Digital Signal Processing (EC 335)

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Lecture 4

Difference Equation

The **convolution sum** expresses the output of a **linear shift-invariant system** in terms of a linear combination of the input values $x[n]$. For example, a system that has a unit sample response $h[n] = \alpha^n u[n]$ is described by the equation .

$$y(n) = \sum_{k=0}^{\infty} \alpha^k x(n-k)$$

Although this equation allows one to compute the output $y[n]$ for an arbitrary input $x[n]$, **from a computational point of view this representation is not very efficient**.

In some cases it may be possible to more efficiently **express the output in terms of past values of the output in addition to the current and past values of the input**. The previous system, for example, may be described more concisely as follows:

$$y(n) = \alpha y(n-1) + x(n)$$

Above equation is a special case of what is known as a linear constant coefficient difference equation, or LCCDE. The general form of a LCCDE is

$$y(n) = \sum_{k=0}^{q} b(k)x(n-k) - \sum_{k=1}^{p} a(k)y(n-k)$$

Difference Equation

Consider system with the following specification

" Add all the samples, wherever you are"

Find the impulse response of the following system? $y[n] = ay[n-1] + x[n]$

If $x[n] = \delta[n] \Rightarrow y[n] = h[n]$

$y[n] = ay[n-1] + x[n]$

$y[0] = a\,\underbrace{y[-1]}_{0} + \underbrace{x[0]}_{1}$

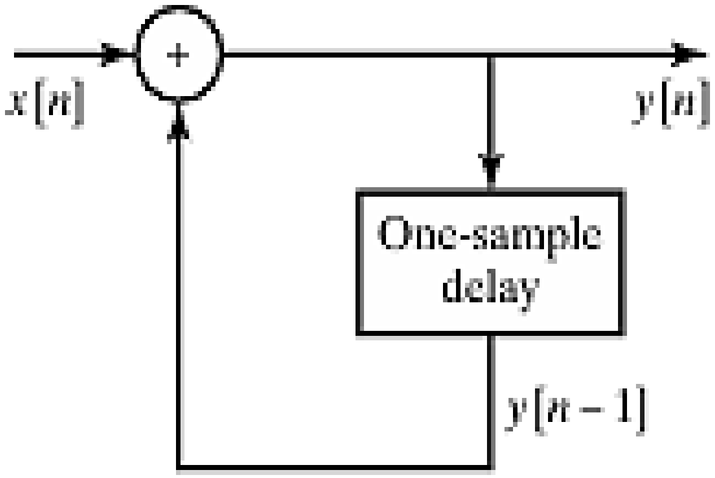
$\Rightarrow y[0] = 1$

$y[1] = a\,\underbrace{y[0]}_{1} + \underbrace{x[1]}_{0}$

$\Rightarrow y[1] = a$

$y[1] = a\,\underbrace{y[1]}_{1} + \underbrace{x[2]}_{0}$

$\Rightarrow y[2] = a^2$



So

$h[n] = \{1, a, a^2, a^3, \ldots\ldots.\}$

$h[n] = a^n u[n]$

Difference Equation

$$y(n) = \boxed{\sum_{k=0}^{q} b(k)x(n-k)} - \sum_{k=1}^{p} a(k)y(n-k)$$

Convolution

- FIR systems are inherently stable.
- IIR system are more effective, few parameters can control the whole system. However, stability might be an issue.
- y = conv(x, h) is Matlab command for Convolution.
- y = filter(b, a, x) is Matlab command to implement difference equation.
- [y, zf] = filter(b, a, x, zi) is Matlab command to implement difference equation with initial and final conditions.

Solution of Difference Equation

- There are several different methods that one may use to solve LCCDEs for a general input $x[n]$.
- The first is to simply set up a table of input and output values and evaluate the difference equation for each value of n. This approach would be appropriate if only a few output values needed to be determined.
- Another approach is to use z-transforms. This approach will be discussed later.
- The third is the classical approach of finding the homogeneous and particular solutions, which we now describe.
- The general solution of LCCDE can be written as

$$y(n) = y_h(n) + y_p(n)$$

- The homogeneous solution is the response of the system to the initial conditions, assuming that the input $x[n] = 0$.
- The particular solution is the response of the system to the input $x[n]$, assuming zero initial conditions.

Solution of Difference Equation

$$y(n) = \sum_{k=0}^{q} b(k)x(n-k) - \sum_{k=1}^{p} a(k)y(n-k)$$

The homogeneous solution is found by solving the homogeneous difference equation.

$$y(n) + \sum_{k=1}^{p} a(k)y(n-k) = 0$$

The solution to above equation may be found by assuming a solution of the form

$$y_h(n) = z^n$$

Substituting this solution, we obtain the polynomial equation

$$z^n + \sum_{k=1}^{p} a(k)z^{n-k} = 0$$

$$z^{n-p}\{z^p + a(1)z^{p-1} + a(2)z^{p-2} + \cdots + a(p-1)z + a(p)\} = 0$$

Solution of Difference Equation

The general solution to the homogeneous difference equation is

$$y_h(n) = \sum_{k=1}^{p} A_k z_k^n$$

For the particular solution, it is necessary to find the sequence $y_p[n]$ that satisfies the difference equation for the given $x[n]$. In general, this requires some creativity and insight. However, for many of the typical inputs that we are interested in, the solution will have the same form as the input. Table lists the particular solution for some commonly encountered inputs. For example, if $x[n] = a^n u[n]$, the particular solution will be of the form

$$y_p(n) = C a^n u(n)$$

provided a is not a root of the characteristic equation

Solution of Difference Equation

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$$y_p(n) = Ca^n u(n)$$

provided a is not a root of the characteristic equation

The constant C is found by substituting the solution into the difference equation.

Solution of Difference Equation

The particular solution to an LCCDE for several different inputs

Term in $x(n)$	Particular Solution
C	C_1
Cn	$C_1 n + C_2$
Ca^n	$C_1 a^n$
$C\cos(n\omega_0)$	$C_1\cos(n\omega_0) + C_2\sin(n\omega_0)$
$C\sin(n\omega_0)$	$C_1\cos(n\omega_0) + C_2\sin(n\omega_0)$
$Ca^n\cos(n\omega_0)$	$C_1 a^n\cos(n\omega_0) + C_2 a^n\sin(n\omega_0)$
$C\delta(n)$	None

Example

Find the solution of $y[n] - 0.25\, y[n-2] = x[n]$
for $x[n] = u[n]$ assuming initial conditions of $y[-1] = 1$ and $y[-2] = 0$.

From Table (previous slide) we see that for $x[n] = u[n]$, the particular solution is

$$y_p(n) = C_1$$

Substituting this solution into the difference equation we find

$$C_1 - 0.25C_1 = 1$$

In order for this to hold, we must have

$$C_1 = \frac{1}{1-0.25} = \frac{4}{3}$$

To find the homogeneous solution, we set $y_h(n) = z^n$, which gives the characteristic polynomial

$$z^2 - 0.25 = 0$$

for repeated roots:
for 2 repeated roots: AS^n + BnS^n
for 3 repeated roots // // + (Cn^2)S^n

or

$$(z+0.5)(z-0.5) = 0$$

for distinct AS1^n + BS2^n

Therefore, the homogeneous solution has the form

$$y_h(n) = A_1(0.5)^n + A_2(-0.5)^n$$

Thus, the total solution is

$$\boxed{y(n) = \tfrac{4}{3} + A_1(0.5)^n + A_2(-0.5)^n \qquad n \geq 0}$$

The constants A, and A2 must now be found so that the total solution satisfies the given initial conditions, y(-1) = 1 and y(-2) = 0. Because the solution given in Eq. (1.17) only applies for n 0, we must derive an equivalent set of initial conditions for y(0) and y(1). Evaluating Eq. (1.16) at n = 0 and n = 1. we have

Example

$$y(n) = \tfrac{4}{3} + A_1(0.5)^n + A_2(-0.5)^n \qquad n \geq 0$$

The constants A1, and A2 must now be found so that the total solution satisfies the given initial conditions, y(-1) = 1 and y(-2) = 0. Because the solution given in above equation only applies for n ≥0, we must derive an equivalent set of initial conditions for y(0) and y(1). Evaluating system equation at n = 0 and n = 1. we have

$$y(0) - 0.25y(-2) = x(0) = 1$$
$$y(1) - 0.25y(-1) = x(1) = 1$$

Substituting these derived initial conditions into above equation, we have

$$y(0) = \tfrac{4}{3} + A_1 + A_2 = 1$$
$$y(1) = \tfrac{4}{3} + \tfrac{1}{2}A_1 - \tfrac{1}{2}A_2 = 1$$

Solving for A_1 and A_2 we find

$$A_1 = -\tfrac{1}{2} \qquad A_2 = \tfrac{1}{6}$$

Thus, the solution is

$$y(n) = \tfrac{4}{3} - (0.5)^{n+1} + \tfrac{1}{6}(-0.5)^n \qquad n \geq 0$$

Example

Consider a system described by the difference equation

$$y(n) = y(n-1) - y(n-2) + 0.5x(n) + 0.5x(n-1)$$

Find the response of this system to the input

$$x(n) = (0.5)^n u(n)$$

with initial conditions $y(-1) = 0.75$ and $y(-2) = 0.25$.

The first step in solving this difference equation is to find the particular solution. With $x(n) = (0.5)^n u(n)$, we assume a solution of the form

$$y_p(n) = C_1(0.5)^n \qquad n \geq 0$$

Substituting this solution into the difference equation, we have

$$C_1(0.5)^n = C_1(0.5)^{n-1} - C_1(0.5)^{n-2} + 0.5(0.5)^n + 0.5(0.5)^{n-1} \qquad n \geq 0$$

Example

Dividing by $(0.5)^n$,

$$C_1 = 2C_1 - 4C_1 + 0.5 + 1$$

which gives

$$C_1 = \tfrac{1}{2}$$

The next step is to find the homogeneous solution. The characteristic equation is

$$z^2 - z + 1 = 0$$

which has roots

$$z = \tfrac{1}{2}(1 \pm j\sqrt{3}) = e^{\pm j\pi/3}$$

Therefore, the form of the homogeneous solution is

$$y_h(n) = A_1 e^{jn\pi/3} + A_2 e^{-jn\pi/3}$$

and the total solution becomes

$$y(n) = (0.5)^{n+1} + A_1 e^{jn\pi/3} + A_2 e^{-jn\pi/3} \qquad n \geq 0 \tag{1.25}$$

Example

The constants A_1 and A_2 must now be found so that the total solution satisfies the given initial conditions, $y(-1) = 0.75$ and $y(-2) = 0.25$. Because the solution given in Eq. (*1.25*) is only applicable for $n \geq 0$, we must derive an equivalent set of initial conditions for $y(0)$ and $y(1)$. Evaluating the difference equation for $n = 0$ and $n = 1$, we have

$$y(0) = y(-1) - y(-2) + 0.5x(0) + 0.5x(-1) = 0.75 - 0.25 + 0.5 = 1$$

and

$$y(1) = y(0) - y(-1) + 0.5x(1) + 0.5x(0) = 1 - 0.75 + 0.25 + 0.5 = 1$$

Example

Now, substituting these *derived initial conditions* into Eq. (*1.25*), we have

$$y(0) = 0.5 + A_1 + A_2 = 1$$
$$y(1) = 0.25 + A_1 e^{j\pi/3} + A_2 e^{-j\pi/3} = 1$$

Writing this pair of equations in the two unknowns A_1 and A_2 in matrix form,

$$\begin{bmatrix} 1 & 1 \\ e^{j\pi/3} & e^{-j\pi/3} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.75 \end{bmatrix}$$

and solving, we find

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = j\frac{\sqrt{3}}{3} \begin{bmatrix} \frac{1}{2}e^{-j\pi/3} - \frac{3}{4} \\ -\frac{1}{2}e^{j\pi/3} + \frac{3}{4} \end{bmatrix}$$

Substituting into Eq. (*1.25*) and simplifying, we find, after a bit of algebra,

$$y(n) = (0.5)^{n+1} + \frac{\sqrt{3}}{2}\sin\left(\frac{n\pi}{3}\right) - \frac{2\sqrt{3}}{2}\sin\left((n-1)\frac{\pi}{3}\right)$$