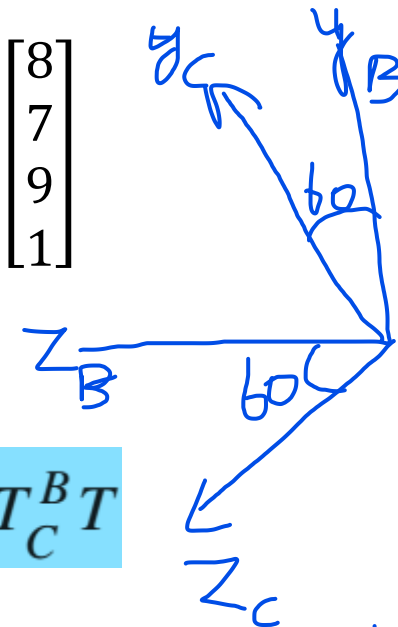


Frame {B} is rotated relative to frame {A} about z-axis by 30 degree and translated 4 units in x-axis and 3 units in y axis. Frame {C} is rotated relative to frame {B} about x-axis by 60 degrees and translated 6 units in x-axis and 5 units in z-axis. Find the position of P relative to frame {A} if ${}^C P = [8 \ 7 \ 9]^T$

$${}^A_B T = \begin{bmatrix} 0.866 & -0.5 & 0 & 4 \\ 0.5 & 0.866 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^B_C T = \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 0.5 & -0.866 & 0 \\ 0 & 0.866 & 0.5 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^C P = \begin{bmatrix} 8 \\ 7 \\ 9 \\ 1 \end{bmatrix}$$



$${}^A_C T = \begin{bmatrix} 0.866 & -0.25 & 0.433 & 9.196 \\ 0.5 & 0.433 & -0.75 & 6 \\ 0 & 0.866 & 0.5 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^A P = {}^A_B T {}^B_C T {}^C P$$

$${}^A_C T = {}^A_B T {}^B_C T$$

$${}^A P = \begin{bmatrix} 0.866 & -0.25 & 0.433 & 9.196 \\ 0.5 & 0.433 & -0.75 & 6 \\ 0 & 0.866 & 0.5 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 7 \\ 9 \\ 1 \end{bmatrix}$$

Handwritten notes showing the sequence of transformations:
 $X_B X_C$ $X_B Y_C$ $X_B Z_C$
 $Y_B X_C$ $Y_B Y_C$ $Y_B Z_C$
 $Z_B X_C$ $Z_B Y_C$ $Z_B Z_C$

$${}^A P = \begin{bmatrix} 18.24 \\ 6.28 \\ 15.56 \\ 1 \end{bmatrix}$$

Handwritten notes showing the rotation matrices for the transformations:
 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 60 & -\sin 60 \\ 0 & \sin 60 & \cos 60 \end{bmatrix}$

Transformation Arithmetic

according to this theorem to invert the transformation matrix from aRb to bRa we need to take transpose of the orientation part and for the translation - (aRb ^AT aPb_org)

$${}^B_A T = \left[\begin{array}{ccc|c} {}^A_B R^T & -{}^A_B R^T {}^A P_{BORG} & & \\ \hline 0 & 0 & 0 & 1 \end{array} \right]$$

Inverting Transform (Proof)

To find ${}^B_A T$, we must compute ${}^B_A R$ and ${}^B P_{AORG}$ from ${}^A_B R$ and ${}^A P_{BORG}$. First, recall from our discussion of rotation matrices that

$${}^B_A R = {}^A_B R^T. \quad (2.42)$$

Next, we change the description of ${}^A P_{BORG}$ into $\{B\}$ by using (2.13):

$${}^B ({}^A P_{BORG}) = {}^B_A R {}^A P_{BORG} + {}^B P_{AORG}. \quad (2.43)$$

The left-hand side of (2.43) must be zero, so we have

$${}^B P_{AORG} = -{}^B_A R {}^A P_{BORG} = -{}^A_B R^T {}^A P_{BORG}. \quad (2.44)$$

Using (2.42) and (2.44), we can write the form of ${}^B_A T$ as

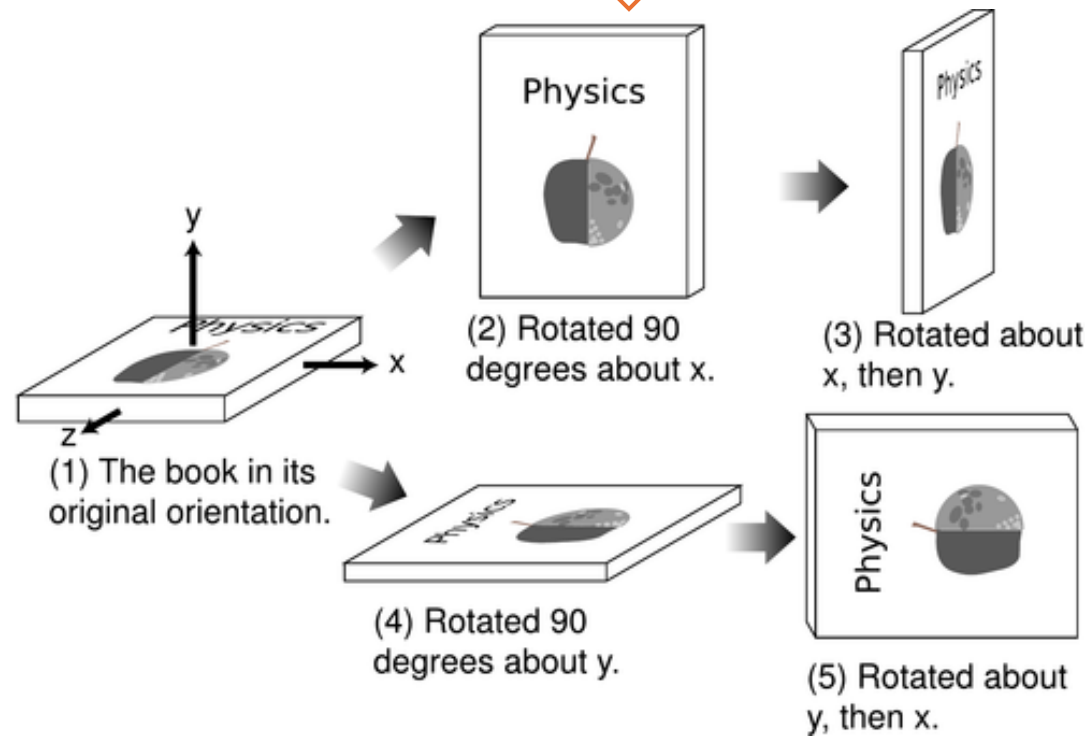
$${}^B_A T = \left[\begin{array}{ccc|c} {}^A_B R^T & -{}^A_B R^T {}^A P_{BORG} & & \\ \hline 0 & 0 & 0 & 1 \end{array} \right]. \quad (2.45)$$

$${}^A_B R = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & - & - \\ 0 & - & - \end{Bmatrix}$$

aRb would be of the form

Transformation Arithmetic

Transformation matrices multiplication is not commutative



EXAMPLE 2.7

Consider two rotations, one about \hat{Z} by 30 degrees and one about \hat{X} by 30 degrees:

$$R_z(30) = \begin{bmatrix} 0.866 & -0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} \quad (2.60)$$

$$R_x(30) = \begin{bmatrix} 1.000 & 0.000 & 0.000 \\ 0.000 & 0.866 & -0.500 \\ 0.000 & 0.500 & 0.866 \end{bmatrix} \quad (2.61)$$

$$R_z(30)R_x(30) = \begin{bmatrix} 0.87 & -0.43 & 0.25 \\ 0.50 & 0.75 & -0.43 \\ 0.00 & 0.50 & 0.87 \end{bmatrix}$$

$$\neq R_x(30)R_z(30) = \begin{bmatrix} 0.87 & -0.50 & 0.00 \\ 0.43 & 0.75 & -0.50 \\ 0.25 & 0.43 & 0.87 \end{bmatrix} \quad (2.62)$$

Transform Equations

$${}^U_T = {}^U_T {}^A_D T;$$

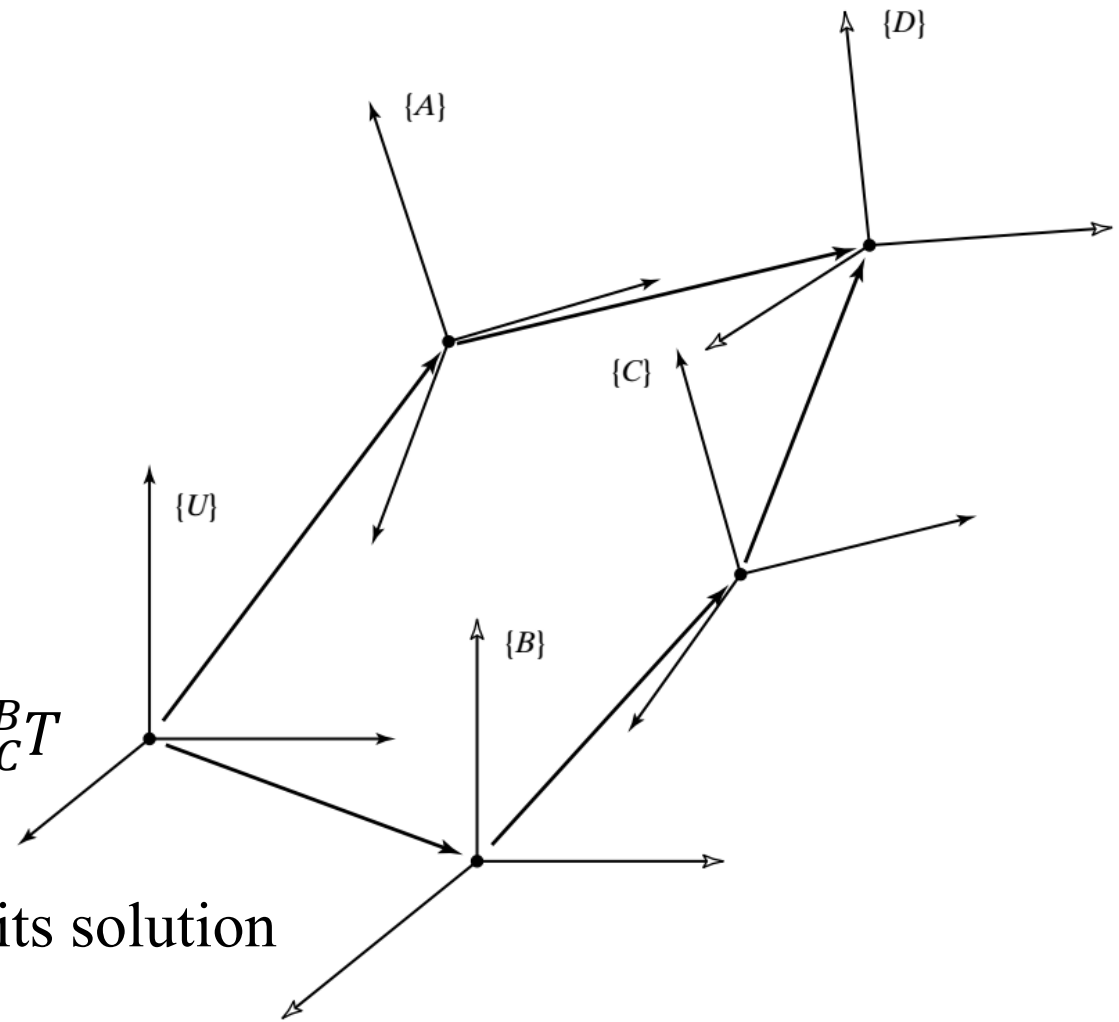
$${}^U_T = {}^U_T {}^B_C T {}^C_D T.$$

$${}^U_T {}^A_D T = {}^U_T {}^B_C T {}^C_D T.$$

Consider that all transforms are known except ${}^B_C T$
Here, we have one transform equation and
one unknown transform; hence, we easily find its solution
to be

$${}^B_C T = {}^U_B T^{-1} {}^U_T {}^A_D T {}^C_D T^{-1}$$

inverting theorem would be used to find these inverses

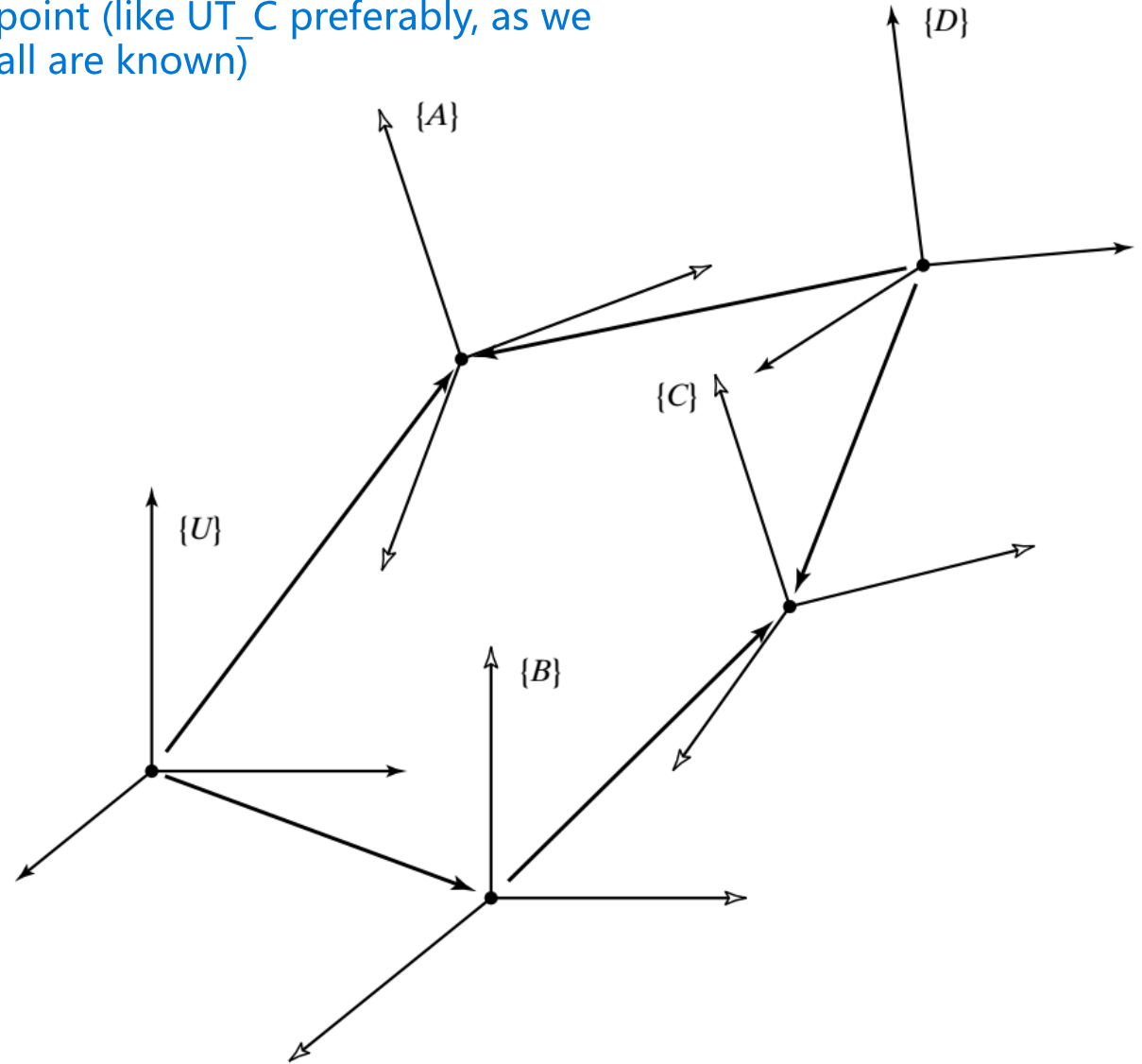


Transform Equations

if we want to calculate ${}^U T_A$ we choose 2 paths to an unknown point (like ${}^U T_C$ preferably, as we don't know its position so it could be cancelled out or known if all are known) and apply equations of the path

$${}^U T_C = {}^U T_A {}^A T_C^{-1} {}^A T_C$$
$${}^U T_C = {}^U T_B {}^B T_C.$$

$${}^U T_A = {}^U T_B {}^B T_C {}^A T_C^{-1} {}^A T_A.$$



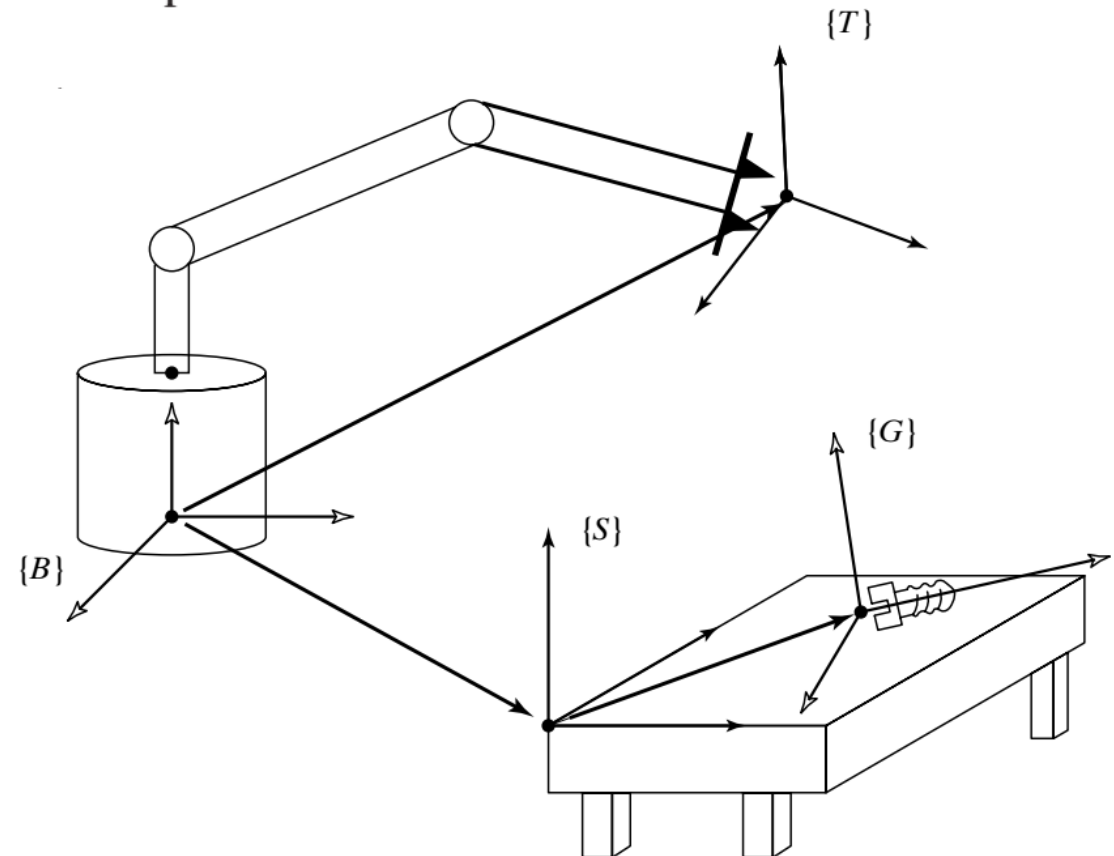
Example

Assume that we know the transform ${}^B_T T$ in Fig. 2.16, which describes the frame at the manipulator's fingertips $\{T\}$ relative to the base of the manipulator, $\{B\}$, that we know where the tabletop is located in space relative to the manipulator's base (because we have a description of the frame $\{S\}$ that is attached to the table as shown, ${}^B_S T$), and that we know the location of the frame attached to the bolt lying on the table relative to the table frame—that is, ${}^S_G T$. Calculate the position and orientation of the bolt relative to the manipulator's hand, ${}^T_G T$.

$${}^B_G T = {}^B_S T {}^S_G T \quad {}^B_G T = {}^B_T T {}^T_G T$$

Equating above two to get the bolt frame
Relative to the tool frame

$${}^T_G T = {}^B_T T^{-1} {}^B_S T {}^S_G T$$



More on Representation of Orientation

- Orientation is by giving a 3×3 rotation matrix.
- Rotation matrices are special in that all columns are mutually orthogonal and have unit magnitude
- Determinant of Rotation matrices is always equal to +1 (Proper Orthonormal)

Cayley's formula for orthonormal matrices

For any proper orthonormal matrix R , there exists a skew-symmetric matrix S such that

$$R = (I_3 - S)^{-1}(I_3 + S) \quad S = \begin{bmatrix} 0 & -s_z & s_y \\ s_z & 0 & -s_x \\ -s_y & s_x & 0 \end{bmatrix}.$$

More on Representation of Orientation

- Therefore any 3×3 rotation matrix can be specified by just three parameters
- This means there are six constraints on the nine elements of a rotation matrix
- A human operator at a computer terminal who wishes to type in the specification of the desired orientation of a robot's hand would have a hard time inputting a nine-element matrix with orthonormal columns
- A representation that requires only three numbers would be simpler

$$|\hat{X}| = 1,$$

$$|\hat{Y}| = 1,$$

$$|\hat{Z}| = 1,$$

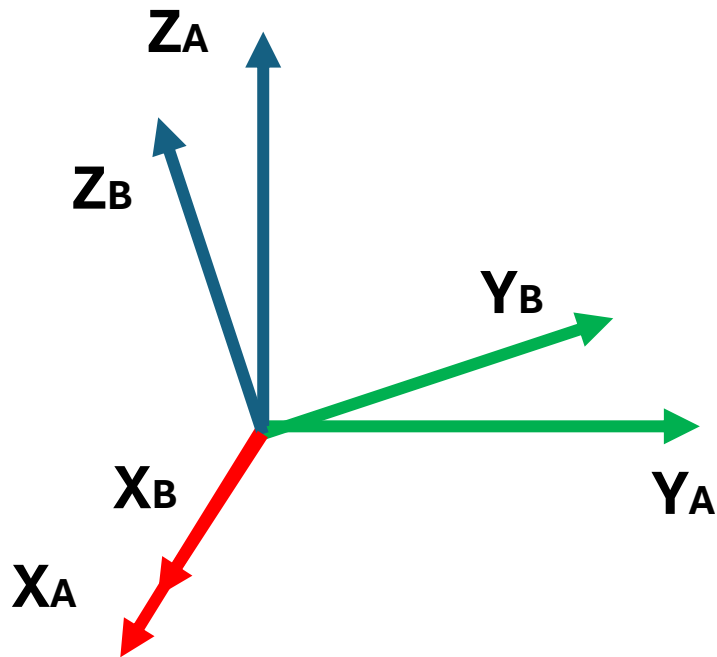
$$\hat{X} \cdot \hat{Y} = 0,$$

$$\hat{X} \cdot \hat{Z} = 0,$$

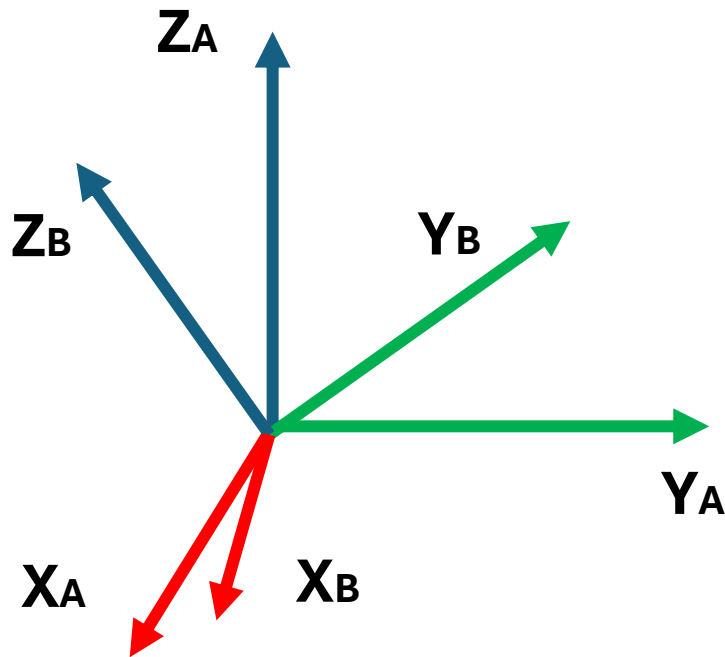
$$\hat{Y} \cdot \hat{Z} = 0.$$

X-Y-Z Fixed Angles

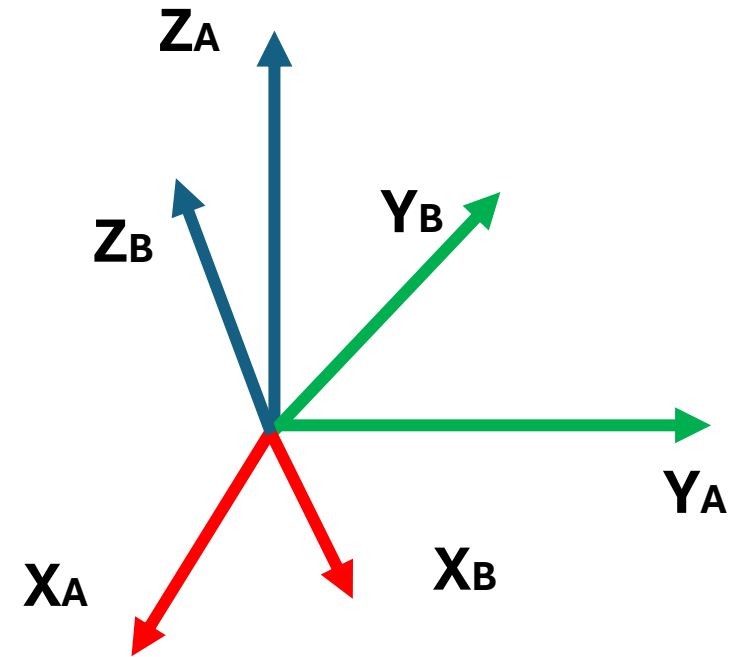
Start with the frame coincident with a known reference frame $\{A\}$. Rotate $\{B\}$ first about \hat{X}_A by an angle γ , then about \hat{Y}_A by an angle β , and, finally, about \hat{Z}_A by an angle α .



Rotation about X_A -axis



Rotation about Y_A -axis



Rotation about Z_A -axis

X-Y-Z Fixed Angles

Start with the frame coincident with a known reference frame $\{A\}$. Rotate $\{B\}$ first about \hat{X}_A by an angle γ , then about \hat{Y}_A by an angle β , and, finally, about \hat{Z}_A by an angle α .

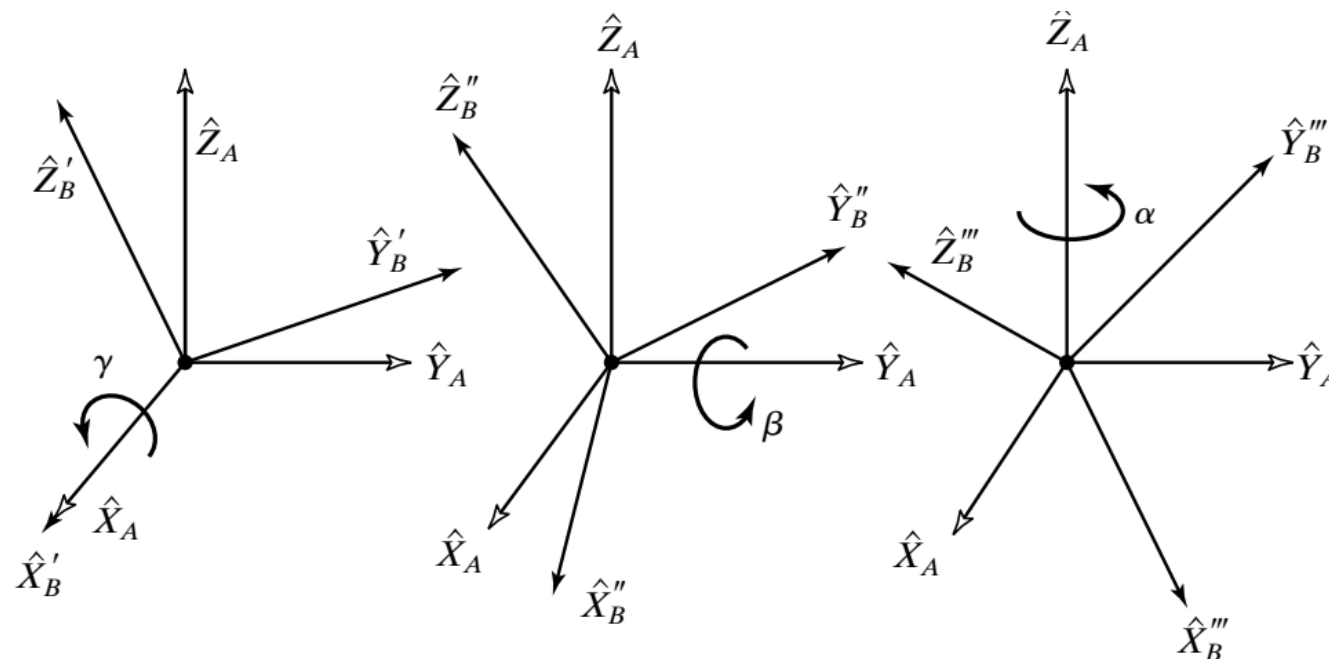


FIGURE 2.17: X–Y–Z fixed angles. Rotations are performed in the order $R_X(\gamma)$, $R_Y(\beta)$, $R_Z(\alpha)$.

X-Y-Z Fixed Angles

$${}^A_B R_{XYZ}(\gamma, \beta, \alpha) = \overset{2^{nd}}{\underbrace{R_Z(\alpha)}} \overset{1^{st}}{\underbrace{R_Y(\beta)}} \underbrace{R_X(\gamma)}$$
$$= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}$$

$${}^A_B R_{XYZ}(\gamma, \beta, \alpha) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}$$

The inverse problem, that of extracting equivalent X-Y-Z fixed angles from a rotation matrix, is often of interest

X-Y-Z Fixed Angles

$${}^A_B R_{XYZ}(\gamma, \beta, \alpha) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$${}^A_B R_{XYZ}(\gamma, \beta, \alpha) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}$$

$$\beta = \text{Atan2}(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}),$$

$$\alpha = \text{Atan2}(r_{21}/c\beta, r_{11}/c\beta),$$

$$\gamma = \text{Atan2}(r_{32}/c\beta, r_{33}/c\beta),$$

X-Y-Z Fixed Angles

Although a second solution exists, by using the positive square root in the formula for β , we always compute the single solution for which $-90.0^\circ \leq \beta \leq 90.0^\circ$.

If $\beta = 90.0^\circ$, then a solution can be calculated to be

$$\beta = 90.0^\circ,$$

$$\alpha = 0.0,$$

$$\gamma = \text{Atan2}(r_{12}, r_{22}).$$

If $\beta = -90.0^\circ$, then a solution can be calculated to be

$$\beta = -90.0^\circ,$$

$$\alpha = 0.0,$$

$$\gamma = -\text{Atan2}(r_{12}, r_{22}).$$

³Atan2(y, x) computes $\tan^{-1}(\frac{y}{x})$ but uses the signs of both x and y to identify the quadrant in which the resulting angle lies. For example, $\text{Atan2}(-2.0, -2.0) = -135^\circ$, whereas $\text{Atan2}(2.0, 2.0) = 45^\circ$, a distinction which would be lost with a single-argument arc tangent function. We are frequently computing angles that can range over a full 360° , so we will make use of the Atan2 function regularly. Note that Atan2 becomes undefined when both arguments are zero. It is sometimes called a “4-quadrant arc tangent,” and some programming-language libraries have it predefined.

Frame {B} was initially coincident with {A}. We then rotated {B} about Z_A -axis by 60. Then we rotated about Y_A -axis by 45 and finally we rotated it about X_A -axis by 30. Calculate the resultant rotation matrix.

$$R_X(30) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C30 & -S30 \\ 0 & S30 & C30 \end{bmatrix}$$

$$R_Y(45) = \begin{bmatrix} C45 & 0 & S45 \\ 0 & 1 & 0 \\ -S45 & 0 & C45 \end{bmatrix}$$

$$R_Z(60) = \begin{bmatrix} C60 & -S60 & 0 \\ S60 & C60 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^A_B R_{ZYX}(60,45,30) = R_X(30) \cdot R_Y(45) \cdot R_Z(60) =$$

$$\begin{bmatrix} 0.35 & -0.61 & 0.71 \\ 0.93 & 0.13 & -0.35 \\ 0.13 & 0.78 & 0.61 \end{bmatrix}$$

Find the Fixed angles of rotation (γ , β , α) for the following XYZ rotation matrix.

$${}^A_B R = \begin{bmatrix} 0.9077 & -0.2946 & 0.2989 \\ 0.3304 & 0.9408 & -0.0760 \\ -0.2588 & 0.1677 & 0.9513 \end{bmatrix}$$

$$\beta = \text{Atan2}(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}),$$

$$\alpha = \text{Atan2}(r_{21}/c\beta, r_{11}/c\beta),$$

$$\gamma = \text{Atan2}(r_{32}/c\beta, r_{33}/c\beta),$$

$$\beta = 15^\circ$$

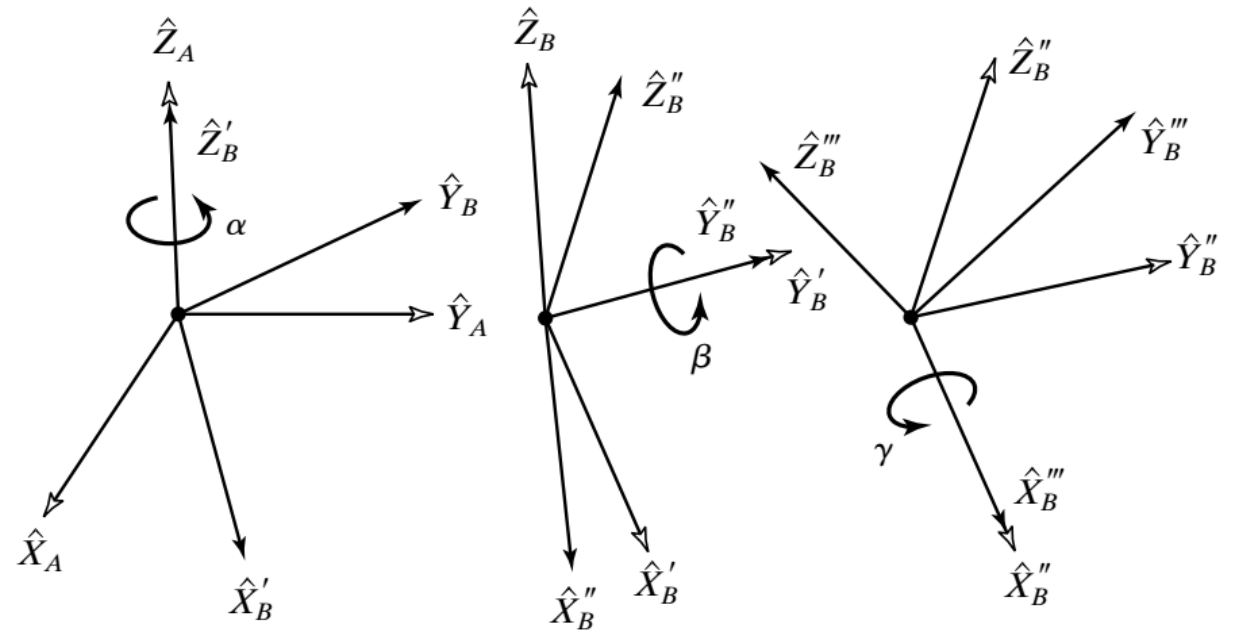
$$\alpha = 20^\circ$$

$$\gamma = 10^\circ$$

Z-Y-X Euler Angles

Start with the frame coincident with a known frame $\{A\}$. Rotate $\{B\}$ first about \hat{Z}_B by an angle α , then about \hat{Y}_B by an angle β , and, finally, about \hat{X}_B by an angle γ .

- In this representation, each rotation is performed about an axis of the moving system $\{B\}$ rather than one of the fixed reference $\{A\}$
- Such sets of three rotations are called **Euler Angles**



Z-Y-X Euler Angles

$${}^A_B R_{Z'Y'X'} = R_Z(\alpha) R_Y(\beta) R_X(\gamma)$$

$$= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}$$

$${}^A_B R_{Z'Y'X'}(\alpha, \beta, \gamma) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}$$

Relations for finding Z-Y-X Euler angles from a given rotation matrix are exactly same as the X-Y-Z Fixed Angles (as the rotation matrix is same)

Z-Y-Z Euler angles

Start with the frame coincident with a known frame $\{A\}$. Rotate $\{B\}$ first about \hat{Z}_B by an angle α , then about \hat{Y}_B by an angle β , and, finally, about Z_b by an angle γ .

$${}^A_B R_{Z'Y'Z'}(\alpha, \beta, \gamma) = \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}$$

if $\sin \beta \neq 0$, it follows that

$$\beta = \text{Atan2}(\sqrt{r_{31}^2 + r_{32}^2}, r_{33}),$$

$$\alpha = \text{Atan2}(r_{23}/s\beta, r_{13}/s\beta),$$

$$\gamma = \text{Atan2}(r_{32}/s\beta, -r_{31}/s\beta).$$

Z-Y-Z Euler angles

If $\beta = 0.0$, then a solution can be calculated to be

$$\beta = 0.0,$$

$$\alpha = 0.0,$$

$$\gamma = \text{Atan2}(-r_{12}, r_{11}).$$

If $\beta = 180.0^\circ$, then a solution can be calculated to be

$$\beta = 180.0^\circ,$$

$$\alpha = 0.0,$$

$$\gamma = \text{Atan2}(r_{12}, -r_{11}).$$