EE-379 Linear Control Systems

Week No. 3: Response and Stability of Systems

- Continuous Time Systems Response
- Response of a First Order System
- Response of a Second Order System
- Stability Analysis

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Chapter 2: Continuous Time Systems Response

- What to expect in this chapter:
  - The **characteristic polynomial** can be factored into first and second-order systems
  - If the behavior of first-order and second-order systems are understood behavior of higher-order systems follows as a combination of the first and second-order building blocks.
  - Definitions will be presented that clarify the quality of performance in terms of a **system's stability.**
  - **Routh and Horwitz Criterion** – even though it is ancient it remains a valuable tool for determining a range of values for an unknown parameter so that stability is ensured.
  - Example, that illustrates the power of the analytical methods

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Chapter 2: Continuous Time Systems Response

- **Standard Inputs**
  - Next important step after a mathematical model of a system is obtained, is to analyze the system's performance.
  - Normally use the standard input signals to identify the characteristics of system's response
    - Step function
    - Ramp function
    - Impulse function
    - Parabolic function
    - Sinusoidal function

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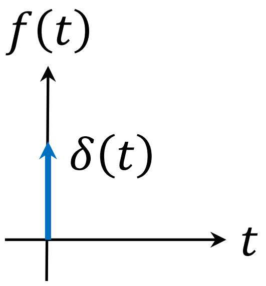
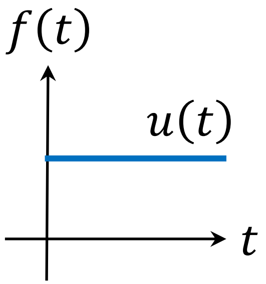
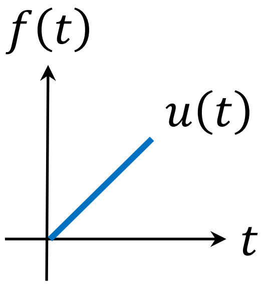
Standard Inputs

Input	Function	Description	Sketch	Use
Impulse	$\delta(t)$	$\delta(t) = \infty \; for \; 0- < t < 0+$ $= 0 \; elsewhere$ $\int_{0-}^{0+} \delta(t)dt = 1$		$Transient \; response \; modeling$
Step	$u(t)$	$u(t) = 1 \; for \; t > 0$ $= 0 \; for \; t < 0$		$Transient \; response$ $Steady - state \; error$
Ramp	$tu(t)$	$u(t) = t \; for \; t \geq 0$ $= elsewhere$		$Steady - state \; error$

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Transient and Steady State Response

- Output response consists of the sum of **forced response** (from the input) and **natural response** (from the nature of the system).
- The **natural response** determines how **good the system** is.
- The **transient response** is the **change in the output response** from the beginning of the response to the final state of the response.
- The steady-state response is the output response as time is approaching infinity or no more changes at the output

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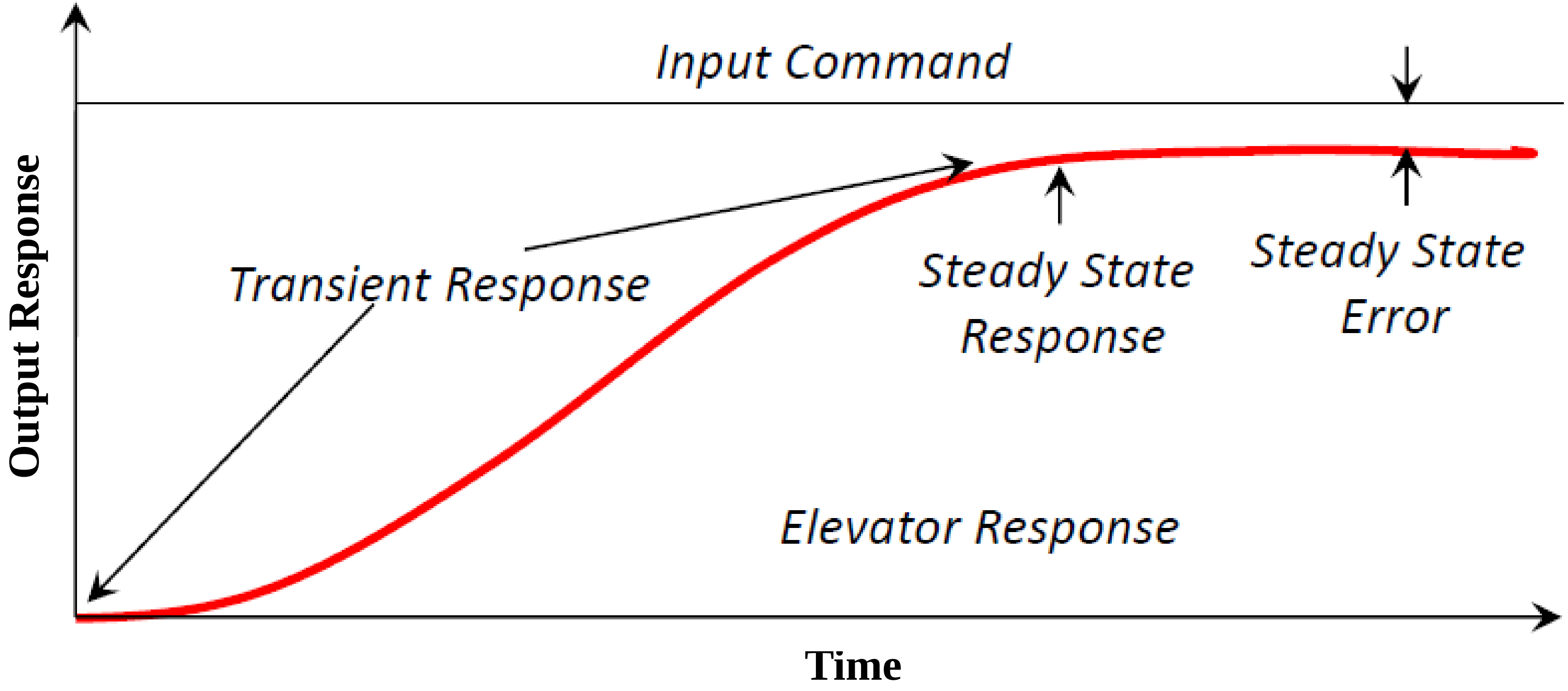
Transient and Steady State Response



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Response of First Order Systems

- In a first-order system, the output **y(t)** and input **r(t)** are related by a differential equation of the form:

$$\frac{d_y}{d_t} + a_0 y = b_0 r \quad \rightarrow \quad \mathrm{T(s)} = \frac{Y(s)}{R(s)} = \frac{b_0}{s + a_0} \cong \frac{k}{1 + \tau s}$$

k = system const.
τ = system time const.

- **System is stable** if the **natural response decays to zero** (roots of the characteristic polynomial must lie in the LHP of s-plane)
- Above-mentioned first order system is stable if and only if $a_0 > 0$.
- Laplace transforming the first-order system equation.

$$sY(s) - y(0^-) + a_0 Y(s) = b_0 R(s)$$

$$Y(s) = \underbrace{\frac{b_0}{s + a_0} R(s)}_{\text{Zero state component}} + \underbrace{\frac{Y(0^-)}{s + a_0}}_{\text{Zero input component}}$$

A system with only one pole

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Basic Equations

$$m\frac{dv}{dt} = f(t) - cv$$

$$L[m\dot{v}] = L[f(t) - cv]$$

$$mL[\dot{v}] = L[f(t)] - cL[v]$$

$$m[sV(s) - v(0)] = F(s) - cV(s)$$

$$V(s) = \frac{F(s)}{ms + c} + \frac{mv(0)}{ms + c}$$

Characteristic Polynomial

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Step Input Response

$$V(s) = \boxed{\frac{F(s)}{ms+c}} + \boxed{\frac{mv(0)}{ms+c}}$$

Zero state response

Zero input response

- Zero state response is the result of a driving function with zero initial conditions.
  - For a step input signal $F(s) = f(\frac{1}{s})$
  - $L^{-1}\left[\frac{F(s)}{ms+c}\right] = L^{-1}\left[\frac{1}{ms+c} \cdot \frac{f}{s}\right] = \boxed{\frac{f}{c}\left(1 - e^{-\frac{ct}{m}}\right)} \longrightarrow using\ partial\ fractions$
- Zero input response is the result of a zero-driving function with only non-zero initial conditions.
  - $L^{-1}\left[\frac{mv(0)}{ms+c}\right] = v(0)L^{-1}\left[\frac{1}{s+\frac{c}{m}}\right] = v(0)e^{-\frac{ct}{m}}$
- Total response or velocity

$$v(t) = \frac{f}{c}\left(1 - e^{-\frac{ct}{m}}\right) + v(0)e^{-\frac{ct}{m}}$$

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Step Input Response

- Transfer function for a first-order system:

$$G(s) = \frac{b_0}{s + a_0} \cong \frac{k}{1 + \tau s} \cong \frac{k/\tau}{s + 1/\tau} \to \begin{matrix} b_0 = k/\tau \\ a_0 = 1/\tau \end{matrix}$$

$$Y(s) = \underbrace{\frac{b_0}{s + a_0} R(s)}_{\text{Zero state component}} + \underbrace{\frac{Y(0^-)}{s + a_0}}_{\text{Zero input component}}$$

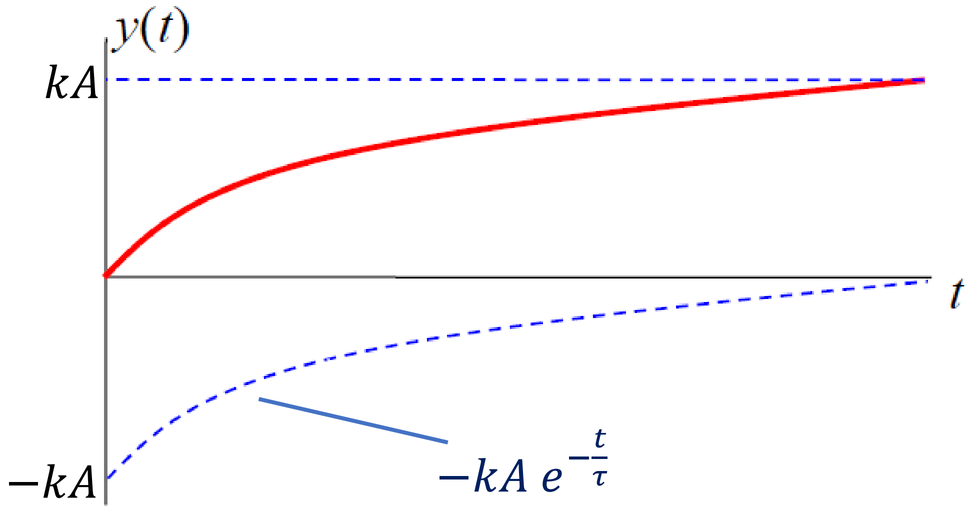
- For a step input signal and zero initial conditions $Y(s)$ contains only **zero state** terms.

$$r(t) = Au(t) \Rightarrow R(s) = \frac{A}{s}$$

$$Y(s) = T(s)R(s) = \frac{b_0}{(s+a_0)} \cdot \frac{A}{s} = \frac{k/\tau}{s+1/\tau} \cdot \frac{A}{s}$$

- Partial fraction and inverse Laplace transform with A=1 give:

$$\boxed{y(t) = \left(k - ke^{-\frac{t}{\tau}}\right)u(t)}$$



- ***k*** represents the steady-state response, when the time approaches infinity the transient response will die out and the system will come to its steady state.

$$\boxed{k = \frac{b_0}{a_0}}$$

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Step Input Response

- Transfer function for a first-order system:

$$G(s) = \frac{b_0}{s + a_0} \cong \frac{k}{1 + \tau s} \cong \frac{k/\tau}{s + 1/\tau} \rightarrow \begin{matrix} b_0 = k/\tau \\ a_0 = 1/\tau \end{matrix}$$

$$Y(s) = \underbrace{\frac{b_0}{s + a_0}}_{\text{Zero state component}} R(s) + \underbrace{\frac{Y(0^-)}{s + a_0}}_{\text{Zero input component}}$$

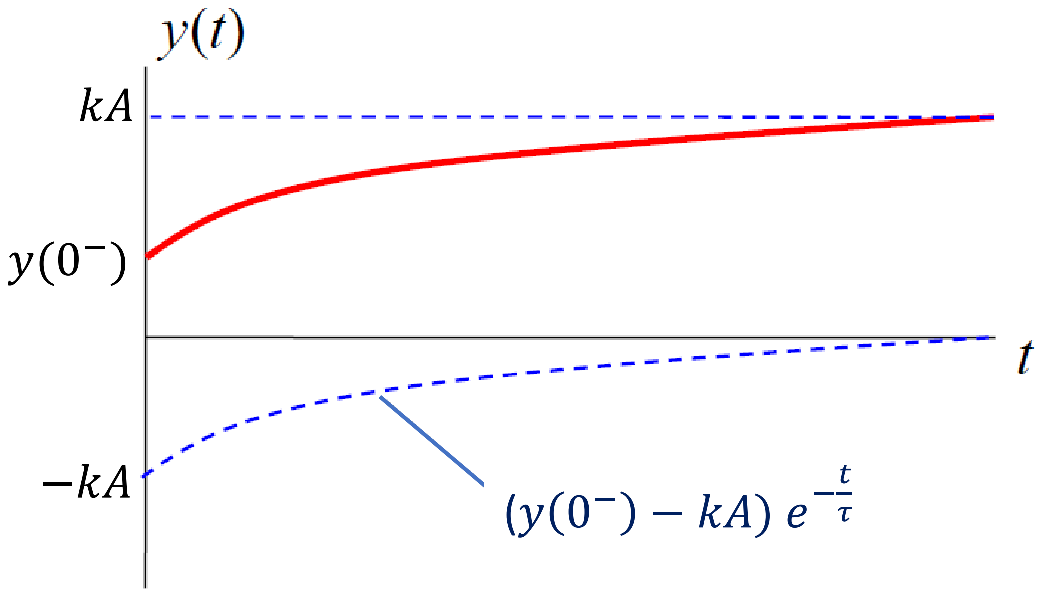
- For a step input signal with initial conditions $Y(s)$ contains **zero state** and **zero input terms**.

$$r(t) = Au(t) \Rightarrow R(s) = \frac{A}{s}$$

$$Y(s) = \left(\frac{b_0}{(s+a_0)}\right) \cdot \frac{A}{s} + \left(\frac{Y(0^-)}{s+a_0}\right) \Rightarrow \frac{k/\tau}{s+1/\tau} \cdot \frac{A}{s} + \left(\frac{Y(0^-)}{s+1/\tau}\right)$$

- Partial fraction and inverse Laplace transform with A=1 give:

$$\boxed{y(t) = \left\{k + [y(0^-) - k]e^{-\frac{t}{\tau}}\right\} u(t)}$$



Amplitude of the exponential term is changed

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Example

- Consider a system with transfer function:

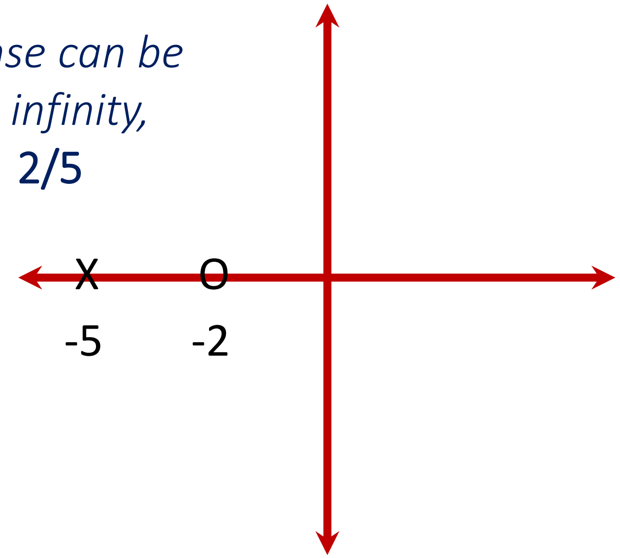
$$G(s) = \frac{Y(s)}{R(s)} = \frac{s+2}{s+5}$$

- Applying a unit step function $R(s)$ and substituting this input into the transfer function and applying the partial fraction, gives

$$Y(s) = \frac{s+2}{s+5} \cdot \frac{1}{s}$$

$$Y(s) = \frac{2}{5} \cdot \frac{1}{s} + \frac{3}{5} \cdot \frac{1}{s+5}$$

The steady-state response can be obtained by putting $\boldsymbol{t}$ *to infinity, which will give* $\boldsymbol{y}(\infty) = 2/5$



- Applying the inverse Laplace transform, gives the output response

$$\boxed{c(t) = \frac{2}{5} + \frac{3}{5} e^{-5t}}$$

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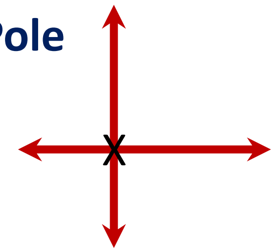
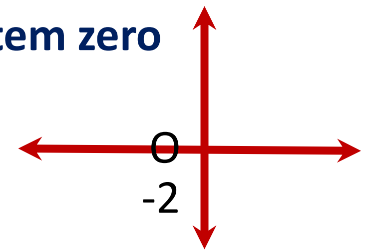
Example

- Output response consists of the sum of **forced response** (from the input) and **natural response** (from the nature of the system)
- Any input to a system will have a forced response at the output.
- The poles in the transfer function of a system will give the natural response at the output

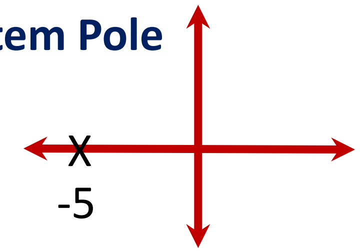
Input Pole



System zero



System Pole



$$Output\ transform: \quad Y(s) = \frac{3/5}{s+5} + \frac{2/5}{s}$$

$$Output\ time\ response = y(t) = \frac{2}{5} + \frac{3}{5}e^{-5t}$$

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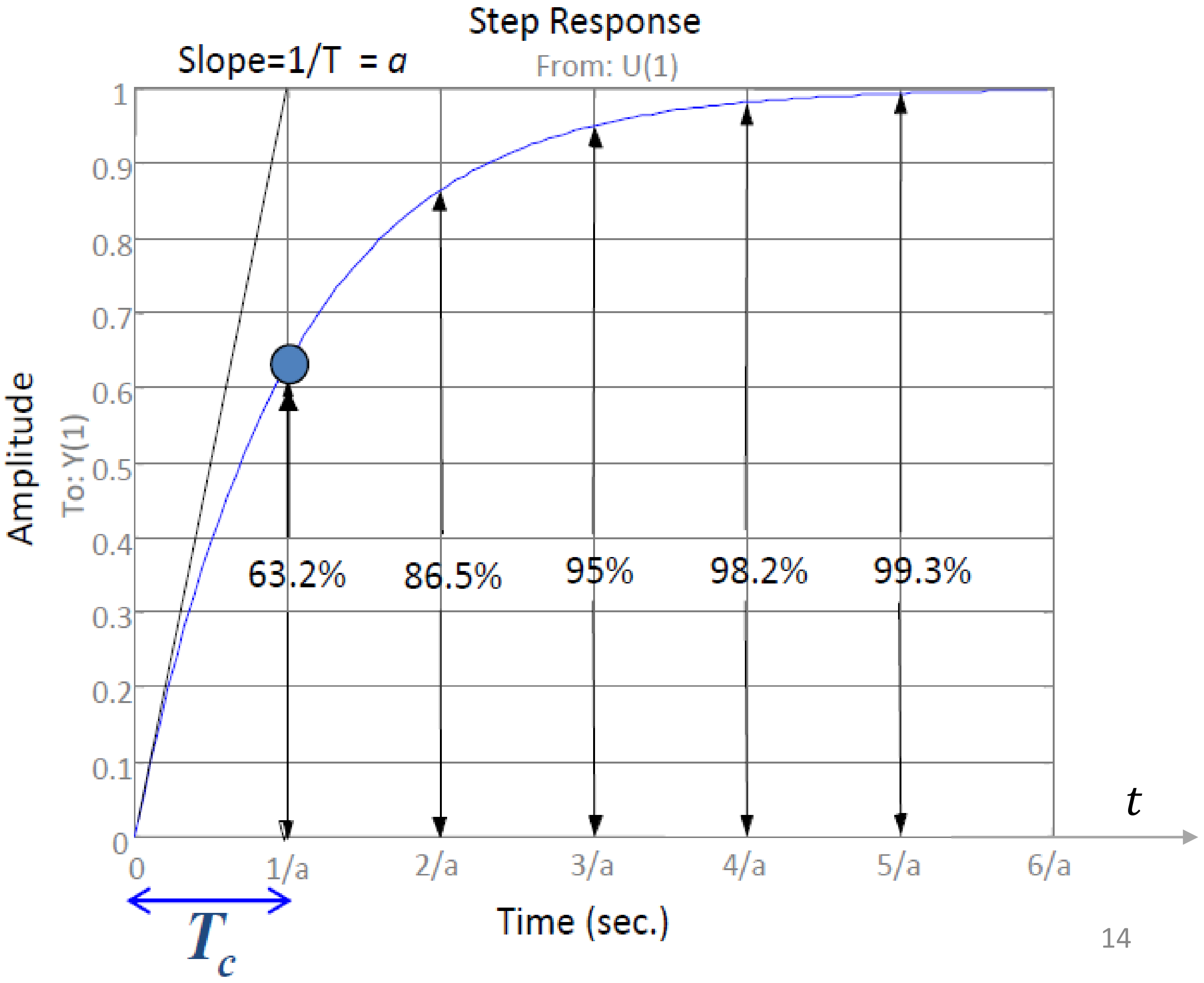
Important Terms – Time Constant

- $\boldsymbol{Time\ constant,\ T_c} = \frac{1}{a}$ *is the time for* e^{-at} *to rise to 63% of its final value, or the time when* $t = \frac{1}{a}$. t= one time const.

$$T_c = \frac{1}{a}$$

$$y(t) = 1 - e^{-at}$$



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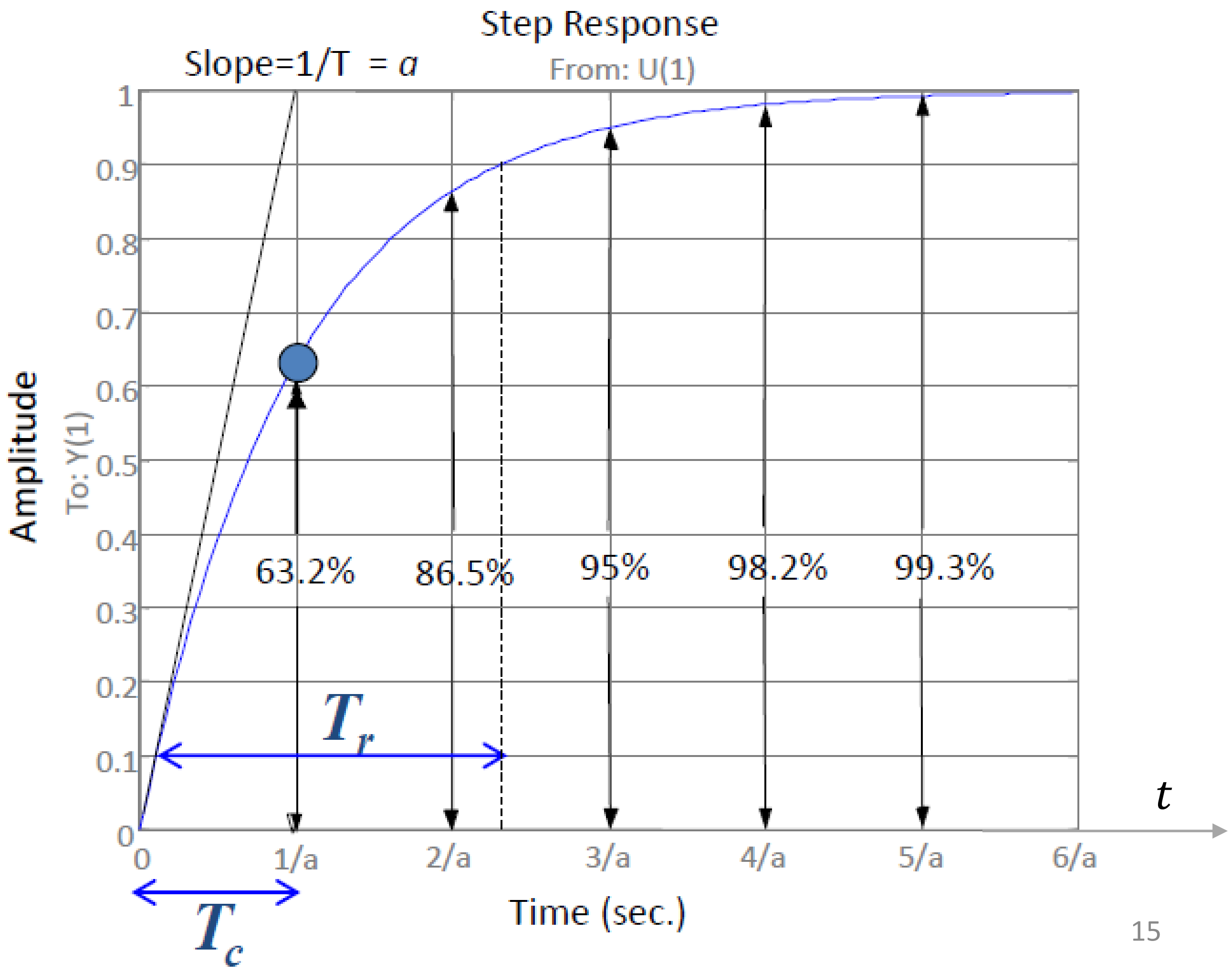
Important Terms – Rise Time

- $\boldsymbol{Rise\ Time, T_r}$ $is\ the\ time\ taken\ for\ the\ output\ waveform\ to\ go\ from\ 10\%\ to\ 90\%\ of\ its\ final\ output\ value.$

$$T_r = \frac{2.2}{a}$$

$$y(t) = 1 - e^{-at}$$



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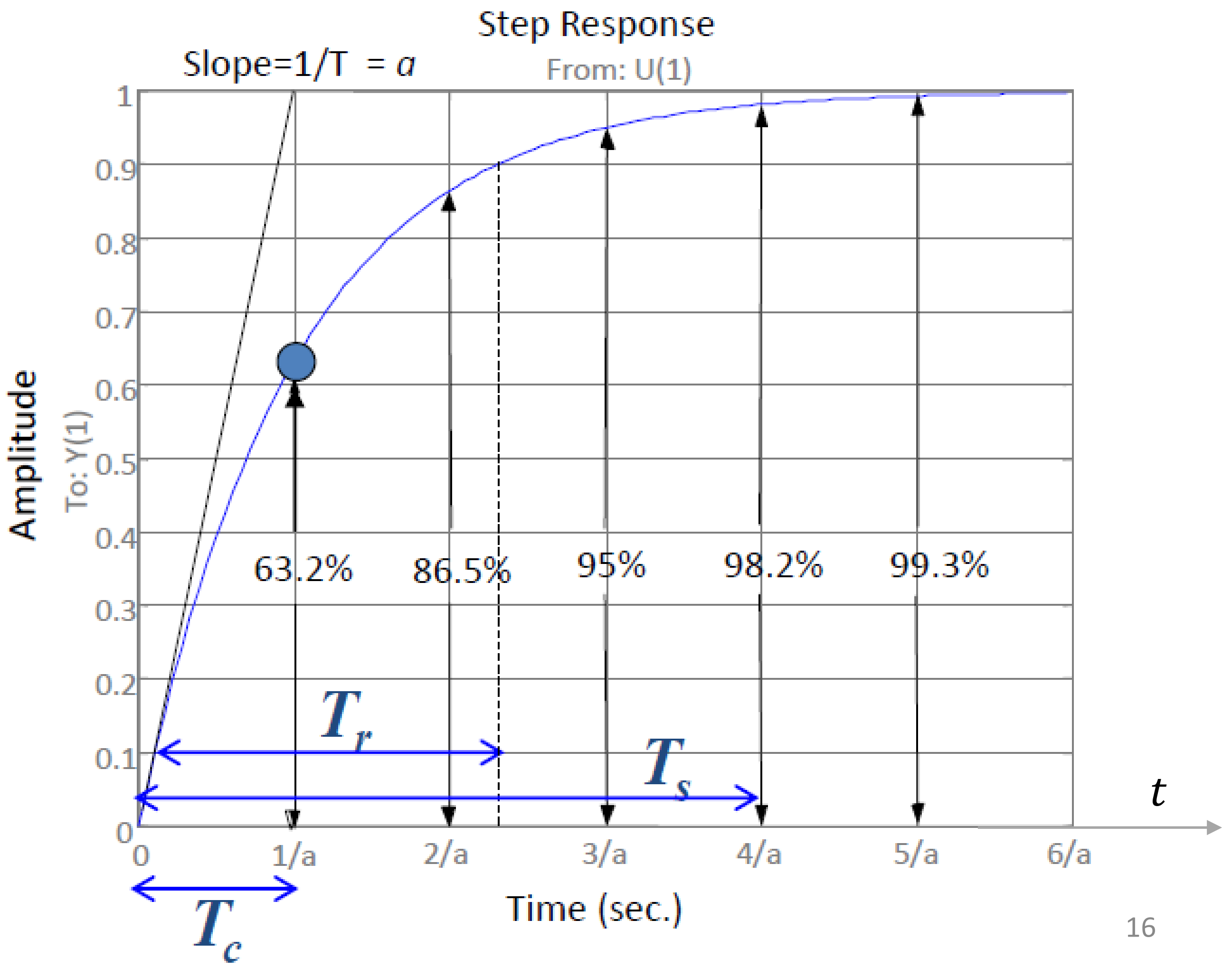
Important Terms – Settling Time

- **Settling Time**, T_s *is the time taken for the output waveform to reach and stay within* 2% *of its output value.*

$$T_s = \frac{4}{a} = 4T_c$$

$$y(t) = 1 - e^{-at}$$



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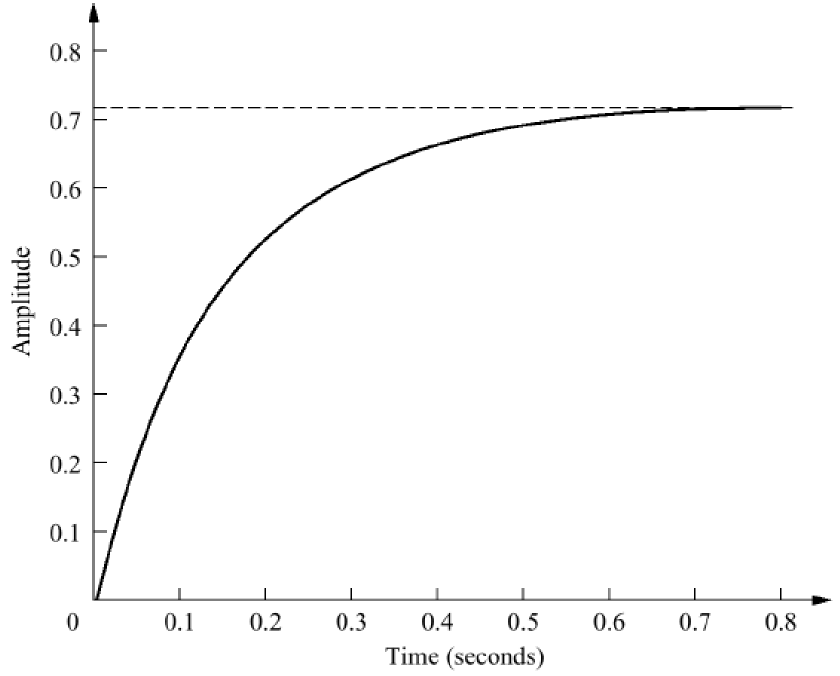
Significance of Response Time

- In some cases, it is hard to obtain a system's **transfer function analytically**.
- We could obtain the transfer function through experiments or testing.
- For example, a simple general first-order system would have a transfer function of

$$\frac{Y(s)}{R(s)} = \frac{K}{s+a} \; for\ step\ input \Rightarrow \frac{k/a}{s} - \frac{k/a}{s+a}$$

- After applying a unit step function, the output response waveform is obtained.

- From the output waveform we could determine the time constant when output rises to **63%** of its final value, which in this case is **0.63 x 0.72 =0.45.** This is about **0.13s** hence **a =1/0.13 = 7.7**



$$\frac{Y(s)}{R(s)} = \frac{K}{s+a} \Rightarrow \frac{5.54}{s+7.7}$$

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Characteristics of Second Order Response

- **A system with two poles.**
- When tested with **a unit step input**, the second-order system will give several types of an output response, which we can analyze.
- This will depend on the location of the system's poles.
- In a second-order system, the output **y(t)** and input **r(t)** are related by a differential equation of the form:

$$\frac{d^2y}{dt^2} + a_1\frac{d_y}{d_t} + a_0 y = b_1\frac{d_r}{d_t} + b_0 r \quad \rightarrow \quad \mathrm{T(s)} = \frac{Y(s)}{R(s)} = \underbrace{\frac{b_1 s + b_0}{s^2 + a_1 s + a_0}}_{\text{Zero state component}} + \underbrace{\frac{Initial\ Conditions}{s^2 + a_1 s + a_0}}_{\text{Zero input component}}$$

- This characteristic polynomial is.

$$s^2 + a_1 s + a_0 = 0 \quad \Rightarrow \quad roots\ \ s_1\ and\ s_2 = \frac{-a \pm \sqrt{a_1^2 - 4a_o}}{2}$$

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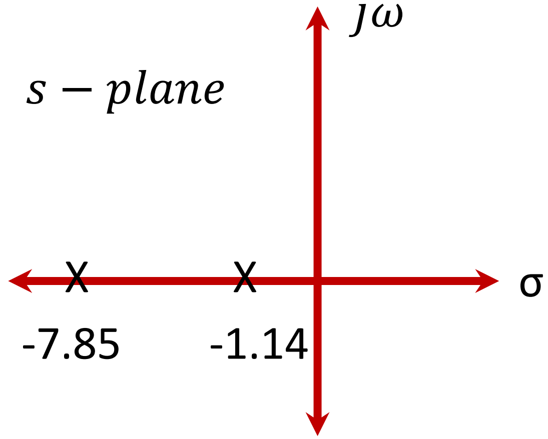
Over-damped Response

$R(s) = \frac{1}{s}$ → G(s): $\frac{b}{s^2 + as + b}$ → C(s)

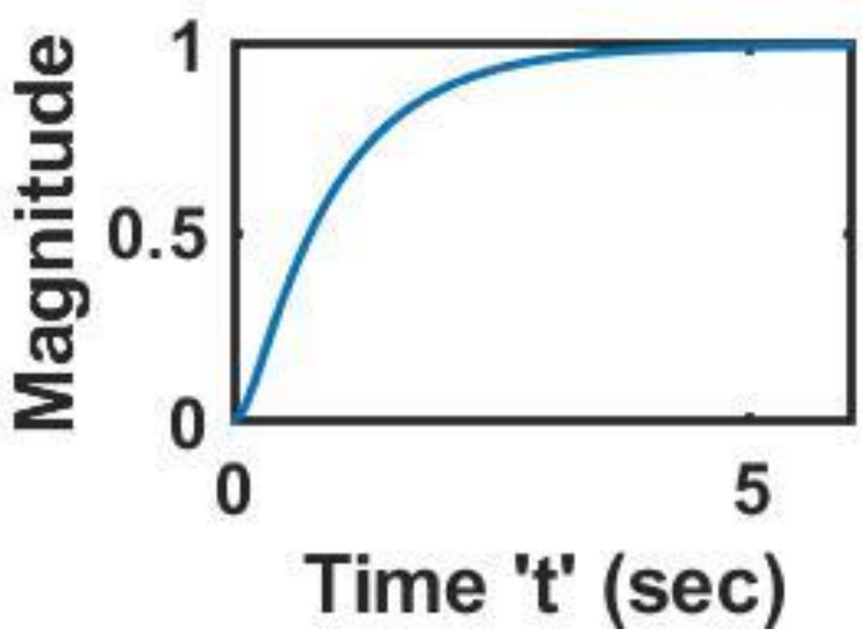
General representation

Roots of the characteristic polynomial are ***real and distinct***



$R(s) = \frac{1}{s}$ → G(s): $\frac{9}{s^2 + 9s + 9}$ → C(s)

Overdamped



$$\boxed{c(t) = 1 + 0.171e^{-7.854t} - 1.171e^{-1.146t}}$$

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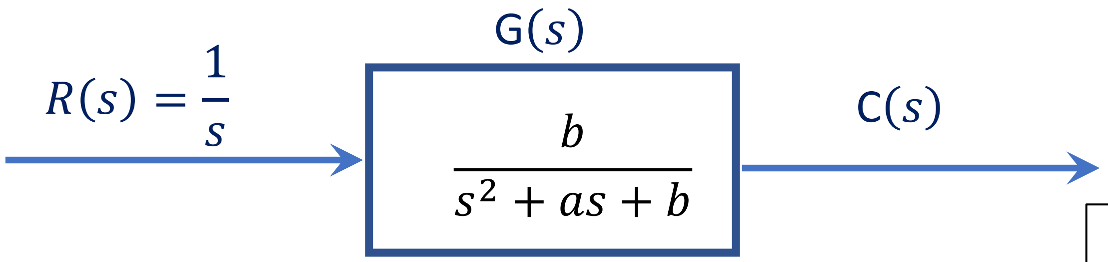
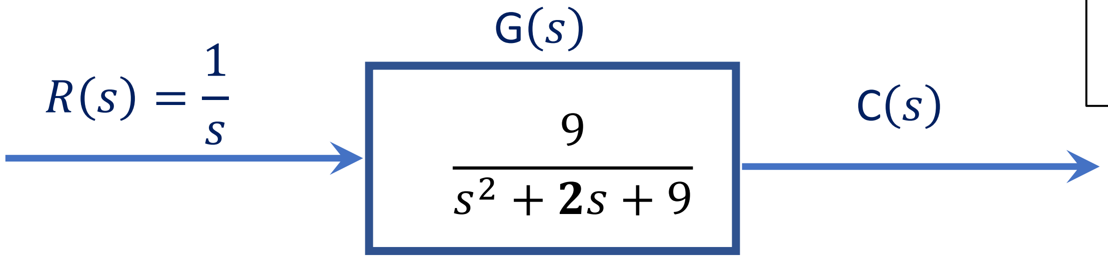
Under-damped Response

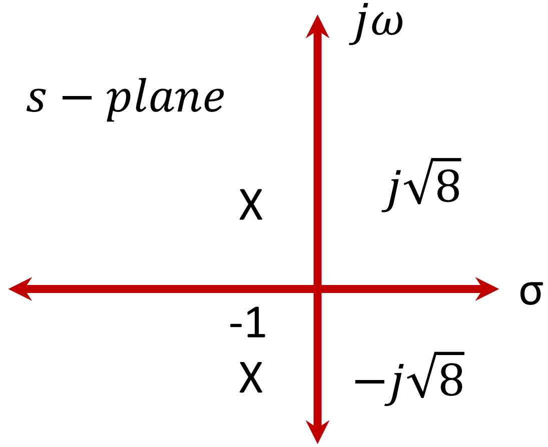


General representation



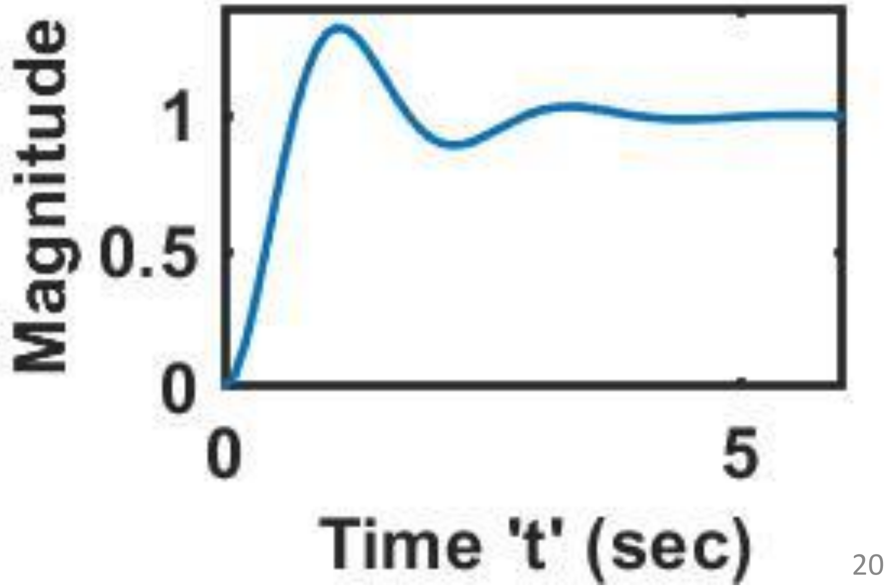
Underdamped

Roots of the characteristic polynomial are ***complex****.* *(real and imaginary part)*



$$c(t) = 1 - e^{-t}\left(cos\sqrt{8}t + \frac{\sqrt{8}}{8}sin\sqrt{8}t\right)$$

$$c(t) = 1 - 1.06e^{-t}\cos(\sqrt{8}t - 19.47^{\circ})$$



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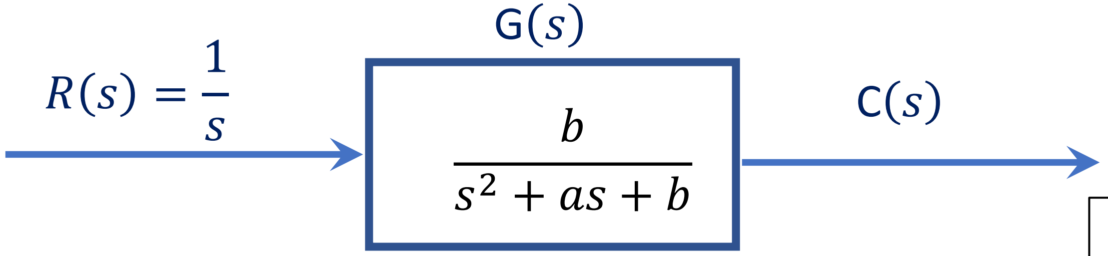
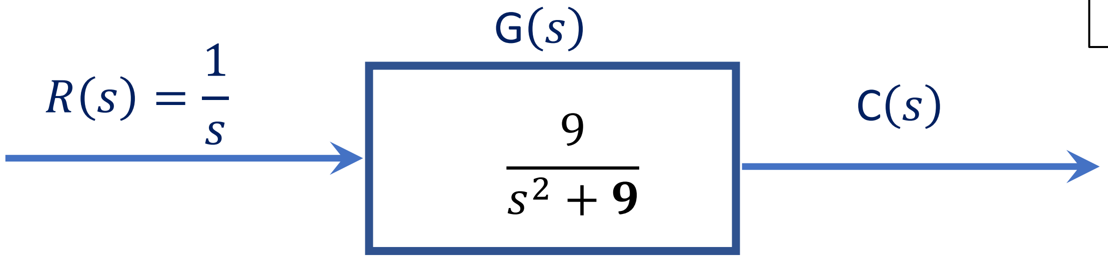
Un-damped Response



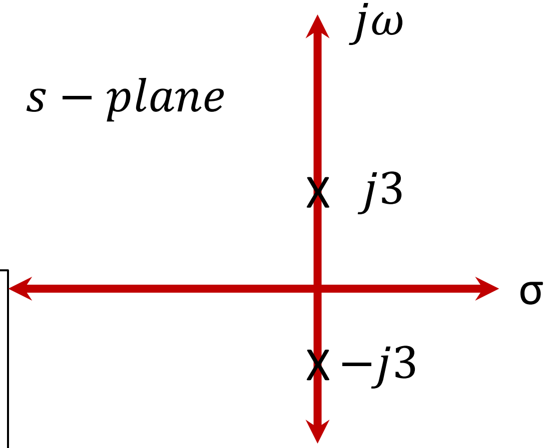
General representation

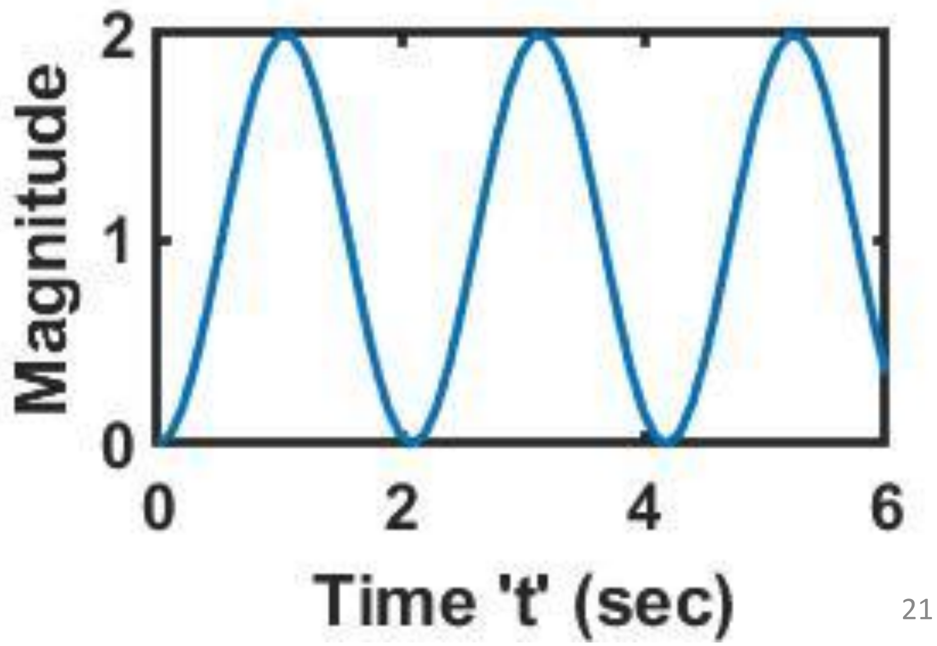


Undamped

$$\boxed{c(t) = 1 - cos3t}$$

Roots of the characteristic polynomial are ***imaginary***





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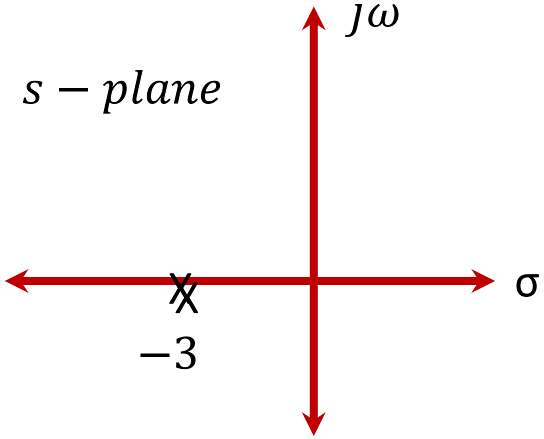
Critically-damped Response

$$R(s) = \frac{1}{s} \longrightarrow \boxed{G(s) = \frac{b}{s^2 + as + b}} \longrightarrow C(s)$$

General representation

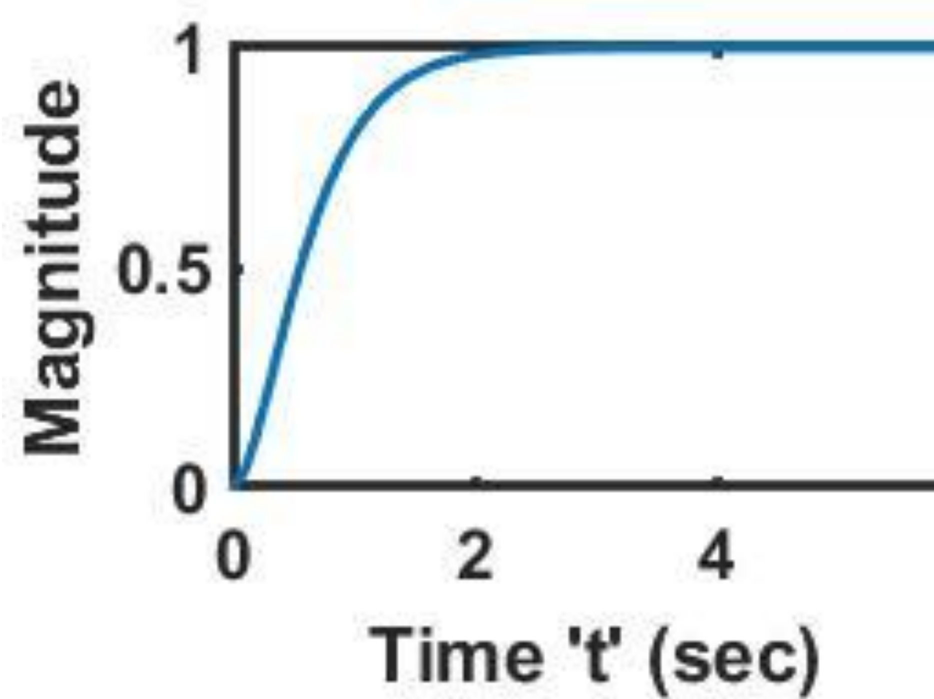
Roots of the characteristic polynomial ***are repeated real numbers***



$$R(s) = \frac{1}{s} \longrightarrow \boxed{G(s) = \frac{\mathbf{9}}{\boldsymbol{s^2 + 6s + 9}}} \longrightarrow C(s)$$

Critically damped

$$\boxed{c(t) = 1 - 3te^{-3t} - e^{-3t}}$$



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General Form

- The **second-order response** can be obtained from the general closed-loop transfer function.

$$\frac{Y(s)}{R(s)} = \frac{b}{s^2 + as + b} \Rightarrow \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

ζ (zeta) is referred to as:

- **damping ratio** of a second-order system, which is a measure of the degree of resistance to change in the system output.
  - $\boldsymbol{a = 2\zeta\omega_n \Rightarrow \zeta = \frac{a}{2\omega_n} = \frac{a}{2\sqrt{b}}}$
  - Equation for pole:

    $s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

    poles are complex if $\zeta < 1$

ω_n is referred to as:

- The ***undamped* natural frequency** of a second order system.
- Frequency of oscillations in the system without damping.
- $\boldsymbol{\omega_n{}^2 = b \quad \Rightarrow \quad \omega_n = \sqrt{b}}$

ω_n determines the systems output response

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General Form – Case 1

$$\frac{Y(s)}{R(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

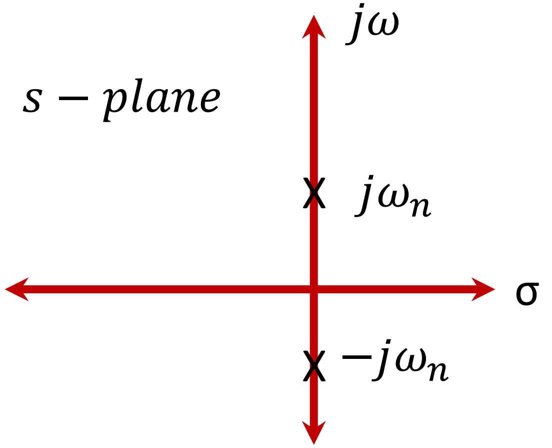
$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

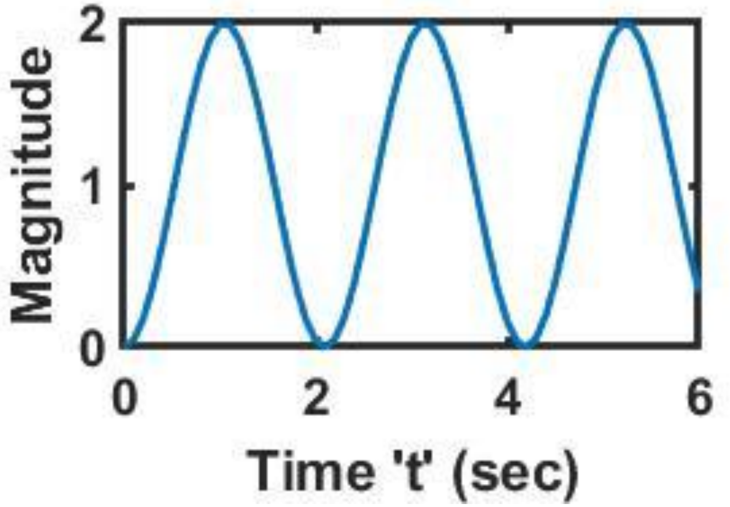
- For $\zeta = 0$ and $K = 1$.
- Roots $s_{1,2} = \pm j\omega_n$
- Applying unit step function at the input gives the output as:

$$Y(s) = \frac{\omega_n^2}{s^2 + \omega_n^2} \cdot \frac{1}{s}$$

$$y(t) = 1 - A\cos(\omega_n t - \emptyset)$$

- the output response is oscillating and in an **un-damped condition**





un-damped

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General Form – Case 2

- For $\mathbf{0 < \zeta < 1}$ and $\mathrm{K} = 1$.

$$s_{1,2} = -\boldsymbol{\zeta\omega_n} \pm \omega_n\sqrt{\boldsymbol{\zeta}^2 - 1}$$

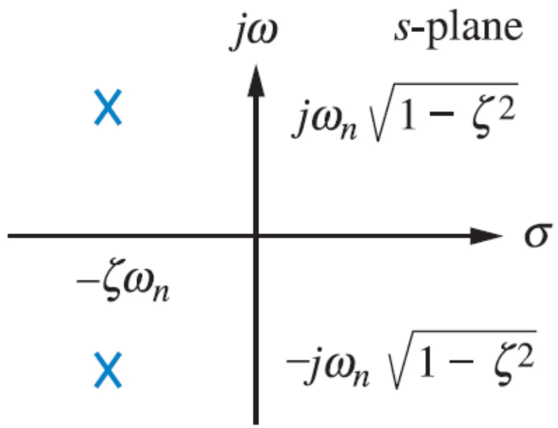
$$s_{1,2} = -\boldsymbol{\zeta\omega_n} \pm \omega_n\sqrt{1 - \boldsymbol{\zeta}^2}$$

- Applying unit step function at the input gives the output as:

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s}$$

$$y(t) = 1 - Ae^{-\sigma_d t}cos(\omega_d t - \phi)$$

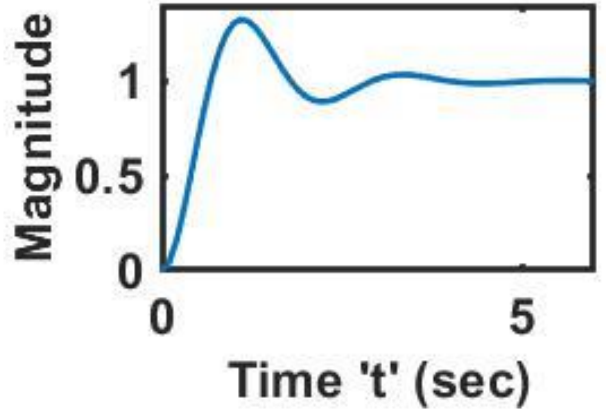
- the output response is oscillating and in an **under-damped** condition



- ❖ $-\sigma_d = \zeta\omega_n,$
- ❖ $\omega_d = \omega_n\sqrt{\boldsymbol{\zeta}^2 - 1}$
- ❖ $A = \frac{1}{\sqrt{1-\zeta^2}}$
- ❖ $\phi = tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$

$$\frac{Y(s)}{R(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$s_{1,2} = -\boldsymbol{\zeta\omega_n} \pm \omega_n\sqrt{\boldsymbol{\zeta}^2 - 1}$$



under-damped

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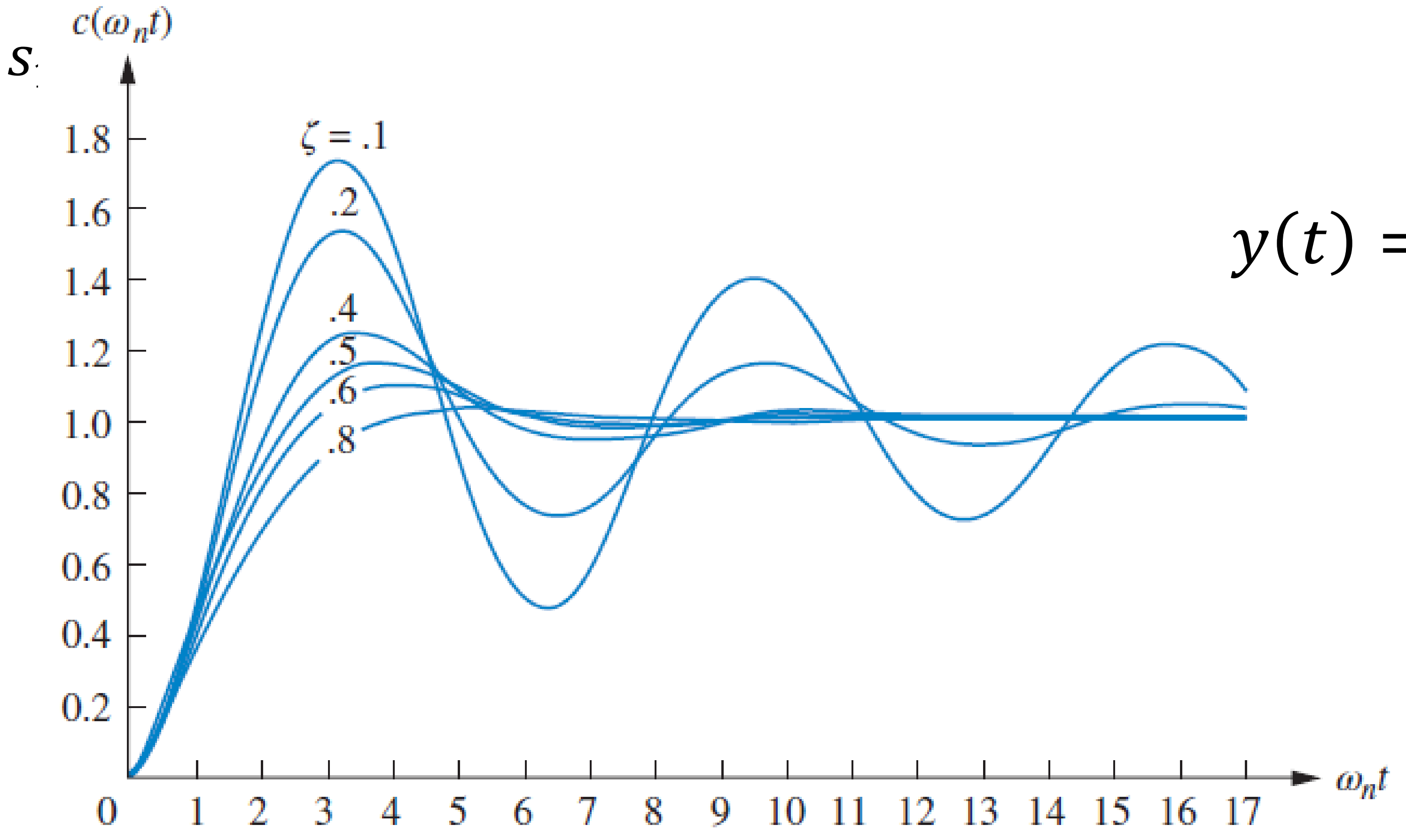
General Form – Case 2

- For $\mathbf{0} < \boldsymbol{\zeta} < \mathbf{1}$ and $\mathrm{K} = 1$.



oscillation (due to the sinusoidal term)

$$y(t) = 1 - Ae^{-\sigma_d t}\cos(\omega_d t - \theta)$$

Decay (due to the exponential term)

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General Form – Case 3

- For $\boldsymbol{\zeta} = \mathbf{1}$ and $\mathrm{K} = 1.$

  $s_{1,2} = -\boldsymbol{\omega_n}$

- Applying unit step function at the input gives the output as:

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s}$$

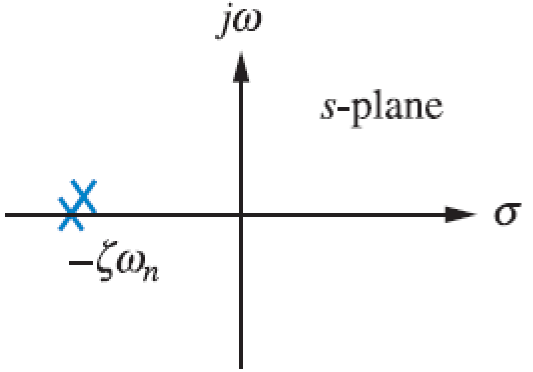
$$= \frac{\omega_n^2}{(s + \omega_n)^2} \cdot \frac{1}{s}$$

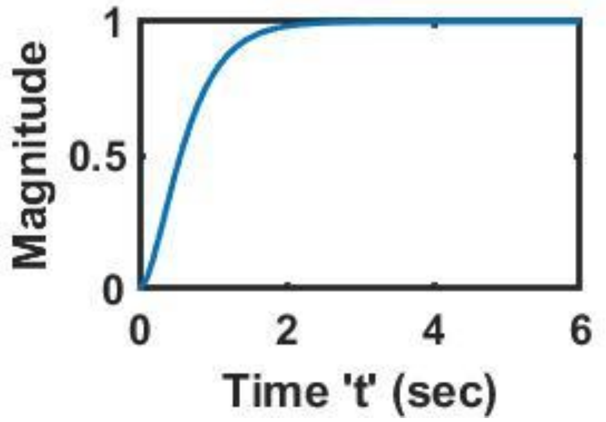
$$y(t) = 1 - K_1 e^{-\omega_n t} + K_2 t e^{-\omega_n t})$$

- the output response is non-oscillating and in **critically-damped** condition

$$\frac{Y(s)}{R(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$s_{1,2} = -\boldsymbol{\zeta}\boldsymbol{\omega_n} \pm \omega_n\sqrt{\boldsymbol{\zeta}^2 - 1}$$





Critically-damped

EE-379 Continuous Time Response

General Form – Case 4

- For $\zeta > 1$ and $K = 1$.

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

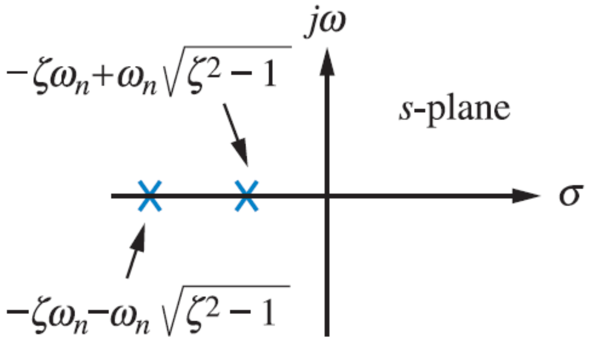
- Applying unit step function at the input gives the output as:

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s}$$

$$= \frac{\omega_n^2}{(s + \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1})(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})} \cdot \frac{1}{s}$$

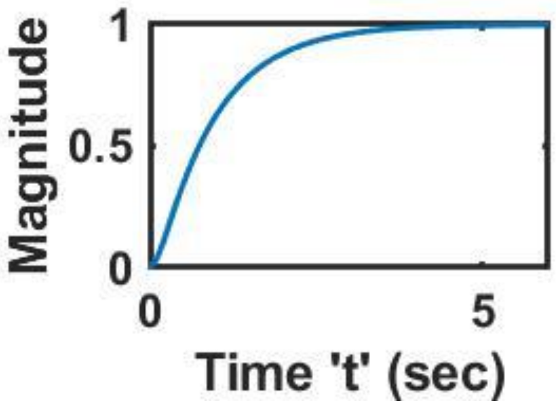
$$y(t) = 1 - (K_1 e^{-s_1 t} + K_2 e^{-s_2 t})$$

- the output response is non-oscillating and in **overdamped-damped** condition



$$\frac{Y(s)}{R(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

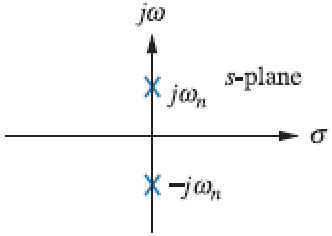
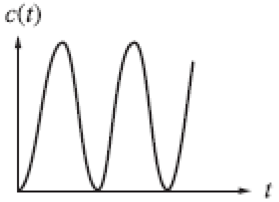
$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

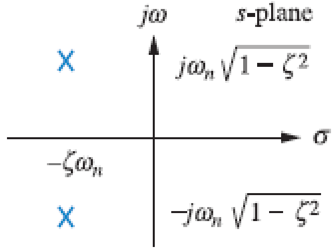
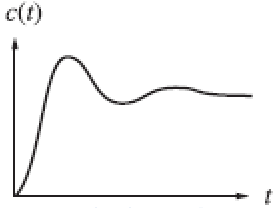


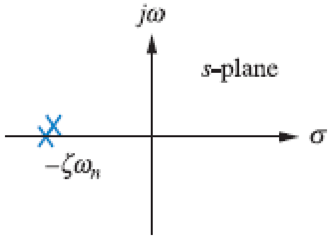
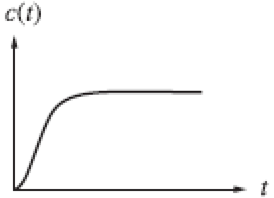
Over-damped

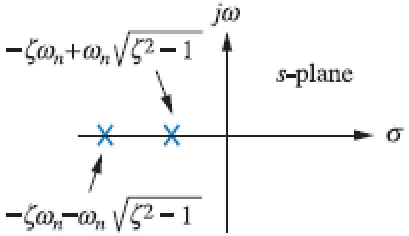
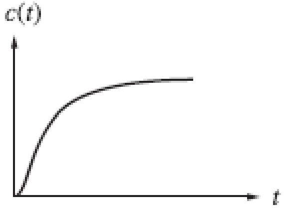
EE-379 Continuous Time Response

General Form – Summary of all cases

<table>
<tr><th>ζ</th><th>Poles</th><th>Step Response</th></tr>
<tr><td>$\zeta = 0$</td><td>
</td><td>

Undamped</td></tr>
<tr><td>$0 < \zeta < 1$</td><td>
</td><td>

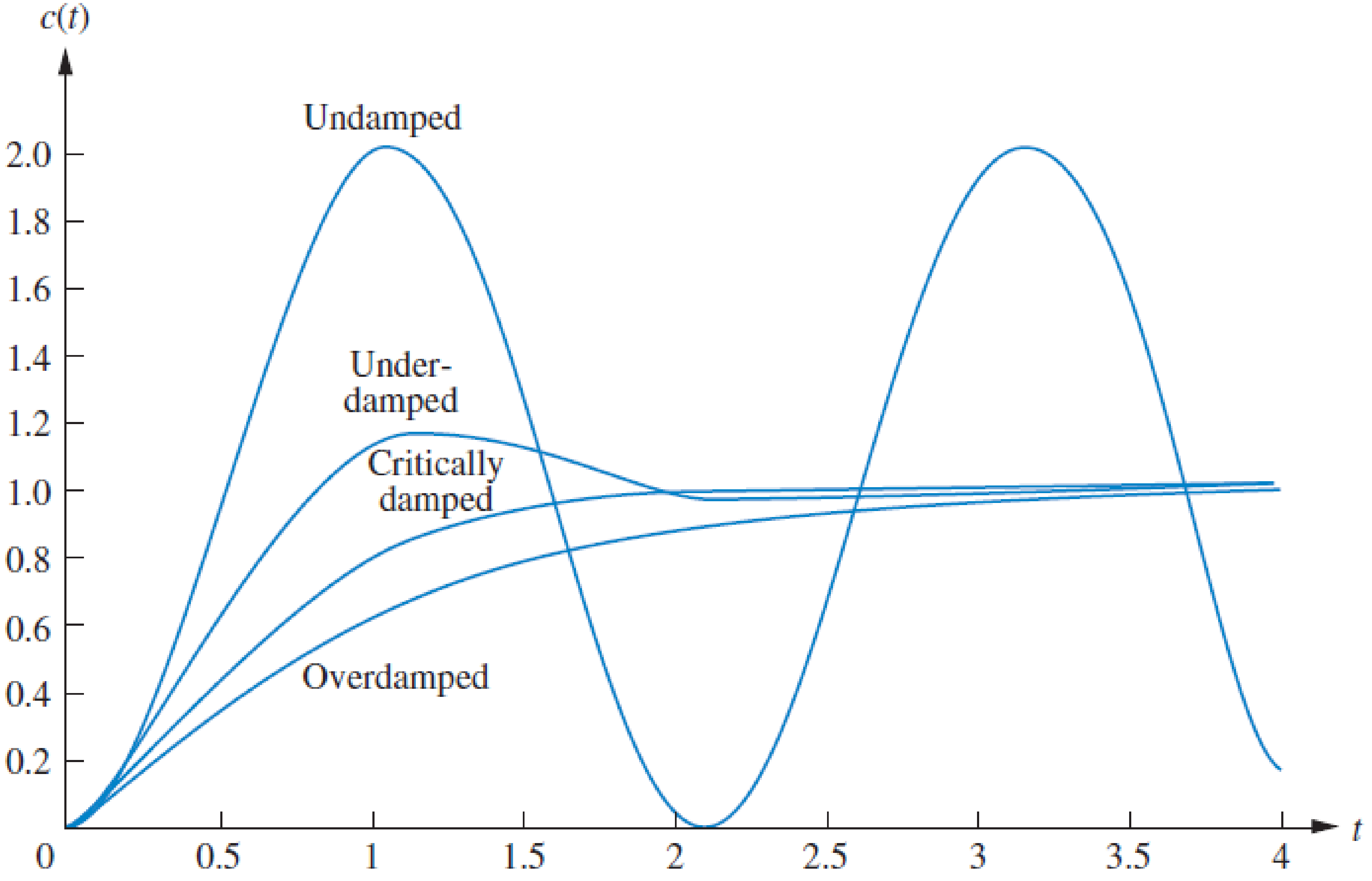
Underdamped</td></tr>
<tr><td>$\zeta = 1$</td><td>
</td><td>

Critically damped</td></tr>
<tr><td>$\zeta > 1$</td><td>
</td><td>

Overdamped</td></tr>
</table>

EE-379 Continuous Time Response

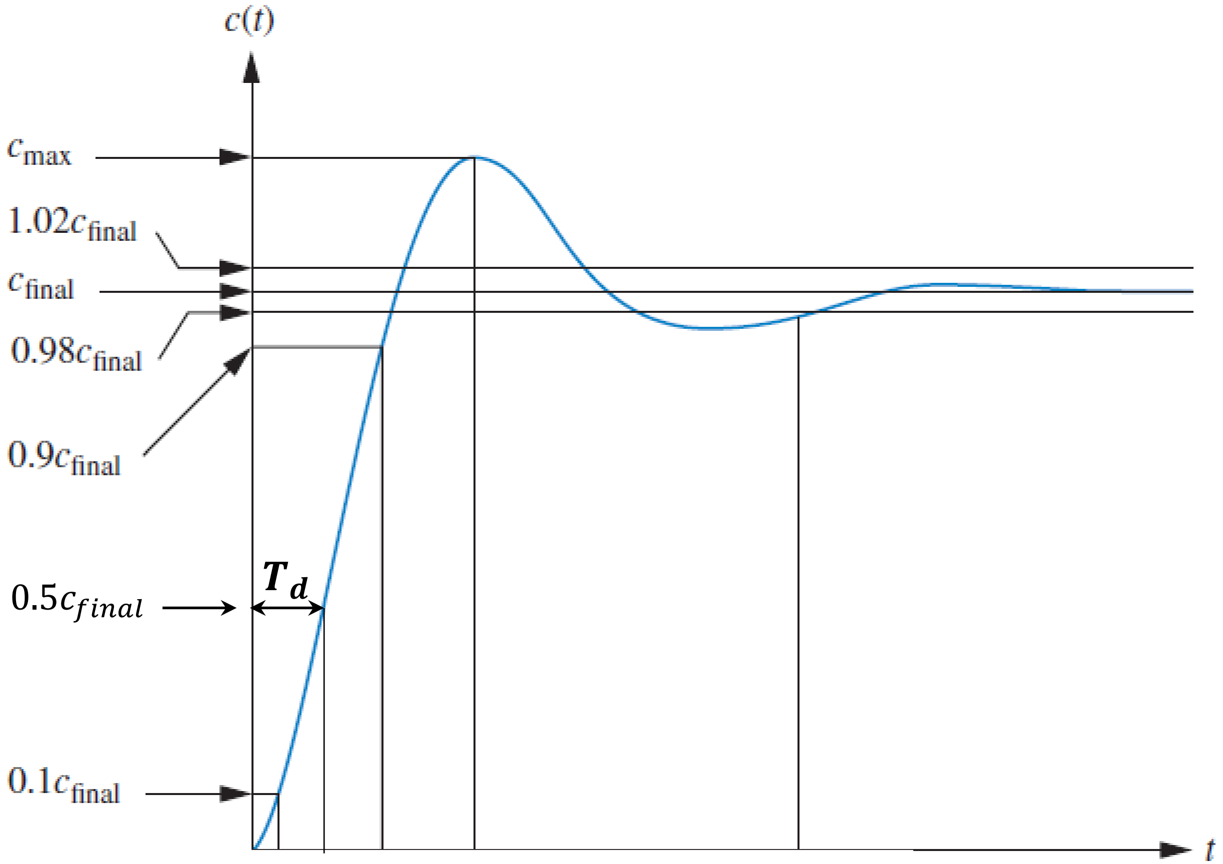
General Form – Summary of all cases



EE-379 Continuous Time Response

Important Terms – Delay Time

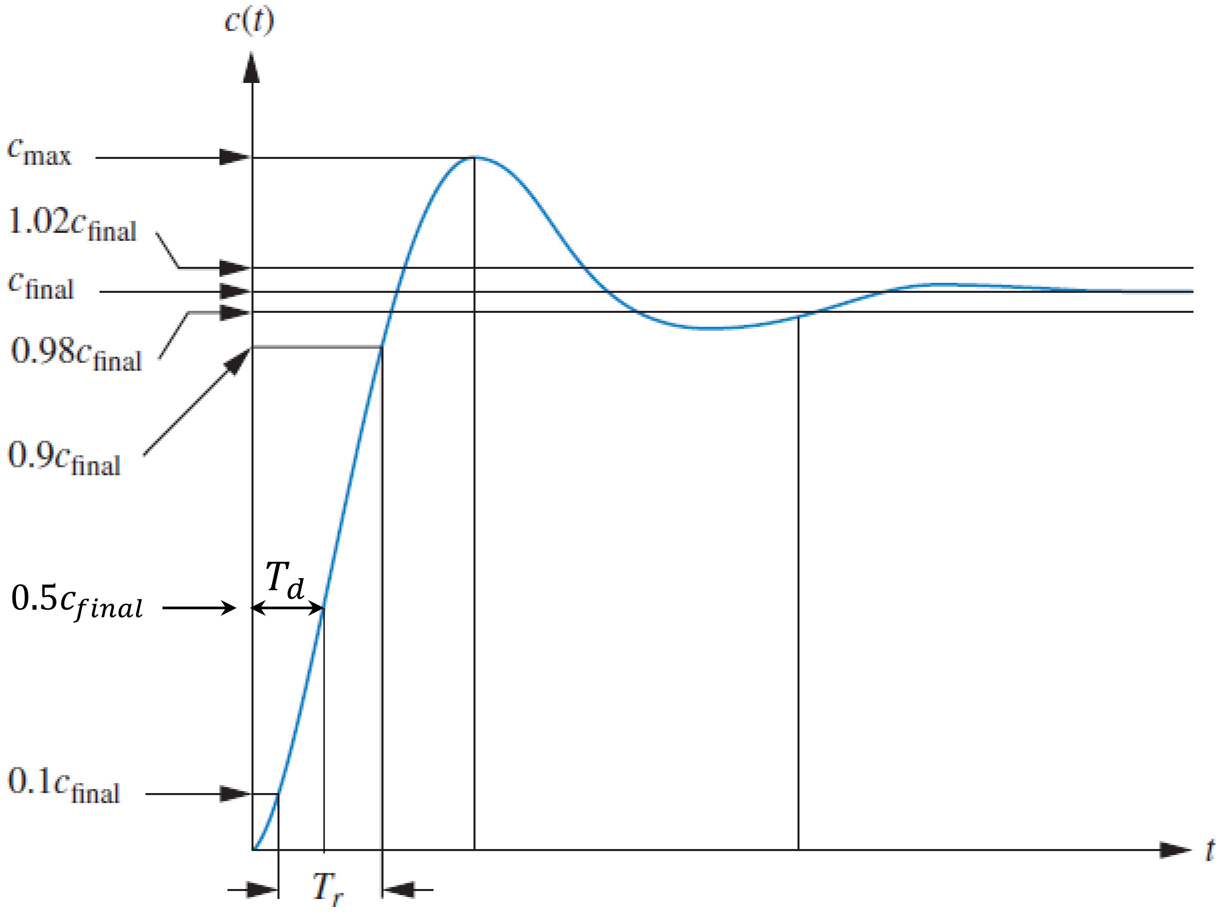
- $\boldsymbol{Delay\,Time,\ T_d}$: The time needed for the output response to reach **50% of the final output** value.



EE-379 Continuous Time Response

Important Terms – Rise Time

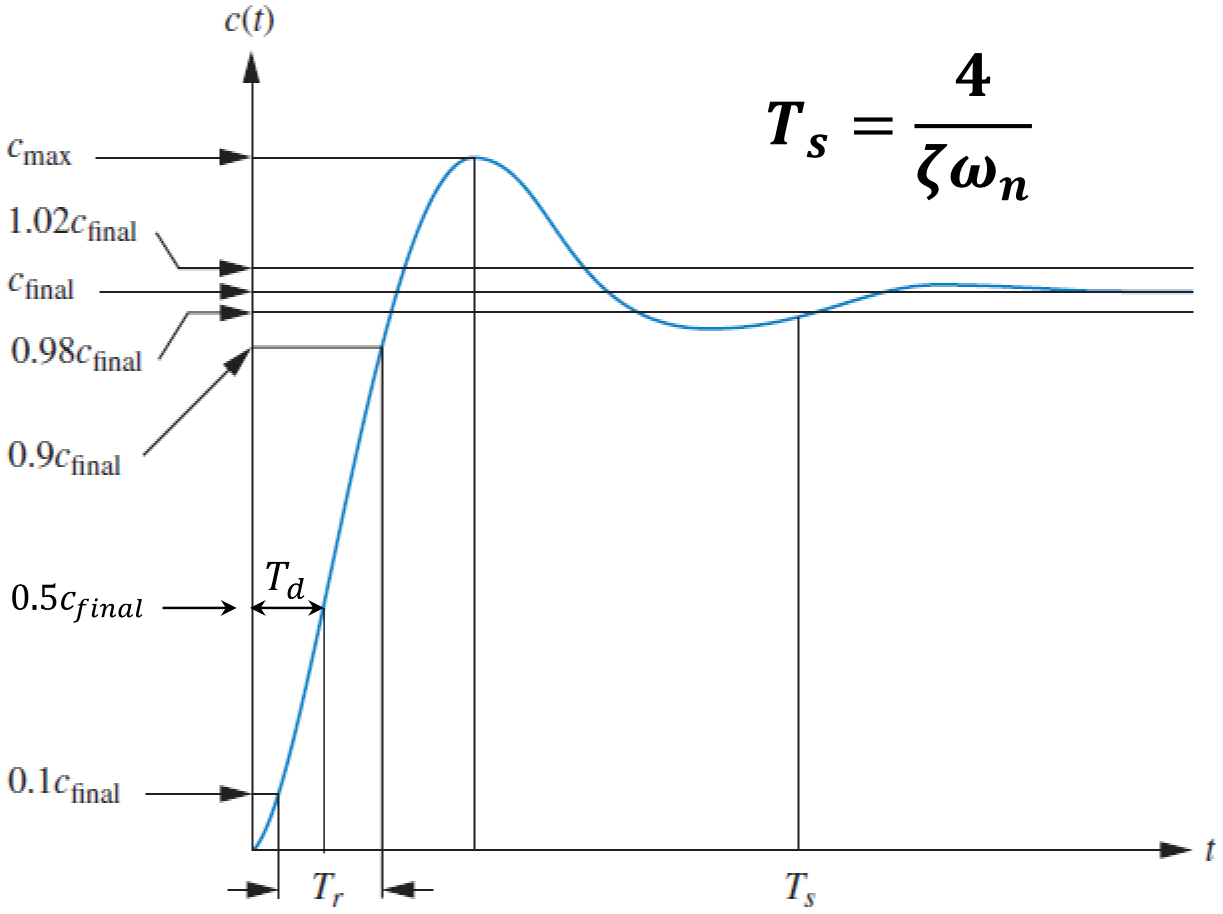
- ***Rise Time,*** $\boldsymbol{T_r}$: The time needed for the output response to go **from 10% to 90% of its final output value.**



EE-379 Continuous Time Response

Important Terms – Settling Time

- $\boldsymbol{Settling\ Time,\ T_s}$: The time taken for the output response to reach and **stay within 2% of its final output value.**

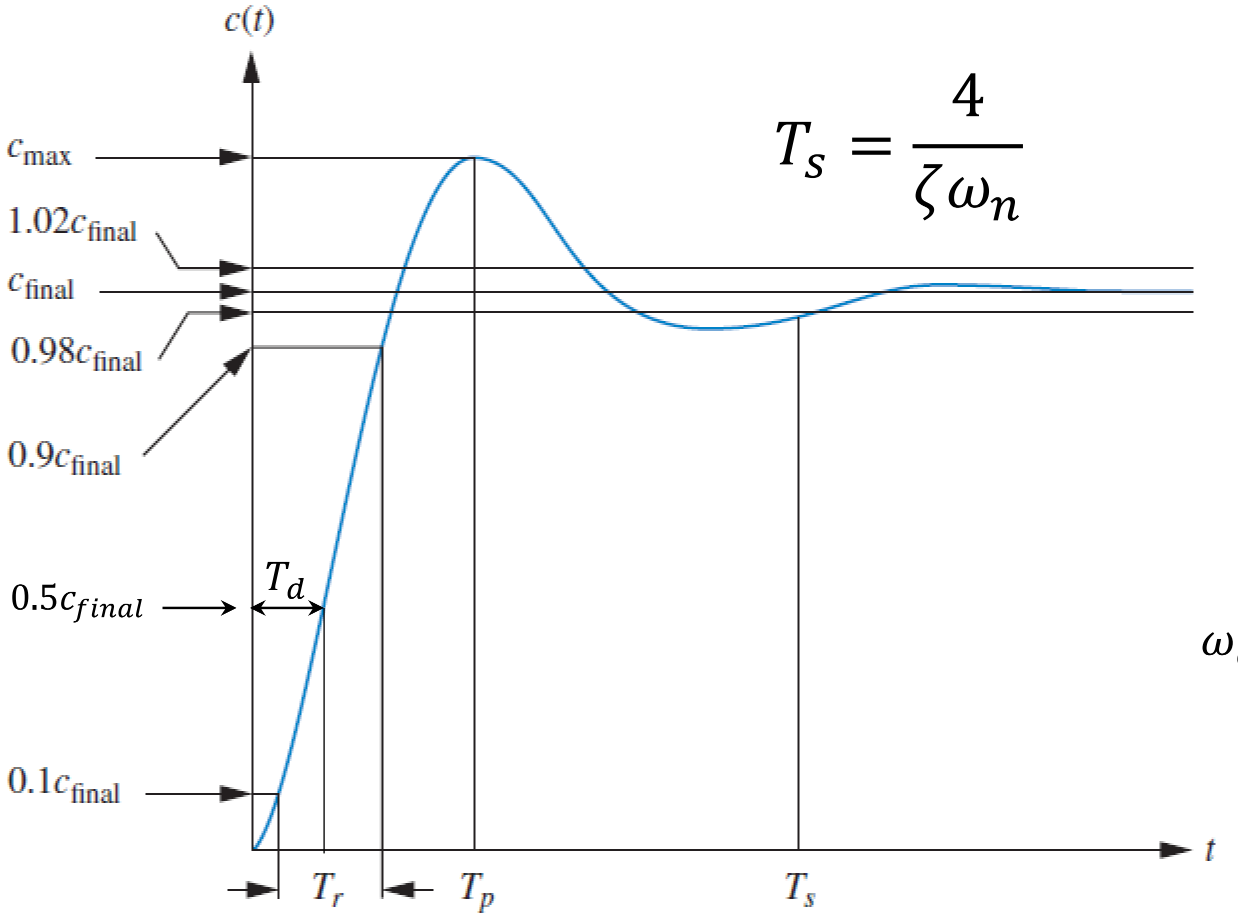


$$T_s = \frac{4}{\zeta\omega_n}$$

EE-379 Continuous Time Response

Important Terms – Peak Time

- ***Peak Time*, T_p:** The time required for the output response to reach the first or maximum peak.



$$T_s = \frac{4}{\zeta\omega_n}$$

$$\boldsymbol{T_p = \frac{\pi}{\omega_d}}$$

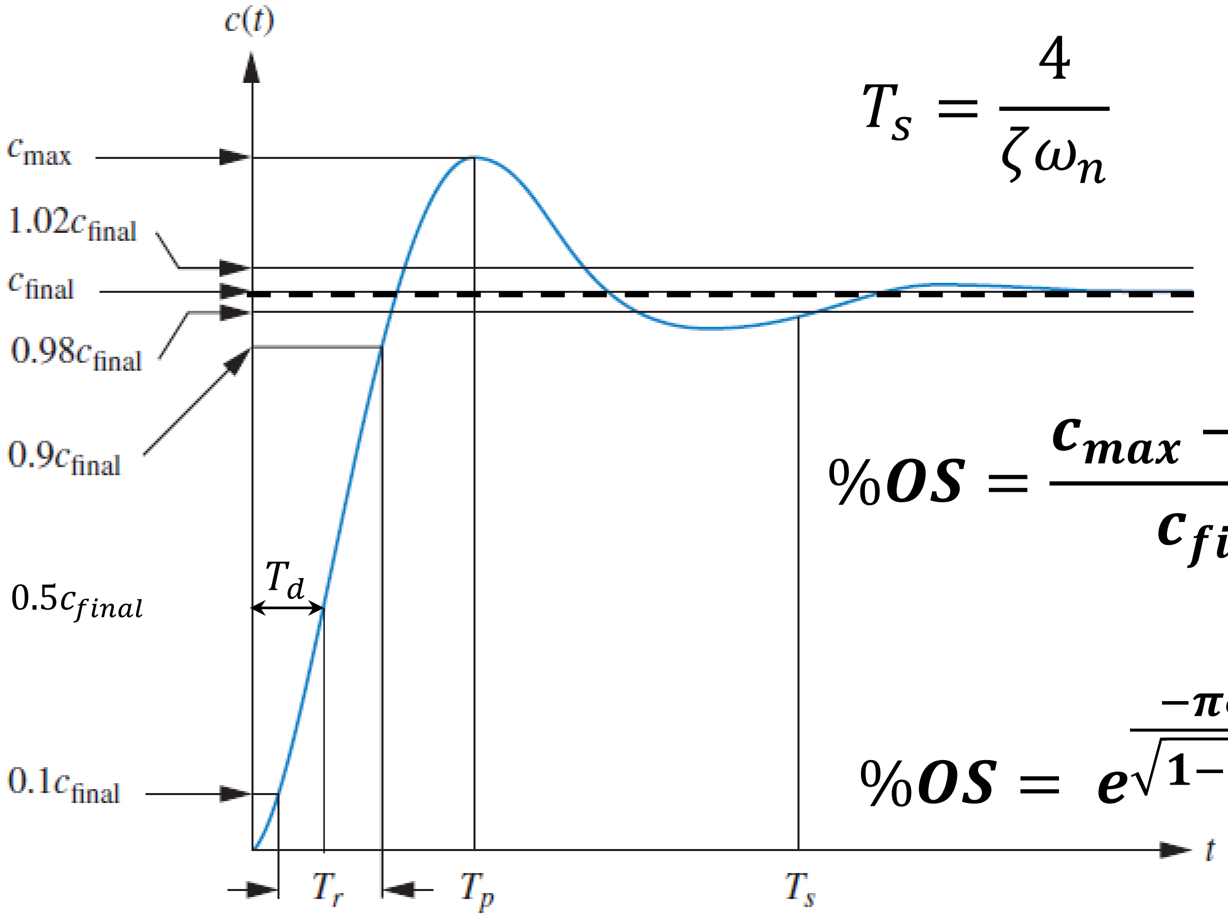
$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

$\omega_d = damped\ natural\ frequency$

EE-379 Continuous Time Response

Important Terms – Percentage Overshoot

- $\boldsymbol{\%age\ Overshoot,\ \%OS}$: The percentage difference between the maximum and the steady state values of the response.



$$T_s = \frac{4}{\zeta\omega_n}$$

$$T_p = \frac{\pi}{\omega_d}$$

$$\boldsymbol{\%OS = \frac{c_{max} - c_{final}}{c_{final}} \times 100}$$

$$\boldsymbol{\%OS = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100}$$

EE-379 Stability Analysis

Types of Stability

Asymptotic Stability

- A system is asymptotically stable when the **zero input response decays to zero** as time approaches ∞, for all possible initial conditions

Bounded Input Bounded Output BIBO) Stability

- A system is BIBO stable if, **for every bounded input, the output remains bounded** with increasing time (all system poles must lie in the left half of the s-plane or be canceled by zeros)

Marginal Stability

- A system is marginally stable if some of the poles lie on the imaginary axis, while all others are in the **LHS** of the s-plane. Some inputs may result in the output becoming unbounded with time.

EE-379 Stability Analysis

Test of Stability

- To test the stability of an LTI system we only **need to examine the poles** of the system, i.e. the roots of the characteristic equation.
- A first or second-order polynomial has **all roots in the LHP** if all polynomial **coefficients have the same algebraic sign**.
- Methods are available for testing for roots with positive real parts, which do not require the actual solution of the characteristic equation.
- Also, methods are available for testing the stability of a closed-loop system based only on the loop transfer function characteristics.
- For higher order systems:

$$3s^2 + s + 10$$

Stable system

$$3s^2 + s - 10$$

Unstable system

Properties of polynomial coefficients	Conclusion about roots
Differing algebraic signs	At least one RHP root
Zero-valued coefficients	Imaginary axis or RHP roots or both
All of the same algebraic sign none zero	No direct information

EE-379 Stability Analysis

Routh-Hurwitz Criterion

- A numerical procedure to determine **the numbers of RHP** and **imaginary axis (IA) roots of polynomial.**
- Assume the characteristic polynomial is:

$$Q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

$$where, a_0 \neq 0$$

- A necessary (but not sufficient) condition for all roots to have non-positive real parts is that all coefficients have the same sign.
- For the necessary and sufficient conditions, we first have to form the Routh Array.

EE-379 Stability Analysis

Routh-Hurwitz Criterion - Array

$$Q(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

$\boldsymbol{s^n}$	a_n	a_{n-2}	a_{n-4}	a_{n-6}	…
$\boldsymbol{s^{n-1}}$	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	…
$\boldsymbol{s^{n-2}}$	b_1	b_2	b_3	b_4	…
$\boldsymbol{s^{n-3}}$	c_1	c_2	c_3	c_4	…
:	:	:	:		
$\boldsymbol{s^2}$	k_1	k_2			
$\boldsymbol{s^1}$	l_1				
$\boldsymbol{s^0}$	m_1				

$$b_1 = \frac{-\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}}{a_{n-1}} = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}$$

$$b_2 = \frac{-\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}}{a_{n-1}} = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$

$$c_1 = \frac{-\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix}}{b_1} = \frac{b_1 a_{n-3} - b_2 a_{n-1}}{b_1}$$

$$c_2 = \frac{-\begin{vmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{vmatrix}}{b_1} = \frac{b_1 a_{n-5} - b_3 a_{n-1}}{b_1}$$

* Elements in the subsequent rows are calculated based on the two previous rows.

EE-379 Stability Analysis

Routh-Hurwitz Criterion

Necessary and sufficient conditions:

- **If all elements in the first column** of the Routh array have the same sign, then all roots of the characteristic equation have **negative real parts (LHP)**.

- If there are **sign changes** in these elements, then the number of roots with non-negative real parts is equal to the number of sign changes

- Elements in the first column which are zero define a special case.

EE-379 Stability Analysis

Routh-Hurwitz Criterion-Example 1

$$Q(s) = 2s^4 + s^3 + 3s^2 + 5s + 10$$

s^4	2	3	10	0
s^3	1	5	0	0
s^2	b_1	b_2	0	
s^1	c_1	0		
s^0	d_1			

$$b_1 = \frac{(1)(3) - (1)(10)}{1} = -7 \qquad c_1 = \frac{(-7)(5) - (1)(10)}{-7} = 6.43$$

$$b_2 = \frac{(1)(10) - (1)(0)}{1} = 10 \qquad d_1 = \frac{(6.43)(10) - (-7)(0)}{6.43} = 10$$

s^4	2	3	10
s^3	1	5	0
s^2	-7	10	0
s^1	6.43	0	
s^0	10		

- The characteristic equation has two roots with positive real parts since the elements of the first column have **two sign** changes. (2,1,**-7**,6.43,10)

EE-379 Stability Analysis

Routh-Hurwitz Criterion-Special Case I

- A zero in the first column:
- Remedy: substitute **ε** for the zero element, finish the Routh array, and then let **ε** → 0 .

$$Q(s) = s^3 - 3s + 2$$

$\boldsymbol{s^3}$	1	-3	0
$\boldsymbol{s^2}$	$0(\varepsilon)$	2	0
$\boldsymbol{s^1}$	$-2/\varepsilon$	0	
$\boldsymbol{s^0}$	2		

$$b_1 = \frac{(\varepsilon)(-3) - (1)(2)}{\epsilon} = \frac{-2}{\varepsilon} \quad (neagtive)$$

$$c_1 = \frac{(b_1)(2) - (\varepsilon)(0)}{b_1} = 2$$

- The characteristic equation has two roots with positive real parts since the elements of the first column have **two sign** changes. (1,ε,**-2/ε**,2)

EE-379 Stability Analysis

Routh-Hurwitz Criterion-Special Case II

- An all-zero row in the Routh array which corresponds to pairs of roots with opposite signs.
- Remedy:
  - **Form an auxiliary polynomial** from the coefficients in the **row above**.
  - Replace the zero coefficients from the coefficients of the differentiated auxiliary polynomial.
  - If there is no sign change, the roots of the auxiliary equation define the roots of the system on the imaginary axis.

EE-379 Stability Analysis

Routh-Hurwitz Criterion-Special Case II-Example I

$$Q(s) = s^4 + s^3 - s - 1$$

s^4	1	0	-1	0
s^3	1	-1	0	0
s^2	1	-1	0	
s^1	$\cancel{0}$ 2	0		
s^0	-1			

$$b_1 = \frac{(1)(0) - (1)(-1)}{1} = 1$$

$$b_2 = \frac{(1)(-1) - (1)(0)}{1} = -1$$

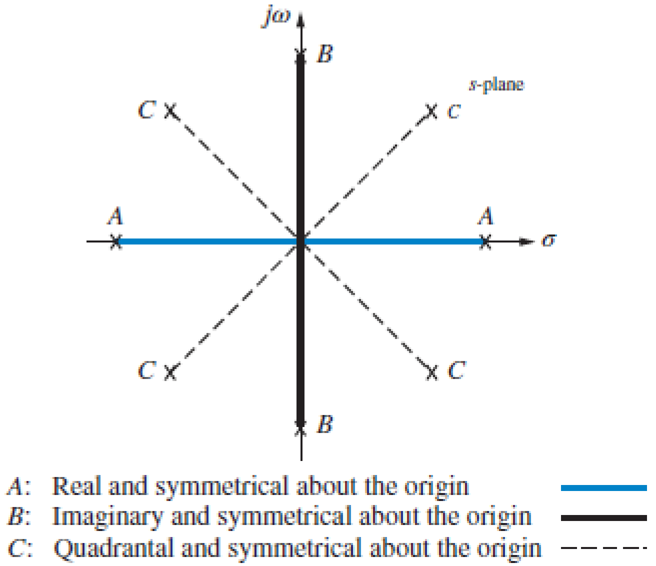
$$c_1 = \frac{(1)(-1) - (1)(-1)}{1} = 0$$

Auxiliary polynomial : $(s^2 - 1)$

$$\frac{d}{d(s)}(s^2 - 1) = 2s$$

$$d_1 = \frac{(2)(-1) - (1)(0)}{2} = \boxed{-1}$$

Concept of Symmetry



- Obtain the **auxiliary polynomial** from the row above the all zero row $\boldsymbol{s^2 - 1}$.
- Differentiate the **auxiliary polynomial** and replace the all zero row with the values obtained from the differentiation
- Complete the array.
- System has one root with a positive real part (1, 1, 1, 2, **-1**).
- The root is found from the auxiliary eq. $\boldsymbol{s^2 - 1 = 0}$, $\boldsymbol{s = \pm 1}$.

EE-379 Stability Analysis

Routh-Hurwitz Criterion-Special Case II-Example II

$$Q(s) = s^8 + s^7 + 12s^6 + 22s^5 + 39s^4 + 59s^3 + 48s^2 + 38s + 20\ (8^{th} - order)$$

s^8	1	12	39	48	20
s^7	1	22	59	38	0
s^6	$-\cancel{10}\ \ -1$	$-\cancel{20}\ \ -2$	$\cancel{10}\ 1$	20 2	0
s^5	$\cancel{20}\ 1$	$\cancel{60}\ 3$	$\cancel{40}\ 2$	0	0
s^4	1	3	2	0	
s^3	$0\ \cancel{4}\ 2$	$0\ \cancel{6}\ 3$	0		
s^2	$\frac{3}{\cancel{2}}\ 3$	$\cancel{2}\ 4$	0		
s^1	$\frac{1}{3}$	0			
s^0	4				

Auxiliary polynomial: $P(s) = s^4 + 3s^2 + 2$

$$\frac{d}{d(s)}P(s) = 4s^3 + 6s$$

Interpretation of the Routh Array

- s^4 is the auxiliary polynomial row.
- $\boldsymbol{s^4}$ *to* $\boldsymbol{s^0} \Rightarrow$ ***even polynomial case***
  - $\boldsymbol{s^4}$ *to* $\boldsymbol{s^0} \Rightarrow$ ***No sign change exist***
    - *no RHP roots and no LHP roots becasue of symmetery*
    - *Even polynomial = all 4 poles on imaginary axis*
- $\boldsymbol{s^8}$ *to* $\boldsymbol{s^4} \Rightarrow$ ***Remaining roots***
  - $\boldsymbol{s^8}$ *to* $\boldsymbol{s^4} \Rightarrow$ ***two sign changes***
    - *two RHP roots and two LHP roots becasue of symmetery*

EE-379 Stability Analysis

Routh-Hurwitz Criterion-Parameter Range

- The Routh Hurwitz stability criterion may be used to find the range of a parameter for which the closed-loop systems is stable.

- Leave the parameter as an unknown coefficient in the characteristic polynomial, form the Routh array, check the range of the parameter such that the first column does not change sign.

EE-379 Stability Analysis

Routh-Hurwitz Criterion-Parameter Range

$$Q(s) = s^4 + 6s^3 + 11s^2 + 6s + K$$

s^4	1	11	K	0
s^3	6	6	0	0
s^2	10	K	0	
s^1	c_1	0		
s^0	d_1			

$$b_1 = \frac{(6)(11) - (1)(6)}{6} = 10$$

$$c_1 = \frac{(10)(6) - (6)(K)}{10} = \frac{60 - 6K}{10}$$

$$b_2 = \frac{(6)(K) - (1)(0)}{6} = K$$

$$d_1 = \frac{(c_1)(K) - (10)(0)}{c_1} = K$$

- Then for stability:
  - c_1 *should be positive i – e.*, $60 - 6K > 0 \Rightarrow K < 10$
  - d_1 *should be positive i – e.*, $K > 0$
  - *Therefore*, $0 < K < 10$