Lecture No.9 Set identities

SET IDENTITIES:

Let A, B, C be subsets of a universal set U.

1. Idempotent Laws
   a. $A \cup A = A$ b. $A \cap A = A$
2. Commutative Laws
   a. $A \cup B = B \cup A$ b. $A \cap B = B \cap A$
3. Associative Laws
   a. $A \cup (B \cup C) = (A \cup B) \cup C$
   b. $A \cap (B \cap C) = (A \cap B) \cap C$
4. Distributive Laws
   a. $A \cup (B \cap C) = (A \cup B) \cap (A \cup B)$
   b. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
5. Identity Laws
   a. $A \cup \varnothing = A$ b. $A \cap \varnothing = \varnothing$
   c. $A \cup U = U$ d. $A \cap U = A$
6. Complement Laws
   a. $A \cup A^c = U$ b. $A \cap A^c = \varnothing$
   c. $U^c = \varnothing$ d. $\varnothing^c = U$
8. Double Complement Law
   $(A^c)^c = A$
9. DeMorgan's Laws
   a. $(A \cup B)^c = A^c \cap B^c$ b. $(A \cap B)^c = A^c \cup B^c$
10. Alternative Representation for Set Difference
   $A - B = A \cap B^c$
11. Subset Laws
   a. $A \cup B \subseteq C$ *iff* $A \subseteq C$ and $B \subseteq C$
   b. $C \subseteq A \cap B$ *iff* $C \subseteq A$ and $C \subseteq B$
12. Absorption Laws
   a. $A \cup (A \cap B) = A$ b. $A \cap (A \cup B) = A$

EXERCISE:

1. **$A \subseteq A \cup B$**
2. **$A - B \subseteq A$**
3. **If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$**
4. **$A \subseteq B$ if, and only if, $B^c \subseteq A^c$**

1. Prove that $A \subseteq A \cup B$

SOLUTION

Here in order to prove the identity you should remember the definition of Subset of a set. We will take the arbitrary element of a set then show that, that element is the member of the other then the first set is the subset of the other. So

Let x be an arbitrary element of A, that is $x \in A$.

$\Rightarrow$ $x \in A$ or $x \in B$

$\Rightarrow \quad x \in A \cup B$

But x is an arbitrary element of A.

$\therefore \quad A \subseteq A \cup B$ (proved)

1. **Prove that $A - B \subseteq A$**

SOLUTION

Let $x \in A - B$

$\Rightarrow \quad x \in A$ and $x \notin B$ (by definition of $A - B$)

$\Rightarrow \quad x \in A$ (in particular)

But x is an arbitrary element of $A - B$

$\therefore \quad A - B \subseteq A$ (proved)

1. **Prove that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$**

SOLUTION

Suppose that $A \subseteq B$ and $B \subseteq C$

Consider $x \in A$

$\Rightarrow \quad x \in B$ (as $A \subseteq B$)

$\Rightarrow \quad x \in C$ (as $B \subseteq C$)

But x is an arbitrary element of A

$\therefore \quad A \subseteq C$ (proved)

1. **Prove that $A \subseteq B$ iff $B^c \subseteq A^c$**

SOLUTION:

Suppose $A \subseteq B$ {To prove $B^c \subseteq A^c$}

Let $x \in B^c$

$\Rightarrow \quad x \notin B$ (by definition of B^c)

$\Rightarrow \quad x \notin A$

$\Rightarrow \quad x \in A^c$ (by definition of A^c)

Now we know that implication and its contrapositivity are logically equivalent and the contrapositive statement of if $x \in A$ then $x \in B$ is: if $x \notin B$ then $x \notin A$ which is the definition of the $A \subseteq B$. Thus if we show for any two sets A and B, if $x \notin B$ then $x \notin A$ it means that

$A \subseteq B$. Hence

But x is an arbitrary element of B^c

$\therefore B^c \subseteq A^c$

Conversely,

Suppose $B^c \subseteq A^c$ {To prove $A \subseteq B$}

Let $x \in A$

$\Rightarrow \quad x \notin A^c$ (by definition of A^c)

$\Rightarrow \quad x \notin B^c$ ($\therefore B^c \subseteq A^c$)

$\Rightarrow \quad x \in B$ (by definition of B^c)

But x is an arbitrary element of A.

$\therefore A \subseteq B$ (proved)

EXERCISE:

Let A and B be subsets of a universal set U.

Prove that $A - B = A \cap B^c$.

SOLUTION

Let $x \in A - B$

$\Rightarrow$ $x \in A$ and $x \notin B$ (definition of set difference)

$\Rightarrow$ $x \in A$ and $x \in B^c$ (definition of complement)

$\Rightarrow$ $x \in A \cap B^c$ (definition of intersection)

But x is an arbitrary element of A – B so we can write

$\therefore A - B \subseteq A \cap B^c$…………(1)

Conversely,

let $y \in A \cap B^c$

$\Rightarrow$ $y \in A$ and $y \in B^c$ (definition of intersection)

$\Rightarrow$ $y \in A$ and $y \notin B$ (definition of complement)

$\Rightarrow$ $y \in A - B$ (definition of set difference)

But y is an arbitrary element of $A \cap B^c$

$\therefore A \cap B^c \subseteq A - B$…………(2)

From (1) and (2) it follows that

$A - B = A \cap B^c$ (as required)

EXERCISE:

Prove the DeMorgan's Law: $(A \cup B)^c = A^c \cap B^c$

PROOF

Let $x \in (A \cup B)^c$

$\Rightarrow$ $x \notin A \cup B$ (definition of complement)

$x \notin A$ and $x \notin B$ (DeMorgan's Law of Logic)

$\Rightarrow$ $x \in A^c$ and $x \in B^c$ (definition of complement)

$\Rightarrow$ $x \in A^c \cap B^c$ (definition of intersection)

But x is an **arbitrary** element of $(A \cup B)^c$ so we have proved that

$\therefore (A \cup B)^c \subseteq A^c \cap B^c$………(1)

Conversely

let $y \in Ac \cap B^c$

$\Rightarrow$ $y \in A^c$ and $y \in B^c$ (definition of intersection)

$\Rightarrow$ $y \notin A$ and $y \notin B$ (definition of complement)

$\Rightarrow$ $y \notin A \cup B$ (DeMorgan's Law of Logic)

$\Rightarrow$ $y \in (A \cup B)^c$ (definition of complement)

But y is an arbitrary element of $A^c \cap B^c$

$\therefore A^c \cap B^c \subseteq (A \cup B)^c$………………(2)

From (1) and (2) we have

$(A \cup B)^c = A^c \cap B^c$

Which is the Demorgan`s Law.

EXERCISE:

Prove the associative law: $A \cap (B \cap C) = (A \cap B) \cap C$

PROOF:

Consider $x \in A \cap (B \cap C)$

$\Rightarrow$ $x \in A$ and $x \in B \cap C$ (definition of intersection)

$\Rightarrow$ $x \in A$ and $x \in B$ and $x \in C$ (definition of intersection)

$\Rightarrow$ $x \in A \cap B$ and $x \in C$ (definition of intersection)

$\Rightarrow$ $x \in (A \cap B) \cap C$ (definition of intersection)

But x is an arbitrary element of $A \cap (B \cap C)$

$$\therefore\ A \cap (B \cap C) \subseteq (A \cap B) \cap C \ldots\ldots(1)$$

Conversely

let $y \in (A \cap B) \cap C$

$\Rightarrow\ y \in A \cap B$ and $y \in C$ (definition of intersection)

$\Rightarrow\ y \in A$ and $y \in B$ and $y \in C$ (definition of intersection)

$\Rightarrow\ y \in A$ and $y \in B \cap C$ (definition of intersection)

$\Rightarrow\ y \in A \cap (B \cap C)$ (definition of intersection)

But y is an arbitrary element of $(A \cap B) \cap C$

$$\therefore\ (A \cap B) \cap C \subseteq A \cap (B \cap C)\ldots\ldots..(2)$$

From (1) & (2), we conclude that

$$A \cap (B \cap C) = (A \cap B) \cap C \qquad \text{(proved)}$$

EXERCISE:

Prove the distributive law: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

PROOF:

Let $x \in A \cup (B \cap C)$

$\Rightarrow\ x \in A$ or $x \in B \cap C$ (definition of union)

Now since we have $x \in A$ or $x \in B \cap C$ it means that either x is in A or in $A \cap B$ it is in the $A \cup (B \cap C)$ so in order to show that $A \cup (B \cap C)$ is the subset of $(A \cup B) \cap (A \cup C)$ we will consider both the cases when x is iu A or x is in $B \cap C$ hence we will consider the two cases.

CASE I:

(when $x \in A$)

$\Rightarrow x \in A \cup B$ and $x \in A \cup C$ (definition of union)

Hence,

$x \in (A \cup B) \cap (A \cup C)$ (definition of intersection)

CASE II:

(when $x \in B \cap C$)

We have $x \in B$ and $x \in C$ (definition of intersection)

Now $x \in B \Rightarrow x \in A \cup B$ (definition of union)

and $x \in C \Rightarrow x \in A \cup C$ (definition of union)

Thus $x \in A \cup B$ and $x \in A \cup C$

$\Rightarrow\ x \in (A \cup B) \cap (A \cup C)$

In both of the cases $x \in (A \cup B) \cap (A \cup C)$

Accordingly,

$$A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)\ldots\ldots..(1)$$

Conversely,

Suppose $x \in (A \cup B) \cap (A \cup C)$

$\Rightarrow\ x \in (A \cup B)$ and $x \in (A \cup C)$ (definition of intersection)

Consider the two cases $x \in A$ and $x \notin A$

CASE I: (when $x \in A$)

We have $x \in A \cup (B \cap C)$ (definition of union)

CASE II: (when $x \notin A$)

Since $x \in A \cup B$ and $x \notin A$, therefore $x \in B$

Also, since $x \in A \cup C$ and $x \notin A$, therefore $x \in C$. Thus $x \in B$ and $x \in C$
That is, $x \in B \cap C$
$\Rightarrow$ $x \in A \cup (B \cap C)$ (definition of union)
Hence in both cases
$x \in A \cup (B \cap C)$
$\therefore (A \cup B) \cap C\ (A \cup C) \subseteq A \cup (B \cap C)$……..(2)
By (1) and (2), it follows that
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (proved)

EXERCISE:

For any sets A and B if $A \subseteq B$ then
(a) $A \cap B = A$ (b) $A \cup B = B$

SOLUTION:

(a) Let $x \in A \cap B$
$\Rightarrow$ $x \in A$ and $x \in B$
$\Rightarrow$ $x \in A$ (in particular)
Hence $A \cap B \subseteq A$…………..(1)

Conversely,
let $x \in A$.
Then $x \in B$ (since $A \subseteq B$)
Now $x \in A$ and $x \in B$, therefore $x \in A \cap B$
Hence, $A \subseteq A \cap B$…………..(2)
From (1) and (2) it follows that
$A = A \cap B$ (proved)

(b) Prove that $A \cup B = B$ when $A \subseteq B$

SOLUTION:

Suppose that $A \subseteq B$. Consider $x \in A \cup B$.
CASE I (when $x \in A$)
Since $A \subseteq B$, $x \in A \Rightarrow x \in B$
CASE II (when $x \notin A$)
Since $x \in A \cup B$, we have $x \in B$
Thus $x \in B$ in both the cases, and we have
$A \cup B \subseteq B$……………(1)

Conversely

let $x \in B$. Then clearly, $x \in A \cup B$
Hence $B \subseteq A \cup B$…………….(2)
Combining (1) and (2), we deduce that
$A \cup B = B$ (proved)

USING SET IDENTITIES:

For all subsets A and B of a universal set U, prove that

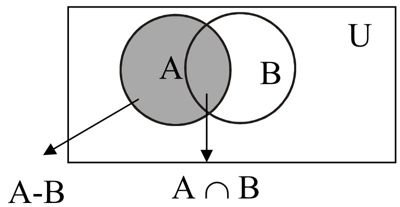
$$(A - B) \cup (A \cap B) = A$$

PROOF:

LHS $= (A - B) \cup (A \cap B)$
$= (A \cap B^c) \cup (A \cap B)$ (Alternative representation for set difference)

$= A \cap (B^c \cup B)$ Distributive Law
$= A \cap U$ Complement Law
$= A$ Identity Law
$= RHS$ (proved)

The result can also be seen by Venn diagram.



EXERCISE:

For any two sets A and B prove that $A - (A - B) = A \cap B$

SOLUTION

LHS $= A - (A - B)$
$= A - (A \cap B^c)$ Alternative representation for set difference
$= A \cap (A \cap B^c)^c$ Alternative representation for set difference
$= A \cap (A^c \cup (B^c)^c)$ DeMorgan's Law
$= A \cap (A^c \cup B)$ Double Complement Law
$= (A \cap A^c) \cup (A \cap B)$ Distributive Law
$= \varnothing \cup (A \cap B)$ Complement Law
$= A \cap B$ Identity Law
$= RHS$ (proved)

EXERCISE:

For all set A, B, and C prove that $(A - B) - C = (A - C) - B$

SOLUTION

LHS $= (A - B) - C$
$= (A \cap B^c) - C$ Alternative representation of set difference
$= (A \cap B^c) \cap C^c$ Alternative representation of set difference
$= A \cap (B^c \cap C^c)$ Associative Law
$= A \cap (C^c \cap B^c)$ Commutative Law
$= (A \cap C^c) \cap B^c$ Associative Law
$= (A - C) \cap B^c$ Alternative representation of set difference
$= (A - C) - B$ Alternative representation of set difference
$= RHS$ (proved)

EXERCISE:

Simplify $(B^c \cup (B^c - A))^c$

SOLUTION

$(B^c \cup (B^c - A))^c = (B^c \cup (B^c \cap A^c))^c$
Alternative representation for set difference
$= (B^c)^c \cap (B^c \cap A^c)^c$ DeMorgan's Law
$= B \cap ((B^c)^c \cup (A^c)^c)$ DeMorgan's Law
$= B \cap (B \cup A)$ Double Complement Law

$= B$ Absorption Law

is the simplified form of the given expression.

PROVING SET IDENTITIES BY MEMBERSHIP TABLE:

Prove the following using Membership Table:

(i) $A - (A - B) = A \cap B$

(ii) $(A \cap B)^c = A^c \cup B^c$

(iii) $A - B = A \cap B^c$

$A - (A - B) = A \cap B$

A	B	A-B	A-(A-B)	A∩B
1	1	0	1	1
1	0	1	0	0
0	1	0	0	0
0	0	0	0	0

$(A \cap B)^c = A^c \cup B^c$

A	**B**	**$A \cap B$**	**$(A \cap B)^c$**	**A^c**	**B^c**	**$A^c \cup B^c$**
1	1	1	0	0	0	0
1	0	0	1	0	1	1
0	1	0	1	1	0	1
0	0	0	1	1	1	1

SOLUTION (iii):

A	B	$A - B$	B^c	$A \cap B^c$
1	1	0	0	0
1	0	1	1	1
0	1	0	0	0
0	0	0	1	0