devoted to root locus:** in chapter 4 we will concentrate on the basic understanding of root locus principles

EE-379 Root Locus Analysis

Background

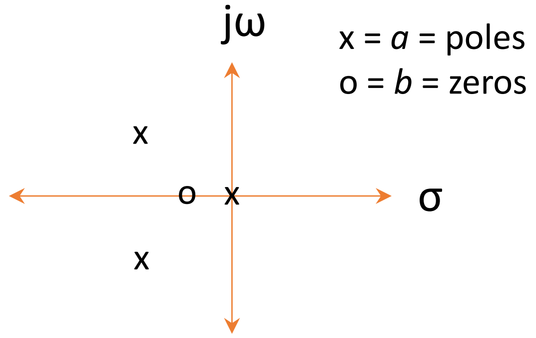
- **Zeros** are the values at which the function is zero (numerator roots)
- **Poles** are the values at which the function is infinite (denominator roots)
- A rational function in the factored form is given by:
- The constant $\boldsymbol{k} = \frac{\boldsymbol{b_m}}{\boldsymbol{a_m}}$ is the **multiplying constant**
- When poles and zeros are plotted on the complex plain the result is a **pole-zero plot**

$$F(s) = \frac{b_m s^m + b_{m-1}\, s^{m-1} + \cdots + b_0}{s^n + a_{n-1}\, s^{m-1} + \cdots + a_0}$$

$$F(s) = \frac{\boldsymbol{K}[(s - z_1)(s - z_2) \ldots (s - z_m)]}{[(s - p_1)(s - p_2) \ldots (s - p_n)]}$$

$z_1, z_2, z_{3,\ldots.} z_m$ *are the zeros of the function*

$p_1, p_2, p_{3,\ldots.} p_n$ *are the poles of the function*



$$F(s) = \frac{4s + 5}{s^3 + 4s^2 + 13s} = \frac{4(s + 5/4)}{s(s + 2 + 3j)(s + 2 - 3j)}$$

EE-379 Root Locus Analysis

Pole-Zero Plot: Graphical Evaluation

- The roots of the closed-loop characteristic equation define the system characteristic responses.
- Their location in the complex s-plane lead to the prediction of the characteristics of the time domain responses in terms of:

  ✓ damping ratio, ζ
  ✓ natural frequency, ω_n

  Second order modes

  ✓ damping constant, b

  first order modes

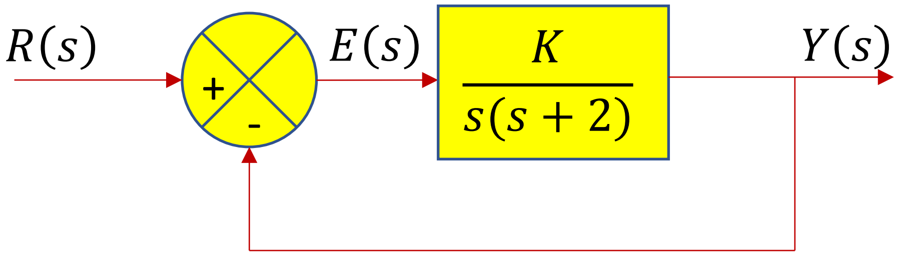
- Consider how these roots change as the loop gain is varied from 0 to ∞

EE-379 Root Locus Analysis

Example

- The closed-loop transfer function is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{K}{s(s+2)+K}$$



- The characteristic equation is

$$s^2 + 2s + K = 0$$

- Consider the characteristic roots as:

$$K = 0 \rightarrow \infty$$

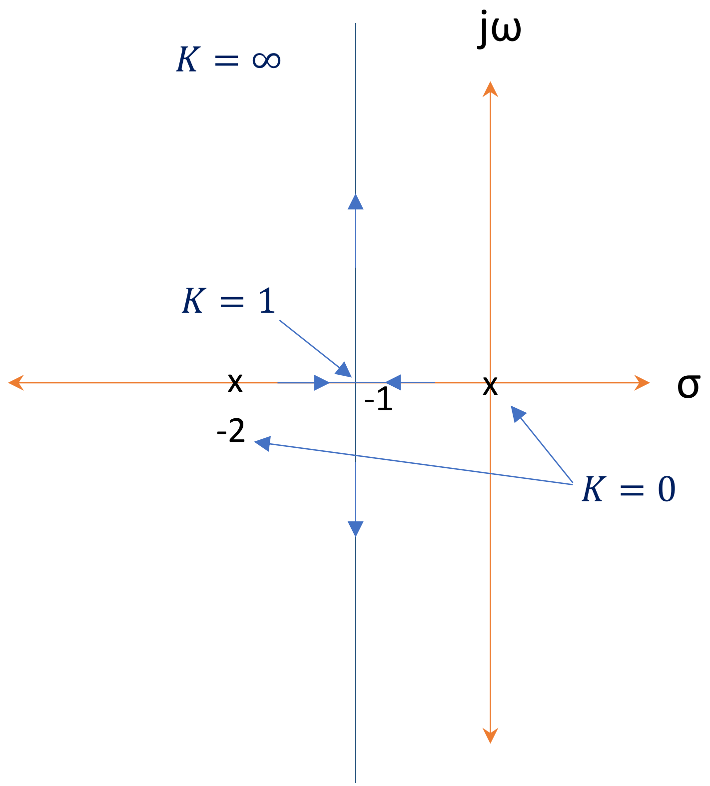
EE-379 Root Locus Analysis

Example

$$s = -1 \pm \sqrt{1 - K}$$

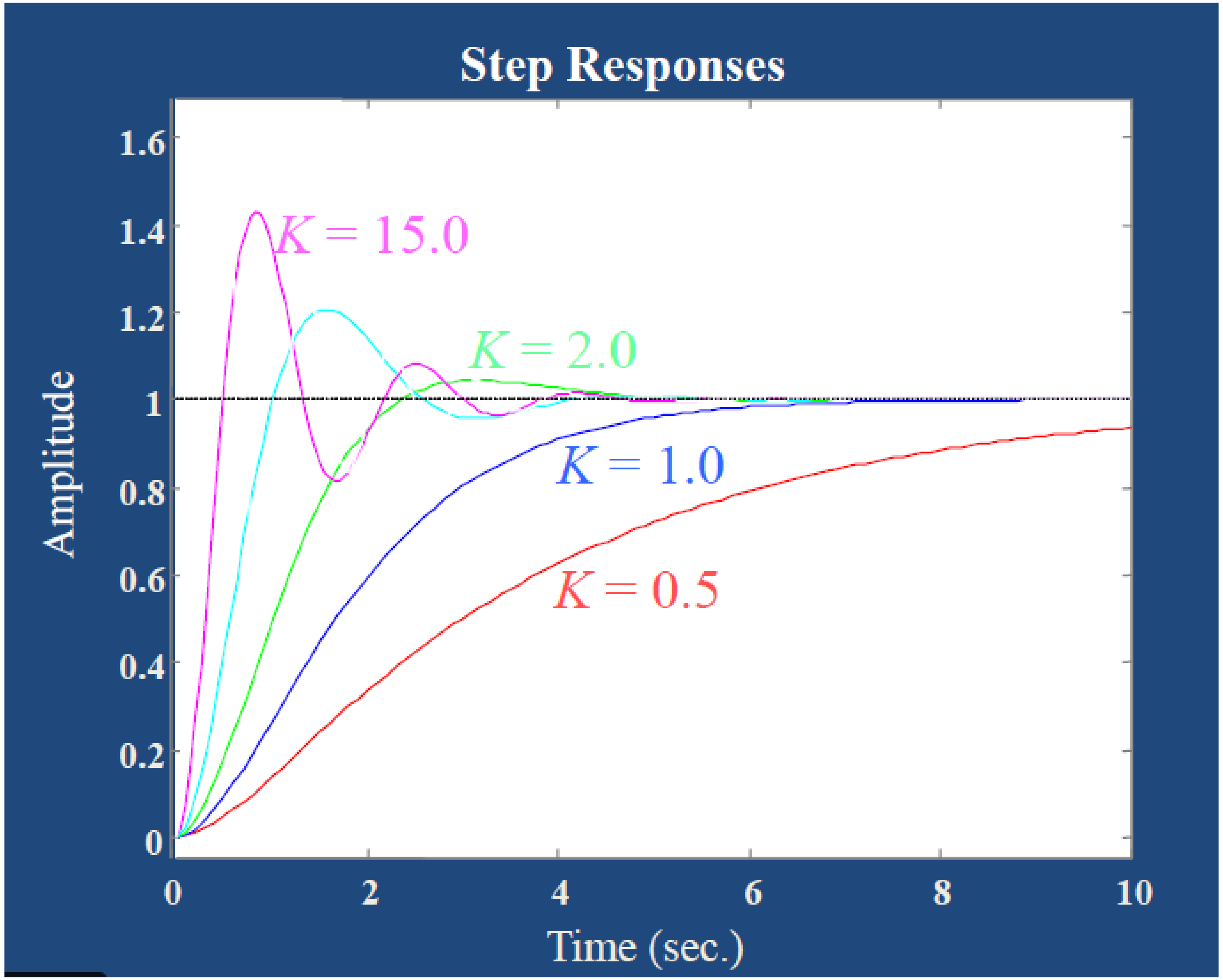
- For $K = 0$ the closed-loop poles are at the **open-loop poles**.
- For $0 < K < 1$ the closed loop poles are on the **real axis**:
- For $K > 1$ the closed-loop poles are complex, with a real value of -1 and an imaginary value increasing with gain K.

Loci of closed-loop roots

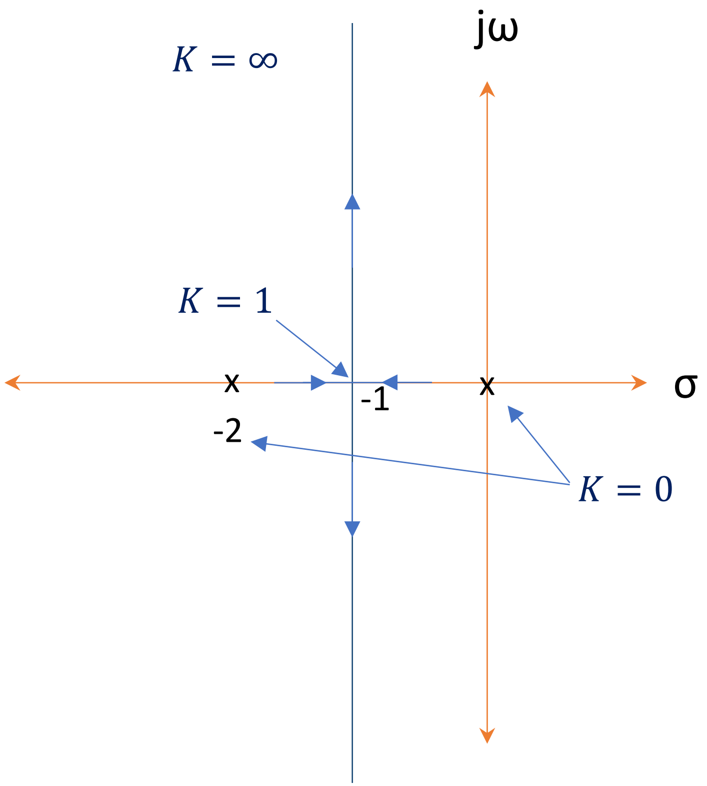


EE-379 Root Locus Analysis

Example



Loci of closed-loop roots

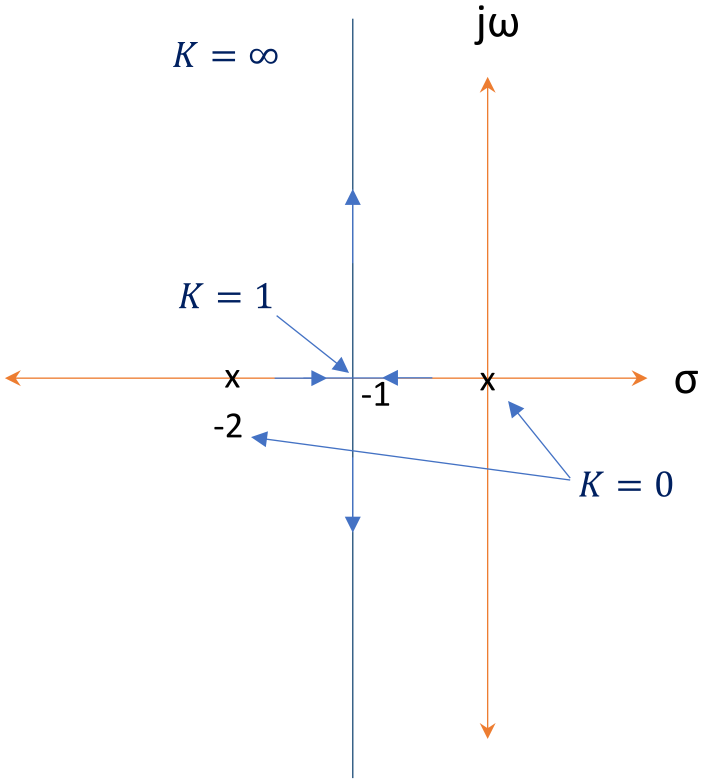


EE-379 Root Locus Analysis

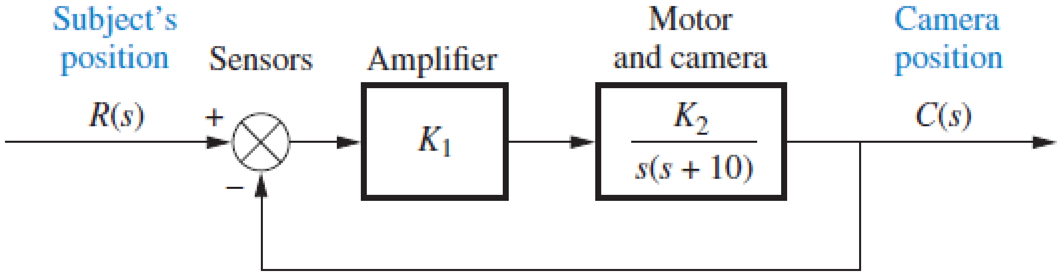
Example-Observations

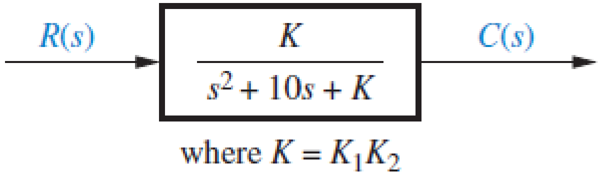
- This is a second-order system and there are **two loci**.
- The root loci start at the open loop poles
- The root loci tend towards the open loop zeros at infinity as $K \to \infty$. (Note: the **number of zeros** is equal to the **number of poles** when the zeros at infinity are included.)
- The relationship between the **characteristic responses** and **the increasing gain** is seen through the **root loci**.

Loci of closed-loop roots



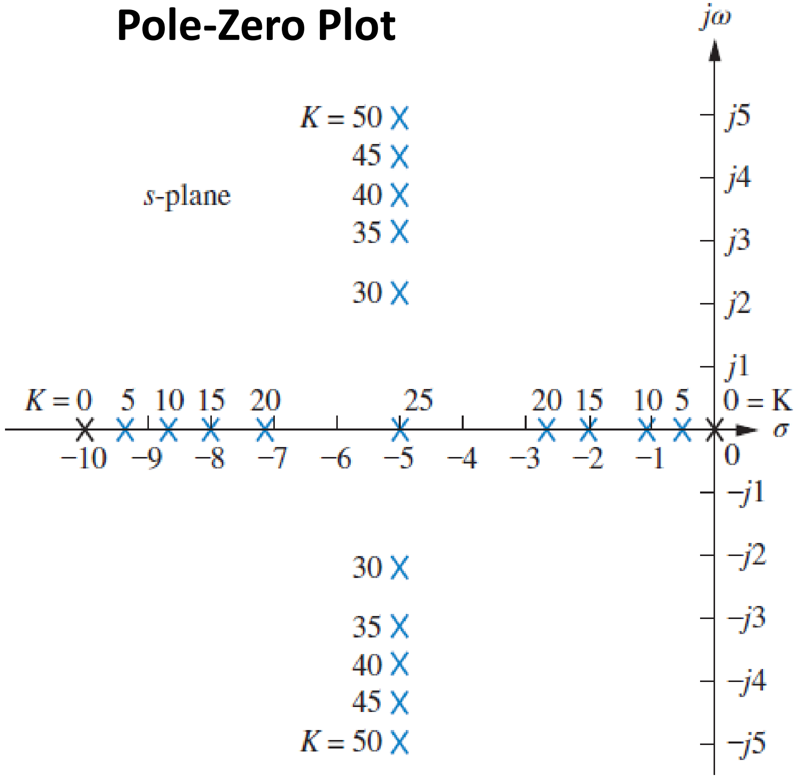
EE-379 Root Locus Analysis



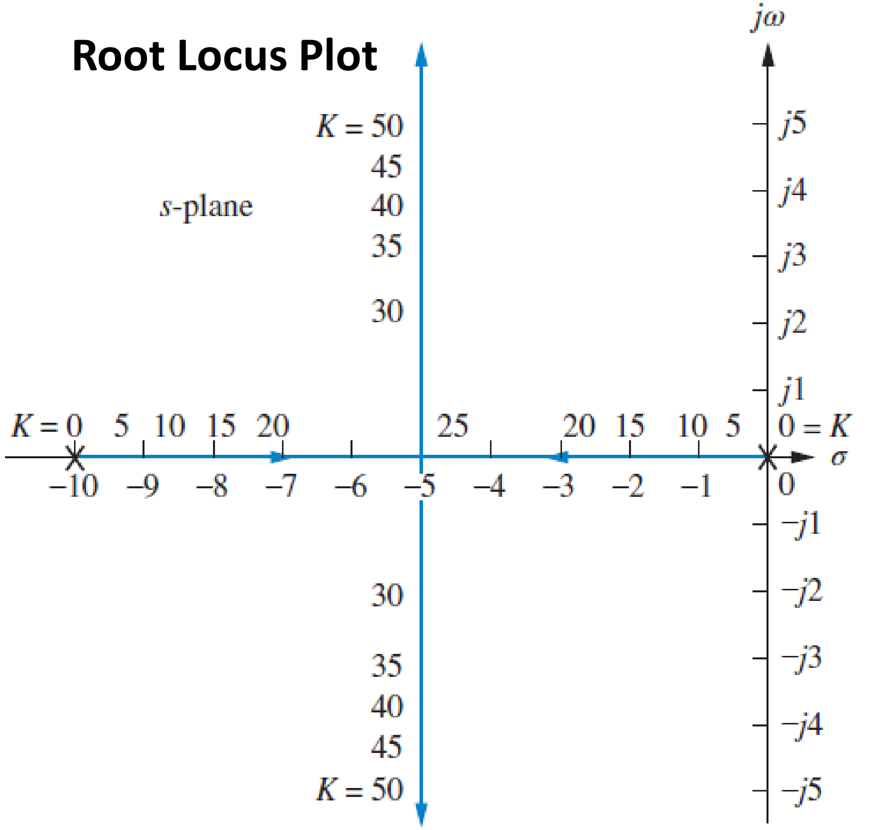


K	Pole 1	Pole 2
0	−10	0
5	−9.47	−0.53
10	−8.87	−1.13
15	−8.16	−1.84
20	−7.24	−2.76
25	−5	−5
30	$-5 + j2.24$	$-5 - j2.24$
35	$-5 + j3.16$	$-5 - j3.16$
40	$-5 + j3.87$	$-5 - j3.87$
45	$-5 + j4.47$	$-5 - j4.47$
50	$-5 + j5$	$-5 - j5$

Pole-Zero Plot



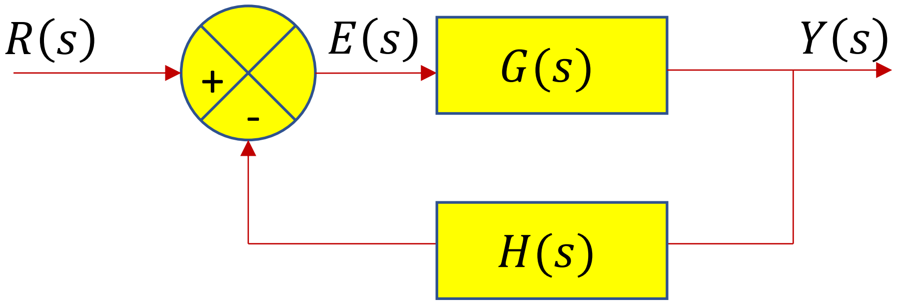
Root Locus Plot



EE-379 Root Locus Analysis

The General Root Locus Method

- Consider the general system



$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + GH(s)}$$

- The characteristic equation is

$$1 + GH(s) = 0 \qquad \Rightarrow \qquad GH(s) = -1$$ → Complex quantity

Therefore, $GH(s) = -1, \mathit{can\ be\ split\ into\ two\ equations}$

1. $|GH(s)| = 1 \qquad \rightarrow \mathit{represents\ magnitude}$
2. $\angle GH(s) = (2k+1)180° \rightarrow \mathit{represents\ angle}$

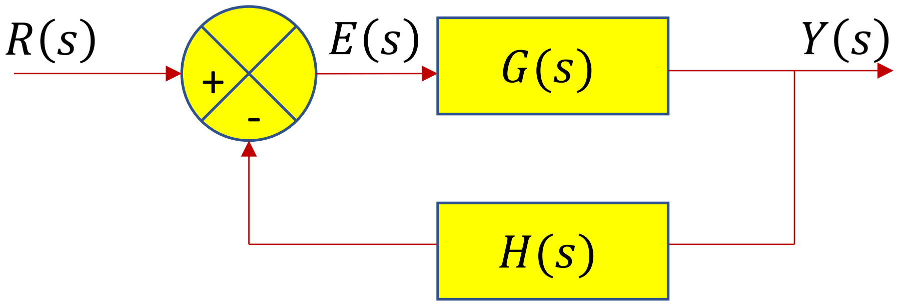
$$k = 0, \pm 1, \pm 2 \ldots$$

EE-379 Root Locus Analysis

The General Root Locus Method

- **All values of ' s ' which satisfy**
  - $|GH(s)| = 1$ **Magnitude criterion**
  - $\angle GH(s) = (2k+1)180°$ **Angle criterion**

    $k = 0, \pm 1, \pm 2 \dots$



are **roots** of the **closed-loop characteristic equation**.

- Consider the following general form

$$GH(s) = \frac{\boldsymbol{K}[(s+z_1)(s+z_2)\dots(s+z_m)]}{[(s+p_1)(s+p_2)\dots(s+p_n)]}$$

$Note: \quad p_i \ may\ be\ zero$

EE-379 Root Locus Analysis

The General Root Locus Method

- Then,

1. $|GH(s)| = \frac{|K| \prod_{i=1}^{m} |s+z_i|}{\prod_{i=1}^{n} |s+p_i|} = 1 \quad \longleftrightarrow \quad |F(s_0)| = \frac{|\boldsymbol{K}| \begin{pmatrix} product\ of\ lengths\ of\ dir \\ segment\ from\ zeros\ to\ s_0 \end{pmatrix}}{\begin{pmatrix} product\ of\ lengths\ of\ dir \\ segment\ from\ poles\ to\ s_0 \end{pmatrix}}$

The **magnitude condition** is that the **point** (which satisfied the angle condition) at which the **magnitude of the open loop transfer function is one**.

2. $\angle GH(s) = \sum_{i=1}^{m} \angle(s+z_i) - \sum_{i=1}^{m} \angle(s+p_i) = (2k+1)180° \qquad k = 0, \pm1, \pm2 \ldots$

$$\angle F(s_0) = (sum\ of\ angles\ of\ dir\ segments)$$

$$\angle F(s_0) = (from\ zeros\ to\ s_0) - (sum\ of\ poles\ angles) + 180° \ \ (if\ \boldsymbol{K} < 0)$$

The **angle condition** is the point at which **the angle of the open loop transfer function is an odd multiple of 180⁰.**

EE-379 Root Locus Analysis

Pole-Zero Plot: Graphical Evaluation

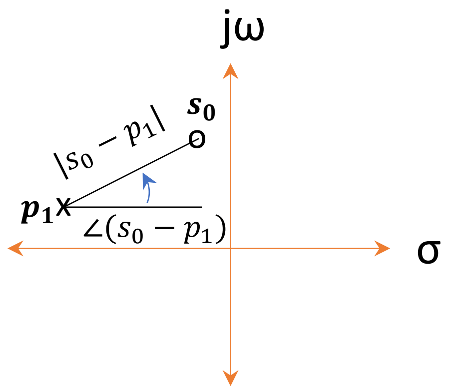
- On a **pole-zero** plot suppose a line is drawn from $\boldsymbol{p_1}$ to $\boldsymbol{s_0}$ where the function is being evaluated.
- This segment has length $|\boldsymbol{s_0} - \boldsymbol{p_1}|$ and angle $\angle(\boldsymbol{s_0} - \boldsymbol{p_1})$ with real axis

$$F(s) = \frac{\boldsymbol{K}[(s-z_1)(s-z_2)\ldots(s-z_m)]}{[(s-p_1)(s-p_2)\ldots(s-p_n)]}$$

$$F(s_0) = \frac{\boldsymbol{K}[(s_o-z_1)(s_0-z_2)\ldots(s_0-z_m)]}{[(s_0-p_1)(s_0-p_2)\ldots(s_0-p_n)]}$$

$$F(s_0) = \frac{\boldsymbol{K}(|s_0-z_1|e^{j\angle(s_0-z_1)})(|s_0-z_2|e^{j\angle(s_0-z_2)})}{(|s_0-p_1|e^{j\angle(s_0-p_1)})(|s_0-p_2|e^{j\angle(s_0-p_2)})}$$

$$|F(s_0)| = \frac{|\boldsymbol{K}|\begin{pmatrix} product\ of\ lengths\ of\ dir \\ segment\ from\ zeros\ to\ s_0 \end{pmatrix}}{\begin{pmatrix} product\ of\ lengths\ of\ dir \\ segment\ from\ poles\ to\ s_0 \end{pmatrix}}$$



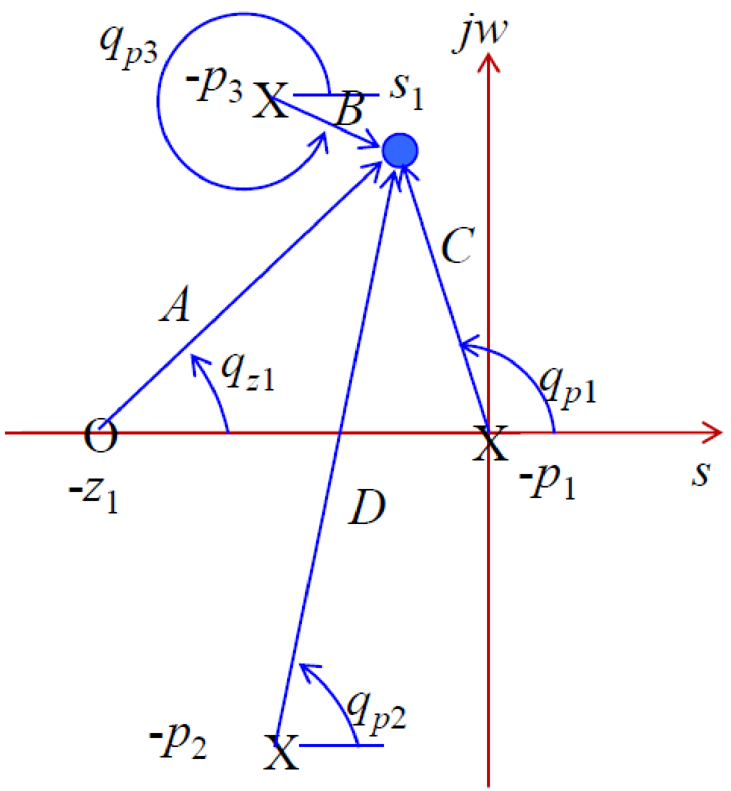
$$\angle F(s_0) = (sum\ of\ angles of\ dir\ segments\ \boldsymbol{from\ zeros\ to\ s_0}) - (sum\ of\ angles of\ dir\ segments\ from\ \boldsymbol{poles\ to\ s_0}) + 180°\ \ (if\ k < 0)$$

EE-379 Root Locus Analysis

General Method – Geometric Interpretation

- Consider the example,

$$GH(s) = \frac{K(s+z_1)}{s(s+p_2)(s+p_3)}$$



- **Then the values of $s = s_1$ which satisfy**

1. $\frac{K|s+z_1|}{|s||s+p_2||s+p_3|} = 1$
2. $\angle(s+z_1) - \left(\angle s + \angle(s+p_2) + \angle(s+p_3)\right) = (2k+1)180°$

are on the loci and are roots of the characteristic equation.

EE-379 Root Locus Analysis

General Method – Geometric Interpretation

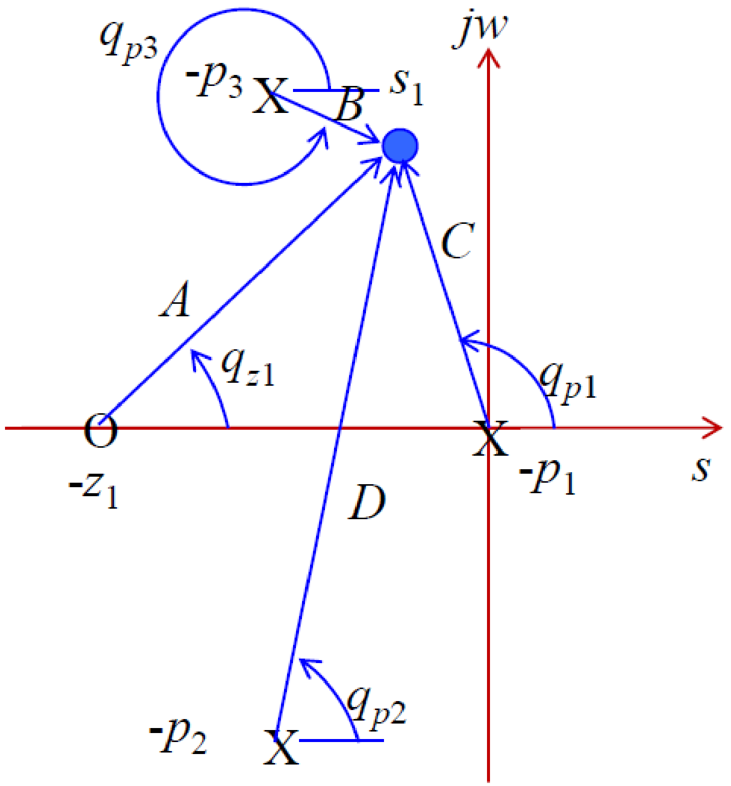
- In terms of vectors, the condition for $s = s_1$ to be on the root loci:

$$\frac{|K|A}{BCD} = 1 \qquad or \qquad \frac{A}{BCD} = \frac{1}{|K|}$$

and,

$$\theta_{z1} - (\theta_{p1} + \theta_{p2} + \theta_{p3}) = (2k+1)180°$$

$$k = 0, \pm 1, \pm 2 \ldots$$



are roots of the characteristic equation.

EE-379 Root Locus Analysis

Analysis

- When plotting the loci of the roots as $\boldsymbol{K = 0 \to \infty}$, **the magnitude condition is always satisfied.** Along the root locus, the magnitude condition can always be satisfied by adjusting K
- Therefore, a value of $\boldsymbol{s = s_1}$ that satisfies **the angle condition, is a point of the root loci**
- The magnitude condition may then be used to determine the gain $\boldsymbol{K}$ corresponding to that value $\boldsymbol{s_1}$.

EE-379 Root Locus Analysis

Analysis – Construction Rules

1. The loci **start** $(\boldsymbol{K = 0})$ at the poles of the open-loop system. There are $\boldsymbol{n-}$ **loci**.

2. The loci **terminate** $(\boldsymbol{K} \rightarrow \infty)$ at the zeroes of the open-loop system (include zeroes at infinity).

- For our example system

$$|GH(s)| = \frac{K|s+z_1|}{|s||s+p_2||s+p_3|} = \frac{1}{K}$$

- Therefore, as $\boldsymbol{K \rightarrow 0}$
  $\boldsymbol{GH(s)} \rightarrow \infty, \textit{the poles of the loop transfer function}$

- As, $\boldsymbol{K} \rightarrow \infty$
  $\boldsymbol{GH(s) \rightarrow 0}, \textit{the zeros of the loop transfer function}$

EE-379 Root Locus Analysis

Analysis – Construction Rules

3. The root loci are **symmetrical** about the real axis.

- The roots with imaginary parts always occur in conjugate complex pairs

4. As $\boldsymbol{K} \rightarrow \infty$ the loci approach **asymptotes**. There are $\boldsymbol{q = n - m}$ asymptotes and they intersect the real axis at **angles** defined by

$$\frac{\mathbf{(2}\boldsymbol{k}\mathbf{+1)180^\circ}}{\boldsymbol{q}}, \qquad k = 0, \pm 1, \pm 2 \ldots (q-1)$$

EE-379 Root Locus Analysis

Analysis – Construction Rules

5. The asymptotes intersection point on the real axis is defined by.

$$\sigma_a = \frac{\sum poles\ of\ GH(s) - \sum zeroes\ of\ GH(s)}{q}$$

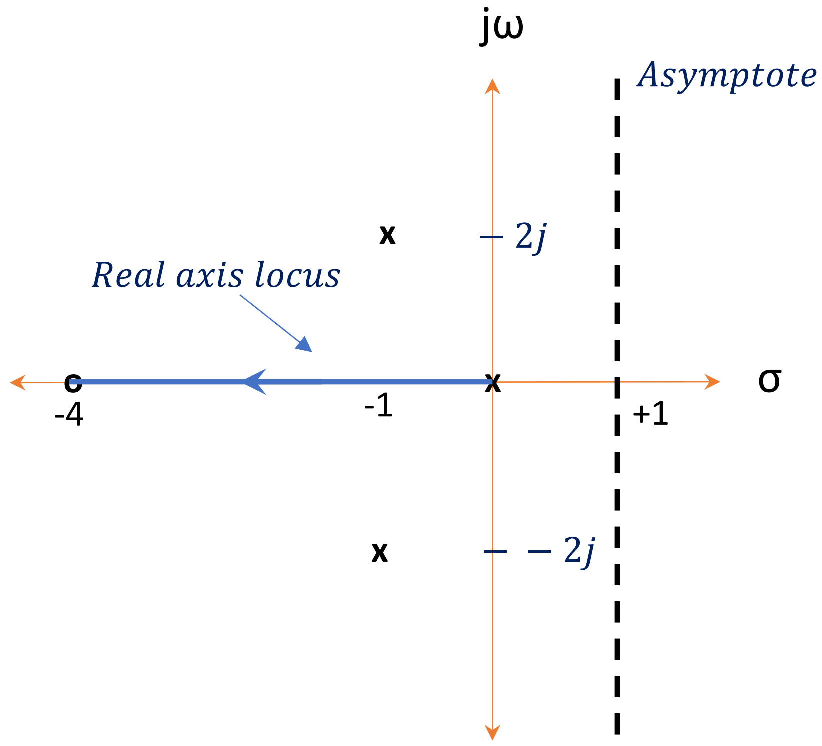
6. Real axis sections of the root loci exist only where there is an odd number of poles and zeroes to the right.

EE-379 Root Locus Analysis

Analysis – Construction Rules: Example

- Consider our example with $z_1 = 4$, $p_{12} = 1 \pm 2j$

$$GH(s) = \frac{K(s+4)}{s(s+1+2j)(s+1-2j)}$$



- **Asymptotes:**

$$Angle = \frac{(2k+1)180°}{3-1} = \pm 90°$$

$$\sigma_a = \frac{[-0-(1+2j)-(1-2j)]-[(-4)]}{3-1} = +1$$

Analysis – Construction Rules

7. The angles of departure, $\boldsymbol{q_d}$ from poles and arrival, $\boldsymbol{q_a}$ to zeroes may be found by applying the angle condition to a point very near the pole or zero.

- The angle of arrival at the zero, $\boldsymbol{-t_1}$ is obtained from

$$\theta_{az1} + \sum_{i=2}^{m} \angle(-z_1 + z_i) - \sum_{i=1}^{n} \angle(-t_1 + p_i) = (2k+1)180^\circ$$

This rule is applicable to “**imaginary poles/zeros**”

EE-379 Root Locus Analysis

Analysis – Construction Rules

- Departure angle from, $\boldsymbol{p_2}$.

$$q_{z1} = \tan^{-1}(2/3) = 33.7°$$

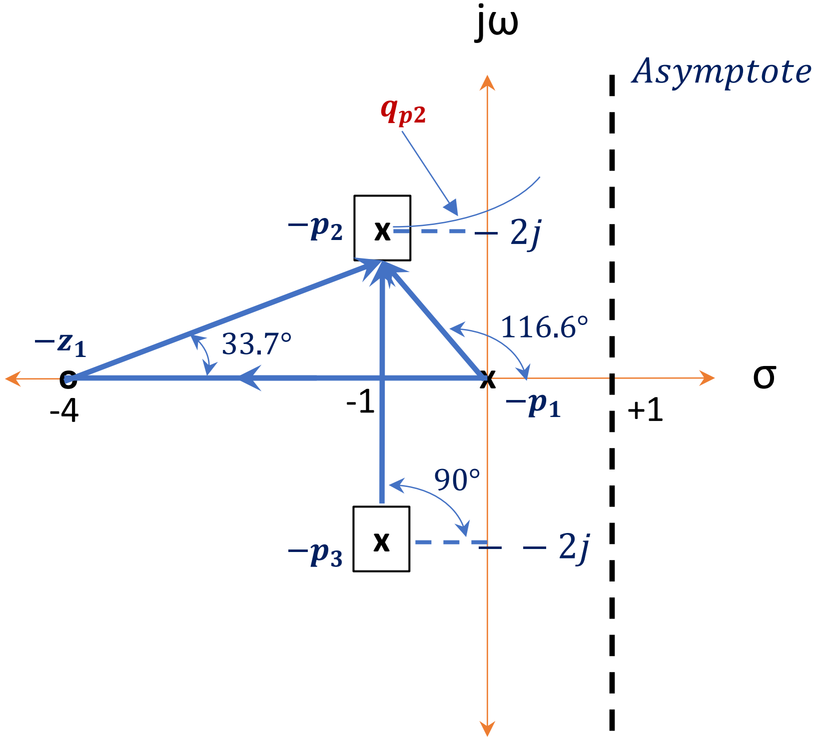
$$q_{p1} = \tan^{-1}(-2/1) = 116.6°$$

$$q_{p3} = 90°$$

- Then,

$$33.7° - \left(90° + 116.6° + q_{p2}\right) = 180°$$

$$q_{p2} = -352.9° = +7.1°$$



EE-379 Root Locus Analysis

Analysis – Construction Rules

8. The **imaginary axis** crossing is obtained by applying the **Routh-Hurwitz criterion** and checking for the gain that results in **marginal stability**. The imaginary components are found from the solution of the resulting **auxiliary equation**

- **Marginal stability** refers to the point where the roots of the closed-loop system are on the **stability boundary, i.e. the imaginary axis**.

EE-379 Root Locus Analysis

Analysis – Construction Rules

Imaginary axis crossing:
Characteristic equation

$$s(s+1+2j)(s+1-2j)+K(s+4)=0$$

$$s^3+2s^2+(5+K)s+4K=0$$

s^3	1	$5+K$	0
s^2	2	$4K$	0
s^1	$5-K$	0	
s^0	$4K$		

- For marginal stability, ***K = 5*** and the auxiliary equation is.

$$2s^2+20=0$$

$$s=\pm\sqrt{10j}=\pm3.16j$$

- Therefore, the **imaginary axis intersection is** $= \pm\mathbf{3.16}\boldsymbol{j}$**.**

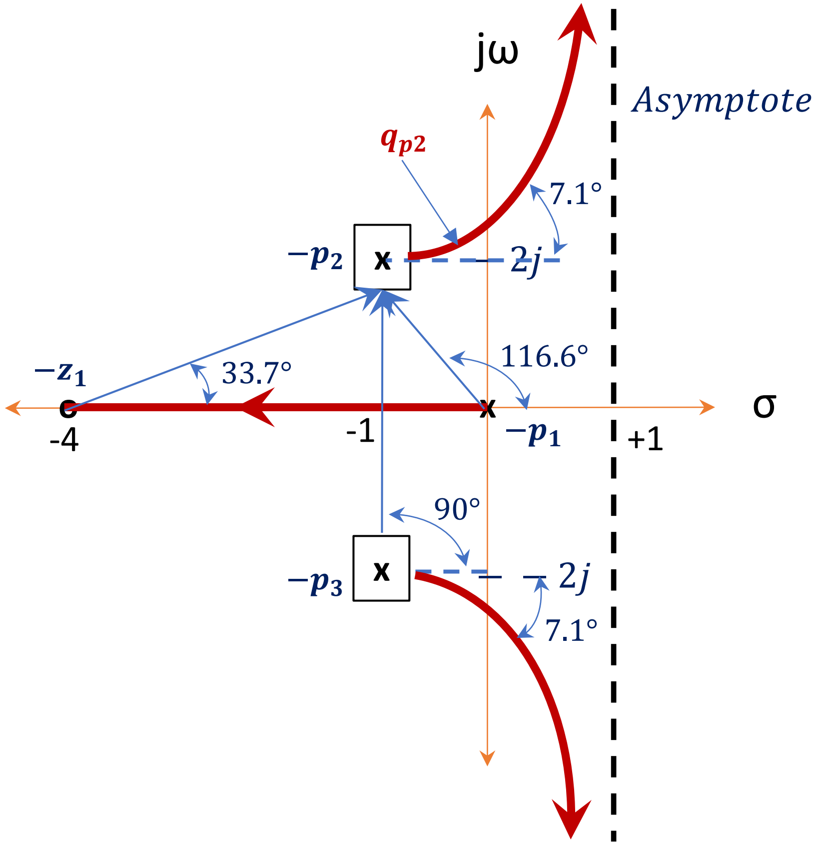
EE-379 Root Locus Analysis

Analysis – Construction Rules

- **Summary:**

  1. There are three root loci.
     I. One on the real axis from $-p_1$ *to* $-z_1$
     II. A pair of loci from $-p_2$ and $-p_3$ to zeroes at infinity along the asymptotes.

  2. The departure angle from these poles is $\pm 7.1°$ and an imaginary axis crossing at $s = \pm 3.16j$

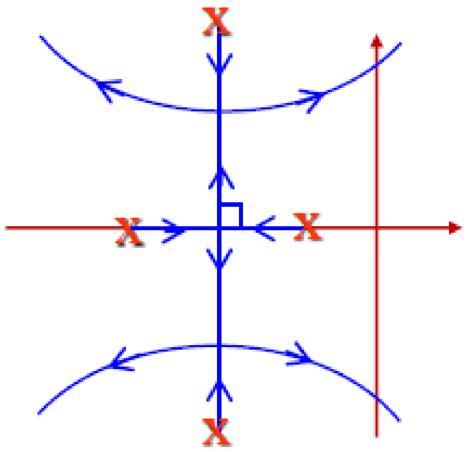


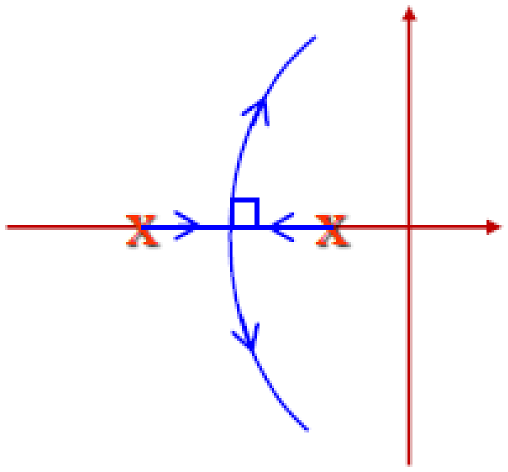
EE-379 Root Locus Analysis

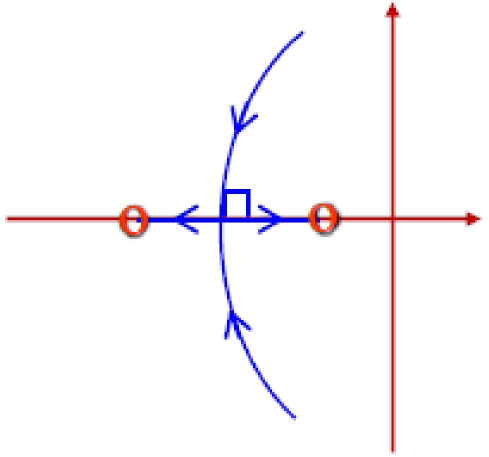
Analysis – Construction Rules

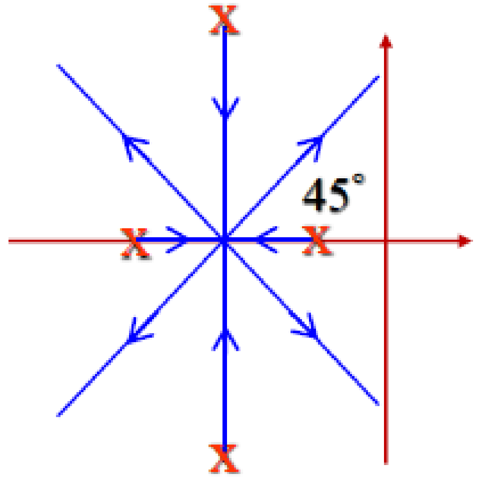
- **Breakaway Points**

When two or more loci meet, they will break away from this point at particular angles. The point is known as a **breakaway point**. It corresponds to multiple roots.









EE-379 Root Locus Analysis

Analysis – Construction Rules

9. The **angle** of breakaway is $\mathbf{180°/k}$ where $\boldsymbol{k}$ is the number of closed-loop poles, departing from the breakaway point on the real axis.

✓ The **location** of the breakaway point is found from.

$$\frac{dK}{ds} = 0 \qquad or \qquad \frac{d[GH(s)]}{ds} = 0$$

- Note:

$$K = -[GH(s)]^{-1}$$

$$\frac{dK}{ds} = [GH(s)]^{-2}\,\frac{d[GH(s)]}{ds} = 0$$

- Also:

$$\frac{d[GH(s)]}{ds} = \frac{d[N(s)/D(s)]}{ds}$$

$$\frac{N'(s)}{D(s)} - \frac{N(s)D'(s)}{D(s)^2} = 0$$

$$\boxed{D(s)N'(s) - N(s)D'(s) = 0}$$

EE-379 Root Locus Analysis

Analysis – Construction Rules

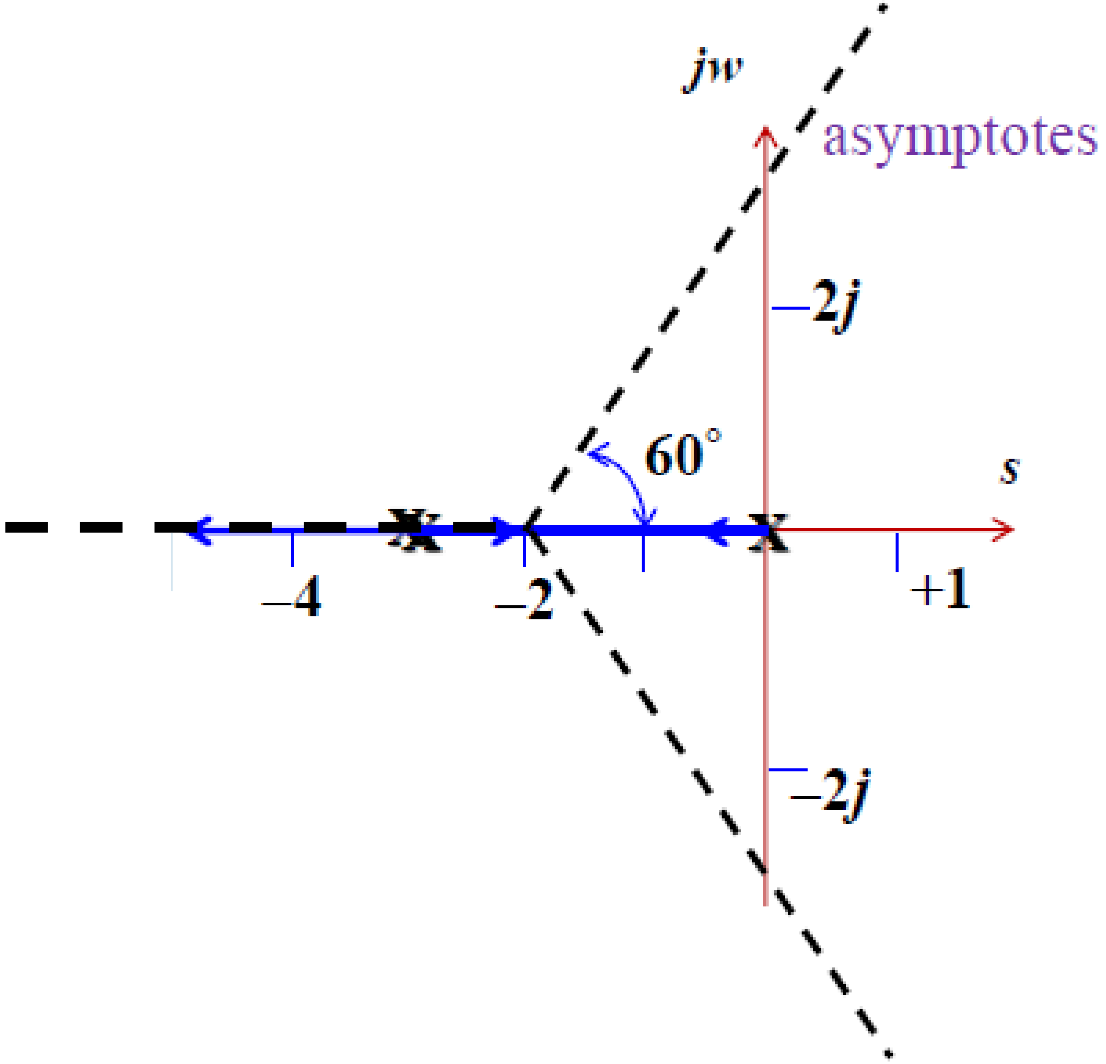
- Consider the following loop

$$GH(s) = \frac{K}{s(s+3)^2}$$

- Real axis loci exist for the full negative axis. the following loop.
- **Asymptotes**:

$$= \frac{(2k+1)180°}{3} = 60°, 180°, 300°$$

$$\sigma_a = \frac{(-3-3-0)-(0)}{3} = -2$$



EE-379 Root Locus Analysis

Analysis – Construction Rules

- Determine the breakaway points from

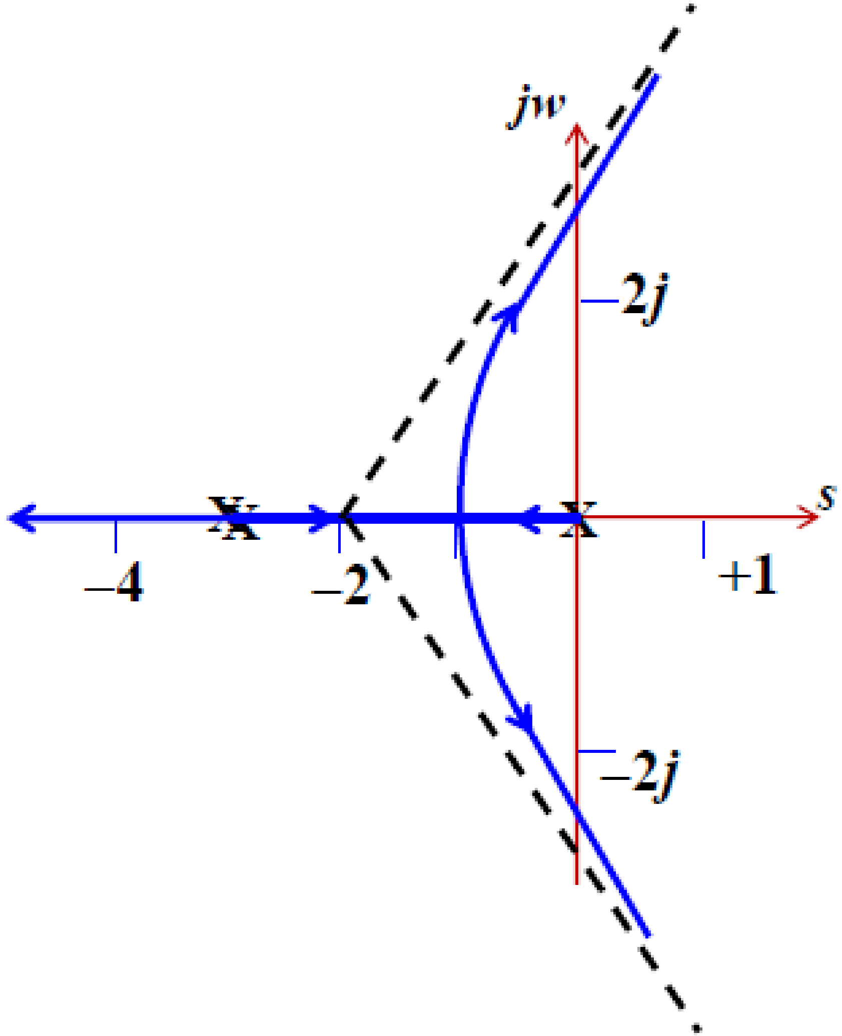
$$\frac{d}{ds}\left[\frac{K}{s(s+3)^2}\right] = \frac{d}{ds}\left[\frac{K}{s^3+6s^2+9s}\right]$$

$$= -\frac{K(3s^2+12s+9)}{(s^3+6s^2+9s)^2} = 0$$

- Then,

$$(s^2+4s+3) = (s+1)(s+3) = 0$$

$$s = -1, \qquad s = -3$$



EE-379 Root Locus Analysis

Analysis – Construction Rules: Summary

1. Plot the poles and zeros of the open-loop system.
2. Find the section of the loci on the real axis (odd number of poles and zeroes to the right).
3. Determine the **asymptote angles** and **intercepts**.

$$angles = \frac{(2k+1)180°}{q} \qquad where, \boldsymbol{q} = \boldsymbol{n} - \boldsymbol{m}, and, \boldsymbol{k} = \boldsymbol{0}, \pm\boldsymbol{1}, \pm\boldsymbol{2}, \pm\boldsymbol{3} \ldots (\boldsymbol{q} - \boldsymbol{1})$$

$$\sigma_a = \frac{\sum Poles - \sum zeros}{q}$$

EE-379 Root Locus Analysis

Analysis – Construction Rules: Summary

180 − Σ∠

4. Determine departure angles in case of **imaginary poles**. For a pole $-\boldsymbol{p_1}$.

$$\angle(-p_1+z_1)+\angle(-p_1+z_2)+\cdots-\theta_{p1}-\angle(-p_1+p_2)-\angle(-p_1+p_3)=(2k+1)180°$$

5. Check for imaginary axis crossings using the Routh-Hurwitz criterion.

6. Determine breakaway points in case of loci originates from two poles together.

$$angle=\frac{180°}{k}, \qquad where\ k = No\ of\ converging\ loci$$

$$\boldsymbol{location\ from} \qquad \frac{d[GH(s)]}{ds}=0$$

7. Complete the plot

EE-379 Root Locus Analysis

Analysis – Construction Rules: Example

- **Loop Transfer function:**

$$GH(s) = \frac{K}{s(s+4)(s^2+4s+20)}$$

- **Roots:**

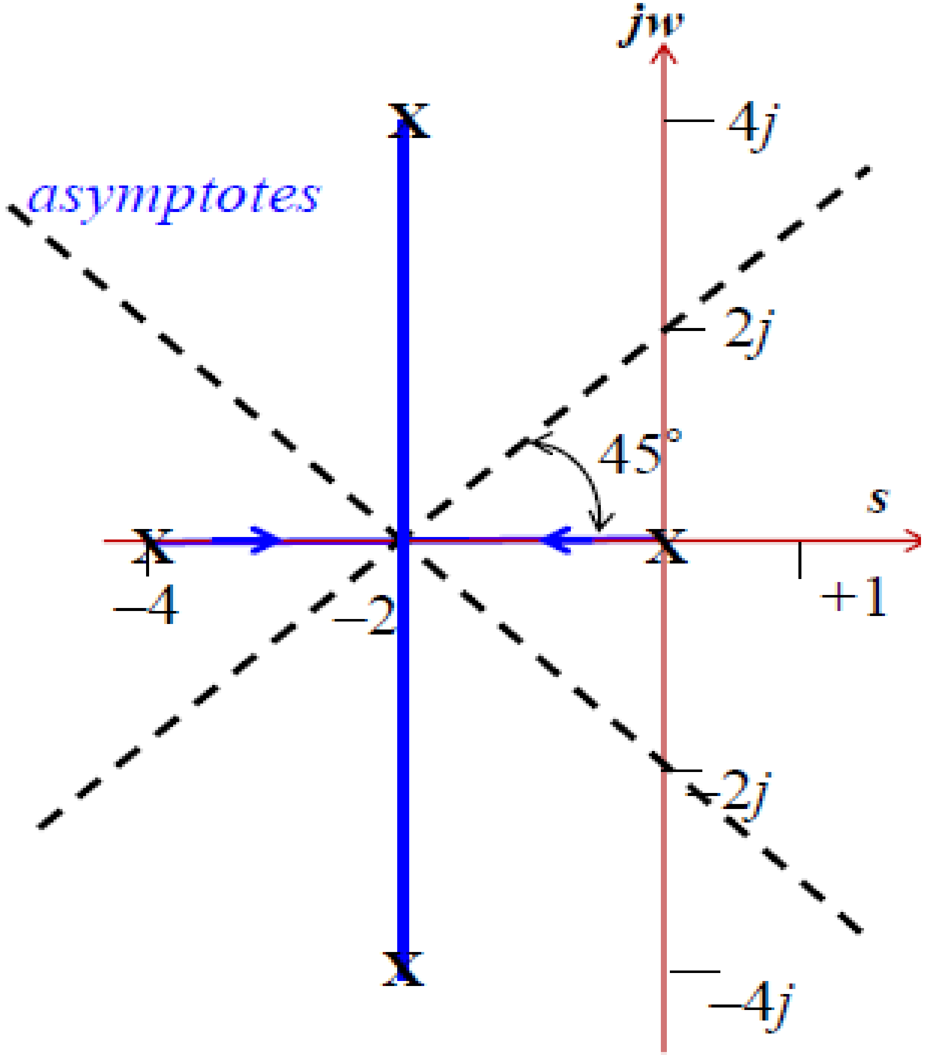
$$s = 0, s = -4, s = -2 \pm 4j$$

- Real axis segments between **0** and **–4**

- **Asymptotes:**

$$angles = \frac{(2k+1)180°}{4-0} = 45°, 135°, 225°, 315°$$

$$\sigma_a = \frac{(-4-2-2-0)}{4} = -2$$



EE-379 Root Locus Analysis

Analysis – Construction Rules: Example

- **Breakaway points:**

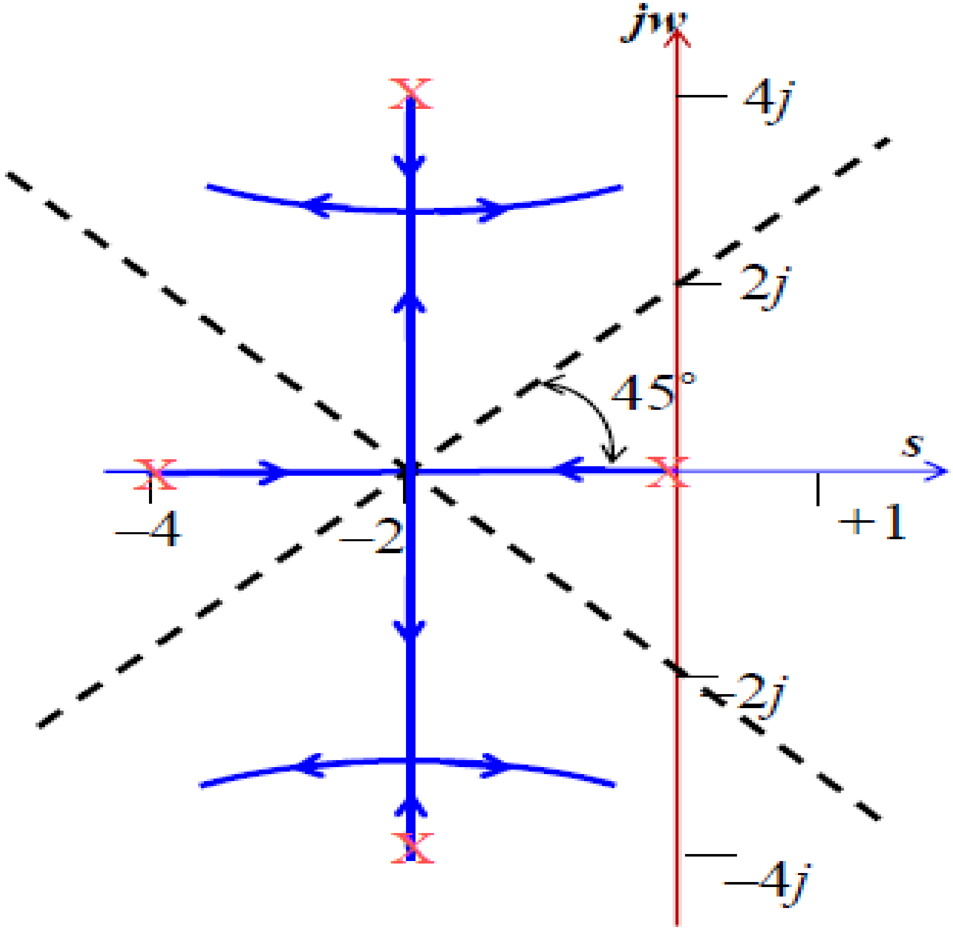
$$\frac{d}{ds}\left[\frac{K}{\underbrace{s^4+8s^3+36s^2+80s}}\right] \quad \text{let } P$$

$$=\frac{K(4s^3+24s^2+72s+80)}{s^4+8s^3+36s^2+80s}=0$$

$$or \qquad s^3+6s^2+18s+20=0$$

$$solving \qquad s_b=-2,-2\pm 2.45j$$

Three points that breakaway at 90°



EE-379 Root Locus Analysis

$1+GH=0 \Rightarrow 1+\frac{K}{P} \Rightarrow \frac{P+K}{P}=0$

Analysis – Construction Rules: Example

- **The imaginary axis crossings:**

  Characteristics equation

  $P+K=0$

  $$s^4 + 8s^3 + 36s^2 + 80s + K = 0$$

s^4	1	36	K
s^3	8	80	0
s^2	26	K	0
s^1	$80 - \frac{8K}{26}$	0	
s^0	K		

- Condition for critical stability

$$80 - \frac{8K}{26} > 0$$

$$K < 260$$

- The auxiliary equation

$$26s^2 + 260 = 0$$

Solving,

$$s = \pm\sqrt{10}j = \pm 3.16j$$

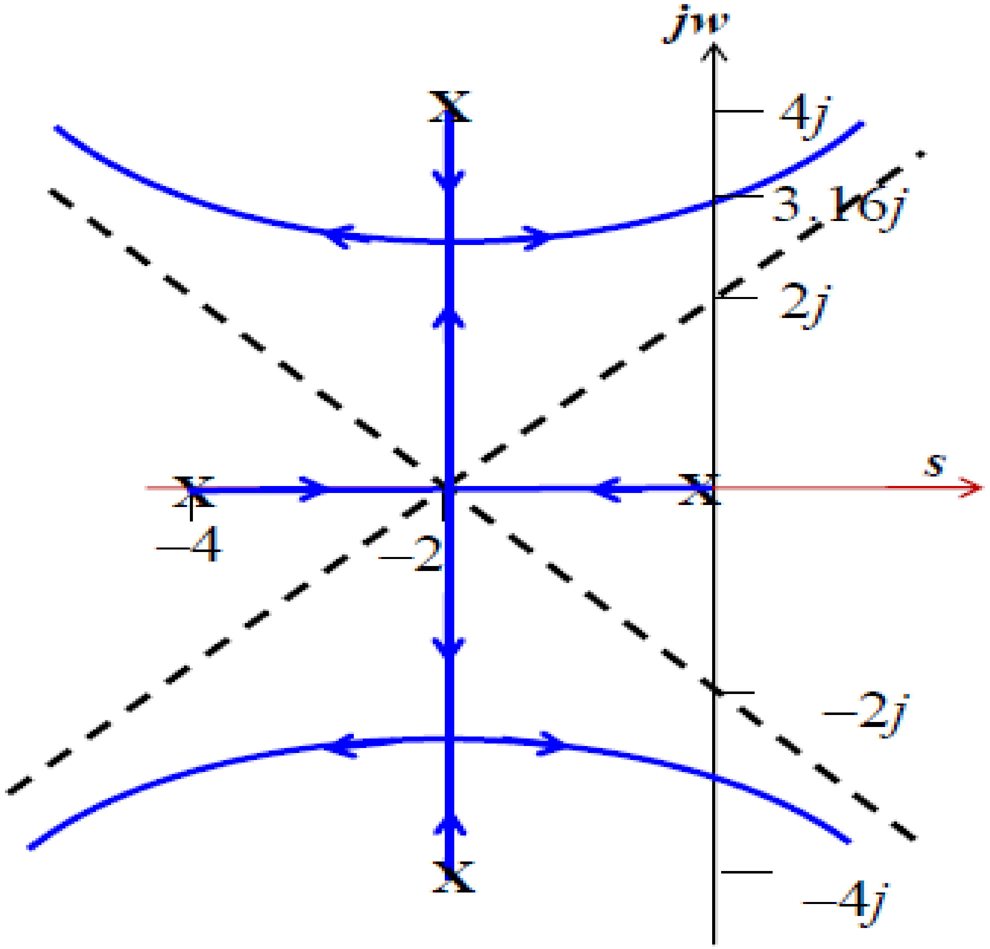
EE-379 Root Locus Analysis

Analysis – Construction Rules: Example

- The final plot is shown on the right.

- What is the value of the gain ***K*** corresponding to the breakaway point at.

$$s_b = -2 \pm 2.45j\ ?$$



EE-379 Root Locus Analysis

Analysis – Construction Rules: Example

- From the general magnitude condition the gain corresponding to the point s_1 on the loci is.

$$K = \frac{\prod_{i=1}^{n}(s + p_i)}{\prod_{i=1}^{m}(s + z_i)}$$

- For the point $s_1 = -2 \pm 2.45j$

$$K = \frac{|-2 + 2.45j + 4||-2 - 2.45j + 4||-2 + 2.45j + 2 + 4j||-2 - 2.45j + 2 - 4j|}{1}$$

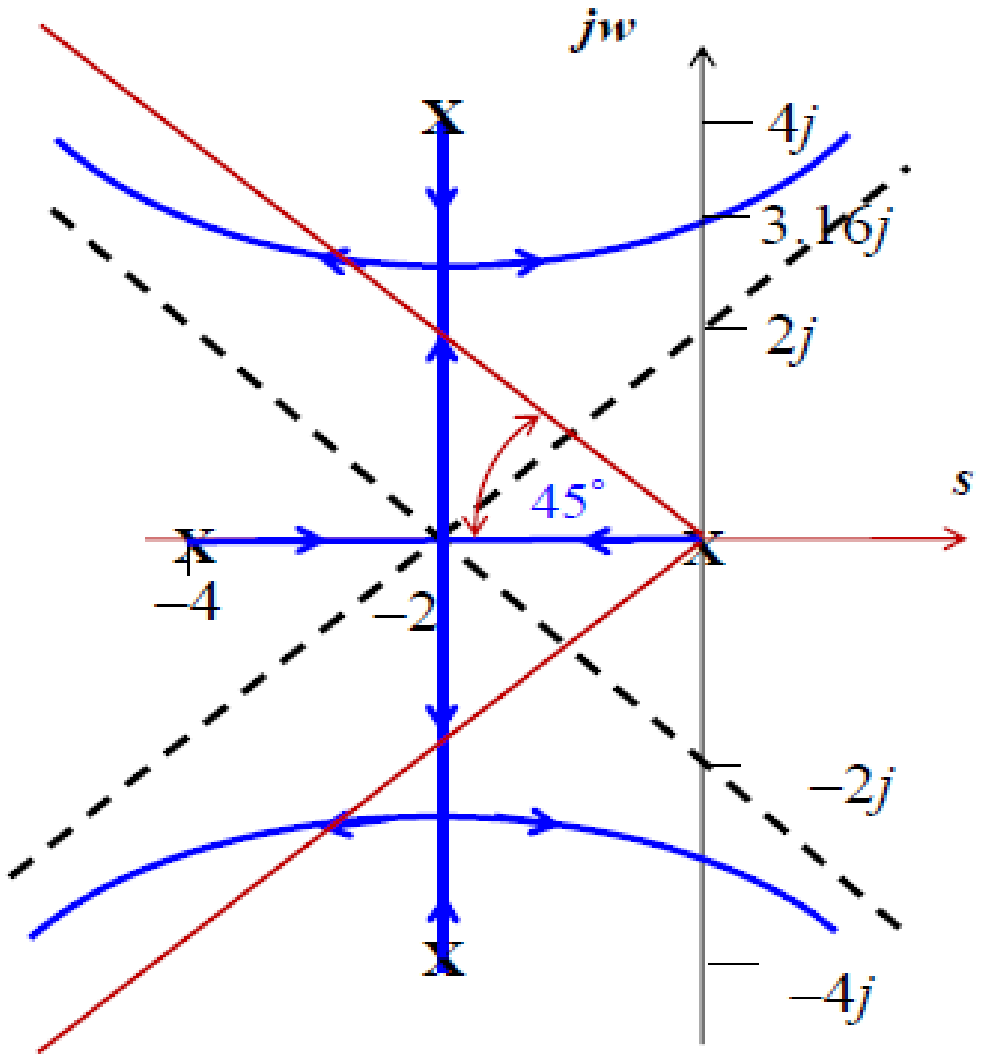
$$K = 3.163 \cdot 3.163 \cdot 6.45 \cdot 1.55 = 100$$

EE-379 Root Locus Analysis

Analysis – Construction Rules: Example

- Is there a gain corresponding to a damping ratio of ***0.707*** or more for all system modes?

$$\zeta = 0.707 = \cos(q)$$
$$q = 45°$$



EE-379 Root Locus Analysis

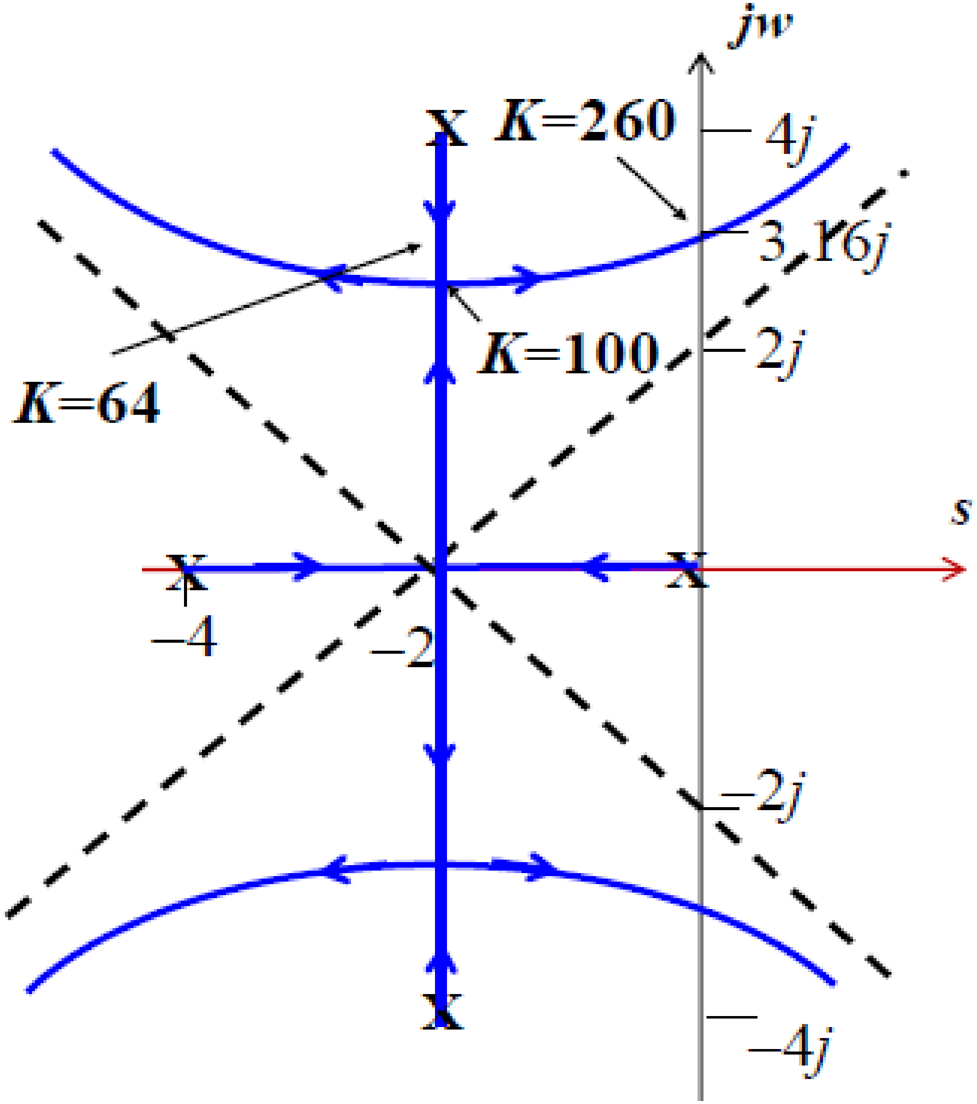
Analysis – Construction Rules: Example

- Examine the responses for the various gains and relate them to the location of the closed loop roots.

  - $K = 64, roots\ are$
    $-2, -2, -2 \pm 3.46j$

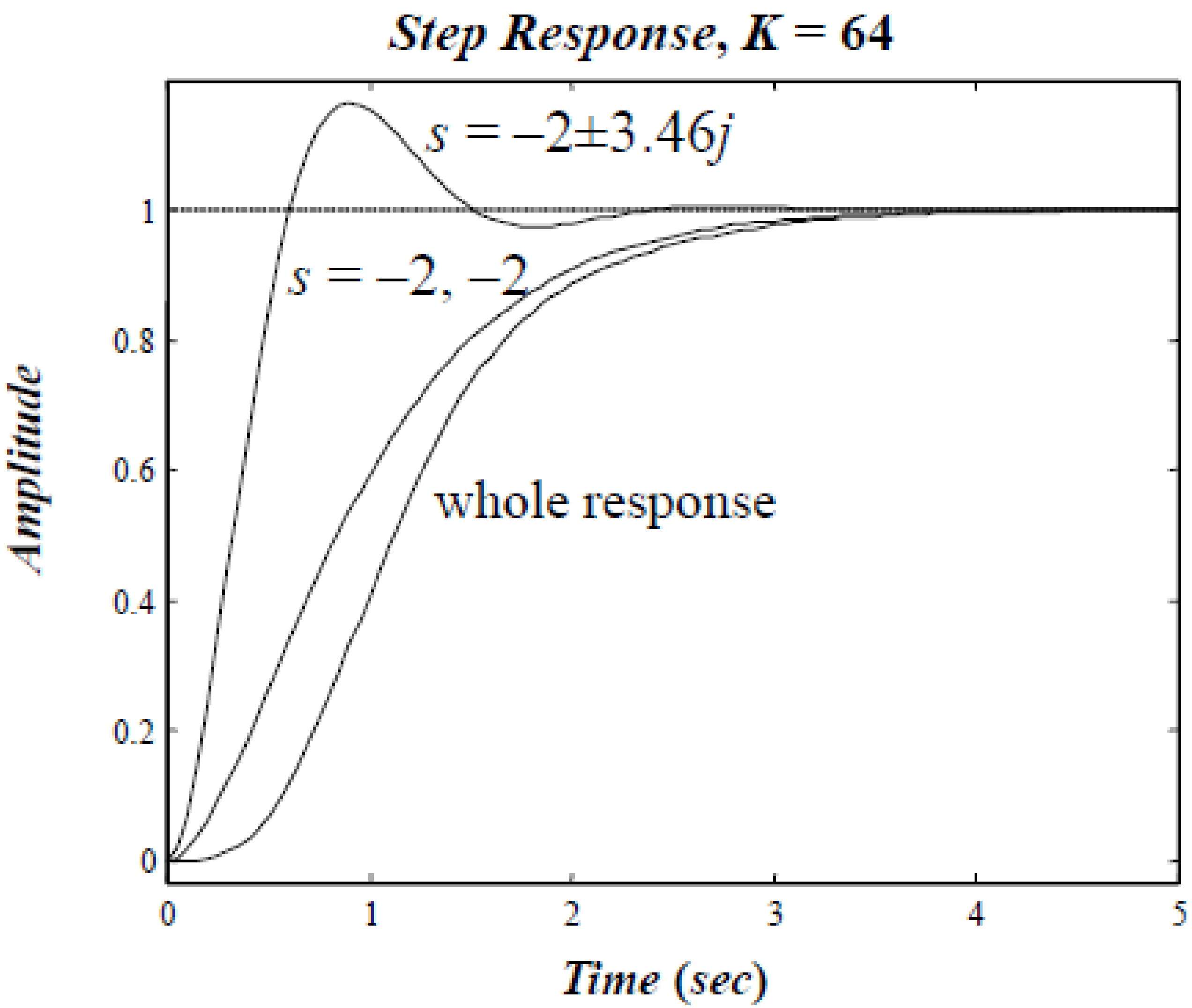
  - $K = 100, roots\ are$
    $-2 + 2.45j, -2 - 2.45j$

  - $K = 260, roots\ are$
    $\pm 3.16j, -4 \pm 3.16j$



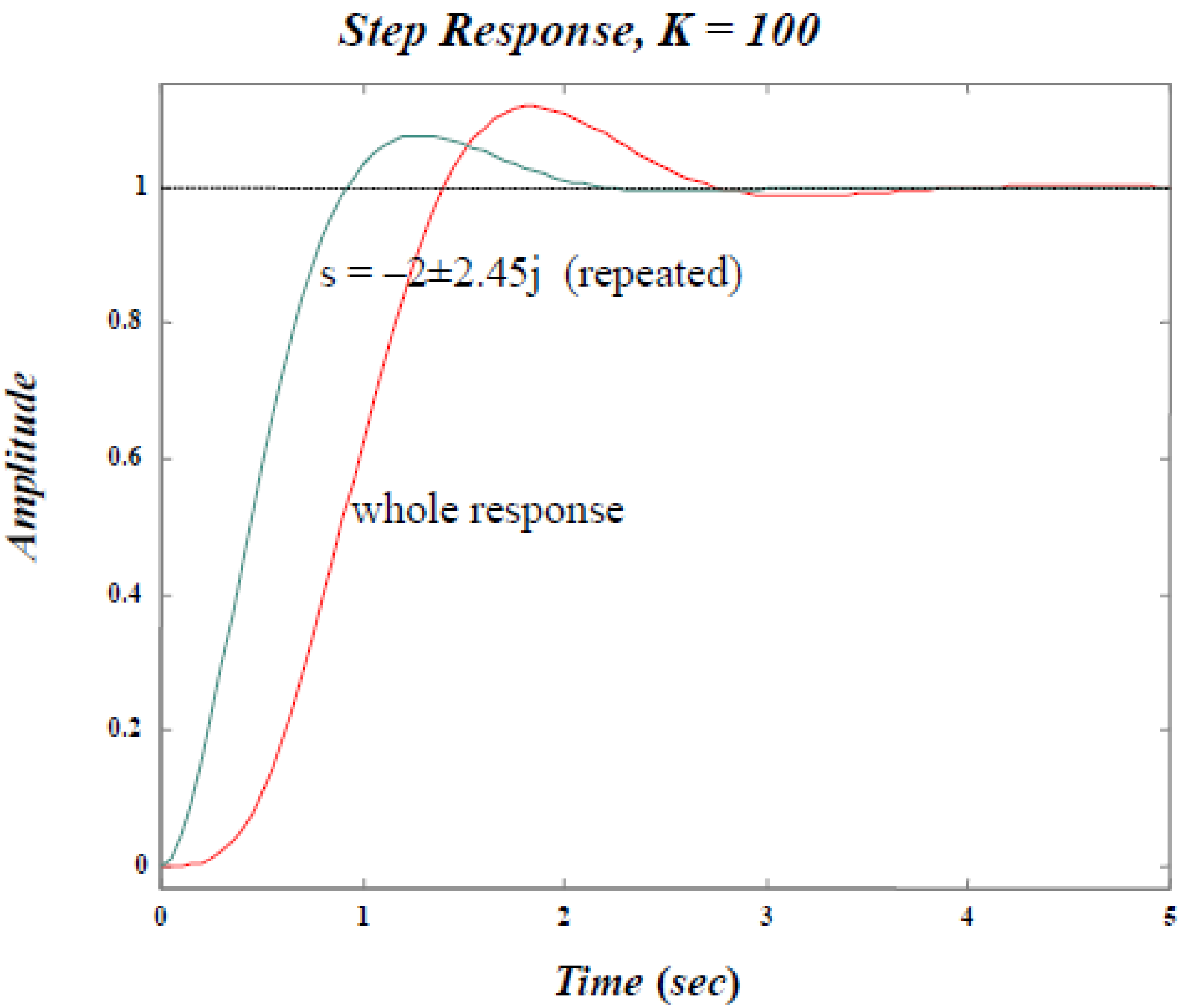
EE-379 Root Locus Analysis

Analysis – Construction Rules: Example



EE-379 Root Locus Analysis

Analysis – Construction Rules: Example



EE-379 Root Locus Analysis

Analysis – Construction Rules: Example