Digital Signal Processing (EC 335)

Dr Zaki Uddin

MTS, CEME, NUST.

Lecture 3

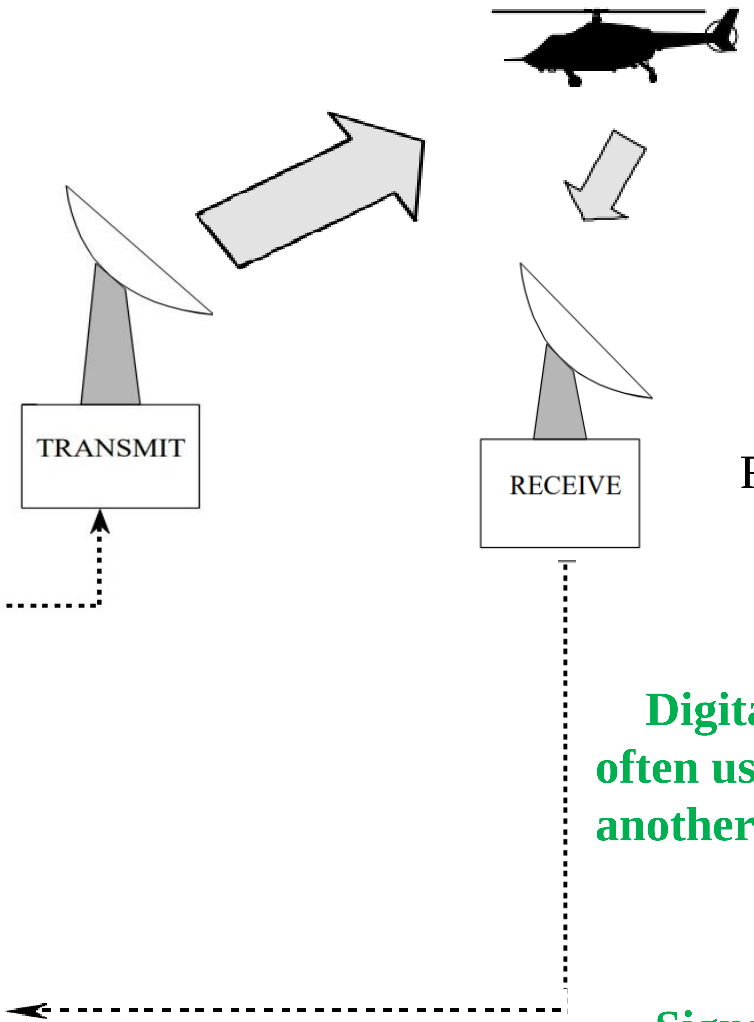
Lecture Targets

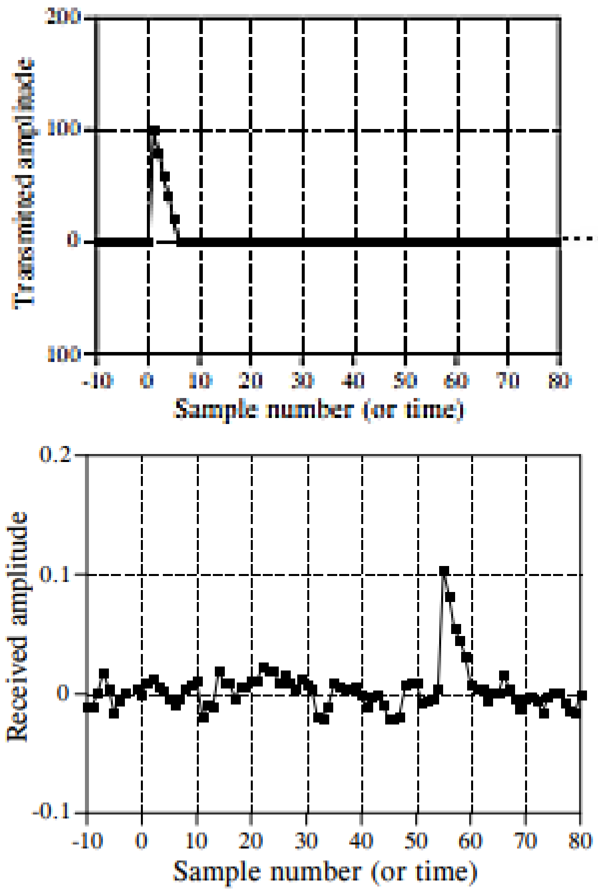
- Correlation
- Cross Correlation
- Auto Correlation
- Properties of Correlation

Correlation

- A signal operation similar to signal convolution, but with completely different physical meaning, is signal correlation.
- The signal correlation operation can be performed either with one signal (autocorrelation) or between two different signals (cross correlation).
- Physically, signal autocorrelation indicates how the signal energy (power) is distributed within the signal, and as such is used to measure the signal power.
- Typical applications of signal autocorrelation are in radar, sonar, satellite, and wireless communications systems.
- Devices that measure signal power using signal correlation are known as signal correlators.
- There are also many applications of signal cross correlation in signal processing systems, especially when the signal is corrupted by another undesirable signal (noise) so that the signal estimation (detection) from a noisy signal has to be performed.
- Signal cross correlation can be also considered as a measure of similarity of two signals.

Correlation





Attenuation factor

Round-trip delay

$$y[n] = \alpha x[n - D] + w[n]$$

Reflected/output signal

Transmitted signal

Additive noise

How to know if the target is present?????

Digital communications is another area where correlation is often used. The information transmitted from one point to another point is usually converted into binary form (0 and 1)

$$x_0[n] \text{ for } 0 \le n \le L-1 \text{ (to tranmit } 0)$$

$$x_1[n] \text{ for } 0 \le n \le L-1 \text{ (to tranmit } 1)$$

Signal received by intended receiver is given by

$$y[n] = x_i[n] + w[n], \quad i = 0,1 \text{ for } 0 \le n \le L-1$$

Receiver will compare the received signal $y[n]$ with both $x_o[n]$ and $x_1[n]$ to determine which of the two signals better matches $y[n]$.

Correlation

Cross Correlation of Energy signals: Let $x(t)$ and $y(t)$ be the two Energy signals, then

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t).y^*(t-\tau)dt = \int_{-\infty}^{\infty} x(t+\tau).y^*(t)dt$$

If $x(t)$ and $y(t)$ are real

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t).y(t-\tau)dt = \int_{-\infty}^{\infty} x(t+\tau).y(t)dt \qquad \text{Eq (1)}$$

$$R_{yx}(\tau) = \int_{-\infty}^{\infty} y(t).x^*(t-\tau)dt = \int_{-\infty}^{\infty} y(t+\tau).x^*(t)dt$$

If $x(t)$ and $y(t)$ are real

$$R_{yx}(\tau) = \int_{-\infty}^{\infty} y(t).x(t-\tau)dt = \int_{-\infty}^{\infty} y(t+\tau).x(t)dt \qquad \text{Eq (2)}$$

Correlation

Put $\tau = -\tau$ in Equation (1), we have

$$R_{xy}(-\tau) = \int_{-\infty}^{\infty} x(t).y(t+\tau)dt \qquad R_{yx}(\tau) = \int_{-\infty}^{\infty} y(t).x(t-\tau)dt = \int_{-\infty}^{\infty} y(t+\tau).x(t)dt$$

$$\boxed{R_{xy}(-\tau) = R_{yx}(\tau)}$$

Let $x(t)$ and $y(t)$ be the two Power signals, then

$$R_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t).y^*(t-\tau)dt = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t+\tau).y^*(t)dt$$

If the signal is power signal and periodic, then we can eliminate this term

Correlation

$$R_{yx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} y(t).x^*(t-\tau)dt = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} y(t+\tau).x^*(t)dt$$

$$\boxed{R_{xy}(-\tau) = R_{yx}(\tau)}$$

Auto Correlation for Energy Signal

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t).x^*(t-\tau)dt = \int_{-\infty}^{\infty} x(t+\tau).x^*(t)dt$$

If $x(t)$ is real

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t).x(t-\tau)dt = \int_{-\infty}^{\infty} x(t+\tau).x(t)dt$$

$$\boxed{R_{xx}(-\tau) = R_{xx}(\tau)}$$

Correlation

Auto Correlation for Power Signal

$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t).x^*(t-\tau)dt = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t+\tau).x^*(t)dt$$

If $x(t)$ is real

$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t).x(t-\tau)dt = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t+\tau).x(t)dt$$

Relation b/w Correlation and Convolution

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t).y(t-\tau)dt$$

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau).y(t-\tau)d\tau$$

$$x(\tau) * y(-\tau) = \int_{-\infty}^{\infty} x(t).y(t-\tau)dt$$

$$\boxed{\begin{aligned} R_{xy}(\tau) &= x(\tau) * y(-\tau) \\ R_{yx}(\tau) &= y(\tau) * x(-\tau) \end{aligned}}$$

Properties of Correlation

1) $$R_{xy}(-\tau) = R_{yx}(\tau) \qquad R_{xx}(-\tau) = R_{xx}(\tau)$$

2) For Energy Signal

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t).x^*(t-\tau)dt$$

$$R_{xx}(0) = \int_{-\infty}^{\infty} x(t).x^*(t)dt$$

$$R_{xx}(0) = \int_{-\infty}^{\infty} \left|x(t)\right|^2 dt$$

$$R_{xx}(0) = E_x$$

3) For Power Signal

$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t).x^*(t-\tau)dt$$

$$R_{xx}(0) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t).x^*(t)dt$$

$$R_{xx}(0) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \left|x(t)\right|^2 dt$$

$$R_{xx}(0) = P_x$$

Properties of Correlation

$$\left|R_{xy}(\tau)\right| \leq \sqrt{R_{xx}(0).R_{yy}(0)} = \sqrt{E_x.E_y}$$

$$\left|R_{xx}(\tau)\right| \leq \sqrt{R_{xx}(0).R_{xx}(0)} = \sqrt{E_x.E_x} = E_x$$

If $x(t)$ is a periodic with a period T, then its ACF will also be periodic with same period T

$$x(t+T) = x(t)$$

$$R_{xx}(\tau+T) = R_{xx}(\tau)$$

DT Energy Signal

$$Cross - Correlation$$

$$R_{xy}[m] = \sum_{n=-\infty}^{\infty} x[n].y^*[n-m] = \sum_{n=-\infty}^{\infty} x[n+m].y^*[n]$$

$$R_{yx}[m] = \sum_{n=-\infty}^{\infty} y[n].x^*[n-m] = \sum_{n=-\infty}^{\infty} y[n+m].x^*[n]$$

$$R_{xy}[-m] = R_{yx}[m]$$

$$Auto - Correlation$$

$$R_{xy}[m] = \sum_{n=-\infty}^{\infty} x[n].x^*[n-m] = \sum_{n=-\infty}^{\infty} x[n+m].x^*[n]$$

DT Power Signal

$Cross-Correlation$

$$R_{xy}[m] = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x[n].y^*[n-m] = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x[n+m].y^*[n]$$

$$R_{yx}[m] = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} y[n].x^*[n-m] = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} y[n+m].x^*[n]$$

$$R_{xy}[-m] = R_{yx}[m]$$

$Auto-Correlation$

$$R_{xx}[m] = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x[n].x^*[n-m]$$

Properties of Correlation

$$\boxed{\begin{aligned} R_{xy}[m] &= x[m] * y[-m] \\ R_{yx}[m] &= y[m] * x[-m] \\ R_{xx}[m] &= x[m] * x[-m] \end{aligned}}$$

$$R_{xy}[-m] = R_{yx}[-m]$$

$$R_{xx}[-m] = R_{xx}[m] (\text{Even})$$

For Energy Signal

$$R_{xx}[0] = E_x$$

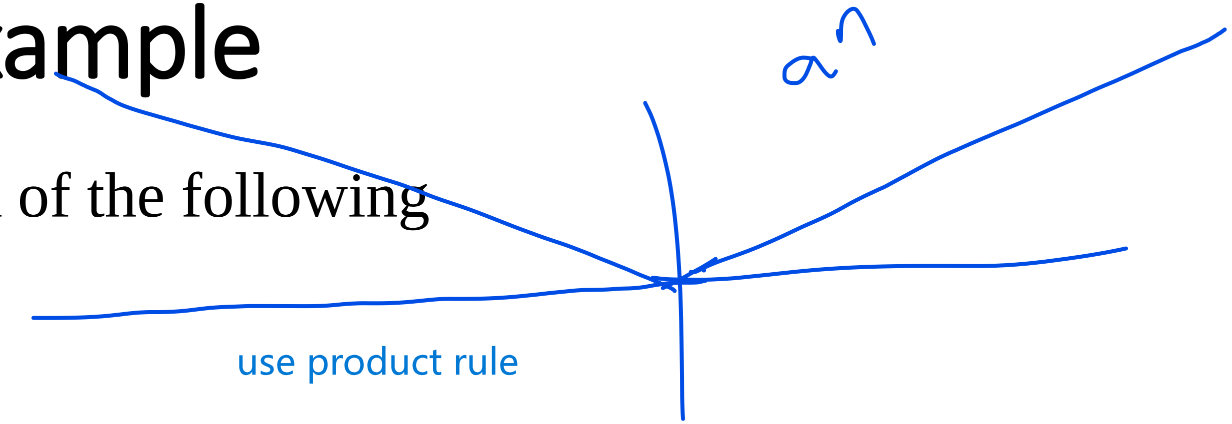
For Power Signal

$$R_{xx}[0] = P_x$$

$$\left|R_{xy}[m]\right| \le \sqrt{R_{xx}[0].R_{yy}[0]} = \sqrt{E_x.E_y}$$

$$\left|R_{xx}[m]\right| \le \sqrt{R_{xx}[0].R_{xx}[0]} = \sqrt{E_x.E_x} = E_x$$

Example



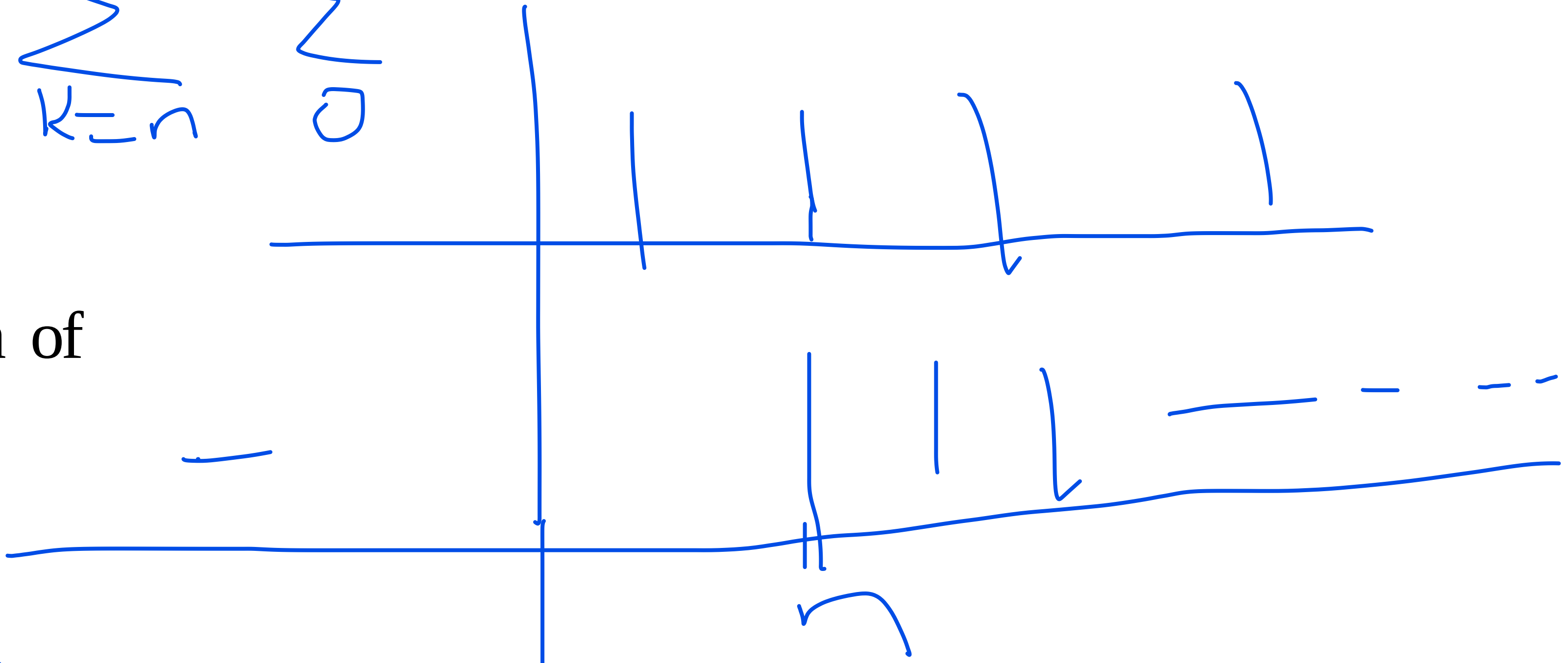
Use graphical method to find cross correlation of the following

$x_1[n]=\{1, 2, 3, 4\}$

$x_2[n]\ ]=\{0, 1, 2, 3\}$

use product rule

$\sum_{k=n}^{\infty}$ $\sum_{0}^{\infty}$



Compute the autocorrelation of

$$x[n] = a^n u[n], 0 < a < 1$$

$h(n)$ $u(-n)$

$h(n-k)$

$a^{n-k}\, u(-(n-k))$ $(k-n)$ $u(k-n)$

$k-n \geq 0$ $k \geq n$

Correlation Examples

$$x[n] \xrightarrow{\text{ACF}} R_{xx}[m]$$

$$x[n] \rightarrow \boxed{\begin{array}{c}\text{D.L.T.I System}\\ h[n]\end{array}} \rightarrow y[n]$$

Find

$R_{yx}[m]$ $\quad$ $R_{xy}[m]$ $\quad$ $R_{yy}[m]$

$$R_{yx}[m] = y[m] * x[-m]$$
$$\therefore\ y[m] = x[m] * h[m]$$
$$R_{yx}[m] = \{x[m] * h[m]\} * x[-m]$$
$$R_{yx}[m] = \left\{ \underbrace{x[m] * x[-m]}_{R_{xx}[m]} \right\} * h[m]$$
$$R_{yx}[m] = R_{xx}[m] * h[m]$$

$$R_{xy}[m] = x[m] * y[-m]$$
$$\therefore\ y[-m] = x[-m] * h[-m]$$
$$R_{yx}[m] = x[m] * \{x[-m] * h[-m]\}$$
$$R_{yx}[m] = \left\{ \underbrace{x[m] * x[-m]}_{R_{xx}[m]} \right\} * h[-m]$$
$$R_{yx}[m] = R_{xx}[m] * h[-m]$$

xy

$$R_{yy}[m] = y[m] * y[-m]$$
$$R_{yx}[m] = \{x[m] * h[m]\} * \{x[-m] * h[-m]\}$$
$$R_{yx}[m] = \{x[m] * x[-m]\} * \{h[m] * h[-m]\}$$
$$R_{yx}[m] = R_{xx}[m] * R_{hh}[m]$$

Wiener-Khintchine Theorem: Correlation Application

Wiener-Khintchine Theorem: The power spectral density (PSD) is equal to the Fourier transform of the autocorrelation function.

Power spectral density (PSD): Power of the signal per unit bandwidth is known as PSD.

$$S_x(e^{j\omega}) = \sum_{k=-\infty}^{\infty} R_{xx}[k]e^{-j\omega k}$$