Chapter 4

GRADIENT , DIVERGENCE , AND CURL

4.1 INTRODUCTION

In this chapter , we shall discuss three physically and geometrically important concepts related to scalar and vector fields , namely , the gradient , the divergence , and the curl . These concepts of vector calculus play an important role in engineering , physics , and several branches of applied mathematics , for example, mechanics , fluid mechanics , elasticity , and electromagnetic theory .

4.2 SCALAR AND VECTOR FIELDS

SCALAR POINT FUNCTION AND SCALAR FIELD

If to each point (x, y, z) of a region R in space there corresponds a unique number or scalar $\phi(x, y, z)$, then ϕ is called a scalar function of position or scalar point function in R. The set of all values of ϕ in R constitutes a scalar field .

Examples of scalar fields are :

(i) $\phi(x, y, z) = x^3 y - z^2$ defines a scalar point function and hence is a scalar field .

(ii) The temperature $T(x, y, z)$ within a body B is a scalar point function because at each point of the body there is one and only one temperature . Hence it defines a scalar field , namely , the temperature field in B .

VECTOR POINT FUNCTION AND VECTOR FIELD

If to each point (x, y, z) of a region R in space there corresponds a unique vector $\vec{A}(x, y, z)$, then $\vec{A}$ is called a vector function of position or vector point function in R and is written as

$$\vec{A} = \vec{A}(x, y\ z)$$
$$= A_1(x, y, z)\hat{i} + A_2(x, y, z)\hat{j} + A_3(x, y, z)\hat{k}$$

The set of all values of $\vec{A}$ in R constitutes a vector field as shown in figure (4.1) .

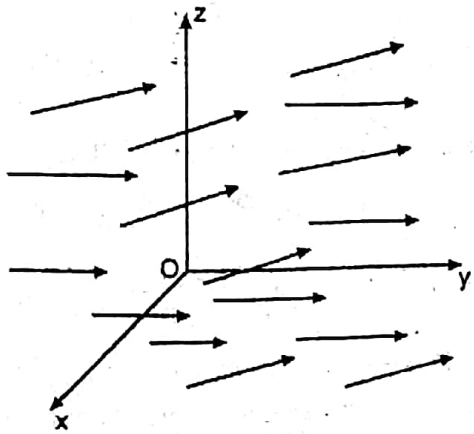

Figure (4.1)

Examples of vector fields are :

(i) $\vec{A}(x, y, z) = x^2 y\hat{i} - 2yz^3\hat{j} + x^2 z\hat{k}$ defines a vector point function and hence is a vector field .

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(ii) The set of tangent vectors of a curve C and the set of normal vectors of a surface S define vector fields as shown in figure (4.2) .

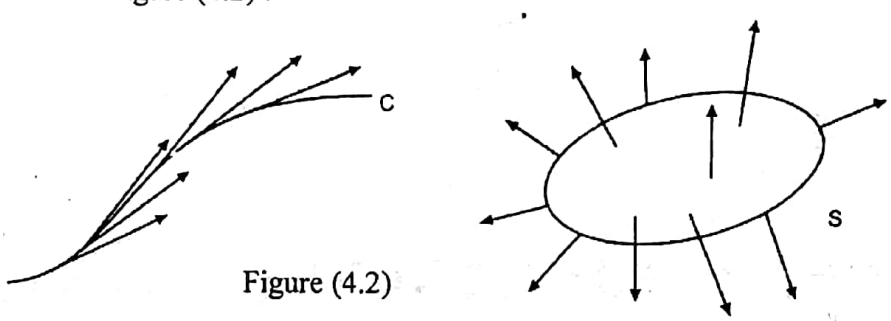

Figure (4.2)

4.3 LEVEL SURFACES

Let $\phi(x, y, z)$ be a scalar point function in a region R of space . Then for each fixed value of the constant C , the equation

Eq of Surface

$$\phi(x, y, z) = C$$

represents a surface in three–dimensional space , and if C is allowed to take a variety of different values , we obtain a family of surfaces , which are called the level surfaces of the function ϕ . On each level surface the function ϕ has the same value at each point on it . For example , for any fixed positive value of C , the equation

$$\phi = x^2 + y^2 + z^2 = C$$

represents a sphere with centre at the origin and radius $\sqrt{C}$. If C varies , the level surfaces of ϕ are the concentric spheres .

If ϕ represents the potential then the level surfaces are called the **equipotential** surfaces and if ϕ represents the temperature then the level surfaces are called the **isothermal** surfaces .

Now we show that one and only one level surface will pass through each point in space . Let $\phi(x, y, z) = C$ and $\phi(x, y, z) = C'$ be two level surfaces of ϕ corresponding to two different values C and C' respectively . If these surfaces intersect each other , then at each point of intersection , the value of ϕ is C and C' as shown in the figure (4.3) .

Since by definition $\phi(x, y, z)$ is a single valued function it is possible only if $C = C'$. Thus no two level surfaces corresponding to two different values of the constant can intersect , i.e. through each point in space there passes one , and only one , level surface of the function ϕ .

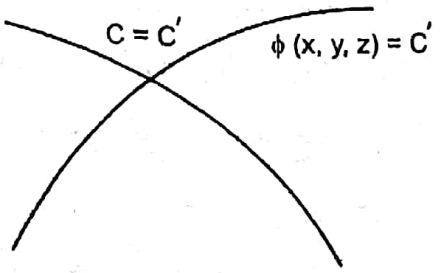

Figure (4.3) $\phi(x, y, z) = C$

4.4 THE OPERATOR DEL

The vector differential operator del , written ∇ , is defined by

3-D vector differential operator

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

This vector operator is useful in defining three quantities which arise in physical applications and are known as the gradient , the divergence , and the curl .

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4.5 GRADIENT OF A SCALAR POINT FUNCTION

Let $\phi(x, y, z)$ be a differentiable scalar point function in a certain region R of space. Then the gradient of ϕ, written $\nabla\phi$ or grad ϕ is defined by

$$\nabla\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

Note that $\nabla\phi$ is a vector quantity. If ϕ is constant, then $\nabla\phi = 0$.

EXAMPLE (1): **If $\phi(x, y, z) = 3x^2y - y^3z^2$, find $\nabla\phi$ at the point $(1, -2, -1)$.**

SOLUTION: We have

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \hat{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \hat{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= 6xy\hat{i} + (3x^2 - 3y^2z^2)\hat{j} - 2y^3z\hat{k}$$

$$(\nabla\phi)_{(1,-2,-1)} = -12\hat{i} - 9\hat{j} - 16\hat{k}$$

$\nabla(0) = 0$

4.6 PROPERTIES OF THE GRADIENT

THEOREM (4.1): **If ϕ and ψ are differentiable scalar point functions and C is a constant, then**

(i) $\nabla(C\phi) = C\nabla\phi$ (ii) $\nabla(\phi^n) = n\phi^{n-1}\nabla\phi$

(iii) $\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$ (iv) $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$

(v) $\nabla\left(\frac{\phi}{\psi}\right) = \frac{\psi\nabla\phi - \phi\nabla\psi}{\psi^2}, \quad \psi \neq 0$

PROOF: We have

(i)
$$\nabla(C\phi) = \hat{i}\frac{\partial}{\partial x}(C\phi) + \hat{j}\frac{\partial}{\partial y}(C\phi) + \hat{k}\frac{\partial}{\partial z}(C\phi)$$

$$= \hat{i}C\frac{\partial\phi}{\partial x} + \hat{j}C\frac{\partial\phi}{\partial y} + \hat{k}C\frac{\partial\phi}{\partial z}$$

$$= C\left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right) = C\nabla\phi$$

(ii)
$$\nabla(\phi^n) = \frac{\partial}{\partial x}(\phi^n)\hat{i} + \frac{\partial}{\partial y}(\phi^n)\hat{j} + \frac{\partial}{\partial z}(\phi^n)\hat{k}$$

$$= n\phi^{n-1}\frac{\partial\phi}{\partial x}\hat{i} + n\phi^{n-1}\frac{\partial\phi}{\partial y}\hat{j} + n\phi^{n-1}\frac{\partial\phi}{\partial z}\hat{k}$$

$$= n\phi^{n-1}\left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right) = n\phi^{n-1}\nabla\phi$$

(iii)
$$\nabla(\phi + \psi) = \hat{i}\frac{\partial}{\partial x}(\phi + \psi) + \hat{j}\frac{\partial}{\partial y}(\phi + \psi) + \hat{k}\frac{\partial}{\partial z}(\phi + \psi)$$

$$= \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right) + \left(\hat{i}\frac{\partial\psi}{\partial x} + \hat{j}\frac{\partial\psi}{\partial y} + \hat{k}\frac{\partial\psi}{\partial z}\right)$$

$$= \nabla\phi + \nabla\psi$$

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(iv) $$\nabla(\phi\psi) = \hat{i}\frac{\partial}{\partial x}(\phi\psi)+\hat{j}\frac{\partial}{\partial y}(\phi\psi)+\hat{k}\frac{\partial}{\partial z}(\phi\psi)$$

$$= \hat{i}\left(\phi\frac{\partial\psi}{\partial x}+\psi\frac{\partial\phi}{\partial x}\right)+\hat{j}\left(\phi\frac{\partial\psi}{\partial y}+\psi\frac{\partial\phi}{\partial y}\right)+\hat{k}\left(\phi\frac{\partial\psi}{\partial z}+\psi\frac{\partial\phi}{\partial z}\right)$$

$$= \phi\left(\hat{i}\frac{\partial\psi}{\partial x}+\hat{j}\frac{\partial\psi}{\partial y}+\hat{k}\frac{\partial\psi}{\partial z}\right)+\psi\left(\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \phi\nabla\psi+\psi\nabla\phi$$

(v) $$\nabla\left(\frac{\phi}{\psi}\right) = \nabla\left(\phi\frac{1}{\psi}\right) = \phi\nabla\left(\frac{1}{\psi}\right)+\frac{1}{\psi}\nabla\phi$$

$$= \phi\left(-\frac{1}{\psi^2}\right)\nabla\psi+\frac{1}{\psi}\nabla\phi$$

$$= \frac{\psi\nabla\phi-\phi\nabla\psi}{\psi^2}, \quad \psi\neq 0$$

THEOREM (4.2): **Prove that**

Imp

(i) $\nabla f(r) = \dfrac{f'(r)\vec{r}}{r}$

(ii) $\nabla(r^n) = n r^{n-2}\vec{r}$, **where n is any real number .**

(iii) $\nabla r = \hat{r}$ (iv) $\nabla\left(\dfrac{1}{r}\right) = -\dfrac{\vec{r}}{r^3}$

PROOF: We know that

(i) $$\nabla f(r) = \frac{\partial}{\partial x}f(r)\hat{i}+\frac{\partial}{\partial y}f(r)\hat{j}+\frac{\partial}{\partial z}f(r)\hat{k}$$

$$= \frac{\partial f}{\partial r}\frac{\partial r}{\partial x}\hat{i}+\frac{\partial f}{\partial r}\frac{\partial r}{\partial y}\hat{j}+\frac{\partial f}{\partial r}\frac{\partial r}{\partial z}\hat{k}$$

$$= \frac{df}{dr}\frac{\partial r}{\partial x}\hat{i}+\frac{df}{dr}\frac{\partial r}{\partial y}\hat{j}+\frac{df}{dr}\frac{\partial r}{\partial z}\hat{k}$$

Since $r=\sqrt{x^2+y^2+z^2}$, it follows that $\dfrac{\partial r}{\partial x}=\dfrac{x}{r}$, $\dfrac{\partial r}{\partial y}=\dfrac{y}{r}$, $\dfrac{\partial r}{\partial z}=\dfrac{z}{r}$

$$\nabla f(r) = f'(r)\frac{x}{r}\hat{i}+f'(r)\frac{y}{r}\hat{j}+f'(r)\frac{z}{r}\hat{k}$$

$$= \frac{f'(r)}{r}(x\hat{i}+y\hat{j}+z\hat{k})$$

$$= \frac{f'(r)\vec{r}}{r}$$

(ii) Using (i) $\nabla(r^n) = \dfrac{n r^{n-1}\vec{r}}{r} = n r^{n-2}\vec{r}$

Note that if $\vec{r} = r\hat{r}$, where $\hat{r}$ is a unit vector in the direction of $\vec{r}$, then $\nabla r^n = n r^{n-1}\hat{r}$.

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ALTERNATIVE METHOD

$$\nabla(r^n) = \nabla(x^2+y^2+z^2)^{n/2}$$
$$= \hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{n/2} + \hat{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)^{n/2} + \hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)^{n/2}$$
$$= n(x^2+y^2+z^2)^{(n/2)-1}(x\hat{i}+y\hat{j}+z\hat{k})$$
$$= n(r^2)^{(n/2)-1}\vec{r}$$
$$= n r^{n-2}\vec{r}$$

(iii) Let $n = 1$ in part (ii) then $\nabla r = \frac{\vec{r}}{r} = \hat{r}$,

(iv) Let $n = -1$ in part (ii) then $\nabla\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$

EXAMPLE (2): **Show that**

(i) $\nabla(\ln r) = \frac{\vec{r}}{r^2}$ (ii) $\nabla(e^{r^2}) = 2e^{r^2}\vec{r}$

(iii) $\nabla(r^3) = 3r\vec{r}$.

SOLUTION: (i) We know that

$$\nabla f(r) = \frac{f'(r)\vec{r}}{r} \qquad (1)$$

Let $f(r) = \ln r$ in equation (1), we get

$$\nabla(\ln r) = \frac{\frac{1}{r}\vec{r}}{r} = \frac{\vec{r}}{r^2}$$

(ii) Let $f(r) = e^{r^2}$ in equation (1), we get

$$\nabla(e^{r^2}) = \frac{e^{r^2}2r\vec{r}}{r} = 2e^{r^2}\vec{r}$$

(iii) We know that

$$\nabla(r^n) = n r^{n-2}\vec{r} \qquad (2)$$

Let $n = 3$ in equation (2) we get

$$\nabla(r^3) = 3r\vec{r}$$

4.7 GEOMETRICAL INTERPRETATION OF GRADIENT

GRADIENT AS A NORMAL VECTOR TO A SURFACE

THEOREM (4.3): **Prove that $\nabla\phi$ is a vector perpendicular to the surface $\phi(x, y, z) = C$, where C is a constant.**

PROOF: Let Γ be a curve through any point $P(x, y, z)$ which lies on the level surface

$$\phi(x, y, z) = C \qquad (1)$$

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Let the parametric equations of the curve Γ be $x = x(s)$, $y = y(s)$, $z = z(s)$.

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of the point P, then $\hat{T} = \frac{d\vec{r}}{ds}$ is the unit vector tangent to the curve Γ at P as shown in figure (4.4).

Since the curve Γ lies on the level surface, therefore, the coordinates of any point on the curve must satisfy equation (1), and so $\phi[x(s), y(s), z(s)] = C$.

Differentiating this equation w.r.t. s using the chain rule,

$$\frac{\partial\phi}{\partial x}\frac{dx}{ds} + \frac{\partial\phi}{\partial y}\frac{dy}{ds} + \frac{\partial\phi}{\partial z}\frac{dz}{ds} = 0$$

or $$\left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)\cdot\left(\frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k}\right) = 0$$

or $$\nabla\phi \cdot \frac{d\vec{r}}{ds} = 0 \quad \text{or} \quad \nabla\phi \cdot \hat{T} = 0$$

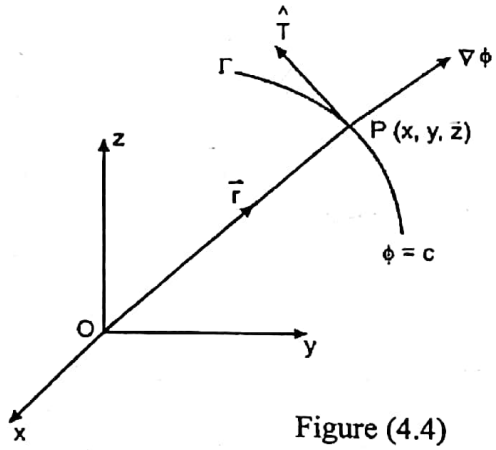

Figure (4.4)

which implies that $\nabla\phi$ is a vector perpendicular to the unit tangent vector $\hat{T}$ and therefore to the surface $\phi(x, y, z) = C$.

EXAMPLE (3): **Find a unit normal vector $\hat{n}$ to the surface given by $z = x^2 + y^2$ at the point (1, 2, 5).**

SOLUTION: Since $z = x^2 + y^2$, the surface is defined as $\phi(x, y, z) = x^2 + y^2 - z = 0$.

Then $$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(x^2+y^2-z) + \hat{j}\frac{\partial}{\partial y}(x^2+y^2-z) + \hat{k}\frac{\partial}{\partial z}(x^2+y^2-z)$$

$$= 2x\hat{i} + 2y\hat{j} - \hat{k}$$

At the point (1, 2, 5) the value of the gradient is

$$\nabla\phi = 2\hat{i} + 4\hat{j} - \hat{k}$$

Hence a unit normal vector $\hat{n}$ to the surface at the given point is

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\hat{i} + 4\hat{j} - \hat{k}}{\sqrt{(2)^2 + (4)^2 + (-1)^2}}$$

$$\hat{n} = \frac{2\hat{i} + 4\hat{j} - \hat{k}}{\sqrt{21}}$$

Another unit normal vector to the surface at the given point in the opposite direction is $-\frac{2\hat{i} + 4\hat{j} - \hat{k}}{\sqrt{21}}$

4.8 DIRECTIONAL DERIVATIVE

We know that the first partial derivatives of $\phi(x, y, z)$ are the rates of change of ϕ in the directions of the coordinate axes. It seems unnatural to restrict attention to these three directions, and we may ask for the rate of change of ϕ in any direction. This leads us to the idea of a directional derivative.

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To define the directional derivative we choose a point $P(x, y, z)$ in space and a direction at P, given by a unit vector $\hat{T}$. Let C be the ray drawn from P in the direction of $\hat{T}$, and let $P'(x+\Delta x, y+\Delta y, z+\Delta z)$ be a neighbouring point on C, whose distance from P is Δs as shown in figure (4.5). Let ϕ be a differentiable scalar point function and $\phi(x, y, z)$ and $\phi(x+\Delta x, y+\Delta y, z+\Delta z)$ be the value of this function at P and P' respectively. Then the limit

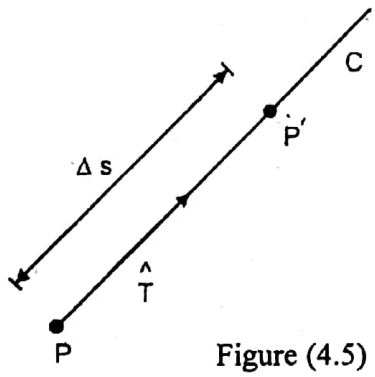

Figure (4.5)

$$\underset{\Delta s \to 0}{\mathrm{Lt}} \frac{\Delta \phi}{\Delta s} = \underset{\Delta s \to 0}{\mathrm{Lt}} \frac{\phi(P') - \phi(P)}{\Delta s} \qquad (1)$$

if it exists, is called the directional derivative of ϕ at P in the direction of $\hat{T}$ and is denoted by $\frac{\partial \phi}{\partial s}$.

Obviously, $\frac{\partial \phi}{\partial s}$ is the rate of change of ϕ at P w.r.t. the distance s measured in the direction of $\hat{T}$. Equation (1) can be written as

$$\frac{\partial \phi}{\partial s} = \underset{\Delta s \to 0}{\mathrm{Lt}} \frac{\phi(x+\Delta x, y+\Delta y, z+\Delta z) - \phi(x, y, z)}{\Delta s} \qquad (2)$$

In this way there are now infinitely many directional derivatives of ϕ at P, each corresponding to a certain direction. Now from elementary calculus, equation (2) can be written as

$$\frac{\partial \phi}{\partial s} = \frac{\partial \phi}{\partial x}\frac{dx}{ds} + \frac{\partial \phi}{\partial y}\frac{dy}{ds} + \frac{\partial \phi}{\partial z}\frac{dz}{ds}$$

$$= \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right) \cdot \left(\frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k}\right)$$

$$= \nabla \phi \cdot \frac{d\vec{r}}{ds} = \nabla \phi \cdot \hat{T} \qquad (3)$$

Since $\hat{T}$ is a unit vector, directional derivative of ϕ $\left(\text{i.e. } \frac{\partial \phi}{\partial s}\right)$ is the component of $\nabla \phi$ in the direction of this unit vector. From equation (3) we have the operator equivalence.

$$\frac{\partial}{\partial s} = \hat{T} \cdot \nabla \qquad (4)$$

This means that the operator $\hat{T} \cdot \nabla$ applied to the scalar function ϕ differentiates it w.r.t. the distance s in the direction of $\hat{T}$.

PARTICULAR CASES

(i) If $\hat{r}$ is a unit vector in the direction of the position vector $\vec{r}$, then the direction derivative of ϕ in this direction is

$$\frac{\partial \phi}{\partial r} = \nabla \phi \cdot \hat{r}$$

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(ii) If , in particular , $\hat{T}$ has the direction of the positive x–axis , then $\hat{T} = \hat{i}$ and from equation (3) , we have

$$\frac{\partial \phi}{\partial s} = \nabla \phi \cdot \hat{i}$$

$$= \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right) \cdot \hat{i} = \frac{\partial \phi}{\partial x}$$

Similarly , $\frac{\partial \phi}{\partial y} = \nabla \phi \cdot \hat{j}$ and $\frac{\partial \phi}{\partial z} = \nabla \phi \cdot \hat{k}$

It therefore , follows that the first partial derivatives $\frac{\partial \phi}{\partial x}$, $\frac{\partial \phi}{\partial y}$, and $\frac{\partial \phi}{\partial z}$ are the directional derivatives in the directions of the coordinates axes .

THEOREM (4.4): **Show that the maximum value of the directional derivative of $\phi(x, y, z)$ is equal to the magnitude of $\nabla \phi$ (i.e. $|\nabla \phi|$) and it takes place in the direction of $\nabla \phi$.**

PROOF: We know that

$$\frac{\partial \phi}{\partial s} = \nabla \phi \cdot \hat{T}$$

$$= |\nabla \phi| \left|\hat{T}\right| \cos\theta = |\nabla \phi| \cos\theta$$

where θ is the angle between $\nabla \phi$ and $\hat{T}$. Since $-1 \le \cos\theta \le 1$, therefore $\frac{\partial \phi}{\partial s}$ is maximum when $\cos\theta = 1$ or $\theta = 0^{\circ}$ i.e. when the direction of $\hat{T}$ is the direction of $\nabla \phi$ and $\left.\frac{\partial \phi}{\partial s}\right|_{max} = |\nabla \phi|$.

Thus the maximum value of the directional derivative takes place in the direction of $\nabla \phi$ and has the magnitude as $|\nabla \phi|$.

NOTE: The directional derivative $\frac{\partial \phi}{\partial s}$ is zero , when $\theta = \frac{\pi}{2}$ i.e. when $\nabla \phi$ and $\hat{T}$ are orthogonal to each other .

EXAMPLE (4): **Find the directional derivative of $\phi(x, y, z) = x^2 + y^2 + z^2$ at the point $(1, 1, 1)$ in the direction of the vector $\hat{i} + \hat{j} + \hat{k}$. Find its maximum value and the direction in which it takes place .**

SOLUTION: Since $\phi(x, y, z) = x^2 + y^2 + z^2$, therefore

$$\nabla \phi = \hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2) + \hat{j}\frac{\partial}{\partial y}(x^2+y^2+z^2) + \hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)$$

$$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

Then at the point $(1, 1, 1)$ the value of this gradient is $\nabla \phi = 2\hat{i} + 2\hat{j} + 2\hat{k}$

The unit vector in the direction of $\hat{i} + \hat{j} + \hat{k}$ is $\hat{T} = \frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{1+1+1}} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$

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Thus the required directional derivative of ϕ at (1 , 1 , 1) is

$$\frac{\partial \phi}{\partial s} = \nabla \phi . \hat{T} = (2\hat{i} + 2\hat{j} + 2\hat{k}) . \left(\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}\right)$$

$$= \frac{2}{\sqrt{3}} + \frac{2}{\sqrt{3}} + \frac{2}{\sqrt{3}} = \frac{6}{\sqrt{3}} = 2\sqrt{3}$$

Now $\quad |\nabla \phi| = \sqrt{(2)^2 + (2)^2 + (2)^2} = \sqrt{12} = 2\sqrt{3}$

The maximum value of the directional derivative $\frac{\partial \phi}{\partial s}$ at the point (1 , 1 , 1) is $|\nabla \phi| = 2\sqrt{3}$, and its direction is that of $\nabla \phi = 2\hat{i} + 2\hat{j} + 2\hat{k}$.

DIRECTIONAL DERIVATIVE ALONG A CURVE

So far , we have defined the directional derivative along a straight line . Instead of a straight line , we can consider an arbitrary smooth curve passing through the point P (x , y , z) . If s measures the distance along the curve as shown in figure (4.6) , then we know that the unit tangent vector $\hat{T}$ to the curve at P is given by

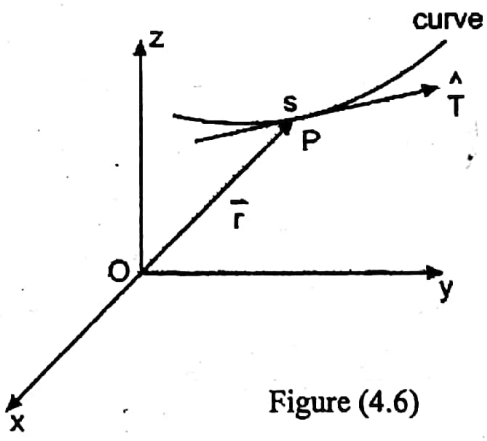

Figure (4.6)

$$\hat{T} = \frac{d\vec{r}}{ds} = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k}$$

and again , in this case , we can show that

$$\frac{\partial \phi}{\partial s} = \nabla \phi . \hat{T} = |\nabla \phi| \cos \theta$$

where θ is the angle between the unit tangent vector $\hat{T}$ to the curve at P and $\nabla \phi$.

Thus we can say that the directional derivative along an arbitrary smooth curve is the same as the directional derivative along a straight line tangent to the curve .

4.9 NORMAL DERIVATIVE

In various applications , the direction along which a directional derivative is formed is that of a specified unit normal vector $\hat{n}$ to a curve in the plane or to a surface in space . The directional derivative is then denoted by $\frac{\partial \phi}{\partial n}$ and is given by

$$\frac{\partial \phi}{\partial n} = \nabla \phi . \hat{n} \qquad (1)$$

i.e. the normal derivative of ϕ is the component of $\nabla \phi$ in the direction of the unit normal vector $\hat{n}$.

From equation (1) we have the operator equivalence $\quad \frac{\partial}{\partial n} = \hat{n} . \nabla$

which when applied to any scalar function always gives the normal derivative .

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4.10 ALTERNATIVE DEFINITION OF GRADIENT

An alternative definition of the gradient can be given as follows :

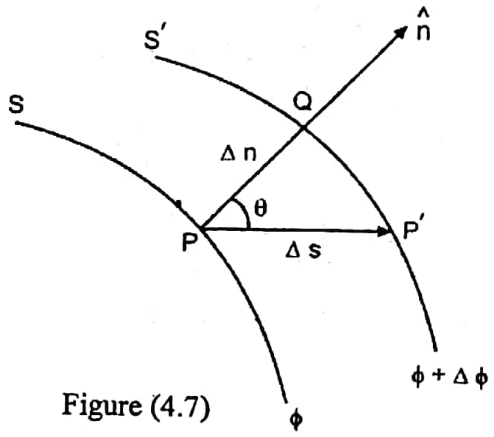

Figure (4.7)

Let S and S' be the level surfaces of the function ϕ through $P(x, y, z)$ and $P'(x + \Delta x, y + \Delta y, z + \Delta z)$ with values ϕ and $\phi + \Delta\phi$ respectively , and let Δs be the distance between the points P and P'. Draw PQ normal to S at P , to meet S' in Q and let Δn be the length PQ . Accordingly , QP' is approximately orthogonal to PQ and if θ is the angle between the normal direction and the direction PP' as shown in figure (4.7) , then $\dfrac{\Delta n}{\Delta s} = \cos\theta$

Since the change in the scalar function ϕ in passing from P to P' is $\Delta\phi$, its average rate of change per unit distance in the direction from P to P' is $\dfrac{\Delta\phi}{\Delta s}$. Now

$$\frac{\Delta\phi}{\Delta s} = \frac{\Delta\phi}{\Delta n}\frac{\Delta n}{\Delta s} = \frac{\Delta\phi}{\Delta n}\cos\theta$$

In the limit as $S' \to S$ so that $P' \to P$ keeping θ constant , this gives

$$\frac{\partial\phi}{\partial s} = \frac{\partial\phi}{\partial n}\cos\theta \qquad (1)$$

If $\hat{n}$ is the unit vector normal to the level surface S at P and having the direction PQ , and $\hat{T}$ is the unit vector in the direction PP' , then $\cos\theta = \hat{n}\cdot\hat{T}$ and equation (1) becomes

$$\frac{\partial\phi}{\partial s} = \frac{\partial\phi}{\partial n}\hat{n}\cdot\hat{T} \qquad (2)$$

Also we know that $\dfrac{\partial\phi}{\partial s} = \nabla\phi\cdot\hat{T}$ (3)

Comparison of equations (2) and (3) shows that

$$\nabla\phi = \frac{\partial\phi}{\partial n}\hat{n} \qquad (4)$$

Thus $\nabla\phi$ is a vector in the direction of the normal to the surface and has magnitude equal to the rate of change of ϕ along this normal i.e. $|\nabla\phi| = \dfrac{\partial\phi}{\partial n}$ (5)

Equation (5) states that the maximum value of the directional derivative takes place along the normal direction .

NOTE: The definition of gradient given in equation (4) is invariant in the sense that $\nabla\phi$ is independent of the choice of the coordinate system .

Also $$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} \qquad (6)$$

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Equations (4) and (6) give alternative forms for $\nabla \phi$ and we see that equation (6) is usually more convenient when ϕ is specified explicitly as a function of x, y, and z.

4.11 DIVERGENCE OF A VECTOR POINT FUNCTION

Let $\vec{A}(x, y, z)$ be a differentiable vector point function in a certain region of space. Then the divergence of $\vec{A}$ written $\nabla \cdot \vec{A}$ or div $\vec{A}$ is defined by

$$\nabla \cdot \vec{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(A_1\hat{i} + A_2\hat{j} + A_3\hat{k}\right)$$

$$= \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

Note that $\nabla \cdot \vec{A}$ is a scalar quantity. Also note that $\nabla \cdot \vec{A} \neq \vec{A} \cdot \nabla$.

If $\vec{A}$ is a constant vector, then $\nabla \cdot \vec{A} = 0$. If $\nabla \cdot \vec{A} = 0$ everywhere in some region R, then $\vec{A}$ is called a **solenoidal** vector point function in that region.

EXAMPLE (5): **If $\vec{A} = x^2 z\hat{i} - 2y^3 z^2\hat{j} + xy^2 z\hat{k}$, find $\nabla \cdot \vec{A}$ at the point $(1, -1, 1)$.**

SOLUTION:

$$\nabla \cdot \vec{A} = \frac{\partial}{\partial x}(x^2 z) + \frac{\partial}{\partial y}(-2y^3 z^2) + \frac{\partial}{\partial z}(xy^2 z),$$

$$= 2xz - 6y^2 z^2 + xy^2$$

Therefore, $(\nabla \cdot \vec{A})_{(1,-1,1)} = -3$

4.12 PROPERTIES OF THE DIVERGENCE

THEOREM (4.5): **If $\vec{A}$ and $\vec{B}$ are differentiable vector point functions, and ϕ is a differentiable scalar point function, then prove that**

(i) $\nabla \cdot (\vec{A} + \vec{B}) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B}$

(ii) $\nabla \cdot (\phi \vec{A}) = \phi(\nabla \cdot \vec{A}) + \vec{A} \cdot (\nabla \phi)$

PROOF: Let $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ and $\vec{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$, then

(i) $\vec{A} + \vec{B} = (A_1 + B_1)\hat{i} + (A_2 + B_2)\hat{j} + (A_3 + B_3)\hat{k}$

Hence $\nabla \cdot (\vec{A} + \vec{B}) = \frac{\partial}{\partial x}(A_1 + B_1) + \frac{\partial}{\partial y}(A_2 + B_2) + \frac{\partial}{\partial z}(A_3 + B_3)$

$$= \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) + \left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}\right)$$

$$= \nabla \cdot \vec{A} + \nabla \cdot \vec{B}$$

(ii) We have $\phi \vec{A} = \phi A_1\hat{i} + \phi A_2\hat{j} + \phi A_3\hat{k}$

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Hence $\nabla.(\phi\vec{A}) = \frac{\partial}{\partial x}(\phi A_1)+\frac{\partial}{\partial y}(\phi A_2)+\frac{\partial}{\partial z}(\phi A_3)$

$$= \phi\frac{\partial A_1}{\partial x}+\frac{\partial\phi}{\partial x}A_1+\phi\frac{\partial A_2}{\partial y}+\frac{\partial\phi}{\partial y}A_2+\phi\frac{\partial A_3}{\partial z}+\frac{\partial\phi}{\partial z}A_3$$

$$= \phi\left(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)+\left(\frac{\partial\phi}{\partial x}A_1+\frac{\partial\phi}{\partial y}A_2+\frac{\partial\phi}{\partial z}A_3\right)$$

$$= \phi(\nabla.\vec{A})+\left(\frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k}\right).(A_1\hat{i}+A_2\hat{j}+A_3\hat{k})$$

$$= \phi(\nabla.\vec{A})+(\nabla\phi).\vec{A}$$

$$= \phi(\nabla.\vec{A})+\vec{A}.\nabla\phi$$

Note that if ϕ is constant , then $\nabla.(\phi\vec{A}) = \phi\,\nabla.\vec{A}$.

THEOREM (4.6): **Prove that**

(i) $\nabla.\vec{r} = 3$ (ii) $\nabla.[f(r)\vec{r}] = 3f(r)+rf'(r)$

(iii) $\nabla.(r^n\vec{r}) = (n+3)r^n$ (iv) $\nabla.\left(\frac{\vec{r}}{r^3}\right) = 0$

PROOF: Let $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$, then

(i) $\nabla.\vec{r} = \frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z) = 1+1+1 = 3$.

(ii) Using theorem (4.5) part (ii) , we have

$$\nabla.[f(r)\vec{r}] = f(r)(\nabla.\vec{r})+\vec{r}.[\nabla f(r)]$$

$$= 3f(r)+\vec{r}.\frac{f'(r)\vec{r}}{r} \quad [\text{using theorem (4.2)}]$$

$$= 3f(r)+\frac{f'(r)}{r}\vec{r}.\vec{r}$$

$$= 3f(r)+rf'(r) \quad (\text{since } \vec{r}.\vec{r} = r^2)$$

(iii) Setting $f(r) = r^n$ in part (ii) , we have

$$\nabla.(r^n\vec{r}) = 3r^n+rnr^{n-1} = 3r^n+nr^n = (n+3)r^n$$

(iv) Let $n = -3$ in part (iii) , then $\nabla.\left(\frac{\vec{r}}{r^3}\right) = (-3+3)r^{-3} = 0$

EXAMPLE (6): **Find the most general differentiable function $f(r)$ so that $f(r)\vec{r}$ is solenoidal.**

SOLUTION: If $f(r)\vec{r}$ is solenoidal , then $\nabla.[f(r)\vec{r}] = 0$

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or $\quad [\nabla f(r)] \cdot \vec{r} + f(r)\, \nabla \cdot \vec{r} = 0 \qquad (1)$

But $\quad \nabla f(r) = \dfrac{f'(r)\,\vec{r}}{r}$, therefore equation (1) becomes

$$\frac{f'(r)\,\vec{r}\cdot\vec{r}}{r} + 3\,f(r) = 0 \quad \text{or} \quad r f'(r) + 3 f(r) = 0$$

or $\quad f'(r) = -\dfrac{3 f(r)}{r}$

or $\quad \dfrac{f'(r)}{f(r)} = -\dfrac{3}{r}$ (by seperating the variables)

Integrating we get $\ln f(r) = -3 \ln r + \ln C = \ln \dfrac{C}{r^3}$

or $\quad f(r) = \dfrac{C}{r^3}$ • where C is an arbitrary constant .

4.13 PHYSICAL INTERPRETATION OF THE DIVERGENCE

Consider the motion of an incompressible fluid (i.e. fluid with constant density say oil or water). Figure (4.8) shows an imaginary small rectangular parallelopiped of dimensions Δx , Δy , Δz with edges parallel to the coordinate axes and having centre at the point $P(x, y, z)$. Let $\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$ be the velocity of the fluid at P .

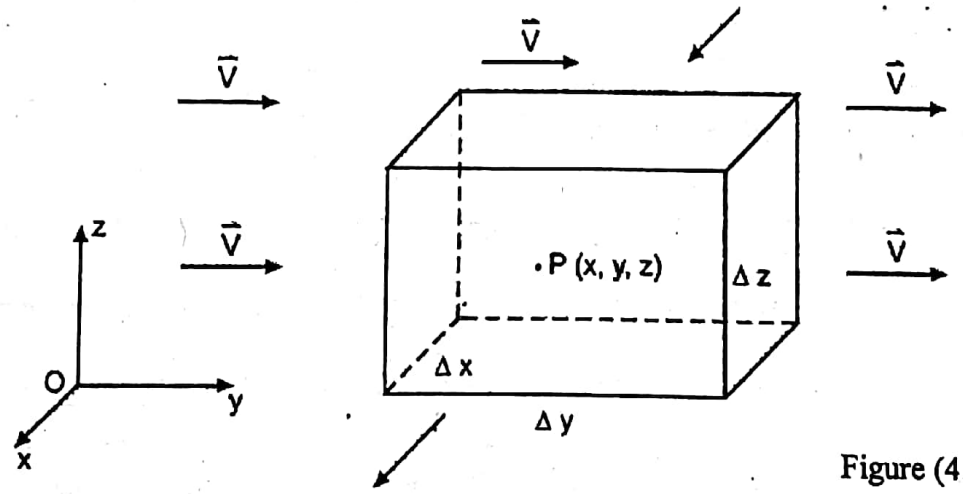

Figure (4.8)

We start by considering the flux (i.e. the amount of fluid crossing in unit time) through opposite faces of the parallelepiped . As the density is constant , either volume or mass can be used as a measure of the amount . For simplicity , volume will be considered here .

Consider first the flux across the faces which are perpendicular to the x–axis . These faces are of area $\Delta y\ \Delta z$ and the flux will only depend on V_1 , this being the component of $\vec{V}$ in the x–direction . Thus flux into the parallelopiped through the back face $= V_1\left(x - \frac{1}{2}\Delta x, y, z\right)\Delta y\,\Delta z$

and the flux out of the parallelopiped through the front face $= V_1\left(x + \frac{1}{2}\Delta x, y, z\right)\Delta y\,\Delta z$.

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Hence the net flux out of the parallelopiped through these two faces

$$= \left[V_1\left(x+\frac{1}{2}\Delta x, y, z\right) - V_1\left(x-\frac{1}{2}\Delta x, y, z\right)\right] \Delta y\, \Delta z$$

Using Taylor's series , we have

$$V_1\left(x+\frac{1}{2}\Delta x, y, z\right) = V_1(x,y,z) + \frac{1}{2}\Delta x \frac{\partial}{\partial x} V_1(x,y,z) + \ldots\ldots$$

and $$V_1\left(x-\frac{1}{2}\Delta x, y, z\right) = V_1(x,y,z) - \frac{1}{2}\Delta x \frac{\partial}{\partial x} V_1(x,y,z) + \ldots\ldots$$

Therefore , $$V_1\left(x+\frac{1}{2}\Delta x, y, z\right) - V_1\left(x-\frac{1}{2}\Delta x, y, z\right) = \Delta x \frac{\partial}{\partial x} V_1(x,y,z) + \ldots\ldots$$

Thus net flux in the x – direction $= \dfrac{\partial V_1}{\partial x}\Delta x\, \Delta y\, \Delta z + \ldots\ldots$ $\left[\text{by writing } \dfrac{\partial V_1}{\partial x} \text{ for } \dfrac{\partial}{\partial x} V_1(x,y,z)\right]$

Similarly , the net flux in the y – direction $= \dfrac{\partial V_2}{\partial y}\Delta x\, \Delta y\, \Delta z + \ldots\ldots$

and the net flux in the z – direction $= \dfrac{\partial V_3}{\partial z}\Delta x\, \Delta y\, \Delta z + \ldots\ldots$

Adding these three contributions , total flux out of the parallelopiped

$$= \left\{\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right\} \Delta x\, \Delta y\, \Delta z + \text{terms involving higher powers of } \Delta x, \Delta y, \Delta z$$

such as $(\Delta x)^2 \Delta y\, \Delta z$, $\Delta x (\Delta y)^3 \Delta z$ etc.

Now as the volume of the parallelopiped is $\Delta x\, \Delta y\, \Delta z$, we have

$$\text{Flux per unit volume} = \left\{\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial Z}\right\} + \text{terms in } \Delta x, \Delta y, \text{ or } \Delta z \text{ and their powers .} \qquad (1)$$

Finally , we let Δx , Δy , Δz all tend to zero so that the parallelopiped shrinks to the point P . The right hand side of equation (1) becomes $\dfrac{\partial V_1}{\partial x} + \dfrac{\partial V_2}{\partial y} + \dfrac{\partial V_3}{\partial z}$, which is precisely the divergence of the vector $\vec{V}$. Thus Fluid Mechanics affords one possible interpretation of the divergence as the amount of outward flux of the velocity field $\vec{V}$ per unit volume . (Note that the divergence of $\vec{V}$ measures the expansion of the fluid at the point .

4.14 LAPLACIAN

If $\phi(x, y, z)$ is a scalar point function , then the divergence of the gradient of ϕ written as $\nabla \cdot \nabla \phi = \nabla^2 \phi$ is called the Laplacian of ϕ , and the equation $\nabla^2 \phi = 0$ is called Laplace's equation . If a scalar function ϕ satisfies the Laplace's equation $\nabla^2 \phi = 0$ in a certain region R , then ϕ is said to be a **harmonic** function in that region .

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THEOREM (4.7): **If ϕ is a differentiable scalar point function , then show that**

$$\nabla . \nabla \phi = \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ **is a Laplacian operator .**

PROOF: We have

$$\nabla . \nabla \phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) . \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z}\right)$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = \nabla^2 \phi$$

EXAMPLE (7): **Find $\nabla^2 \phi$ if $\phi = 2x^3 y^2 z^4$.**

SOLUTION: We know that

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \qquad (1)$$

Now $\frac{\partial \phi}{\partial x} = 6x^2 y^2 z^4$, $\frac{\partial \phi}{\partial y} = 4x^3 y z^4$ $\frac{\partial \phi}{\partial z} = 8x^3 y^2 z^3$

and $\frac{\partial^2 \phi}{\partial x^2} = 12xy^2 z^4$, $\frac{\partial^2 \phi}{\partial y^2} = 4x^3 z^4$, $\frac{\partial^2 \phi}{\partial z^2} = 24x^3 y^2 z^2$

Thus from equation (1) $\nabla^2 \phi = 12xy^2 z^4 + 4x^3 z^4 + 24x^3 y^2 z^2$

THEOREM (4.8): **Prove that**

(i) $\nabla^2 f(r) = \frac{2}{r} f'(r) + f''(r)$

(ii) $\nabla^2 r^n = n(n+1) r^{n-2}$, **where n is a real constant .**

(iii) $\nabla^2 \left(\frac{1}{r}\right) = 0$

PROOF: (i) $\nabla^2 f(r) = \nabla . \nabla f(r) = \nabla . \left[\frac{f'(r)\vec{r}}{r}\right]$

$$= \frac{f'(r)}{r} \nabla . \vec{r} + \vec{r} . \nabla \left[\frac{f'(r)}{r}\right] \quad [\text{ Theorem (4.5) (ii) }]$$

$$= \frac{f'(r)}{r}(3) + \vec{r} \left\{\frac{1}{r}\frac{d}{dr}\left[\frac{f'(r)}{r}\right]\vec{r}\right] \quad [\text{ Theorem (4.2) }]$$

$$= \frac{3}{r} f'(r) + \frac{1}{r}\left[\frac{r f''(r) - f'(r)}{r^2}\right] \vec{r} . \vec{r}$$

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$$= \frac{3}{r} f'(r) + \frac{1}{r}\left[\frac{r f''(r) - f'(r)}{r^2}\right] r^2$$

$$= \frac{3}{r} f'(r) - \frac{1}{r} f'(r) + f''(r)$$

$$= \frac{2}{r} f'(r) + f''(r)$$

(ii) Setting $f(r) = r^n$ in part (i), we get

$$\nabla^2 r^n = \frac{2}{r} n r^{n-1} + n(n-1) r^{n-2}$$

$$= 2 n r^{n-2} - n r^{n-2} + n^2 r^{n-2}$$

$$= n(n+1) r^{n-2}$$

(iii) Let $n = -1$ in part (ii), then

$$\nabla^2\left(\frac{1}{r}\right) = -1(-1+1) r^{-3} = 0$$

Any function ϕ which satisfies the Laplace's equation $\nabla^2 \phi = 0$ is called a solution of this equation. It follows that $\phi = \frac{1}{r}$ is a solution of Laplace's equation and is therefore **harmonic**.

EXAMPLE (8): **Show that**

(i) $\nabla^2(\ln r) = \frac{1}{r^2}$ (ii) $\nabla^2(e^r) = e^r\left(1 + \frac{2}{r}\right)$

(iii) $\nabla^4(e^r) = \nabla^2(\nabla^2 e^r) = e^r\left(1 + \frac{4}{r}\right)$

SOLUTION: We know that $\nabla^2 f(r) = \frac{2}{r} f'(r) + f''(r)$ (1)

(i) Let $f(r) = \ln r$, then $f'(r) = \frac{1}{r}$, $f''(r) = -\frac{1}{r^2}$ and from equation (1), we get

$$\nabla^2(\ln r) = \frac{2}{r}\left(\frac{1}{r}\right) - \frac{1}{r^2} = \frac{2}{r^2} - \frac{1}{r^2} = \frac{1}{r^2}$$

(ii) Let $f(r) = e^r$, then $f'(r) = e^r$, $f''(r) = e^r$ and from equation (1), we get

$$\nabla^2(e^r) = \frac{2}{r} e^r + e^r = e^r\left(1 + \frac{2}{r}\right) \qquad (2)$$

(iii) $\nabla^4(e^r) = \nabla^2(\nabla^2 e^r) = \nabla^2\left[e^r\left(1 + \frac{2}{r}\right)\right]$ [using equation (2)]

Here $f(r) = e^r\left(1 + \frac{2}{r}\right)$

$$f'(r) = e^r\left(-\frac{2}{r^2}\right) + \left(1 + \frac{2}{r}\right) e^r = e^r\left(1 + \frac{2}{r} - \frac{2}{r^2}\right)$$

$$f''(r) = e^r\left(-\frac{2}{r^2} + \frac{4}{r^3}\right) + \left(1 + \frac{2}{r} - \frac{2}{r^2}\right) e^r = \left(1 + \frac{2}{r} - \frac{4}{r^2} + \frac{4}{r^3}\right) e^r$$

Thus from equation (1), we get

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$$\nabla^4(e^r) = \nabla^2(\nabla^2 e^r) = \frac{2}{r}\left(1+\frac{2}{r}-\frac{2}{r^2}\right)e^r + \left(1+\frac{2}{r}-\frac{4}{r^2}+\frac{4}{r^3}\right)e^r$$

$$= \left(\frac{2}{r}+\frac{4}{r^2}-\frac{4}{r^3}+1+\frac{2}{r}-\frac{4}{r^2}+\frac{4}{r^3}\right)e^r$$

$$= e^r\left(1+\frac{4}{r}\right)$$

EXAMPLE (9): **Find the function $f(r)$ such that $\nabla^2 f(r) = 0$.**

SOLUTION: We have $\nabla^2 f(r) = 0$

or $f''(r) + \frac{2}{r} f'(r) = 0$ [using theorem (4.8)]

or $\frac{f''(r)}{f'(r)} = -\frac{2}{r}$

Integrating, we get $\ln f'(r) = -2\ln r + \ln C$

$$= \ln\frac{C}{r^2}$$

where C is the constant of integration.

Taking antilog, we have $f'(r) = \frac{C}{r^2}$

Integrating again, $f(r) = -\frac{C}{r} + D = A + \frac{B}{r}$

where $A = D$ and $B = -C$ are arbitrary constants.

4.15 CURL OF A VECTOR POINT FUNCTION

Let $\vec{A}(x, y, z)$ be a differentiable vector point function in a certain region of space. Then the curl or rotation of $\vec{A}$, written as $\nabla \times \vec{A}$ or curl $\vec{A}$, is defined by

$$\nabla \times \vec{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times \left(A_1\hat{i} + A_2\hat{j} + A_3\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\hat{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\hat{k}$$

Note that in the expansion of the determinant the operators $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ must precede A_1, A_2, A_3.

If $\vec{A}$ is a constant vector, then $\nabla \times \vec{A} = \vec{0}$. If $\nabla \times \vec{A} = \vec{0}$ in some region R, then $\vec{A}$ is called an **irrotational** vector point function in that region.

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EXAMPLE (10): If $\vec{A} = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$, find $\nabla \times \vec{A}$ at the point $(1,-1,1)$.

SOLUTION: We have

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix} = (2z^4 + 2x^2y)\hat{i} + 3xz^2\hat{j} - 4xyz\hat{k}$$

$$(\nabla \times \vec{A})_{(1,-1,1)} = 3\hat{j} + 4\hat{k}$$

4.16 PROPERTIES OF THE CURL

THEOREM (4.9): If $\vec{A}$ and $\vec{B}$ are differentiable vector point functions , and ϕ is a differentiable scalar point function , then prove that

(i) $\nabla \times (\vec{A} + \vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B}$

(ii) $\nabla \times (\phi \vec{A}) = \phi(\nabla \times \vec{A}) + (\nabla \phi) \times \vec{A}$

(iii) $\nabla \times (\nabla \phi) = \vec{0}$ (curl grad $\phi = \vec{0}$)

(iv) $\nabla \cdot (\nabla \times \vec{A}) = 0$ (div curl $\vec{A} = 0$)

PROOF: Let $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ and $\vec{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$, then

(i) $\vec{A} + \vec{B} = (A_1 + B_1)\hat{i} + (A_2 + B_2)\hat{j} + (A_3 + B_3)\hat{k}$

Hence, $$\nabla \times (A + \vec{B}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 + B_1 & A_2 + B_2 & A_3 + B_3 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_1 & B_2 & B_3 \end{vmatrix} = \nabla \times \vec{A} + \nabla \times \vec{B}$$

(ii) $\phi\vec{A} = \phi A_1\hat{i} + \phi A_2\hat{j} + \phi A_3\hat{k}$, then

$$\nabla \times (\phi\vec{A}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi A_1 & \phi A_2 & \phi A_3 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(\phi A_3) - \frac{\partial}{\partial z}(\phi A_2)\right]\hat{i} + \left[\frac{\partial}{\partial z}(\phi A_1) - \frac{\partial}{\partial x}(\phi A_3)\right]\hat{j} + \left[\frac{\partial}{\partial x}(\phi A_2) - \frac{\partial}{\partial y}(\phi A_1)\right]\hat{k}$$

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$$= \left[\phi\frac{\partial A_3}{\partial y}+\frac{\partial \phi}{\partial y}A_3-\phi\frac{\partial A_2}{\partial z}-\frac{\partial \phi}{\partial z}A_2\right]\hat{i}+\left[\phi\frac{\partial A_1}{\partial z}+\frac{\partial \phi}{\partial z}A_1-\phi\frac{\partial A_3}{\partial x}-\frac{\partial \phi}{\partial x}A_3\right]\hat{j}$$

$$+\left[\phi\frac{\partial A_2}{\partial x}+\frac{\partial \phi}{\partial x}A_2-\phi\frac{\partial A_1}{\partial y}-\frac{\partial \phi}{\partial y}A_1\right]\hat{k}$$

$$= \phi\left[\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)\hat{i}+\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)\hat{j}+\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)\hat{k}\right]$$

$$+\left[\left(\frac{\partial \phi}{\partial y}A_3-\frac{\partial \phi}{\partial z}A_2\right)\hat{i}+\left(\frac{\partial \phi}{\partial z}A_1-\frac{\partial \phi}{\partial x}A_3\right)\hat{j}+\left(\frac{\partial \phi}{\partial x}A_2-\frac{\partial \phi}{\partial y}A_3\right)\hat{k}\right]$$

$$= \phi(\nabla \times \vec{A})+\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z}\\ A_1 & A_2 & A_3\end{vmatrix} = \phi(\nabla \times \vec{A})+(\nabla\phi)\times\vec{A}$$

(iii) $$\nabla \times (\nabla\phi) = \nabla \times\left(\frac{\partial \phi}{\partial x}\hat{i}+\frac{\partial \phi}{\partial y}\hat{j}+\frac{\partial \phi}{\partial z}\hat{k}\right) = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z}\end{vmatrix}$$

$$= \left(\frac{\partial^2\phi}{\partial y\partial z}-\frac{\partial^2\phi}{\partial z\partial y}\right)\hat{i}+\left(\frac{\partial^2\phi}{\partial z\partial x}-\frac{\partial^2\phi}{\partial x\partial z}\right)\hat{j}+\left(\frac{\partial^2\phi}{\partial x\partial y}-\frac{\partial^2\phi}{\partial y\partial x}\right)\hat{k}$$

$$= \vec{0}$$

provided we assume that ϕ has continuous second partial derivatives so that the order of differentiation is immaterial ,

i.e. $$\frac{\partial^2\phi}{\partial y\partial z}=\frac{\partial^2\phi}{\partial z\partial y}\ ,\quad \frac{\partial^2\phi}{\partial z\partial x}=\frac{\partial^2\phi}{\partial x\partial z}\ ,\quad \frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}.$$

(iv) Since $$\nabla \times \vec{A} = \left[\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)\hat{i}+\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)\hat{j}+\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)\hat{k}\right]$$

Hence $$\nabla\cdot(\nabla \times \vec{A}) = \frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)$$

$$= \frac{\partial^2 A_3}{\partial x\partial y}-\frac{\partial^2 A_2}{\partial x\partial z}+\frac{\partial^2 A_1}{\partial y\partial z}-\frac{\partial^2 A_3}{\partial y\partial x}+\frac{\partial^2 A_2}{\partial z\partial x}-\frac{\partial^2 A_1}{\partial z\partial y} = 0$$

assuming that $\vec{A}$ has continuous second partial derivatives .

THEOREM (4.10): Prove that

(i) $\nabla \times \vec{r} = \vec{0}$ **(ii)** $\nabla \times [f(r)\vec{r}] = \vec{0}$

(iii) $\nabla \times (r^n\vec{r}) = \vec{0}$ **(iv)** $\nabla \times \left(\frac{\vec{r}}{r^2}\right) = \vec{0}$

where $\vec{r}$ is the position vector .

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PROOF: Since $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, therefore

(i) $$\nabla \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$$

(ii) Using theorem (4.9) part (ii) , we have

$$\nabla \times (\phi \vec{A}) = \phi (\nabla \times \vec{A}) + \nabla \phi \times \vec{A} \text{ , therefore}$$

$$\nabla \times [f(r)\vec{r}] = f(r)(\nabla \times \vec{r}) + [\nabla f(r)] \times \vec{r}$$

$$= f(r)\vec{0} + \frac{f'(r)}{r}\vec{r} \times \vec{r} = \vec{0} \quad (\text{since } \vec{r} \times \vec{r} = \vec{0}).$$

(iii) Setting $f(r) = r^n$ in part (ii) , we get

$$\nabla \times (r^n \vec{r}) = \vec{0}$$

(iv) Let $n = -2$ in part (iii) , we get

$$\nabla \times \left(\frac{\vec{r}}{r^2}\right) = \vec{0}.$$

4.17 GEOMETRICAL INTERPRETATION OF THE CURL

To find a possible interpretation of the curl , let us consider a body rotating with uniform angular speed ω about an axis ℓ. Let us define the angular velocity vector $\vec{\omega}$ to be a vector of length ω extending along ℓ in the direction in which a right–handed screw would move if given the same rotation as the body . Finally , let $\vec{r}$ be the vector drawn from any point O on the axis ℓ to an arbitrary point P (x , y , z) on the body as shown in figure (4.9) .

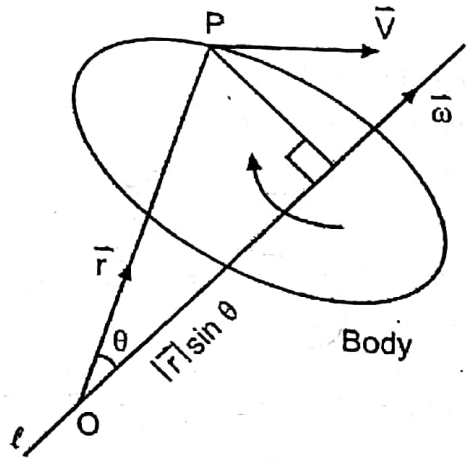

Figure (4.9)

It is clear that the radius at which P rotates is $|\vec{r}| \sin\theta$.
Hence , the linear speed of P is

$$|\vec{V}| = \omega |\vec{r}| \sin\theta$$

$$= |\vec{\omega}||\vec{r}| \sin\theta = |\vec{\omega} \times \vec{r}|$$

Moreover , the velocity vector $\vec{V}$ is directed perpendicular to the plane of $\vec{\omega}$ and $\vec{r}$, so that $\vec{\omega}$, $\vec{r}$, and $\vec{V}$ form a right handed system . Hence , the cross product $\vec{\omega} \times \vec{r}$ gives not only the magnitude of $\vec{V}$ but the direction as well , i.e. $\vec{V} = \vec{\omega} \times \vec{r}$

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If we now take the point O as the origin of coordinates , we can write

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \quad \text{and} \quad \vec{\omega} = \omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k}$$

Hence , the equation $\vec{V} = \vec{\omega} \times \vec{r}$ can be written as

$$\vec{V} = (\omega_2 z - \omega_3 y)\hat{i} - (\omega_1 z - \omega_3 x)\hat{j} + (\omega_1 y - \omega_2 x)\hat{k}$$

If we take the curl of $\vec{V}$, we have

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & -(\omega_1 z - \omega_3 x) & \omega_1 y - \omega_2 x \end{vmatrix}$$

Expanding this , remembering that $\vec{\omega}$ is a constant vector , we find

$$\nabla \times \vec{V} = 2\omega_1\hat{i} + 2\omega_2\hat{j} + 2\omega_3\hat{k} = 2\vec{\omega}$$

or $$\vec{\omega} = \frac{1}{2}\nabla \times \vec{V}$$

which says that the angular velocity at any point of a uniformly rotating body is equal to one–half the curl of the linear velocity at that point of the body . This justifies the name rotation used for curl . It is also motivation of the term **irrotational** for a vector field whose curl is the zero vector. In fluid dynamics , $\nabla \times \vec{V}$ is called vorticity vector and measures the degree to which a fluid swirls , or rotates about a given direction – much as the angular velocity vector measures the rate of rotation of a rigid body .

4.18 OPERATIONS WITH ∇

Here we consider the various combinations of the operator ∇ with vector and scalar functions .

THEOREM (4.11): If $\vec{A}$ and $\vec{B}$ are two vector point functions and ϕ a scalar point function , then show that

(i) $(\vec{A}\cdot\nabla)\phi = \vec{A}\cdot\nabla\phi$

(ii) $(\vec{A}\times\nabla)\phi = \vec{A}\times\nabla\phi$

(iii) $(\vec{A}\cdot\nabla)\vec{B} = A_1\dfrac{\partial\vec{B}}{\partial x} + A_2\dfrac{\partial\vec{B}}{\partial y} + A_3\dfrac{\partial\vec{B}}{\partial z}$

(iv) $(\vec{A}\cdot\nabla)\vec{r} = \vec{A}$

(v) **Give possible meaning to $(\vec{A}\times\nabla)\vec{B}$.**

PROOF: Let $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, then

$$\vec{A}\cdot\nabla = (A_1\hat{i} + A_2\hat{j} + A_3\hat{k})\cdot\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial y}\right) = A_1\frac{\partial}{\partial x} + A_2\frac{\partial}{\partial y} + A_3\frac{\partial}{\partial z}$$

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(i) $$(\vec{A}\cdot\nabla)\phi = \left(A_1\frac{\partial}{\partial x}+A_2\frac{\partial}{\partial y}+A_3\frac{\partial}{\partial z}\right)\phi$$

$$= A_1\frac{\partial\phi}{\partial x}+A_2\frac{\partial\phi}{\partial y}+A_3\frac{\partial\phi}{\partial z}$$

$$= (A_1\hat{i}+A_2\hat{j}+A_3\hat{k})\cdot\left(\frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k}\right)$$

$$= \vec{A}\cdot\nabla\phi$$

(ii) $$(\vec{A}\times\nabla)\phi = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ A_1 & A_2 & A_3\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{vmatrix}\phi$$

$$= \left[\hat{i}\left(A_2\frac{\partial}{\partial z}-A_3\frac{\partial}{\partial y}\right)+\hat{j}\left(A_3\frac{\partial}{\partial x}-A_1\frac{\partial}{\partial z}\right)+\hat{k}\left(A_1\frac{\partial}{\partial y}-A_2\frac{\partial}{\partial x}\right)\right]\phi$$

$$= \left(A_2\frac{\partial\phi}{\partial z}-A_3\frac{\partial\phi}{\partial y}\right)\hat{i}+\left(A_3\frac{\partial\phi}{\partial x}-A_1\frac{\partial\phi}{\partial z}\right)\hat{j}+\left(A_1\frac{\partial\phi}{\partial y}-A_2\frac{\partial\phi}{\partial x}\right)\hat{k}$$

$$= \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ A_1 & A_2 & A_3\\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z}\end{vmatrix} = \vec{A}\times\nabla\phi$$

(iii) $$(\vec{A}\cdot\nabla)\vec{B} = \left(A_1\frac{\partial}{\partial x}+A_2\frac{\partial}{\partial y}+A_3\frac{\partial}{\partial z}\right)\vec{B}$$

$$= A_1\frac{\partial\vec{B}}{\partial x}+A_2\frac{\partial\vec{B}}{\partial y}+A_3\frac{\partial\vec{B}}{\partial z}$$

(iv) $$(\vec{A}\cdot\nabla)\vec{r} = \left(A_1\frac{\partial}{\partial x}+A_2\frac{\partial}{\partial y}+A_3\frac{\partial}{\partial z}\right)(x\hat{i}+y\hat{j}+z\hat{k})$$

$$= A_1\hat{i}+A_2\hat{j}+A_3\hat{k} = \vec{A}$$

(v) No definition or meaning can be assigned to $(\vec{A}\times\nabla)\vec{B}$, because it is a kind of differential operator with vector quantities.

EXAMPLE (11): If $\vec{A} = 2yz\hat{i}-x^2y\hat{j}+xz^2\hat{k}$, $\vec{B} = x^2\hat{i}+yz\hat{j}-xy\hat{k}$ and $\phi = 2x^2yz^3$, find

(i) $(\vec{A}\cdot\nabla)\phi$ (ii) $(\vec{A}\times\nabla)\phi$

(iii) $(\vec{A}\cdot\nabla)\vec{B}$

SOLUTION: Since $\phi = 2x^2yz^3$, we have

$$\nabla\phi = 4xyz^3\hat{i}+2x^2z^3\hat{j}+6x^2yz^2\hat{k}$$

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(i) $$(\vec{A}\cdot\nabla)\phi = \vec{A}\cdot\nabla\phi$$
$$= (2yz\hat{i} - x^2y\hat{j} + xz^2\hat{k})\cdot(4xyz^3\hat{i} + 2x^2z^3\hat{j} + 6x^2yz^2\hat{k})$$
$$= 8xy^2z^4 - 2x^4yz^3 + 6x^3yz^4$$

(ii) $$(\vec{A}\times\nabla)\phi = \vec{A}\times\nabla\phi$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2yz & -x^2y & xz^2 \\ 4xyz^3 & 2x^2z^3 & 6x^2yz^2 \end{vmatrix}$$
$$= -(6x^4y^2z^2 + 2x^3z^5)\hat{i} + (4x^2yz^5 - 12x^2y^2z^3)\hat{j}$$
$$+ (4x^2yz^4 + 4x^3y^2z^3)\hat{k}$$

(iii) Since $\vec{A}\cdot\nabla = 2yz\frac{\partial}{\partial x} - x^2y\frac{\partial}{\partial y} + xz^2\frac{\partial}{\partial z}$, therefore
$$(\vec{A}\cdot\nabla)\vec{B} = \left(2yz\frac{\partial}{\partial x} - x^2y\frac{\partial}{\partial y} + xz^2\frac{\partial}{\partial z}\right)(x^2\hat{i} + yz\hat{j} - xy\hat{k})$$
$$= 2yz(2x\hat{i} - y\hat{k}) - x^2y(z\hat{j} - x\hat{k}) + xz^2(y\hat{j})$$
$$= 4xyz\hat{i} + (xyz^2 - x^2yz)\hat{j} + (x^3y - 2y^2z)\hat{k}$$

THEOREM (4.12): If $\vec{A}$ and $\vec{B}$ are two vector functions, prove that
$$(\vec{A}\times\nabla)\cdot\vec{B} = \vec{A}\cdot(\nabla\times\vec{B})$$

PROOF: We have
$$\vec{A}\times\nabla = \left(A_2\frac{\partial}{\partial z} - A_3\frac{\partial}{\partial y}\right)\hat{i} + \left(A_3\frac{\partial}{\partial x} - A_1\frac{\partial}{\partial z}\right)\hat{j} + \left(A_1\frac{\partial}{\partial y} - A_2\frac{\partial}{\partial x}\right)\hat{k}$$, therefore
$$(\vec{A}\times\nabla)\cdot\vec{B} = \left(A_2\frac{\partial}{\partial z} - A_3\frac{\partial}{\partial y}\right)(\hat{i}\cdot\vec{B}) + \left(A_3\frac{\partial}{\partial x} - A_1\frac{\partial}{\partial z}\right)(\hat{j}\cdot\vec{B}) + \left(A_1\frac{\partial}{\partial y} - A_2\frac{\partial}{\partial x}\right)(\hat{k}\cdot\vec{B})$$

Since $\hat{i}\cdot\vec{B} = B_1$, $\hat{j}\cdot\vec{B} = B_2$, $\hat{k}\cdot\vec{B} = B_3$, therefore
$$(\vec{A}\times\nabla)\cdot\vec{B} = \left(A_2\frac{\partial B_1}{\partial z} - A_3\frac{\partial B_1}{\partial y}\right) + \left(A_3\frac{\partial B_2}{\partial x} - A_1\frac{\partial B_2}{\partial z}\right) + \left(A_1\frac{\partial B_3}{\partial y} - A_2\frac{\partial B_3}{\partial x}\right)$$
$$= A_1\left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}\right) + A_2\left(\frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial z}\right) + A_3\left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\right)$$
$$= (A_1\hat{i} + A_2\hat{j} + A_3\hat{k})\cdot\left[\left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}\right)\hat{i} + \left(\frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial x}\right)\hat{j} + \left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\right)\hat{k}\right]$$
$$= \vec{A}\cdot(\nabla\times\vec{B})$$

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4.19 VECTOR IDENTITIES

THEOREM (4.13): If $\vec{A}$ and $\vec{B}$ are two differentiable vector point functions , prove that

(i) $\nabla.(\vec{A}\times\vec{B}) = \vec{B}.(\nabla\times\vec{A}) - \vec{A}.(\nabla\times\vec{B})$

(ii) $\nabla\times(\vec{A}\times\vec{B}) = (\vec{B}.\nabla)\vec{A} - \vec{B}(\nabla.\vec{A}) - (\vec{A}.\nabla)\vec{B} + \vec{A}(\nabla.\vec{B})$

(iii) $\nabla(\vec{A}.\vec{B}) = (\vec{B}.\nabla)\vec{A} + (\vec{A}.\nabla)\vec{B} + \vec{B}\times(\nabla\times\vec{A}) + \vec{A}\times(\nabla\times\vec{B})$

(iv) $\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla.\vec{A}) - \nabla^2\vec{A}$

PROOF: Let $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ and $\vec{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$, then

(i) $\vec{A}\times\vec{B} = (A_2B_3 - A_3B_2)\hat{i} + (A_3B_1 - A_1B_3)\hat{j} + (A_1B_2 - A_2B_1)\hat{k}$

and
$$\nabla.(\vec{A}\times\vec{B}) = \frac{\partial}{\partial x}(A_2B_3 - A_3B_2) + \frac{\partial}{\partial y}(A_3B_1 - A_1B_3) + \frac{\partial}{\partial z}(A_1B_2 - A_2B_1)$$
$$= A_2\frac{\partial B_3}{\partial x} + B_3\frac{\partial A_2}{\partial x} - A_3\frac{\partial B_2}{\partial x} - B_2\frac{\partial A_3}{\partial x}$$
$$+ A_3\frac{\partial B_1}{\partial y} + B_1\frac{\partial A_3}{\partial y} - A_1\frac{\partial B_3}{\partial y} - B_3\frac{\partial A_1}{\partial y}$$
$$+ A_1\frac{\partial B_2}{\partial z} + B_2\frac{\partial A_1}{\partial z} - A_2\frac{\partial B_1}{\partial z} - B_1\frac{\partial A_2}{\partial z}$$
$$= B_1\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) + B_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) + B_3\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$
$$- A_1\left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}\right) - A_2\left(\frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial x}\right) - A_3\left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\right)$$
$$= (B_1\hat{i} + B_2\hat{j} + B_3\hat{k}).\left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\hat{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\hat{k}\right]$$
$$- (A_1\hat{i} + A_2\hat{j} + A_3\hat{k}).\left[\left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}\right)\hat{i} + \left(\frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial x}\right)\hat{j} + \left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\right)\hat{k}\right]$$
$$= \vec{B}.(\nabla\times\vec{A}) - \vec{A}.(\nabla\times\vec{B})$$

(ii) We know that
$$\nabla\times\vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\vec{V}$$
$$= \hat{i}\times\frac{\partial\vec{V}}{\partial x} + \hat{j}\times\frac{\partial\vec{V}}{\partial y} + \hat{k}\times\frac{\partial\vec{V}}{\partial z} = \sum\hat{i}\times\frac{\partial\vec{V}}{\partial x}$$
$$\nabla\times(\vec{A}\times\vec{B}) = \sum\left[\hat{i}\times\frac{\partial}{\partial x}(\vec{A}\times\vec{B})\right]$$
$$= \sum\left[\hat{i}\times\left(\vec{A}\times\frac{\partial\vec{B}}{\partial x} + \frac{\partial\vec{A}}{\partial x}\times\vec{B}\right)\right]$$

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$$= \sum\left[\hat{i}\times\left(\vec{A}\times\frac{\partial\vec{B}}{\partial x}\right)\right]+\sum\left[\hat{i}\times\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right)\right]$$

$$= \sum\left[\hat{i}\times\left(\vec{A}\times\frac{\partial\vec{B}}{\partial x}\right)\right]-\sum\left[\hat{i}\times\left(\vec{B}\times\frac{\partial\vec{A}}{\partial x}\right)\right] \qquad (1)$$

Now $\hat{i}\times\left(\vec{A}\times\frac{\partial\vec{B}}{\partial x}\right)=\left(\hat{i}\cdot\frac{\partial\vec{B}}{\partial x}\right)\vec{A}-(\hat{i}\cdot\vec{A})\frac{\partial\vec{B}}{\partial x}$

and so $\sum\left[\hat{i}\times\left(\vec{A}\times\frac{\partial\vec{B}}{\partial x}\right)\right]=\left[\sum\left(\hat{i}\cdot\frac{\partial\vec{B}}{\partial x}\right)\right]\vec{A}-\vec{A}\cdot\left(\sum\hat{i}\frac{\partial}{\partial x}\right)\vec{B}$

$$= (\nabla\cdot\vec{B})\vec{A}-(\vec{A}\cdot\nabla)\vec{B} \qquad (2)$$

Similarly, on interchanging $\vec{A}$ and $\vec{B}$ in equation (2), we get

$$\sum\left[\hat{i}\times\left(\vec{B}\times\frac{\partial\vec{A}}{\partial x}\right)\right]=(\nabla\cdot\vec{A})\vec{B}-(\vec{B}\cdot\nabla)\vec{A} \qquad (3)$$

Substitution of equations (2) and (3) in equation (1) gives

$$\nabla\times(\vec{A}\times\vec{B})=(\vec{B}\cdot\nabla)\vec{A}-(\vec{A}\cdot\nabla)\vec{B}+\vec{A}(\nabla\cdot\vec{B})-\vec{B}(\nabla\cdot\vec{A})$$

(iii) $\nabla(\vec{A}\cdot\vec{B})=\sum\left[\hat{i}\frac{\partial}{\partial x}(\vec{A}\cdot\vec{B})\right]$

$$=\sum\left[\hat{i}\left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}+\frac{\partial\vec{A}}{\partial x}\cdot\vec{B}\right)\right]$$

$$=\sum\left[\left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)\hat{i}\right]+\sum\left[\left(\vec{B}\cdot\frac{\partial\vec{A}}{\partial x}\right)\hat{i}\right] \qquad (4)$$

We know that

$$\vec{A}\times(\vec{B}\times\vec{C})=(\vec{A}\cdot\vec{C})\vec{B}-(\vec{A}\cdot\vec{B})\vec{C}$$

or $(\vec{A}\cdot\vec{B})\vec{C}=(\vec{A}\cdot\vec{C})\vec{B}-\vec{A}\times(\vec{B}\times\vec{C})$

Thus $\left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)\hat{i}=(\vec{A}\cdot\hat{i})\frac{\partial\vec{B}}{\partial x}-\vec{A}\times\left(\frac{\partial\vec{B}}{\partial x}\times\hat{i}\right)$

$$=(\vec{A}\cdot\hat{i})\frac{\partial\vec{B}}{\partial x}+\vec{A}\times\left(\hat{i}\times\frac{\partial\vec{B}}{\partial x}\right)$$

and so $\sum\left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)\hat{i}=\vec{A}\cdot\left(\sum\hat{i}\frac{\partial}{\partial x}\right)\vec{B}+\vec{A}\times\sum\left(\hat{i}\times\frac{\partial\vec{B}}{\partial x}\right)$

$$=(\vec{A}\cdot\nabla)\vec{B}+\vec{A}\times(\nabla\times\vec{B}) \qquad (5)$$

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Similarly, interchanging $\vec{A}$ and $\vec{B}$ in equation (5), we get

$$\sum\left(\vec{B}\cdot\frac{\partial\vec{A}}{\partial x}\right)\hat{i} = (\vec{B}\cdot\nabla)\vec{A}+\vec{B}\times(\nabla\times\vec{A}) \qquad (6)$$

Substitution of equations (5) and (6) in equation (4) gives

$$\nabla(\vec{A}\cdot\vec{B}) = (\vec{B}\cdot\nabla)\vec{A}+(\vec{A}\cdot\nabla)\vec{B}+\vec{B}\times(\nabla\times\vec{A})+\vec{A}\times(\nabla\times\vec{B})$$

(iv) We know that $\vec{B}\times(\vec{C}\times\vec{D}) = (\vec{B}\cdot\vec{D})\vec{C}-(\vec{B}\cdot\vec{C})\vec{D}$

Setting $\vec{B} = \vec{C} = \nabla$ and $\vec{D} = \vec{A}$, we get

$$\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A})-(\nabla\cdot\nabla)\vec{A}$$

$$= \nabla(\nabla\cdot\vec{A})-\nabla^2\vec{A}$$

Note that rather than writing $(\nabla\cdot\vec{A})\nabla$, we must write $\nabla(\nabla\cdot\vec{A})$ to make sure that ∇ operates on $\nabla\cdot\vec{A}$.

EXAMPLE (12): **If $\vec{A}$ is a constant vector, prove that**

(i) $\nabla(\vec{A}\cdot\vec{r}) = \vec{A}$ **(ii)** $\nabla\cdot(\vec{A}\times\vec{r}) = 0$

(iii) $\nabla\times(\vec{A}\times\vec{r}) = 2\vec{A}$

SOLUTION: By theorem [4.13 (iii)], we have

(i) $$\nabla(\vec{A}\cdot\vec{r}) = (\vec{r}\cdot\nabla)\vec{A}+(\vec{A}\cdot\nabla)\vec{r}+\vec{r}\times(\nabla\times\vec{A})+\vec{A}\times(\nabla\times\vec{r})$$

Since $\vec{A}$ is constant vector, we have

$(\vec{r}\cdot\nabla)\vec{A} = \vec{0}$, $\nabla\times\vec{A} = \vec{0}$. Furthermore, $\nabla\times\vec{r} = \vec{0}$.

Thus $\nabla(\vec{A}\cdot\vec{r}) = (\vec{A}\cdot\nabla)\vec{r} = \vec{A}$ [using theorem (4.11) (iv)]

ALTERNATIVE METHOD

Let $\vec{A} = A_1\hat{i}+A_2\hat{j}+A_3\hat{k}$ and $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$, then

$\vec{A}\cdot\vec{r} = A_1x+A_2y+A_3z$

and $$\nabla(\vec{A}\cdot\vec{r}) = \hat{i}\frac{\partial}{\partial x}(A_1x+A_2y+A_3z)+\hat{j}\frac{\partial}{\partial y}(A_1x+A_2y+A_3\hat{z})$$
$$+\hat{k}\frac{\partial}{\partial z}(A_1x+A_2y+A_3\hat{z})$$

$$= A_1\hat{i}+A_2\hat{j}+A_3\hat{k} = \vec{A}$$

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(ii) By theorem [4.13 (i)]

$$\nabla.(\vec{A}\times\vec{r}) = \vec{r}.(\nabla\times\vec{A}) - \vec{A}.(\nabla\times\vec{r}) = \vec{r}\,.\,\vec{0} - \vec{A}\,.\,\vec{0} = 0$$

(iii) By theorem [4.13 (ii)]

$$\nabla\times(\vec{A}\times\vec{r}) = (\vec{r}.\nabla)\vec{A} - \vec{r}(\nabla.\vec{A}) - (\vec{A}.\nabla)\vec{r} + \vec{A}(\nabla.\vec{r})$$

$$= -(\vec{A}.\nabla)\vec{r} + \vec{A}(\nabla.\vec{r})$$

$$= -\vec{A} + 3\vec{A} = 2\vec{A} \qquad [\text{since } (\vec{A}.\nabla)\vec{r} = \vec{A}]$$

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4.20 SOLVED PROBLEMS

PROBLEMS ON THE GRADIENT

PROBLEM (1): Let R be the distance from a fixed point $A(a, b, c)$ to any point $P(x, y, z)$. Show that $\nabla R = \hat{R}$ is a unit vector in the direction $\vec{AP} = \vec{R}$.

SOLUTION: If $\vec{r}_A$ and $\vec{r}_P$ are the position vectors of the points A and P respectively, then

$$\vec{r}_A = a\hat{i} + b\hat{j} + c\hat{k} \text{ and } \vec{r}_P = x\hat{i} + y\hat{j} + z\hat{k}$$

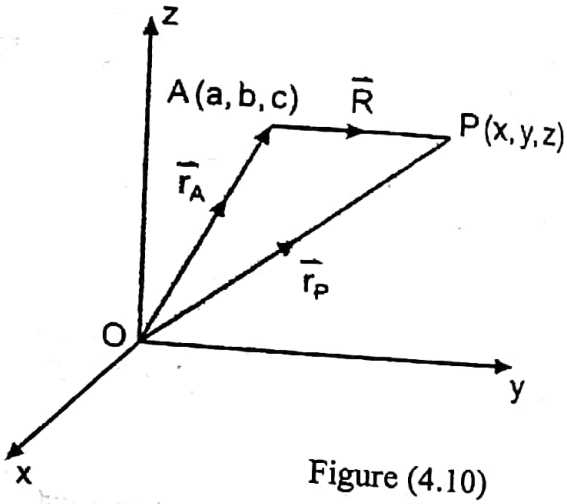

Figure (4.10)

Thus $\vec{R} = \vec{r}_P - \vec{r}_A$

$$= (x-a)\hat{i} + (y-b)\hat{j} + (z-c)\hat{k}$$

so that $R = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$

Then $\nabla R = \nabla\left[\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}\right]$

$$= \frac{(x-a)\hat{i} + (y-b)\hat{j} + (z-c)\hat{k}}{\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}}$$

$$= \frac{\vec{R}}{R} = \hat{R}, \text{ a unit vector in the direction of } \vec{R}$$

PROBLEM (2): If $\phi = x^n + y^n + z^n$, show that $\vec{r} \cdot \nabla\phi = n\phi$.

SOLUTION: Since $\phi = x^n + y^n + z^n$, therefore

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(x^n + y^n + z^n) + \hat{j}\frac{\partial}{\partial y}(x^n + y^n + z^n) + \hat{k}\frac{\partial}{\partial z}(x^n + y^n + z^n)$$

$$= nx^{n-1}\hat{i} + ny^{n-1}\hat{j} + nz^{n-1}\hat{k}$$

and so $\vec{r} \cdot \nabla\phi = (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (nx^{n-1}\hat{i} + ny^{n-1}\hat{j} + nz^{n-1}\hat{k})$

$$= nx^n + ny^n + nz^n$$

$$= n(x^n + y^n + z^n) = n\phi$$

PROBLEM (3): If $\phi = (x^2 + y^2 + z^2)\, e^{-\sqrt{x^2+y^2+z^2}}$, find $\nabla\phi$.

SOLUTION: Since $r = \sqrt{x^2 + y^2 + z^2}$, therefore, $\phi = r^2 e^{-r}$

Now $\nabla\phi = \dfrac{\partial\phi}{\partial r}\hat{r}$

$$= (2re^{-r} - r^2 e^{-r})\hat{r}$$

$$= (2-r)re^{-r}\hat{r} = (2-r)e^{-r}\vec{r}$$

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PROBLEM (4): If $\nabla\phi = 2xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k}$, find $\phi(x,y,z)$ if $\phi(1,-2,2) = 4$.

SOLUTION: We know that

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} \qquad (1)$$

Since $\nabla\phi = 2xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k}$ (2)

therefore comparing equations (1) and (2), we get

$$\frac{\partial\phi}{\partial x} = 2xyz^3 \qquad (3)$$

$$\frac{\partial\phi}{\partial y} = x^2z^3 \qquad (4)$$

$$\frac{\partial\phi}{\partial z} = 3x^2yz^2 \qquad (5)$$

Integrating equation (3) partially w.r.t. x keeping y and z constants,

$$\phi = x^2yz^3 + f(y,z) \qquad (6)$$

Similarly, from equations (4) and (5), we get

$$\phi = x^2yz^3 + g(x,z) \qquad (7)$$

$$\phi = x^2yz^3 + h(x,y) \qquad (8)$$

Comparison of equations (6), (7), and (8) shows that there will be a common value of ϕ if we choose

$f(y,z) = g(x,z) = h(x,y) = C$ where C is an arbitrary constant.

Thus $\phi = x^2yz^3 + C$ (9)

Using $\phi(1,-2,2) = 4$, we find from equation (9)

$$4 = (1)^2(-2)(2)^3 + C$$

or $4 = -16 + C$ or $C = 20$

Hence from equation (9), we get

$$\phi = x^2yz^3 + 20$$

PROBLEM (5): If $\nabla\phi = 2r^4\vec{r}$, find ϕ.

SOLUTION: We know that the directional derivative of ϕ w.r.t. the distance r in the direction of the unit vector $\hat{r}$ is the component of $\nabla\phi$ in the direction of this unit vector,

i.e. $$\frac{\partial\phi}{\partial r} = \nabla\phi\cdot\hat{r}$$

$$= 2r^4\vec{r}\cdot\hat{r} = 2r^5\hat{r}\cdot\hat{r} = 2r^5 \qquad (\text{since } \hat{r}\cdot\hat{r} = 1)$$

Integrating w.r.t. r, we get

$$\phi = \frac{r^6}{3} + \text{constant}$$

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PROBLEM (6): If $\phi = \phi(x, y, z)$ is a differentiable function, show that $d\phi = \nabla\phi \cdot d\vec{r}$.

SOLUTION: We know that if $\phi = \phi(x, y, z)$ is a differentiable function of x, y, z, then

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\vec{k}\right).\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$$

$$= \nabla\phi \cdot d\vec{r}$$

PROBLEM (7): If $\vec{A}(x, y, z) = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, show that

$$d\vec{A} = (\nabla A_1 . d\vec{r})\hat{i} + (\nabla A_2 . d\vec{r})\hat{j} + (\nabla A_3 . d\vec{r})\hat{k}$$

SOLUTION: Since $\vec{A}(x, y, z) = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, therefore

$$d\vec{A} = dA_1\hat{i} + dA_2\hat{j} + dA_3\hat{k} \qquad (1)$$

Now $$dA_1 = \frac{\partial A_1}{\partial x}dx + \frac{\partial A_1}{\partial y}dy + \frac{\partial A_1}{\partial z}dz$$

$$= \left(\frac{\partial A_1}{\partial x}\hat{i} + \frac{\partial A_1}{\partial y}\hat{j} + \frac{\partial A_1}{\partial z}\hat{k}\right).\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$$

$$= \nabla A_1 . d\vec{r}$$

Similarly, $dA_2 = \nabla A_2 . d\vec{r}$ and $dA_3 = \nabla A_3 . d\vec{r}$

Thus from equation (1), we get

$$d\vec{A} = (\nabla A_1 . d\vec{r})\hat{i} + (\nabla A_2 . d\vec{r})\hat{j} + (\nabla A_3 . d\vec{r})\hat{k}$$

PROBLEM (8): If $\phi = \phi(u)$, where $u = u(x, y, z)$, then prove that

(i) $\nabla\phi = \nabla\phi(u) = \phi'(u)\nabla u$ (ii) $\nabla\int f(u)\,du = f(u)\nabla u$

PROOF: (i) $$\nabla\phi = \nabla\phi(u) = \hat{i}\frac{\partial}{\partial x}\phi(u) + \hat{j}\frac{\partial}{\partial y}\phi(u) + \hat{k}\frac{\partial}{\partial z}\phi(u)$$

$$= \hat{i}\frac{d\phi}{du}\frac{\partial u}{\partial x} + \hat{j}\frac{d\phi}{du}\frac{\partial u}{\partial y} + \hat{k}\frac{d\phi}{du}\frac{\partial u}{\partial z}$$

$$= \frac{d\phi}{du}\left(\frac{\partial u}{\partial x}\hat{i} + \frac{\partial u}{\partial y}\hat{j} + \frac{\partial u}{\partial z}\hat{k}\right) = \phi'(u)\nabla u$$

(ii) Let $\phi(u) = \int f(u)\,du$, then $\phi'(u) = f(u)$. Hence

$$\nabla\int f(u)\,du = \nabla\phi(u) = \phi'(u)\nabla u \quad [\text{using part (i)}]$$

$$= f(u)\nabla u$$

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PROBLEM (9): If $\phi = \phi(u, v, w)$, where $u = u(x, y, z)$, $v = v(x, y, z)$, $w = w(x, y, z)$, show that

$$\nabla \phi = \frac{\partial \phi}{\partial u} \nabla u + \frac{\partial \phi}{\partial v} \nabla v + \frac{\partial \phi}{\partial w} \nabla w.$$

SOLUTION: By definition $\nabla \phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$ (1)

Since $\phi = \phi(u, v, w)$, therefore

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial \phi}{\partial w}\frac{\partial w}{\partial x}$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial \phi}{\partial w}\frac{\partial w}{\partial y}$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial \phi}{\partial w}\frac{\partial w}{\partial z}$$

Substituting these in equation (1), we get

$$\nabla \phi = \left(\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial \phi}{\partial w}\frac{\partial w}{\partial x}\right)\hat{i} + \left(\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial \phi}{\partial w}\frac{\partial w}{\partial y}\right)\hat{j}$$
$$+ \left(\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial \phi}{\partial w}\frac{\partial w}{\partial z}\right)\hat{k}$$

$$= \frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}\hat{i} + \frac{\partial u}{\partial y}\hat{j} + \frac{\partial u}{\partial z}\hat{k}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}\hat{i} + \frac{\partial v}{\partial y}\hat{j} + \frac{\partial v}{\partial z}\hat{k}\right) + \frac{\partial \phi}{\partial w}\left(\frac{\partial w}{\partial x}\hat{i} + \frac{\partial w}{\partial y}\hat{j} + \frac{\partial w}{\partial z}\hat{k}\right)$$

$$= \frac{\partial \phi}{\partial u}\nabla u + \frac{\partial \phi}{\partial v}\nabla v + \frac{\partial \phi}{\partial w}\nabla w$$

PROBLEM (10): Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point $(1, -2, -1)$ in the direction of the vector $2\hat{i} - \hat{j} - 2\hat{k}$.

SOLUTION: We have

$$\nabla \phi = \hat{i}\frac{\partial}{\partial x}(x^2yz + 4xz^2) + \hat{j}\frac{\partial}{\partial y}(x^2yz + 4xz^2) + \hat{k}\frac{\partial}{\partial z}(x^2yz + 4xz^2)$$

$$= (2xyz + 4z^2)\hat{i} + x^2z\hat{j} + (x^2y + 8xz)\hat{k}$$

Thus at the point $(1, -2, -1)$, the value of the gradient is

$$(\nabla \phi)_{(1,-2,-1)} = 8\hat{i} - \hat{j} - 10\hat{k}$$

The unit vector in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$ is

$$\hat{T} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{4+1+4}} = \frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}$$

Then the required directional derivative of ϕ at $(1, -2, 1)$ is

$$\frac{\partial \phi}{\partial s} = \nabla \phi \cdot \hat{T} = (8\hat{i} - \hat{j} - 10\hat{k}) \cdot \left(\frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}\right) = \frac{16}{3} + \frac{1}{3} + \frac{20}{3} = \frac{37}{3}$$

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PROBLEM (11): **(i) In what direction from the point $(2, 1, -1)$ is the directional derivativ $\phi = x^2 y z^3$ a maximum ? (ii) What is the magnitude of this maximum ?**

SOLUTION: We know that

$$\nabla \phi = \hat{i} \frac{\partial}{\partial x}(x^2 y z^3) + \hat{j} \frac{\partial}{\partial y}(x^2 y z^3) + \hat{k} \frac{\partial}{\partial z}(x^2 y z^3)$$

$$= 2xyz^3 \hat{i} + x^2 z^3 \hat{j} + 3x^2 y z^2 \hat{k}$$

Thus at the point $(2, 1, -1)$ the values of the gradient is

$$\nabla \phi = -4\hat{i} - 4\hat{j} + 12\hat{k}$$

(i) The directional derivative is maximum in the direction

$$\nabla \phi = -4\hat{i} - 4\hat{j} + 12\hat{k}$$

(ii) The magnitude of this maximum is

$$|\nabla \phi| = \sqrt{16 + 16 + 144} = \sqrt{176} = 4\sqrt{11}$$

PROBLEM (12): **Find the values of the constants a, b, c so that the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at $(1, 2, -1)$ has a maximum of magnitude 64 in a direction parallel to the z–axis .**

SOLUTION: Since $\phi = axy^2 + byz + cz^2x^3$, therefore

$$\nabla \phi = (ay^2 + 3cz^2x^2)\hat{i} + (2axy + bz)\hat{j} + (by + 2czx^3)\hat{k}$$

At the point $(1, 2, -1)$, the value of this gradient is

$$= (4a + 3c)\hat{i} + (4a - b)\hat{j} + (2b - 2c)\hat{k}$$

We know that the maximum directional derivative takes place in the direction of $\nabla \phi$ and has the magnitude of $\nabla \phi$. This maximum directional derivative will be parallel to the z–axis if

$$4a + 3c = 0 \qquad (1)$$

and

$$4a - b = 0 \qquad (2)$$

Therefore $\nabla \phi = (2b - 2c)\hat{k}$

Since the magnitude of this maximum directional derivative is 64 , therefore

$$2b - 2c = 64 \qquad (3)$$

Solving equations (1) , (2) , and (3) , we find that

$$a = 6, \; b = 24, \; c = -8.$$

PROBLEM (13): **Find the normal derivative $\left(\text{i.e. } \frac{\partial \phi}{\partial n} = \nabla \phi \cdot \hat{n}\right)$ of $\phi = xy^2 + yz^3$ at $(-1, 2, 1)$, where $\hat{n}$ is a unit normal vector to the surface $x \ln z - y^3 + 4 = 0$ at $(-1, 2, 1)$.**

SOLUTION: Since $\phi = xy^2 + yz^3$, therefore

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$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(xy^2+yz^3)+\hat{j}\frac{\partial}{\partial y}(xy^2+yz^3)+\hat{k}\frac{\partial}{\partial z}(xy^2+yz^3)$$

$$= y^2\hat{i}+(2xy+z^3)\hat{j}+3yz^2\hat{k}$$

Thus at the point $(-1,2,1)$ the value of the gradient is $\nabla\phi = 4\hat{i}-3\hat{j}+6\hat{k}$

Let the surface be represented by $\phi_1(x,y,z) = x\ln z - y^3 + 4 = 0$

Then $$\nabla\phi_1 = \hat{i}\frac{\partial}{\partial x}(x\ln z-y^3+4)+\hat{j}\frac{\partial}{\partial y}(x\ln z-y^3+4)+\hat{k}\frac{\partial}{\partial z}(x\ln z-y^3+4)$$

$$= \ln z\,\hat{i}-3y^2\hat{j}+\frac{x}{z}\hat{k}$$

At the point $(-1,2,1)$ the value of this gradient is $\nabla\phi_1 = -12\hat{j}-\hat{k}$

and $|\nabla\phi_1| = \sqrt{(-12)^2+(1)^2} = \sqrt{145}$

The unit normal vector $\hat{n}$ to this surface is

$$\hat{n} = \frac{\nabla\phi_1}{|\nabla\phi_1|} = \frac{-12\hat{j}-\hat{k}}{\sqrt{145}}$$

Thus the required normal derivative of ϕ at $(-1,2,1)$ is given by

$$\frac{\partial\phi}{\partial n} = \nabla\phi\cdot\hat{n}$$

$$= (4\hat{i}-3\hat{j}+6\hat{k})\cdot\left(\frac{-12\hat{j}-\hat{k}}{\sqrt{145}}\right) = \frac{36-6}{\sqrt{145}} = \frac{30}{\sqrt{145}}$$

PROBLEM (14): **Find the angle between the normals to the surface $xy = z^2$ at the points $(1,4,2)$ and $(-3,-3,3)$.**

SOLUTION: Let the surface be represented by $\phi(x,y,z) = xy - z^2 = 0$

A normal to this surface is given by

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(xy-z^2)+\hat{j}\frac{\partial}{\partial y}(xy-z^2)+\hat{k}\frac{\partial}{\partial z}(xy-z^2) = y\hat{i}+x\hat{j}-2z\hat{k}$$

Let $\vec{N}_1$ be the normal to the surface at $(1,4,2)$, then

$$\vec{N}_1 = (\nabla\phi)_{(1,4,2)} = 4\hat{i}+\hat{j}-4\hat{k}$$

Let $\vec{N}_2$ be the normal to the surface at $(-3,-3,3)$, then

$$\vec{N}_2 = (\nabla\phi)_{(-3,-3,3)} = -3\hat{i}-3\hat{j}-6\hat{k}$$

Let θ be the angle between the normals to the surface at $(1,4,2)$ and $(-3,-3,3)$,

then $$\theta = \cos^{-1}\left(\frac{\vec{N}_1\cdot\vec{N}_2}{|\vec{N}_1||\vec{N}_2|}\right) \qquad (1)$$

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Now $\vec{N}_1 \cdot \vec{N}_2 = (4\hat{i}+\hat{j}-4\hat{k}).(-3\hat{i}-3\hat{j}-6\hat{k}) = -12-3+24 = 9$

$$|\vec{N}_1| = \sqrt{(4)^2+(1)^2+(-4)^2} = \sqrt{16+1+16} = \sqrt{33}$$

$$|\vec{N}_2| = \sqrt{(-3)^2+(-3)^2+(-6)^2} = \sqrt{9+9+36} = \sqrt{54}$$

then from equation (1) $\theta = \cos^{-1}\left(\dfrac{9}{\sqrt{33}\,\sqrt{54}}\right) = 77.69^{\circ}$ Approximately

PROBLEM (15): **Find the angle between the surfaces $x^2+y^2+z^2 = 9$ and $z = x^2+y^2-3$ at the point $(2,-1,2)$.**

SOLUTION: The angle between the surfaces at the point is the angle between the normals to the surfaces at that point. A normal to the surface $\phi_1(x,y,z) = x^2+y^2+z^2 = 9$ is

$$\nabla\phi_1 = 2x\hat{i}+2y\hat{j}+2z\hat{k}$$

$$= 4\hat{i}-2\hat{j}+4\hat{k} \quad \text{at the point } (2,-1,2)$$

A normal to the surface $\phi_2(x,y,z) = x^2+y^2-z = 3$

is $\nabla\phi_2 = 2x\hat{i}+2y\hat{j}-\hat{k}$

$$= 4\hat{i}-2\hat{j}-\hat{k} \quad \text{at the point } (2,-1,2).$$

Now at the point $(2,-1,2)$, we have

$$(\nabla\phi_1).(\nabla\phi_2) = |\nabla\phi_1||\nabla\phi_2|\cos\theta$$

where θ is the required angle.

or $(4\hat{i}-2\hat{j}+4\hat{k}).(4\hat{i}-2\hat{j}-\hat{k})$

$$= |4\hat{i}-2\hat{j}+4\hat{k}|\,|4\hat{i}-2\hat{j}-\hat{k}|\cos\theta$$

or $16 = 6\sqrt{21}\cos\theta$

and $\cos\theta = \dfrac{8}{3\sqrt{21}} = 0.5819.$

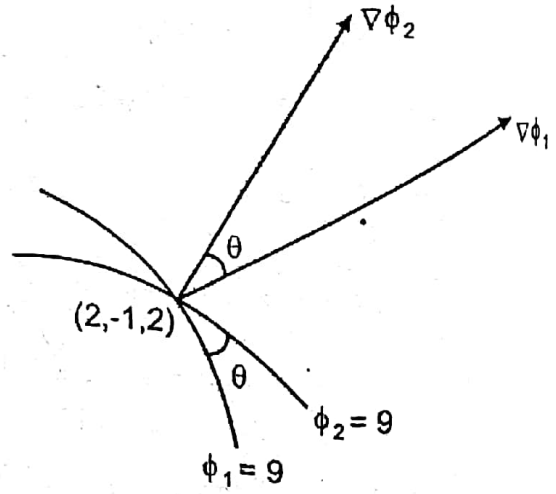

Figure (4.11)

Thus the acute angle is $\theta = \cos^{-1} 0.5819 = 54^{\circ}$ approximately.

PROBLEM (16): **Find the constants a and b so that the surface $ax^2-byz = (a+2)x$ will be orthogonal to the surface $4x^2y+z^3 = 4$ at the point $(1,-1,2)$.**

SOLUTION: Let $\phi_1(x,y,z) = ax^2-byz-(a+2)x = 0$ and $\phi_2(x,y,z) = 4x^2y+z^3 = 4$

The two surfaces will be orthogonal to each other at a given point if the normals to these surfaces are orthogonal to each other at that point.

A normal to the surface $\phi_1 = 0$ is

$$\nabla\phi_1 = [2ax-(a+2)]\hat{i}-bz\hat{j}-by\hat{k}$$

$$= (a-2)\hat{i}-2b\hat{j}+b\hat{k} \quad \text{at } (1,-1,2)$$

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A normal to the surface $\phi_2 = 4$ is

$$\nabla \phi_2 = 8xy\hat{i} + 4x^2\hat{j} + 3z^2\hat{k}$$
$$= -8\hat{i} + 4\hat{j} + 12\hat{k} \text{ at } (1,-1,2)$$

The two normals will be orthogonal to each other at the given point if $\nabla\phi_1 . \nabla\phi_2 = 0$

i.e. $(a-2)(-8)+(-2b)(4)+b(12) = 0$

or $-2a+b = -4$ (1)

Another equation in a and b follows from the surface $\phi_1 = 0$ at the given point,

i.e. $b = 1$ (2)

Solving equations (1) and (2), we find that $a = \frac{5}{2}$, $b = 1$

PROBLEMS ON THE DIVERGENCE

PROBLEM (17): **Determine the constant a so that $\vec{V} = (x+3y)\hat{i} + (y-2x)\hat{j} + (x+az)\hat{k}$ is solenoidal.**

SOLUTION: A vector $\vec{V}$ is solenoidal if its divergence is zero. Now

$$\nabla . \vec{V} = \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2x) + \frac{\partial}{\partial z}(x+az) = 1+1+a$$

Then $\nabla . \vec{V} = a+2 = 0$ implies $a = -2$

PROBLEM (18): **If ϕ and ψ are scalar point functions, show that**

$$\nabla . (\phi\nabla\psi - \psi\nabla\phi) = \phi\nabla^2\psi - \psi\nabla^2\phi$$

SOLUTION: We know that

$$\nabla . (\phi\vec{A}) = \phi(\nabla . \vec{A}) + \vec{A} . \nabla\phi \quad (1)$$

Let. $\vec{A} = \nabla\psi$ in equation (1), we get

$$\nabla . (\phi\nabla\psi) = \phi(\nabla . \nabla\psi) + \nabla\psi . \nabla\phi = \phi\nabla^2\psi + \nabla\psi . \nabla\phi \quad (2)$$

Interchanging ϕ and ψ yields.

$$\nabla . (\psi\nabla\phi) = \psi\nabla^2\phi + \nabla\phi . \nabla\psi \quad (3)$$

Subtracting equation (3) from equation (2), we get

$$\nabla . (\phi\nabla\psi) - \nabla . (\psi\nabla\phi) = \phi\nabla^2\psi - \psi\nabla^2\phi$$

or $\nabla . (\phi\nabla\psi - \psi\nabla\phi) = \phi\nabla^2\psi - \psi\nabla^2\phi$

PROBLEM (19): **For a constant vector $\vec{A}$, show that $\nabla . [(\vec{A} . \vec{r})\vec{r}] = 4(\vec{A} . \vec{r})$**

SOLUTION: $\nabla . [(\vec{A} . \vec{r})\vec{r}] = (\vec{A} . \vec{r})(\nabla . \vec{r}) + \vec{r} . \nabla(\vec{A} . \vec{r})$

$$= 3(\vec{A} . \vec{r}) + \vec{r} . \vec{A} \quad [\text{since } \nabla(\vec{A} . \vec{r}) = \vec{A}]$$
$$= 4(\vec{A} . \vec{r})$$

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PROBLEM (20): If $\vec{A} = \vec{A}(x,y,z)$, show that $(d\vec{r}\,.\nabla)\vec{A} = d\vec{A}$.

SOLUTION: Since $\vec{A} = \vec{A}(x,y,z)$, therefore

$$d\vec{A} = \frac{\partial \vec{A}}{\partial x}dx + \frac{\partial \vec{A}}{\partial y}dy + \frac{\partial \vec{A}}{\partial z}dz$$

$$= \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z}\right)\vec{A}$$

$$= \left[(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}).\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\right]\vec{A}$$

$$= (d\vec{r}\,.\nabla)\vec{A}$$

PROBLEM (21): If $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, show that $\nabla.\vec{A} = \nabla A_1.\hat{i} + \nabla A_2.\hat{j} + \nabla A_3.\hat{k}$

SOLUTION: Since $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, therefore

$$\nabla.\vec{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \qquad (1)$$

Now $\nabla A_1.\hat{i} = \left(\frac{\partial A_1}{\partial x}\hat{i} + \frac{\partial A_1}{\partial y}\hat{j} + \frac{\partial A_1}{\partial z}\hat{k}\right).\hat{i} = \frac{\partial A_1}{\partial x}$

Similarly $\nabla A_2.\hat{j} = \frac{\partial A_2}{\partial y}$ and $\nabla A_3.\hat{k} = \frac{\partial A_3}{\partial z}$

Thus equation (1) becomes

$$\nabla.\vec{A} = \nabla A_1.\hat{i} + \nabla A_2.\hat{j} + \nabla A_3.\hat{k}$$

PROBLEM (22): Prove that $\nabla.\left[\frac{f(r)\,\vec{r}}{r}\right] = \frac{2}{r}f(r) + f'(r)$.

SOLUTION:

$$\nabla.\left[\frac{f(r)\,\vec{r}}{r}\right] = \nabla.\left[\frac{f(r)}{r}\vec{r}\right]$$

$$= \frac{f(r)}{r}(\nabla.\vec{r}) + \vec{r}.\nabla\left[\frac{f(r)}{r}\right] \qquad \text{[Theorem (4.5) (ii)]}$$

$$= \frac{f(r)}{r}(3) + \vec{r}.\frac{1}{r}\frac{d}{dr}\left[\frac{f(r)}{r}\right]\vec{r} \qquad \text{[Theorem (4.2)]}$$

$$= \frac{3}{r}f(r) + \vec{r}.\frac{1}{r}\left[\frac{rf'(r) - f(r)}{r^2}\right]\vec{r}$$

$$= \frac{3}{r}f(r) + \frac{1}{r}\left[\frac{rf'(r) - f(r)}{r^2}\right](\vec{r}.\vec{r})$$

$$= \frac{3}{r}f(r) + \frac{1}{r}\left[\frac{rf'(r) - f(r)}{r^2}\right]r^2$$

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$$= \frac{3}{r}f(r)+f'(r)-\frac{1}{r}f(r)$$

$$= \frac{2}{r}f(r)+f'(r)$$

PROBLEM (23): Show that

(i) $\nabla.\left[r\,\nabla\left(\frac{1}{r^3}\right)\right]=\frac{3}{r^4}$ (ii) $\nabla.\left[\frac{1}{r}\nabla\left(\frac{1}{r}\right)\right]=\frac{1}{r^4}$

(iii) $\nabla\left[\nabla.\left(\frac{\vec{r}}{r}\right)\right]=-\frac{2\vec{r}}{r^3}$

SOLUTION: We have

(i) $\nabla.\left[r\nabla\left(\frac{1}{r^3}\right)\right] = \nabla.[r(-3r^{-5}\vec{r})]$ [using theorem (4.2)]

$$= -3\nabla.(r^{-4}\vec{r})$$

$$= -3(-4+3)r^{-4} = \frac{3}{r^4}$$ [using theorem (4.6)]

(ii) $\nabla.\left[\frac{1}{r}\nabla\left(\frac{1}{r}\right)\right] = \nabla.\left[\frac{1}{r}\left(-\frac{\vec{r}}{r^3}\right)\right]$ [using theorem (4.2)]

$$= -\nabla.(r^{-4}\vec{r})$$

$$= -(-4+3)r^{-4} = \frac{1}{r^4}$$ [using theorem (4.6)]

(iii) Using theorem (4.6), we have $\nabla.\left(\frac{\vec{r}}{r}\right)=\frac{2}{r}$, therefore

$$\nabla\left[\nabla.\left(\frac{\vec{r}}{r}\right)\right] = \nabla\left(\frac{2}{r}\right)$$

$$= -2r^{-3}\vec{r} = -\frac{2\vec{r}}{r^3}$$ [using theorem (4.2)]

PROBLEMS ON LAPLACIAN

PROBLEM (24): Prove that $\nabla^2(\phi\psi) = \phi\nabla^2\psi+2\nabla\phi.\nabla\psi+\psi\nabla^2\phi$

SOLUTION: $\nabla^2(\phi\psi) = \nabla.\nabla(\phi\psi)$

$$= \nabla.(\phi\nabla\psi+\psi\nabla\phi)$$

$$= \nabla.(\phi\nabla\psi)+\nabla.(\psi\nabla\phi)$$

$$= \phi(\nabla.\nabla\psi)+(\nabla\phi).(\nabla\psi)+\psi(\nabla.\nabla\phi)+(\nabla\psi).(\nabla\phi)$$

$$= \phi\nabla^2\psi+2\nabla\phi.\nabla\psi+\psi\nabla^2\phi$$

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PROBLEM (25): Show that $\nabla^2\left[\nabla\cdot\left(\frac{\vec{r}}{r^2}\right)\right] = \frac{2}{r^4}$

SOLUTION: Using theorem (4.7) , we have

$$\nabla\cdot\left(\frac{\vec{r}}{r^2}\right) = \nabla\cdot\left(r^{-2}\,\vec{r}\right) = (-2+3)\,r^{-2} = \frac{1}{r^2}\text{ , therefore}$$

$$\nabla^2\left[\nabla\cdot\left(\frac{\vec{r}}{r^2}\right)\right] = \nabla^2\left(\frac{1}{r^2}\right) = \nabla^2(r^{-2})$$

$$= -2(-2+1)r^{-4} \qquad \text{[using theorem (4.8)]}$$

$$= \frac{2}{r^4}$$

PROBLEMS ON THE CURL

PROBLEM (26): If $\vec{A} = x^2y\,\hat{i} - 2xz\,\hat{j} + 2yz\,\hat{k}$ find $\nabla\times\nabla\times\vec{A}$.

SOLUTION: We have

$$\nabla\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & -2xz & 2yz \end{vmatrix} = (2x+2z)\,\hat{i} - (x^2+2z)\,\hat{k}$$

$$\nabla\times\nabla\times\vec{A} = \nabla\times\left[(2x+2z)\,\hat{i} - (x^2+2z)\,\hat{k}\right]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+2z & 0 & -(x^2+2z) \end{vmatrix} = (2x+2)\,\hat{j}$$

PROBLEM (27): If $\vec{A} = A_1\,\hat{i} + A_2\,\hat{j} + A_3\,\hat{k}$, show that

$$\nabla\times\vec{A} = \nabla A_1\times\hat{i} + \nabla A_2\times\hat{j} + \nabla A_3\times\hat{k}$$

SOLUTION: Since $\vec{A} = A_1\,\hat{i} + A_2\,\hat{j} + A_3\,\hat{k}$, then

$$\nabla\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\hat{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\hat{k} \qquad (1)$$

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Now $\nabla A_1 \times \hat{i} = \left(\frac{\partial A_1}{\partial x}\hat{i} + \frac{\partial A_1}{\partial y}\hat{j} + \frac{\partial A_1}{\partial z}\hat{k}\right) \times \hat{i}$

$$= \frac{\partial A_1}{\partial y}(\hat{j} \times \hat{i}) + \frac{\partial A_1}{\partial z}(\hat{k} \times \hat{i})$$

$$= -\frac{\partial A_1}{\partial y}\hat{k} + \frac{\partial A_1}{\partial z}\hat{j} \qquad (2)$$

$$\nabla A_2 \times \hat{j} = \left(\frac{\partial A_2}{\partial x}\hat{i} + \frac{\partial A_2}{\partial y}\hat{j} + \frac{\partial A_2}{\partial z}\hat{k}\right) \times \hat{j}$$

$$= \frac{\partial A_2}{\partial x}(\hat{i} \times \hat{j}) + \frac{\partial A_2}{\partial z}(\hat{k} \times \hat{j})$$

$$= \frac{\partial A_2}{\partial x}\hat{k} - \frac{\partial A_2}{\partial z}\hat{i} \qquad (3)$$

$$\nabla A_3 \times \hat{k} = \left(\frac{\partial A_3}{\partial x}\hat{i} + \frac{\partial A_3}{\partial y}\hat{j} + \frac{\partial A_3}{\partial z}\hat{k}\right) \times \hat{k}$$

$$= \frac{\partial A_3}{\partial x}(\hat{i} \times \hat{k}) + \frac{\partial A_3}{\partial y}(\hat{j} \times \hat{k})$$

$$= -\frac{\partial A_3}{\partial x}\hat{j} + \frac{\partial A_3}{\partial y}\hat{i} \qquad (4)$$

Adding equations (2), (3), and (4), we get

$$\nabla A_1 \times \hat{i} + \nabla A_2 \times \hat{j} + \nabla A_3 \times \hat{k} = \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\hat{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\hat{k} \qquad (5)$$

From equations (1) and (5), we have

$$\nabla \times \vec{A} = \nabla A_1 \times \hat{i} + \nabla A_2 \times \hat{j} + \nabla A_3 \times \hat{k}$$

PROBLEM (28): **Prove that**

(i) $\nabla \times (\phi \nabla \phi) = \vec{0}$ **(ii)** $\nabla \times (\phi \nabla \psi) + \nabla \times (\psi \nabla \phi) = \vec{0}$

SOLUTION: We have

(i) $\nabla \times (\phi \nabla \phi) = \phi(\nabla \times \nabla \phi) + \nabla \phi \times \nabla \phi$

$$= \phi(\vec{0}) + \vec{0} = \vec{0} \qquad [\text{using theorem (4.9)}]$$

(ii) $\nabla \times (\phi \nabla \psi) + \nabla \times (\psi \nabla \phi) = \phi(\nabla \times \nabla \psi) + \nabla \phi \times \nabla \psi + \psi(\nabla \times \nabla \phi) + \nabla \psi \times \nabla \phi$

$$= \phi(\vec{0}) + \nabla \phi \times \nabla \psi + \psi(\vec{0}) - \nabla \phi \times \nabla \psi = \vec{0}$$

PROBLEM (29): **If $\vec{A}$ is a constant vector, show that $\nabla \times [(\vec{A} \cdot \vec{r})\vec{A}] = \vec{0}$.**

SOLUTION: We have

$$\nabla \times [(\vec{A} \cdot \vec{r})\vec{A}] = (\vec{A} \cdot \vec{r})(\nabla \times \vec{A}) + \nabla(\vec{A} \cdot \vec{r}) \times \vec{A} \qquad [\text{using theorem (4.9)}]$$

$$= (\vec{A} \cdot \vec{r})(\nabla \times \vec{A}) + \vec{A} \times \vec{A} \qquad [\text{since } \nabla(\vec{A} \cdot \vec{r}) = \vec{A}] \qquad (1)$$

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Since $\vec{A}$ is a constant vector , therefore $\nabla \times \vec{A} = \vec{0}$. Also $\vec{A} \times \vec{A} = \vec{0}$.
Thus equation (1) becomes

$$\nabla \times [(\vec{A} \cdot \vec{r})\vec{A}] = (\vec{A} \cdot \vec{r})(\vec{0}) + \vec{0} = \vec{0} + \vec{0} = \vec{0}$$

PROBLEM (30): **(i) Find constants a , b , c so that**

$$\vec{V} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k} \text{ is irrotational.}$$

(ii) Show that $\vec{V}$ can be expressed as the gradient of a scalar function .

SOLUTION: We have

(i) $$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$$= (c+1)\hat{i} + (a-4)\hat{j} + (b-2)\hat{k} .$$

This equals zero when $a = 4$, $b = 2$, $c = -1$ and thus

$$\vec{V} = (x+2y+4z)\hat{i} + (2x-3y-z)\hat{j} + (4x-y+2z)\hat{k} \quad (1)$$

(ii) Assume that $\vec{V} = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$ (2)

Comparing equations (1) and (2) , we get

$$\frac{\partial\phi}{\partial x} = x+2y+4z \quad (3)$$

$$\frac{\partial\phi}{\partial y} = 2x-3y-z \quad (4)$$

$$\frac{\partial\phi}{\partial z} = 4x-y+2z \quad (5)$$

Integrating equation (3) partially w.r.t. x keeping y and z constants ,

$$\phi = \frac{x^2}{2} + 2xy + 4xz + f(y,z) \quad (6)$$

Similarly , from equations (4) and (5) , we get

$$\phi = 2xy - \frac{3y^2}{2} - yz + g(x,z) \quad (7)$$

$$\phi = 4xz - yz + z^2 + h(x,y) \quad (8)$$

Comparison of equations (6) , (7) , and (8) shows that there will be a common value of ϕ if we choose

$$f(y,z) = -\frac{3y^2}{2} + z^2, \quad g(x,z) = \frac{x^2}{2} + z^2, \quad h(x,y) = \frac{x^2}{2} - \frac{3y^2}{2}$$ so that

$$\phi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4xz - yz + \text{constant}$$

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PROBLEM (31): **Show that $\vec{V} = \frac{\vec{r}}{r^2}$ is irrotational. Find ϕ such that $\vec{V} = -\nabla\phi$ and such that $\phi(a) = 0$, where $a > 0$.**

SOLUTION: A vector $\vec{V}$ is called irrotational if $\nabla \times \vec{V} = \vec{0}$, otherwise rotational.

Since $\nabla \times \vec{V} = \nabla \times \left(\frac{\vec{r}}{r^2}\right) = \nabla \times (r^{-2}\vec{r}) = \vec{0}$, therefore $\vec{V}$ is irrotational.

Now $\vec{V} = -\nabla\phi = \frac{\vec{r}}{r^2}$ or $\nabla\phi = -\frac{\vec{r}}{r^2}$

or $\frac{\partial\phi}{\partial r} = \nabla\phi \cdot \hat{r} = -\frac{\vec{r}}{r^2} \cdot \hat{r} = -\frac{1}{r}\hat{r} \cdot \hat{r} = -\frac{1}{r}$ $\left(\text{since } \hat{r} \cdot \hat{r} = 1\right)$

Integrating w.r.t. r, we get $\phi = -\ln r + C$

Using $\phi(a) = 0$, we have $C = \ln a$

therefore $\phi = -\ln r + \ln a = \ln\frac{a}{r}$

PROBLEM (32): **Show that if $\phi(x, y, z)$ is any solution of Laplace's equation, then $\nabla\phi$ is a vector which is both solenoidal and irrotational.**

SOLUTION: Since ϕ satisfies Laplace's equation therefore, $\nabla^2\phi = 0$ or $\nabla \cdot (\nabla\phi) = 0$.

Thus $\nabla\phi$ is solenoidal. Also by theorem [4.9 (iii)] $\nabla \times (\nabla\phi) = \vec{0}$ so that $\nabla\phi$ is irrotational.

PROBLEM (33): **Show that** $\nabla \times \left[\frac{1}{r}\nabla\left(\frac{1}{r}\right)\right] = \vec{0}$

SOLUTION: We have

$$\nabla \times \left[\frac{1}{r}\nabla\left(\frac{1}{r}\right)\right] = \nabla \times \left[\frac{1}{r}\left(-\frac{\vec{r}}{r^3}\right)\right] \qquad \text{[using theorem (4.2)]}$$

$$= -\nabla \times (r^{-4}\vec{r}) = \vec{0} \qquad \text{[using theorem (4.10)]}$$

PROBLEM (34): **Prove that**

(i) $(\vec{A} \times \nabla) \times \vec{r} = -2\vec{A}$ **(ii)** $(\vec{A} \times \nabla) \cdot \vec{r} = 0$

SOLUTION: (i) Let $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, then

$$\vec{A} \times \nabla = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$$

$$= \left(A_2\frac{\partial}{\partial z} - A_3\frac{\partial}{\partial y}\right)\hat{i} + \left(A_3\frac{\partial}{\partial x} - A_1\frac{\partial}{\partial z}\right)\hat{j} + \left(A_1\frac{\partial}{\partial y} - A_2\frac{\partial}{\partial x}\right)\hat{k}$$

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Thus $$(A \times \nabla) \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_2 \frac{\partial}{\partial z} - A_3 \frac{\partial}{\partial y} & A_3 \frac{\partial}{\partial x} - A_1 \frac{\partial}{\partial z} & A_1 \frac{\partial}{\partial y} - A_2 \frac{\partial}{\partial x} \\ x & y & z \end{vmatrix}$$

$$= (-A_1 - A_1)\hat{i} + (-A_2 - A_2)\hat{j} + (-A_3 - A_3)\hat{k}$$

$$= -2A_1\hat{i} - 2A_2\hat{j} - 2A_3\hat{k}$$

$$= -2(A_1\hat{i} + A_2\hat{j} + A_3\hat{k}) = -2\vec{A}$$

(ii) We know from theorem (4.12) that

$$(\vec{A} \times \nabla) . \vec{B} = \vec{A} . (\nabla \times \vec{B}) \qquad (1)$$

Let $\vec{B} = \vec{r}$ in equation (1) , then

$$(\vec{A} \times \nabla) . \vec{r} = \vec{A} . (\nabla \times \vec{r})$$

$$= 0 \qquad [\text{using theorem (4.10)}]$$

PROBLEMS ON VECTOR IDENTITIES

PROBLEM (35): **Prove that** $(\vec{V} . \nabla)\vec{V} = \frac{1}{2}\nabla V^2 - \vec{V} \times (\nabla \times \vec{V})$

SOLUTION: We have

$$\nabla V^2 = \nabla(\vec{V} . \vec{V})$$

$$= (\vec{V} . \nabla)\vec{V} + (\vec{V} . \nabla)\vec{V} + \vec{V} \times (\nabla \times \vec{V}) + \vec{V} \times (\nabla \times \vec{V})$$

$$= 2(\vec{V} . \nabla)\vec{V} + 2\vec{V} \times (\nabla \times \vec{V}) \qquad (\text{Using theorem [4.13 (iii)]})$$

or $$(\vec{V} . \nabla)\vec{V} = \frac{1}{2}\nabla V^2 - \vec{V} \times (\nabla \times \vec{V})$$

PROBLEM (36): **If $\vec{A}$ is a constant vector , prove that**

(i) $\nabla(\vec{A} . \vec{u}) = (\vec{A} . \nabla)\vec{u} + \vec{A} \times (\nabla \times \vec{u})$

(ii) $\nabla . (\vec{A} \times \vec{u}) = -\vec{A} . (\nabla \times \vec{u})$

(iii) $\nabla \times (\vec{A} \times \vec{u}) = \vec{A}(\nabla . \vec{u}) - (\vec{A} . \nabla)u$

SOLUTION: Since $\vec{A}$ is a constant vector , we have

$$\nabla . \vec{A} = 0, \quad \nabla \times \vec{A} = \vec{0} \quad \text{and} \quad (\vec{u} . \nabla)\vec{A} = \vec{0}$$

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(i) $$\nabla(\vec{A}\cdot\vec{u}) = (\vec{u}\cdot\nabla)\vec{A}+(\vec{A}\cdot\nabla)\vec{u}+\vec{u}\times(\nabla\times\vec{A})+\vec{A}\times(\nabla\times\vec{u})$$
$$= \vec{0}+(\vec{A}\cdot\nabla)\vec{u}+\vec{u}\times\vec{0}+\vec{A}\times(\nabla\times\vec{u})$$
$$= (\vec{A}\cdot\nabla)\vec{u}+\vec{A}\times(\nabla\times\vec{u})$$

(ii) $$\nabla\cdot(\vec{A}\times\vec{u}) = \vec{u}\cdot(\nabla\times\vec{A})-\vec{A}\cdot(\nabla\times\vec{u})$$
$$= \vec{u}\cdot\vec{0}-\vec{A}\cdot(\nabla\times\vec{u}) = -\vec{A}\cdot(\nabla\times\vec{u})$$

(iii) $$\nabla\times(\vec{A}\times\vec{u}) = (\vec{u}\cdot\nabla)\vec{A}-\vec{u}(\nabla\cdot\vec{A})-(\vec{A}\cdot\nabla)\vec{u}+\vec{A}(\nabla\cdot\vec{u})$$
$$= \vec{0}-\vec{0}-(\vec{A}\cdot\nabla)\vec{u}+\vec{A}(\nabla\cdot\vec{u}) = \vec{A}(\nabla\cdot\vec{u})-(\vec{A}\cdot\nabla)\vec{u}$$

PROBLEM (37): **If $\vec{A}$ is a constant vector, show that**

(i) $$\nabla\left(\frac{\vec{A}\cdot\vec{r}}{r^n}\right) = \frac{\vec{A}}{r^n}-\frac{n(\vec{A}\cdot\vec{r})\vec{r}}{r^{n+2}}$$

(ii) $$\nabla\cdot\left(\frac{\vec{A}\times\vec{r}}{r^n}\right) = 0$$

(iii) $$\nabla\times\left(\frac{\vec{A}\times\vec{r}}{r^n}\right) = \frac{2-n}{r^n}\vec{A}+\frac{n(\vec{A}\cdot\vec{r})\vec{r}}{r^{n+2}}$$

SOLUTION: (i) By theorem [4.1 (iv)], we have

$$\nabla\left(\frac{\vec{A}\cdot\vec{r}}{r^n}\right) = \nabla\left(\frac{1}{r^n}\right)(\vec{A}\cdot\vec{r})+\frac{1}{r^n}\nabla(\vec{A}\cdot\vec{r})$$
$$= -nr^{-n-1}\nabla r(\vec{A}\cdot\vec{r})+\frac{1}{r^n}\vec{A} \qquad [\text{since } \nabla(\vec{A}\cdot\vec{r}) = \vec{A}]$$
$$= \frac{\vec{A}}{r^n}-nr^{-n-1}\frac{\vec{r}}{r}(\vec{A}\cdot\vec{r}) \qquad \left[\text{since } \nabla r = \hat{r} = \frac{\vec{r}}{r}\right]$$
$$= \frac{\vec{A}}{r^n}-\frac{n(\vec{A}\cdot\vec{r})\vec{r}}{r^{n+2}}$$

(ii) By theorem [4.5 (ii)], we have

$$\nabla\cdot\left(\frac{\vec{A}\times\vec{r}}{r^n}\right) = \nabla\left(\frac{1}{r^n}\right)\cdot(\vec{A}\times\vec{r})+\frac{1}{r^n}\nabla\cdot(\vec{A}\times\vec{r})$$
$$= -nr^{-n-1}\nabla r\cdot(\vec{A}\times\vec{r})+0 \qquad [\text{since } \nabla\cdot(\vec{A}\times\vec{r}) = 0]$$
$$= -nr^{-n-1}\frac{\vec{r}}{r}\cdot(\vec{A}\times\vec{r})$$

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$$= -\frac{n}{r^{n+2}}\vec{r}\cdot\vec{A}\times\vec{r}$$

$$= \frac{n}{r^{n+2}}\vec{r}\cdot\vec{r}\times\vec{A} \qquad [\text{since } \vec{A}\times\vec{r} = -\vec{r}\times\vec{A}]$$

$$= \frac{n}{r^{n+2}}\vec{r}\times\vec{r}\cdot\vec{A} = 0 \qquad [\text{since } \vec{r}\times\vec{r} = \vec{0}]$$

ALTERNATIVE METHOD

By theorem [4.13 (ii)] , we get

$$\nabla\cdot\left(\frac{\vec{A}\times\vec{r}}{r^n}\right) = \nabla\cdot\left(\vec{A}\times\frac{\vec{r}}{r^n}\right) = \frac{\vec{r}}{r^n}\cdot(\nabla\times\vec{A}) + \vec{A}\cdot\left(\nabla\times\frac{\vec{r}}{r^n}\right) = 0.$$

[since $\nabla\times\vec{A} = \vec{0}$ and $\nabla\times[f(r)\vec{r}] = \vec{0}$]

(iii) By theorem [4 9 (ii)] , we have

$$\nabla\times\left(\frac{\vec{A}\times\vec{r}}{r^n}\right) = \nabla\left(\frac{1}{r^n}\right)\times(\vec{A}\times\vec{r}) + \frac{1}{r^n}[\nabla\times(\vec{A}\times\vec{r})]$$

$$= -nr^{-n-1}\nabla r\times(\vec{A}\times\vec{r}) + \frac{2\vec{A}}{r^n} \qquad [\text{since } \nabla\times(\vec{A}\times\vec{r}) = 2\vec{A}]$$

$$= -nr^{-n-1}\frac{\vec{r}}{r}\times(\vec{A}\times\vec{r}) + \frac{2\vec{A}}{r^n} \qquad \left[\text{since } \nabla r = \hat{r} = \frac{\vec{r}}{r}\right]$$

$$= -\frac{n}{r^{n+2}}\vec{r}\times(\vec{A}\times\vec{r}) + \frac{2\vec{A}}{r^n}$$

$$= -\frac{n}{r^{n+2}}[(\vec{r}\cdot\vec{r})\vec{A} - (\vec{r}\cdot\vec{A})\vec{r}] + \frac{2\vec{A}}{r^n}$$

$$= -\frac{n}{r^{n+2}}r^2\vec{A} + \frac{n}{r^{n+2}}(\vec{r}\cdot\vec{A})\vec{r} + \frac{2\vec{A}}{r^n}$$

$$= \frac{2-n}{r^n}\vec{A} + \frac{n}{r^{n+2}}(\vec{A}\cdot\vec{r})\vec{r}$$

PROBLEM (38): **If $\vec{A}$ and $\vec{B}$ are constant vectors , prove that**

(i) $\nabla\cdot[(\vec{r}\times\vec{A})\times\vec{B}] = -2\vec{B}\cdot\vec{A}$

(ii) $\nabla\times[(\vec{r}\times\vec{A})\times\vec{B}] = \vec{B}\times\vec{A}$

SOLUTION: We have

(i) $$\nabla\cdot[(\vec{r}\times\vec{A})\times\vec{B}] = \nabla\cdot[(\vec{r}\cdot\vec{B})\vec{A} - (\vec{A}\cdot\vec{B})\vec{r}]$$

$$= \nabla\cdot[(\vec{r}\cdot\vec{B})\vec{A}] - \nabla\cdot[(\vec{A}\cdot\vec{B})\vec{r}]$$

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$$= (\vec{r}\cdot\vec{B})\nabla\cdot\vec{A}+\nabla(\vec{r}\cdot\vec{B})\cdot\vec{A}$$
$$-(\vec{A}\cdot\vec{B})\nabla\cdot\vec{r}-\nabla(\vec{A}\cdot\vec{B})\cdot\vec{r} \qquad (1)$$

Since $\vec{A}$ is a constant vector, therefore $\nabla\cdot\vec{A}=\vec{0}$. Also, $\nabla\cdot\vec{r}=3$.
Thus equation (1) reduces to

$$\nabla\cdot[(\vec{r}\times\vec{A})\times\vec{B}] = \nabla(\vec{r}\cdot\vec{B})\cdot\vec{A}-3(\vec{A}\cdot\vec{B})-\nabla(\vec{A}\cdot\vec{B})\cdot\vec{r}$$
$$= \nabla(\vec{B}\cdot\vec{r})\cdot\vec{A}-3(\vec{A}\cdot\vec{B})-\nabla(\vec{A}\cdot\vec{B})\cdot\vec{r} \qquad (2)$$

Now $\nabla(\vec{B}\cdot\vec{r})=\vec{B}$ and $\nabla(\vec{A}\cdot\vec{B})=\vec{0}$, since $\vec{A}\cdot\vec{B}$ is constant.
Thus equation (2) reduces to

$$\nabla[(\vec{r}\times\vec{A})\times\vec{B}] = \vec{B}\cdot\vec{A}-3(\vec{A}\cdot\vec{B}) = \vec{B}\cdot\vec{A}-3(\vec{B}\cdot\vec{A}) = -2(\vec{B}\cdot\vec{A}).$$

(ii) $$\nabla\times[(\vec{r}\times\vec{A})\times\vec{B}] = \nabla\times[(\vec{r}\cdot\vec{B})\vec{A}-(\vec{A}\cdot\vec{B})\vec{r}]$$
$$= \nabla\times[(\vec{r}\cdot\vec{B})\vec{A}]-\nabla\times[(\vec{A}\cdot\vec{B})\vec{r}]$$
$$= (\vec{r}\cdot\vec{B})\nabla\times\vec{A}+\nabla(\vec{r}\cdot\vec{B})\times\vec{A}$$
$$-(\vec{A}\cdot\vec{B})\nabla\times\vec{r}-\nabla(\vec{A}\cdot\vec{B})\times\vec{r} \qquad (3)$$

Since $\vec{A}$ is a constant vector, therefore $\nabla\times\vec{A}=\vec{0}$. Also, $\nabla\times\vec{r}=\vec{0}$. Thus equation (3) reduces to

$$\nabla\times[(\vec{r}\times\vec{A})\times\vec{B}] = \nabla(\vec{r}\cdot\vec{B})\times\vec{A}-\nabla(\vec{A}\cdot\vec{B})\times\vec{r}$$
$$= \nabla(\vec{B}\cdot\vec{r})\times\vec{A}-\nabla(\vec{A}\cdot\vec{B})\times\vec{r}$$

Now $\nabla(\vec{B}\cdot\vec{r})=\vec{B}$ and $\nabla(\vec{A}\cdot\vec{B})=\vec{0}$, since $\vec{A}\cdot\vec{B}$ is constant. Thus equation (4) reduces to $\nabla\times[(\vec{r}\times\vec{A})\times\vec{B}]=\vec{B}\times\vec{A}$.

PROBLEM (39): **If $\vec{A}$ is a solenoidal vector (i.e. one whose divergence vanishes identically), show that $\nabla\times\nabla\times\nabla\times\nabla\times\vec{A}=\nabla^2\nabla^2\vec{A}=\nabla^4\vec{A}$.**

SOLUTION: We know from theorem [4.13 (iv)] that

$$\nabla\times\nabla\times\vec{A} = \nabla(\nabla\cdot\vec{A})-\nabla^2\vec{A} \qquad (1)$$

Since $\vec{A}$ is solenoidal, therefore $\nabla\cdot\vec{A}=0$. Thus equation (1) reduces to

$$\nabla\times\nabla\times\vec{A} = -\nabla^2\vec{A}$$

Hence $$\nabla\times\nabla\times\nabla\times\nabla\times\vec{A} = \nabla\times\nabla\times(\nabla\times\nabla\times\vec{A})$$
$$= \nabla\times\nabla\times(-\nabla^2\vec{A})$$

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$$= -\nabla^2(\nabla \times \nabla \times \vec{A})$$

$$= -\nabla^2(-\nabla^2 \vec{A}) = \nabla^2 \nabla^2 \vec{A} = \nabla^4 \vec{A}$$

PROBLEM (40): **If $\vec{A}$, $\vec{B}$, $\vec{C}$ are constant vectors , prove that**

$$\nabla(\vec{A}.\vec{r}\ \vec{B}.\vec{r}\ \vec{C}.\vec{r}) = \vec{B}.\vec{r}\ \vec{C}.\vec{r}\ \vec{A} + \vec{A}.\vec{r}\ \vec{C}.\vec{r}\ \vec{B} + \vec{A}.\vec{r}\ \vec{B}.\vec{r}\ \vec{C}$$

SOLUTION: We know that

$$\nabla(UVW) = UV\nabla W + UW\nabla V + VW\nabla U$$

$$\nabla(\vec{A}.\vec{r}\ \vec{B}.\vec{r}\ \vec{C}.\vec{r}) = \vec{A}.\vec{r}\ \vec{B}.\vec{r}\ \nabla(\vec{C}.\vec{r}) + \vec{A}.\vec{r}\ \vec{C}.\vec{r}\ \nabla(\vec{B}.\vec{r}) + \vec{B}.\vec{r}\ \vec{C}.\vec{r}\ \nabla(\vec{A}.\vec{r}) \qquad (1)$$

Also , we know from solved example (12) that

$\nabla(\vec{A}.\vec{r}) = \vec{A}$, $\nabla(\vec{B}.\vec{r}) = \vec{B}$ and $\nabla(\vec{C}.\vec{r}) = \vec{C}$. Thus equation (1) becomes

$$\nabla(\vec{A}.\vec{r}\ \vec{B}.\vec{r}\ \vec{C}.\vec{r}) = \vec{B}.\vec{r}\ \vec{C}.\vec{r}\ \vec{A} + \vec{A}.\vec{r}\ \vec{C}.\vec{r}\ \vec{B} + \vec{A}.\vec{r}\ \vec{B}.\vec{r}\ \vec{C}$$

PROBLEM (41): **If $\nabla.\vec{E} = 0$, $\nabla.\vec{H} = 0$, $\nabla \times \vec{E} = -\frac{\partial \vec{H}}{\partial t}$, $\nabla \times \vec{H} = \frac{\partial \vec{E}}{\partial t}$,**

show that $\vec{E}$ and $\vec{H}$ satisfy $\nabla^2 u = \frac{\partial^2 u}{\partial t^2}$.

SOLUTION:

$$\nabla \times (\nabla \times \vec{E}) = \nabla \times \left(-\frac{\partial \vec{H}}{\partial t}\right)$$

$$= -\frac{\partial}{\partial t}(\nabla \times \vec{H}) = -\frac{\partial}{\partial t}\left(\frac{\partial \vec{E}}{\partial t}\right) = -\frac{\partial^2 \vec{E}}{\partial t^2} \qquad (1)$$

By theorem [4.13 (iv)] , we have

$$\nabla \times (\nabla \times \vec{E}) = -\nabla^2 \vec{E} + \nabla(\nabla.\vec{E}) = -\nabla^2 \vec{E} \qquad (2)$$

From equations (1) and (2) , we get $\nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial t^2}$

Similarly , $\nabla \times (\nabla \times \vec{H}) = \nabla \times \left(\frac{\partial \vec{E}}{\partial t}\right) = \frac{\partial}{\partial t}(\nabla \times \vec{E})$

$$= \frac{\partial}{\partial t}\left(-\frac{\partial \vec{H}}{\partial t}\right) = -\frac{\partial^2 \vec{H}}{\partial t^2} \qquad (3)$$

Again by theorem [4.13 (iv)] , we have

$$\nabla \times (\nabla \times \vec{H}) = -\nabla^2 \vec{H} + \nabla(\nabla.\vec{H}) = -\nabla^2 \vec{H} \qquad (4)$$

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From equations (3) and (4), we get $\nabla^2 \vec{H} = \frac{\partial^2 \vec{H}}{\partial t^2}$

The given equations are related to Maxwell's equations of electromagnetic theory. The equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial t^2}$$

is called the **three–dimensional wave equation**.

PROBLEM (42): **From the equations** $\nabla . \vec{D} = \rho$, $\nabla . \vec{H} = 0$,

$$\nabla \times \vec{H} = \frac{1}{C}\left(\frac{\partial \vec{D}}{\partial t} + \rho \vec{V}\right), \quad \nabla \times \vec{D} = -\frac{1}{C}\frac{\partial \vec{H}}{\partial t}$$

in which t is the time variable, C is a constant, and the other symbols denote point functions, prove that

$$\nabla^2 \vec{D} - \frac{1}{C^2}\frac{\partial^2 \vec{D}}{\partial t^2} = \nabla \rho + \frac{1}{C^2}\frac{\partial}{\partial t}(\rho \vec{V})$$

$$\nabla^2 \vec{H} - \frac{1}{C^2}\frac{\partial^2 \vec{H}}{\partial t^2} = -\frac{1}{C}\nabla \times (\rho \vec{V})$$

SOLUTION: Given that

$$\nabla . \vec{D} = \rho \tag{1}$$

$$\nabla . \vec{H} = 0 \tag{2}$$

$$\nabla \times \vec{H} = \frac{1}{C}\left(\frac{\partial \vec{D}}{\partial t} + \rho \vec{V}\right) \tag{3}$$

$$\nabla \times \vec{D} = -\frac{1}{C}\frac{\partial \vec{H}}{\partial t} \tag{4}$$

(i) $$\nabla \times \nabla \times \vec{D} = \nabla \times \left(-\frac{1}{C}\frac{\partial \vec{H}}{\partial t}\right) \quad [\text{using equation (4)}]$$

$$= -\frac{1}{C}\left(\nabla \times \frac{\partial \vec{H}}{\partial t}\right) = -\frac{1}{C}\frac{\partial}{\partial t}(\nabla \times \vec{H})$$

$$= -\frac{1}{C}\frac{\partial}{\partial t}\left[\frac{1}{C}\left(\frac{\partial \vec{D}}{\partial t} + \rho \vec{V}\right)\right] \quad [\text{using equation (3)}]$$

$$= -\frac{1}{C^2}\frac{\partial^2 \vec{D}}{\partial t^2} - \frac{1}{C^2}\frac{\partial}{\partial t}(\rho \vec{V}) \tag{5}$$

But $\nabla \times \nabla \times \vec{D} = \nabla(\nabla . \vec{D}) - \nabla^2 \vec{D}$. Thus equation (5) becomes

$$\nabla(\nabla . \vec{D}) - \nabla^2 \vec{D} = -\frac{1}{C^2}\frac{\partial^2 \vec{D}}{\partial t^2} - \frac{1}{C^2}\frac{\partial}{\partial t}(\rho \vec{V})$$

$$\nabla \rho - \nabla^2 \vec{D} = -\frac{1}{C^2}\frac{\partial^2 \vec{D}}{\partial t^2} - \frac{1}{C^2}\frac{\partial}{\partial t}(\rho \vec{V}) \quad [\text{using equation (1)}]$$

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or $$\nabla^2 \vec{D} - \frac{1}{C^2}\frac{\partial^2 \vec{D}}{\partial t^2} = \nabla \rho + \frac{1}{C^2}\frac{\partial}{\partial t}(\rho \vec{V})$$

(ii) $$\nabla \times \nabla \times \vec{H} = \nabla \times \frac{1}{C}\left(\frac{\partial \vec{D}}{\partial t} + \rho \vec{V}\right) \quad \text{[using equation (3)]}$$

$$= \frac{1}{C}\left(\nabla \times \frac{\partial \vec{D}}{\partial t}\right) + \frac{1}{C}\nabla \times (\rho \vec{V})$$

$$= \frac{1}{C}\frac{\partial}{\partial t}(\nabla \times \vec{D}) + \frac{1}{C}\nabla \times (\rho \vec{V})$$

$$= \frac{1}{C}\frac{\partial}{\partial t}\left(-\frac{1}{C}\frac{\partial \vec{H}}{\partial t}\right) + \frac{1}{C}\nabla \times (\rho \vec{V}) \quad \text{[using equation (4)]}$$

$$= -\frac{1}{C^2}\frac{\partial^2 \vec{H}}{\partial t^2} + \frac{1}{C}\nabla \times (\rho \vec{V}) \qquad (6)$$

But $\nabla \times \nabla \times \vec{H} = \nabla(\nabla . \vec{H}) - \nabla^2 \vec{H}$. Thus equation (6) becomes

or $$\nabla(\nabla . \vec{H}) - \nabla^2 \vec{H} = -\frac{1}{C^2}\frac{\partial^2 \vec{H}}{\partial t^2} + \frac{1}{C}\nabla \times (\rho \vec{V})$$

$$\nabla(0) - \nabla^2 \vec{H} = -\frac{1}{C^2}\frac{\partial^2 \vec{H}}{\partial t^2} + \frac{1}{C}\nabla \times (\rho \vec{V}) \quad \text{[using equation (2)]}$$

or $$\nabla^2 \vec{H} - \frac{1}{C^2}\frac{\partial^2 \vec{H}}{\partial t^2} = -\frac{1}{C}\nabla \times (\rho \vec{V})$$

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