

EE-379 Linear Control Systems

Week No. 6: Root Locus Analysis

- Background
- Pole Zero Plots
- General Method
- Construction Rules
- Examples

EE-379 Root Locus Analysis

Background

- **Chapter 1:** D.E can be written for various systems and can be solved, and the response could be divided into **forced** and **natural** components
- **Chapter 2:** Established definitions related to **natural response** and **types of stability** (stability is generally considered to be the property of the natural response)
- **Chapter 3:** The other response component (**steady state**) was considered that is the tendency of a device to follow (track) a command.
- **Chapter 4:** **Return to stability** and provide a very broad and useful measure as compared to the Routh Hurwitz criterion.

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Background

- **Logical approach to determine stability** would be to
 - ✓ Extract roots of the characteristic polynomial as the adjustable gain varies (computational packages i.e., MATLAB can readily perform this task)
 - ✓ Walter Evans (1940) developed a set of rules by which the path traced by roots of a closed loop characteristic equation can be sketched this plot is the root locus.
 - ✓ **There are two chapters devoted to root locus:** in chapter 4 we will concentrate on the basic understanding of root locus principles

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Background

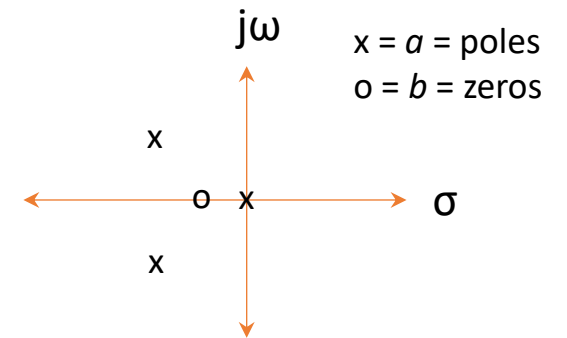
- **Zeros** are the values at which the function is zero (numerator roots)
- **Poles** are the values at which the function is infinite (denominator roots)
- A rational function in the factored form is given by:
- The constant $k = \frac{b_m}{a_m}$ is the **multiplying constant**
- When poles and zeros are plotted on the complex plain the result is a **pole-zero plot**

$$F(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_0}{s^n + a_{n-1} s^{m-1} + \dots + a_0}$$

$$F(s) = \frac{K[(s - z_1)(s - z_2) \dots (s - z_m)]}{[(s - p_1)(s - p_2) \dots (s - p_n)]}$$

$z_1, z_2, z_3, \dots, z_m$ are the zeros of the function

$p_1, p_2, p_3, \dots, p_n$ are the poles of the function



$$F(s) = \frac{4s + 5}{s^3 + 4s^2 + 13s} = \frac{4(s + 5/4)}{s(s + 2 + 3j)(s + 2 - 3j)}$$

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Pole-Zero Plot: Graphical Evaluation

- The roots of the **closed-loop** characteristic equation define the system characteristic responses.
- Their location in the **complex s-plane** lead to the prediction of the characteristics of the time domain responses in terms of:
 - ✓ damping ratio, ζ
 - ✓ natural frequency, ω_n
 - ✓ damping constant, b

Second order modes

first order modes
- Consider how these roots change as the **loop gain** is varied from **0** to ∞

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Example

- The closed-loop transfer function is

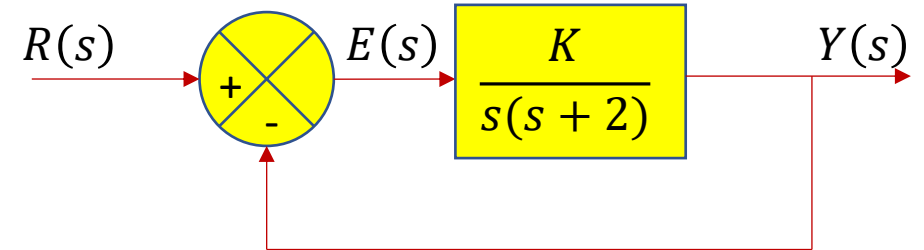
$$T(s) = \frac{Y(s)}{R(s)} = \frac{K}{s(s+2) + K}$$

- The characteristic equation is

$$s^2 + 2s + K = 0$$

- Consider the characteristic roots as:

$$K = 0 \rightarrow \infty$$



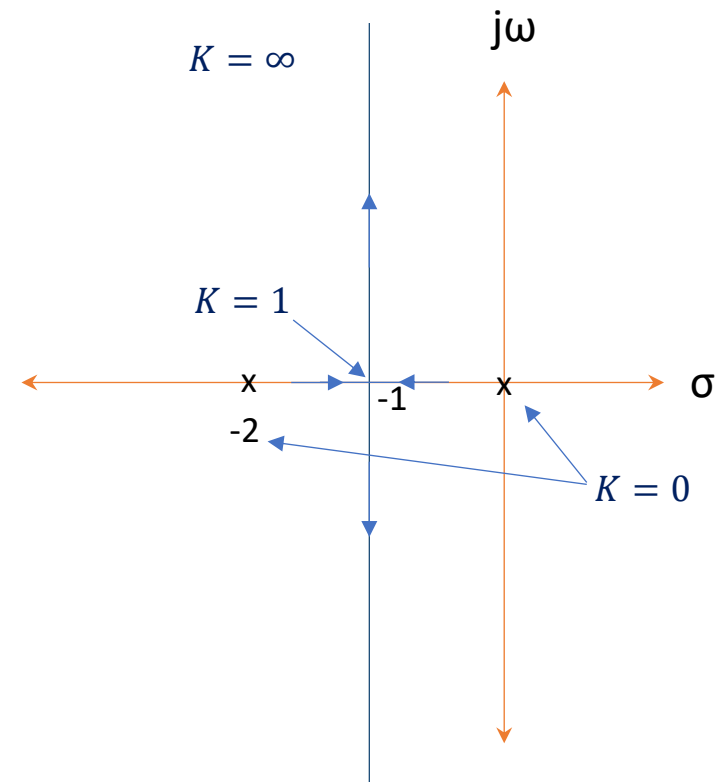
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Example

$$s = -1 \pm \sqrt{1 - K}$$

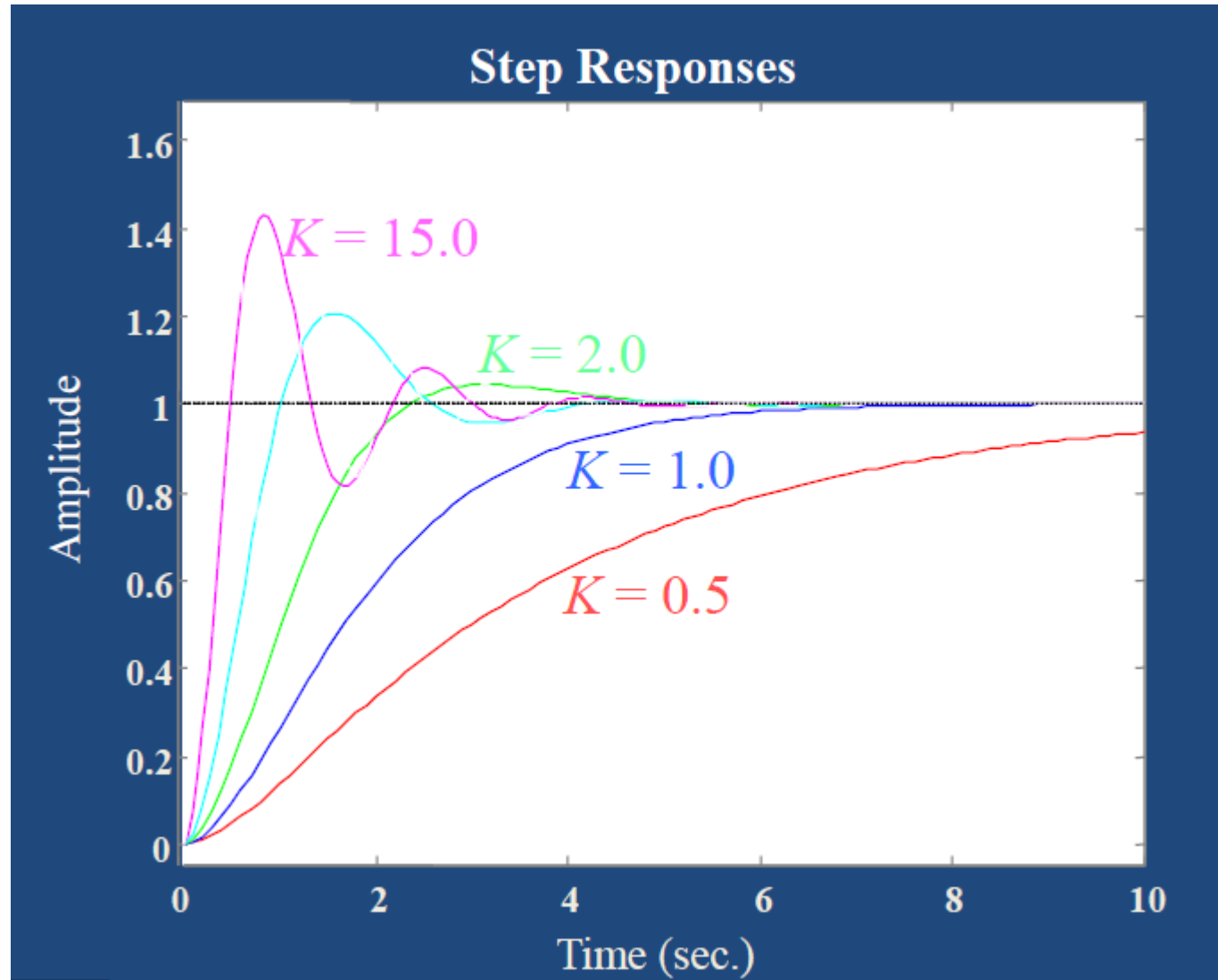
- For $K = 0$ the closed-loop poles are at the **open-loop poles**.
- For $0 < K < 1$ the closed loop poles are on the **real axis**:
- For $K > 1$ the closed-loop poles are complex, with a real value of -1 and an imaginary value increasing with gain K .

Loci of closed-loop roots

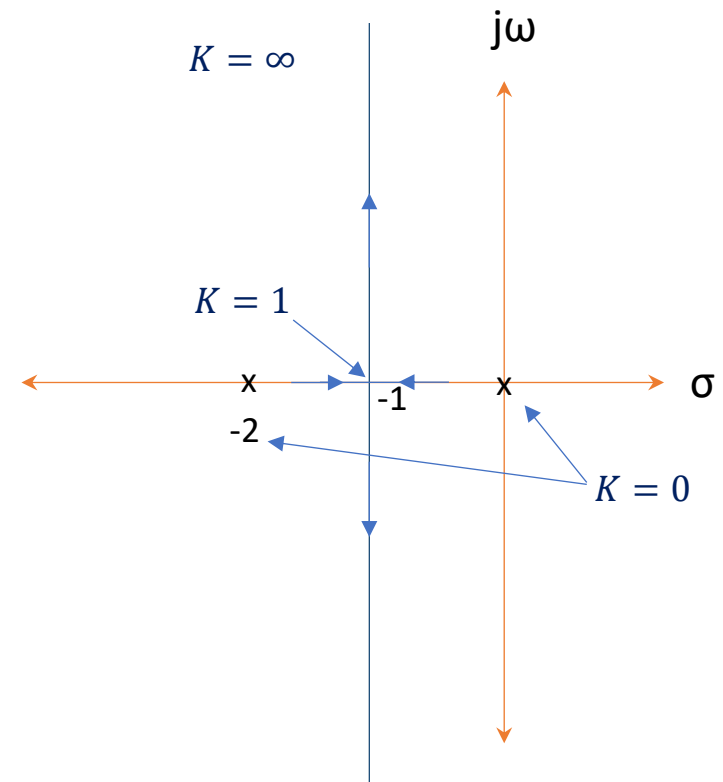


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Example



Loci of closed-loop roots

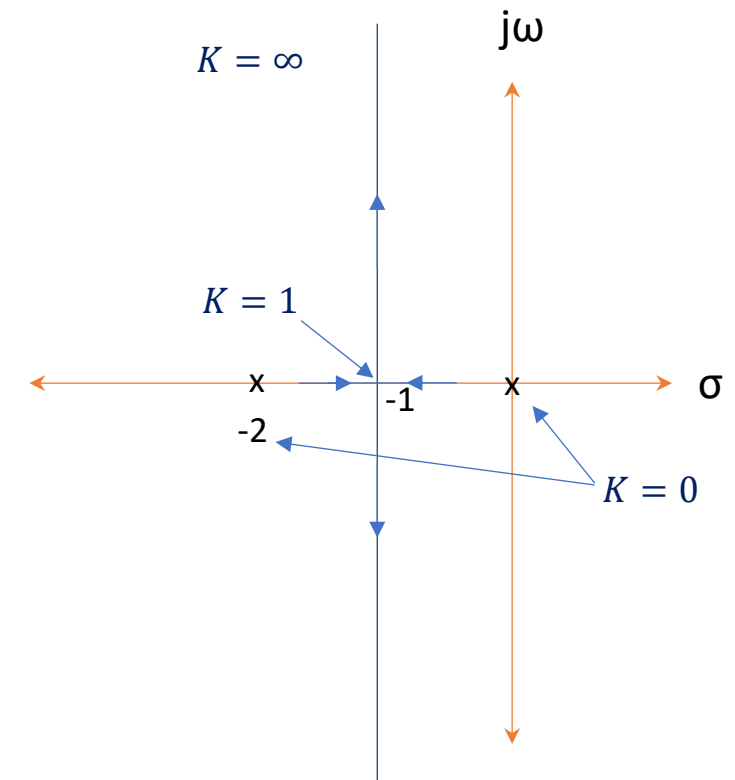


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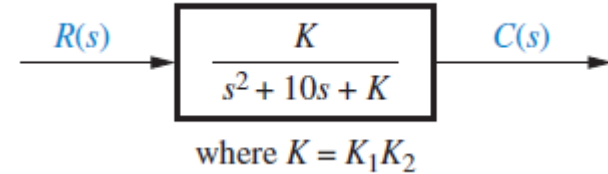
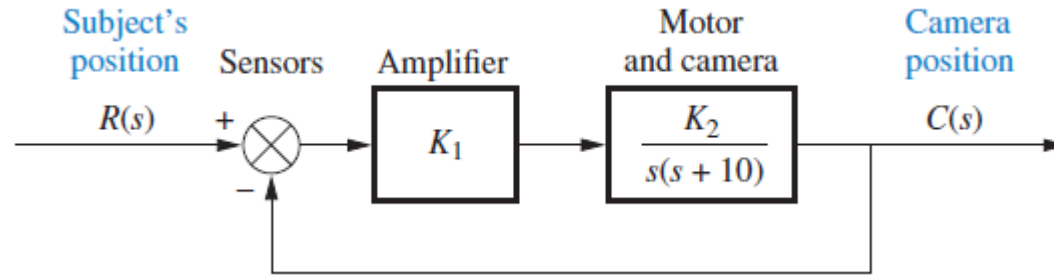
Example-Observations

- This is a second-order system and there are **two loci**.
- The root loci **start** at the **open loop poles**
- The root loci tend towards the open loop zeros at infinity as $K \rightarrow \infty$. (Note: the **number of zeros** is equal to the **number of poles** when the zeros at infinity are included.)
- The relationship between the **characteristic responses** and **the increasing gain** is seen through the **root loci**.

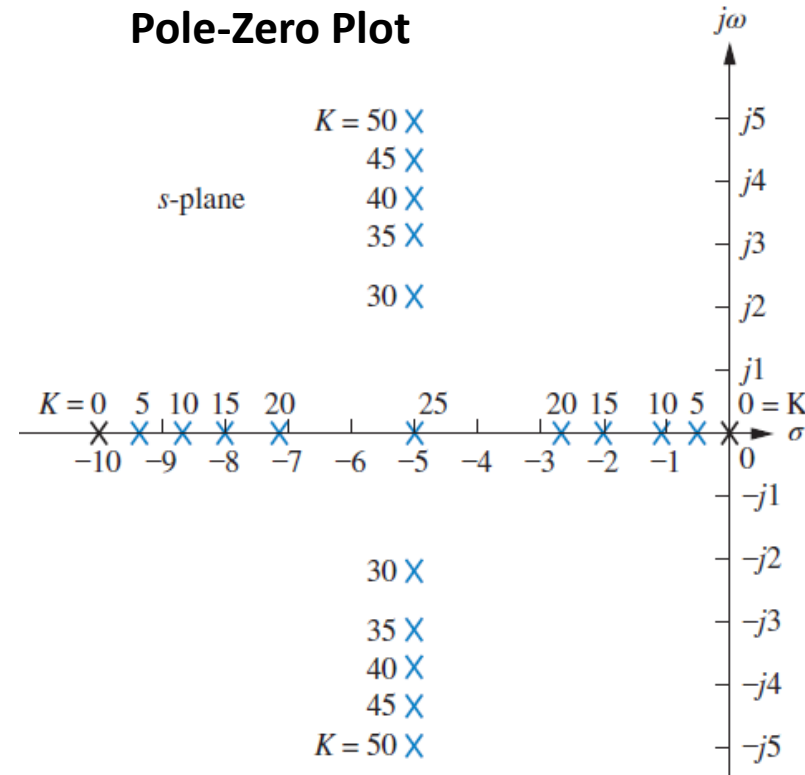
Loci of closed-loop roots



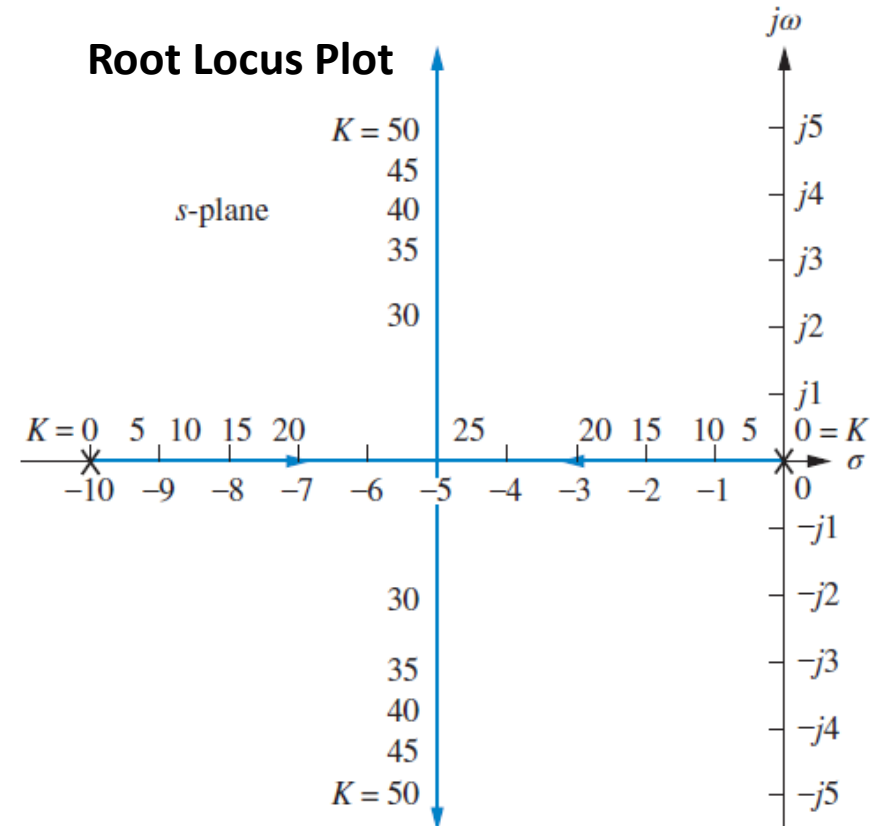
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Pole-Zero Plot



Root Locus Plot



K	Pole 1	Pole 2
0	-10	0
5	-9.47	-0.53
10	-8.87	-1.13
15	-8.16	-1.84
20	-7.24	-2.76
25	-5	-5
30	$-5 + j2.24$	$-5 - j2.24$
35	$-5 + j3.16$	$-5 - j3.16$
40	$-5 + j3.87$	$-5 - j3.87$
45	$-5 + j4.47$	$-5 - j4.47$
50	$-5 + j5$	$-5 - j5$

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The General Root Locus Method

- Consider the general system

$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + GH(s)}$$

- The characteristic equation is

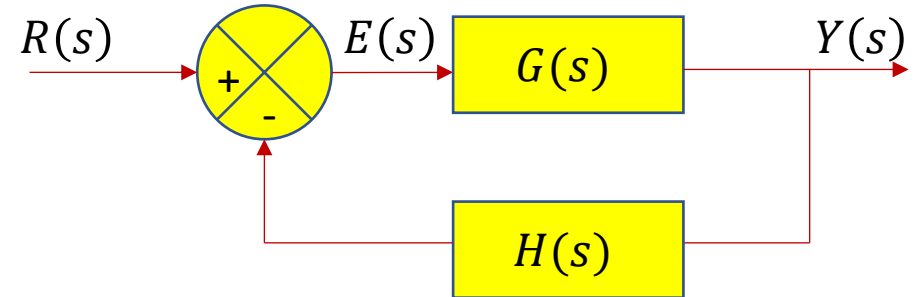
$$1 + GH(s) = 0 \quad \Rightarrow \quad GH(s) = -1 \rightarrow \text{Complex quantity}$$

Therefore, $GH(s) = -1$, can be split into two equations

1. $|GH(s)| = 1 \rightarrow \text{represents magnitude}$

2. $\angle GH(s) = (2k + 1)180^\circ \rightarrow \text{represents angle}$

$$k = 0, \pm 1, \pm 2 \dots$$



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The General Root Locus Method

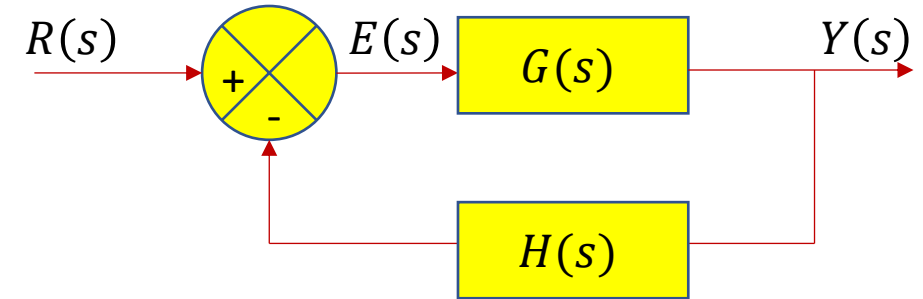
- All values of 's' which satisfy

- $|GH(s)| = 1$ **Magnitude criterion**
- $\angle GH(s) = (2k + 1)180^\circ$ **Angle criterion**
 $k = 0, \pm 1, \pm 2 \dots$

are **roots** of the **closed-loop characteristic equation**.

- Consider the following general form

$$GH(s) = \frac{K[(s + z_1)(s + z_2) \dots (s + z_m)]}{[(s + p_1)(s + p_2) \dots (s + p_n)]}$$



Note: p_i may be zero

EE-379 Root Locus Analysis

The General Root Locus Method

- Then,

$$1. \quad |GH(s)| = \frac{|K| \prod_{i=1}^m |s+z_i|}{\prod_{i=1}^n |s+p_i|} = 1 \quad \longleftrightarrow \quad |F(s_0)| = \frac{|K| \left(\begin{smallmatrix} \text{product of lengths of dir} \\ \text{segment from zeros to } s_0 \end{smallmatrix} \right)}{\left(\begin{smallmatrix} \text{product of lengths of dir} \\ \text{segment from poles to } s_0 \end{smallmatrix} \right)}$$

The **magnitude condition** is that the **point** (which satisfied the angle condition) at which the **magnitude of the open loop transfer function is one**.

$$2. \quad \angle GH(s) = \sum_{i=1}^m \angle(s + z_i) - \sum_{i=1}^n \angle(s + p_i) = (2k + 1)180^\circ \quad k = 0, \pm 1, \pm 2 \dots$$

$\angle F(s_0)$ = (sum of angles of dir segments)

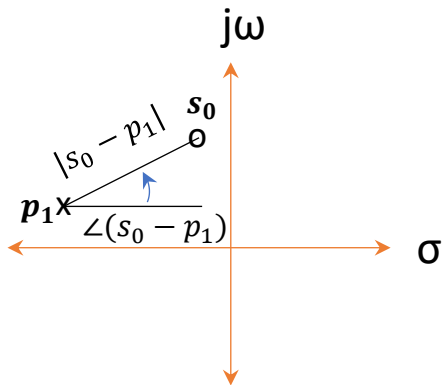
$\angle F(s_0)$ = (from zeros to s_0) – (sum of poles angles) + 180° (if $K < 0$)

The **angle condition** is the point at which **the angle of the open loop transfer function is an odd multiple of 180°** .

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Pole-Zero Plot: Graphical Evaluation

- On a **pole-zero** plot suppose a line is drawn from p_1 to s_0 where the function is being evaluated.
- This segment has length $|s_0 - p_1|$ and angle $\angle(s_0 - p_1)$ with real axis



$$F(s) = \frac{K[(s - z_1)(s - z_2) \dots (s - z_m)]}{[(s - p_1)(s - p_2) \dots (s - p_n)]}$$

$$F(s_0) = \frac{K[(s_0 - z_1)(s_0 - z_2) \dots (s_0 - z_m)]}{[(s_0 - p_1)(s_0 - p_2) \dots (s_0 - p_n)]}$$

$$F(s_0) = \frac{K(|s_0 - z_1|e^{j\angle(s_0 - z_1)})(|s_0 - z_2|e^{j\angle(s_0 - z_2)})}{(|s_0 - p_1|e^{j\angle(s_0 - p_1)})(|s_0 - p_2|e^{j\angle(s_0 - p_2)})}$$

$$|F(s_0)| = \frac{|K| \left(\text{product of lengths of dir segment from zeros to } s_0 \right)}{\left(\text{product of lengths of dir segment from poles to } s_0 \right)}$$

$$\angle F(s_0) = (\text{sum of angles of dir segments from zeros to } s_0) - (\text{sum of angles of dir segments from poles to } s_0) + 180^\circ \text{ (if } k < 0)$$

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General Method – Geometric Interpretation

- Consider the example,

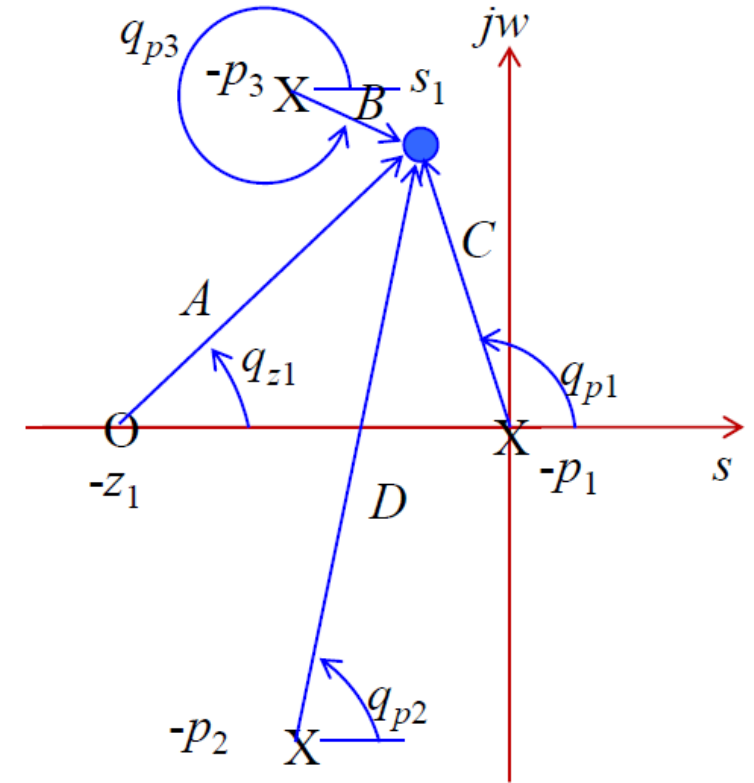
$$GH(s) = \frac{K(s + z_1)}{s(s + p_2)(s + p_3)}$$

- Then the values of $s = s_1$ which satisfy

$$1. \quad \frac{K|s+z_1|}{|s||s+p_2||s+p_3|} = 1$$

$$2. \quad \angle(s + z_1) - (\angle s + \angle(s + p_2) + \angle(s + p_3)) = (2k + 1)180^\circ$$

are on the loci and are roots of the characteristic equation.



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General Method – Geometric Interpretation

- In terms of vectors, the condition for $s = s_1$ to be on the root loci:

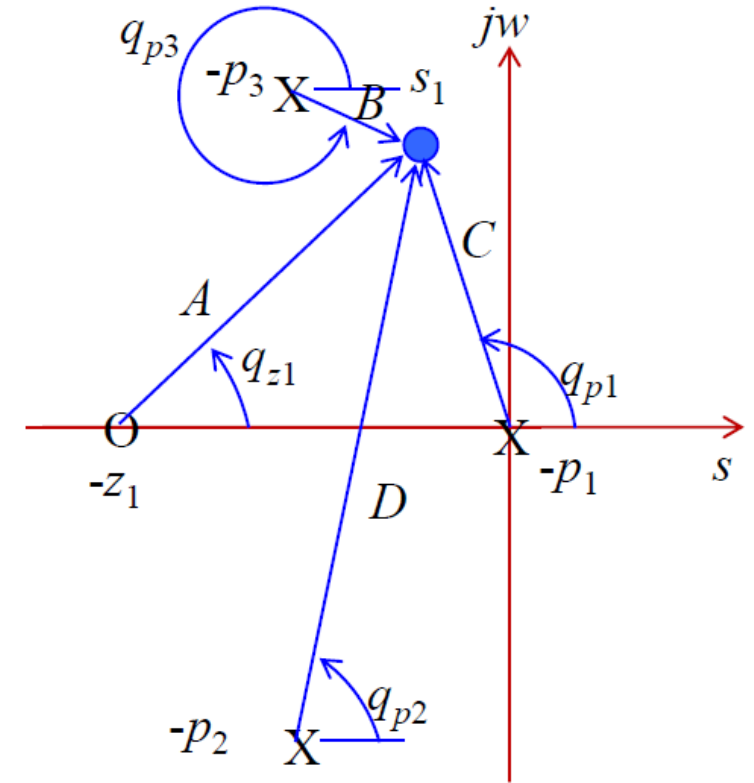
$$\frac{|K|A}{BCD} = 1 \quad \text{or} \quad \frac{A}{BCD} = \frac{1}{|K|}$$

and,

$$\theta_{z1} - (\theta_{p1} + \theta_{p2} + \theta_{p3}) = (2k + 1)180^\circ$$

$$k = 0, \pm 1, \pm 2 \dots$$

are roots of the characteristic equation.



EE-379 Root Locus Analysis

Analysis

- When plotting the loci of the roots as $K = 0 \rightarrow \infty$, the magnitude condition is always satisfied.
- Therefore, a value of $s = s_1$ that satisfies the angle condition, is a point of the root loci
- The magnitude condition may then be used to determine the gain K corresponding to that value s_1 .

EE-379 Root Locus Analysis

Analysis – Construction Rules

1. The loci **start** ($K = 0$) at the **poles** of the open-loop system. There are $n -$ **loci**.
2. The loci **terminate** ($K \rightarrow \infty$) at the **zeroes** of the open-loop system (include zeroes at infinity).

- For our example system

$$|GH(s)| = \frac{K|s + z_1|}{|s||s + p_2||s + p_3|} = \frac{1}{K}$$

- Therefore, as $K \rightarrow 0$
 $GH(s) \rightarrow \infty$, the poles of the loop transfer function
- As, $K \rightarrow \infty$
 $GH(s) \rightarrow 0$, the zeros of the loop transfer function

EE-379 Root Locus Analysis

Analysis – Construction Rules

3. The root loci are **symmetrical** about the real axis.
 - The roots with imaginary parts always occur in conjugate complex pairs
4. As $K \rightarrow \infty$ the loci approach **asymptotes**. There are $q = n - m$ asymptotes and they intersect the real axis at **angles** defined by

$$\frac{(2k + 1)180^\circ}{q}, \quad k = 0, \pm 1, \pm 2 \dots (q - 1)$$

EE-379 Root Locus Analysis

Analysis – Construction Rules

5. The asymptotes **intersection point** on the real axis is defined by.

$$\sigma_a = \frac{\sum \text{poles of } GH(s) - \sum \text{zeroes of } GH(s)}{q}$$

6. **Real axis** sections of the root loci exist only where there is an odd number of poles and zeroes to the **right**.

EE-379 Root Locus Analysis

Analysis – Construction Rules: Example

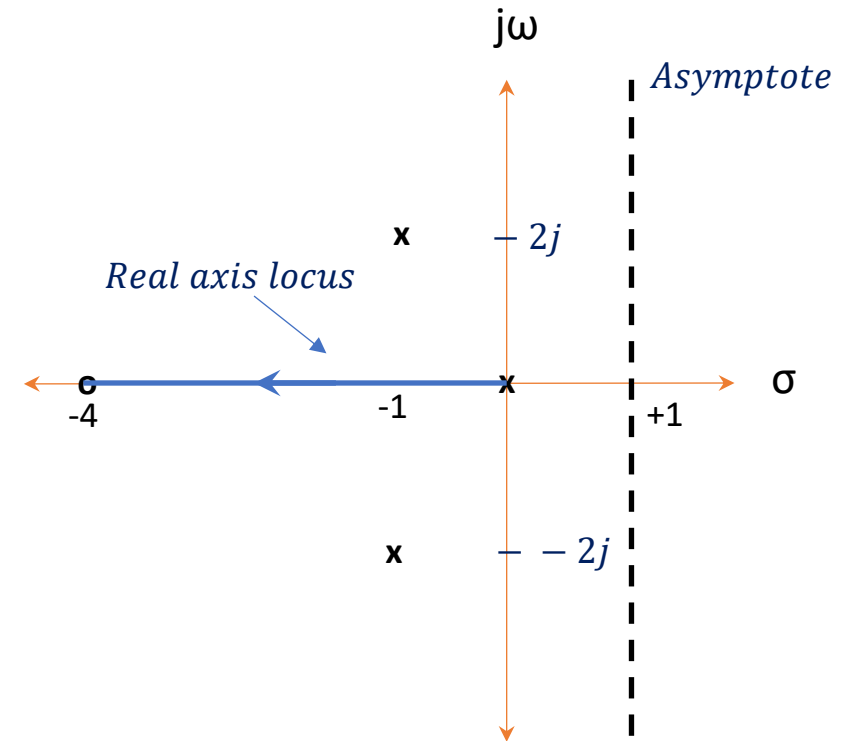
- Consider our example with $z_1 = 4$, $p_{12} = 1 \pm 2j$

$$GH(s) = \frac{K(s + 4)}{s(s + 1 + 2j)(s + 1 - 2j)}$$

- Asymptotes:**

$$Angle = \frac{(2k + 1)180^\circ}{3 - 1} = \pm 90^\circ$$

$$\sigma_a = \frac{[-0 - (1 + 2j) - (1 - 2j)] - [(-4)]}{3 - 1} = +1$$



EE-379 Root Locus Analysis

Analysis – Construction Rules

7. The angles of departure, q_d from poles and arrival, q_a to zeroes may be found by applying the angle condition to a point very near the pole or zero.

 - The angle of arrival at the zero, $-t_1$ is obtained from

$$\theta_{az1} + \sum_{i=2}^m \angle(-z_1 + z_i) - \sum_{i=1}^n \angle(-t_1 + p_i) = (2k + 1)180^\circ$$

This rule is applicable to “imaginary poles/zeros”

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Analysis – Construction Rules

- Departure angle from, p_2 .

$$q_{z1} = \tan^{-1}(2/3) = 33.7^\circ$$

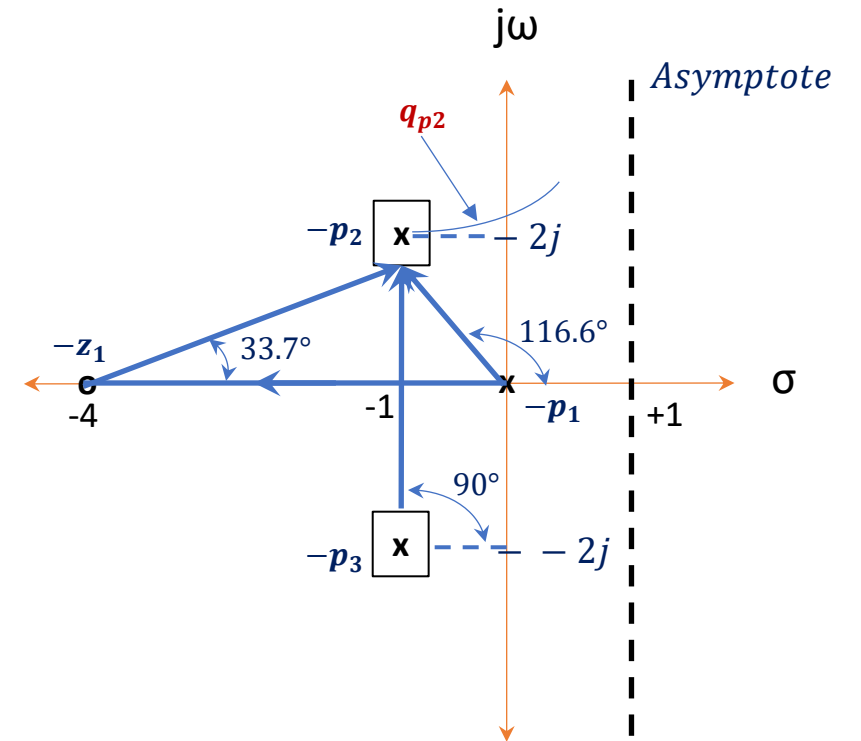
$$q_{p1} = \tan^{-1}(-2/1) = 116.6^\circ$$

$$q_{p3} = 90^\circ$$

- Then,

$$33.7^\circ - (90^\circ + 116.6^\circ + q_{p2}) = 180^\circ$$

$$q_{p2} = -352.9^\circ = +7.1^\circ$$



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Analysis – Construction Rules

8. The **imaginary axis** crossing is obtained by applying the **Routh-Hurwitz criterion** and checking for the gain that results in **marginal stability**. The imaginary components are found from the solution of the resulting **auxiliary equation**
- **Marginal stability** refers to the point where the roots of the closed-loop system are on the **stability boundary, i.e. the imaginary axis**.

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Analysis – Construction Rules

Imaginary axis crossing:

Characteristic equation

$$s(s + 1 + 2j)(s + 1 - 2j) + K(s + 4) = 0$$

$$s^3 + 2s^2 + (5 + K)s + 4K = 0$$

s^3	1	$5 + K$	0
s^2	2	$4K$	0
s^1	$5 - K$	0	
s^0	$4K$		

- For marginal stability, $K = 5$ and the auxiliary equation is.

$$2s^2 + 20 = 0$$

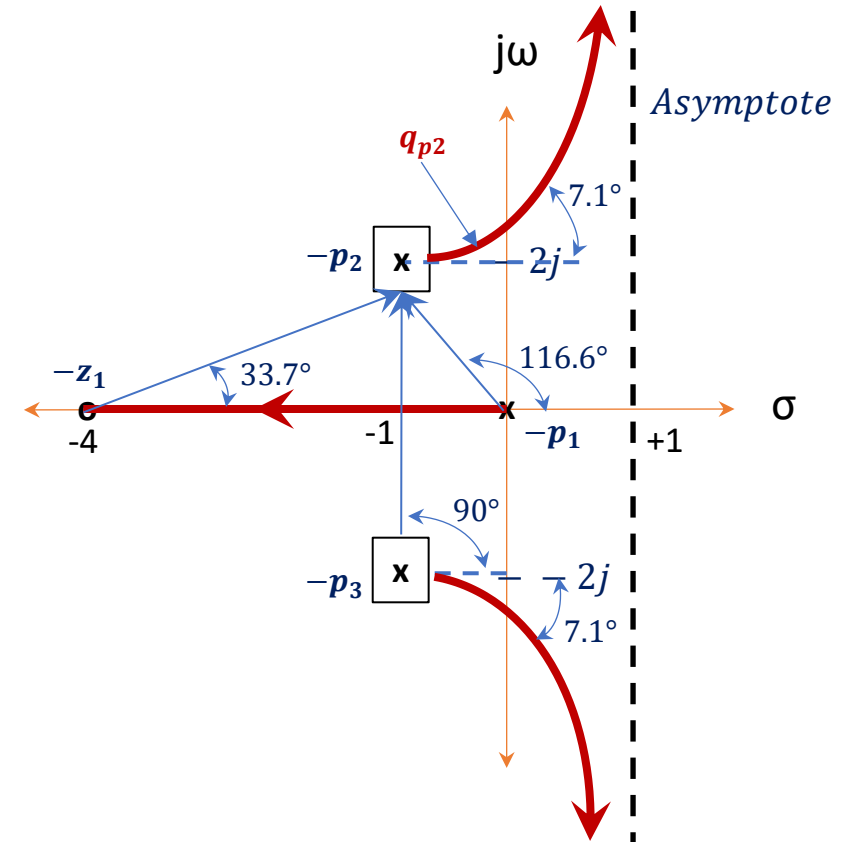
$$s = \pm\sqrt{10}j = \pm 3.16j$$

- Therefore, the imaginary axis intersection is $\pm 3.16j$.

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Analysis – Construction Rules

- **Summary:**
 1. There are three root loci.
 - I. One on the real axis from $-p_1$ to $-z_1$
 - II. A pair of loci from $-p_2$ and $-p_3$ to zeroes at infinity along the asymptotes.
 2. The departure angle from these poles is $\pm 7.1^\circ$ and an imaginary axis crossing at $s = \pm 3.16j$

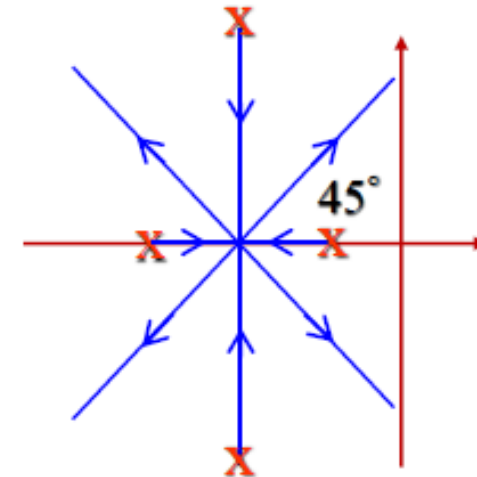
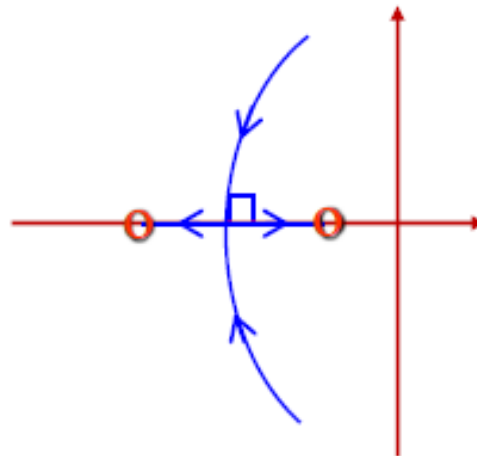
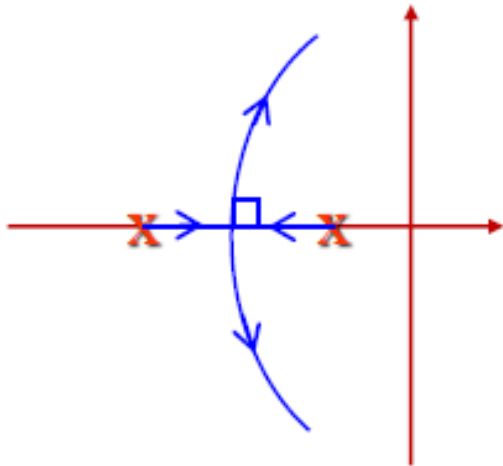
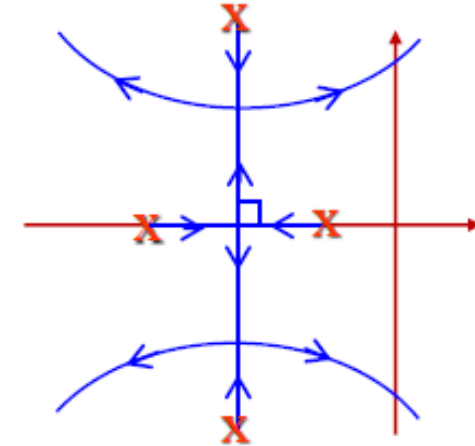


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Analysis – Construction Rules

- **Breakaway Points**

When two or more loci meet, they will break away from this point at particular angles. The point is known as a **breakaway point**. It corresponds to multiple roots.



EE-379 Root Locus Analysis

Analysis – Construction Rules

9. The **angle** of breakaway is $180^\circ/k$ where k is the number of closed-loop poles, departing from the breakaway point on the real axis.

✓ The **location** of the breakaway point is found from.

$$\frac{dK}{ds} = 0 \quad \text{or} \quad \frac{d[GH(s)]}{ds} = 0$$

• Note:

$$K = -[GH(s)]^{-1}$$
$$\frac{dK}{ds} = [GH(s)]^{-2} \frac{d[GH(s)]}{ds} = 0$$

• Also:

$$\frac{d[GH(s)]}{ds} = \frac{d[N(s)/D(s)]}{ds}$$

$$\frac{N'(s)}{D(s)} - \frac{N(s)D'(s)}{D(s)^2} = 0$$

$$D(s)N'(s) - N(s)D'(s) = 0$$

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Analysis – Construction Rules

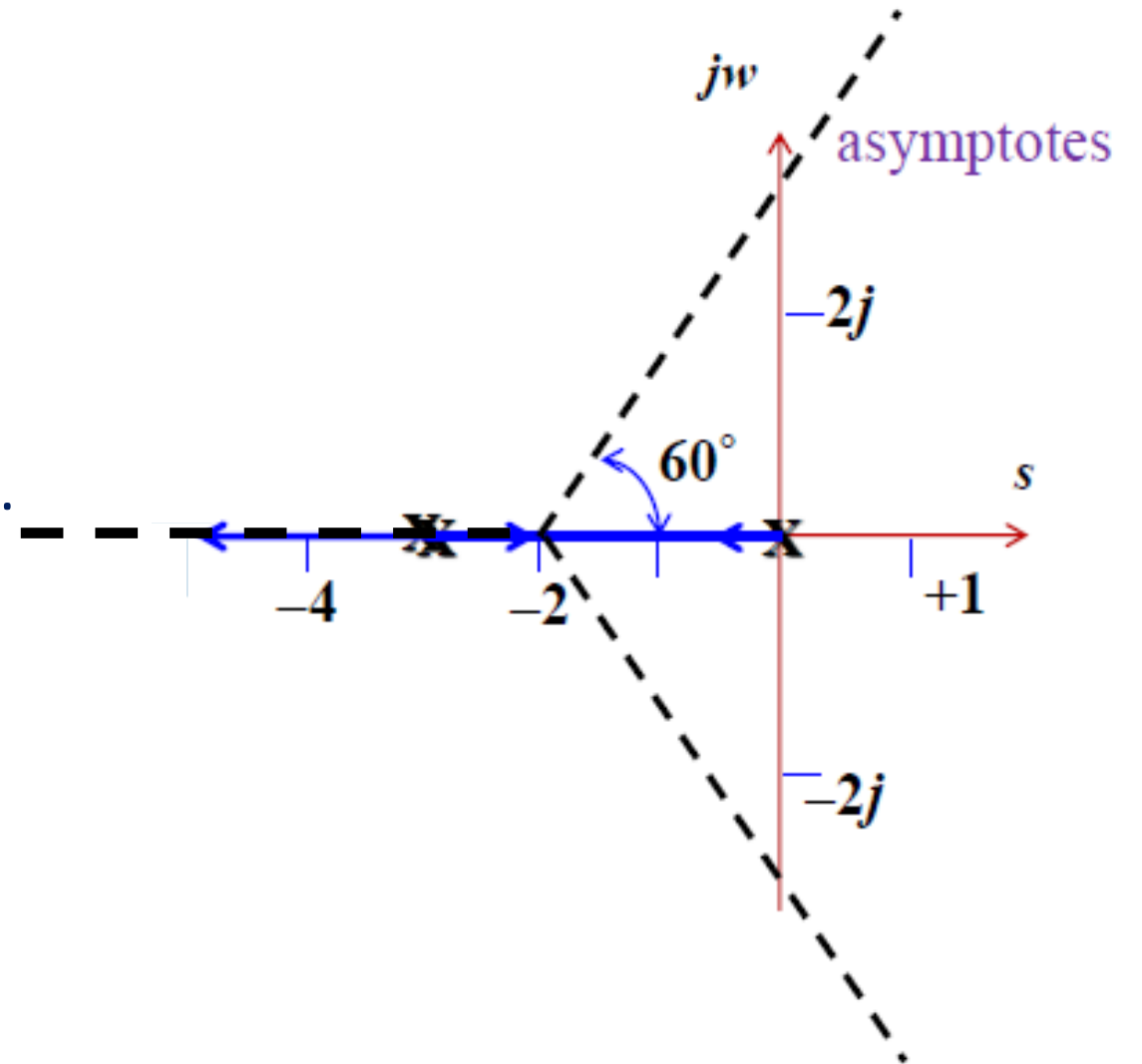
- Consider the following loop

$$GH(s) = \frac{K}{s(s+3)^2}$$

- Real axis loci exist for the full negative axis. the following loop.
- Asymptotes:**

$$= \frac{(2k+1)180^\circ}{3} = 60^\circ, 180^\circ, 300^\circ$$

$$\sigma_a = \frac{(-3 - 3 - 0) - (0)}{3} = -2$$



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Analysis – Construction Rules

- Determine the breakaway points from

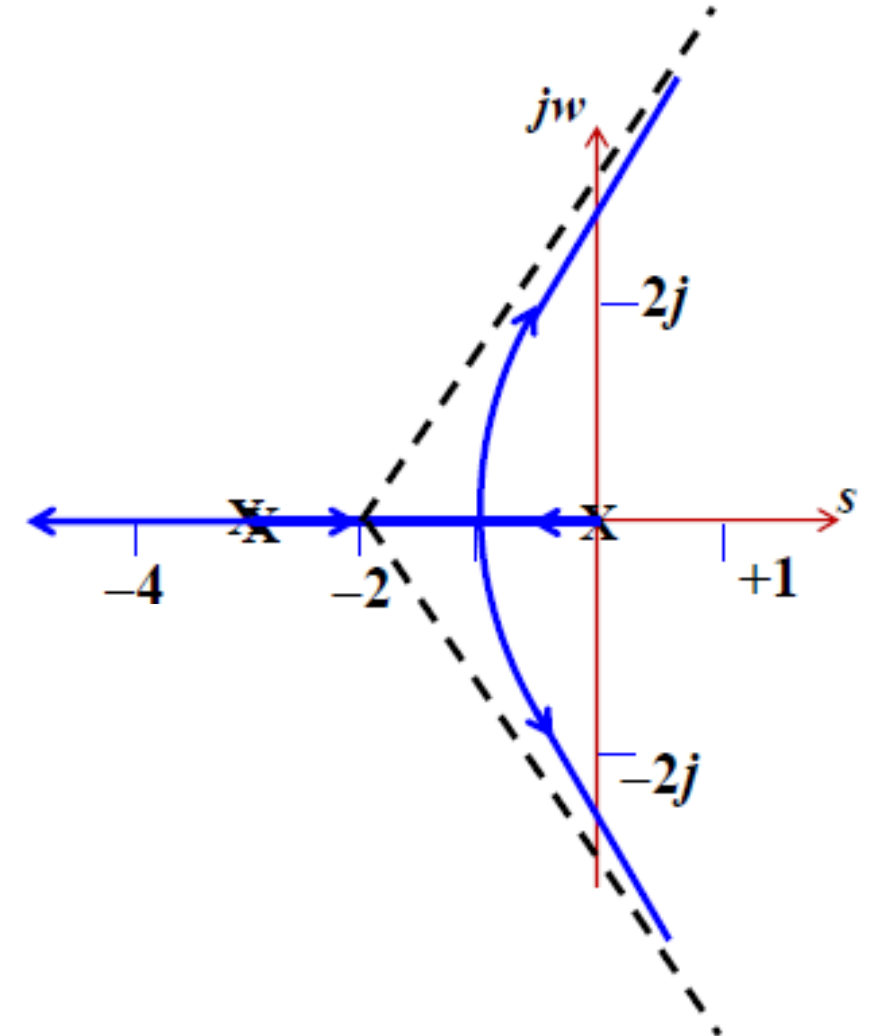
$$\frac{d}{ds} \left[\frac{K}{s(s+3)^2} \right] = \frac{d}{ds} \left[\frac{K}{s^3 + 6s^2 + 9s} \right]$$

$$= -\frac{K(3s^2 + 12s + 9)}{(s^3 + 6s^2 + 9s)^2} = 0$$

- Then,

$$(s^2 + 4s + 3) = (s + 1)(s + 3) = 0$$

$$s = -1, \quad s = -3$$



EE-379 Root Locus Analysis

Analysis – Construction Rules: Summary

1. Plot the poles and zeros of the open-loop system.
2. Find the section of the loci on the real axis (odd number of poles and zeroes to the right).
3. Determine the **asymptote angles** and **intercepts**.

$$angles = \frac{(2k + 1)180^\circ}{q} \quad \text{where, } q = n - m, \text{ and, } k = 0, \pm 1, \pm 2, \pm 3 \dots (q - 1)$$

$$\sigma_a = \frac{\sum Poles - \sum zeros}{q}$$

EE-379 Root Locus Analysis

Analysis – Construction Rules: Summary $180^\circ - \sum \angle$

4. Determine departure angles in case of **imaginary poles**. For a pole $-p_1$.

$$\angle(-p_1 + z_1) + \angle(-p_1 + z_2) + \cdots - \theta_{p1} - \angle(-p_1 + p_2) - \angle(-p_1 + p_3) = (2k + 1)180^\circ$$

5. Check for imaginary axis crossings using the Routh-Hurwitz criterion.

6. Determine breakaway points in case of loci originates from two poles together.

$$\text{angle} = \frac{180^\circ}{k}, \quad \text{where } k = \text{No of converging loci}$$

$$\text{location from} \quad \frac{d[GH(s)]}{ds} = 0$$

7. Complete the plot

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Analysis – Construction Rules: Example

- Loop Transfer function:

$$GH(s) = \frac{K}{s(s+4)(s^2+4s+20)}$$

- Roots:

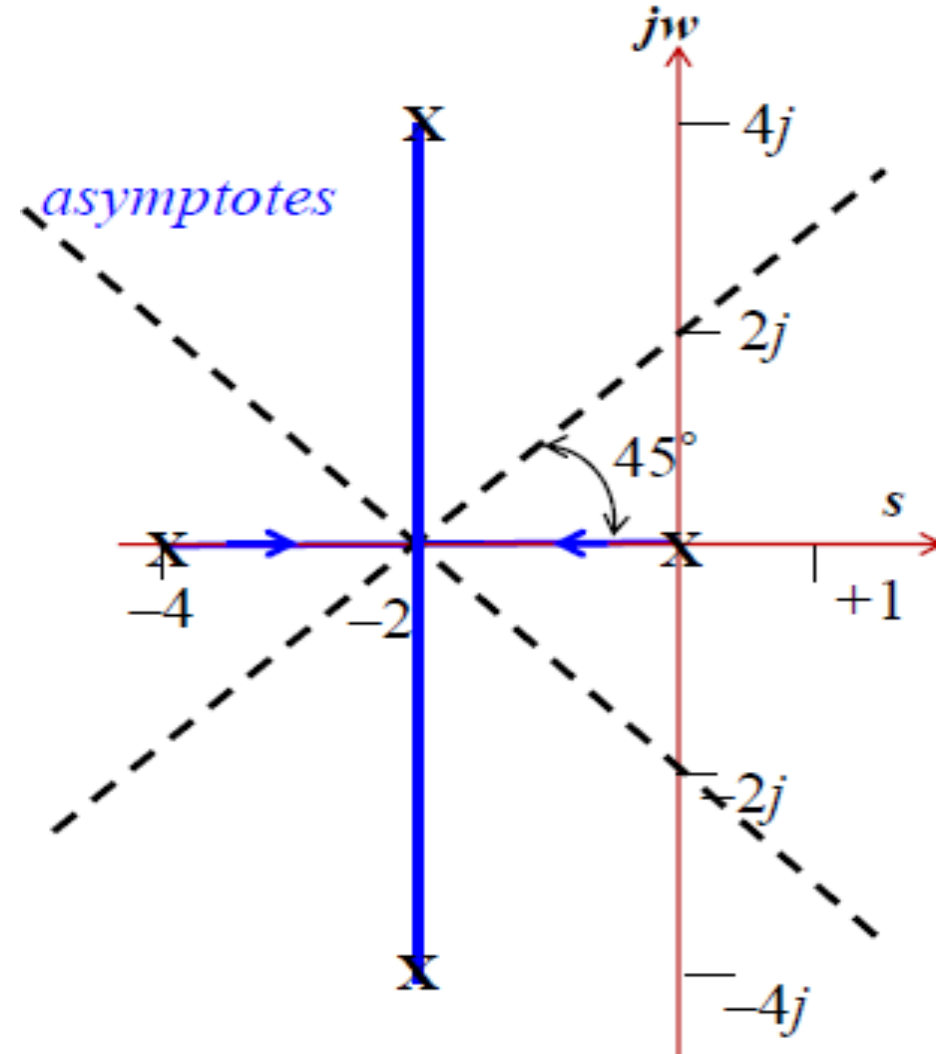
$$s = 0, s = -4, s = -2 \pm 4j$$

- Real axis segments between **0** and **-4**

- Asymptotes:

$$angles = \frac{(2k+1)180^\circ}{4-0} = 45^\circ, 135^\circ, 225^\circ, 315^\circ$$

$$\sigma_a = \frac{(-4 - 2 - 2 - 0)}{4} = -2$$



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Analysis – Construction Rules: Example

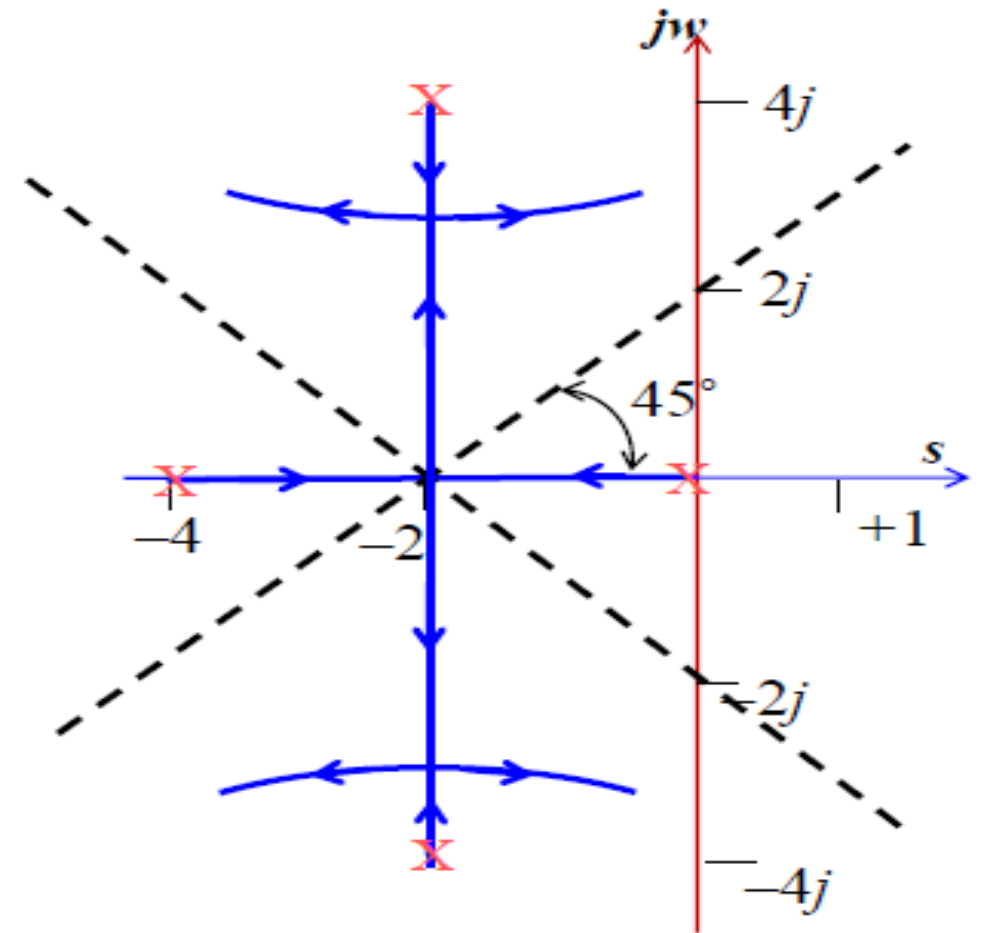
- Breakaway points:

$$\frac{d}{ds} \left[\frac{K}{s^4 + 8s^3 + 36s^2 + 80s} \right] \quad \text{let } P$$
$$= \frac{K(4s^3 + 24s^2 + 72s + 80)}{s^4 + 8s^3 + 36s^2 + 80s} = 0$$

$$\text{or } s^3 + 6s^2 + 18s + 20 = 0$$

$$\text{solving } s_b = -2, -2 \pm 2.45j$$

Three points that breakaway at 90°



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Analysis – Construction Rules: Example

- The imaginary axis crossings:

Characteristics equation

$$s^4 + 8s^3 + 36s^2 + 80s + K = 0$$

$$P+K=0$$

$$1+GH=0 \Rightarrow$$

$$1 + \frac{K}{P} \Rightarrow \frac{P+K}{P} = 0$$

- Condition for critical stability

$$80 - \frac{8K}{26} > 0$$

$$K < 260$$

- The auxiliary equation

$$26s^2 + 260 = 0$$

Solving,

$$s = \pm\sqrt{10}j = \pm3.16j$$

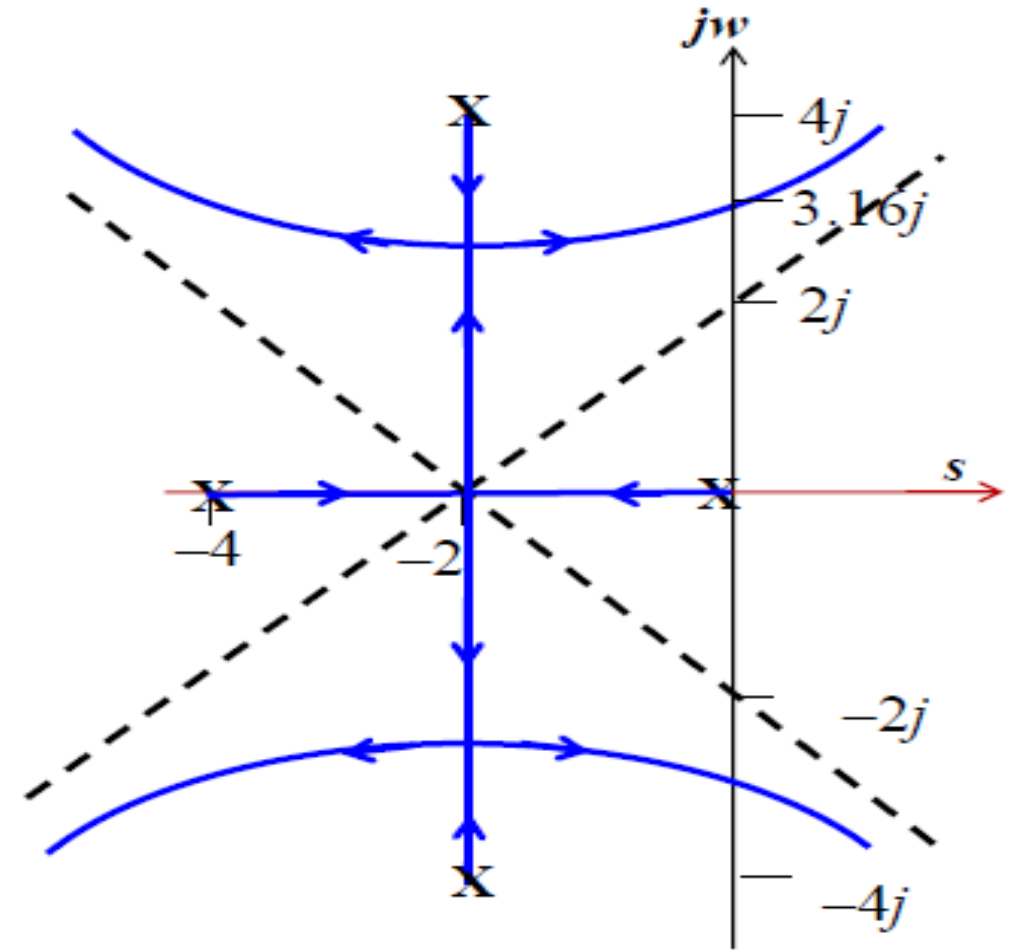
s^4	1	36	K
s^3	8	80	0
s^2	26	K	0
s^1	$80 - \frac{8K}{26}$	0	
s^0	K		

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Analysis – Construction Rules: Example

- The final plot is shown on the right.
- What is the value of the gain K corresponding to the breakaway point at.

$$s_b = -2 \pm 2.45j ?$$



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Analysis – Construction Rules: Example

- From the general magnitude condition the gain corresponding to the point s_1 on the loci is.

$$K = \frac{\prod_{i=1}^n (s + p_i)}{\prod_{i=1}^m (s + z_i)}$$

- For the point $s_1 = -2 \pm 2.45j$

$$K = \frac{|-2 + 2.45j + 4| |-2 - 2.45j + 4| |-2 + 2.45j + 2 + 4j| |-2 - 2.45j + 2 - 4j|}{1}$$

$$K = 3.163 \cdot 3.163 \cdot 6.45 \cdot 1.55 = 100$$

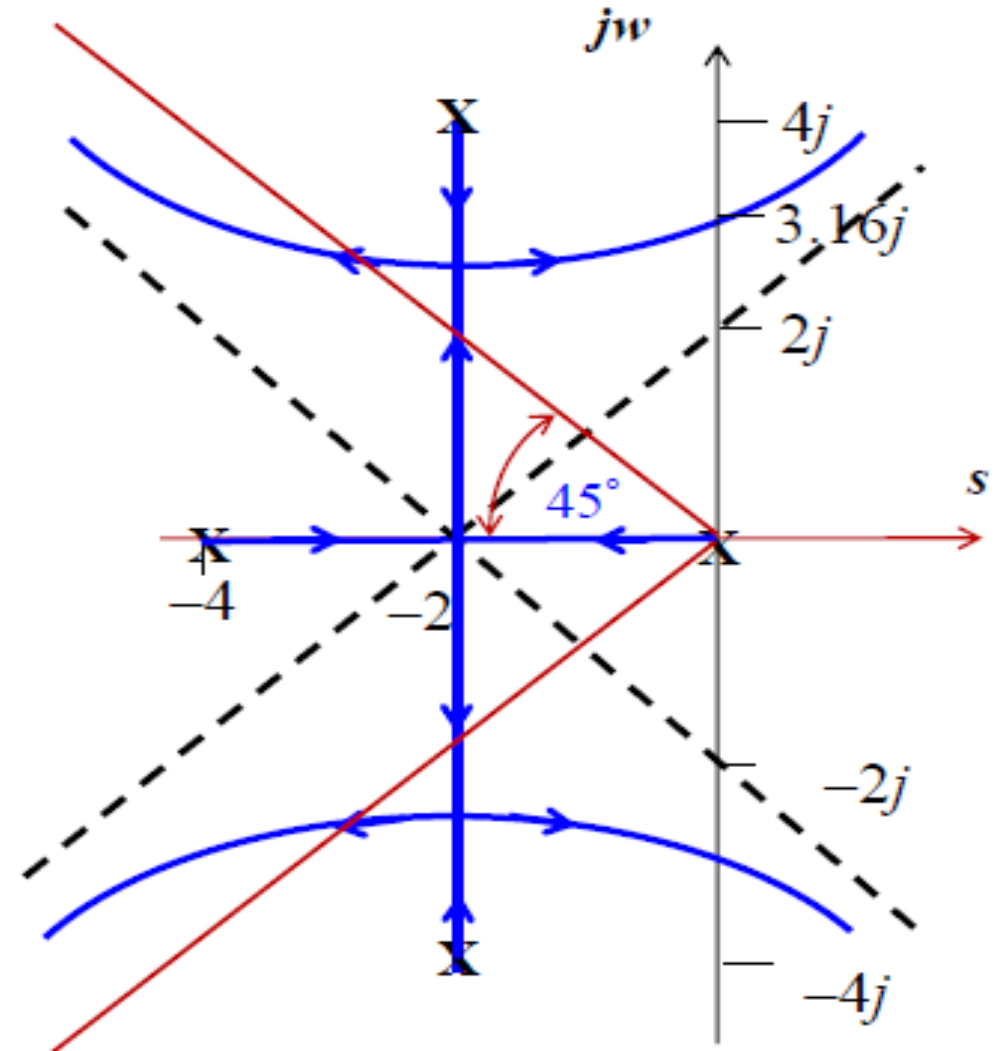
EE-379 Root Locus Analysis

Analysis – Construction Rules: Example

- Is there a gain corresponding to a damping ratio of **0.707** or more for all system modes?

$$\zeta = 0.707 = \cos(q)$$

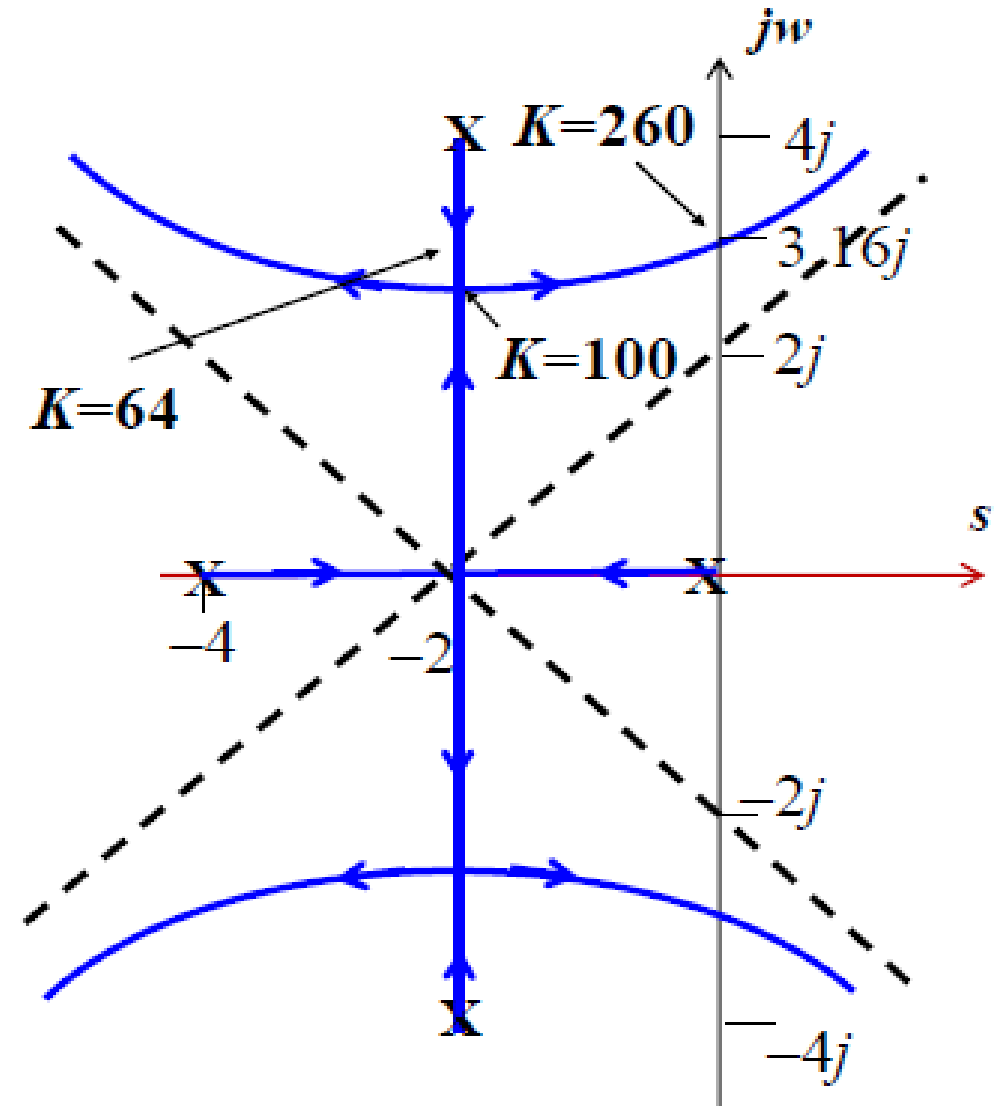
$$q = 45^\circ$$



EE-379 Root Locus Analysis

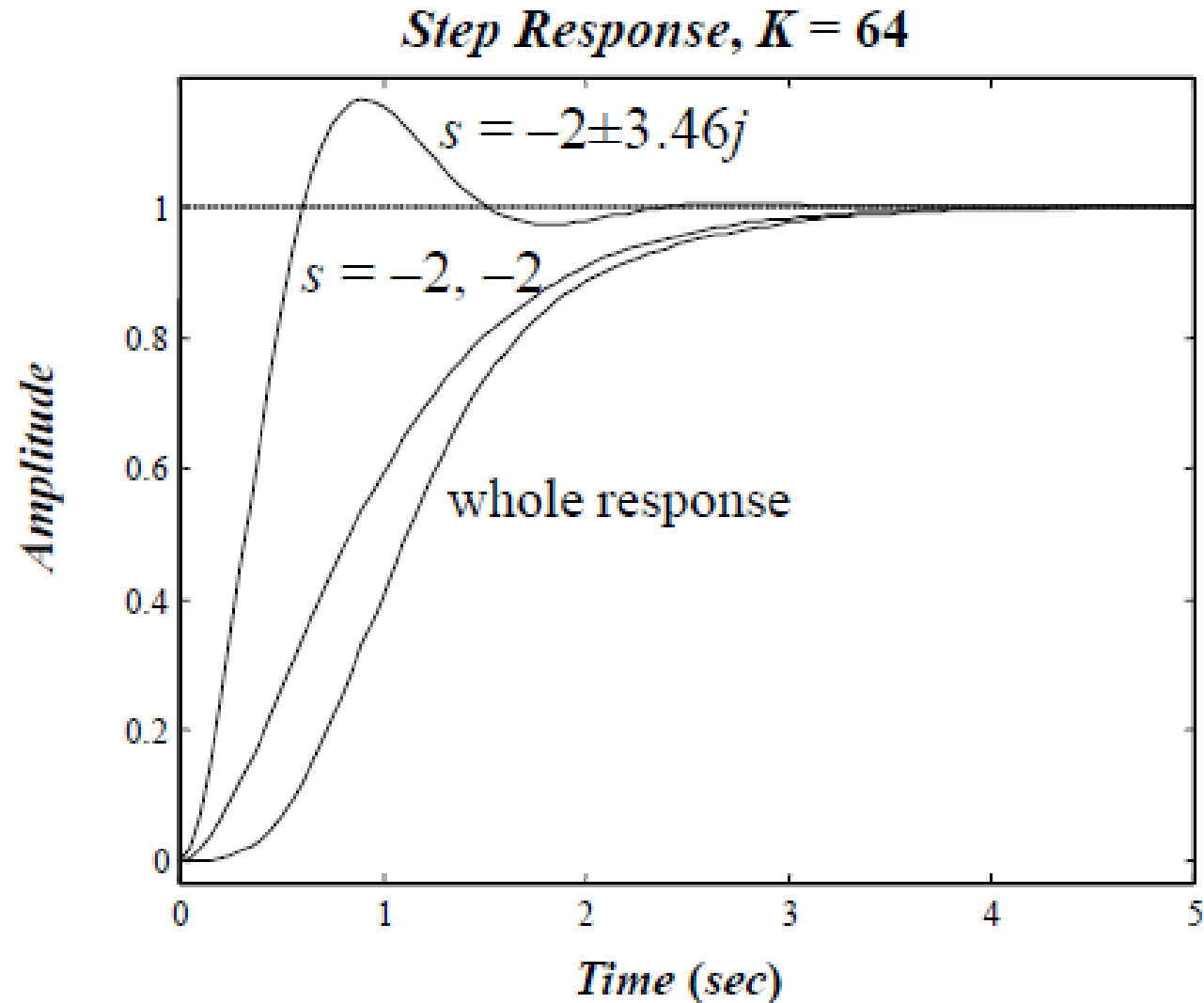
Analysis – Construction Rules: Example

- Examine the responses for the various gains and relate them to the location of the closed loop roots.
- $K = 64$, roots are $-2, -2, -2 \pm 3.46j$
- $K = 100$, roots are $-2 + 2.45j, -2 - 2.45j$
- $K = 260$, roots are $\pm 3.16j, -4 \pm 3.16j$



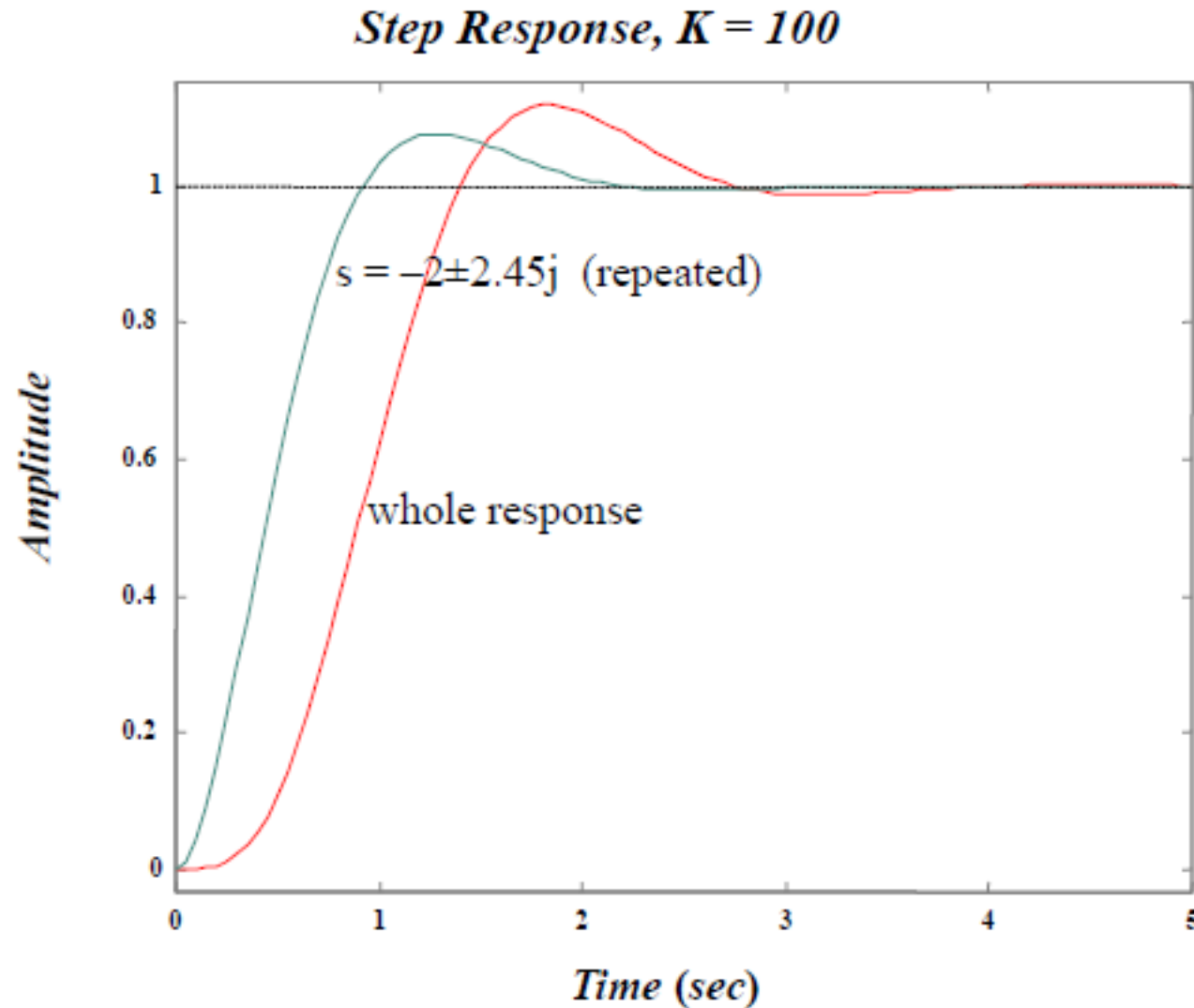
EE-379 Root Locus Analysis

Analysis – Construction Rules: Example



EE-379 Root Locus Analysis

Analysis – Construction Rules: Example



EE-379 Root Locus Analysis

Analysis – Construction Rules: Example

