EE-379 Linear Control Systems

Week No. 4 & 5: Performance Specifications

- Steady State Error
- System Sensitivity
- Sensitivity Functions
- Effect of Disturbances
- Effect of Measurement Noise
- Ziegler Nichols Compensation

EE-379 Performance Specification

Tracking Systems

- Two aspects of performance are often considered when a control system is designed
  - Transient Performance
  - Steady state performance

- Previously (Ch 1 and 2) we have concentrated on defining the differential equations, transfer function, and stability **in terms of the natural response**.

- Now we will analyze the tendency of the system to follow a desired command. Emphasis will shift to the steady state performance of a closed-loop system.

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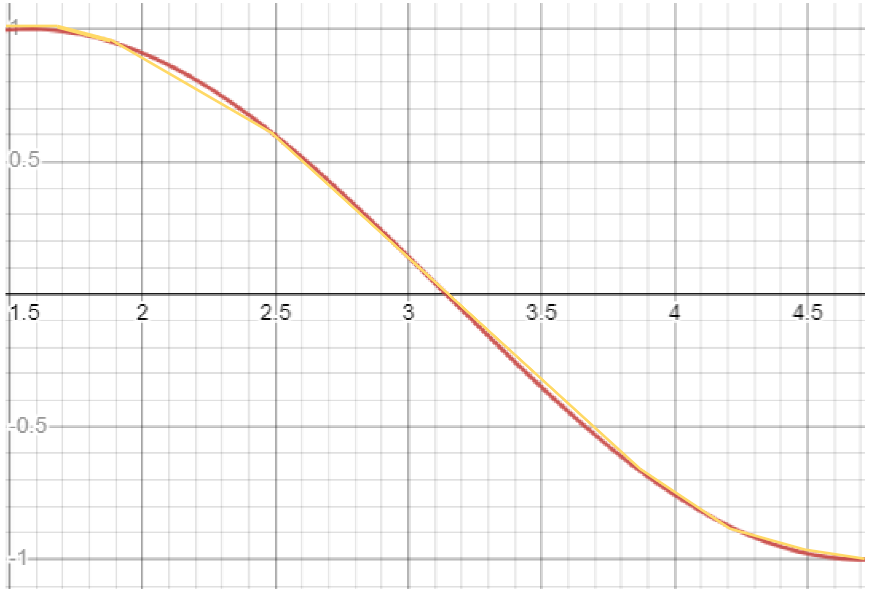
Tracking Systems - Analysis

- Why tracking systems?
- In general, the input *r(t)* can be written as a power series in terms of powers of *t*.

$$r(t) = r(a) + \frac{dr}{dt}_{t=0}(t-a) + \frac{\frac{d^2r}{dt^2}_{t=a}}{2!}(t-a)^2 + \frac{\frac{d^3r}{dt^3}_{t=a}}{3!}(t-a)^3 + \cdots$$

$$r(t) = A_o + A_1 t + A_2 t^2 + A_3 t^3 + \cdots$$



- In this chapter we will examine how a control system responds to commands (particularly those that are powers of ***t***).

- More complicated commands are expressible in powers of ***t***

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Tracking Systems – Analysis

- A tracking system creates an output that tracks (follows) an input with some tolerance
- The **elevation control system** for a **shipboard satellite dish antenna** may have a transfer function as;

$$T(s) = \frac{50}{s^2 + 4s + 50}$$

- **Unit step response** of this function is:

$$Y(s) = T(s)\left(\frac{1}{s}\right) = \frac{50}{s(s^2 + 4s + 50)}$$

$$= \underbrace{\frac{1}{s}}_{\text{forced}} + \underbrace{\frac{-s-4}{s^2 + 4s + 50}}_{\text{natural}}$$

- This is a function of time after *t = 0.*

$$y(t) = 1 + 1.04e^{-2t}\cos(6.78t + 163.6°)$$

- Good tracking system has a rapidly decreasing **natural response** (depending on the initial conditions).
- **Forced response** component should then be able to accurately track reference inputs

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Tracking Systems – Analysis

- **Analysis and design of tracking systems** can be separated into following major parts
  - Locate **characteristics roots (poles)** of the transfer function. This determines the **natural response** which should decay quickly and have well-damped oscillatory terms.
  - Tracking the reference input by the **forced response** of the system.
  - What happens to performance **if the model is inaccurate**?
  - Tracking system response due to **unwanted, inaccessible disturbance inputs**

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Relative Stability

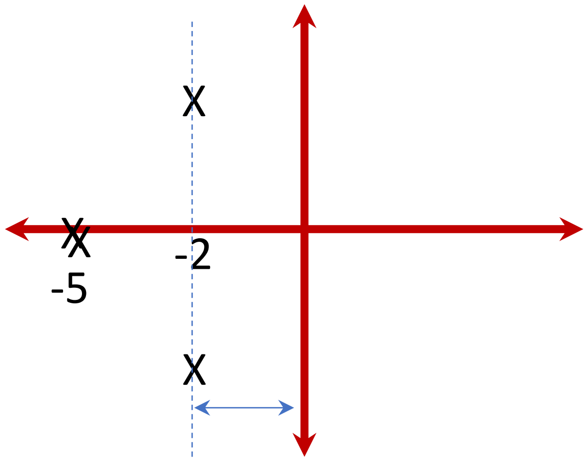
- Relative stability is the **distance** into the **left half** of the complex plane from the **imaginary axis** to the **nearest pole**.

- Pair of complex conjugate roots ***s1, s2*** $= -\sigma \pm jw$ gives rise to a **damped oscillatory** term in the **natural response**.

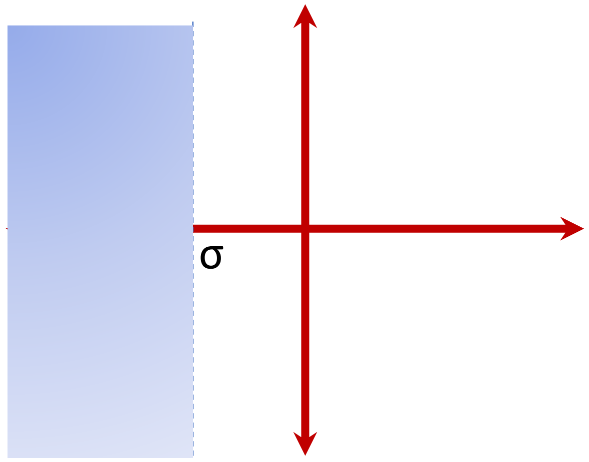
$$y_i(t) = Ae^{-at}\cos(\omega t + \theta)$$

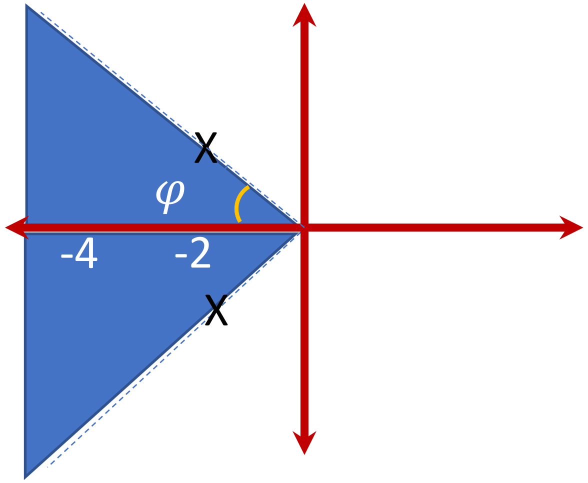
where A and θ depends on initial conditions.

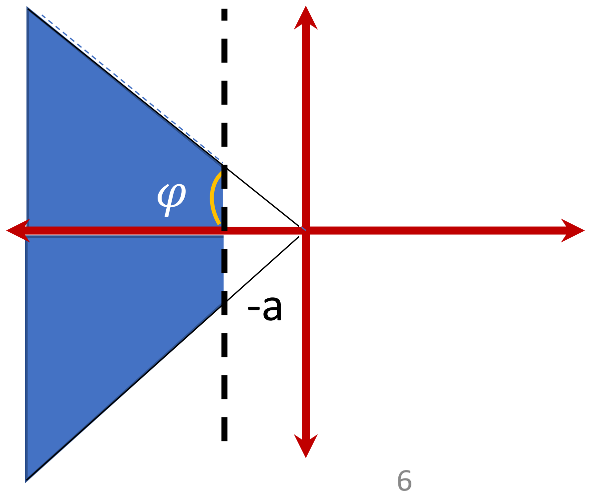
- The damping ratio of this term is $\zeta = cos(\varphi)$ where φ is the damping angle.



Relative stability = 2 units







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Steady State Error – Initial/Final Value

- The **initial value** of a function of time *y(t)* is related to the function's Laplace transform by:

$$y(0) = \lim_{s\to\infty}[sY(s)]$$

- For example:

$$Y(s) = \frac{-4s^4 + 3s^3 + s^2 - s + 1}{3s^5 - 2s^4 + s^3 - s + 10}$$

$$y(0) = \lim_{s\to\infty}[sY(s)] = -\frac{4}{3}$$

The initial value theorem is often used to find **the initial conditions of a system**

- The **final value** of a function of time y(t) is related to the function's Laplace transform by:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0}[sY(s)]$$

- For example:

$$Y(s) = \frac{-4s^3 - s^2 + 7s + 3}{s^3 + 9s^2 + 2s}$$

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0}[sY(s)] = \frac{3}{2}$$

The final value theorem is commonly used to determine the **steady-state behavior** of a system in response to a **specific input**

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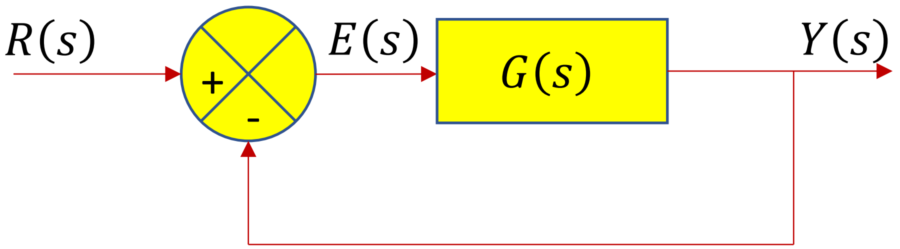
Steady State Error - Definition

- The steady-state error is defined as the **difference between the input and the output** for a prescribed test input as $t \rightarrow infinity$



$$E(s) = input \ - output$$

$$= R(s) \ - Y(s)$$

- **Steady-state error analysis only applicable to stable systems**, as the unstable systems represent the loss of control in steady state

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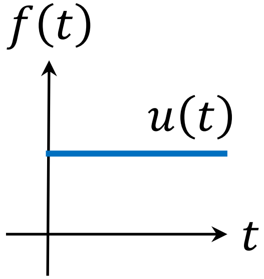
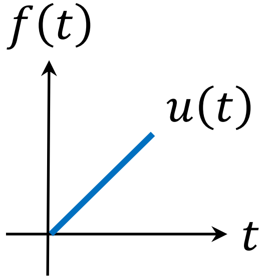
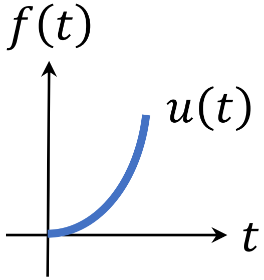
Steady State Error – Test Inputs

- Some **common test inputs** used for **steady-state error analysis** and design are:

Input	Waveform	Physical Interpretation	Time Function	Laplace transform
Step		*Constant position*	$\mathbf{1}$	$\dfrac{1}{s}$
Ramp		*Constant velocity*	$\boldsymbol{t}$	$\dfrac{1}{s^2}$
Parabola		*Constant velocity*	$\dfrac{1}{2}t^2$	$\dfrac{1}{s^3}$

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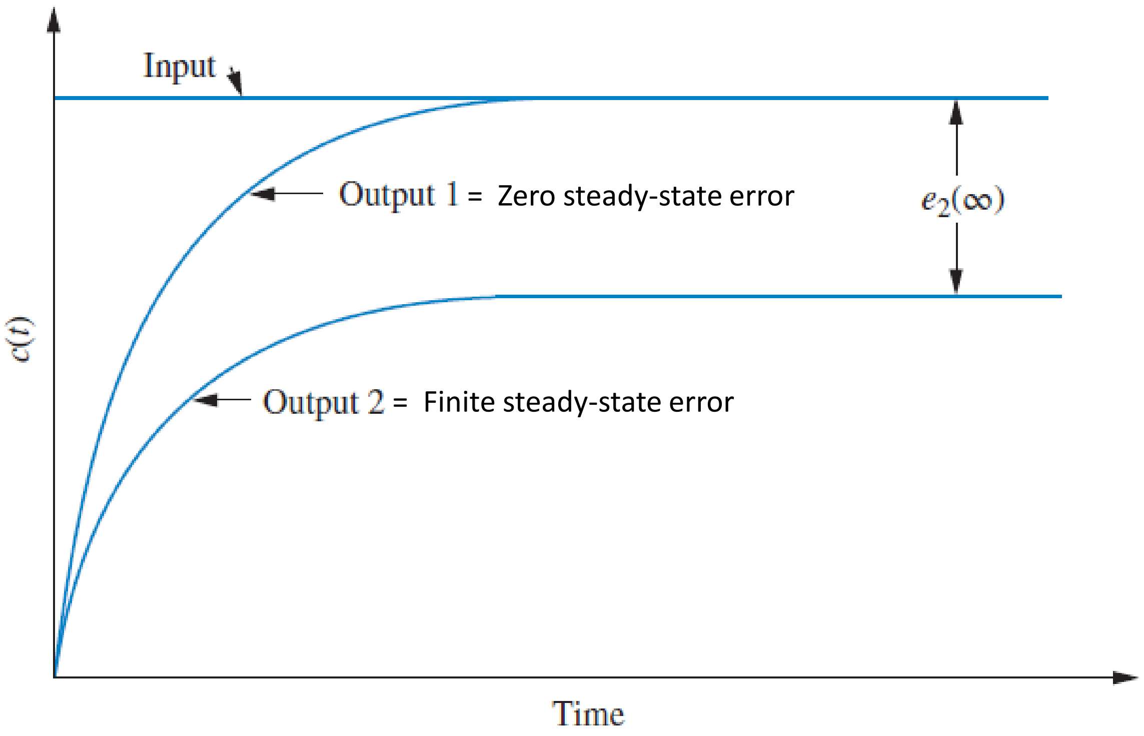
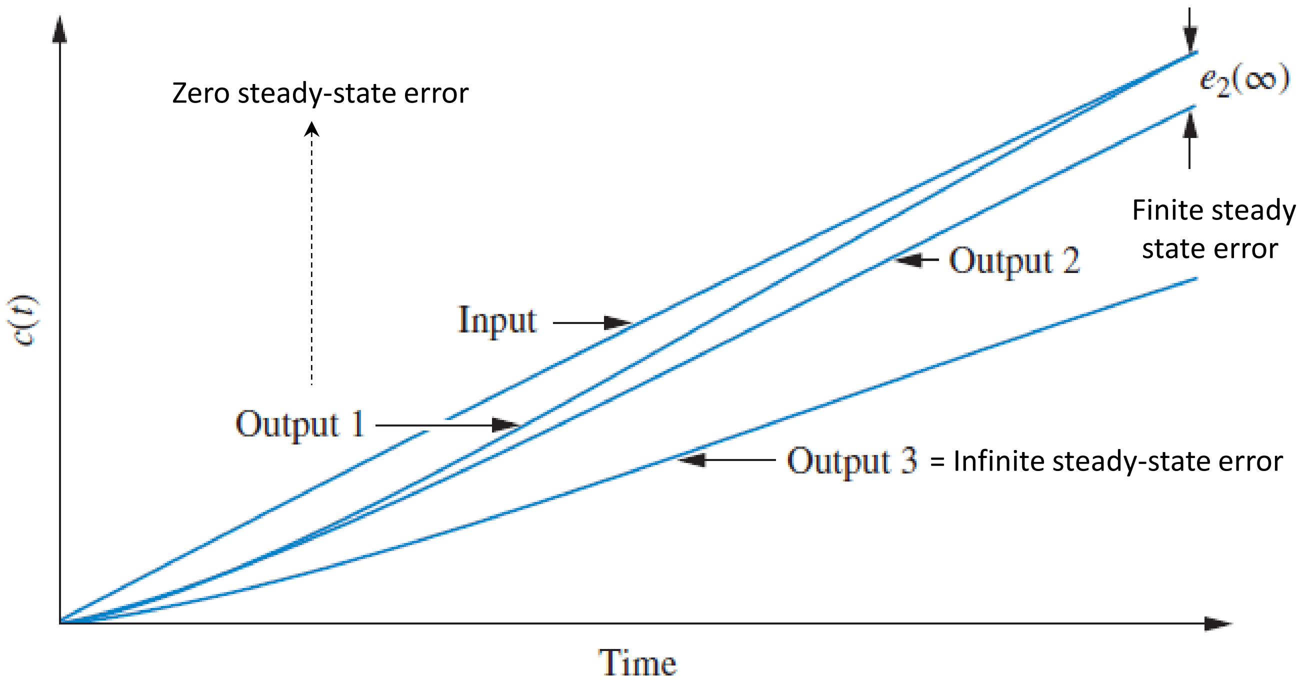
Steady State Error – Graphical Representation





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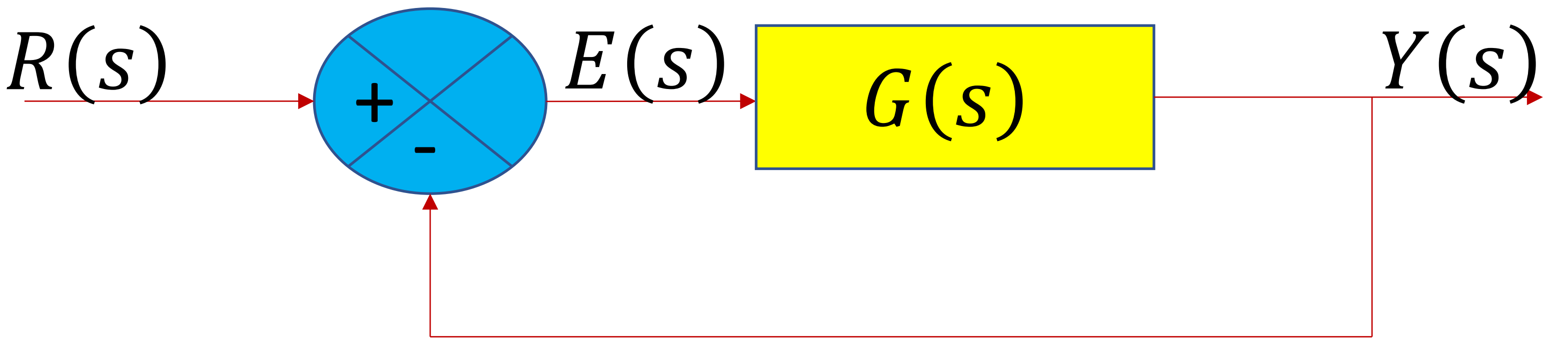
Steady State Error – Mathematical Expression

- The steady-state error is the difference between the input and output, assume a closed loop transfer function, ***T(s)***. The general representation of **steady-state error for a unity feedback system** is:



$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

$$\begin{aligned} E(s) &= input \ - output \\ &= R(s) \ - Y(s) \end{aligned}$$

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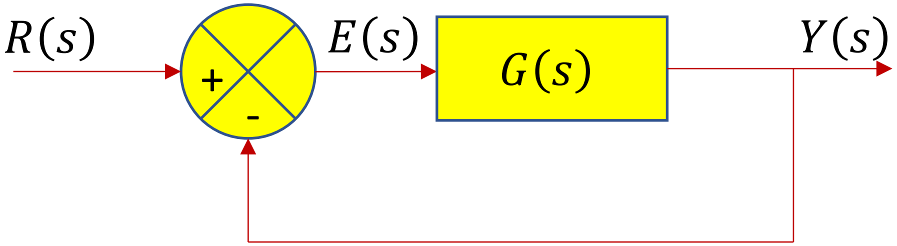
Steady State Error – Mathematical Expression

- The steady-state error for a **unity feedback system**:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

$$E(s) = input - output$$

$$= R(s) - Y(s)$$



$$E(s) = R(s) - \frac{G(s)}{1 + G(s)} \cdot R(s)$$

$$= R(s)\left[1 - \frac{G(s)}{1 + G(s)}\right]$$

$$= R(s)\left[\frac{1 + G(s) - G(s)}{1 + G(s)}\right]$$

$$\boxed{E(s) = \left[\frac{1}{1 + G(s)}\right] R(s)}$$

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Steady State Error – Mathematical Expression

- The **steady-state error for a unity** feedback system using **final value theorem:**

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)}$$

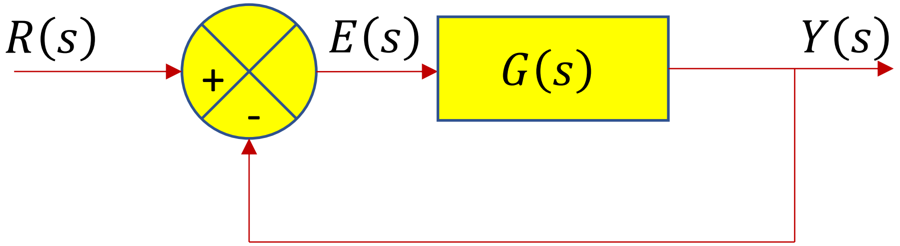
Need to analyze the steady state error for different R(s)and G(s)

$$E(s) = R(s) - \frac{G(s)}{1 + G(s)} \cdot R(s)$$

$$= R(s) \left[1 - \frac{G(s)}{1 + G(s)} \right]$$

$$= R(s) \left[\frac{1 + G(s) - G(s)}{1 + G(s)} \right]$$

$$\boxed{E(s) = \left[\frac{1}{1 + G(s)} \right] R(s)}$$



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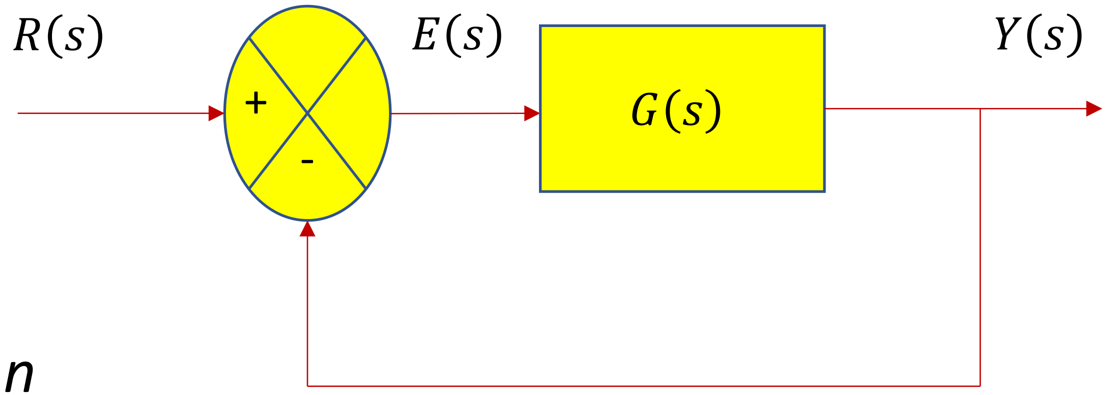
Steady State Error – System Types

- In order to **simplify** the **analysis of steady-state error**, systems can be classified by **system type.**

$$G(s) = \frac{K(s+z_1)(s+z_2)\ldots(s+z_m)}{s^N(s+p_1)(s+p_2)\ldots(s+p_n)}$$

$m \leq n$



- The system type can be determined by identifying the value for **N** at the denominator of the transfer function.

  - If **N = 0**, the system is of **Type 0**
  - If **N = 1**, the system is of **Type 1**
  - If **N = 2**, the system is of **Type 2**

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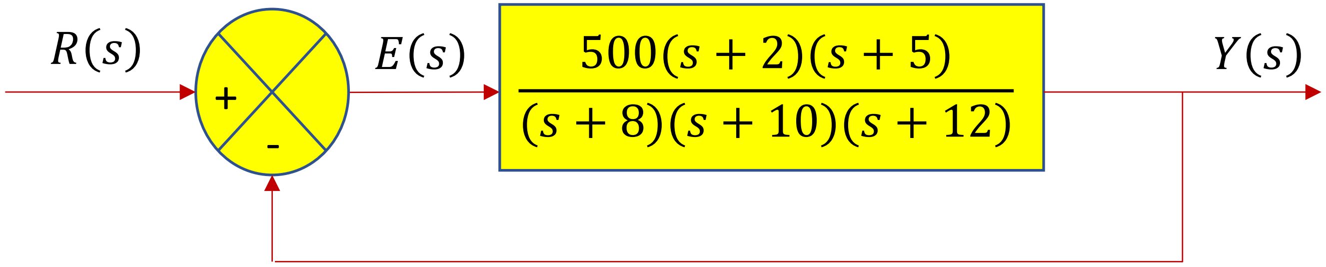
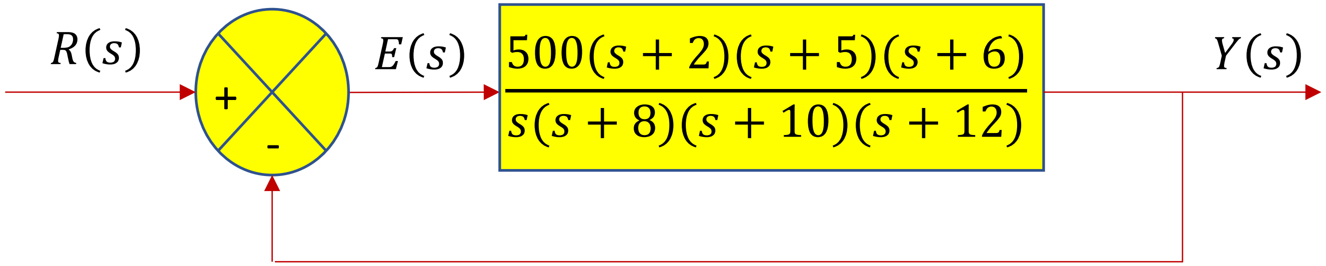
Steady State Error – System Types

Examples of System Type.

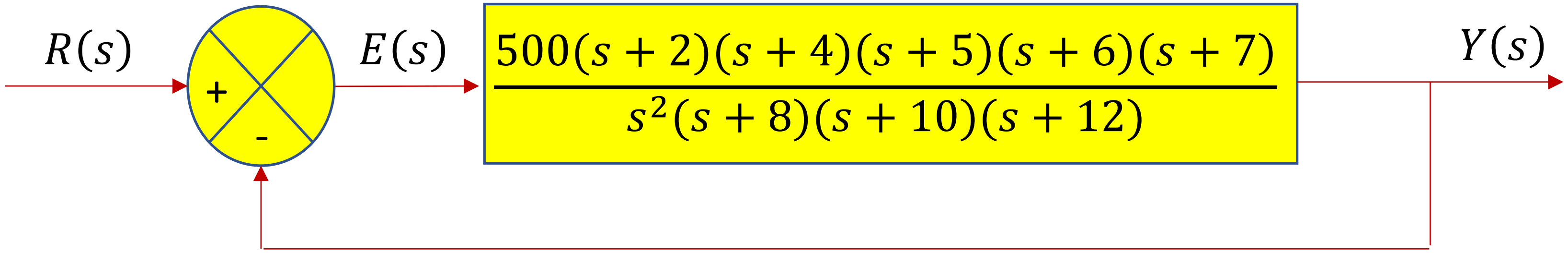
Type 0 →



Type 1 →



Type 2 →



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Steady State Error – Unit Step Input

$$\boxed{e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)}}$$

$$e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)} \cdot \frac{1}{s}$$

$$= \frac{1}{1 + \lim_{s \to 0} G(s)}$$

$$e_{ss} = \frac{1}{1 + K_p}$$

$where,$

$$K_p = \lim_{s \to 0} G(s)$$

K_p = position error constant

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Steady State Error – Unit Step Input

- **For system of Type 0.**

$$G(s) = \frac{K(s+z_1)(s+z_2)\dots(s+z_m)}{s^N(s+p_1)(s+p_2)\dots(s+p_n)}$$

$$for\ N = 0$$

$$K_p = \lim_{s\to 0} \frac{K(s+z_1)(s+z_2)\dots(s+z_m)}{s^N(s+p_1)(s+p_2)\dots(s+p_n)}$$

$$K_p = \frac{K z_1 . z_2 \dots z_m}{p_1 . p_2 \dots p_n}$$

$$\boxed{e_{ss} = \frac{1}{1+K_p}}$$

- **For system of Type 1 and above.**

$$G(s) = \frac{K(s+z_1)(s+z_2)\dots(s+z_m)}{s^N(s+p_1)(s+p_2)\dots(s+p_n)}$$

$$for\ N \geq 1$$

$$K_p = \lim_{s\to 0} \frac{K(s+z_1)(s+z_2)\dots(s+z_m)}{s^N(s+p_1)(s+p_2)\dots(s+p_n)}$$

$$K_p = \infty$$

$$\boxed{e_{ss} = \frac{1}{1+K_p} = 0}$$

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Steady State Error – Unit Step Input

Conclusion

Those systems from **Type 1 and above** will have **zero steady-state error for step input**.

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Steady State Error – Unit Ramp Input

$$e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)} \cdot \frac{1}{s^2}$$

$$= \lim_{s \to 0} \frac{1}{s + sG(s)}$$

$$= \frac{1}{\lim_{s \to 0} sG(s)}$$

$$e_{ss} = \frac{1}{K_v}$$

$where,$

$$K_v = \lim_{s \to 0} sG(s)$$

K_v=velocity error constant

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Steady State Error – Unit Ramp Input

- **For system of Type 0.**

$$G(s) = \frac{K(s+z_1)(s+z_2)\dots(s+z_m)}{s^N(s+p_1)(s+p_2)\dots(s+p_n)}$$

$$for\ N = 0$$

$$K_v = \lim_{s\to 0} s\,\frac{K(s+z_1)(s+z_2)\dots(s+z_m)}{s^N(s+p_1)(s+p_2)\dots(s+p_n)}$$

$$K_v = 0$$

$$\boxed{e_{ss} = \frac{1}{K_v} = \infty}$$

- **For system of Type 1.**

$$G(s) = \frac{K(s+z_1)(s+z_2)\dots(s+z_m)}{s^N(s+p_1)(s+p_2)\dots(s+p_n)}$$

$$for\ N = 1$$

$$K_v = \lim_{s\to 0} s\,\frac{K(s+z_1)(s+z_2)\dots(s+z_m)}{s^1(s+p_1)(s+p_2)\dots(s+p_n)}$$

$$K_v = \frac{K z_1 . z_2 \dots z_m}{p_1 . p_2 \dots p_n}$$

$$\boxed{e_{ss} = \frac{1}{K_v}}$$

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Steady State Error – Unit Ramp Input

- **For system of Type 2 and above.**

$$G(s) = \frac{K(s+z_1)(s+z_2)\ldots(s+z_m)}{s^N(s+p_1)(s+p_2)\ldots(s+p_n)}$$

$$for\ N \geq 2$$

$$K_v = \lim_{s\to 0} s\,\frac{K(s+z_1)(s+z_2)\ldots(s+z_m)}{s^2(s+p_1)(s+p_2)\ldots(s+p_n)}$$

$$K_v = \infty$$

$$\boxed{e_{ss} = \frac{1}{K_v} = 0}$$

Conclusion:

- Systems of **type 0** will have **infinity steady state error**.
- Systems of **type 1** will have **finite steady state error**.
- Those systems from **type 2** and above will have **zero steady state error** for **ramp input**

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Steady State Error – Unit Parabolic Input

$$e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)} \cdot \frac{1}{s^3}$$

$$= \lim_{s \to 0} \frac{1}{s^2 + s^2 G(s)}$$

$$= \frac{1}{\lim_{s \to 0} s^2 G(s)}$$

$$e_{ss} = \frac{1}{K_a}$$

$where,$

$$K_a = \lim_{s \to 0} s^2 G(s)$$

$K_a =$ acceleration error constant

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Steady State Error – Unit Parabolic Input

- **For system of Type 0 & 1.**

$$G(s) = \frac{K(s+z_1)(s+z_2)\dots(s+z_m)}{s^N(s+p_1)(s+p_2)\dots(s+p_n)}$$

$$for\ N \leq 1$$

$$K_a = \lim_{s\to 0} s^2 \frac{K(s+z_1)(s+z_2)\dots(s+z_m)}{s^N(s+p_1)(s+p_2)\dots(s+p_n)}$$

$$K_a = 0$$

$$\boxed{e_{ss} = \frac{1}{K_a} = \infty}$$

- **For system of Type 2.**

$$G(s) = \frac{K(s+z_1)(s+z_2)\dots(s+z_m)}{s^N(s+p_1)(s+p_2)\dots(s+p_n)}$$

$$for\ N = 1$$

$$K_a = \lim_{s\to 0} s^2 \frac{K(s+z_1)(s+z_2)\dots(s+z_m)}{s^2(s+p_1)(s+p_2)\dots(s+p_n)}$$

$$K_a = \frac{K z_1 . z_2 \dots z_m}{p_1 . p_2 \dots p_n}$$

$$\boxed{e_{ss} = \frac{1}{a}}$$

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Steady State Error – Unit Ramp Input

- **For system of Type 3 and above.**

$$G(s) = \frac{K(s+z_1)(s+z_2)\dots(s+z_m)}{s^N(s+p_1)(s+p_2)\dots(s+p_n)}$$

$$for\ N \geq 3$$

$$K_a = \lim_{s\to 0} s^2 \frac{K(s+z_1)(s+z_2)\dots(s+z_m)}{s^3(s+p_1)(s+p_2)\dots(s+p_n)}$$

$$K_a = \infty$$

$$\boxed{e_{ss} = \frac{1}{K_a} = 0}$$

Conclusion:

- Systems of **type 0** and **1** will have **infinity steady state error**
- Systems of **type 2** will have **finite steady state error**
- Those systems from **type 3 and above** will have **zero steady state error for parabolic input.**

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Steady State Error – Summary

- The steady-state error for a system with **unity feedback** can be summarized as given below:

Input	Steady State Error Formula	Type 0		Type 1		Type 2	
		Static Error Constant	Error	Static Error Constant	Error	Static Error Constant	Error
Step $u(t)$	$\frac{1}{1+K_p}$	$K_p = constant$	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v = constant$	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a = constant$	$\frac{1}{K_a}$

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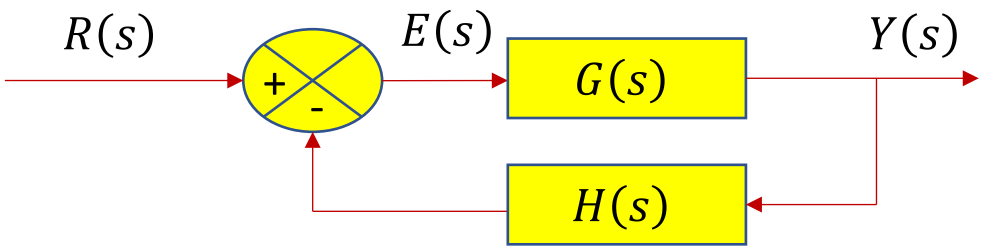
Steady State Error – Non-Unity Feedback Systems

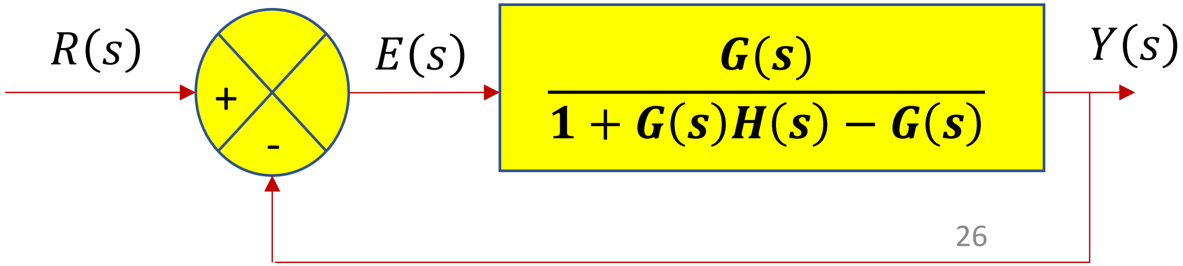
- The steady-state error with a non-unity feedback system can be determined in two ways:
  - ✓ By solving the problem using the fundamental definition of steady-state error
  - ✓ By changing the block diagram into the equivalent unity feedback system, and the respective formula to calculate the respective steady-state errors

$$E(s) = input \; - output$$

$$= R(s) \; - Y(s)$$

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$





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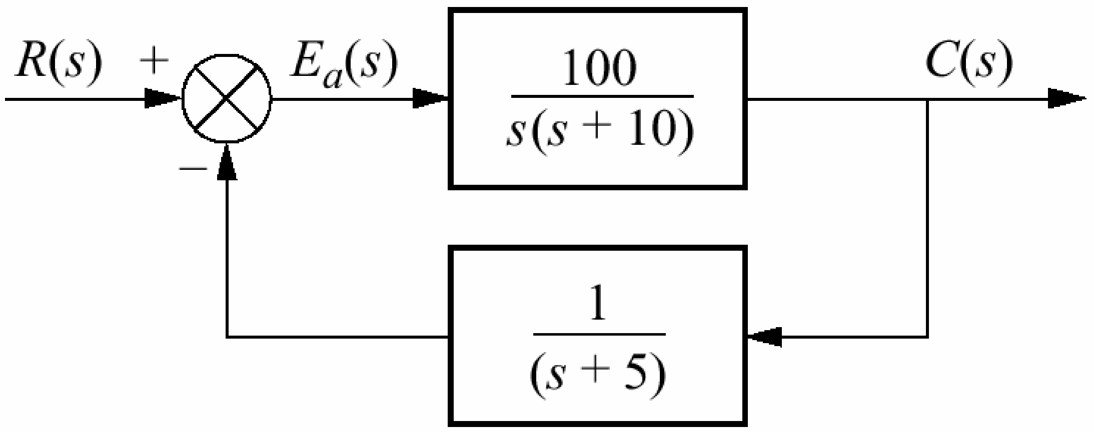
Steady State Error – Example

- For the system shown, the steady-state error for a **unit step input**. (**non-unity feedback**)



- Using the equivalent unity feedback system block diagram:

$$G_e(s) = \frac{G(s)}{1 + G(s)H(s) - G(s)}$$

$$= \frac{100(s+5)}{s^3 + s^2 - 50s - 400} \qquad (Type - 0\ system)$$

- The appropriate static error constant is then K_p , whose value is:

$$K_p = \lim_{s \to 0} G_e(s) = \frac{100(5)}{-400} = -\frac{5}{4}$$

- The steady-state error, $e(\infty)\ is$

$$e(\infty) = \frac{1}{1+K_p} = \frac{1}{1-(5/4)} = -4$$

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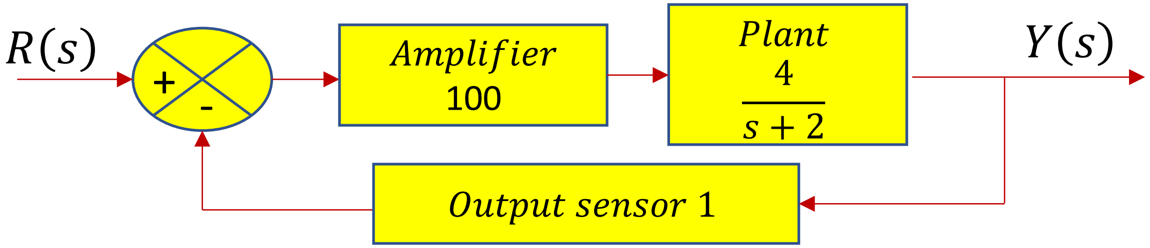
System Sensitivity – Plant Variations

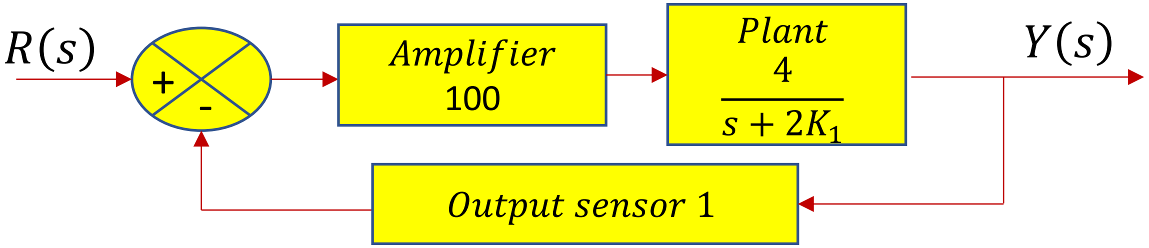
- Feedback can be used to make the **response** of a system relatively **independent** of certain type of **changes or inaccuracies**



$$T(s) = \frac{400/s + 2)}{1 + 400/(s+2)} = \frac{400}{s + 402}$$

- Suppose we have a system

- Now, suppose one of the plant parameters change or is inaccurately modelled



$$T(s) = \frac{400/(s + 2K_1)}{1 + 400/(s+2K_1)} = \frac{400}{s + 400 + 2K_1}$$

- For $K_1 = 1$ the plant is the same; other values of K_1 will cause perturbations from the nominal plant

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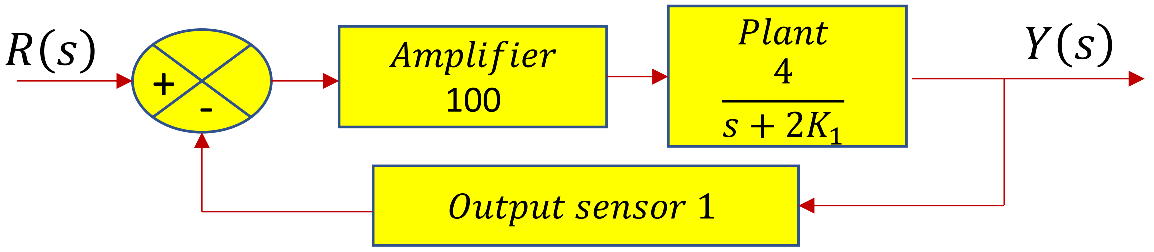
System Sensitivity – Plant Variations



- Even **50%** changes in the parameter from $\boldsymbol{K_1 = 1/2}$ to $\boldsymbol{K_1 = 3/2}$ results in a relatively minor change in the $T(s)$

$$T(s) = \frac{400/(s+2K_1)}{1+400/(s+2K_1)} = \frac{400}{s+400+2K_1}$$

- Even negative value $\boldsymbol{K_1 = -1}$ gives the same stable overall $T(s)$
- The system steady-state error for a unit step input is

$$e_{ss} = \lim_{s\to 0} s\left(\frac{1}{s}\right)[1 - T(s)] = \lim_{s\to 0} \frac{s+2K_1}{s+400+2K_1}$$

input - ouput)

$$\left(\frac{1}{s} - \frac{1}{s}T(s)\right)$$

$$s\,\frac{1}{s}\left(1 - T(s)\right)$$

$$= \frac{2K_1}{400+2K_1}$$

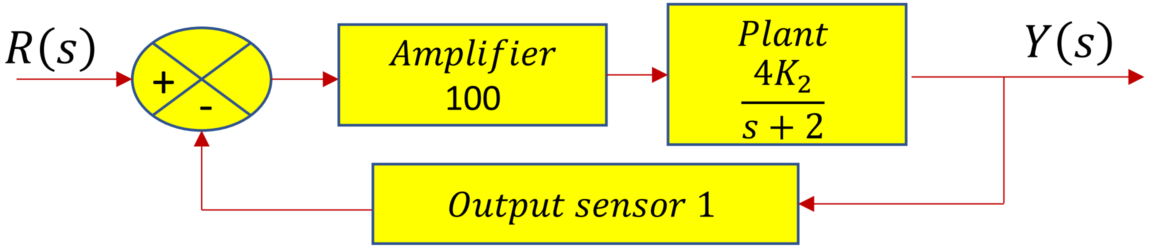
- Steady State Error is dominated by the factor of 400 and is proportional to $\boldsymbol{K_1}$

$$\boxed{E(s) = R(s)\left[1 - \frac{G(s)}{1+G(s)}\right] = R(s)[1 - \mathrm{T}(s)]}$$

EE-379 Performance Specification

System Sensitivity – Plant Variations

- Suppose the inaccuracy is of the form $\boldsymbol{K_2}$
- The Transfer function becomes
- The systems **steady-state error** for a unit step input is:
  - Dominated by the factor of $400\boldsymbol{K_2}$ and is inversely proportional to $\boldsymbol{K_2}$
- Changes in amplifier gain of 400 will produce the same effect on T(s) and its step response



$$T(s) = \frac{400K_2/(s+2)}{1+400K_2/(s+2)} = \frac{400K_2}{s+400K_2+2}$$

$$e_{ss} = \lim_{s\to 0} s\left(\frac{1}{s}\right)[1-T(s)] = \lim_{s\to 0} \frac{s+2}{s+400K_2+2}$$

$$= \frac{2}{400K_2+2}$$

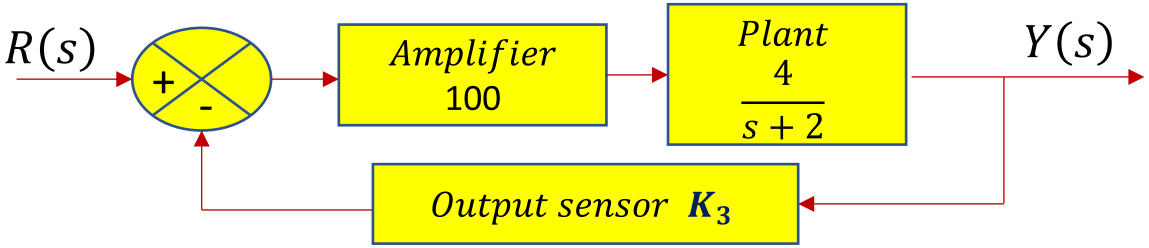
EE-379 Performance Specification

System Sensitivity – Sensor Variations

- Suppose the inaccuracy is in the sensor gain i.e., $\boldsymbol{K_3}$
- The Transfer function becomes
- The system's steady-state error for a unit step input is:
- This can become quite large for comparable percentage parameter changes
- Is this result expected ??

Why?

- **Error** by the sensor in the perceived output is **indistinguishable** from actual output error



$$T(s) = \frac{400/(s+2)}{1 + 400K_3/(s+2)} = \frac{400}{s + 400K_3 + 2}$$

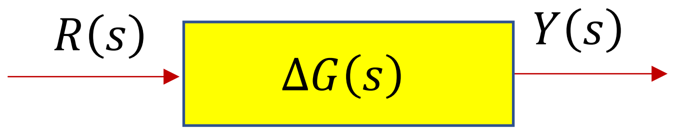
$$e_{ss} = \lim_{s\to 0} s\left(\frac{1}{s}\right)[1 - T(s)] = \lim_{s\to 0} \frac{s + 400(K_3 - 1) + 2}{s + 400K_3 + 2}$$

$$= \frac{400(K_3 - 1) + 2}{400K_3 + 2}$$

EE-379 Performance Specification

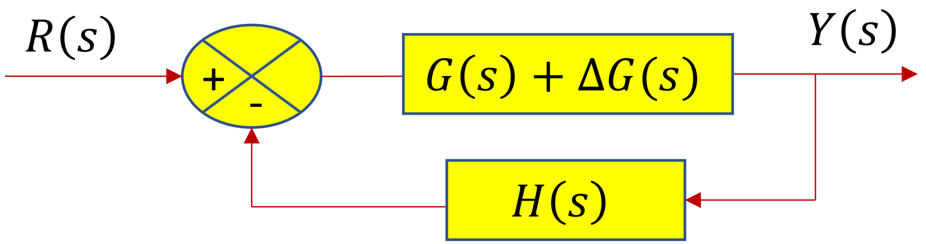
Sensitivity Functions

- A process, represented by $G(s)$, is subject to a **changing environment**, **aging**, and **ignorance** of the exact values of the process parameters.
- In the **open-loop system**, all these errors and changes result in a **changing and inaccurate output**.
- In the **open-loop case**, the change in the output is:



$$\Delta Y(s) = \Delta G(s) R(s)$$

- However, a **closed-loop system** senses the changes in the output due to process changes and attempts to correct the output.



- Consider a change in the process as:

R(s) x dtf

$$G(s) + \Delta G(s)$$

$$Y(s) + \Delta Y(s) = \frac{G(s)+\Delta G(s)}{1+\left(G(s)+\Delta G(s)\right)H(s)} R(s)$$

$$\Delta Y(s) = \frac{\Delta G(s)}{(1+G(s)H(s)+\Delta G(s)H(s))(1+G(s)H(s))} R(s)$$

EE-379 Performance Specification

Sensitivity Functions

$$\Delta Y(s) = \frac{\Delta G(s)}{[1+G(s)H(s)+\Delta G(s)H(s)][1+G(s)H(s)]} R(s)$$

$$G(s)H(s) >>> \Delta G(s)H(s)$$

$$\Delta Y(s) = \frac{\Delta G(s)}{[1+G(s)H(s)]^2} R(s)$$

- The change in the output of the closed-loop system is reduced by a factor $[1 + G(s)H(s)]^2$

EE-379 Performance Specification

Sensitivity Functions

- During the design process, the engineer may want to consider the extent to which **changes in system parameters affect the behavior of a system**.
- **The degree to which changes in system parameters affect** system transfer functions, and hence performance, is called **Sensitivity**.
- The sensitivity function of ***T*** with respect to changes in a parameter ***a*** is defined as

$$S_a^T = \lim_{\Delta a \to 0} \frac{Percentage\ change\ in\ the\ function, \boldsymbol{T}\ due\ to\ parameter\ \boldsymbol{a}}{Percentage\ change\ in\ the\ parameter, \boldsymbol{a}}$$

$$S_a^T = \lim_{\Delta a \to 0} \frac{\Delta T/T}{\Delta a/a}$$

$$S_a^T = \lim_{\Delta a \to 0} \frac{a\ \Delta T}{T\ \Delta a}$$

$$S_a^T = \frac{a\ \delta T}{T\ \delta a}$$

EE-379 Performance Specification

Sensitivity Functions

- Assume that the function ***G*** depends upon a parameter ***a*** then for the closed-loop system,

$$T(s) = \frac{G}{1+GH}$$

- The sensitivity function of ***T*** with respect to ***a*** is:

$$S_a^T = \frac{\delta T}{\delta a} \times \frac{a}{T} = \frac{\delta T}{\delta G} \cdot \frac{G}{T} \times \frac{\delta G}{\delta a} \cdot \frac{a}{G} = S_G^T S_a^G$$

Where,

$$\frac{\delta \frac{G}{1+GH}}{\delta G}$$

$$S_G^T = \frac{\delta T}{\delta G} \cdot \frac{G}{T} = \frac{1}{(1+GH)^2} \cdot \frac{G}{G/(1+GH)}$$

$$S_G^T = \frac{\delta T}{\delta G} \cdot \frac{G}{T} = \frac{1}{(1+GH)}$$

- As the loop gain $\boldsymbol{G(s)H(s)}$ becomes large, the sensitivity of the closed-loop characteristics is reduced.

- **Result:** Sensitivity of the system may be reduced below that of the open-loop system by **increasing** $\boldsymbol{G(s)H(s)}$ over the frequency range of interest.

- The sensitivity of the feedback system to changes in the feedback element $\boldsymbol{H(s)}$ is:

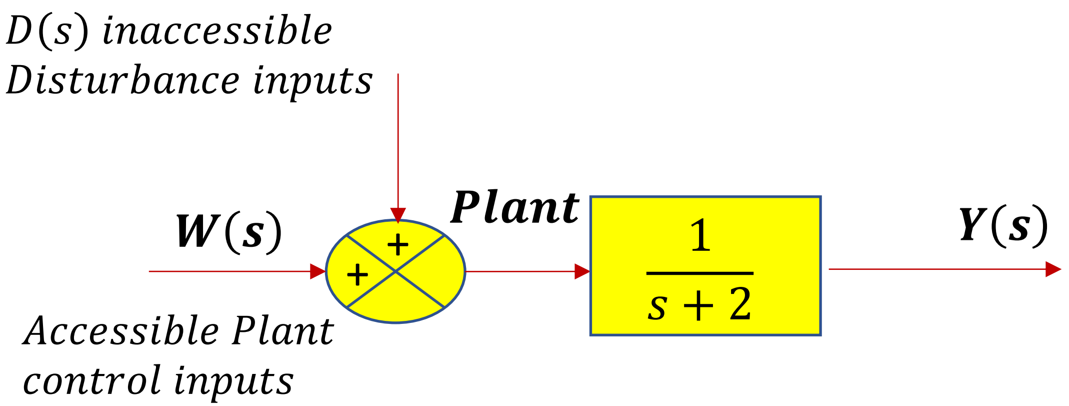
$$S_H^T = \frac{\delta T}{\delta H} \cdot \frac{H}{T} = \left(\frac{G}{1+GH}\right)^2 \cdot \frac{-H}{G/(1+GH)}$$

$$S_H^T = \frac{-GH}{(1+GH)}$$

EE-379 Performance Specification

Effect of Disturbances

- Another major advantage of a feedback system is that it can be used to reduce the effects of disturbance inputs upon system response



- In this system the disturbance signal ***D(s)*** affects the plant but is **not accessible to the designer**

- The **transfer function** relating ***Y(s)*** to ***D(s)*** is

$$T_D(s) = \frac{Y(s)}{D(s)} = \frac{1}{s+2}$$

- For a unit step disturbance input the **final value** of the output due to the disturbance is

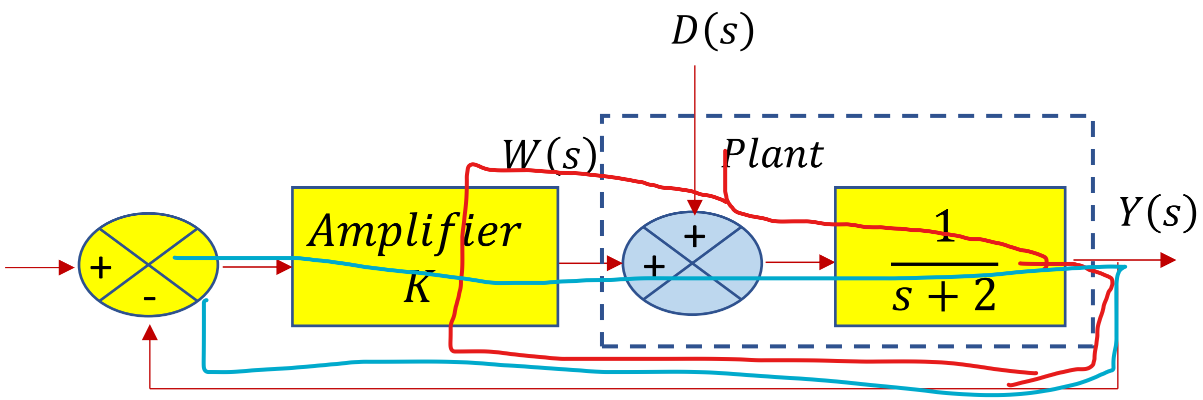
$$Y(s) = \frac{1}{s} \cdot \frac{1}{s+2}$$

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s) = \frac{1}{2}$$

EE-379 Performance Specification

Effect of Disturbances

- If the plant is driven in the feedback arrangement as shown



- The **transfer function** relating ***Y(s)*** to ***D(s)*** is

$$T_D(s) = \frac{Y(s)}{D(s)} <=> \frac{\frac{1}{s+2}}{1-\left[-\frac{K}{s+2}\right]} = \frac{1}{s+2+K}$$

- For a **unit step disturbance input** to the feedback system, the resulting **steady state output is**

$$Y(s) = D(s)T_D(s) = \frac{1}{s} \cdot \frac{1}{s+2+K}$$

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \frac{1}{2+K}$$

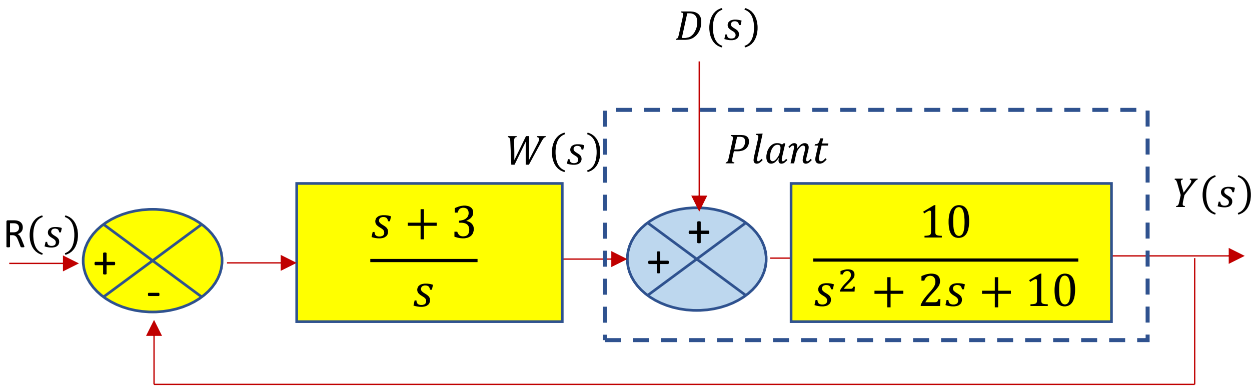
- This error can be made arbitrarily small by making **K** sufficiently large

EE-379 Performance Specification

Effect of Disturbances

- Another type of system is shown below
- The two system transfer functions are
- Routh Array shows that the system is stable
- **Unit step disturbance** produces **zero contribution**

$$\lim_{s\to 0} s\left(\frac{1}{s}\right)T_D(s) = \lim_{s\to 0}\frac{10s}{s^3+2s^2+20s+30} = 0$$



s^3	1	20
s^2	2	30
s^1	5	0
s^0	30	

$$T_R(s) = \frac{Y(s)}{R(s)} = \frac{10(s+3)/s(s^2+2s+10)}{1+10(s+3)/s(s^2+2s+10)} = \frac{10(s+3)}{s^3+2s^2+20s+30}$$

$$T_D(s) = \frac{Y(s)}{D(s)} = \frac{10/(s^2+2s+10)}{1+10(s+3)/s(s^2+2s+10)} = \frac{10s}{s^3+2s^2+20s+30}$$

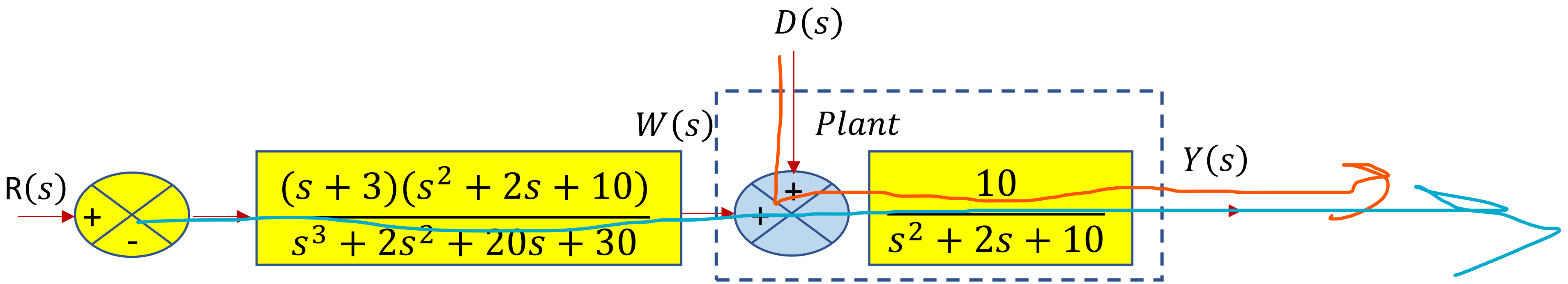
EE-379 Performance Specification

Effect of Disturbances

- Now consider the open loop (non-feedback) system as shown
- This system has the same transfer function relating ***Y(s)*** and ***R(s)***
- Relationship between **output Y(s)** and **disturbance D(s)** is not modified by feedback
- **Unit step disturbance produces** a **unit contribution** to the steady state output.

$$T_D(s) = \frac{Y(s)}{D(s)} = \frac{10}{s^2 + 2s + 10}$$

$$\lim_{s \to 0} s\left(\frac{1}{s}\right) T_D(s) = \lim_{s \to 0} \frac{10}{s^2 + 2s + 10} = 1$$



EE-379 Performance Specification

Effect of Disturbances

- Effect of $\boldsymbol{D(s)}$ for an **open-loop system**.

$$Y(s) = G_c(s)G_p(s)R(s) + G_p(s)D(s)$$

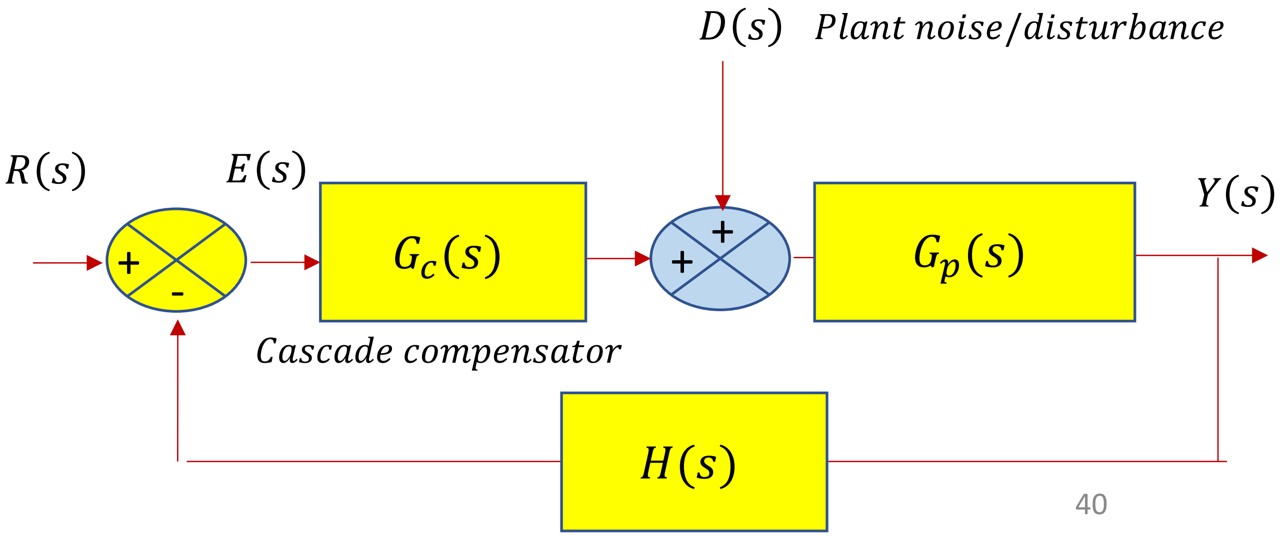
- Effect of $\boldsymbol{D(s)}$ for the **closed-loop system.**

$$Y(s) = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)H(s)}R(s) + \frac{G_p(s)}{1 + G_c(s)G_p(s)H(s)}D(s)$$

- For large loop gain, $G_c G_p \mathrm{H}(s)$

$$Y(s) \approx \frac{1}{H(s)}R(s) + \frac{1}{G_c H(s)}D(s)$$

High gain controllers can significantly reduce the effect of disturbance inputs while maintaining the desired $Y(s)/R(s)$ relationship.



EE-379 Performance Specification

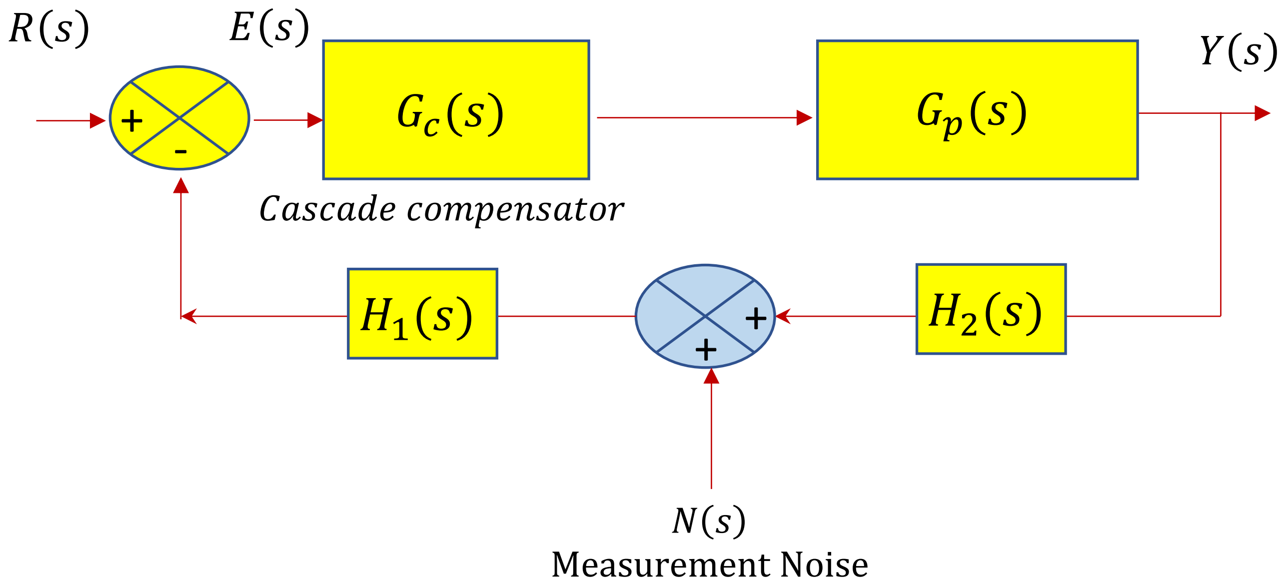
Effect of Measurement Noise

- **Measurement noise** may be represented by a **signal injected in the feedback path**.
- For a closed-loop system.

$$Y(s) = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)H_1H_2(s)}R(s) + \frac{G_c(s)G_p(s)H_1(s)}{1 + G_c(s)G_p(s)H_1H_2(s)}N(s)$$

- For large loop gain,

$$Y(s) \approx \frac{1}{H_1H_2(s)}R(s) + \frac{1}{H_2(s)}N(s)$$



EE-379 Performance Specification

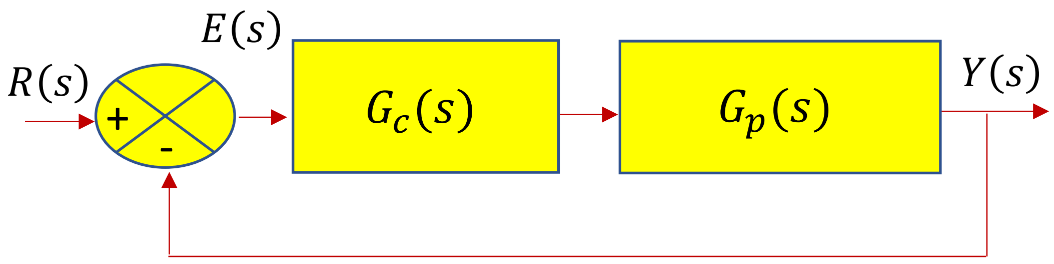
Ziegler Nichols Compensation

- A **typical process control plant** has real poles and zeros and is **type 0**.
- The compensator to the plant is generally either Proportional (P), Proportional Integral (PI) or Proportional Integral Derivative (PID) having the following form.

$$G_c(s) = K_p + \frac{K_i}{s} + K_d s = \frac{K_d s^2 + K_p s + K_i}{s}$$

- The Ziegler Nichols Method has the following two steps:
  1. Step I: Set the true plant under proportional control, with a very small gain so that $G_c(s) = K_p$. This gain is adjusted until the system becomes marginally stable. $\text{Adjusted Gain} = K_{p_o}$
     Period of oscillation = T_o
  1. Step II: The compensator is defined by

$$G_c(s) = K_p\left(1 + \frac{1}{T_i s} + T_d s\right)$$



once values of $T_d, and\ T_i$ have been calculated

$$K_i = \frac{K_p}{T_i}, \qquad K_d = K_p T_d$$

Design Equations for Ziegler Nichols

	K_p	T_i	T_d
P	$K_p = 0.5K_{po}$		
PI	$K_p = 0.45K_{po}$	$T_i = 0.83T_o$	
PID	$K_p = 0.6K_{po}$	$T_i = 0.5T_o$	$T_d = 0.125T_o$

EE-379 Performance Specification

Ziegler Nichols Compensation-Example

- Consider the process control plant

$$G_p(s) = \frac{64}{s^3 + 14s^2 + 56s + 64}$$

- First step is to let $G_c(s) = K_p$ and find the value of $K_p = K_{po}$ such that $1 + G_c(s)G_p(s)$ is **marginally stable.**

$$T(s) = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)} = \frac{64}{s^3 + 14s^2 + 56s + 64(1 + K_p)}$$

- The Routh Array is formed as

s^3	1	56
s^2	14	$64(1 + K_p)$
s^1	$(1/14)[784-64(1+K_p)]$	0
s^0	$64(1 + K_p)$	

- The value of K_{po} is found by setting the first term in row s^1 to 0 this gives K_{po} = 11.25

- At this value of K_{po} the complex conjugate roots are obtained from the row s^2 i.e., $14(s^2 + 56)$ which gives the roots as $\pm j7.483$

- The characteristic polynomial is then divided by $(s^2 + 56)$ which gives the remaining root -14.

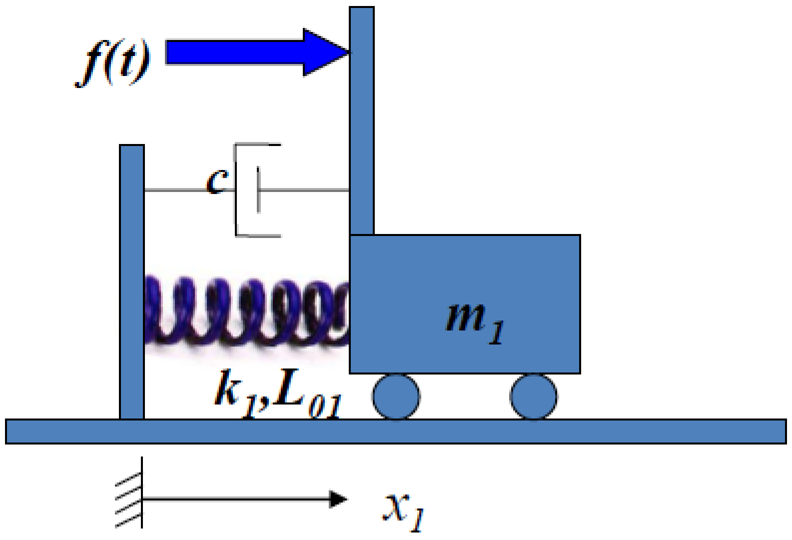
- This gives $T_o = \frac{2\pi}{\omega_o} = \frac{2\pi}{7.483} = 0.84$ and the remaining values can be easily found from the design table.

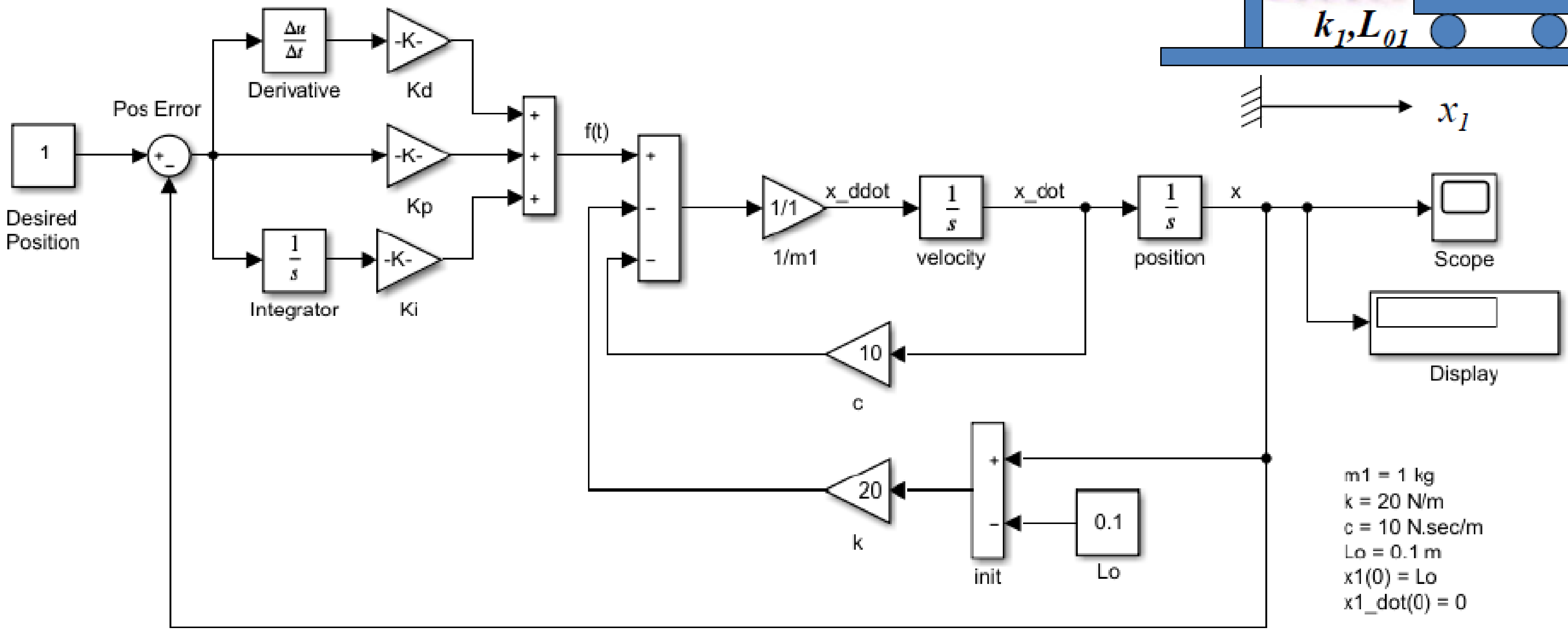
	K_p	T_i	T_d
P	$K_p = 0.5K_{po}$		
PI	$K_p = 0.45K_{po}$	$T_i = 0.83T_o$	
PID	$K_p = 0.6K_{po}$	$T_i = 0.5T_o$	$T_d = 0.125T_o$

EE-379 Performance Specification

Ziegler Nichols Compensation-Example

- The Simulink model of a simple mass damper spring (mck) system with a PID controller





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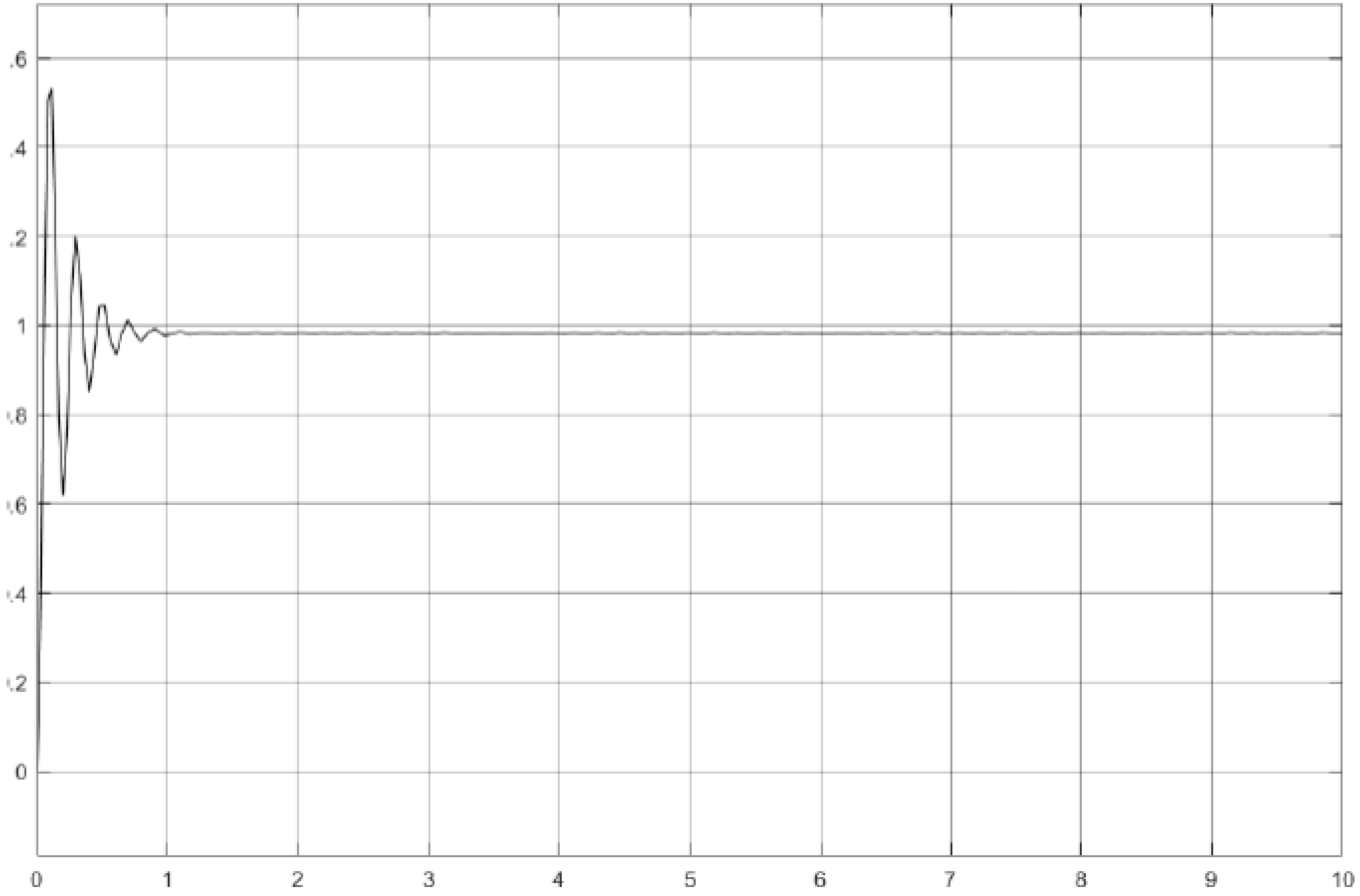
Ziegler Nichols Compensation-Example

- Response of uncompensated System



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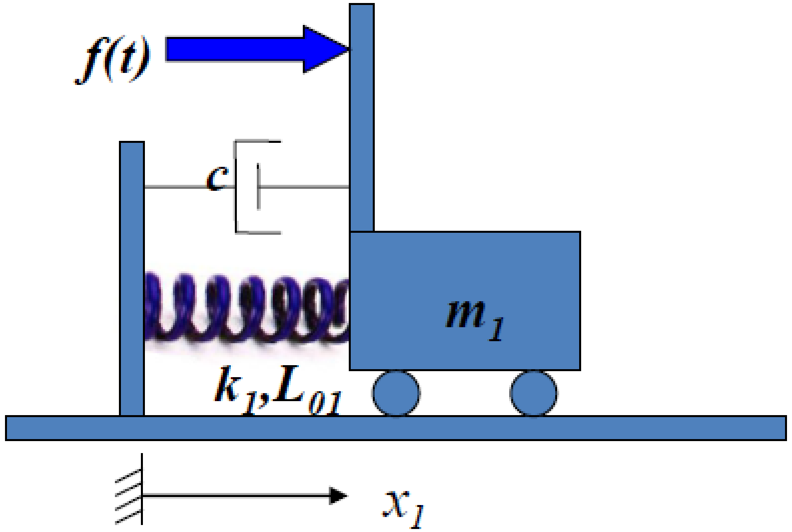
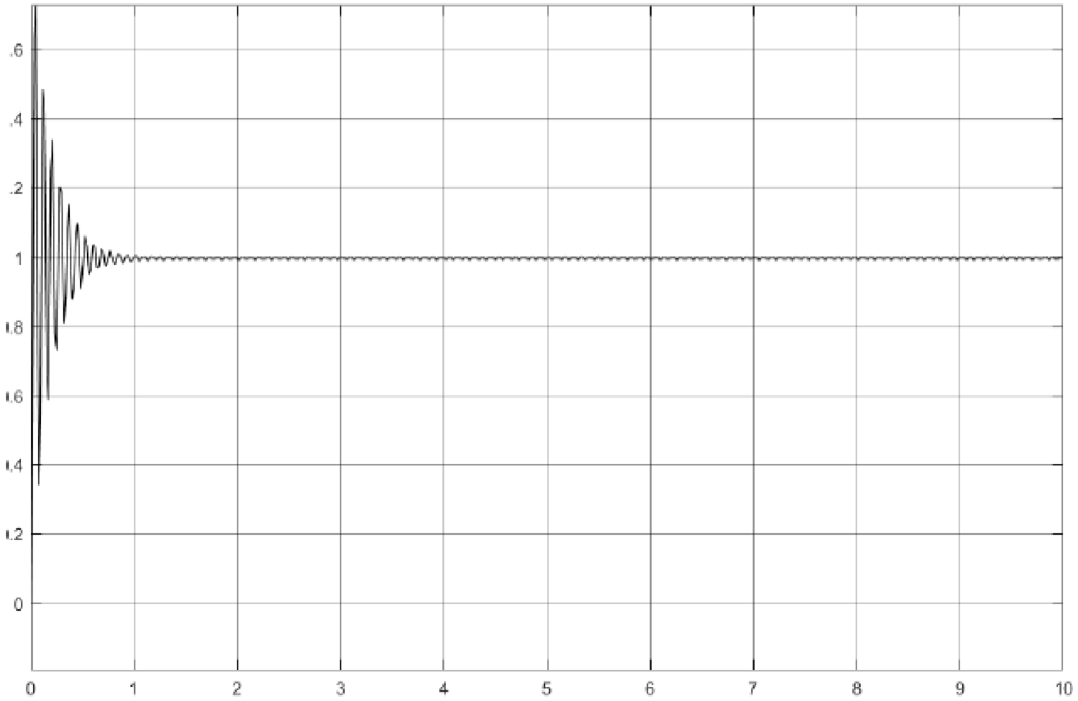
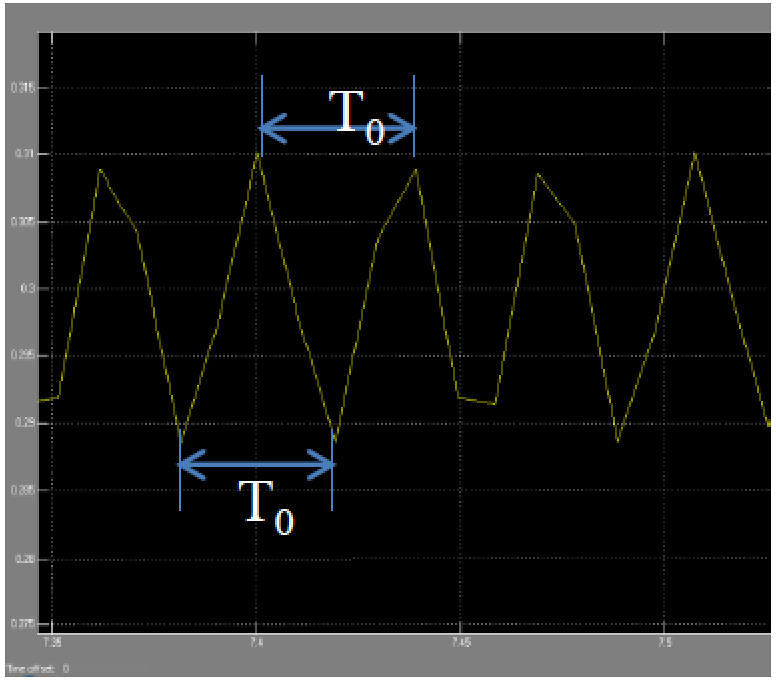
Ziegler Nichols Compensation-Example

- Set up the system with only proportional control and provide a step input
- start with a small value for K_p and increase until the output gives constant amplitude oscillations
- This value of the proportional constant will be called, K_u , the ultimate gain







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Ziegler Nichols Compensation-Example

- The controller equation can be written as

$$G_c(s) = K_p\left(1 + \frac{1}{T_i s} + T_d s\right)$$

- Select $K_p, T_i\ and T_d\ according\ to.$

	K_p	T_i	T_d
P	$K_p = 0.5K_{po}$		
PI	$K_p = 0.45K_{po}$	$T_i = 0.83T_o$	
PID	$K_p = 0.6K_{po}$	$T_i = 0.5T_o$	$T_d = 0.125T_o$

$$K_{po} = 6000\ N/m$$

$$T_o = 0.08\ sec$$

$$\Downarrow$$

$$K_p = 3600\ N/m$$

$$T_i = 0.04\ sec$$

$$T_d = 0.01\ sec$$

$$\Downarrow$$

$$K_p = 3600\ N/m$$

$$K_i = 90{,}000\ N/(m\ sec)$$

$$K_d = 36\ N\ sec/m$$

EE-379 Performance Specification

Ziegler Nichols Compensation-Example

- Response of Ziegler Nichols compensated System