Discrete Mathematics for Computer Science

Department of Computer Science

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Reference Book: Discrete Mathematics and its applications BY Kenneth H. Rosen – 8th edition

Lecture 10

Chapter 2. Basic Structures

2.3 Functions

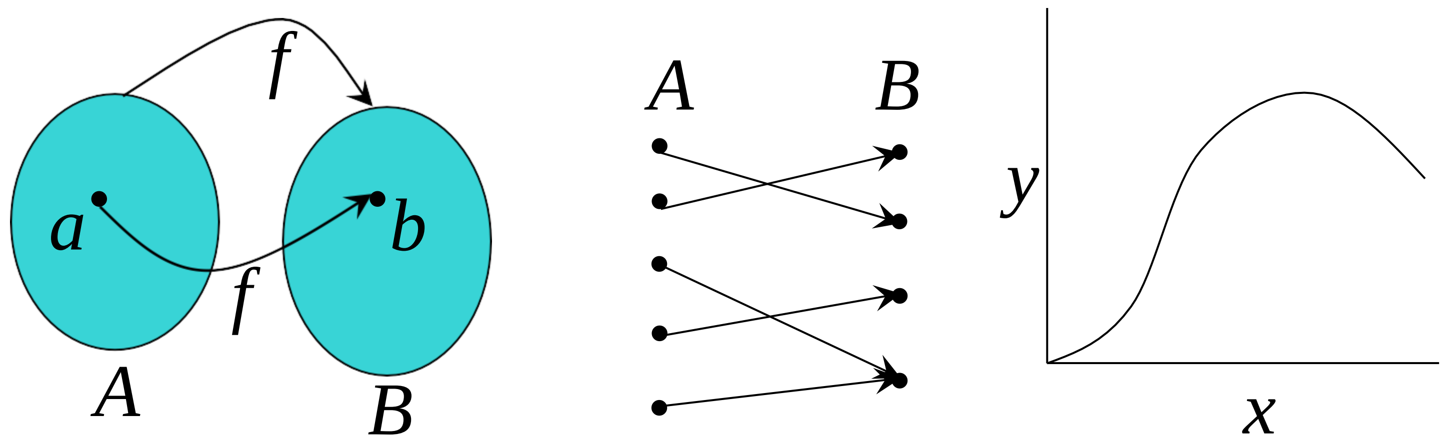
2.3 Functions

- From calculus, you are familiar with the concept of a real-valued function *f*, which assigns to each number $x \in \mathbf{R}$ a value $y = f(x)$, where $y \in \mathbf{R}$.

- But, the notion of a function can also be naturally generalized to the concept of assigning elements of *any* set to elements of *any* set. (Also known as a *map*.)

Function: Formal Definition

- For any sets A and B, we say that a ***function* (*or "mapping"*) *f from A to B* ($f : A \rightarrow B$)** is a particular assignment of **exactly one element** $f(x) \in B$ to **each element** $x \in A$.
- Functions can be represented graphically in several ways:



Like Venn diagrams   Bipartite Graph   Plot

Some Function Terminology

- If it is written that $f : A \rightarrow B$, and $f(a) = b$ (where $a \in A$ and $b \in B$), then we say:
  - A is the ***domain*** of f
  - B is the ***codomain*** of f
  - b is the ***image*** of a under f
    - a can not have more than 1 image
  - a is a ***pre-image*** of b under f
    - b may have more than 1 pre-image
  - The ***range*** $R \subseteq B$ of f is $R = \{b \mid \exists a\ f(a) = b\}$

Range versus Codomain

- The range of a function might *not* be its whole codomain.
- The codomain is the set that the function is *declared* to map all domain values into.
- The range is the *particular* set of values in the codomain that the function *actually* maps elements of the domain to.

Range vs. Codomain: Example

- Suppose I declare that: "*f* is a function mapping students in this class to the set of grades {A, B, C, D, F}."

- At this point, you know *f* 's codomain is: {A, B, C, D , F }, and its range is unknown!

- Suppose the grades turn out all As and Bs.

- Then the range of *f* is {A, B}, but its codomain is still {A, B, C, D , F } !

Function Operators

- $+$, $\times$ ("plus","times") are binary operators over **R**. (Normal addition & multiplication.)
- Therefore, we can also add and multiply two *real-valued functions* f,g: $\mathbf{R} \to \mathbf{R}$:
  - $(f + g)$: $\mathbf{R} \to \mathbf{R}$, where $(f + g)(x) = f(x) + g(x)$
  - (fg): $\mathbf{R} \to \mathbf{R}$, where $(fg)(x) = f(x)g(x)$
- Example 6:

  Let f and g be functions from **R** to **R** such that $f(x) = x^2$ and $g(x) = x - x^2$. What are the functions $f + g$ and fg?

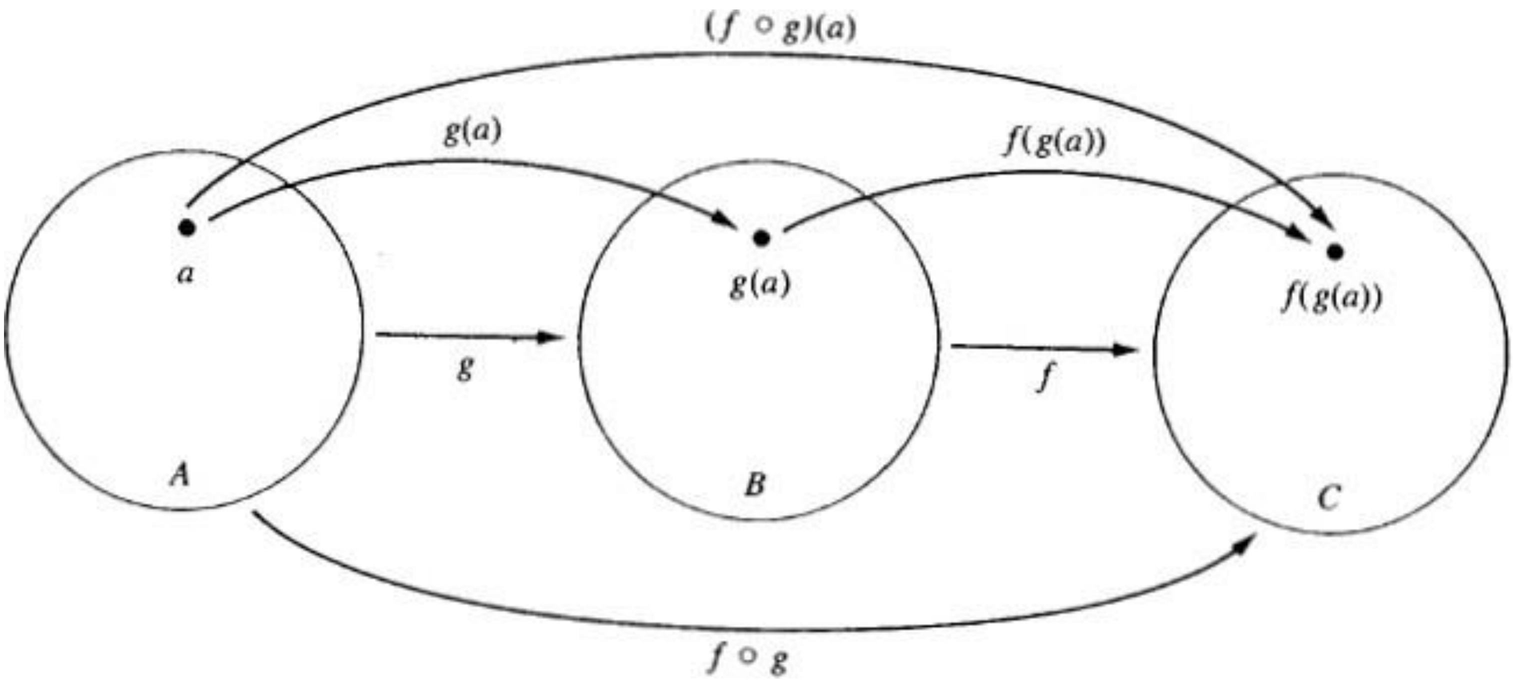
Function Composition Operator

Note the match here. It's necessary!

- For functions $g: A \to B$ and $f: B \to C$, there is a special operator called ***compose*** ("$\circ$").
  - It <u>composes</u> (creates) a new function from *f* and *g* by applying *f* to the result of applying *g*.
  - We say $(f \circ g): A \to C$, where $(f \circ g)(a) = f(g(a))$.
  - Note: $f \circ g$ cannot be defined unless range of *g* is a subset of the domain of *f*.
  - Note $g(a) \in B$, so $f(g(a))$ is defined and $\in C$.
  - Generally, $f \circ g \neq g \circ f$.

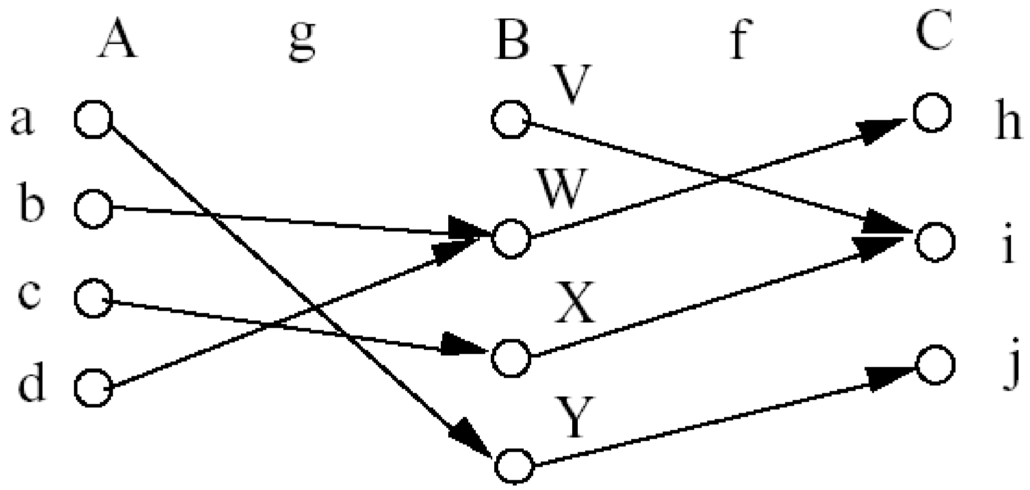
Function Composition Illustration

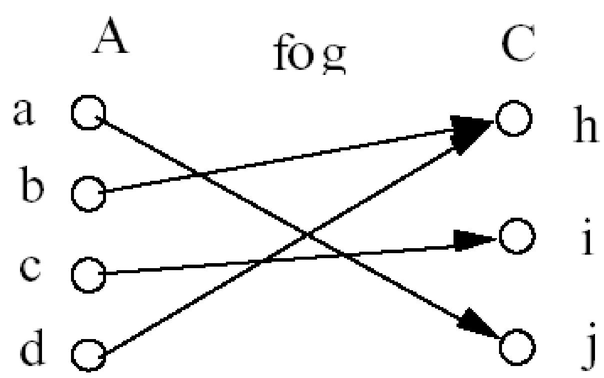
- $g: A \to B, \quad f : B \to C$



Function Composition: Example

- $g: A \to B, \quad f : B \to C$





Function Composition: Example

- Example 20: Let $g: \{a, b, c\} \to \{a, b, c\}$ such that

$$g(a) = b,\ g(b) = c,\ g(c) = a.$$

Let $f : \{a, b, c\} \to \{1, 2, 3\}$ such that

$$f(a) = 3,\ f(b) = 2,\ f(c) = 1.$$

What is the composition of f and g, and what is the composition of g and f ?

  - $f \circ g: \{a, b, c\} \to \{1, 2, 3\}$ such that

$$(f \circ g)(a) = 2,\ (f \circ g)(b) = 1,\ (f \circ g)(c) = 3.$$

$$\boxed{\begin{aligned} (f \circ g)(a) &= f(g(a)) = f(b) = 2 \\ (f \circ g)(b) &= f(g(b)) = f(c) = 1 \\ (f \circ g)(c) &= f(g(c)) = f(a) = 3 \end{aligned}}$$

  - $g \circ f$ is not defined (why?)

Function Composition: Example

- If $f(x) = x^2$ and $g(x) = 2x + 1$, then what is the composition of f and g, and what is the composition of g and f ?
  - $(f \circ g)(x) = f(g(x))$
    $= f(2x+1)$
    $= (2x+1)^2$
  - $(g \circ f)(x) = g(f(x))$
    $= g(x^2)$
    $= 2x^2 + 1$

Note that $f \circ g \neq g \circ f$. $(4x^2+4x+1 \neq 2x^2+1)$

Images of Sets under Functions

- Given $f : A \rightarrow B$, and $S \subseteq A$,
- The ***image*** of S under f is simply the set of all images (under f) of the elements of S.

$$f(S) = \{f(t) \mid t \in S\}$$
$$= \{b \mid \exists t \in S: f(t) = b\}.$$

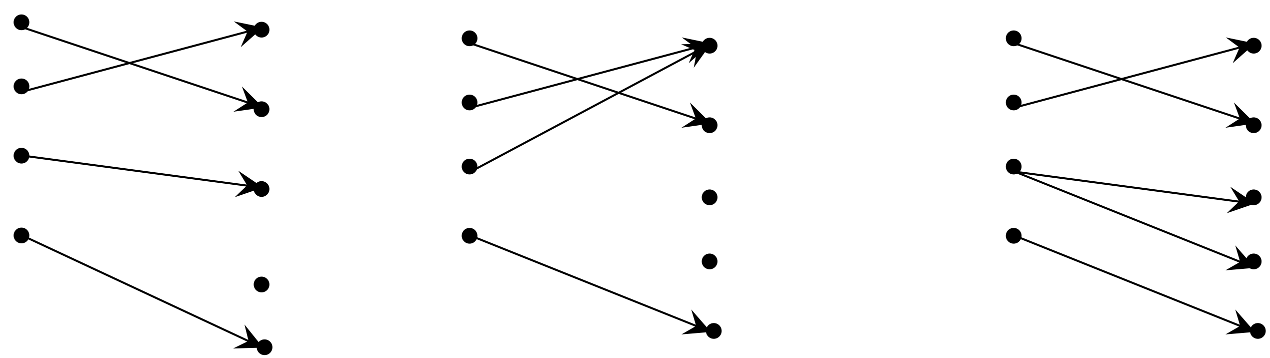
- Note the range of f can be defined as simply the image (under f) of f's domain.
- Let $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4\}$ with $f(a) = 2$, $f(b) = 1$, $f(c) = 4$, $f(d) = 1$, and $f(e) = 1$. The image of the subset $S = \{b, c, d\}$ is the set $f(S) = \{1, 4\}$.

One-to-One Functions

- A function f is ***one-to-one*** (*1–1*), or ***injective***, or an ***injection***, iff $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f (i.e. every element of its range has *only* 1 pre-image).
  - Formally, given $f : A \rightarrow B$,
    "f is injective": $\forall a,b\ (f(a) = f(b) \rightarrow a = b)$ or equivalently $\forall a,b\ (a \neq b \rightarrow f(a) \neq f(b))$
- Only <u>one</u> element of the domain is mapped <u>to</u> any given <u>one</u> element of the range.
  - Domain & range have the same cardinality. What about codomain?

One-to-One Illustration

- Bipartite (2-part) graph representations of functions that are (or not) one-to-one:



One-to-one Not one-to-one Not even a function!

- <u>Example 8</u>:
  Is the function $f : \{a, b, c, d\} \rightarrow \{1, 2, 3, 4, 5\}$ with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, and $f(d) = 3$ one-to-one?
- <u>Example 9</u>:
  Let $f : \mathbf{Z} \rightarrow \mathbf{Z}$ such that $f(x) = x^2$. Is f one-to-one? NO

Sufficient Conditions for 1–1ness

- For functions *f* over numbers, we say:
  - *f* is ***strictly*** (or ***monotonically***) ***increasing*** iff $x > y \rightarrow f(x) > f(y)$ for all x, y in domain;
  - *f* is ***strictly*** (or ***monotonically***) ***decreasing*** iff $x > y \rightarrow f(x) < f(y)$ for all x, y in domain;

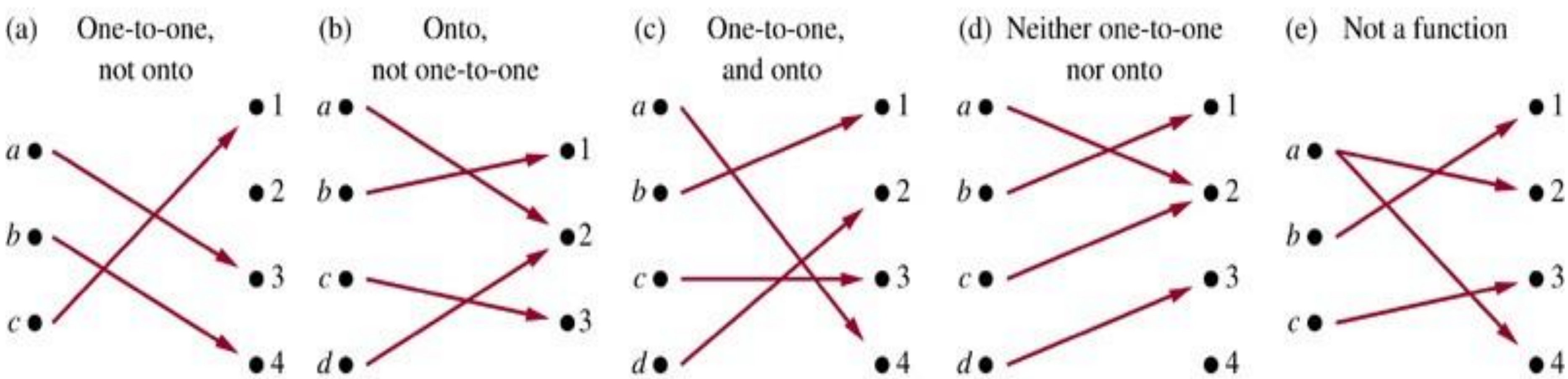
- If *f* is either strictly increasing or strictly decreasing, then *f* is one-to-one.
  - *E.g.* x^3

Onto (Surjective) Functions

- A function $f : A \rightarrow B$ is ***onto*** or ***surjective*** or a ***surjection*** iff <u>for every element $b \in B$</u> there is an element $a \in A$ with $f(a) = b$ ($\forall b \in B, \exists a \in A: f(a) = b$) (i.e. its range is equal to its codomain).
- Think: An *onto* function maps the set A <u>onto</u> (over, covering) the *entirety* of the set B, not just over a piece of it.

Illustration of Onto

- Some functions that are, or are not, *onto* their codomains:



- Example13: Is the function $f(x) = x + 1$ from the set of integers to the set of integers onto?

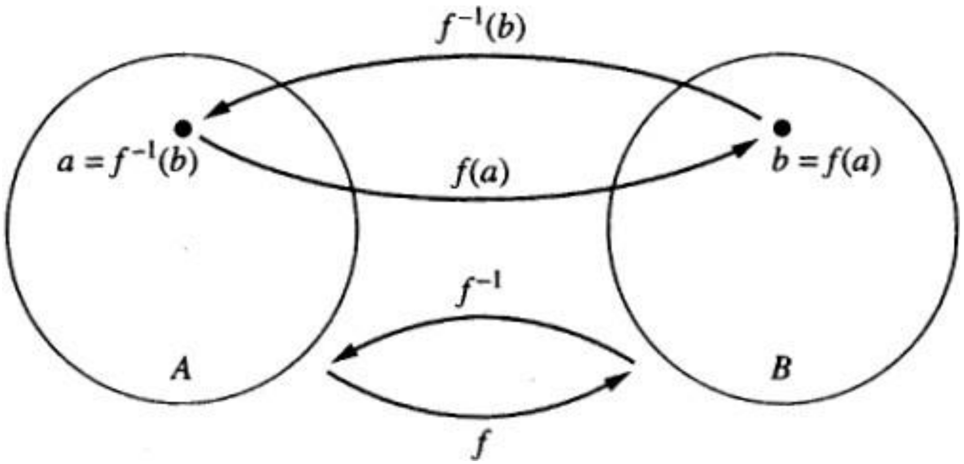
Bijections and Inverse Function

- A function f is said to be *a* ***one-to-one correspondence***, or *a* ***bijection***, or *reversible*, or *invertible*, iff it is both one-to-one and onto.

- Let $f : A \rightarrow B$ be a bijection.
  The ***inverse function*** of f is the function that assigns to an element $b \in B$ the unique element $a \in A$ such that $f(a) = b$.
  The inverse function of f is denoted by $f^{-1}: B \rightarrow A$.
  Hence, $f^{-1}(b) = a$ when $f(a) = b$.

Inverse Function Illustration

- Let $f: A \to B$ be a bijection



- Example 16: Let $f : \{a, b, c\} \to \{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, $f(c) = 1$. Is f invertible, and if it is, what is its inverse?

  Yes. $f^{-1}(1) = c$, $f^{-1}(2) = a$, $f^{-1}(3) = b$

- Example 18: Let f be the function from **R** to **R** with $f(x) = x^2$. Is f invertible?

  No. f is not a one-to-one function. So it's not invertible.

The Identity Function

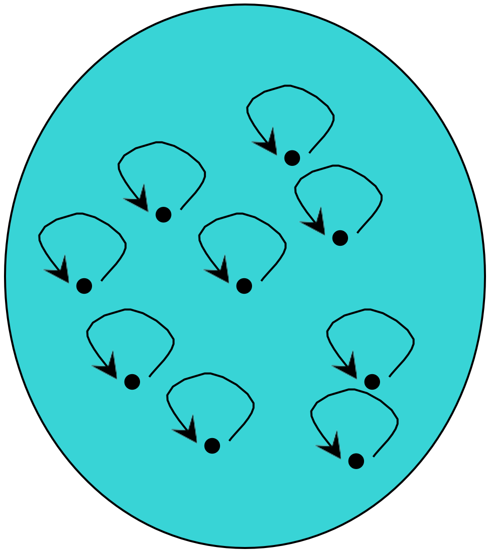
- For any domain A, the ***identity function*** $I: A \to A$ (also written as I_A, **1**, $\mathbf{1}_A$) is the unique function such that $\forall a \in A$: $I(a) = a$.
- Note that the identity function is always both one-to-one and onto (i.e., bijective).
- For a bijection $f : A \to B$ and its inverse function $f^{-1}: B \to A$,

$$f^{-1} \circ f = I_A$$

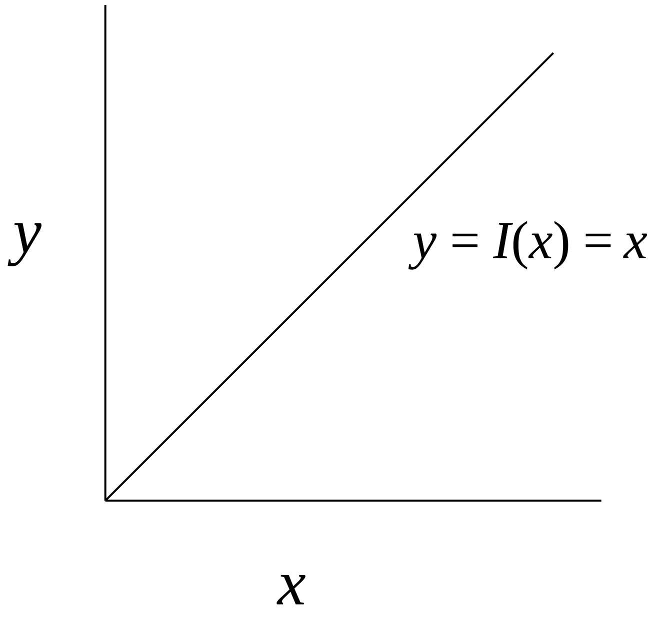
- Some identity functions you've seen:
  - $+\ 0$, $\times\ 1$, $\wedge$ **T**, $\vee$ **F**, $\cup\ \varnothing$, $\cap\ U$.

Identity Function Illustrations

- The identity function:



Domain and range



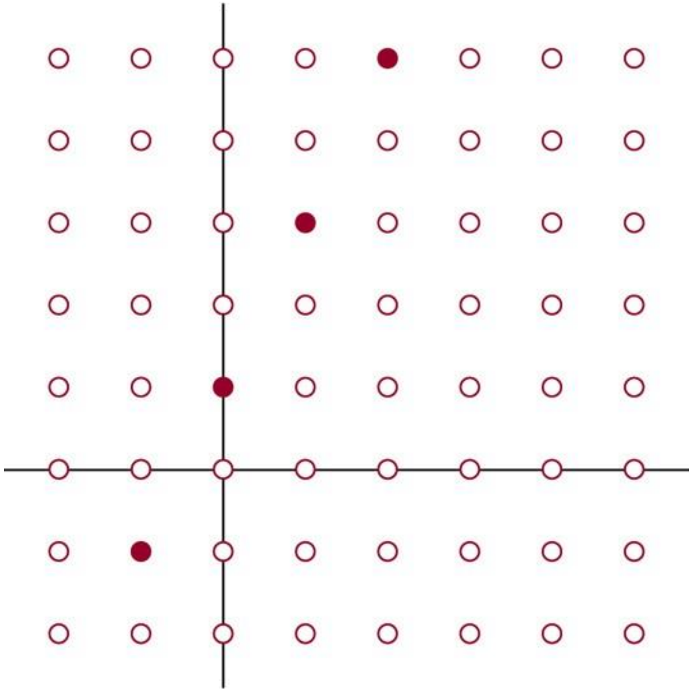
Graphs of Functions

- We can represent a function $f : A \rightarrow B$ as a set of ordered pairs $\{(a, f(a)) \mid a \in A\}$.

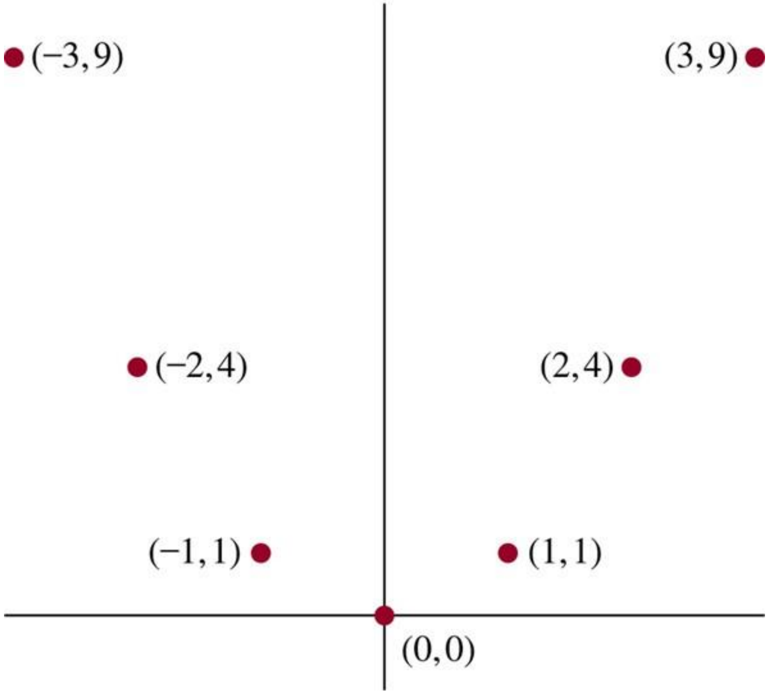
← The function's *graph*.

- Note that $\forall a \in A$, there is only 1 pair (a, b).
  - Later (ch.9): *relations* loosen this restriction.
- For functions over numbers, we can represent an ordered pair (x, y) as a point on a plane.
  - A function is then drawn as a curve (set of points), with only one y for each x.

Graphs of Functions: Examples

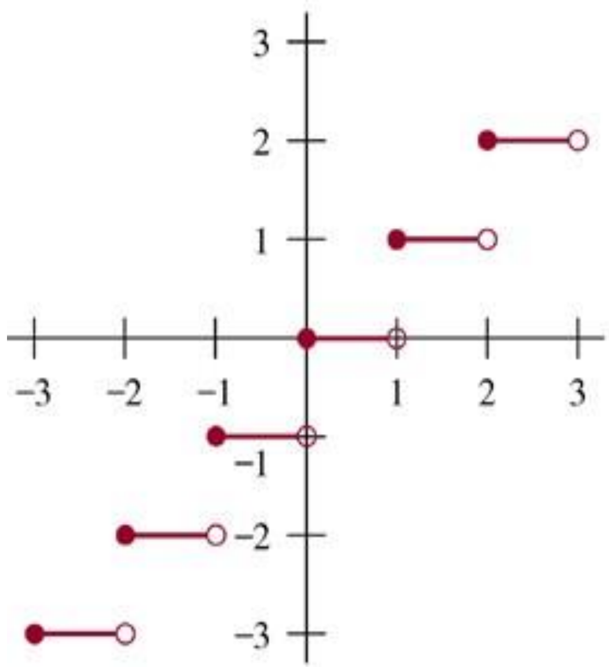


The graph of $f(n) = 2n + 1$ from **Z** to **Z**

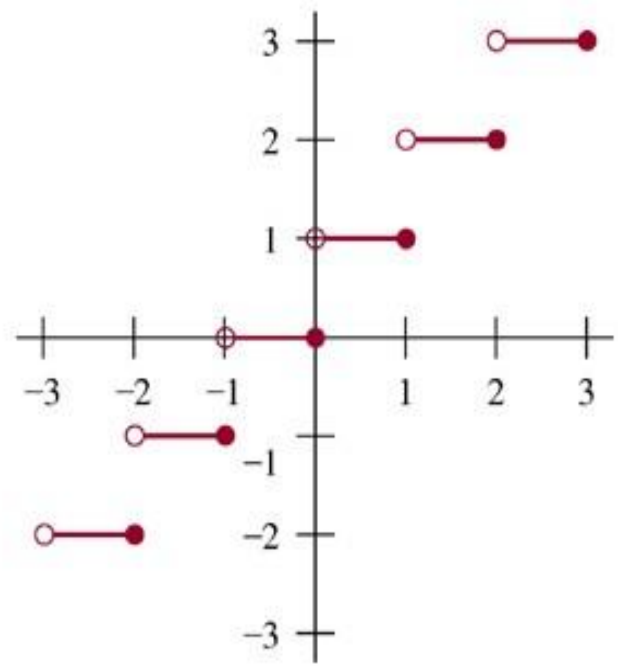


The graph of $f(x) = x^2$ from **Z** to **Z**

Plots with Floor/Ceiling: Example



$y = \lfloor x \rfloor$



$y = \lceil x \rceil$

Applications of Floor/Ceiling:

- Data stored on a computer disk or transmitted over a data network are usually represented as a string of bytes. Each byte is made up of 8 bits. How many bytes are required to encode 100 bits of data?

Solution: To determine the number of bytes needed, we determine the smallest integer that is at least as large as the quotient when 100 is divided by 8, the number of bits in a byte. Consequently, 100/8 = 12.5 = 13 bytes are required.