



Digital Signal Processing (EC 335)

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Lecture 5

Lecture Targets

- Fourier series representation of DT periodic signals (DTFS)
- Properties of DTFS

CTFS

Synthesis Equation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Analysis Equation

$$a_k = \frac{1}{T} \int_0^T x(t) \cdot e^{-jk\omega_0 t} dt$$

DTFS

Synthesis Equation

$$x[n] = \sum_{k=0}^{N-1} c[k] e^{\frac{jk2\pi n}{N}}$$

for sinusoidal (sin, cos)

The frequency domain signal $c[k]$ of the DT signal $x[n]$ will be also periodic.

Analysis Equation

$$c[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-\frac{jk2\pi n}{N}}$$

$$c[k+N] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-\frac{j2\pi(k+N)n}{N}}$$

$\cos 2\pi n - j \sin 2\pi n$

Periodicity property of $c[k]$

$$c[k+N] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-\frac{j2\pi kn}{N}} \cdot \underbrace{e^{-\frac{j2\pi Nn}{N}}}_1$$

$$c[k+N] = c[k]$$

DTFS

Significance of the term

$$\frac{2\pi k}{N}$$

in DTFS equations

FS coefficient of DT signal will repeat itself after N sample, where N is the Period of the signal.

$$\frac{2\pi}{N} \text{ rad/sample}$$

What will be the gap (Frequency spacing) between the harmonics????

Can we find the location of each sample???

Properties of DTFS

Linearity

$$x_1[n] \rightarrow c_1(k)$$

$$x_2[n] \rightarrow c_2(k)$$

$$\alpha x_1[n] + \beta x_2[n] \rightarrow \alpha c_1(k) + \beta c_2(k)$$

Time-shifting

$$x[n] \rightarrow c(k)$$

$$x[n - n_o] \rightarrow e^{j2\pi k n_o / N} \times c(k)$$

$$\therefore \frac{2\pi k}{N} = \omega_o$$

What is the difference b/w periodic and Linear convolution?????

Time-scaling

$$x[n] \rightarrow c(k)$$

$$x[\alpha n] \rightarrow \alpha \times c(k)$$

Convolution property

$$x_1[n] * x_2[n] \rightarrow N \times c_1(k) \times c_2(k)$$



Periodic convolution

$$x_1[n] * x_2[n] = \sum_{m=0}^{N-1} x_1[m].x_2[n-m]$$

$$x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k].x_2[n-k]$$

Properties of DTFS

Multiplication Property

$$x_1[n] \times x_2[n] \rightarrow c_1(k) * c_2(k)$$

$*$ \rightarrow Periodic or linear?

Time reversal

$$x[n] \rightarrow c(k)$$

$$x[-n] \rightarrow c(-k)$$

Frequency-shifting

$$x[n] \rightarrow c(k)$$

$$e^{j \frac{2\pi k_o n}{N}} \times x[n] \rightarrow c(k - k_o)$$

Example

Find the FS coefficient of

$$x[n] = \cos \frac{\pi}{3} n$$

Sol :

$$c[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-\frac{jk2\pi n}{N}}$$

$$x[n] = \sum_{n=0}^{N-1} c[k] e^{\frac{jk2\pi n}{N}}$$

$$x[n] = \cos \frac{\pi}{3} n$$

$$\omega_o = \frac{\pi}{3}$$

$$N = \left(\frac{2\pi}{\omega_o} \right) m, m = \text{smallest integer}$$

$$N = \left(\frac{2\pi \times 3}{\pi} \right) \times m \Rightarrow N = 6$$

$$x[n] = \sum_{k=-3}^2 c[k] e^{\frac{j2\pi kn}{6}} \rightarrow \frac{j\pi kn}{3}$$

$$x[n] = c[-3]e^{j\frac{\pi}{3}n(-3)} + c[-2]e^{j\frac{\pi}{3}n(-2)} + c[-1]e^{j\frac{\pi}{3}n(-1)} + c[0] + c[1]e^{j\frac{\pi}{3}n(1)} + c[2]e^{j\frac{\pi}{3}n(2)} \dots \dots \dots (1)$$

$$x[n] = \cos \frac{\pi}{3} n = \frac{1}{2} e^{j\frac{\pi}{3}n} + \frac{1}{2} e^{-j\frac{\pi}{3}n} \dots \dots \dots (2)$$

Comparing (1) and (2), we have

$$c[-1] = \frac{1}{2}, c[1] = \frac{1}{2}$$

$$c[-3] = c[-2] = c[0] = c[2] = 0$$

Plot $c[k]$?????

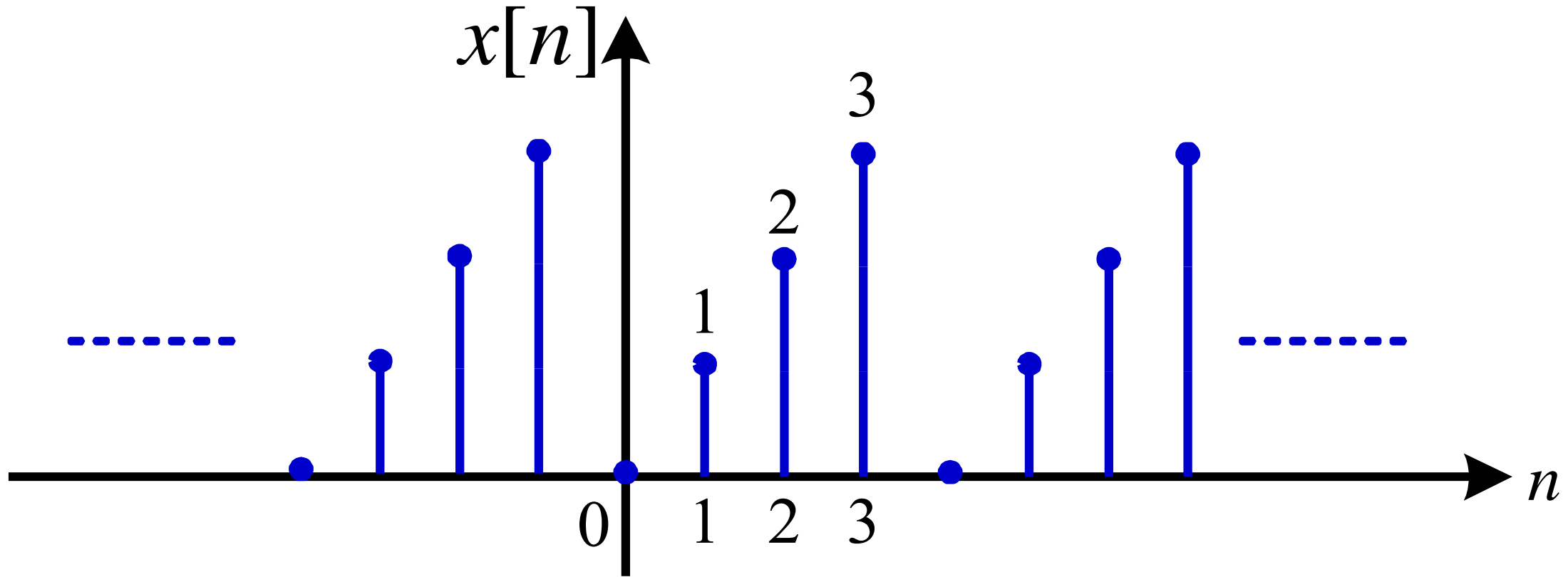
$$\sin \theta = \frac{e^{j\theta}}{2j} - \frac{e^{-j\theta}}{2j}$$

Find FS representation of

$$x[n] = \sin \left(\frac{2\pi}{7} \right) n$$

Example

Find the FS coefficient of the following signal



Example

$$C[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j \frac{k 2\pi}{N} n}$$

$$N = 4$$

$$C[k] = \frac{1}{4} \sum_{n=0}^{N-1} x[n] e^{-j \frac{k 2\pi}{4} n}$$

$$C[k] = \frac{1}{4} \left\{ \underbrace{x[0]}_0 e^{-j \frac{k\pi}{2} (0)} + \underbrace{x[1]}_1 e^{-j \frac{k\pi}{2} (1)} \right. \\ \left. \underbrace{x[2]}_2 e^{-j \frac{k\pi}{2} (2)} + \underbrace{x[3]}_3 e^{-j \frac{k\pi}{2} (3)} \right\}$$

$$C[k] = \frac{1}{4} \left\{ \left(e^{-j \frac{\pi}{2}} \right)^k + 2 \left(e^{-j\pi} \right)^k + 3 \left(e^{-j \frac{3\pi}{2}} \right)^k \right\}$$

$$\left(e^{-j \frac{\pi}{2}} \right)^k = (-j)^k, \left(e^{-j\pi} \right)^k = (-1)^k, \left(e^{-j \frac{3\pi}{2}} \right)^k = (j)^k$$

$$C[k] = \frac{1}{4} \left\{ (-j)^k + 2(-1)^k + 3(j)^k \right\}$$

Find C[3], C[7], C[-1]

Questions

Find the FS coefficient of the periodic impulse train

$$x[n] = \sum_{l=-\infty}^{\infty} \delta(n - Nl)$$

N should be strictly positive integer

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} \delta(n) e^{-j(2\pi/N)kn}$$

From the sifting property of the *impulse*, we have

$$c_k = \frac{1}{N} e^0$$

$$c_k = \frac{1}{N} \quad \text{for all } k \in \mathbb{Z}.$$

Example

Find the FS coefficient of $x[n] = 1 + \sin\left(\frac{2\pi}{N}n\right) + 3\cos\left(\frac{2\pi}{N}n\right) + 2\cos\left(\frac{4\pi}{N}n + \frac{\pi}{2}\right)$

$$\omega_o = \frac{2\pi}{N}$$

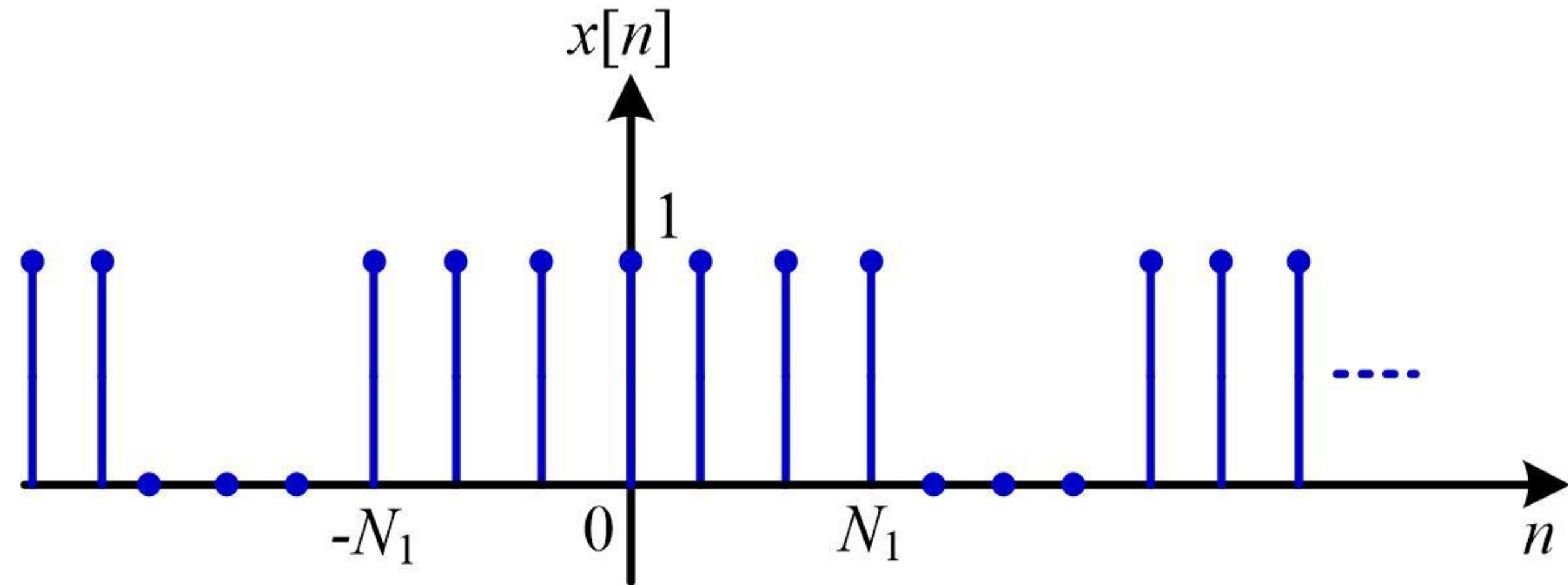
$$x[n] = 1 + \frac{1}{2j}e^{j\frac{2\pi}{N}n} - \frac{1}{2j}e^{-j\frac{2\pi}{N}n} + 3 \cdot \frac{1}{2}e^{j\frac{2\pi}{N}n} + 3 \cdot \frac{1}{2}e^{-j\frac{2\pi}{N}n} + 2 \cdot \frac{1}{2}e^{j\left(\frac{4\pi}{N}n + \frac{\pi}{2}\right)} + 2 \cdot \frac{1}{2}e^{-j\left(\frac{4\pi}{N}n + \frac{\pi}{2}\right)}$$

$$x[n] = 1 + e^{j\frac{2\pi}{N}n} \left[\frac{3}{2} + \frac{1}{2j} \right] + e^{-j\frac{2\pi}{N}n} \left[\frac{3}{2} - \frac{1}{2j} \right] + \underbrace{e^{j\frac{\pi}{2}}}_j e^{j\frac{4\pi}{N}n} + \underbrace{e^{-j\frac{\pi}{2}}}_{-j} e^{-j\frac{4\pi}{N}n}$$

$$c[0] = 1, c[1] = \left[\frac{3}{2} + \frac{1}{2j} \right], c[-1] = \left[\frac{3}{2} - \frac{1}{2j} \right], c[2] = j, c[-2] = -j$$

Example

Find the FS coefficient of



Example

$$c[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j \frac{2\pi k}{N} n}$$

$$c[k] = \frac{1}{N} \sum_{n=-N_1}^{N_1} \underbrace{x[n]}_1 e^{-j \frac{2\pi k}{N} n}$$

$$c[k] = \frac{1}{N} \sum_{m=0}^{2N_1} e^{-j \frac{2\pi k (m-N_1)}{N}}$$

$$\sum_{n=0}^N (a)^n = \frac{1-a^{N+1}}{1-a}$$

$$c[k] = \frac{1}{N} \sum_{m=0}^{2N_1} \underbrace{e^{-j \frac{2\pi k N_1}{N}}}_{\text{constant}} e^{-j \frac{2\pi k m}{N}}$$

$$c[k] = \frac{e^{j \frac{2\pi k N_1}{N}}}{N} \sum_{m=0}^{2N_1} e^{-j \frac{2\pi k m}{N}}$$

$$c[k] = \frac{e^{j \frac{2\pi k N_1}{N}}}{N} \sum_{m=0}^{2N_1} \left(e^{-j \frac{2\pi k}{N}} \right)^m$$



Example

Find the FS representation of the 5-periodic sequence given by

$$x[n] = \begin{cases} -\frac{1}{2}, n = -1 \\ 1, n = 0 \\ \frac{1}{2}, n = 1 \\ 0, n \in \{-2, 2\} \end{cases}$$

$$c_k = \frac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{-j(2\pi/N)kn}$$

$$\begin{aligned} c_k &= \frac{1}{5} \sum_{n=-2}^2 x(n) e^{-j(2\pi/5)kn} \\ &= \frac{1}{5} \left(-\frac{1}{2} e^{-j(2\pi/5)(-1)k} + e^{-j(2\pi/5)(0)k} + \frac{1}{2} e^{-j(2\pi/5)(1)k} \right) \end{aligned}$$

$$\begin{aligned} c_k &= \frac{1}{5} \left(-\frac{1}{2} e^{j(2\pi/5)k} + 1 + \frac{1}{2} e^{-j(2\pi/5)k} \right) \\ &= \frac{1}{5} \left(1 - \frac{1}{2} (e^{j(2\pi/5)k} - e^{-j(2\pi/5)k}) \right) \\ &= \frac{1}{5} \left(1 - \frac{1}{2} [2j \sin(\frac{2\pi}{5}k)] \right) \\ &= \frac{1}{5} \left[1 - j \sin(\frac{2\pi}{5}k) \right]. \end{aligned}$$

Practice Questions

Find the FS representation of the 8-periodic sequence given by

$$x[n] = \begin{cases} 1, n \in \{0..3\} \\ 0, n \in \{4..7\} \end{cases}$$

Find the FS representation of the 8-periodic sequence given by

$$x[n] = \begin{cases} n, n \in \{-2..2\} \\ 0, n = -3 \text{ or } n \in \{3..4\} \end{cases}$$

Convergence of DTFS

Since the analysis and synthesis equations for DTFS involve only finite sums (as opposed to infinite series), convergence is not a significant concern.

If an N -periodic sequence is bounded (i.e., is finite in value), its Fourier-series coefficient sequence will exist and be bounded and the Fourier series analysis and synthesis equations must converge.