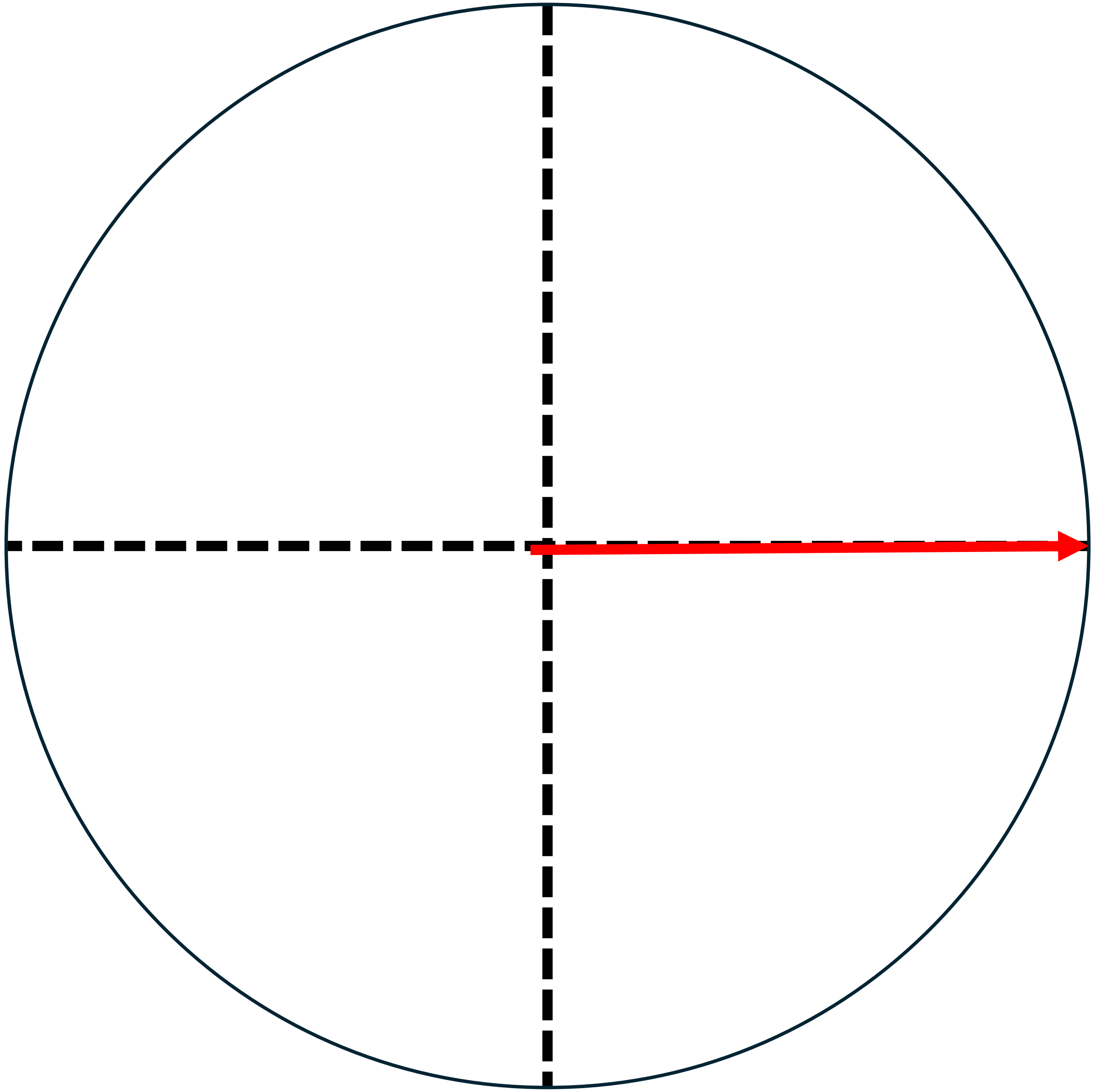
Why inverse of tangent gives better angle representation



$$Sin\beta = 0.5$$

$$\beta = 30°$$

$$\beta = 150°$$

To know the correct quadrant we need the value of $Cos\beta$ as well

$$Cos\beta = 0.866$$

$$\beta = Tan^{-1}\left(\frac{0.5}{0.866}\right) = 30°$$

$$Cos\beta = -0.866$$

$$\beta = Tan^{-1}\left(\frac{0.5}{-0.866}\right) = 150°$$

Equivalent angle-axis representations

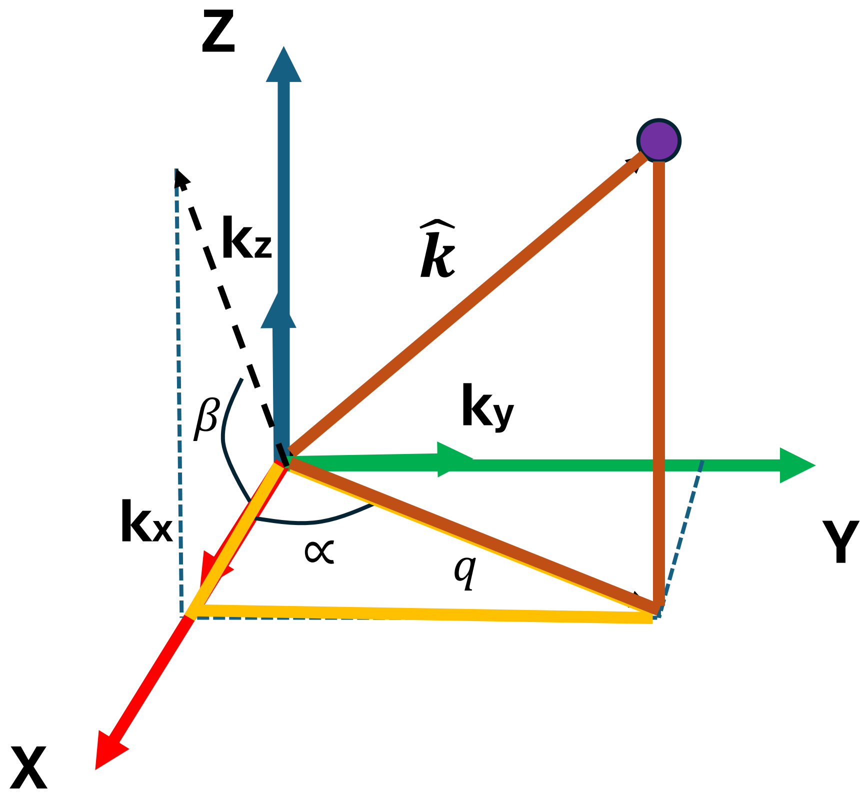
Start with the frame coincident with a known frame $\{A\}$; then rotate $\{B\}$ about the vector ${}^{A}\hat{K}$ by an angle θ according to the right-hand rule.

- Vector $\widehat{K}$ is sometimes called the equivalent axis of a finite rotation.

- A general orientation of {*B*} relative to {*A*} may be written as ${}^{A}_{B}R(\widehat{K},\theta)$ or $R_K(\theta)$ and will be called the equivalent angle–axis representation.

- The specification of the vector ${}^{A}\widehat{K}$ requires only two parameters, because its length is always taken to be one.

- The angle specifies a third parameter. Often, we will multiply the unit direction, $\widehat{K}$ with the amount of rotation, θ, to form a compact 3 × 1 vector description of orientation, denoted by *K* (no "hat'')

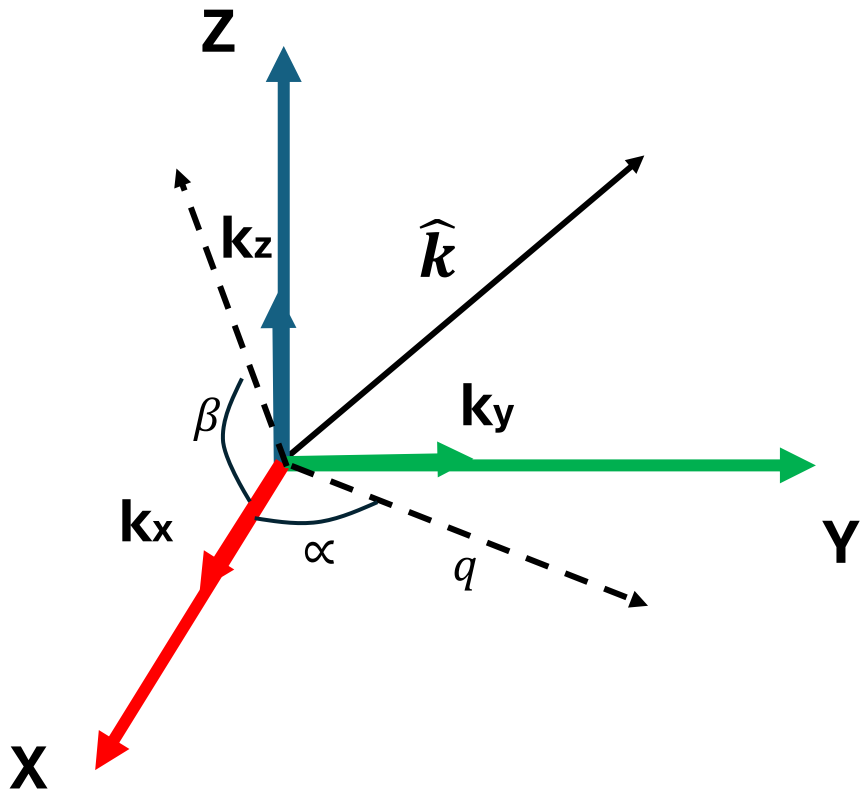


$$\hat{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} \qquad \hat{q} = \begin{bmatrix} k_x \\ k_y \\ 0 \end{bmatrix}$$

β is equal to the angle between $\hat{k}$ and $\hat{q}$

$$Cos(\propto) = \frac{k_x}{q} \qquad Sin(\propto) = \frac{k_y}{q}$$

$$Cos(\beta) = \frac{q}{k} = q \qquad Sin(\beta) = \frac{k_z}{k} = q$$



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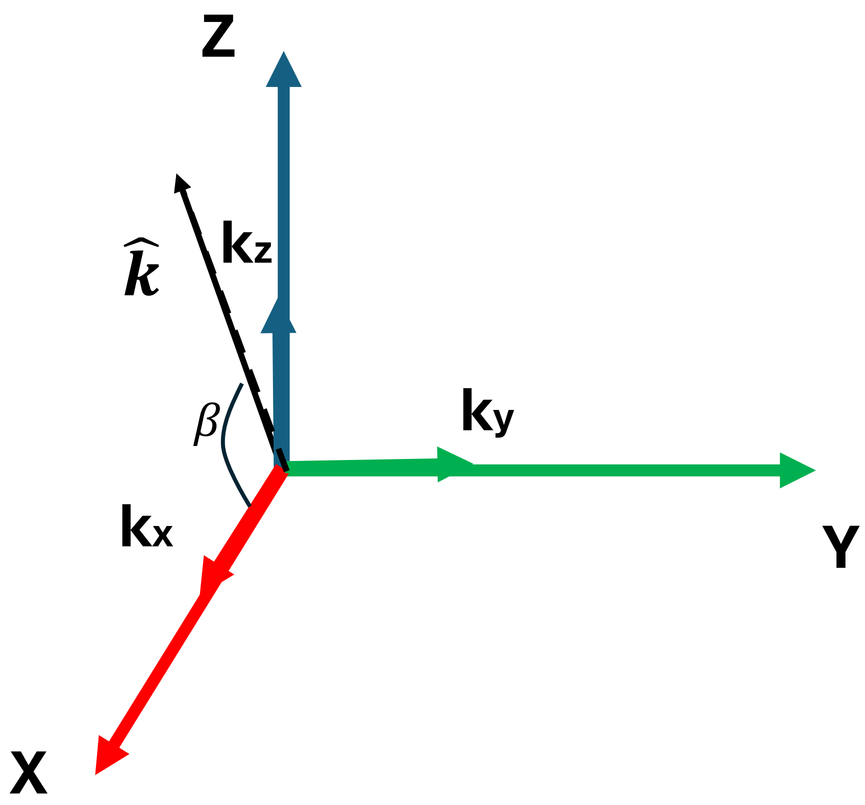

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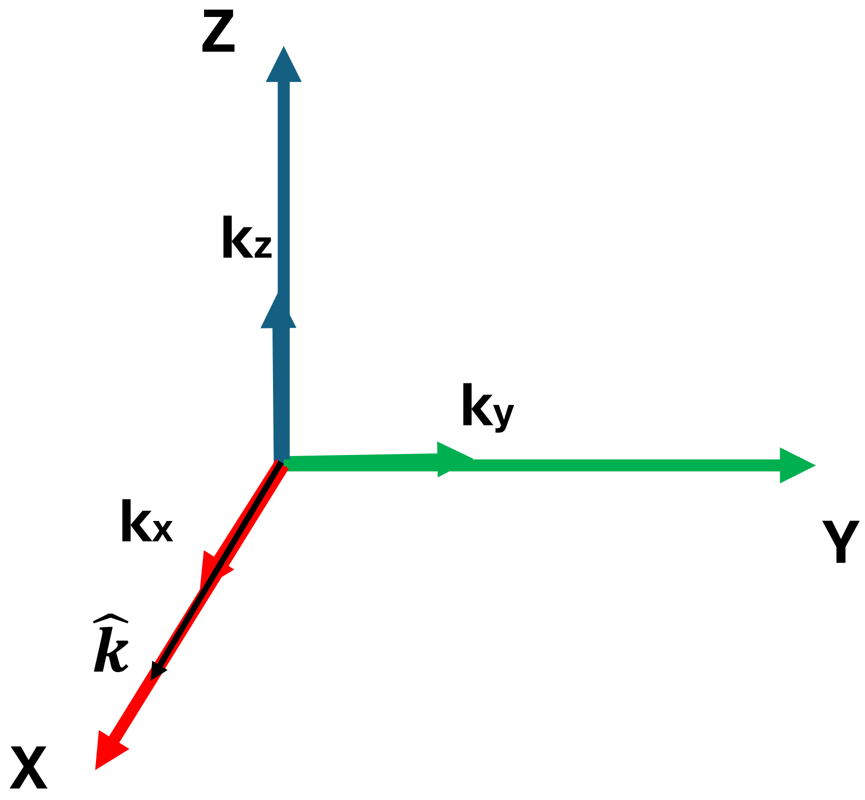
$$R = R_z(-\propto)$$



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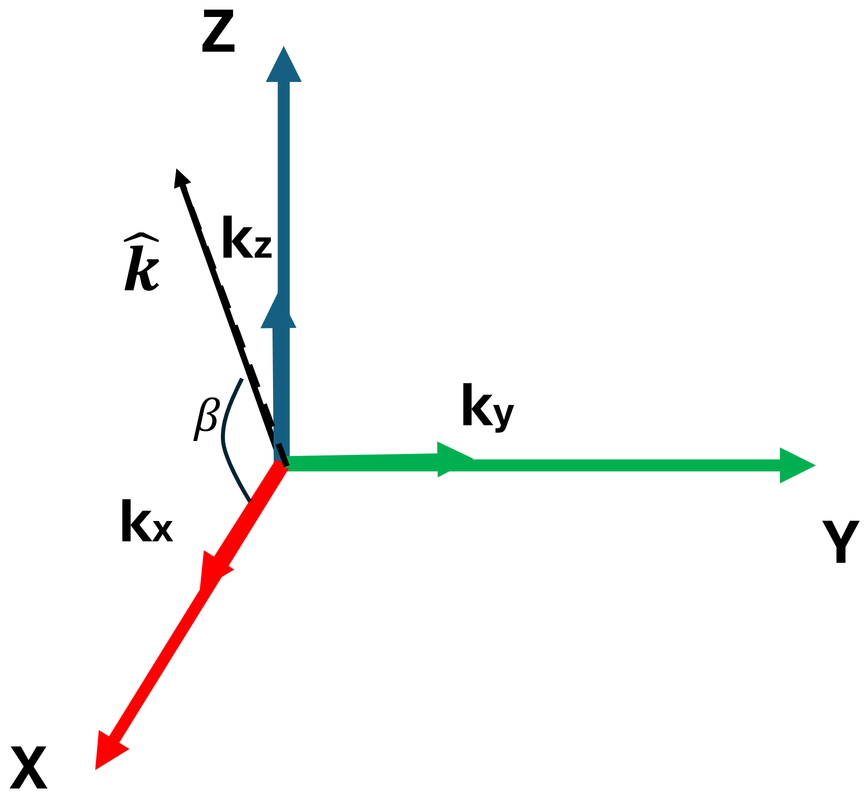
$$R = R_y(\beta)R_z(-\propto)$$
$$R = R_x(\theta)R_y(\beta)R_z(-\propto)$$



$$\hat{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} \qquad \hat{q} = \begin{bmatrix} k_x \\ k_y \\ 0 \end{bmatrix}$$

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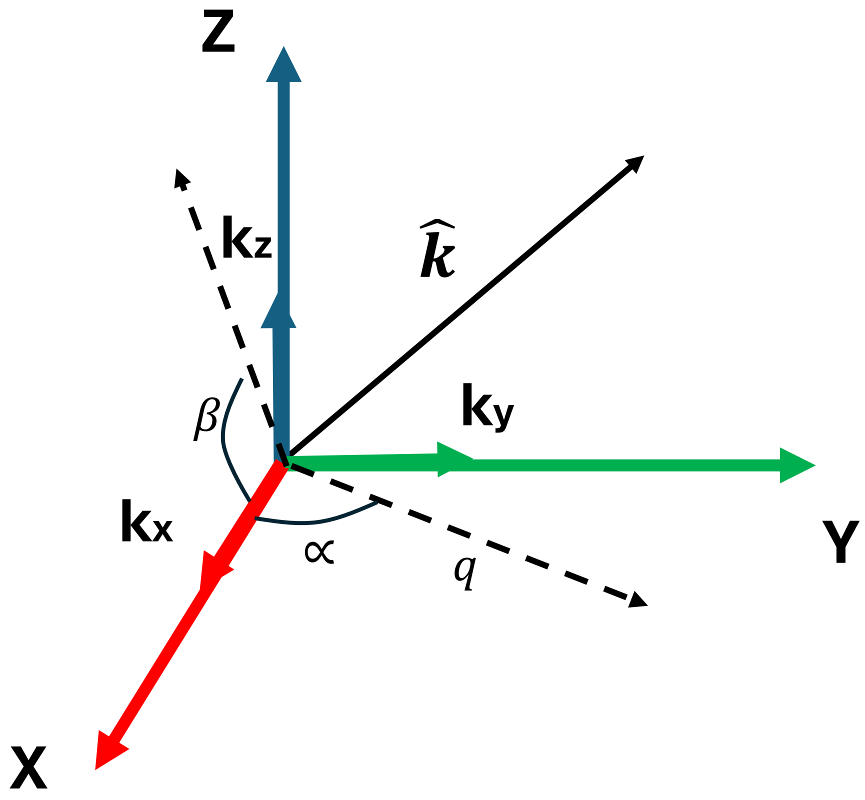
$$R = R_y(-\beta)R_x(\theta)R_y(\beta)R_z(-\propto)$$

$$\hat{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} \qquad \hat{q} = \begin{bmatrix} k_x \\ k_y \\ 0 \end{bmatrix}$$



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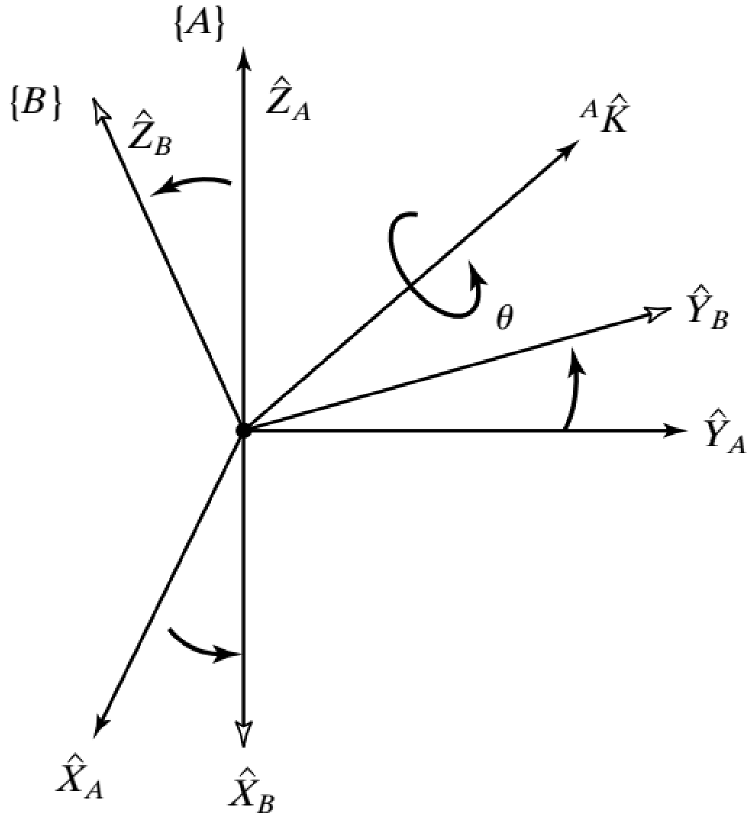
$$R = R_z(\propto)R_y(-\beta)R_x(\theta)R_y(\beta)R_z(-\propto)$$

Equivalent angle-axis representations



$$R_k(\theta) = R_z(\propto)R_y(-\beta)R_x(\theta)R_y(\beta)R_z(-\propto)$$

$$R_K(\theta) = \begin{bmatrix} k_x k_x v\theta + c\theta & k_x k_y v\theta - k_z s\theta & k_x k_z v\theta + k_y s\theta \\ k_x k_y v\theta + k_z s\theta & k_y k_y v\theta + c\theta & k_y k_z v\theta - k_x s\theta \\ k_x k_z v\theta - k_y s\theta & k_y k_z v\theta + k_x s\theta & k_z k_z v\theta + c\theta \end{bmatrix}$$

where $c\theta = \cos\theta$, $s\theta = \sin\theta$, $v\theta = 1 - \cos\theta$, and ${}^A\hat{K} = [k_x k_y k_z]^T$. The sign of θ is determined by the right-hand rule, with the thumb pointing along the positive sense of ${}^A\hat{K}$.

Equivalent angle-axis representations Inverse

$$R_K(\theta) = \begin{bmatrix} k_x k_x v\theta + c\theta & k_x k_y v\theta - k_z s\theta & k_x k_z v\theta + k_y s\theta \\ k_x k_y v\theta + k_z s\theta & k_y k_y v\theta + c\theta & k_y k_z v\theta - k_x s\theta \\ k_x k_z v\theta - k_y s\theta & k_y k_z v\theta + k_x s\theta & k_z k_z v\theta + c\theta \end{bmatrix}$$

To calculate inverse, we first need to find the trace of the matrix. Trace is the sum of the diagonals of the square matrix.

$$Trace = (k_x^2 v\theta + c\theta) + (k_y^2 v\theta + c\theta) + (k_z^2 \, v\theta + c\theta)$$

$$Trace = (k_x^2 + k_y^2 + k_z^2) v\theta + 3c\theta)$$

Since $\hat{k}$ is a unit vector, $k_x^2 + k_y^2 + k_z^2 = 1$.

$$Trace = 1 - c\theta + 3c\theta$$

$$Trace = 1 + 2c\theta$$

$$\theta = \text{Acos}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

$$\hat{K} = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

Equivalent angle-axis representation

Example

- $\hat{K} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$

K axis is along the standard x axis of the body. Find the equivalent transformation matrix using Equivalent angle-axis representation

$$R_K(\theta) = \begin{bmatrix} k_x k_x v\theta + c\theta & \cancel{k_x k_y v\theta} - \cancel{k_z s\theta} & \cancel{k_x k_z v\theta} + \cancel{k_y s\theta} \\ \cancel{k_x k_y v\theta} + \cancel{k_z s\theta} & \cancel{k_y k_y v\theta} + c\theta & \cancel{k_y k_z v\theta} - k_x s\theta \\ \cancel{k_x k_z v\theta} - \cancel{k_y s\theta} & \cancel{k_y k_z v\theta} + k_x s\theta & \cancel{k_z k_z v\theta} + c\theta \end{bmatrix}$$

$$R_K(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix}$$

Equivalent angle-axis representations

$$
{}^{A}_{B}R_K(\theta) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix},
$$

$$
\theta = \mathrm{Acos}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right) \qquad \hat{K} = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}
$$

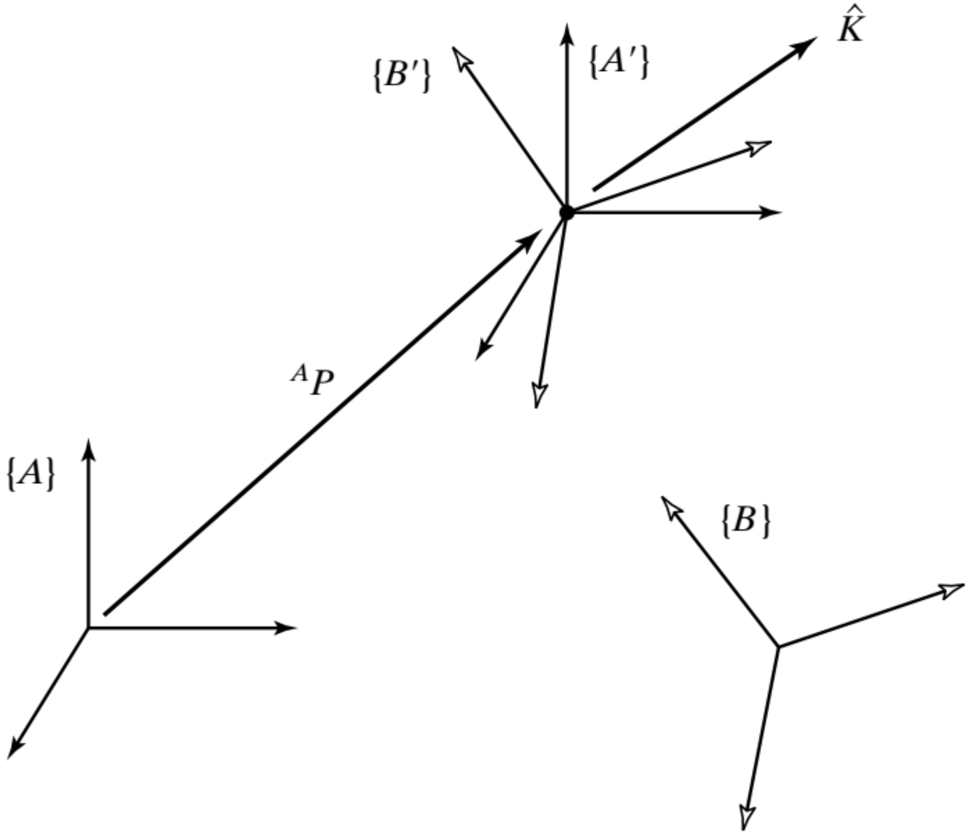
Example

A frame $\{B\}$ is described as initially coincident with $\{A\}$. We then rotate $\{B\}$ about the vector ${}^{A}\hat{K} = [0.707\ 0.707\ 0.0]^T$ (passing through the point ${}^{A}P = [1.0\ 2.0\ 3.0]$) by an amount $\theta = 30$ degrees. Give the frame description of $\{B\}$.

A frame $\{B\}$ is described as initially coincident with $\{A\}$. We then rotate $\{B\}$ about the vector ${}^A\hat{K} = [0.707\ 0.707\ 0.0]^T$ (passing through the point ${}^AP = [1.0\ 2.0\ 3.0]$) by an amount $\theta = 30$ degrees. Give the frame description of $\{B\}$.

Before the rotation, $\{A\}$ and $\{B\}$ are coincident. As is shown in Fig. 2.20, we define two new frames, $\{A'\}$ and $\{B'\}$, which are coincident with each other and have the same orientation as $\{A\}$ and $\{B\}$ respectively, but are translated relative to $\{A\}$ by an offset that places their origins on the axis of rotation. We will choose

$$
{}^{A}_{A'}T = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 1.0 \\ 0.0 & 1.0 & 0.0 & 2.0 \\ 0.0 & 0.0 & 1.0 & 3.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}
$$



Similarly, the description of $\{B\}$ in terms of $\{B'\}$ is

$$
{}^{B'}_{B}T = \begin{bmatrix} 1.0 & 0.0 & 0.0 & -1.0 \\ 0.0 & 1.0 & 0.0 & -2.0 \\ 0.0 & 0.0 & 1.0 & -3.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}
$$

Now, keeping other relationships fixed, we can rotate $\{B'\}$ relative to $\{A'\}$. This is a rotation about an axis that passes through the origin, so we can use (2.80) to compute $\{B'\}$ relative to $\{A'\}$. Substituting into (2.80) yields the rotation-matrix part of the frame description. There was no translation of the origin, so the position vector is $[0, 0, 0]^T$. Thus, we have

$$
{}^{A'}_{B'}T = \begin{bmatrix} 0.933 & 0.067 & 0.354 & 0.0 \\ 0.067 & 0.933 & -0.354 & 0.0 \\ -0.354 & 0.354 & 0.866 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}
$$

Finally, we can write a transform equation to compute the desired frame,

$$
{}^{A}_{B}T = {}^{A}_{A'}T \; {}^{A'}_{B'}T \; {}^{B'}_{B}T,
$$

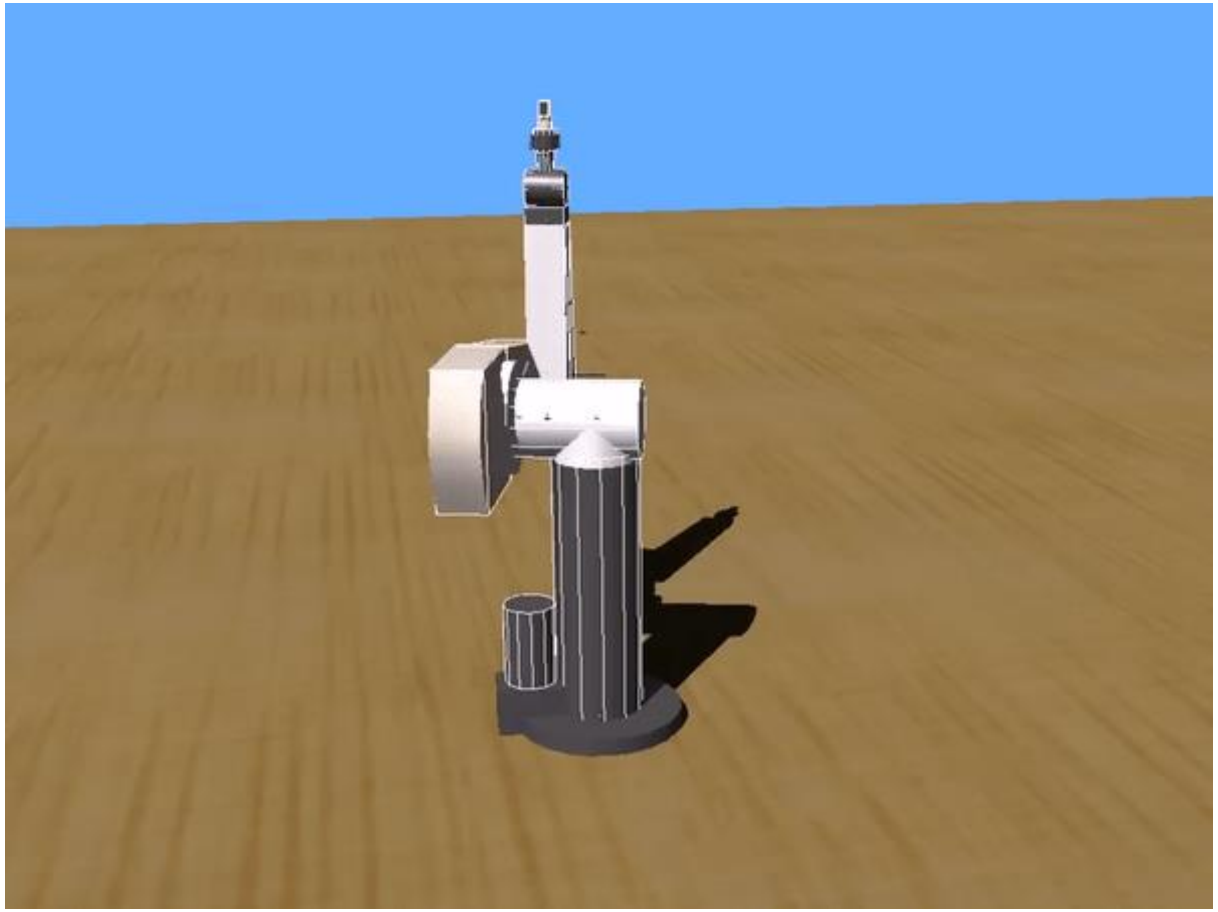
which evaluates to

$$
{}^{A}_{B}T = \begin{bmatrix} 0.933 & 0.067 & 0.354 & -1.13 \\ 0.067 & 0.933 & -0.354 & 1.13 \\ -0.354 & 0.354 & 0.866 & 0.05 \\ 0.000 & 0.000 & 0.000 & 1.00 \end{bmatrix}.
$$

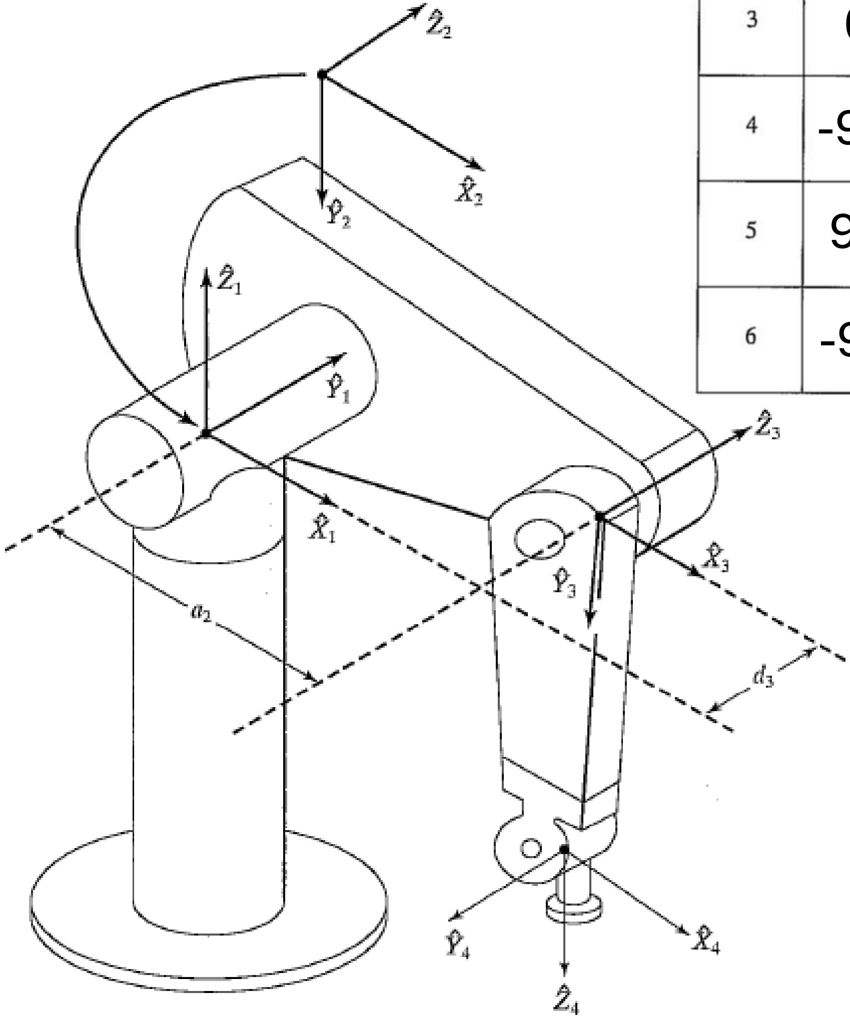
MANIPULATOR KINEMATICS

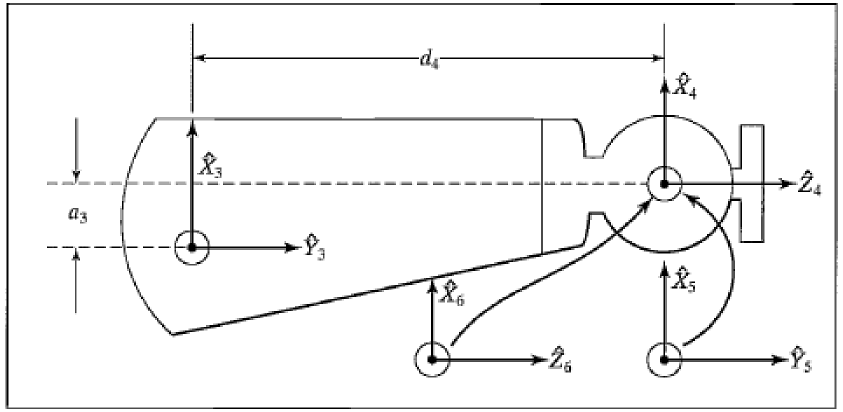
Forward kinematics

1. Identify joints and links of a robot.
2. Assign coordinate systems.
3. Find joint and link parameters (Denavit Hartenberg Table).
4. Derive Transformation matrices using the DH parameters.
5. Compute forward kinematics.



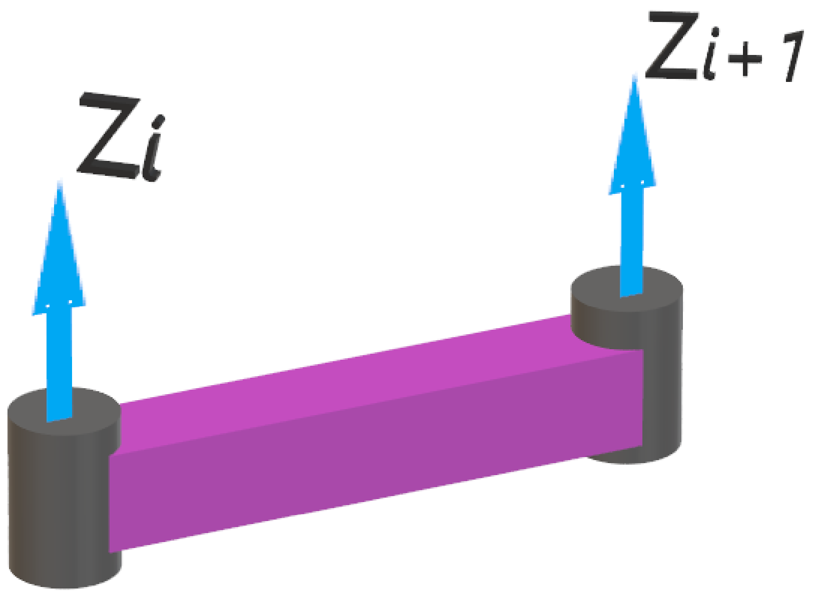
i	$\alpha_i - 1$	$a_i - 1$	d_i	θi
1	0	0	0	θ_1
2	-90	0	0	θ_2
3	0	a_2	d_3	θ_3
4	-90	a_3	d_4	θ_4
5	90	0	0	θ_5
6	-90	0	0	θ_6





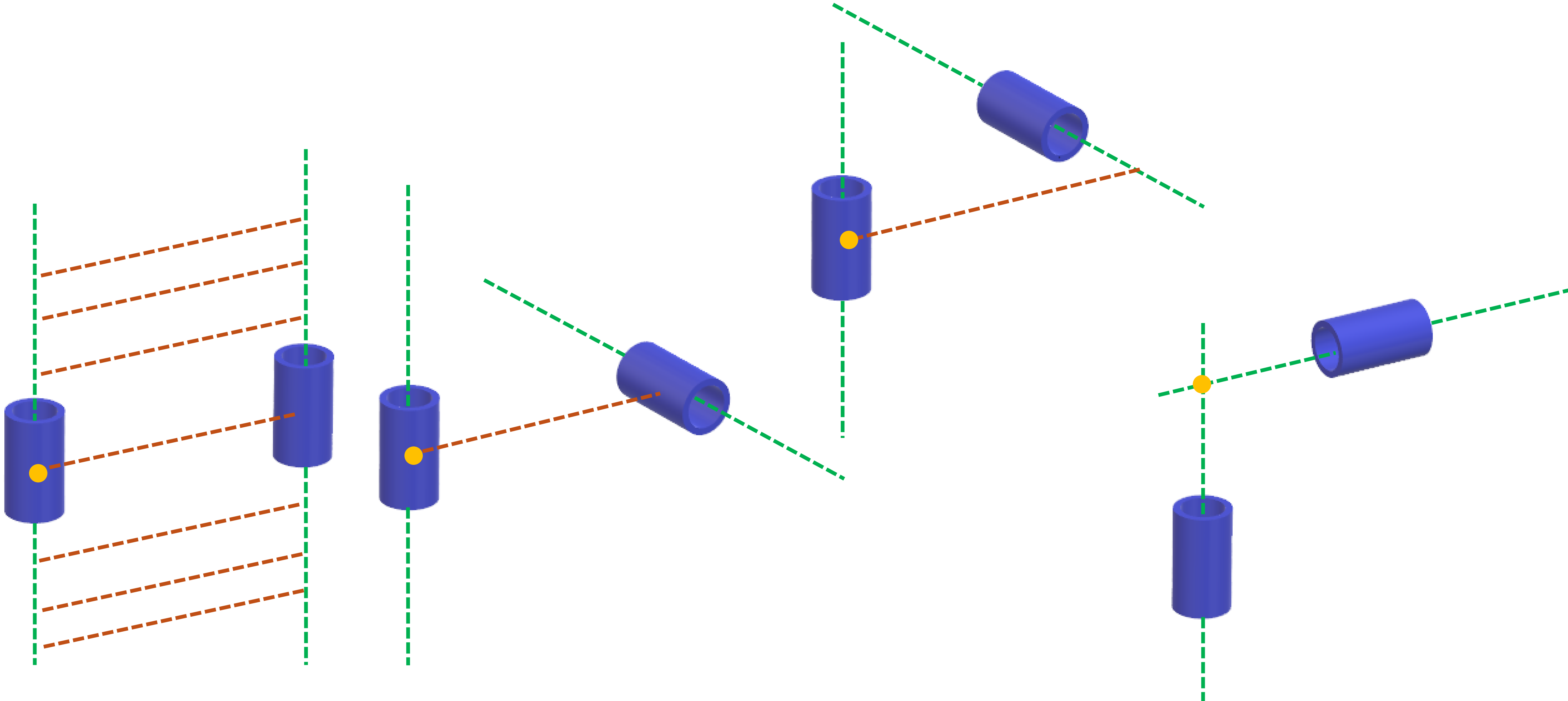
3.4 Convention for Affixing Frames to Links

Z axes: Assign Z-axes along the axis of rotation for revolute joints and axis of translation for prismatic joints.



3.4 Convention for Affixing Frames to Links

Origin of Frame: It is the common perpendicular or the point of intersection.

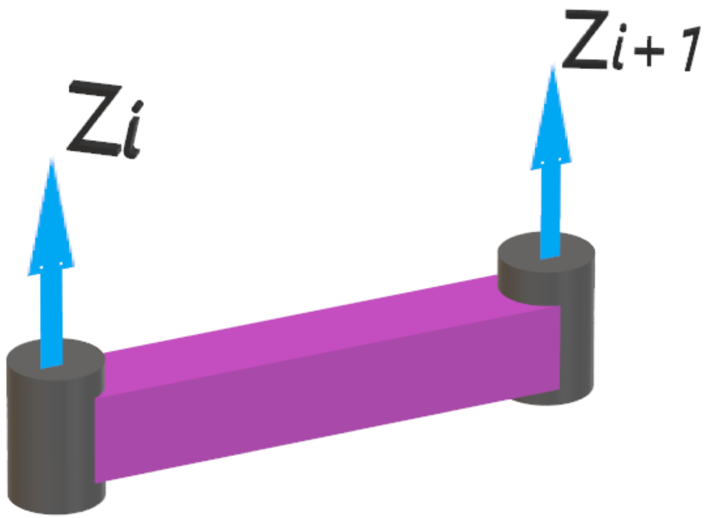


Assigning Frames to Robot Manipulator

To set the origin of the ith frame.

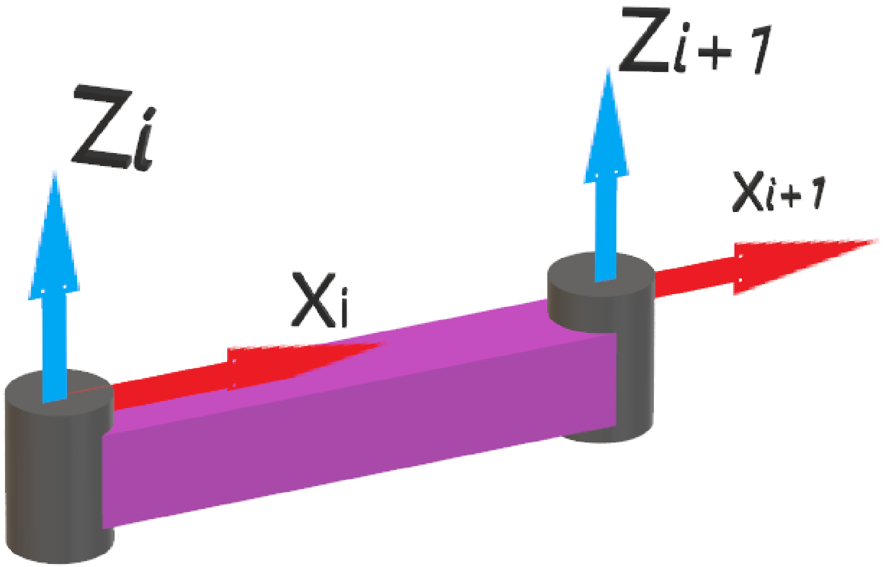
Point of intersection between joint axes (i and i+1) or at the

start of common normal between the joint (I and i+1)



Assigning Frames to Robot Manipulator

Assign Xi axis along the common normal and if the axes intersect, then Xi would be normal to the plane containing the two axes.

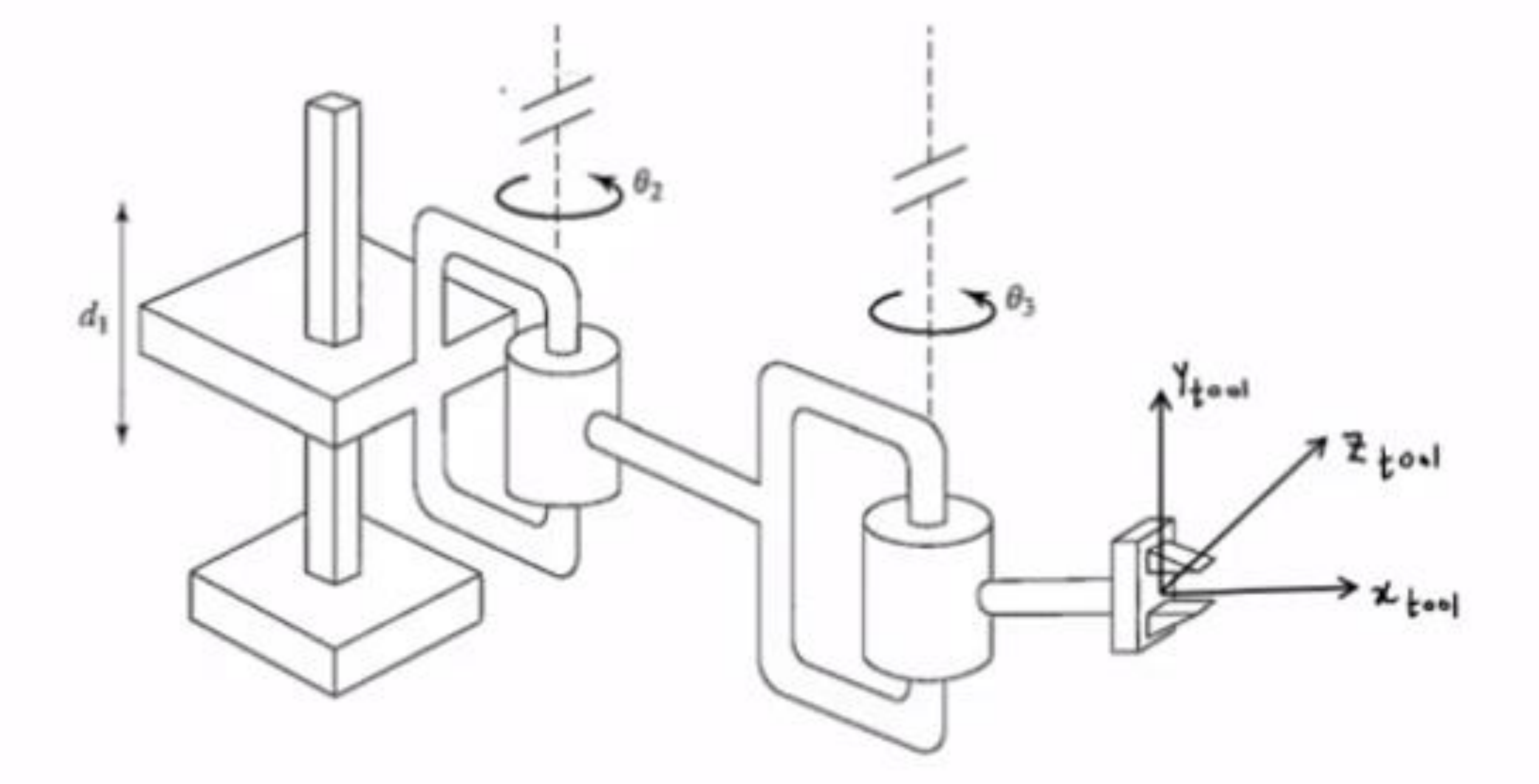


Assigning Frames to Robot Manipulator

Frame {0} and Frame {n}:

Frame {0} is generally attached to base of the robot and to match frame{1} such that most of the parameters can be made to zero.

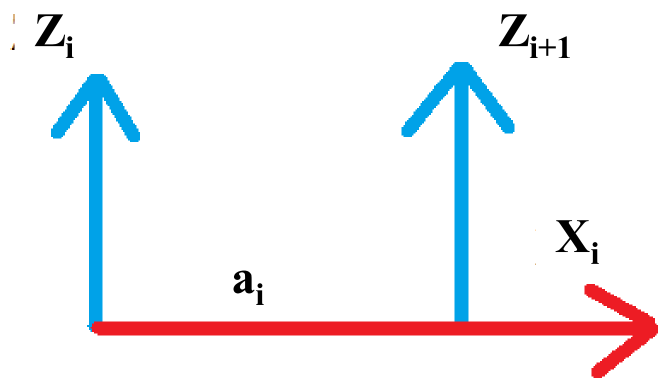
Frame {n} is attached to the end-effector of the robot.


θ_2
θ_3
d_1
y_{tool}
z_{tool}
x_{tool}

DH Parameters

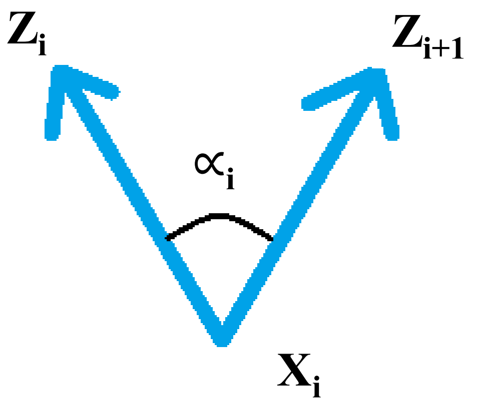
Link length:

a_i = Distance from Z_i to Z_{i+1} along X_i



Link twist:

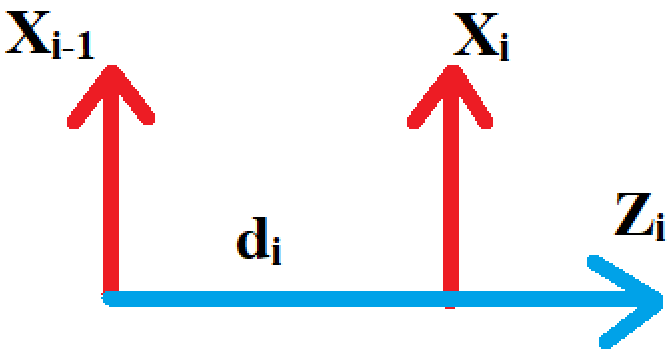
α_i = Angle from Z_i to Z_{i+1} about X_i



DH Parameters:

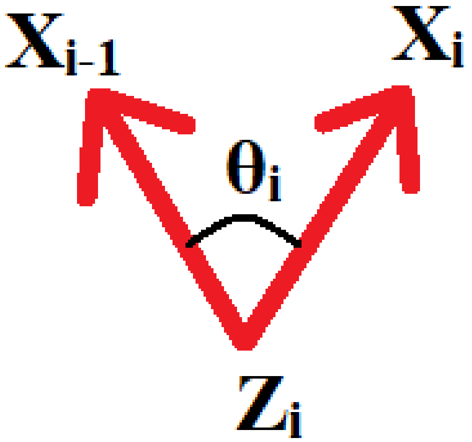
Joint offset:

d_i = Distance from X_{i-1} to X_i along Z_i



Joint angle:

θ_i = Angle from X_{i-1} to X_i about Z_i



$a_i =$ *the distance from* $\hat{Z}_i$ *to* $\hat{Z}_{i+1}$ *measured along* $\hat{X}_i$;

$\alpha_i =$ *the angle from* $\hat{Z}_i$ *to* $\hat{Z}_{i+1}$ *measured about* $\hat{X}_i$;

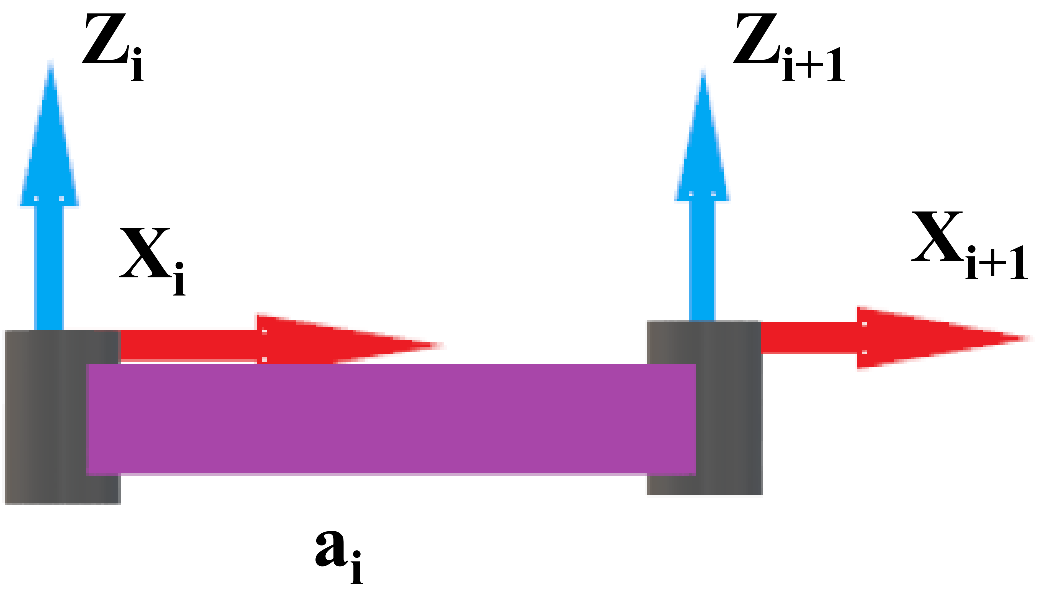
$d_i =$ *the distance from* $\hat{X}_{i-1}$ *to* $\hat{X}_i$ *measured along* $\hat{Z}_i$; and

$\theta_i =$ *the angle from* $\hat{X}_{i-1}$ *to* $\hat{X}_i$ *measured about* $\hat{Z}_i$.

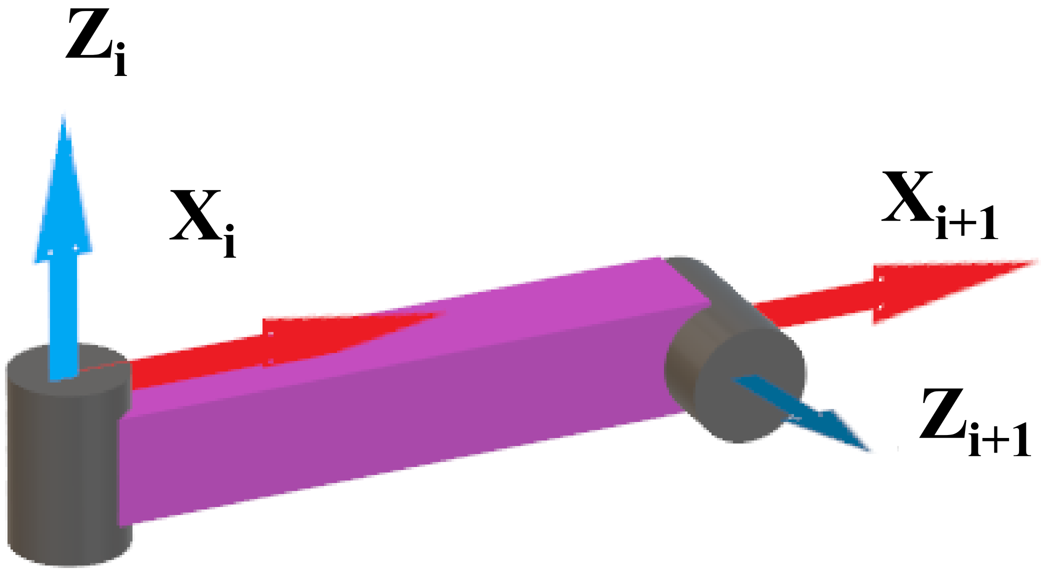
Link Description

Link length (a) and link twist (α) are the two parameters used to describe a link.

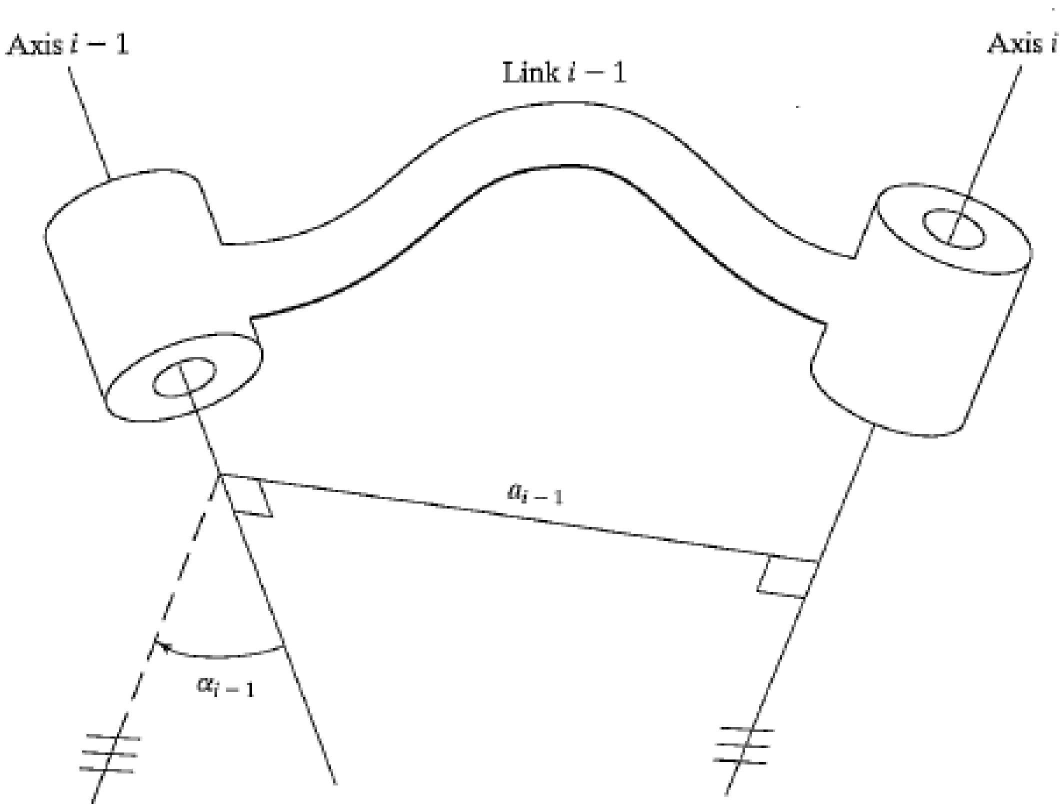
Link Length:



Twist Angle: The angle about the common perpendicular line between the joint axes

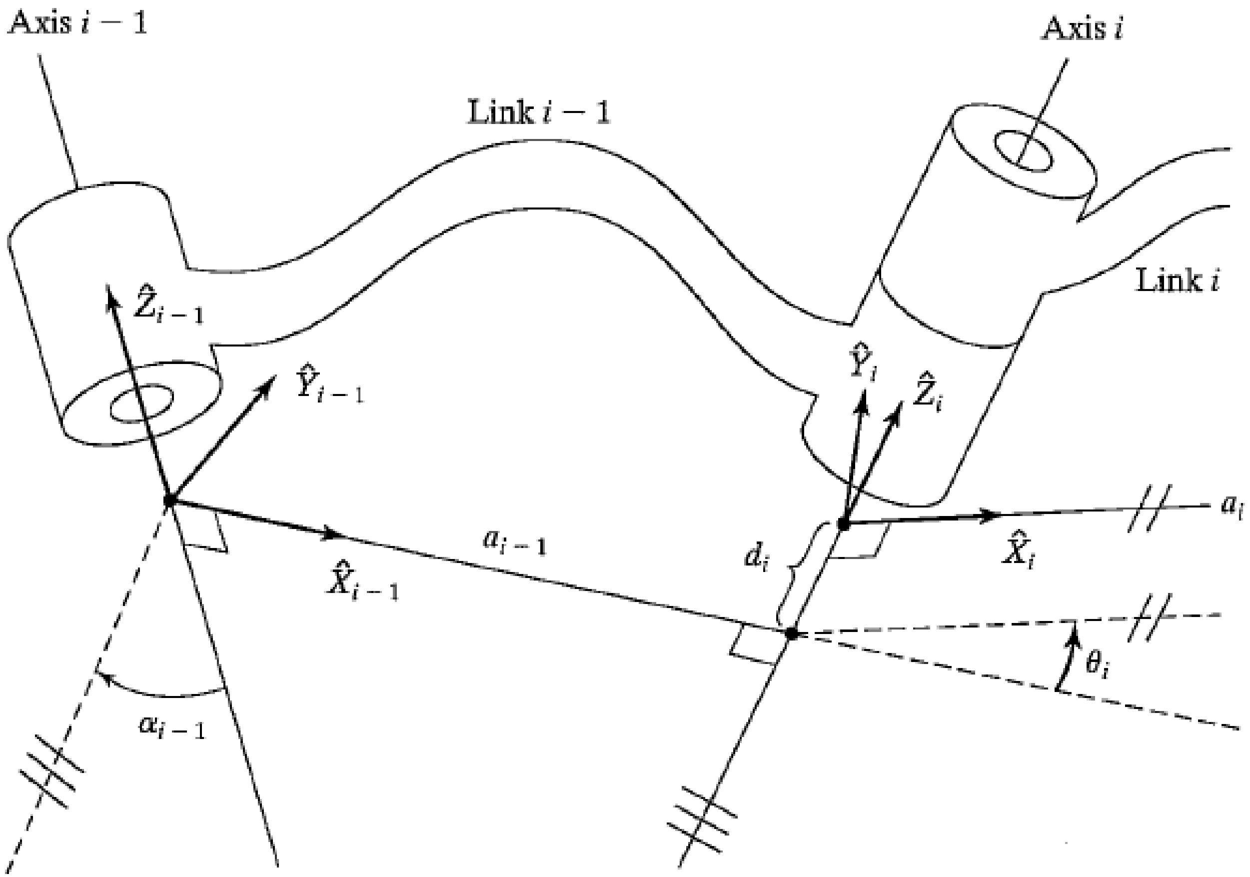


Link Description

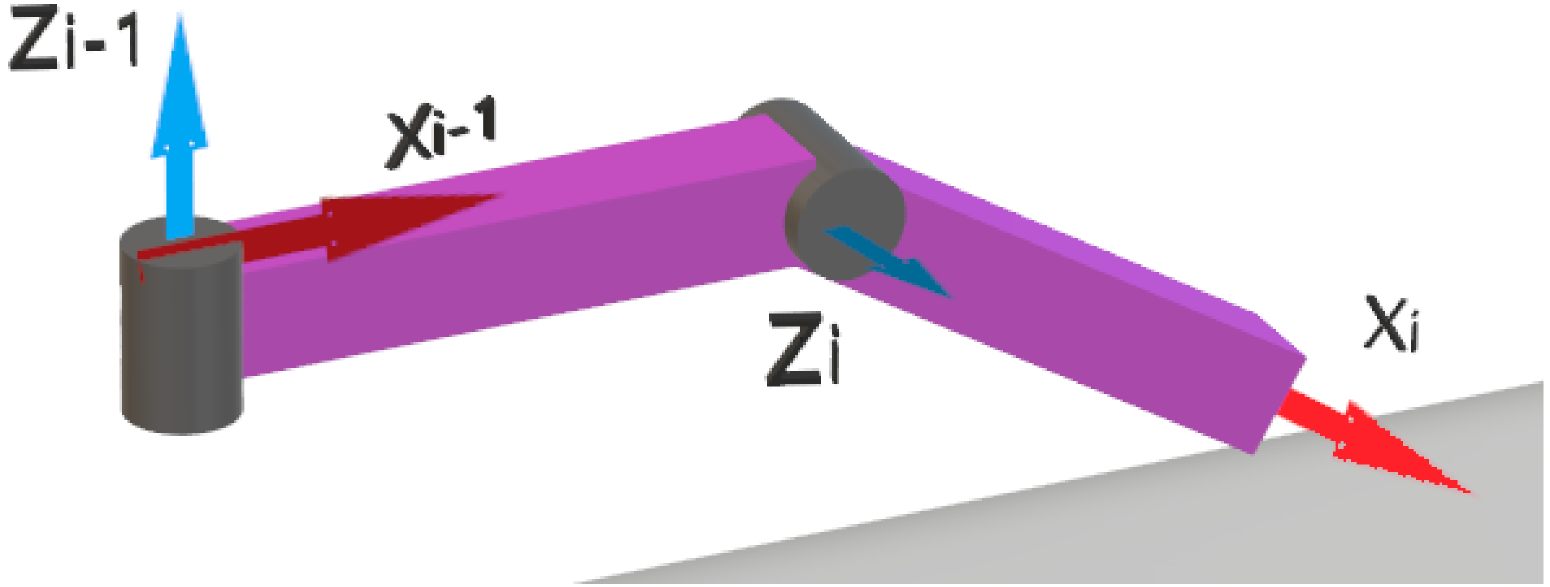


Joint Description: Two parameters are used to describe the joint. Joint offset (d_i) and joint angle (θ_i)

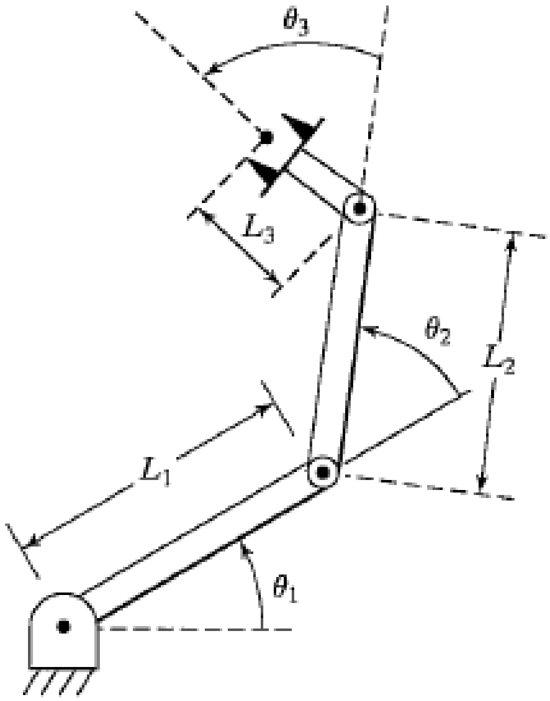
Joint Offset: The length along axis of motion between two common perpendiculars



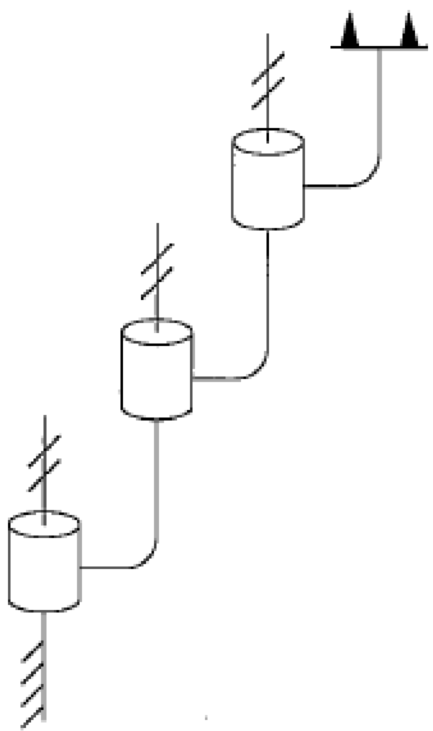
Joint Angle: The angle about the axis of motion between two common perpendiculars.



Example 3.3

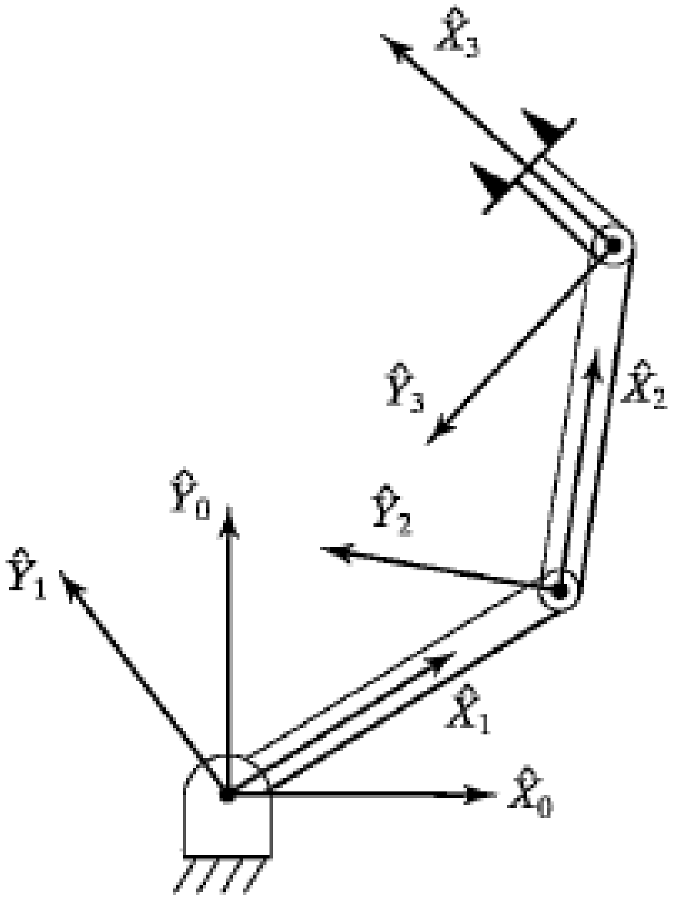


(a)

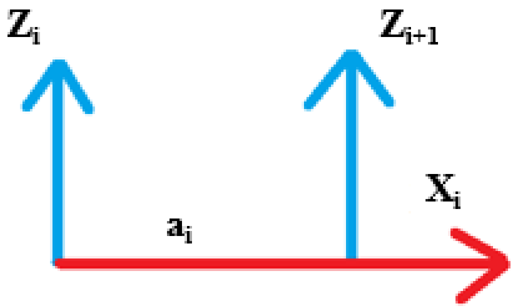


(b)

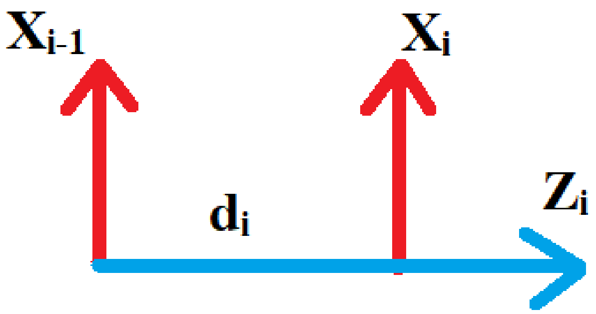
Example 3.3



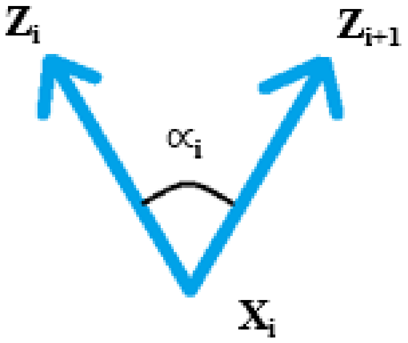
Link Length



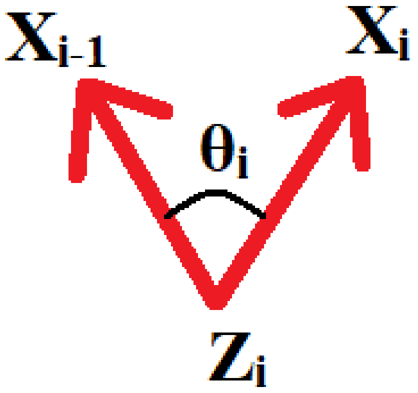
Joint Offset



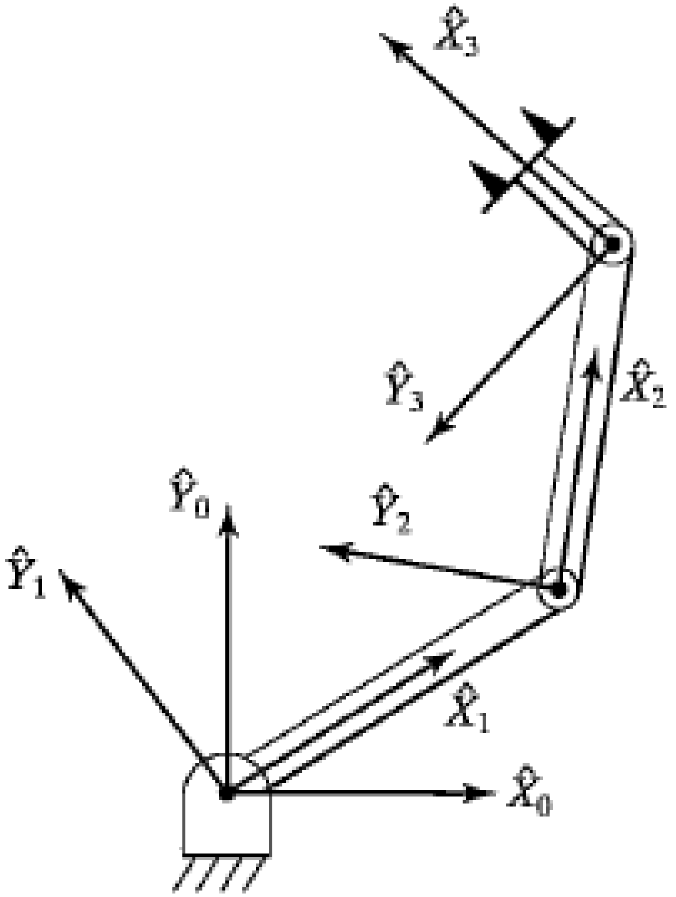
Link Twist



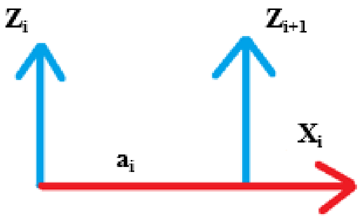
Joint Angle



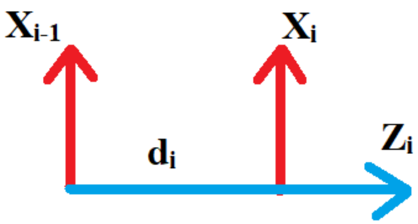
Example 3.3



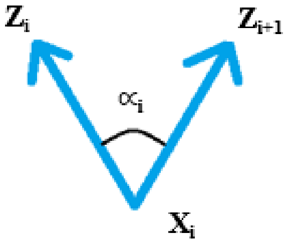
Link Length



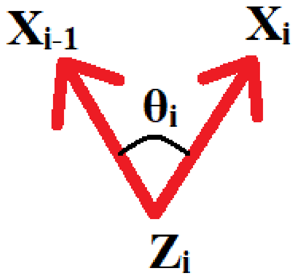
Joint Offset



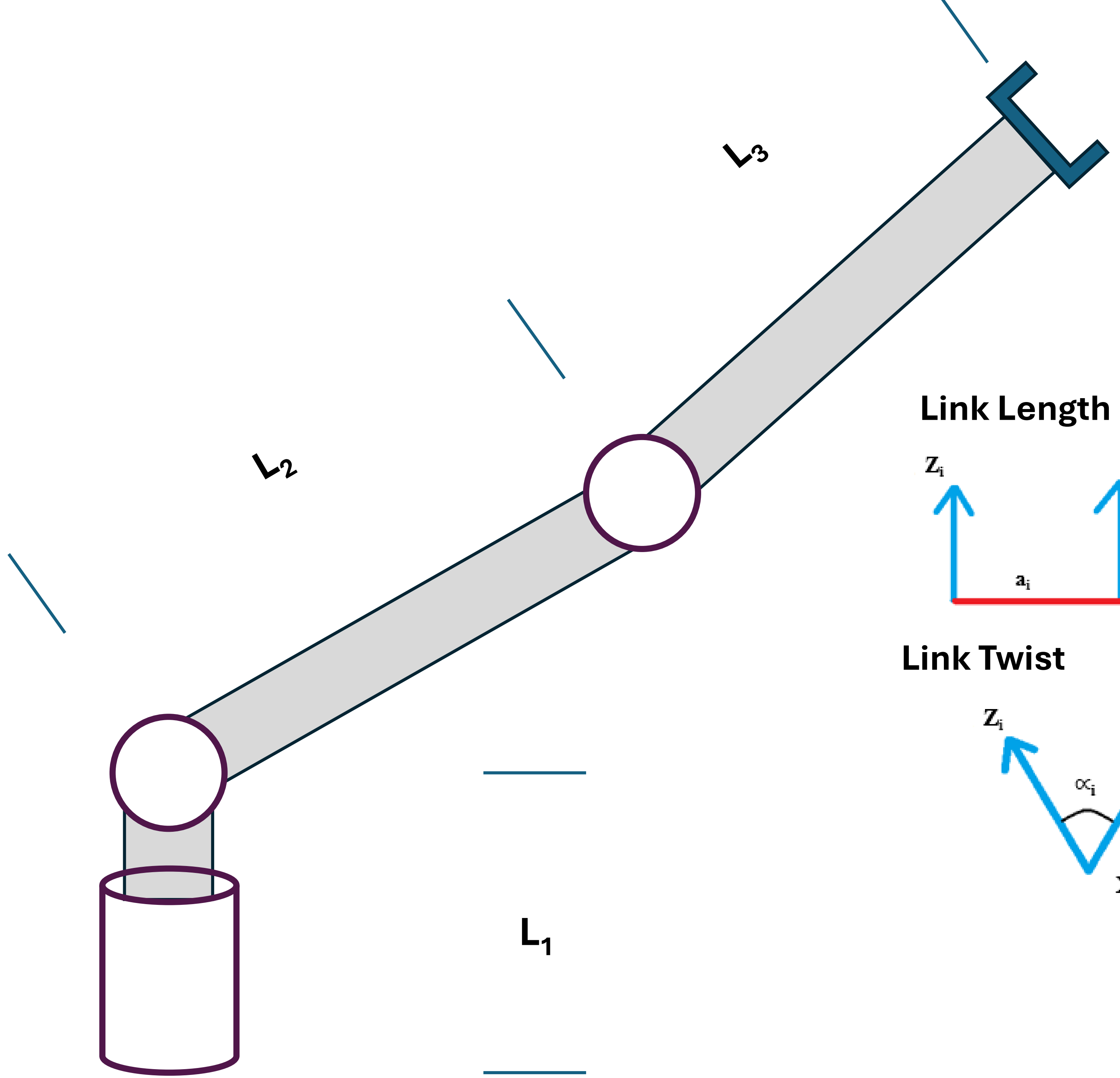
Link Twist



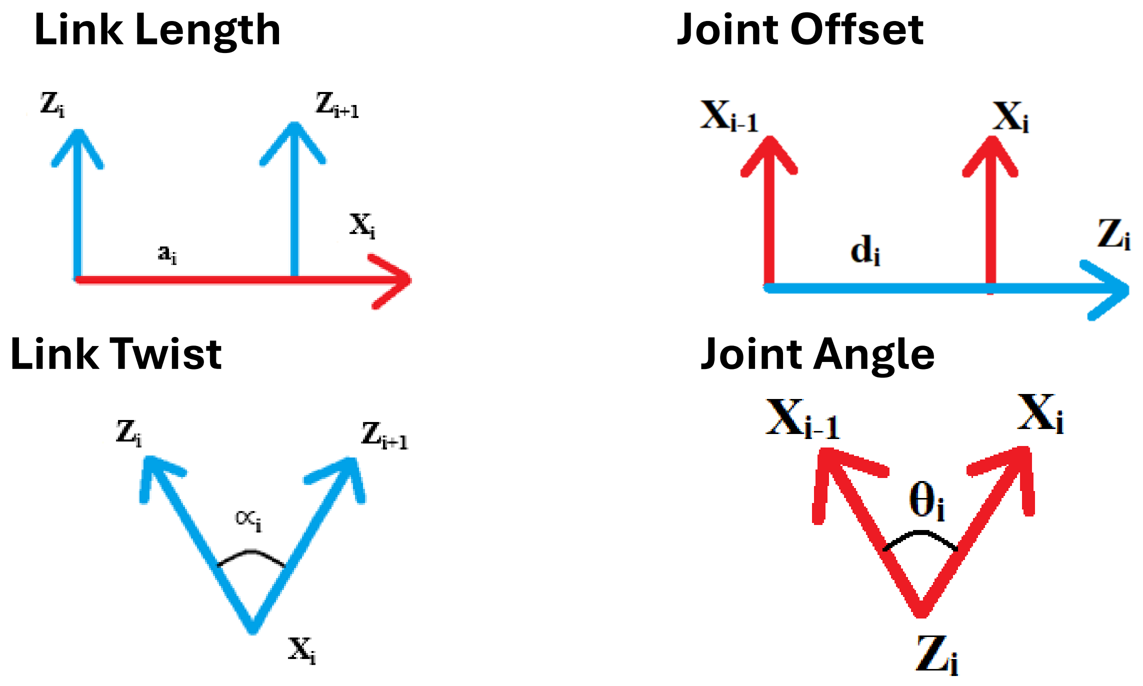
Joint Angle



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	L_1	0	θ_2
3	0	L_2	0	θ_3



L3
L2
L1



Link Length
Zi
Zi+1
Xi
ai
Joint Offset
Xi-1
Xi
Zi
di
Link Twist
Zi
Zi+1
∝i
Xi
Joint Angle
Xi-1
Xi
θi
Zi