Chapter 3

VECTOR DIFFERENTIATION AND INTEGRATION

3.1 INTRODUCTION

In this chapter ; we shall be concerned with vector functions in three–dimensional space and the application of differential and integral calculus to them . We shall extend the basic concepts of calculus in a simple and natural way . The study of vector calculus makes the vectors much more useful in geometrical , physical , and engineering applications , especially in connection with forces and velocities of motions . Here , we shall discuss scalar and vector functions , limits and continuity , differentiation and integration of vector functions .

3.2 SCALAR AND VECTOR FUNCTIONS OF ONE VARIABLE

SCALAR FUNCTION

If to each value of a scalar variable t in some interval , there corresponds a unique scalar 'ϕ , then ϕ is called a scalar function of t denoted by $\phi(t)$ and we write $\phi = \phi(t)$. A real–valued function is often called a scalar function . For example , $\phi(t) = t^2 + \sin t$ defines a scalar function .

VECTOR FUNCTION

If to each value of a scalar variable t in some interval , there corresponds a unique vector $\vec{A}$, then $\vec{A}$ is called a vector function of t denoted by $\vec{A}(t)$ and we write $\vec{A} = \vec{A}(t)$.

If $\hat{i}, \hat{j}, \hat{k}$ are the unit vectors associated with a rectangular coordinate system , then a vector function $\vec{A}(t)$ is written as $\vec{A}(t) = A_1(t)\hat{i} + A_2(t)\hat{j} + A_3(t)\hat{k}$

where $A_1(t), A_2(t)$, and $A_3(t)$ are scalar functions of t and are called the components of $\vec{A}(t)$.

We say that the vector function $\vec{A}(t)$ is defined if all its components A_1, A_2, and A_3 are defined .

EXAMPLE (1): If $\vec{A}(t) = \cos t\,\hat{i} - 2t^2\hat{j} + e^t\hat{k}$ then find $\vec{A}(0)$ and $\vec{A}(\pi)$.

SOLUTION: We have $\vec{A}(t) = \cos t\,\hat{i} - 2t^2\hat{j} + e^t\hat{k}$

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Since $\cos t$, t^2, and e^t are defined for all values of t, therefore $\vec{A}(t)$ is defined for all t.

We see that for any given value of t, $\vec{A}(t)$ is a vector. Thus

$$\vec{A}(0) = \cos(0)\hat{i} - 2(0)^2\hat{j} + e^0\hat{k} = \hat{i} + \hat{k}$$

and $$\vec{A}(\pi) = \cos\pi\,\hat{i} - 2(\pi)^2\hat{j} + e^{\pi}\hat{k} = -\hat{i} - 2\pi^2\hat{j} + e^{\pi}\hat{k}$$

3.3 DOMAIN AND RANGE

The set of all possible values of t for which $\vec{A}$ is defined is called the **domain of** $\vec{A}$ and the set of corresponding vectors is called the **range of** $\vec{A}$.

EXAMPLE (2): **Find the domain of the vector function**

$$\vec{A}(t) = \frac{1}{t}\hat{i} + \sqrt{t+1}\,\hat{j} + \sin t\,\hat{k}.$$

SOLUTION: We have $\vec{A}(t) = \frac{1}{t}\hat{i} + \sqrt{t+1}\,\hat{j} + \sin t\,k$

Then the domain of $\vec{A}(t)$ is the set of all t for which the functions $\frac{1}{t}$, $\sqrt{t+1}$, and $\sin t$ are defined. Since $\frac{1}{t}$ is defined for $t \neq 0$, $\sqrt{t+1}$ is defined for $t \geq -1$, and $\sin t$ is defined for all t, we see that the domain of $\vec{A}(t)$ is the set $D = \{t;\ t \geq -1 \text{ and } t \neq 0\}$.

3.4 LIMIT AND CONTINUITY OF A VECTOR FUNCTION

LIMIT OF A VECTOR FUNCTION

Let a vector function $\vec{A}(t)$ be defined for all values of t in some neighbourhood about a point $t = t_0$, except possibly at t_0 itself, and let $\vec{L}$ be a constant vector called the limit vector. The function $\vec{A}(t)$ is said to approach the limit vector $\vec{L}$ as t approaches t_0, if for any given real number $\epsilon > 0$, however small, we can find some real number $\delta > 0$ such that

$$\left|\vec{A}(t) - \vec{L}\right| < \epsilon \text{ whenever } 0 < |t - t_0| < \delta.$$

Symbolically, we write $\underset{t \to t_0}{\mathrm{Lt}}\ \vec{A}(t) = \vec{L}$.

Note that if $\vec{A}(t) = A_1(t)\hat{i} + A_2(t)\hat{j} + A_3(t)\hat{k}$ and $\vec{L} = L_1\hat{i} + L_2\hat{j} + L_3\hat{k}$, then

$$\underset{t \to t_0}{\mathrm{Lt}}\ \vec{A}(t) = \vec{L} \text{ if and only if } \underset{t \to t_0}{\mathrm{Lt}}\ A_i(t) = L_i,\ i = 1, 2, 3.$$

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EXAMPLE (3): If $\vec{A}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + t\hat{k}$, find $\underset{t \to \pi/4}{\text{Lt}} \vec{A}(t)$.

SOLUTION: We have $\vec{A}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + t\hat{k}$,

Then $$\underset{t \to \pi/4}{\text{Lt}} \vec{A}(t) = \left(\underset{t \to \pi/4}{\text{Lt}} \cos t\right)\hat{i} + \underset{t \to \pi/4}{\text{Lt}} \sin t)\hat{j} + \underset{t \to \pi/4}{\text{Lt}} (t)\hat{k}$$

$$= \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \frac{\pi}{4}\hat{k}$$

PROPERTIES OF LIMITS OF VECTOR FUNCTIONS

If $\underset{t \to t_0}{\text{Lt}} \vec{A}(t) = \vec{L}$, $\underset{t \to t_0}{\text{Lt}} \vec{B}(t) = \vec{M}$ and $\underset{t \to t_0}{\text{Lt}} \phi(t) = C$, where $\vec{L}$ and $\vec{M}$ are constant vectors and C is a scalar constant, then

(i) $$\underset{t \to t_0}{\text{Lt}} [\vec{A}(t) + \vec{B}(t)] = \underset{t \to t_0}{\text{Lt}} \vec{A}(t) + \underset{t \to t_0}{\text{Lt}} \vec{B}(t) = \vec{L} + \vec{M}$$

i.e. the limit of the sum of two vector functions is the sum of their limits.

(ii) $$\underset{t \to t_0}{\text{Lt}} [\vec{A}(t) - \vec{B}(t)] = \underset{t \to t_0}{\text{Lt}} \vec{A}(t) - \underset{t \to t_0}{\text{Lt}} \vec{B}(t) = \vec{L} - \vec{M}$$

i.e. the limit of the difference of two vector functions is the difference of their limits.

(iii) $$\underset{t \to t_0}{\text{Lt}} [\vec{A}(t) \cdot \vec{B}(t)] = \underset{t \to t_0}{\text{Lt}} \vec{A}(t) \cdot \underset{t \to t_0}{\text{Lt}} \vec{B}(t) = \vec{L} \cdot \vec{M}$$

i.e. the limit of the dot product of two vector functions is the dot product of their limits.

(iv) $$\underset{t \to t_0}{\text{Lt}} [\vec{A}(t) \times \vec{B}(t)] = \underset{t \to t_0}{\text{Lt}} \vec{A}(t) \times \underset{t \to t_0}{\text{Lt}} \vec{B}(t) = \vec{L} \times \vec{M}$$

i.e. the limit of the cross product of two vector functions is the cross product of their limits.

(v) $$\underset{t \to t_0}{\text{Lt}} [\phi(t)\vec{A}(t)] = \underset{t \to t_0}{\text{Lt}} \phi(t) \underset{t \to t_0}{\text{Lt}} \vec{A}(t) = C\vec{L}$$

i.e. the limit of the product of a scalar function and a vector function is the product of their limits.

(vi) $$\underset{t \to t_0}{\text{Lt}} |\vec{A}(t)| = \left|\underset{t \to t_0}{\text{Lt}} \vec{A}(t)\right| = |\vec{L}|$$

i.e. the limit of the absolute value of a vector function is the absolute value of the limit.

CONTINUITY OF A VECTOR FUNCTION

Let a vector function $\vec{A}(t)$ be defined at a point $t = t_0$ in its domain. The function $\vec{A}(t)$ is said to be continuous at $t = t_0$ if $\underset{t \to t_0}{\text{Lt}} \vec{A}(t) = \vec{A}(t_0)$.

Thus the function $\vec{A}(t)$ is said to be continuous at $t = t_0$, if for any given real number $\epsilon > 0$ we can find a real number $\delta > 0$, such that $\left|\vec{A}(t) - \vec{A}(t_0)\right| < \epsilon$ whenever $|t - t_0| < \delta$.

The vector function $\vec{A}(t)$ is called continuous if it is continuous at every point in its domain.

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COMPONENT TEST FOR CONTINUITY AT A POINT

THEOREM (3.1): **Prove that a vector function $\vec{A}(t) = A_1(t)\hat{i} + A_2(t)\hat{j} + A_3(t)\hat{k}$ is continuous at $t = t_0$ if each scalar function $A_i(t)$ $(i = 1, 2, 3)$ is continuous at $t = t_0$.**

PROOF: If $A_i(t)$ $(i = 1, 2, 3)$ is continuous at $t = t_0$, then we have

$$\underset{t \to t_0}{\text{Lt}} A_i(t) = A_i(t_0), \quad i = 1, 2, 3 \qquad (1)$$

Now $\vec{A}(t) = A_1(t)\hat{i} + A_2(t)\hat{j} + A_3(t)\hat{k}$

and so $\vec{A}(t_0) = A_1(t_0)\hat{i} + A_2(t_0)\hat{j} + A_3(t_0)\hat{k}$ (2)

Using property (1) of limits, we have

$$\underset{t \to t_0}{\text{Lt}} \vec{A}(t) = \underset{t \to t_0}{\text{Lt}} \left[A_1(t)\hat{i} + A_2(k)\hat{j} + A_3(t)\hat{k}\right]$$

$$= \underset{t \to t_0}{\text{Lt}} A_1(t)\hat{i} + \underset{t \to t_0}{\text{Lt}} A_2(t)\hat{j} + \underset{t \to t_0}{\text{Lt}} A_3(t)\hat{k}$$

Since $\hat{i}, \hat{j}, \hat{k}$ are constant vectors, from property (v) of limits, we have

$$\underset{t \to t_0}{\text{Lt}} \vec{A}(t) = \hat{i} \underset{t \to t_0}{\text{Lt}} A_1(t) + \hat{j} \underset{t \to t_0}{\text{Lt}} A_2(t) + \hat{k} \underset{t \to t_0}{\text{Lt}} A_3(t)$$

Using equations (1) and (2), we get

$$\underset{t \to t_0}{\text{Lt}} \vec{A}(t) = \hat{i} A_1(t_0) + \hat{j} A_2(t_0) + \hat{k} A_3(t_0) = \vec{A}(t_0)$$

Hence by definition, $\vec{A}(t)$ is continuous at $t = t_0$.

EXAMPLE (4): **Test the continuity of the following vector functions:**

(i) $\vec{A}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + e^t\hat{k}$ **(ii)** $\vec{A}(t) = t\hat{i} + \frac{1}{t}\hat{j} - \ln(t)\hat{k}$

SOLUTION: (i) $\vec{A}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + e^t\hat{k}$

Since the functions $\cos t$, $\sin t$, and e^t are continuous for all values of t, therefore the vector function $\vec{A}(t)$ is continuous for all values of t.

(ii) $\vec{A}(t) = t\hat{i} + \frac{1}{t}\hat{j} - \ln(t)\hat{k}$

The i – component function t is defined and continuous for all values of t. The j – component function $\frac{1}{t}$ is not defined at $t = 0$ and hence is not continuous at $t = 0$. The k – component function $\ln(t)$ is defined and continuous for all values of $t > 0$. Hence the given vector function $\vec{A}(t)$ is continuous for all values of $t > 0$.

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3.5 ORDINARY DERIVATIVE OF A VECTOR FUNCTION

Let $\vec{A}(t)$ be a vector function depending on a single scalar variable t. The ordinary derivative of $\vec{A}$ w.r.t. t is defined as

$$\frac{d\vec{A}}{dt} = \underset{\Delta t \to 0}{\mathrm{Lt}} \frac{\Delta \vec{A}}{\Delta t} = \underset{\Delta t \to 0}{\mathrm{Lt}} \frac{\vec{A}(t+\Delta t) - \vec{A}(t)}{\Delta t}$$

provided the limit exists. If $\frac{d\vec{A}}{dt}$ exists at $t = t_0$, $\vec{A}(t)$ is said to be differentiable at $t = t_0$.

Since $\frac{d\vec{A}}{dt}$ is itself a vector function depending on t, we can define its derivative w.r.t. t. If this derivative exists it is denoted by $\frac{d^2\vec{A}}{dt^2}$. Higher order derivatives such as $\frac{d^3\vec{A}}{dt^3}$, etc. can similarly be defined.

A scalar or vector function of t is called differentiable of order n if its nth derivative exists. From elementary calculus we know that a differentiable function is necessarily continuous but that the converse is not always true. Unless otherwise stated we assume that all functions considered are differentiable to any order needed in a particular discussion.

3.6 GEOMETRICAL INTERPRETATION OF VECTOR DERIVATIVE

Consider the motion of a particle along a curve C in space.

Let $\vec{A}(t)$ and $\vec{A}(t+\Delta t)$ be the vectors of the points P and P' corresponding to t and $t+\Delta t$ respectively, then from figure (3.1).

$$\Delta\vec{A} = \vec{A}(t+\Delta t) - \vec{A}(t)$$

Since Δt is a scalar, the quotient

$$\frac{\Delta\vec{A}}{\Delta t} = \frac{\vec{A}(t+\Delta t) - \vec{A}(t)}{\Delta t}$$

is a vector in the direction of $\Delta\vec{A}$.

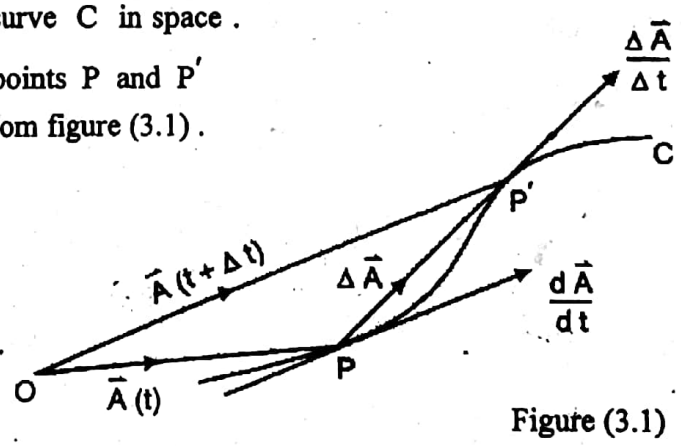

Figure (3.1)

As $\Delta t \to 0$, $P' \to P$ along C and $\underset{\Delta t \to 0}{\mathrm{Lt}} \frac{\Delta\vec{A}}{\Delta t} = \frac{d\vec{A}}{dt}$ represents the vector in the direction of the tangent to the curve C at P.

3.7 VELOCITY AND ACCELERATION

If $\vec{A}(t)$ is the position vector $\vec{r}(t)$ of a particle P moving along the curve C, then $\frac{d\vec{r}}{dt}$ represents the velocity vector of the particle along the curve and is denoted by $\vec{V}$, i.e. $\vec{V} = \frac{d\vec{r}}{dt}$.

Similarly, the rate of change of $\vec{V}$ w.r.t. t is called the **acceleration** vector of the particle P along the curve and is denoted by $\vec{a}$ i.e. $\vec{a} = \frac{d\vec{V}}{dt} = \frac{d^2\vec{r}}{dt^2}$.

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THEOREM (3.2): If $\vec{A}(t) = A_1(t)\hat{i} + A_2(t)\hat{j} + A_3(t)\hat{k}$, where A_1, A_2, and A_3 are differentiable functions of a scalar variable t, prove that

$$\frac{d\vec{A}}{dt} = \frac{dA_1}{dt}\hat{i} + \frac{dA_2}{dt}\hat{j} + \frac{dA_3}{dt}\hat{k}$$

PROOF: By definition, $\frac{d\vec{A}}{dt} = \underset{\Delta t \to 0}{\text{Lt}} \frac{\vec{A}(t+\Delta t) - \vec{A}(t)}{\Delta t}$

$$= \underset{\Delta t \to 0}{\text{Lt}} \frac{[A_1(t+\Delta t)\hat{i} + A_2(t+\Delta t)\hat{j} + A_3(t+\Delta t)\hat{k}] - [A_1(t)\hat{i} + A_2(t)\hat{j} + A_3(t)\hat{k}]}{\Delta t}$$

$$= \underset{\Delta t \to 0}{\text{Lt}} \left[\frac{A_1(t+\Delta t) - A_1(t)}{\Delta t}\hat{i} + \frac{A_2(t+\Delta t) - A_2(t)}{\Delta t}\hat{j} + \frac{A_3(t+\Delta t) - A_3(t)}{\Delta t}\hat{k}\right]$$

$$= \frac{dA_1}{dt}\hat{i} + \frac{dA_2}{dt}\hat{j} + \frac{dA_3}{dt}\hat{k}.$$

EXAMPLE (5): Given $\vec{R} = \sin t\,\hat{i} + \cos t\,\hat{j} + t\hat{k}$, find

(i) $\frac{d\vec{R}}{dt}$ (ii) $\frac{d^2\vec{R}}{dt^2}$ (iii) $\left|\frac{d\vec{R}}{dt}\right|$ (iv) $\left|\frac{d^2\vec{R}}{dt^2}\right|$.

SOLUTION: Since $\vec{R} = \sin t\,\hat{i} + \cos t\,\hat{j} + t\hat{k}$, therefore

(i) $\frac{d\vec{R}}{dt} = \frac{d}{dt}(\sin t)\hat{i} + \frac{d}{dt}(\cos t)\hat{j} + \frac{d}{dt}(t)\hat{k} = \cos t\,\hat{i} - \sin t\,\hat{j} + \hat{k}$

(ii) $\frac{d^2\vec{R}}{dt^2} = \frac{d}{dt}\left(\frac{d\vec{R}}{dt}\right) = \frac{d}{dt}(\cos t)\hat{i} - \frac{d}{dt}(\sin t)\hat{j} + \frac{d}{dt}(1)\hat{k} = -\sin t\,\hat{i} - \cos t\,\hat{j}$

(iii) $\left|\frac{d\vec{R}}{dt}\right| = \sqrt{(\cos t)^2 + (-\sin t)^2 + (1)^2} = \sqrt{\cos^2 t + \sin^2 t + 1} = \sqrt{1+1} = \sqrt{2}$

(iv) $\left|\frac{d^2\vec{R}}{dt^2}\right| = \sqrt{(-\sin t)^2 + (-\cos t)^2} = \sqrt{\sin^2 t + \cos^2 t} = 1$

3.8 DIFFERENTIATION FORMULAS

THEOREM (3.3): If $\vec{A}$, $\vec{B}$, and $\vec{C}$ are differentiable vector functions of a scalar variable t, and ϕ is a differentiable scalar function of t, then

(i) $\frac{d}{dt}(\vec{A} \pm \vec{B}) = \frac{d\vec{A}}{dt} \pm \frac{d\vec{B}}{dt}$

(ii) $\frac{d}{dt}(\vec{A} \cdot \vec{B}) = \vec{A} \cdot \frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt} \cdot \vec{B}$

(iii) $\frac{d}{dt}(\vec{A} \times \vec{B}) = \vec{A} \times \frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt} \times \vec{B}$

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(iv) $$\frac{d}{dt}(\phi\vec{A}) = \phi\frac{d\vec{A}}{dt} + \frac{d\phi}{dt}\vec{A}$$

(v) $$\frac{d}{dt}[\vec{A}.(\vec{B}\times\vec{C})] = \vec{A}.\left(\vec{B}\times\frac{d\vec{C}}{dt}\right) + \vec{A}.\left(\frac{d\vec{B}}{dt}\times\vec{C}\right) + \frac{d\vec{A}}{dt}.(\vec{B}\times\vec{C})$$

(vi) $$\frac{d}{dt}[\vec{A}\times(\vec{B}\times\vec{C})] = \vec{A}\times\left(\vec{B}\times\frac{d\vec{C}}{dt}\right) + \vec{A}\times\left(\frac{d\vec{B}}{dt}\times\vec{C}\right) + \frac{d\vec{A}}{dt}\times(\vec{B}\times\vec{C})$$

PROOF: (i) $$\frac{d}{dt}(\vec{A}+\vec{B}) = \underset{\Delta t\to 0}{Lt}\frac{[\vec{A}(t+\Delta t)+\vec{B}(t+\Delta t)]-[\vec{A}(t)+\vec{B}(t)]}{\Delta t}$$

$$= \underset{\Delta t\to 0}{Lt}\left[\frac{\vec{A}(t+\Delta t)-\vec{A}(t)}{\Delta t} + \frac{\vec{B}(t+\Delta t)-\vec{B}(t)}{\Delta t}\right]$$

$$= \frac{d\vec{A}}{dt} + \frac{d\vec{B}}{dt}$$

(ii) $$\frac{d}{dt}(\vec{A}.\vec{B}) = \underset{\Delta t\to 0}{Lt}\frac{[\vec{A}(t+\Delta t).\vec{B}(t+\Delta t)]-[\vec{A}(t).\vec{B}(t)]}{\Delta t}$$

$$= \underset{\Delta t\to 0}{Lt}\frac{[\vec{A}(t+\Delta t).\vec{B}(t+\Delta t)-\vec{A}(t+\Delta t).\vec{B}(t)]}{\Delta t}$$

$$+\underset{\Delta t\to 0}{Lt}\frac{[\vec{A}(t+\Delta t).\vec{B}(t)-\vec{A}(t).\vec{B}(t)]}{\Delta t}$$

$$= \underset{\Delta t\to 0}{Lt}\ \vec{A}(t+\Delta t).\left[\frac{\vec{B}(t+\Delta t)-\vec{B}(t)}{\Delta t}\right]$$

$$+\underset{\Delta t\to 0}{Lt}\left[\frac{\vec{A}(t+\Delta t)-\vec{A}(t)}{\Delta t}\right].\vec{B}(t)$$

$$= \vec{A}.\frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt}.\vec{B}$$

ANOTHER METHOD

Let $\vec{A} = A_1\hat{i}+A_2\hat{j}+A_3\hat{k}$ and $\vec{B} = B_1\hat{i}+B_2\hat{j}+B_3\hat{k}$. Then

$$\frac{d}{dt}(\vec{A}.\vec{B}) = \frac{d}{dt}(A_1B_1+A_2B_2+A_3B_3)$$

$$= \left(A_1\frac{dB_1}{dt}+A_2\frac{dB_2}{dt}+A_3\frac{dB_3}{dt}\right)+\left(\frac{dA_1}{dt}B_1+\frac{dA_2}{dt}B_2+\frac{dA_3}{dt}B_3\right)$$

$$= \vec{A}.\frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt}.\vec{B}$$

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(iii) $$\frac{d}{dt}(\vec{A}\times\vec{B}) = \underset{\Delta t\to 0}{Lt}\frac{[\vec{A}(t+\Delta t)\times\vec{B}(t+\Delta t)]-[\vec{A}(t)\times\vec{B}(t)]}{\Delta t}$$

$$= \underset{\Delta t\to 0}{Lt}\frac{[\vec{A}(t+\Delta t)\times\vec{B}(t+\Delta t)-\vec{A}(t+\Delta t)\times\vec{B}(t)]}{\Delta t}$$

$$+\underset{\Delta t\to 0}{Lt}\frac{[\vec{A}(t+\Delta t)\times\vec{B}(t)-\vec{A}(t)\times\vec{B}(t)]}{\Delta t}$$

$$= \underset{\Delta t\to 0}{Lt}\ \vec{A}(t+\Delta t)\times\left[\frac{\vec{B}(t+\Delta t)-\vec{B}(t)}{\Delta t}\right]$$

$$+\underset{\Delta t\to 0}{Lt}\left[\frac{\vec{A}(t+\Delta t)-\vec{A}(t)}{\Delta t}\right]\times\vec{B}(t)$$

$$= \vec{A}\times\frac{d\vec{B}}{dt}+\frac{d\vec{A}}{dt}\times\vec{B}$$

(iv) $$\frac{d}{dt}(\phi\vec{A}) = \underset{\Delta t\to 0}{Lt}\left[\frac{\phi(t+\Delta t)\vec{A}(t+\Delta t)-\phi(t)\vec{A}(t)}{\Delta t}\right]$$

$$= \underset{\Delta t\to 0}{Lt}\left[\frac{\phi(t+\Delta t)\vec{A}(t+\Delta t)-\phi(t+\Delta t)\vec{A}(t)}{\Delta t}\right]$$

$$+\underset{\Delta t\to 0}{Lt}\left[\frac{\phi(t+\Delta t)\vec{A}(t)-\phi(t)\vec{A}(t)}{\Delta t}\right]$$

$$= \underset{\Delta t\to 0}{Lt}\ \phi(t+\Delta t)\left[\frac{\vec{A}(t+\Delta t)-\vec{A}(t)}{\Delta t}\right]$$

$$+\underset{\Delta t\to 0}{Lt}\left[\frac{\phi(t+\Delta t)-\phi(t)}{\Delta t}\right]\vec{A}(t)$$

$$= \phi\frac{d\vec{A}}{dt}+\frac{d\phi}{dt}\vec{A}.$$

(v) $$\frac{d}{dt}[\vec{A}.(\vec{B}\times\vec{C})] = \vec{A}.\frac{d}{dt}(\vec{B}\times\vec{C})+\frac{d\vec{A}}{dt}.(\vec{B}\times\vec{C})$$

$$= \vec{A}.\left[\vec{B}\times\frac{d\vec{C}}{dt}+\frac{d\vec{B}}{dt}\times\vec{C}\right]+\frac{d\vec{A}}{dt}.(\vec{B}\times\vec{C})$$

$$= \vec{A}.\left(\vec{B}\times\frac{d\vec{C}}{dt}\right)+\vec{A}.\left(\frac{d\vec{B}}{dt}\times\vec{C}\right)+\frac{d\vec{A}}{dt}.(\vec{B}\times\vec{C})$$

(vi) $$\frac{d}{dt}[\vec{A}\times(\vec{B}\times\vec{C})] = \vec{A}\times\frac{d}{dt}(\vec{B}\times\vec{C})+\frac{d\vec{A}}{dt}\times(\vec{B}\times\vec{C})$$

$$= \vec{A}\times\left(\vec{B}\times\frac{d\vec{C}}{dt}+\frac{d\vec{B}}{dt}\times\vec{C}\right)+\frac{d\vec{A}}{dt}\times(\vec{B}\times\vec{C})$$

$$= \vec{A}\times\left(\vec{B}\times\frac{d\vec{C}}{dt}\right)+\vec{A}\times\left(\frac{d\vec{B}}{dt}\times\vec{C}\right)+\frac{d\vec{A}}{dt}\times(\vec{B}\times\vec{C})$$

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3.9 SPACE CURVES

If the coordinates (x, y, z) of a point P in space are described by

$$x = x(t), \quad y = y(t), \quad z = z(t) \qquad (1)$$

where x, y, z are continuous functions of a scalar variable t, then as t varies over an interval on the t-axis, the point P moves in space tracing out a curve C called the **space curve**. It follows that equations (1) are the parametric equations of a curve C in space. The scalar variable t is called the **parameter** which may be time or something else.

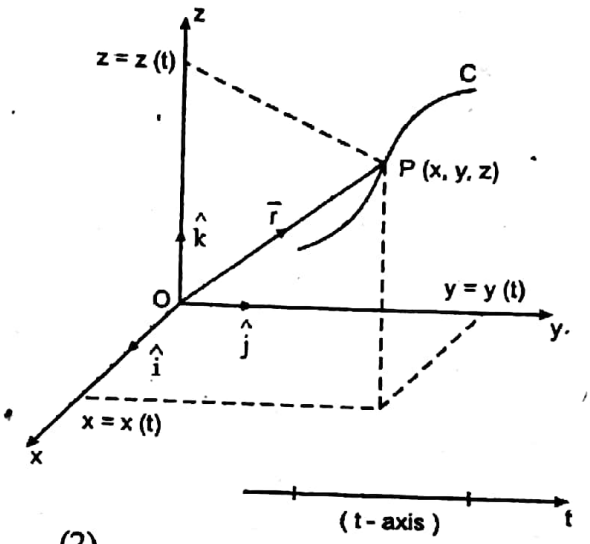

Figure (3.2)

Let $\vec{r}$ be the position vector of a point P on C and if $\hat{i}$, $\hat{j}$, and $\hat{k}$ are the unit vectors, then

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} \qquad (2)$$

is a vector function of a scalar variable t represented by $\vec{r} = \vec{r}(t)$. Equation (2) is called the vector equation for the space curve C. This equation is a mapping of the interval located on the t – axis, onto the curve C in space. It maps every t in that interval onto the point of C with position vector $\vec{r}(t)$.

EXAMPLE (6): **Determine the space curve C represented by the vector function**

$$\vec{r}(t) = (a\cos t)\hat{i} + (a\sin t)\hat{j}, \quad 0 \le t \le 2\pi.$$

SOLUTION: The parametric equations of the space curve C represented by $\vec{r}(t)$ are

$$x = a\cos t, \quad y = a\sin t, \quad z = 0.$$

Since $z = 0$, we know that curve is in the xy–plane. Now

$$x^2 + y^2 = a^2(\cos^2 t + \sin^2 t) = a^2$$

At $t = 0$, $x = a$, $y = 0$, $z = 0$

and at $t = 2\pi$, $x = a$, $y = 0$, $z = 0$.

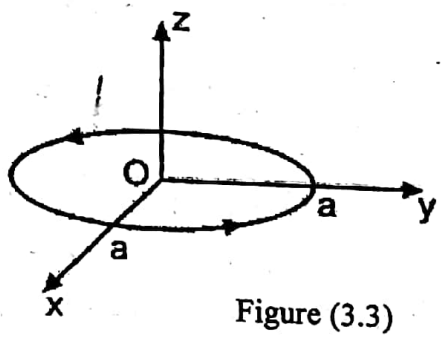

Figure (3.3)

Thus the curve C is a circle in the xy–plane having center at the origin and radius a. It begins and terminates at the point $(a, 0, 0)$ as t changes from 0 to 2π as shown in figure (3.3).

THEOREM (3.4) **A space curve C is defined by the parametric equations**

$$x = x(s), \quad y = y(s), \quad z = z(s),$$

where s is the arc length of C measured from a fixed point on C. If $\vec{r}$ is the position vector of any point P on C, show that $\frac{d\vec{r}}{ds} = \hat{T}$ is a unit tangent vector to C at P.

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PROOF: The position vector of any point P on C is

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$= x(s)\hat{i} + y(s)\hat{j} + z(s)\hat{k}$$

Then $\dfrac{d\vec{r}}{ds} = \dfrac{dx}{ds}\hat{i} + \dfrac{dy}{ds}\hat{j} + \dfrac{dz}{ds}\hat{k}$

is a vector tangent to the curve C at P. To show that it has unit magnitude, we note that

$$\left|\frac{d\vec{r}}{ds}\right| = \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2}$$
$$= \sqrt{\frac{(dx)^2 + (dy)^2 + (dz)^2}{(ds)^2}} = 1$$

since $(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$ from calculus.

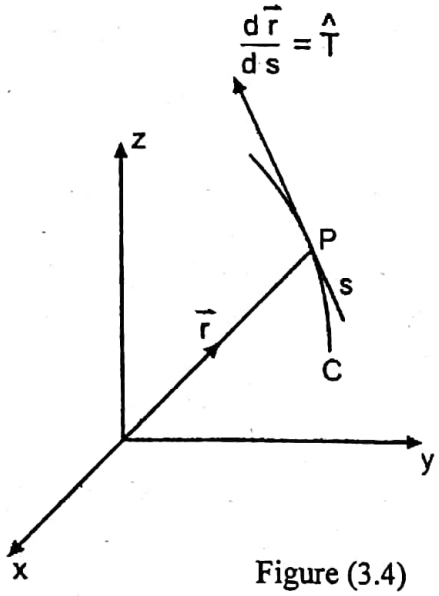

Figure (3.4)

NOTE: Velocity $\vec{V} = \dfrac{d\vec{r}}{dt} = \dfrac{d\vec{r}}{ds}\dfrac{ds}{dt} = \dfrac{ds}{dt}\hat{T}$ or $\vec{V} = V\hat{T}$ where $V = \dfrac{ds}{dt}$

Thus we see that the magnitude of $\vec{V}$, often called the speed is $V = \dfrac{ds}{dt}$ or $\left|\dfrac{d\vec{r}}{dt}\right| = \dfrac{ds}{dt}$.

EXAMPLE (7): **Find the unit tangent vector $\hat{T}$ to the curve represented by**

$$\vec{r}(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k} \text{ at } t = 1.$$

SOLUTION: Since $\vec{r}(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$

therefore $\vec{V} = \dfrac{d\vec{r}}{dt} = \hat{i} + 2t\hat{j} + 3t^2\hat{k}$

and $V = |\vec{V}| = \sqrt{(1)^2 + (2t)^2 + (3t^2)^2} = \sqrt{1 + 4t^2 + 9t^4}$

Then the unit tangent vector $\hat{T}$ is given by $\hat{T} = \dfrac{\vec{V}}{V} = \dfrac{\hat{i} + 2t\hat{j} + 3t^2\hat{k}}{\sqrt{1 + 4t^2 + 9t^4}}$

At $t = 1$, the unit tangent vector is $\hat{T} = \dfrac{\hat{i} + 2\hat{j} + 3\hat{k}}{\sqrt{1 + 4 + 9}} = \dfrac{\hat{i} + 2\hat{j} + 3\hat{k}}{\sqrt{14}}$

THEOREM (3.5): **Prove that the vector defined by $\dfrac{d\hat{T}}{ds}$ is normal to the unit tangent vector $\hat{T}$ at any point on the curve C.**

PROOF: Since $\hat{T}$ is the unit tangent vector, therefore $\hat{T}\cdot\hat{T} = 1$

Differentiating both the sides w.r.t. s, we get $\dfrac{d}{ds}(\hat{T}\cdot\hat{T}) = 0$

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or $\quad \dfrac{d\hat{T}}{ds}\cdot\hat{T}+\hat{T}\cdot\dfrac{d\hat{T}}{ds}=0$

or $\quad 2\left(\dfrac{d\hat{T}}{ds}\cdot\hat{T}\right)=0 \quad$ or $\quad \dfrac{d\hat{T}}{ds}\cdot\hat{T}=0$

which implies that $\dfrac{d\hat{T}}{ds}$ is normal to $\hat{T}$.

THEOREM (3.6): Prove that the necessary and sufficient condition for a vector function $\vec{A}(t)$ to be constant is $\dfrac{d\vec{A}}{dt}=\vec{0}$.

PROOF: We know that a vector is said to be constant if both its magnitude and direction do not change. Let $\vec{A}(t)$ be a constant vector function of a scalar variable t. Then

$$\vec{A}(t+\Delta t)=\vec{A}(t) \quad \text{or} \quad \vec{A}(t+\Delta t)-\vec{A}(t)=\vec{0}$$

Thus $\quad \dfrac{d\vec{A}}{dt}=\underset{\Delta t\to 0}{\mathrm{Lt}}\dfrac{\vec{A}(t+\Delta t)-\vec{A}(t)}{\Delta t}$

$$=\underset{\Delta t\to 0}{\mathrm{Lt}}\frac{\vec{0}}{\Delta t}=\vec{0} \quad \text{which is the necessary condition.}$$

Conversely, let $\dfrac{d\vec{A}}{dt}=\vec{0}$

If $\quad \vec{A}(t)=A_1(t)\hat{i}+A_2(t)\hat{j}+A_3(t)\hat{k}$

then $\quad \dfrac{d\vec{A}}{dt}=\dfrac{dA_1}{dt}\hat{i}+\dfrac{dA_2}{dt}\hat{j}+\dfrac{dA_3}{dt}\hat{k}$

or $\quad \vec{0}=\dfrac{dA_1}{dt}\hat{i}+\dfrac{dA_2}{dt}\hat{j}+\dfrac{dA_3}{dt}\hat{k}$

Equating the coefficients of $\hat{i}$, $\hat{j}$, and $\hat{k}$, we get

$$\frac{dA_1}{dt}=0,\quad \frac{dA_2}{dt}=0,\quad \frac{dA_3}{dt}=0.$$

These equations imply that A_1, A_2, and A_3 are independent of t, i.e. they are constant scalars.

Hence $\vec{A}(t)$ is a constant vector function, which is the sufficient condition. Hence the theorem.

THEOREM (3.7): Prove that the necessary and sufficient condition for a vector $\vec{A}$ to have a constant magnitude is $\vec{A}\cdot\dfrac{d\vec{A}}{dt}=0$.

PROOF: Let $\vec{A}=\vec{A}(t)$ be a vector function of a scalar variable t.

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Let the magnitude of $\vec{A}$ be constant i.e. $A = |\vec{A}| = \text{constant}$, then $\vec{A}\cdot\vec{A} = A^2 = \text{constant}$

and $\frac{d}{dt}(\vec{A}\cdot\vec{A}) = 0$ or $\vec{A}\cdot\frac{d\vec{A}}{dt} + \frac{d\vec{A}}{dt}\cdot\vec{A} = 0$

or $2\left(\vec{A}\cdot\frac{d\vec{A}}{dt}\right) = 0$ or $\vec{A}\cdot\frac{d\vec{A}}{dt} = 0$. Thus the condition is necessary.

Conversely, suppose that $\vec{A}\cdot\frac{d\vec{A}}{dt} = 0$, then $\vec{A}\cdot\frac{d\vec{A}}{dt} + \frac{d\vec{A}}{dt}\cdot\vec{A} = 0$

or $\frac{d}{dt}(\vec{A}\cdot\vec{A}) = 0$ or $\vec{A}\cdot\vec{A} = \text{constant}$

or $A^2 = \text{constant}$ or $A = |\vec{A}| = \text{constant}$. Thus the condition is sufficient.

EXAMPLE (8): **Show that $\vec{A}(t) = \sin t\,\hat{i} + \cos t\,\hat{j} + \sqrt{3}\,\hat{k}$ has constant magnitude and is orthogonal to its derivative.**

SOLUTION: We have $\vec{A}(t) = \sin t\,\hat{i} + \cos t\,\hat{j} + \sqrt{3}\,\hat{k}$

Therefore, $|\vec{A}(t)| = \sqrt{\sin^2 t + \cos^2 t + 3} = \sqrt{1+3} = 2$ and $\frac{d\vec{A}}{dt} = \cos t\,\hat{i} - \sin t\,\hat{j}$

so that $\vec{A}\cdot\frac{d\vec{A}}{dt} = \sin t\cos t - \sin t\cos t = 0$

Thus $\vec{A}(t)$ has constant magnitude and is orthogonal to $\frac{d\vec{A}}{dt}$.

THEOREM (3.8): **Prove that the necessary and sufficient condition for a vector $\vec{A}$ to have a constant direction is $\vec{A}\times\frac{d\vec{A}}{dt} = \vec{0}$**

PROOF: Let $\vec{A} = \vec{A}(t)$ be a vector function of a scalar variable t. Let $\hat{a}$ be the unit vector in the direction of $\vec{A}$. Then

$$\vec{A}\times\frac{d\vec{A}}{dt} = (A\hat{a})\times\frac{d}{dt}(A\hat{a}) = (A\hat{a})\times\left(A\frac{d\hat{a}}{dt} + \frac{dA}{dt}\hat{a}\right)$$

$$= A^2\left(\hat{a}\times\frac{d\hat{a}}{dt}\right) + A\frac{dA}{dt}(\hat{a}\times\hat{a})$$

$$= A^2\left(\hat{a}\times\frac{d\hat{a}}{dt}\right) + \vec{0} \quad (\text{since } \hat{a}\times\hat{a} = \vec{0})$$

$$= A^2\left(\hat{a}\times\frac{d\hat{a}}{dt}\right) \qquad (1)$$

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Let the vector $\vec{A}$ have a constant direction, so that $\frac{d\hat{a}}{dt} = \vec{0}$. Thus from equation (1)

$$\vec{A} \times \frac{d\vec{A}}{dt} = \vec{0}$$. Therefore, the condition is necessary.

Conversely, let $\vec{A} \times \frac{d\vec{A}}{dt} = \vec{0}$ or $(A\hat{a}) \times \frac{d}{dt}(A\hat{a}) = \vec{0}$ or $A^2\left(\hat{a} \times \frac{d\hat{a}}{dt}\right) = \vec{0}$

or $$\hat{a} \times \frac{d\hat{a}}{dt} = \vec{0} \qquad (2)$$

Since $\hat{a}$ is of constant magnitude, therefore

$$\hat{a} \cdot \frac{d\hat{a}}{dt} = 0 \qquad (3)$$

Equations (2) and (3) hold simultaneously only if $\frac{d\hat{a}}{dt} = \vec{0}$ or $\hat{a}$ = constant.

Consequently, the direction of $\vec{A}$ is constant. Hence the condition is sufficient.

3.10 VECTOR FUNCTIONS OF MORE THAN ONE VARIABLE

If u and v are two scalar variables, and if to each pair (u, v) in some region R, there corresponds a unique vector $\vec{A}$, then $\vec{A}$ is called a vector function of u and v denoted by $\vec{A}(u, v)$ and we write $\vec{A} = \vec{A}(u, v)$. In a rectangular coordinate system this becomes

$$\vec{A}(u, v) = A_1(u, v)\hat{i} + A_2(u, v)\hat{j} + A_3(u, v)\hat{k}$$

where A_1, A_2, and A_3 are scalar functions of u, v, and are called the **components** of $\vec{A}(u, v)$.

Similarly, $\vec{A} = \vec{A}(u, v, w)$ is a vector function of three variables u, v, and w. In a rectangular coordinate system $\vec{A}(u, v, w) = A_1(u, v, w)\hat{i} + A_2(u, v, w)\hat{j} + A_3(u, v, w)\hat{k}$

where A_1, A_2, and A_3 are scalar functions of u, v, and w and are called the **components** of

$$\vec{A}(u, v, w)$$

3.11 PARTIAL DERIVATIVES OF A VECTOR FUNCTION

Let $\vec{A} = \vec{A}(u, v, w)$ be a vector function depending on three scalar variables u, v and w. The partial derivative of $\vec{A}$ w.r.t. u is defined as

$$\frac{\partial \vec{A}}{\partial u} = \underset{\Delta u \to 0}{\mathrm{Lt}} \frac{\vec{A}(u + \Delta u, v, w) - \vec{A}(u, v, w)}{\Delta u}$$

provided the limit exists. Similarly, the partial derivatives of $\vec{A}$ w.r.t. v and w are defined respectively

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by $$\frac{\partial \vec{A}}{\partial v} = \underset{\Delta v \to 0}{\text{Lt}} \frac{\vec{A}(u, v+\Delta v, w) - \vec{A}(u, v, w)}{\Delta v}$$

and $$\frac{\partial \vec{A}}{\partial w} = \underset{\Delta w \to 0}{\text{Lt}} \frac{\vec{A}(u, v, w+\Delta w) - \vec{A}(u, v, w)}{\Delta w}$$, provided that these limits exist .

For functions of two or more variables we use the term **differentiable** to mean that the function has continuous first partial derivatives .

Higher – order partial derivatives can also be defined in the same way . For example ,

$$\frac{\partial^2 \vec{A}}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial \vec{A}}{\partial u}\right), \quad \frac{\partial^2 \vec{A}}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial \vec{A}}{\partial v}\right), \quad \frac{\partial^2 \vec{A}}{\partial w^2} = \frac{\partial}{\partial w}\left(\frac{\partial \vec{A}}{\partial w}\right)$$

$$\frac{\partial^2 \vec{A}}{\partial u \partial v} = \frac{\partial}{\partial u}\left(\frac{\partial \vec{A}}{\partial v}\right), \quad \frac{\partial^2 \vec{A}}{\partial v \partial u} = \frac{\partial}{\partial v}\left(\frac{\partial \vec{A}}{\partial u}\right), \quad \frac{\partial^3 \vec{A}}{\partial u \partial w^2} = \frac{\partial}{\partial u}\left(\frac{\partial^2 \vec{A}}{\partial w^2}\right)$$

If $\vec{A}$ has continuous partial derivatives of the second order at least , then $\frac{\partial^2 \vec{A}}{\partial u \partial v} = \frac{\partial^2 \vec{A}}{\partial v \partial u}$

i.e. the order of differentiation does not matter .

THEOREM (3.9): **If $\vec{A}(u, v, w) = A_1(u, v, w)\hat{i} + A_2(u, v, w)\hat{j} + A_3(u, v, w)\hat{k}$, where A_1, A_2, and A_3 are differentiable functions of three scalar variables u, v, and w, then prove that** $\frac{\partial \vec{A}}{\partial u} = \frac{\partial A_1}{\partial u}\hat{i} + \frac{\partial A_2}{\partial u}\hat{j} + \frac{\partial A_3}{\partial u}\hat{k}$

PROOF: By definition

$$\frac{\partial \vec{A}}{\partial u} = \underset{\Delta u \to 0}{\text{Lt}} \frac{\vec{A}(u+\Delta u, v, w) - \vec{A}(u, v, w)}{\Delta u}$$

$$= \underset{\Delta u \to 0}{\text{Lt}} \frac{\left[A_1(u+\Delta u,v,w)\hat{i}+A_2(u+\Delta u,v,w)\hat{j}+A_3(u+\Delta u,v,w)\hat{k}\right] - \left[A_1(u,v,w)\hat{i}+A_2(u,v,w)\hat{j}+A_3(u,v,w)\hat{k}\right]}{\Delta u}$$

$$= \underset{\Delta u \to 0}{\text{Lt}} \left[\frac{A_1(u+\Delta u, v, w) - A_1(u, v, w)}{\Delta u}\hat{i} + \frac{A_2(u+\Delta u, v, w) - A_2(u, v, w)}{\Delta u}\hat{j} + \frac{A_3(u+\Delta u, v, w) - A_3(u, v, w)}{\Delta u}\hat{k}\right]$$

$$= \frac{\partial A_1}{\partial u}\hat{i} + \frac{\partial A_2}{\partial u}\hat{j} + \frac{\partial A_3}{\partial u}\hat{k}$$

RULES FOR PARTIAL DIFFERENTIATION OF VECTOR FUNCTIONS

Rules for partial differentiation of vector functions are similar to those used in elementary calculus. Thus if $\vec{A}$ and $\vec{B}$ are differentiable vector functions of u , v , and w , and ϕ is a differentiable scalar function of u , v , and w , then (using u as an example)

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(i) $\frac{\partial}{\partial u}(\vec{A} \pm \vec{B}) = \frac{\partial \vec{A}}{\partial u} \pm \frac{\partial \vec{B}}{\partial u}$

(ii) $\frac{\partial}{\partial u}(\vec{A} \cdot \vec{B}) = \vec{A} \cdot \frac{\partial \vec{B}}{\partial u} + \frac{\partial \vec{A}}{\partial u} \cdot \vec{B}$

(iii) $\frac{\partial}{\partial u}(\vec{A} \times \vec{B}) = \vec{A} \times \frac{\partial \vec{B}}{\partial u} + \frac{\partial \vec{A}}{\partial u} \times \vec{B}$

(iv) $\frac{\partial}{\partial u}(\phi \vec{A}) = \phi \frac{\partial \vec{A}}{\partial u} + \frac{\partial \phi}{\partial u} \vec{A}$

3.12 SURFACES IN SPACE

We know that the vector equation $\vec{r} = \vec{r}(t)$ represents a space curve, where the parameter t represents any value on the t – axis. The parametric representation of the space curve is

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

We now show that if u and v are the parameters, then the vector equation $\vec{r} = \vec{r}(u, v)$ represents a surface in space. If u has a fixed value say u_0, then $\vec{r} = \vec{r}(u_0, v)$ becomes a one–parameter equation describing a space curve along which v varies. This curve is denoted by the equation $u = u_0$. Similarly,

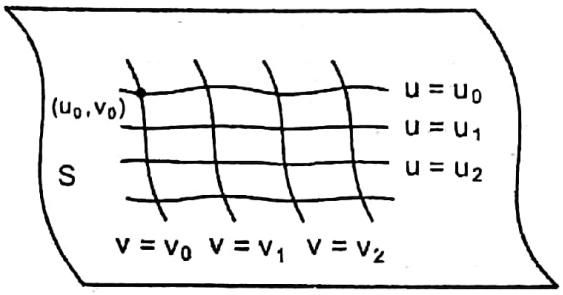

Figure (3.5)

$u = u_1$ defines another curve $\vec{r} = \vec{r}(u_1, v)$. As u varies, the equation $\vec{r} = \vec{r}(u, v)$ represents a curve which moves in space and generates a surfaces S. Then $\vec{r} = \vec{r}(u, v)$ represents a surface S thus generated as shown in figure (3.5). The curves $u = u_0$, $u = u_1$, ... represent definite curves on the surface. Similarly, $v = v_0$, $v = v_1$, ... represent curves on the surface.

By assigning definite values to u and v, we obtain a point on the surface. Thus curves $u = u_0$ and $v = v_0$, for example, intersect and define the point (u_0, v_0) on the surface. The pair of numbers (u, v) are called the **curvilinear** coordinates of the points on the surface, and the u – curves and v – curves are called parametric curves. If all the curves u = constant and v = constant are perpendicular at each point of intersection, we call the curvilinear coordinate system orthogonal.

The position vector $\vec{r}$ of any point on the surface S can then be written as

$$\vec{r} = \vec{r}(u, v) = x(u, v)\hat{i} + y(u, v)\hat{j} + z(u, v)\hat{k}; \quad (u, v) \text{ in } R. \qquad (1)$$

where R is some region in the uv – plane.

The **parametric** equations of this surface S are

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v)$$

Equation (1) is a mapping of the region R in the uv – plane, onto the surface S in space. It maps every point (u, v) in R onto the point of S with position vector $\vec{r}(u, v)$ as shown in figure (3.6).

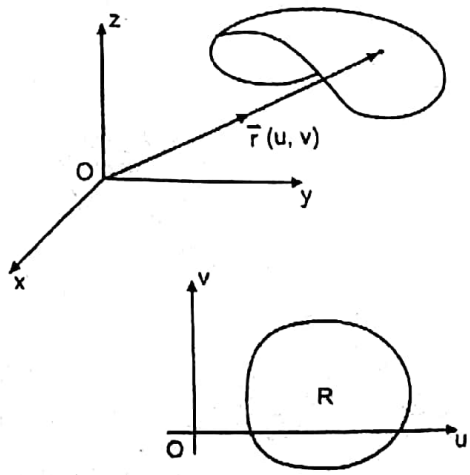

Figure (3.6)

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THEOREM (3.9): If $\vec{r} = \vec{r}(u, v)$ represents a surface S in space, then show that the unit normal vector $\hat{n}$ to the surface S at any point is given by $\hat{n} = \dfrac{\dfrac{\partial \vec{r}}{\partial u} \times \dfrac{\partial \vec{r}}{\partial v}}{\left| \dfrac{\partial \vec{r}}{\partial u} \times \dfrac{\partial \vec{r}}{\partial v} \right|}$.

PROOF: Let $P(u_0, v_0)$ be a point on the surface S. Then $\dfrac{\partial \vec{r}}{\partial u}$ at P is obtained by differentiating $\vec{r}$ w.r.t. u, keeping v = constant $= v_0$.

We know that $\dfrac{\partial \vec{r}}{\partial u}$ at P represents a vector tangent to the curve $v = v_0$ at P as shown in figure (3.7).

Similarly, $\dfrac{\partial \vec{r}}{\partial v}$ at P represents a vector tangent to the curve u = constant $= u_0$.

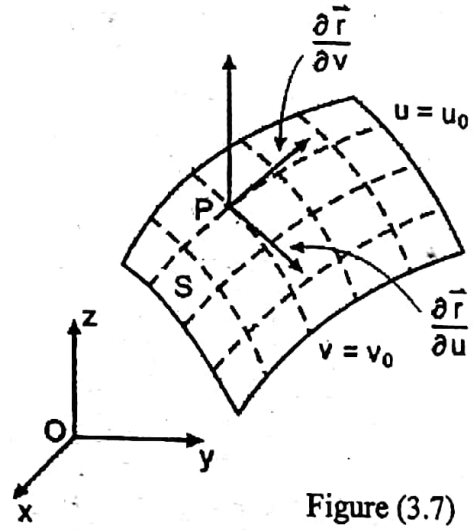

Figure (3.7)

Since $\dfrac{\partial \vec{r}}{\partial u}$ and $\dfrac{\partial \vec{r}}{\partial v}$ represent vectors at P tangent to the curves which lie on the surface S at P, it follows that these vectors are tangent to the surface at P. Hence it follows that $\dfrac{\partial \vec{r}}{\partial u} \times \dfrac{\partial \vec{r}}{\partial v}$ is a vector normal to the surface S at P. Then since $\left| \dfrac{\partial \vec{r}}{\partial u} \times \dfrac{\partial \vec{r}}{\partial v} \right|$ is the magnitude of this vector,

$$\hat{n} = \frac{\dfrac{\partial \vec{r}}{\partial u} \times \dfrac{\partial \vec{r}}{\partial v}}{\left| \dfrac{\partial \vec{r}}{\partial u} \times \dfrac{\partial \vec{r}}{\partial v} \right|}$$

is a unit normal vector to the surface S at the point P.

EXAMPLE (9): **Find a unit normal vector $\hat{n}$ to the surface defined by**

$$\vec{r}(u, v) = u \cos v\, \hat{i} + u \sin v\, \hat{j} + u\, \hat{k}.$$

SOLUTION: Since $\vec{r} = \vec{r}(u, v) = u \cos v\, \hat{i} + u \sin v\, \hat{j} + u\, \hat{k}$, therefore

$$\frac{\partial \vec{r}}{\partial u} = \cos v\, \hat{i} + \sin v\, \hat{j} + \hat{k} \quad \text{and} \quad \frac{\partial \vec{r}}{\partial v} = -u \sin v\, \hat{i} + u \cos v\, \hat{j}$$

Then $$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & 1 \\ -u \sin v & u \cos v & 0 \end{vmatrix}$$

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$$= -u\cos v\,\hat{i} + u\sin v\,\hat{j} + u(\cos^2 v + \sin^2 v)\hat{k}$$

$$= -u\cos v\,\hat{i} + u\sin v\,\hat{j} + u\hat{k}$$

represents a normal vector to the surface at any point (u, v).

Now $$\left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| = \sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2}$$

$$= \sqrt{u^2(\cos^2 v + \sin^2 v) + u^2} = \sqrt{2u^2} = \sqrt{2}\,u$$

Thus a unit normal vector $\hat{n}$ is given by

$$\hat{n} = \frac{\dfrac{\partial \vec{r}}{\partial u} \times \dfrac{\partial \vec{r}}{\partial v}}{\left|\dfrac{\partial \vec{r}}{\partial u} \times \dfrac{\partial \vec{r}}{\partial v}\right|} = \frac{-u\cos v\,\hat{i} + u\sin v\,\hat{j} + u\hat{k}}{\sqrt{2}\,u} = \frac{1}{\sqrt{2}}(-\cos v\,\hat{i} + \sin v\,\hat{j} + \hat{k})$$

3.13 VECTOR INTEGRATION

If $\vec{f}(t)$ is a given continuous vector function of a scalar variable t, then the process of finding another vector function $\vec{F}(t)$, whose derivative w.r.t. 't' is equal to $\vec{f}(t)$ $\left[\text{i.e. } \frac{d}{dt}\vec{F}(t) = \vec{f}(t)\right]$ is called **integration**. Thus integration is defined as the inverse process to differentiation.

The vector function $\vec{F}(t)$ so determined is called the (indefinite) integral of $\vec{f}(t)$ w.r.t. t and is written $\int \vec{f}(t)\,dt = \vec{F}(t)$

Since the derivative of a constant vector is zero, the vector function $\vec{F}(t) + \vec{C}$, where $\vec{C}$ is an arbitrary constant vector is also an integral of $\vec{f}(t)$. It therefore follows that

$$\int \vec{f}(t)\,dt = \vec{F}(t) + \vec{C}$$

The arbitrary constant vector $\vec{C}$ is called the constant of integration.

If further $\vec{f}(t)$ is given in its component form i.e. $\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$,

then $$\int \vec{f}(t)\,dt = \int [f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}]\,dt$$

$$= \left[\int f_1(t)\,dt + C_1\right]\hat{i} + \left[\int f_2(t)\,dt + C_2\right]\hat{j} + \left[\int f_3(t)\,dt + C_3\right]\hat{k}$$

$$= \hat{i}\int f_1(t)\,dt + \hat{j}\int f_2(t)\,dt + \hat{k}\int f_3(t)\,dt + \vec{C}$$

where $\vec{C} = C_1\hat{i} + C_2\hat{j} + C_3\hat{k}$.

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The definite integral of $\vec{f}(t)$ between the limits $t = a$ and $t = b$ can be written as

$$\int_a^b \vec{f}(t)\,dt = \left|\vec{F}(t)+\vec{C}\right|_a^b = \vec{F}(b)-\vec{F}(a)$$

EXAMPLE (10): If $\vec{f}(t) = (t-t^2)\hat{i}+2t^3\hat{j}-3\hat{k}$, find

(i) $\int \vec{f}(t)\,dt$ (ii) $\int_1^2 \vec{f}(t)\,dt$

SOLUTION: (i) $\int \vec{f}(t)dt = \int\left[(t-t^2)\hat{i}+2t^3\hat{j}-3\hat{k}\right]dt$

$$= \left(\frac{t^2}{2}-\frac{t^3}{3}\right)\hat{i}+\frac{t^4}{2}\hat{j}-3t\hat{k}+\vec{C}$$

(ii) From part (a) we have

$$\int_1^2 \vec{f}(t)dt = \left|\left(\frac{t^2}{2}-\frac{t^3}{3}\right)\hat{i}+\frac{t^4}{2}\hat{j}-3t\hat{k}\right|_1^2$$

$$= \left[\left(2-\frac{8}{3}\right)\hat{i}+8\hat{j}-6\hat{k}\right]-\left[\left(\frac{1}{2}-\frac{1}{3}\right)\hat{i}+\frac{1}{2}\hat{j}-3\hat{k}\right]$$

$$= -\frac{5}{6}\hat{i}+\frac{15}{2}\hat{j}-3\hat{k}$$

PROPERTIES OF INTEGRABLE VECTOR FUNCTIONS

THEOREM (3.10): **If $\vec{f}(t)$ is a vector function of a scalar variable t, and $\vec{A}$ is a constant vector, then**

(i) $\int K\vec{f}(t)\,dt = K\int \vec{f}(t)dt$ **(K any scalar constant)**

(ii) $\int \vec{A}\cdot\vec{f}(t)\,dt = \vec{A}\cdot\int \vec{f}(t)dt+\vec{C}$

(iii) $\int \vec{A}\times\vec{f}(t)\,dt = \vec{A}\times\int \vec{f}(t)dt+\vec{C}$

(iv) $\int \vec{A}\times\frac{d\vec{f}}{dt}\,dt = \vec{A}\times\vec{f}(t)+\vec{C}$

(v) $\int 2\frac{d\vec{f}}{dt}\cdot\frac{d^2\vec{f}}{dt^2}\,dt = \left(\frac{d\vec{f}}{dt}\right)^2+\vec{C}$

(vi) $\int \vec{f}\times\frac{d^2\vec{f}}{dt^2}\,dt = \vec{f}\times\frac{d\vec{f}}{dt}+\vec{C}$

where $\vec{C}$ is an arbitrary constant vector of integration.

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PROOF: (i) This result is obvious .

(ii) $$\frac{d}{dt}\left[\vec{A}\cdot\int\vec{f}(t)\,dt\right] = \vec{A}\cdot\vec{f}(t)+\frac{d\vec{A}}{dt}\cdot\int\vec{f}(t)\,dt$$

$$= \vec{A}\cdot\vec{f}(t) \quad \left(\text{since } \frac{d\vec{A}}{dt} = \vec{0}\right)$$

Integrating both sides w.r.t. t , we get

$$\int\vec{A}\cdot\vec{f}(t)\,dt = \vec{A}\cdot\int\vec{f}(t)\,dt+\vec{C}$$

(iii) $$\frac{d}{dt}\left[\vec{A}\times\int\vec{f}(t)\,dt\right] = \vec{A}\times\vec{f}(t)+\frac{d\vec{A}}{dt}\times\int\vec{f}(t)\,dt$$

$$= \vec{A}\times\vec{f}(t) \quad \left(\text{since } \frac{d\vec{A}}{dt} = \vec{0}\right)$$

Integrating both sides w.r.t. t , we get

$$\int\vec{A}\times\vec{f}(t)\,dt = \vec{A}\times\int\vec{f}(t)\,dt+\vec{C}$$

(iv) $$\frac{d}{dt}\left[\vec{A}\times\vec{f}(t)\right] = \vec{A}\times\frac{d\vec{f}}{dt}+\frac{d\vec{A}}{dt}\times\vec{f}(t)$$

$$= \vec{A}\times\frac{d\vec{f}}{dt} \quad \left(\text{since } \frac{d\vec{A}}{dt} = \vec{0}\right).$$

Integrating both sides w.r.t. t , we get

$$\int\vec{A}\times\frac{d\vec{f}}{dt}\,dt = \vec{A}\times\vec{f}(t)+\vec{C}$$

(v) $$\frac{d}{dt}\left(\frac{d\vec{f}}{dt}\right)^2 = \frac{d}{dt}\left(\frac{d\vec{f}}{dt}\cdot\frac{d\vec{f}}{dt}\right)$$

$$= \frac{d\vec{f}}{dt}\cdot\frac{d^2\vec{f}}{dt^2}+\frac{d^2\vec{f}}{dt^2}\cdot\frac{d\vec{f}}{dt} = 2\frac{d\vec{f}}{dt}\cdot\frac{d^2\vec{f}}{dt^2}$$

Integrating both sides w.r.t. t , we get

$$\int 2\frac{d\vec{f}}{dt}\cdot\frac{d^2\vec{f}}{dt^2}\,dt = \left(\frac{d\vec{f}}{dt}\right)^2+\vec{C}$$

(vi) $$\frac{d}{dt}\left(\vec{f}\times\frac{d\vec{f}}{dt}\right) = \vec{f}\times\frac{d^2\vec{f}}{dt^2}+\frac{d\vec{f}}{dt}\times\frac{d\vec{f}}{dt}$$

$$= \vec{f}\times\frac{d^2\vec{f}}{dt^2} \quad \left(\text{since } \frac{d\vec{f}}{dt}\times\frac{d\vec{f}}{dt} = \vec{0}\right)$$

Integrating both sides w.r.t. t , we get

$$\int\vec{f}\times\frac{d^2\vec{f}}{dt^2}\,dt = \vec{f}\times\frac{d\vec{f}}{dt}+\vec{C}$$

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3.14 SOLVED PROBLEMS

PROBLEM (1): For what values of a and b is the vector function

$$\vec{F}(t) = \begin{cases} \dfrac{(a+b)\sin t}{t}\hat{i} + 3\cos t\,\hat{j} + 16\,b\dfrac{\tan t}{t}\hat{k} & , t \neq 0 \\ 6\hat{i} + 3\hat{j} + 4\hat{k} & , t = 0 \end{cases}$$

continuous at $t = 0$?

SOLUTION: By definition, the vector function $\vec{F}(t)$ is continuous at $t = 0$ if

$$\underset{t\to 0}{\mathrm{Lt}}\ \vec{F}(t) = \vec{F}(0) \qquad (1)$$

Now $\underset{t\to 0}{\mathrm{Lt}}\ \vec{F}(t) = \underset{t\to 0}{\mathrm{Lt}}\left[\dfrac{(a+b)\sin t}{t}\hat{i} + 3\cos t\,\hat{j} + 16\,b\dfrac{\tan t}{t}\hat{k}\right]$

$$= (a+b)\left(\underset{t\to 0}{\mathrm{Lt}}\ \frac{\sin t}{t}\right)\hat{i} + 3\left(\underset{t\to 0}{\mathrm{Lt}}\ \cos t\right)\hat{j} + 16\,b\left(\underset{t\to 0}{\mathrm{Lt}}\ \frac{\tan t}{t}\right)\hat{k}$$

$$= (a+b)\hat{i} + 3\hat{j} + 16\,b\hat{k}$$

Also $\vec{F}(0) = 6\hat{i} + 3\hat{j} + 4\hat{k}$

Thus from equation (1), we have

$$(a+b)\hat{i} + 3\hat{j} + 16\,b\hat{k} = 6\hat{i} + 3\hat{j} + 4\hat{k}$$

This implies $a + b = 6$ and $16\,b = 4$

or $a = \dfrac{23}{4}$, $b = \dfrac{1}{4}$

PROBLEM (2): If $\vec{A} = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$ and $\vec{B} = \sin t\,\hat{i} - \cos t\,\hat{j}$, find

(i) $\dfrac{d}{dt}(\vec{A}\cdot\vec{B})$ (ii) $\dfrac{d}{dt}(\vec{A}\times\vec{B})$ (iii) $\dfrac{d}{dt}(\vec{A}\cdot\vec{A})$.

SOLUTION: (i) $\dfrac{d}{dt}(\vec{A}\cdot\vec{B}) = \vec{A}\cdot\dfrac{d\vec{B}}{dt} + \dfrac{d\vec{A}}{dt}\cdot\vec{B}$

$$= (5t^2\hat{i} + t\hat{j} - t^3\hat{k})\cdot(\cos t\,\hat{i} + \sin t\,\hat{j}) + (10t\,\hat{i} + \hat{j} - 3t^2\hat{k})\cdot(\sin t\,\hat{i} - \cos t\,\hat{j})$$

$$= 5t^2\cos t + t\sin t + 10t\sin t - \cos t = (5t^2 - 1)\cos t + 11t\sin t$$

ALTERNATIVE METHOD

We have $(\vec{A}\cdot\vec{B}) = 5t^2\sin t - t\cos t$. Then

$$\frac{d}{dt}(\vec{A}\cdot\vec{B}) = \frac{d}{dt}(5t^2\sin t - t\cos t)$$

$$= 5t^2\cos t + 10t\sin t + t\sin t - \cos t = (5t^2 - 1)\cos t + 11t\sin t$$

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(ii) $$\frac{d}{dt}(\vec{A} \times \vec{B}) = \vec{A} \times \frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt} \times \vec{B}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5t^2 & t & -t^3 \\ \cos t & \sin t & 0 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 10t & 1 & -3t^2 \\ \sin t & -\cos t & 0 \end{vmatrix}$$

$$= [t^3 \sin t\,\hat{i} - t^3 \cos t\,\hat{j} + (5t^2 \sin t - t\cos t)\hat{k}]$$

$$+ [-3t^2 \cos t\,\hat{i} - 3t^2 \sin t\,\hat{j} + (-10t\cos t - \sin t)\hat{k}]$$

$$= (t^3 \sin t - 3t^2 \cos t)\hat{i} - (t^3 \cos t + 3t^2 \sin t)\hat{j} + (5t^2 \sin t - \sin t - 11t\cos t)\hat{k}$$

ALTERNATIVE METHOD

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5t^2 & t & -t^3 \\ \sin t & -\cos t & 0 \end{vmatrix} = -t^3 \cos t\,\hat{i} - t^3 \sin t\,\hat{j} + (-5t^2 \cos t - t\sin t)\hat{k}$$

Then $\frac{d}{dt}(\vec{A} \times \vec{B}) = (t^3 \sin t - 3t^2 \cos t)\hat{i} - (t^3 \cos t + 3t^2 \sin t)\hat{j} + (5t^2 \sin t - 11t\cos t - \sin t)\hat{k}$

(iii) $$\frac{d}{dt}(\vec{A} \cdot \vec{A}) = \vec{A} \cdot \frac{d\vec{A}}{dt} + \frac{d\vec{A}}{dt} \cdot \vec{A} = 2\vec{A} \cdot \frac{d\vec{A}}{dt}$$

$$= 2(5t^2\hat{i} + t\hat{j} - t^3\hat{k}) \cdot (10t\hat{i} + \hat{j} - 3t^2\hat{k}) = 100t^3 + 2t + 6t^5$$

ALTERNATIVE METHOD

$$\vec{A} \cdot \vec{A} = (5t^2)^2 + (t)^2 + (-t^3)^2 = 25t^4 + t^2 + t^6$$

then $$\frac{d}{dt}(25t^4 + t^2 + t^6) = 100t^3 + 2t + 6t^5.$$

PROBLEM (3): **Show that** $\vec{A} \cdot \frac{d\vec{A}}{dt} = A\frac{dA}{dt}$.

SOLUTION: Since $\vec{A} \cdot \vec{A} = A^2$, we have $\frac{d}{dt}(\vec{A} \cdot \vec{A}) = \frac{d}{dt}(A^2)$

or $$2\vec{A} \cdot \frac{d\vec{A}}{dt} = 2A\frac{dA}{dt} \quad \text{or} \quad \vec{A} \cdot \frac{d\vec{A}}{dt} = A\frac{dA}{dt}$$

Note that if $\vec{A}$ is a constant vector $\vec{A} \cdot \frac{d\vec{A}}{dt} = 0$.

PROBLEM (4): **Prove that** $$\frac{d}{dt}\left(\vec{A} \times \frac{d\vec{B}}{dt} - \frac{d\vec{A}}{dt} \times \vec{B}\right) = A \times \frac{d^2\vec{B}}{dt^2} - \frac{d^2\vec{A}}{dt^2} \times \vec{B}$$

SOLUTION: $$\frac{d}{dt}\left(\vec{A} \times \frac{d\vec{B}}{dt} - \frac{d\vec{A}}{dt} \times \vec{B}\right) = \frac{d}{dt}\left(\vec{A} \times \frac{d\vec{B}}{dt}\right) - \frac{d}{dt}\left(\frac{d\vec{A}}{dt} \times \vec{B}\right)$$

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$$= \left(\vec{A} \times \frac{d^2\vec{B}}{dt^2} + \frac{d\vec{A}}{dt} \times \frac{d\vec{B}}{dt}\right) - \left(\frac{d\vec{A}}{dt} \times \frac{d\vec{B}}{dt} + \frac{d^2\vec{A}}{dt^2} \times \vec{B}\right)$$

$$= \vec{A} \times \frac{d^2\vec{B}}{dt^2} - \frac{d^2\vec{A}}{dt^2} \times \vec{B}$$

PROBLEM (5): **Show that**

(i) $$\frac{d}{dt}\left[\vec{A} \cdot \frac{d\vec{A}}{dt} \times \frac{d^2\vec{A}}{dt^2}\right] = \vec{A} \cdot \frac{d\vec{A}}{dt} \times \frac{d^3\vec{A}}{dt^3}$$

(ii) $$\frac{d^2}{dt^2}\left[\vec{A} \cdot \frac{d\vec{A}}{dt} \times \frac{d^2\vec{A}}{dt^2}\right] = \vec{A} \cdot \frac{d^2\vec{A}}{dt^2} \times \frac{d^3\vec{A}}{dt^3} + \vec{A} \cdot \frac{d\vec{A}}{dt} \times \frac{d^4\vec{A}}{dt^4}$$

SOLUTION: (i) Using the formula for differentiation of scalar triple product, we have

$$\frac{d}{dt}\left[\vec{A} \cdot \frac{d\vec{A}}{dt} \times \frac{d^2\vec{A}}{dt^2}\right] = \vec{A} \cdot \frac{d\vec{A}}{dt} \times \frac{d^3\vec{A}}{dt^3} + \vec{A} \cdot \frac{d^2\vec{A}}{dt^2} \times \frac{d^2\vec{A}}{dt^2} + \frac{d\vec{A}}{dt} \cdot \frac{d\vec{A}}{dt} \times \frac{d^2\vec{A}}{dt^2}$$

$$= \vec{A} \cdot \frac{d\vec{A}}{dt} \times \frac{d^3\vec{A}}{dt^3} + \vec{A} \cdot \frac{d^2\vec{A}}{dt^2} \times \frac{d^2\vec{A}}{dt^2} + \frac{d\vec{A}}{dt} \times \frac{d\vec{A}}{dt} \cdot \frac{d^2\vec{A}}{dt^2}$$

$$= \vec{A} \cdot \frac{d\vec{A}}{dt} \times \frac{d^3\vec{A}}{dt^3} + 0 + 0 = \vec{A} \cdot \frac{d\vec{A}}{dt} \times \frac{d^3\vec{A}}{dt^3}$$

(ii) $$\frac{d^2}{dt^2}\left[\vec{A} \cdot \frac{d\vec{A}}{dt} \times \frac{d^2\vec{A}}{dt^2}\right] = \frac{d}{dt}\left[\frac{d}{dt}\left(\vec{A} \cdot \frac{d\vec{A}}{dt} \times \frac{d^2\vec{A}}{dt^2}\right)\right]$$

$$= \frac{d}{dt}\left(\vec{A} \cdot \frac{d\vec{A}}{dt} \times \frac{d^3\vec{A}}{dt^3}\right)$$

$$= \vec{A} \cdot \frac{d\vec{A}}{dt} \times \frac{d^4\vec{A}}{dt^4} + \vec{A} \cdot \frac{d^2\vec{A}}{dt^2} \times \frac{d^3\vec{A}}{dt^3} + \frac{d\vec{A}}{dt} \cdot \frac{d\vec{A}}{dt} \times \frac{d^3\vec{A}}{dt^3}$$

$$= \vec{A} \cdot \frac{d\vec{A}}{dt} \times \frac{d^4\vec{A}}{dt^4} + \vec{A} \cdot \frac{d^2\vec{A}}{dt^2} \times \frac{d^3\vec{A}}{dt^3} + \frac{d\vec{A}}{dt} \times \frac{d\vec{A}}{dt} \cdot \frac{d^3\vec{A}}{dt^3}$$

$$= \vec{A} \cdot \frac{d^2\vec{A}}{dt^2} \times \frac{d^3\vec{A}}{dt^3} + \vec{A} \cdot \frac{d\vec{A}}{dt} \times \frac{d^4\vec{A}}{dt^4}.$$

PROBLEM (6): **Find the space curve represented by the vector function**

$$\vec{r}(t) = a\cos t\,\hat{i} + b\sin t\,\hat{j}.$$

SOLUTION: The parametric equations of the curve represented by $\vec{r}(t)$ are $x = a\cos t$, $y = b\sin t$, $z = 0$

Since $z = 0$, the curve lies in the xy–plane.

Now, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 t + \sin^2 t = 1$, $z = 0$

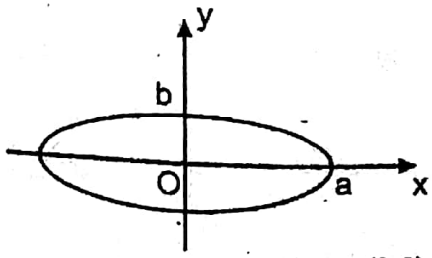

Figure (3.8)

At $t = 0$, $x = a$, $y = 0$, $z = 0$, and at $t = \pi$, $x = -a$, $y = 0$, $z = 0$.

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Thus the curve represented by $\vec{r}(t)$ is an ellipse in the xy–plane with centre at the origin and semi–major and semi–minor axes as a and b respectively. It begins and terminates at the point (a , 0 , 0) as t changes from 0 to 2π as shown in figure (3.8).

PROBLEM (7): **A particle moves along a curve whose parametric equations are $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$, where t is the time.**

(i) Determine its velocity and acceleration at any time.

(ii) Find the magnitudes of the velocity and acceleration at $t = 0$.

SOLUTION: (i) The position vector of the particle is

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = e^{-t}\hat{i} + 2\cos 3t\,\hat{j} + 2\sin 3t\,\hat{k}$$

Then the velocity is $\vec{V} = \dfrac{d\vec{r}}{dt} = -e^{-t}\hat{i} - 6\sin 3t\,\hat{j} + 6\cos 3t\,\hat{k}$

and the acceleration is $\vec{a} = \dfrac{d^2\vec{r}}{dt^2} = e^{-t}\hat{i} - 18\cos 3t\,\hat{j} - 18\sin 3t\,\hat{k}$.

(ii) At $t = 0$, $\vec{V} = -\hat{i} + 6\hat{k}$ and $\vec{a} = \hat{i} - 18\hat{j}$.

Then magnitude of velocity at $t = 0$ is $|\vec{V}| = \sqrt{(-1)^2 + (6)^2} = \sqrt{37}$

and magnitude of acceleration at $t = 0$ is $|\vec{a}| = \sqrt{(1)^2 + (-18)^2} = \sqrt{325}$

PROBLEM (8) **A particle moves along the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$, where t is the time. Find the components of its velocity $\vec{V}$ and acceleration $\vec{a}$ at time $t = 1$ in the direction $\vec{B} = \hat{i} - 3\hat{j} + 2\hat{k}$.**

SOLUTION: The position vector of the particle is $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$= 2t^2\hat{i} + (t^2 - 4t)\hat{j} + (3t - 5)\hat{k}$$

Therefore, velocity $\vec{V} = \dfrac{d\vec{r}}{dt} = 4t\,\hat{i} + (2t - 4)\hat{j} + 3\hat{k}$

At $t = 1$, $\vec{V} = 4\hat{i} - 2\hat{j} + 3\hat{k}$

The unit vector in the direction of $\vec{B}$ is $\hat{b} = \dfrac{\hat{i} - 3\hat{j} + 2\hat{k}}{\sqrt{14}}$

Then the component of velocity $\vec{V}$ at $t = 1$ in the given direction is

$$\text{Comp}_{\vec{B}}\,\vec{V} = \vec{V}\cdot\hat{b} = (4\hat{i} - 2\hat{j} + 3\hat{k})\cdot\frac{\hat{i} - 3\hat{j} + 2\hat{k}}{\sqrt{14}} = \frac{4 + 6 + 6}{\sqrt{14}} = \frac{16}{\sqrt{14}}$$

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The acceleration vector $\vec{a}$ is given by

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\left[4t\hat{i}+(2t-4)\hat{j}+3\hat{k}\right] = 4\hat{i}+2\hat{j}+0\hat{k}$$

At $t = 1$, $\vec{a} = 4\hat{i}+2\hat{j}+0\hat{k}$

Then the component of acceleration $\vec{a}$ in the given direction is

$$\text{Comp}_{\vec{B}}\,\vec{a} = \vec{a}\cdot\hat{b} = (4\hat{i}+2\hat{j}+0\hat{k})\cdot\frac{(\hat{i}-3\hat{j}+2\hat{k})}{\sqrt{14}}$$

$$= \frac{4-6+0}{\sqrt{14}} = -\frac{2}{\sqrt{14}}$$

PROBLEM (9) **(i) Find the unit tangent vector to any point on the curve**

$$x = t^2+1,\ y = 4t-3,\ z = 2t^2-6t.$$

(ii) Determine the unit tangent vector at the point where $t = 2$.

SOLUTION: (i) Here $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$

$$= (t^2+1)\hat{i}+(4t-3)\hat{j}+(2t^2-6t)\hat{k}$$

Therefore, $\vec{V} = \frac{d\vec{r}}{dt} = 2t\hat{j}+4\hat{j}+(4t-6)\hat{k}$

and $V = |\vec{V}| = \sqrt{(2t)^2+(4)^2+(4t-6)^2}$

Then the required unit tangent vector $\hat{T} = \frac{\vec{V}}{V} = \frac{2t\hat{i}+4\hat{j}+(4t-6)\hat{k}}{\sqrt{(2t)^2+(4)^2+(4t-6)^2}}$

(ii) At $t = 2$, the unit tangent vector is $\hat{T} = \frac{4\hat{i}+4\hat{j}+2\hat{k}}{\sqrt{(4)^2+(4)^2+(2)^2}}$

$$= \frac{4\hat{i}+4\hat{j}+2\hat{k}}{6} = \frac{2}{3}\hat{i}+\frac{2}{3}\hat{j}+\frac{1}{3}\hat{k}$$

PROBLEM (10): **The position vector of a particle at time t is**

$$\vec{r} = \cos(t-1)+\sin(t-1)+at^3\hat{k}$$

find the value of a such that at time $t = 1$, the acceleration is normal to the position vector.

SOLUTION: We have $\vec{r} = \cos(t-1)\hat{i}+\sin(t-1)\hat{j}+at^3\hat{k}$

$$\frac{d\vec{r}}{dt} = -\sin(t-1)\hat{i}+\cos(t-1)\hat{j}+3at^2\hat{k}$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = -\cos(t-1)\hat{i}-\sin(t-1)\hat{j}+6at\hat{k}$$

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At $t = 1$, $\vec{r} = \hat{i} + a\hat{k}$ and $\vec{a} = -\hat{i} + 6a\hat{k}$

If $\vec{a}$ and $\vec{r}$ are normal, then $\vec{a} \cdot \vec{r} = 0$ or $(-\hat{i} + 6a\hat{k}) \cdot (\hat{i} + a\hat{k}) = 0$

$$-1 + 6a^2 = 0 \quad \text{or} \quad a^2 = \frac{1}{6} \quad \text{or} \quad a = \pm\frac{1}{\sqrt{6}}$$

PROBLEM (11): **A particle moves so that its position vector is given by $\vec{r} = \cos wt\,\hat{i} + \sin wt\,\hat{j}$ where w is a constant. Show that (i) the velocity $\vec{V}$ of the particle is perpendicular to $\vec{r}$ (ii) the acceleration $\vec{a}$ is directed towards the origin and has magnitude proportional to the distance from the origin.**

(iii) $\vec{r} \times \vec{V} = w\hat{k}$, a constant vector.

SOLUTION: (i) Since $\vec{r} = \cos wt\,\hat{i} + \sin wt\,\hat{j}$, therefore

$$\vec{V} = \frac{d\vec{r}}{dt} = -w\sin wt\,\hat{i} + w\cos wt\,\hat{j}$$

Then $\vec{r} \cdot \vec{V} = (\cos wt\,\hat{i} + \sin wt\,\hat{j}) \cdot (-w\sin wt\,\hat{i} + w\cos wt\,\hat{j})$

$$= (\cos wt)(-w\sin wt) + (\sin wt)(w\cos wt) = 0.$$

Thus $\vec{r}$ and $\vec{V}$ are perpendicular.

(ii) $\vec{a} = \dfrac{d^2\vec{r}}{dt^2} = \dfrac{d\vec{V}}{dt} = -w^2\cos wt\,\hat{i} - w^2\sin wt\,\hat{j}$

$$= -w^2(\cos wt\,\hat{i} + \sin wt\,\hat{j}) = -w^2\vec{r}$$

Then the acceleration is opposite to the direction of $\vec{r}$ i.e. it is directed towards the origin. Its magnitude is proportional to $|\vec{r}|$ which is the distance from the origin.

(iii) $\vec{r} \times \vec{V} = (\cos wt\,\hat{i} + \sin wt\,\hat{j}) \times (-w\sin wt\,\hat{i} + w\cos wt\,\hat{j})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos wt & \sin wt & 0 \\ -w\sin wt & w\cos wt & 0 \end{vmatrix} = w(\cos^2 wt + \sin^2 wt)\hat{k} = w\hat{k}, \quad \text{a constant vector.}$$

PROBLEM (12): **If $\hat{r}$ is a unit vector in the direction of $\vec{r}$, prove that $\hat{r} \times \dfrac{d\hat{r}}{dt} = \dfrac{1}{r^2}\vec{r} \times \dfrac{d\vec{r}}{dt}$.**

SOLUTION: Here $\hat{r} = \dfrac{\vec{r}}{r}$

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Differentiating both sides w.r.t. t, we get

$$\frac{d\hat{r}}{dt} = \frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \frac{1}{r}\frac{d\vec{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\vec{r}$$

Thus $$\hat{r} \times \frac{d\hat{r}}{dt} = \frac{\vec{r}}{r} \times \left(\frac{1}{r}\frac{d\vec{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\vec{r}\right)$$

$$= \frac{1}{r^2}\vec{r} \times \frac{d\vec{r}}{dt} - \frac{1}{r^3}\frac{dr}{dt}\vec{r} \times \vec{r}$$

$$= \frac{1}{r^2}\vec{r} \times \frac{d\vec{r}}{dt} \qquad (\text{since } \vec{r} \times \vec{r} = \vec{0})$$

PROBLEM (13): **If $\hat{a}(t)$ is a unit vector, then show that $\hat{a}.\left(\hat{a} + \frac{d^2\hat{a}}{dt^2}\right) + \left(\frac{d\hat{a}}{dt}\right)^2 = 1$.**

SOLUTION: Since $\hat{a}(t)$ is a unit vector, so

$$\hat{a}(t).\hat{a}(t) = 1 \qquad (1)$$

Differentiating both sides w.r.t. t, we get $\hat{a}.\frac{d\hat{a}}{dt} + \frac{d\hat{a}}{dt}.\hat{a} = 0$ or $2\hat{a}.\frac{d\hat{a}}{dt} = 0$

or $$\hat{a}.\frac{d\hat{a}}{dt} = 0$$

Differentiating again w.r.t. t, we get

$$\hat{a}.\frac{d^2\hat{a}}{dt^2} + \frac{d\hat{a}}{dt}.\frac{d\hat{a}}{dt} = 0$$

or $$\hat{a}.\frac{d^2\hat{a}}{dt^2} + \left(\frac{d\hat{a}}{dt}\right)^2 = 0 \qquad (2)$$

Adding equations (1) and (2), we get

$$\hat{a}.\hat{a} + \hat{a}.\frac{d^2\hat{a}}{dt^2} + \left(\frac{d\hat{a}}{dt}\right)^2 = 1$$

or $$\hat{a}.\left(\hat{a} + \frac{d^2\hat{a}}{dt^2}\right) + \left(\frac{d\hat{a}}{dt}\right)^2 = 1$$

PROBLEM (14): **(i) If $\vec{r}.d\vec{r} = 0$, show that r = constant**

(ii) If $\vec{r} \times d\vec{r} = \vec{0}$, show that $\hat{r}$ = constant.

SOLUTION: (i) We have $\vec{r}.d\vec{r} = 0$ or $\frac{1}{2}d(\vec{r}.\vec{r}) = 0$

or $$d\left(\frac{1}{2}r^2\right) = 0$$

Integrating, we get $\frac{1}{2}r^2$ = constant. This implies r = constant.

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(ii) Since $\vec{r} = x\hat{i} + y\hat{j} + dz\hat{k}$

and $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$, therefore (1)

$$\vec{r} \times d\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ dx & dy & dz \end{vmatrix} = \vec{0}$$

or $(y\,dz - z\,dy)\hat{i} + (z\,dx - x\,dz)\hat{j} + (x\,dy - y\,dx)\hat{k} = \vec{0}$

Comparing the coefficients of $\hat{i}$, $\hat{j}$, and $\hat{k}$, we get

$$y\,dz - z\,dy = 0, \quad z\,dx - x\,dz = 0, \quad x\,dy - y\,dx = 0$$

These equations give $\dfrac{dy}{y} = \dfrac{dz}{z}$, $\dfrac{dz}{z} = \dfrac{dx}{x}$ and $\dfrac{dx}{x} = \dfrac{dy}{y}$

or $\dfrac{dx}{x} = \dfrac{dy}{y} = \dfrac{dz}{z}$

From $\dfrac{dy}{y} = \dfrac{dx}{x}$, we get

$\ln y = \ln x + \ln a = \ln ax$, where a is a scalar constant

or $y = ax$ (2)

Similarly, from $\dfrac{dz}{z} = \dfrac{dx}{x}$, we get

$\ln z = \ln x + \ln b = \ln bx$, where b is a scalar constant.

or $z = bx$ (3)

Thus from equations (1), (2), and (3) we get

$$\vec{r} = x\hat{i} + ax\hat{j} + bx\hat{k}$$

Hence $\hat{r} = \dfrac{x\hat{i} + ax\hat{j} + bx\hat{k}}{\sqrt{x^2 + a^2x^2 + b^2x^2}} = \dfrac{\hat{i} + a\hat{j} + b\hat{k}}{\sqrt{1 + a^2 + b^2}}$ which is a constant vector.

PROBLEM (15): **If $\vec{f}$, $\vec{g}$, and $\vec{h}$ are vector functions of a scalar variable t, and if $\dfrac{d\vec{f}}{dt} = \vec{h} \times \vec{f}$ and $\dfrac{d\vec{g}}{dt} = \vec{h} \times \vec{g}$, prove that $\dfrac{d}{dt}(\vec{f} \times \vec{g}) = \vec{h} \times (\vec{f} \times \vec{g})$.**

SOLUTION: We have

$$\frac{d}{dt}(\vec{f} \times \vec{g}) = \frac{d\vec{f}}{dt} \times \vec{g} + \vec{f} \times \frac{d\vec{g}}{dt}$$

$$= (\vec{h} \times \vec{f}) \times \vec{g} + \vec{f} \times (\vec{h} \times \vec{g})$$

$$= (\vec{h} \cdot \vec{g})\vec{f} - (\vec{f} \cdot \vec{g})\vec{h} + (\vec{f} \cdot \vec{g})\vec{h} - (\vec{f} \cdot \vec{h})\vec{g}$$

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$$= (\vec{h}\cdot\vec{g})\vec{f} - (\vec{f}\cdot\vec{h})\vec{g}$$

$$= (\vec{h}\cdot\vec{g})\vec{f} - (\vec{h}\cdot\vec{f})\vec{g} = \vec{h}\times(\vec{f}\times\vec{g})$$

PROBLEM (16): **If $\vec{f}$ is a vector function of t, prove that**

$$\frac{d}{dt}\left(\frac{\vec{f}}{f}\right) = \frac{(\vec{f}\cdot\vec{f})\vec{f}' - \vec{f}(\vec{f}\cdot\vec{f}')}{(\vec{f}\cdot\vec{f})^{3/2}}$$

SOLUTION: We have

$$\frac{d}{dt}\left(\frac{\vec{f}}{f}\right) = \frac{d}{dt}\left[\frac{\vec{f}}{(\vec{f}\cdot\vec{f})^{1/2}}\right]$$

$$= \frac{(\vec{f}\cdot\vec{f})^{1/2}\frac{d\vec{f}}{dt} - \vec{f}\frac{d}{dt}(\vec{f}\cdot\vec{f})^{1/2}}{\vec{f}\cdot\vec{f}}$$

$$= \frac{(\vec{f}\cdot\vec{f})^{1/2}\vec{f}' - \frac{1}{2}\vec{f}(\vec{f}\cdot\vec{f})^{-1/2}\frac{d}{dt}(\vec{f}\cdot\vec{f})}{\vec{f}\cdot\vec{f}}$$

$$= \frac{(\vec{f}\cdot\vec{f})^{1/2}\vec{f}' - \frac{1}{2}\vec{f}(\vec{f}\cdot\vec{f})^{-1/2}(2\vec{f}\cdot\vec{f}')}{\vec{f}\cdot\vec{f}}$$

$$= \frac{(\vec{f}\cdot\vec{f})\vec{f}' - \vec{f}(\vec{f}\cdot\vec{f}')}{(\vec{f}\cdot\vec{f})^{3/2}}$$

PROBLEM (17): **Differentiate the following expressions w.r.t. t, where $\vec{r}$ is a vector function of a scalar variable t, and $\vec{A}$ and $\vec{B}$ are constant vectors.**

(i) $\vec{r}\cdot\vec{A}$ (ii) $\vec{r}\times\vec{A}$ (iii) $\vec{r}\cdot\frac{d\vec{r}}{dt}$

(iv) $\vec{r}\times\frac{d\vec{r}}{dt}$ (v) $\vec{r}^{\,2} + \frac{1}{\vec{r}^{\,2}}$ (vi) $(A\vec{r} + r\vec{B})^2$

(vii) $r^3\vec{r} + \vec{A}\times\frac{d\vec{r}}{dt}$ (viii) $\frac{\vec{r}+\vec{A}}{\vec{r}^{\,2}+\vec{A}^{\,2}}$ (ix) $\frac{\vec{r}\times\vec{A}}{\vec{r}\cdot\vec{A}}$

SOLUTION: (i) $\frac{d}{dt}(\vec{r}\cdot\vec{A}) = \vec{r}\cdot\frac{d\vec{A}}{dt} + \frac{d\vec{r}}{dt}\cdot\vec{A} = \frac{d\vec{r}}{dt}\cdot\vec{A}$ $\left(\text{since } \frac{d\vec{A}}{dt} = \vec{0}\right)$

(ii) $\frac{d}{dt}(\vec{r}\times\vec{A}) = \vec{r}\times\frac{d\vec{A}}{dt} + \frac{d\vec{r}}{dt}\times\vec{A} = \frac{d\vec{r}}{dt}\times\vec{A}$ $\left(\text{since } \frac{d\vec{A}}{dt} = \vec{0}\right)$

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(iii) $$\frac{d}{dt}\left(\vec{r}\cdot\frac{d\vec{r}}{dt}\right)=\vec{r}\cdot\frac{d^2\vec{r}}{dt^2}+\frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt}=\vec{r}\cdot\frac{d^2\vec{r}}{dt^2}+\left(\frac{d\vec{r}}{dt}\right)^2$$

(iv) $$\frac{d}{dt}\left(\vec{r}\times\frac{d\vec{r}}{dt}\right)=\vec{r}\times\frac{d^2\vec{r}}{dt^2}+\frac{d\vec{r}}{dt}\times\frac{d\vec{r}}{dt}=\vec{r}\times\frac{d^2\vec{r}}{dt^2}\qquad\left(\text{since }\frac{d\vec{r}}{dt}\times\frac{d\vec{r}}{dt}=\vec{0}\right)$$

(v) $$\frac{d}{dt}\left(\vec{r}^{\,2}+\frac{1}{\vec{r}^{\,2}}\right)=\frac{d}{dt}(\vec{r}^{\,2})+\frac{d}{dt}\left(\frac{1}{\vec{r}^{\,2}}\right)=\frac{d}{dt}(r^2)+\frac{d}{dt}\left(\frac{1}{r^2}\right)=2r\frac{dr}{dt}-\frac{2}{r^3}\frac{dr}{dt}$$

(vi) $$\frac{d}{dt}(A\vec{r}+r\vec{B})^2=2(2\vec{r}+r\vec{B})\cdot\frac{d}{dt}(A\vec{r}+r\vec{B})\qquad\left(\text{since }\frac{d}{dt}(\vec{A})^2=2\vec{A}\cdot\frac{d\vec{A}}{dt}\right)$$

$$=2(A\vec{r}+r\vec{B})\left(A\frac{d\vec{r}}{dt}+\frac{dA}{dt}\vec{r}+r\frac{d\vec{B}}{dt}+\frac{dr}{dt}\vec{B}\right)$$

$$=2(A\vec{r}+r\vec{B})\left(A\frac{d\vec{r}}{dt}+\frac{dr}{dt}\vec{B}\right)\qquad\left(\text{since }\frac{d\vec{A}}{dt}=0\text{ and }\frac{d\vec{B}}{dt}=0\right)$$

(vii) $$\frac{d}{dt}\left(r^3\vec{r}+\vec{A}\times\frac{d\vec{r}}{dt}\right)=\frac{d}{dt}(r^3\vec{r})+\frac{d}{dt}\left(\vec{A}\times\frac{d\vec{r}}{dt}\right)$$

$$=r^3\frac{d\vec{r}}{dt}+\vec{r}\frac{d}{dt}(r^3)+\vec{A}\times\frac{d^2\vec{r}}{dt^2}+\frac{d\vec{A}}{dt}\times\frac{d\vec{r}}{dt}$$

$$=r^3\frac{d\vec{r}}{dt}+\left(3r^2\frac{dr}{dt}\right)\vec{r}+\vec{A}\times\frac{d^2\vec{r}}{dt}\qquad\left(\text{since }\frac{d\vec{A}}{dt}=\vec{0}\right)$$

(viii) $$\frac{d}{dt}\left(\frac{\vec{r}+\vec{A}}{\vec{r}^{\,2}+\vec{A}^{\,2}}\right)=\frac{(\vec{r}^{\,2}+\vec{A}^{\,2})\frac{d}{dt}(\vec{r}+\vec{A})-(\vec{r}+\vec{A})\frac{d}{dt}(\vec{r}^{\,2}+\vec{A}^{\,2})}{(\vec{r}^{\,2}+\vec{A}^{\,2})^2}$$

$$=\frac{(\vec{r}^{\,2}+\vec{A}^{\,2})\left(\frac{d\vec{r}}{dt}+\frac{d\vec{A}}{dt}\right)-(\vec{r}+\vec{A})\left(\frac{d}{dt}\vec{r}^{\,2}+\frac{d}{dt}\vec{A}^{\,2}\right)}{(\vec{r}^{\,2}+\vec{A}^{\,2})^2}$$

$$=\frac{(\vec{r}^{\,2}+\vec{A}^{\,2})\frac{d\vec{r}}{dt}-(\vec{r}+\vec{A})\frac{d}{dt}(\vec{r}^{\,2})}{(\vec{r}^{\,2}+\vec{A}^{\,2})^2}\qquad\left(\text{since }\frac{d\vec{A}}{dt}=\frac{d\vec{A}^{\,2}}{dt}=\vec{0}\right)$$

$$=\frac{(\vec{r}^{\,2}+\vec{A}^{\,2})\frac{d\vec{r}}{dt}-(\vec{r}+\vec{A})\left(2\vec{r}\cdot\frac{d\vec{r}}{dt}\right)}{(\vec{r}^{\,2}+\vec{A}^{\,2})^2}$$

$$=\frac{\frac{d\vec{r}}{dt}}{\vec{r}^{\,2}+\vec{A}^{\,2}}-\frac{(\vec{r}+\vec{A})\left(2\vec{r}\cdot\frac{d\vec{r}}{dt}\right)}{(\vec{r}^{\,2}+\vec{A}^{\,2})^2}$$

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(ix) $$\frac{d}{dt}\left(\frac{\vec{r}\times\vec{A}}{\vec{r}\cdot\vec{A}}\right) = \frac{(\vec{r}\cdot\vec{A})\frac{d}{dt}(\vec{r}\times\vec{A})-(\vec{r}\times\vec{A})\frac{d}{dt}(\vec{r}\cdot\vec{A})}{(\vec{r}\cdot\vec{A})^2}$$

$$= \frac{(\vec{r}\cdot\vec{A})\left(\frac{d\vec{r}}{dt}\times\vec{A}\right)-(\vec{r}\times\vec{A})\left(\frac{d\vec{r}}{dt}\cdot\vec{A}\right)}{(\vec{r}\cdot\vec{A})^2} \quad [\text{ using (i) and (ii) }]$$

PROBLEM (18): If $\vec{A} = (2x^2y - x^4)\hat{i} + (e^{xy} - y\sin x)\hat{j} + (x^2\cos y)\hat{k}$, find

$$\frac{\partial\vec{A}}{\partial x},\ \frac{\partial\vec{A}}{\partial y},\ \frac{\partial^2\vec{A}}{\partial x^2},\ \frac{\partial^2\vec{A}}{\partial y^2},\ \frac{\partial^2\vec{A}}{\partial x\partial y},\ \frac{\partial^2\vec{A}}{\partial y\partial x}$$

SOLUTION: We have

$$\frac{\partial\vec{A}}{\partial x} = \frac{\partial}{\partial x}(2x^2y-x^4)\hat{i} + \frac{\partial}{\partial x}(e^{xy}-y\sin x)\hat{j} + \frac{\partial}{\partial x}(x^2\cos y)\hat{k}$$

$$= (4xy-4x^3)\hat{i} + (ye^{xy}-y\cos x)\hat{j} + 2x\cos y\,\hat{k}$$

$$\frac{\partial\vec{A}}{\partial y} = \frac{\partial}{\partial y}(2x^2y-x^4)\hat{i} + \frac{\partial}{\partial y}(e^{xy}-y\sin x)\hat{j} + \frac{\partial}{\partial y}(x^2\cos y)\hat{k}$$

$$= 2x^2\hat{i} + (xe^{xy}-\sin x)\hat{j} - x^2\sin y\,\hat{k}$$

$$\frac{\partial^2\vec{A}}{\partial x^2} = \frac{\partial}{\partial x}(4xy-4x^3)\hat{i} + \frac{\partial}{\partial x}(ye^{xy}-y\cos x)\hat{j} + \frac{\partial}{\partial x}(2x\cos y)\hat{k}$$

$$= (4y-12x^2)\hat{i} + (y^2e^{xy}+y\sin x)\hat{j} + 2\cos y\,\hat{k}$$

$$\frac{\partial^2\vec{A}}{\partial y^2} = \frac{\partial}{\partial y}(2x^2)\hat{i} + \frac{\partial}{\partial y}(xe^{xy}-\sin x)\hat{j} - \frac{\partial}{\partial y}(x^2\sin y)\hat{k}$$

$$= 0\,\hat{i} + x^2e^{xy}\hat{j} - x^2\cos y\,\hat{k} = x^2e^{xy}\hat{j} - x^2\cos y\,\hat{k}$$

$$\frac{\partial^2\vec{A}}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial\vec{A}}{\partial y}\right) = \frac{\partial}{\partial x}(2x^2)\hat{i} + \frac{\partial}{\partial x}(xe^{xy}-\sin x)\hat{j} - \frac{\partial}{\partial x}(x^2\sin y)\hat{k}$$

$$= 4x\,\hat{i} + (xye^{xy}+e^{xy}-\cos x)\hat{j} - 2x\sin y\,\hat{k}$$

$$\frac{\partial^2\vec{A}}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial\vec{A}}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}(4xy-4x^3)\hat{i} + \frac{\partial}{\partial y}(ye^{xy}-y\cos x)\hat{j} + \frac{\partial}{\partial y}(2x\cos y)\hat{k}$$

$$= 4x\,\hat{i} + (xye^{xy}+e^{xy}-\cos x)\hat{j} - 2x\sin y\,\hat{k}$$

Note that $\frac{\partial^2\vec{A}}{\partial x\partial y} = \frac{\partial\vec{A}}{\partial x\partial y}$, i.e. the order of differentiation is immaterial. This is true in general if $\vec{A}$ has continuous partial derivatives of the second order atleast.

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PROBLEM (19): If $\phi(x,y,z) = xy^2z$ and $\vec{A} = xz\hat{i} - xy^2\hat{j} + yz^2\hat{k}$, find $\frac{\partial^3}{\partial x^2 \partial z}(\phi\vec{A})$ at the point $(2,-1,1)$.

SOLUTION: Since $\phi\vec{A} = (xy^2z)(xz\hat{i} - xy^2\hat{j} + yz^2\hat{k})$

$$= x^2y^2z^2\hat{i} - x^2y^4z\hat{j} + xy^3z^3\hat{k}$$

$$\frac{\partial}{\partial z}(\phi\vec{A}) = \frac{\partial}{\partial z}(x^2y^2z^2\hat{i} - x^2y^4z\hat{j} + xy^3z^3\hat{k})$$

$$= 2x^2y^2z\hat{i} - x^2y^4\hat{j} + 3xy^3z^2\hat{k}$$

$$\frac{\partial^2}{\partial x \partial z}(\phi\vec{A}) = \frac{\partial}{\partial x}(2x^2y^2z\hat{i} - x^2y^4\hat{j} + 3xy^3z^2\hat{k})$$

$$= 4xy^2z\hat{i} - 2xy^4\hat{j} + 3y^3z^2\hat{k}$$

$$\frac{\partial^3}{\partial x^2 \partial z}(\phi\vec{A}) = \frac{\partial}{\partial x}(4xy^2z\hat{i} - 2xy^4\hat{j} + 3y^3z^2\hat{k}) = 4y^2z\hat{i} - 2y^4\hat{j}$$

Therefore, $\frac{\partial^3}{\partial x^2 \partial z}(\phi\vec{A})_{(2,-1,1)} = 4(-1)^2(1)\hat{i} - 2(-1)^4\hat{j} = 4\hat{i} - 2\hat{j}$

PROBLEM (20): Let $\vec{F}$ depend on x, y, z, t where x, y, and z depend on t. Prove that

$$\frac{d\vec{F}}{dt} = \frac{\partial\vec{F}}{\partial t} + \frac{\partial\vec{F}}{\partial x}\frac{dx}{dt} + \frac{\partial\vec{F}}{\partial y}\frac{dy}{dt} + \frac{\partial\vec{F}}{\partial z}\frac{dz}{dt}$$

under suitable assumptions of differentiality.

SOLUTION: Suppose that $\vec{F} = F_1(x,y,z,t)\hat{i} + F_2(x,y,z,t)\hat{j} + F_3(x,y,z,t)\hat{k}$

Then
$$d\vec{F} = dF_1\hat{i} + dF_2\hat{j} + dF_3\hat{k}$$
$$= \left(\frac{\partial F_1}{\partial x}dx + \frac{\partial F_1}{\partial y}dy + \frac{\partial F_1}{\partial z}dz + \frac{\partial F_1}{\partial t}dt\right)\hat{i}$$
$$+ \left(\frac{\partial F_2}{\partial x}dx + \frac{\partial F_2}{\partial y}dy + \frac{\partial F_2}{\partial z}dz + \frac{\partial F_2}{\partial t}dt\right)\hat{j}$$
$$+ \left(\frac{\partial F_3}{\partial x}dx + \frac{\partial F_3}{\partial y}dy + \frac{\partial F_3}{\partial z}dz + \frac{\partial F_3}{\partial t}dt\right)\hat{k}$$
$$= \left(\frac{\partial F_1}{\partial t}\hat{i} + \frac{\partial F_2}{\partial t}\hat{j} + \frac{\partial F_3}{\partial t}\hat{k}\right)dt + \left(\frac{\partial F_1}{\partial x}\hat{i} + \frac{\partial F_2}{\partial x}\hat{j} + \frac{\partial F_3}{\partial x}\hat{k}\right)dx$$
$$+ \left(\frac{\partial F_1}{\partial y}\hat{i} + \frac{\partial F_2}{\partial y}\hat{j} + \frac{\partial F_3}{\partial y}\hat{k}\right)dy + \left(\frac{\partial F_1}{\partial z}\hat{i} + \frac{\partial F_2}{\partial z}\hat{j} + \frac{\partial F_3}{\partial z}\hat{k}\right)dz$$
$$= \frac{\partial\vec{F}}{\partial t}dt + \frac{\partial\vec{F}}{\partial x}dx + \frac{\partial\vec{F}}{\partial y}dy + \frac{\partial\vec{F}}{\partial z}dz$$

and so
$$\frac{d\vec{F}}{dt} = \frac{\partial\vec{F}}{\partial t} + \frac{\partial\vec{F}}{\partial x}\frac{dx}{dt} + \frac{\partial\vec{F}}{\partial y}\frac{dy}{dt} + \frac{\partial\vec{F}}{\partial z}\frac{dz}{dt}$$

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PROBLEM (21): **Find a unit normal vector $\hat{n}$ to the surface S represented by**

$$\vec{r} = a\cos u \sin v\,\hat{i} + a\sin u \sin v\,\hat{j} + a\cos v\,\hat{k}\,; \quad a > 0.$$

SOLUTION: Since $\vec{r} = a\cos u\sin v\,\hat{i} + a\sin u\sin v\,\hat{j} + a\cos v\,\hat{k}$

Therefore $\dfrac{\partial \vec{r}}{\partial u} = -a\sin u\sin v\,\hat{i} + a\cos u\sin v\,\hat{j}$

$$\frac{\partial \vec{r}}{\partial v} = a\cos u\cos v\,\hat{i} + a\sin u\cos v\,\hat{j} - a\sin v\,\hat{k}$$

Then
$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin u\sin v & a\cos u\sin v & 0 \\ a\cos u\cos v & a\sin u\cos v & -a\sin v \end{vmatrix}$$

$$= -a^2\cos u\sin^2 v\,\hat{i} - a^2\sin u\sin^2 v\,\hat{j} - (a^2\sin^2 u\sin v\cos v + a^2\cos^2 u\sin v\cos v)\,\hat{k}$$

$$= -a^2\cos u\sin^2 v\,\hat{i} - a^2\sin u\sin^2 v\,\hat{j} - a^2\sin v\cos v\,\hat{k}$$

represents a vector normal to the surface at any point (u, v).

Now
$$\left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| = \sqrt{a^4\cos^2 u\sin^4 v + a^4\sin^2 u\sin^4 v + a^4\sin^2 v\cos^2 v}$$

$$= \sqrt{a^4(\cos^2 u + \sin^2 u)\sin^4 v + a^4\sin^2 v\cos^2 v}$$

$$= \sqrt{a^4\sin^4 v + a^4\sin^2 v\cos^2 v} = \sqrt{a^4(\sin^2 v + \cos^2 v)\sin^2 v}$$

$$= \sqrt{a^4\sin^2 v} = \begin{cases} a^2\sin v & \text{if } \sin v > 0 \\ -a^2\sin v & \text{if } \sin v < 0 \end{cases}$$

Thus a unit normal vector $\hat{n}$ is given by

$$\hat{n} = \frac{\dfrac{\partial \vec{r}}{\partial u} \times \dfrac{\partial \vec{r}}{\partial v}}{\left|\dfrac{\partial \vec{r}}{\partial u} \times \dfrac{\partial \vec{r}}{\partial v}\right|} = \pm\left(\cos u\sin v\,\hat{i} + \sin u\sin v\,\hat{j} + \cos v\,\hat{k}\right)$$

It should be noted that the given surface is defined by $x = a\cos u\sin v$, $y = a\sin u\sin v$, $z = a\cos v$ from which it is seen that $x^2 + y^2 + z^2 = a^2$, which is a sphere with centre at the origin and radius a. Since $\vec{r} = a\hat{n}$, it follows that $\hat{n} = \cos u\sin v\,\hat{i} + \sin u\sin v\,\hat{j} + \cos v\,\hat{k}$ is the outward drawn unit normal to the sphere at the point (u, v).

PROBLEM (22): **Show that a unit normal vector $\hat{n}$ to a surface S represented by $x = x$, $y = y$, $z = z(x, y)$**

is given by $$\hat{n} = \frac{-\dfrac{\partial z}{\partial x}\hat{i} - \dfrac{\partial z}{\partial y}\hat{j} + \hat{k}}{\sqrt{\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1}}$$

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SOLUTION: We see that the given equations represent a usual surface in three–dimensional rectangular Cartesian coordinates system. We can represent the surface S by

$$\vec{r} = \vec{r}(x,y) = x\hat{i} + y\hat{j} + z(x,y)\hat{k}$$

so the partial derivatives are

$$\frac{\partial \vec{r}}{\partial x} = \hat{i} + \frac{\partial z}{\partial x}\hat{k} \quad \text{and} \quad \frac{\partial \vec{r}}{\partial y} = \hat{j} + \frac{\partial z}{\partial y}\hat{k}$$

Thus
$$\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \frac{\partial z}{\partial x} \\ 0 & 1 & \frac{\partial z}{\partial y} \end{vmatrix} = -\frac{\partial z}{\partial x}\hat{i} - \frac{\partial z}{\partial y}\hat{j} + \hat{k}$$

and
$$\left| \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right| = \sqrt{\left(-\frac{\partial z}{\partial x}\right)^2 + \left(-\frac{\partial z}{\partial y}\right)^2 + 1} = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

Hence the unit normal vector $\hat{n}$ to the surface S is given by

$$\hat{n} = \frac{\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y}}{\left| \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right|} = \frac{-\frac{\partial z}{\partial x}\hat{i} - \frac{\partial z}{\partial y}\hat{j} + \hat{k}}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}$$

PROBLEM (23): **Show that a unit normal vector $\hat{n}$ to a surface S represented by**

$$\phi(x,y,z) = 0 \quad \textbf{is given by} \quad \hat{n} = \frac{\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}}{\sqrt{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2}}$$

SOLUTION: If the equation $\phi(x,y,z) = 0$ defines implicitly z as a function of x and y, then from calculus we have

$$\frac{\partial z}{\partial x} = -\frac{\phi_x}{\phi_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\phi_y}{\phi_z}; \quad \phi_z \neq 0$$

Thus from problem (22)

$$\frac{\partial \vec{r}}{\partial x} = \hat{i} + \frac{\partial z}{\partial x}\hat{k} = \hat{i} - \frac{\phi_x}{\phi_z}\hat{k} \quad \text{and} \quad \frac{\partial \vec{r}}{\partial y} = \hat{j} + \frac{\partial z}{\partial y}\hat{k} = \hat{j} - \frac{\phi_y}{\phi_z}\hat{k}$$

and
$$\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -\frac{\phi_x}{\phi_z} \\ 0 & 1 & -\frac{\phi_y}{\phi_z} \end{vmatrix} = \frac{\phi_x}{\phi_z}\hat{i} + \frac{\phi_y}{\phi_z}\hat{j} + \hat{k} = \frac{1}{\phi_z}\left(\phi_x\hat{i} + \phi_y\hat{j} + \phi_z\hat{k}\right)$$

Also,
$$\left| \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right| = \frac{1}{\phi_z}\sqrt{(\phi_x)^2 + (\phi_y)^2 + (\phi_z)^2}$$

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Thus the unit normal vector $\hat{n}$ to the surface S is given by

$$\hat{n} = \frac{\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y}}{\left| \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right|} = \frac{\phi_x \hat{i} + \phi_y \hat{j} + \phi_z \hat{k}}{\sqrt{(\phi_x)^2 + (\phi_y)^2 + (\phi_z)^2}} = \frac{\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}}{\sqrt{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2}}$$

NOTE: If $\phi_x \neq 0$ or $\phi_y \neq 0$, a similar procedure would yield the same result.

PROBLEM (24): **Find a unit normal vector $\hat{n}$ for a surface S that is represented by $x^2 + y^2 + z^2 = a^2$.**

SOLUTION: We know that the given equation represents the surface S of a sphere with centre at the origin and radius a. The sphere S can be represented by $\phi(x, y, z) = 0$

where $\phi(x, y, z) = x^2 + y^2 + z^2 - a^2$

Now $\frac{\partial \phi}{\partial x} = 2x, \quad \frac{\partial \phi}{\partial y} = 2y, \quad \frac{\partial \phi}{\partial z} = 2z$

and $\sqrt{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2} = 2\sqrt{x^2 + y^2 + z^2} = 2a$

Then a unit normal vector $\hat{n}$ to the surface S is given by

$$\hat{n} = \frac{\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}}{\sqrt{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2}} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{2a} = \frac{x}{a}\hat{i} + \frac{y}{a}\hat{j} + \frac{z}{a}\hat{k}$$

PROBLEM (25): **Evaluate $\int_2^3 \vec{A} \cdot \frac{d\vec{A}}{dt}\, dt$ if $\vec{A}(2) = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{A}(3) = 4\hat{i} - 2\hat{j} + 3\hat{k}$.**

SOLUTION: Since $\vec{A} \cdot \frac{d\vec{A}}{dt} = \frac{1}{2}\frac{d}{dt}(\vec{A} \cdot \vec{A})$

Therefore $\int_2^3 \vec{A} \cdot \frac{d\vec{A}}{dt}\, dt = \frac{1}{2}\left|(\vec{A} \cdot \vec{A})\right|_{t=2}^{3} = \frac{1}{2}\left[\vec{A}(3) \cdot \vec{A}(3) - \vec{A}(2) \cdot \vec{A}(2)\right]$ (1)

Now $\vec{A}(3) \cdot \vec{A}(3) = (4\hat{i} - 2\hat{j} + 3\hat{k}) \cdot (4\hat{i} - 2\hat{j} + 3\hat{k}) = 16 + 4 + 9 = 29$

and $\vec{A}(2) \cdot \vec{A}(2) = (2\hat{i} - \hat{j} + 2\hat{k}) \cdot (2\hat{i} - \hat{j} + 2\hat{k}) = 4 + 1 + 4 = 9$

Thus from equation (1), we get

$$\int_2^3 \vec{A} \cdot \frac{d\vec{A}}{dt}\, dt = \frac{1}{2}[29 - 9] = 10$$

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PROBLEM (26): If $\vec{r} = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$, evaluate $\int_1^2 \vec{r} \times \frac{d^2\vec{r}}{dt^2}\,dt$.

SOLUTION: Since $\vec{r} = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$, therefore

$$\frac{d\vec{r}}{dt} = 10t\hat{i} + \hat{j} - 3t^2\hat{k} \quad \text{and} \quad \frac{d^2\vec{r}}{dt^2} = 10\hat{i} - 6t\hat{k}$$

and
$$\vec{r} \times \frac{d^2\vec{r}}{dt^2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5t^2 & t & -t^3 \\ 10 & 0 & -6t \end{vmatrix} = -6t^2\hat{i} + 20t^3\hat{j} - 10t\hat{k}$$

Thus
$$\int_1^2 \vec{r} \times \frac{d^2\vec{r}}{dt^2}\,dt = \int_1^2 \left(-6t^2\hat{i} + 20t^3\hat{j} - 10t\hat{k}\right)dt = \left|-2t^3\hat{i} + 5t^4\hat{j} - 5t^2\hat{k}\right|_1^2$$
$$= \left(-16\hat{i} + 80\hat{j} - 20\hat{k}\right) - \left(-2\hat{i} + 5\hat{j} - 5\hat{k}\right) = -14\hat{i} + 75\hat{j} - 15\hat{k}$$

ANOTHER METHOD

We know from theorem (3.10), that

$$\int_1^2 \vec{r} \times \frac{d^2\vec{r}}{dt^2}\,dt = \left|\vec{r} \times \frac{d\vec{r}}{dt}\right|_1^2 \qquad (1)$$

Since $\vec{r} = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$, therefore $\frac{d\vec{r}}{dt} = 10t\hat{i} + \hat{j} - 3t^2\hat{k}$

and
$$\vec{r} \times \frac{d\vec{r}}{dt} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5t^2 & t & -t^3 \\ 10t & 1 & -3t^2 \end{vmatrix} = -2t^3\hat{i} + 5t^4\hat{j} - 5t^2\hat{k}$$

Thus from equation (1), we get

$$\int_1^2 \vec{r} \times \frac{d^2\vec{r}}{dt^2}\,dt = \left|-2t^3\hat{i} + 5t^4\hat{j} - 5t^2\hat{k}\right|_1^2$$
$$= \left(-16\hat{i} + 80\hat{j} - 20\hat{k}\right) - \left(-2\hat{i} + 5\hat{j} - 5\hat{k}\right) = -14\hat{i} + 75\hat{j} - 15\hat{k}$$

PROBLEM (27): Solve $\vec{A} \times \frac{d^2\vec{f}}{dt^2} = \vec{B}$ where $\vec{A}$ and $\vec{B}$ are constant vectors and $\vec{f}$ is a vector function of t.

SOLUTION: We have $\vec{A} \times \frac{d^2\vec{f}}{dt^2} = \vec{B}$

Integrating both sides w.r.t. t, we get

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$$\int \vec{A} \times \frac{d^2 \vec{f}}{dt^2} dt = \int \vec{B}\, dt \qquad (1)$$

Now $\frac{d}{dt}\left(\vec{A} \times \frac{d\vec{f}}{dt}\right) = \vec{A} \times \frac{d^2 \vec{f}}{dt^2} + \frac{d\vec{A}}{dt} \times \frac{d\vec{f}}{dt}$

$$= \vec{A} \times \frac{d^2 \vec{f}}{dt^2} \qquad \left(\text{since } \frac{d\vec{A}}{dt} = \vec{0}\right)$$

Thus from equation (1) we get

$$\int \frac{d}{dt}\left(\vec{A} \times \frac{d\vec{f}}{dt}\right) dt = \int \vec{B}\, dt$$

or $\vec{A} \times \frac{d\vec{f}}{dt} = \vec{B}\, t + \vec{C}$ where $\vec{C}$ is an arbitrary constant vector of integration .

Again , integrating both sides w.r.t. t and using theorem (3.10) , we get

$$\vec{A} \times \vec{f} = \frac{1}{2}\vec{B}\, t^2 + \vec{C}\, t + \vec{D}$$ where $\vec{D}$ is an arbitrary constant vector .

PROBLEM (28): **Integrate the equation $\frac{d^2 \vec{r}}{dt^2} = -n^2 \vec{r}$.**

SOLUTION: We have $\frac{d^2 \vec{r}}{dt^2} = -n^2 \vec{r}$.

Taking the dot product of both sides with $2\frac{d\vec{r}}{dt}$, we get

$$2\frac{d\vec{r}}{dt} \cdot \frac{d^2 \vec{r}}{dt^2} = -n^2\left(2\vec{r} \cdot \frac{d\vec{r}}{dt}\right)$$

or $$2\frac{d\vec{r}}{dt} \cdot \frac{d^2 \vec{r}}{dt^2} = -n^2 \frac{d}{dt}(\vec{r} \cdot \vec{r}) = -n^2 \frac{d}{dt}(r^2)$$

Integrating both sides w.r.t. t and using theorem (3.10) , we get

$$\int 2\frac{d\vec{r}}{dt} \cdot \frac{d^2 \vec{r}}{dt^2} dt = -n^2 \int \frac{d}{dt}(r^2)\, dt$$

or $\left(\frac{d\vec{r}}{dt}\right)^2 = -n^2 r^2 + \vec{C}$ where $\vec{C}$ is an arbitrary constant vector of integration .

PROBLEM (29): **If $\frac{d^2 \vec{r}}{dt^2} = 4\hat{i}$ and $\vec{r}(t) = \vec{0}$ and $\frac{d\vec{r}}{dt} = 4\hat{j}$, when $t = 0$, show that the position vector $\vec{r}(t)$ describes a parabola .**

SOLUTION: We have $\frac{d^2 \vec{r}}{dt^2} = 4\hat{i}$

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Integrating both sides w.r.t. t, we get

$$\frac{d\vec{r}}{dt} = 4t\hat{i} + \vec{C}_1 \qquad (1)$$

where $\vec{C}_1$ is an arbitrary constant vector of integration.

Using $\frac{d\vec{r}}{dt} = 4\hat{j}$ when $t = 0$, we find from equation (1) $\vec{C}_1 = 4\hat{j}$

Then equation (1) becomes $\frac{d\vec{r}}{dt} = 4t\hat{i} + 4\hat{j}$

Again, integrating w.r.t. t, we get

$$\vec{r}(t) = 2t^2\hat{i} + 4t\hat{j} + \vec{C}_2 \qquad (2)$$

Using $\vec{r}(t) = \vec{0}$, when $t = 0$, we find from equation (2) $\vec{C}_2 = \vec{0}$

Thus equation (2) becomes

$$\vec{r}(t) = 2t^2\hat{i} + 4t\hat{j} \qquad (3)$$

But the position vector of an arbitrary point is

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \qquad (4)$$

Comparing equations (3) and (4), we have

$$x = 2t^2, \quad y = 4t, \quad z = 0$$

Eliminating the parameter t, we get

$$x = 2\left(\frac{y}{4}\right)^2 \quad \text{or} \quad y^2 = 8x \text{ which is a parabola.}$$

PROBLEM (30): **The acceleration of a particle at any time $t \geq 0$ is given by**

$$\vec{a} = \frac{d\vec{V}}{dt} = 12\cos 2t\,\hat{i} - 8\sin 2t\,\hat{j} + 16t\,\hat{k}$$

If the velocity $\vec{V}$ and displacement $\vec{r}$ are zero at $t = 0$, find $\vec{V}$ and $\vec{r}$ at any time.

SOLUTION: $\vec{a} = \frac{d\vec{V}}{dt} = 12\cos 2t\,\hat{i} - 8\sin 2t\,\hat{j} + 16t\,\hat{k}$

Integrating w.r.t. t, we get

$$\vec{V} = 6\sin 2t\,\hat{i} + 4\cos 2t\,\hat{j} + 8t^2\hat{k} + \vec{C}_1$$

Using $\vec{V} = \vec{0}$ when $t = 0$, we find $\vec{0} = 4\hat{j} + \vec{C}_1$ or $\vec{C}_1 = -4\hat{j}$

Therefore, $\vec{V} = 6\sin 2t\,\hat{i} + (4\cos 2t - 4)\hat{j} + 8t^2\hat{k}$

Integrating again w.r.t. t, we get

$$\vec{r} = -3\cos 2t\,\hat{i} + (2\sin 2t - 4t)\hat{j} + \frac{8}{3}t^3\hat{k} + \vec{C}_2$$

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Using $\vec{r} = \vec{0}$ when $t = 0$, we get $\vec{0} = -3\hat{i} + \vec{C}_2$ or $\vec{C}_2 = 3\hat{i}$

Then $\vec{r} = (3 - 3\cos 2t)\hat{i} + (2\sin 2t - 4t)\hat{j} + \frac{8}{3}t^3\hat{k}$

PROBLEM (31): **The equation of motion of a particle P of mass m is given by $m\frac{d^2\vec{r}}{dt^2} = f(r)\hat{r}$ where $\vec{r}$ is the position vector of P measured from an origin O, $\hat{r}$ is a unit vector in the direction $\vec{r}$, and $f(r)$ is a function of the distance of P from O**

(i) **Show that $\vec{r} \times \frac{d\vec{r}}{dt} = \vec{C}$, where $\vec{C}$ is a constant vector.**

(ii) **Interpret physically the cases $f(r) < 0$ and $f(r) > 0$.**

SOLUTION: (i) Multiplying both sides of the equation $m\frac{d^2\vec{r}}{dt^2} = f(r)\hat{r}$ by $\vec{r} \times$, we get

Then $m\vec{r} \times \frac{d^2\vec{r}}{dt^2} = f(r)\,\vec{r} \times \hat{r}$

$$= f(r)\,r\hat{r} \times \hat{r} = \vec{0} \quad (\text{since } \hat{r} \times \hat{r} = \vec{0})$$

Thus $\vec{r} \times \frac{d^2\vec{r}}{dt^2} = \vec{0}$ and $\frac{d}{dt}\left(\vec{r} \times \frac{d\vec{r}}{dt}\right) = \vec{0}$

Integrating, $\vec{r} \times \frac{d\vec{r}}{dt} = \vec{C}$, where $\vec{C}$ is a constant vector.

(ii) If $f(r) < 0$ the acceleration $\frac{d^2\vec{r}}{dt^2}$ has direction opposite to $\hat{r}$; hence the force is directed towards O and the particle is always attracted towards O.

If $f(r) > 0$ the force is directed away from O and the particle is under the influence of a repulsive force at O.

PROBLEM (32): **Given $\vec{F}(t) = e^{2t}\vec{C}_1 + e^{3t}\vec{C}_2$, where $\vec{C}_1$ and $\vec{C}_2$ are constant vectors, show that $\frac{d^2\vec{F}}{dt^2} - 5\frac{d\vec{F}}{dt} + 6\vec{F} = \vec{0}$.**

SOLUTION: Since $\vec{F}(t) = e^{2t}\vec{C}_1 + e^{3t}\vec{C}_2$, therefore

$$\frac{d\vec{F}}{dt} = 2e^{2t}\vec{C}_1 + 3e^{3t}\vec{C}_2 \text{ and } \frac{d^2\vec{F}}{dt^2} = 4e^{2t}\vec{C}_1 + 9e^{3t}\vec{C}_2$$

Thus $\frac{d^2\vec{F}}{dt^2} - 5\frac{d\vec{F}}{dt} + 6\vec{F} = 4e^{2t}\vec{C}_1 + 9e^{3t}\vec{C}_2 - 10e^{2t}\vec{C}_1 - 15e^{3t}\vec{C}_2 + 6e^{2t}\vec{C}_1 + 6e^{3t}\vec{C}_2$

$$= (4 - 10 + 6)e^{2t}\vec{C}_1 + (9 - 15 + 6)e^{3t}\vec{C}_2 = \vec{0}$$

NOTE: This is an example of a vector differential equation being satisfied by a vector function.

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PROBLEM (33): **If $\vec{r}(t) = \vec{A}\cos wt + \vec{B}\sin wt$, where $\vec{A}$ and $\vec{B}$ are constant vectors and w being a scalar constant, then show that**

(i) $\dfrac{d^2\vec{r}}{dt^2} + w^2\vec{r} = 0$ **(ii)** $\vec{r} \times \dfrac{d\vec{r}}{dt} = w(\vec{A} \times \vec{B})$

SOLUTION: Since $\vec{r}(t) = \vec{A}\cos wt + \vec{B}\sin wt$, therefore

$$\frac{d\vec{r}}{dt} = -w\vec{A}\sin wt + w\vec{B}\cos wt \quad \text{and} \quad \frac{d^2\vec{r}}{dt^2} = -w^2\vec{A}\cos wt - w^2\vec{B}\sin wt$$

(i) $$\frac{d^2\vec{r}}{dt^2} + w^2\vec{r} = -w^2\vec{A}\cos wt - w^2\vec{B}\sin wt + w^2\vec{A}\cos wt + w^2\vec{B}\sin wt = 0$$

(ii) $$\vec{r} \times \frac{d\vec{r}}{dt} = (\vec{A}\cos wt + \vec{B}\sin wt) \times (-w\vec{A}\sin wt + w\vec{B}\cos wt)$$

$$= -w\cos wt\sin wt(\vec{A} \times \vec{A}) + w\cos^2 wt(\vec{A} \times \vec{B})$$

$$-w\sin^2 wt(\vec{B} \times \vec{A}) + w\sin wt\cos wt(\vec{B} \times \vec{B})$$

$$= w\cos^2 wt(\vec{A} \times \vec{B}) - w\sin^2 wt(\vec{B} \times \vec{A}) \quad (\text{since } \vec{A} \times \vec{A} = \vec{B} \times \vec{B} = \vec{0})$$

$$= w(\cos^2 wt + \sin^2 wt)(\vec{A} \times \vec{B}) \quad (\text{since } \vec{B} \times \vec{A} = -\vec{A} \times \vec{B})$$

$$= w(\vec{A} \times \vec{B})$$

PROBLEM (34): **Solve the differential equation**

$$\frac{d^3\vec{r}}{dt^3} - \frac{d^2\vec{f}}{dt^2} - 2\frac{d\vec{f}}{dt} = \vec{0}$$

such that $\vec{f}(t) = \hat{i}$, $\dfrac{d\vec{f}}{dt} = \hat{j}$, $\dfrac{d^2\vec{f}}{dt^2} = \hat{k}$, when $t = 0$.

SOLUTION: Write the given equation in operator notation as

$$(D^3 - D^2 - 2D)\vec{f} = \vec{0}$$

The auxiliary equation is $D^3 - D^2 - 2D = 0$, whose roots are $D = 0, -1, 2$.

The general solution is given by

$$\vec{f}(t) = \vec{C}_1 + \vec{C}_2 e^{-t} + \vec{C}_3 e^{2t} \qquad (1)$$

Using $\vec{f}(t) = \hat{i}$, when $t = 0$, we get from equation (1)

$$\vec{C}_1 + \vec{C}_2 + \vec{C}_3 = \hat{i} \qquad (2)$$

Now $$\frac{d\vec{f}}{dt} = -\vec{C}_2 e^{-t} + 2\vec{C}_3 e^{2t} \qquad (3)$$

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Using $\frac{d\vec{f}}{dt} = \hat{j}$, when $t = 0$, we get from equation (3)

$$-\vec{C}_2 + 2\vec{C}_3 = \hat{j} \tag{4}$$

Also from equation (3)

$$\frac{d^2\vec{f}}{dt^2} = \vec{C}_2 e^{-t} + 4\vec{C}_3 e^{2t} \tag{5}$$

Using $\frac{d^2\vec{f}}{dt^2} = \hat{k}$, when $t = 0$, we get from equation (5)

$$\vec{C}_2 + 4\vec{C}_3 = \hat{k} \tag{6}$$

From equations (4) and (6), we get

$$\vec{C}_2 = -\frac{2}{3}\hat{j} + \frac{1}{3}\hat{k} \text{ and } \vec{C}_3 = \frac{1}{6}(\hat{j} + \hat{k})$$

Then from equation (2) we get

$$\vec{C}_1 = \hat{i} + \frac{1}{2}\hat{j} - \frac{1}{2}\hat{k}$$

Substituting the values of $\vec{C}_1$, $\vec{C}_2$, and $\vec{C}_3$ in equation (1), we get

$$\vec{f}(t) = \left(\hat{i} + \frac{1}{2}\hat{j} - \frac{1}{2}\hat{k}\right) + \left(-\frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}\right)e^{-t} + \frac{1}{6}(\hat{j} + \hat{k})e^{2t}$$

$$= \hat{i} + \left(\frac{1}{2} - \frac{2}{3}e^{-t} + \frac{1}{6}e^{2t}\right)\hat{j} + \left(-\frac{1}{2} + \frac{1}{3}e^{-t} + \frac{1}{6}e^{2t}\right)\hat{k}$$

PROBLEM (35): Solve $\frac{d\vec{y}}{dt} = \vec{x}$, $\frac{d\vec{x}}{dt} = -\vec{y}$

SOLUTION: Since $\frac{d\vec{x}}{dt} = -\vec{y}$, therefore $\frac{d^2\vec{x}}{dt^2} = -\frac{d\vec{y}}{dt} = -\vec{x}$

or $\frac{d^2\vec{x}}{dt^2} + \vec{x} = \vec{0}$ whose solution can be found to be $\vec{x} = \vec{C}_1 \cos t + \vec{C}_2 \sin t$

Now $\frac{d\vec{y}}{dt} = \vec{x} = \vec{C}_1 \cos t + \vec{C}_2 \sin t$

Integrating, we get $\vec{y} = \vec{C}_1 \sin t - \vec{C}_2 \cos t$.

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3.15 EXERCISE

PROBLEM (1): Find the domains of the following vector functions :

(i) $\vec{A}(t) = \frac{1}{t}\hat{i} + \frac{1}{t-1}\hat{j}$ (ii) $\vec{A}(t) = \frac{1}{t^2-1}\hat{i} + (t^2-1)\hat{j}$

(iii) $\vec{A}(t) = \ln(t)\hat{i} + \ln(1-t)\hat{j}$

PROBLEM (2): Find the following limits :

(i) $\underset{t\to 0}{\text{Lt}}\left[\cos^2 t\,\hat{i} + 25t^3\hat{j} + \tan t\,\hat{k}\right]$ (ii) $\underset{t\to 0}{\text{Lt}}\left[\frac{\sin t}{t}\hat{i} + \ln(3+t)\hat{j}\right]$

(iii) $\underset{t\to \pi}{\text{Lt}}\left(\sec t\,\hat{i} + \cos t\,\hat{j} + \tan t\,\hat{k}\right)$

PROBLEM (3): Test the continuity of the following vector functions :

(i) $\vec{A}(t) = \cos(t)\hat{i} - 2t^2\hat{j} + e^t\hat{k}$ (ii) $\vec{A}(t) = t\hat{i} + t^2\hat{j} + \frac{1}{t}\hat{k}$

PROBLEM (4): For what values of a, b, and c is the following vector function continuous at $t = 2$?

$$\vec{A}(t) = \begin{cases} (a+2b+3c)t^2\hat{i} + (2a-b)t^3\hat{j} + c\hat{k}, & t \neq 2 \\ \hat{i} + 2\hat{j} + 3\hat{k}, & t = 2 \end{cases}$$

PROBLEM (5): If $\vec{A} = e^{-t}\hat{i} + \ln(t^2+1)\hat{j} - \tan t\,\hat{k}$, find at $t = 0$

(i) $\frac{d\vec{A}}{dt}$ (ii) $\frac{d^2\vec{A}}{dt^2}$ (iii) $\left|\frac{d\vec{A}}{dt}\right|$ (iv) $\left|\frac{d^2\vec{A}}{dt^2}\right|$.

PROBLEM (6): If $\vec{A}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + 8\hat{k}$, show that $\frac{d\vec{A}}{dt}\cdot\frac{d^2\vec{A}}{dt^2} = 0$

PROBLEM (7): If $\vec{A} = t^3\hat{i} + \left(2t^3 - \frac{1}{5t^2}\right)\hat{j}$, then show that $\vec{A}\times\frac{d\vec{A}}{dt} = \hat{k}$.

PROBLEM (8): If a and b are constants and $\vec{A}(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j} + bt\hat{k}$, then show that

(i) $\left|\frac{d\vec{A}}{dt}\times\frac{d^2\vec{A}}{dt^2}\right| = a\sqrt{a^2+b^2}$ (ii) $\vec{A}\cdot\frac{d^2\vec{A}}{dt^2}\times\frac{d^3\vec{A}}{dt^3} = a^2bt$

PROBLEM (9): If $\vec{A} = t^2\hat{i} - t\hat{j} + (2t+1)\hat{k}$ and $\vec{B} = (2t-3)\hat{i} + \hat{j} - t\hat{k}$, find at $t = 1$

(i) $\frac{d}{dt}(\vec{A}\cdot\vec{B})$ (ii) $\frac{d}{dt}(\vec{A}\times\vec{B})$ (iii) $\frac{d}{dt}|\vec{A} + \vec{B}|$ (iv) $\frac{d}{dt}\left(\vec{A}\times\frac{d\vec{B}}{dt}\right)$

PROBLEM (10): Find $\frac{d}{dt}\left[\phi(t)\vec{A}(t)\right]$ if

if $\phi(t) = 2t+3$, $\vec{A}(t) = 3t\hat{i} + t^4\hat{j} - t\hat{k}$

PROBLEM (11): If $\vec{A} = \sin t\,\hat{i} + \cos t\,\hat{j} + t\hat{k}$, $\vec{B} = \cos t\,\hat{i} - \sin t\,\hat{j} - 3\hat{k}$, $\vec{C} = 2\hat{i} + 3\hat{j} - \hat{k}$,

find $\frac{d}{dt}\left[\vec{A}\times(\vec{B}\times\vec{C})\right]$ at $t = 0$.

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PROBLEM (12): If $\vec{A}(t) = 3t^2\hat{i} - (t+4)\hat{j} + (t^2-2t)\hat{k}$ and $\vec{B}(t) = \sin t\,\hat{i} + 3e^{-t}\hat{j} - 3\cos t\,\hat{k}$, find $\frac{d^2}{dt^2}(\vec{A}\times\vec{B})$ at $t = 0$.

PROBLEM (13): Find (i) $\frac{d}{dt}\left[\vec{r}\times\left(\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right)\right]$ (ii) $\frac{d^2}{dt^2}\left[\vec{r}\times\left(\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right)\right]$

PROBLEM (14): Find the space curve represented by the vector function :

$\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + \sin t\,\hat{k}, \quad 0 \le t \le 2\pi$

PROBLEM (15): Find the velocity and acceleration of a particle which moves along the curve $x = 2\sin 3t$, $y = 2\cos 3t$, $z = 8t$ at any time $t > 0$.
Find the magnitude of the velocity and acceleration .

PROBLEM (16): A particle moves along the curve $x = 4\cos t$, $y = 4\sin t$, $z = 6t$, where t is the time . Find the velocity and acceleration at time $t = 0$ and $t = \pi/2$.

PROBLEM (17): A particle moves along the curve $x = t^3+1$, $y = t^2$, $z = 2t+5$, where t is the time . Find the components of its velocity $\vec{V}$ and acceleration $\vec{a}$ at time $t = 1$ in the direction of $\hat{i}+\hat{j}+3\hat{k}$.

PROBLEM (18): Find the unit tangent vectors $\hat{T}$ to the curves represented by the following vector functions at the specified points :

(i) $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\hat{k}$, at $t = \pi$

(ii) $\vec{r}(t) = \left(t - \frac{t^3}{3}\right)\hat{i} + t^2\hat{j} + \left(t + \frac{t^3}{3}\right)\hat{k}$, at $t = 1$

(iii) $\vec{r}(t) = a(t - \sin t)\,\hat{i} + a(1 - \cos t)\,\hat{j}$, at any point .

PROBLEM (19): Find a unit tangent vector to any point on the curve

$x = a\cos wt$, $y = a\sin wt$, $z = bt$ where a , b , and w are constants .

PROBLEM (20): Find the angle between the tangents to the curve

$\vec{r} = t^2\hat{i} + 2t\hat{j} - t^3\hat{k}$ at the points $t = \pm 1$

PROBLEM (21): If $\vec{A} = \cos xy\,\hat{i} + (3xy - 2x^2)\hat{j} - (3x+2y)\hat{k}$, find

$\frac{\partial\vec{A}}{\partial x}$, $\frac{\partial\vec{A}}{\partial y}$, $\frac{\partial^2\vec{A}}{\partial x^2}$, $\frac{\partial^2\vec{A}}{\partial y^2}$, $\frac{\partial^2\vec{A}}{\partial x\partial y}$, $\frac{\partial^2\vec{A}}{\partial y\partial x}$

PROBLEM (22): If $\vec{A}(x,y,z) = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$, show that

$\frac{\partial\vec{A}}{\partial x}$, $\frac{\partial\vec{A}}{\partial y}$, and $\frac{\partial\vec{A}}{\partial z}$ are perpendicular to one another .

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PROBLEM (23): If $\vec{A} = uvw\hat{i} + uw^2\hat{j} - v^3\hat{k}$ and $\vec{B} = u^3\hat{i} - uvw\hat{j} - u^2w\hat{k}$, calculate

(i) $\dfrac{\partial^2 \vec{A}}{\partial u \partial v}$ at the origin (ii) $\dfrac{\partial^2 \vec{A}}{\partial v^2} \times \dfrac{\partial^2 \vec{B}}{\partial u^2}$ at the point $(1, 1, 0)$

PROBLEM (24) If $\vec{A} = x^2yz\hat{i} - 2xz^3\hat{j} + xz^2\hat{k}$ and $\vec{B} = 2z\hat{i} + y\hat{j} - x^2\hat{k}$, find $\dfrac{\partial^2}{\partial x \partial y}(\vec{A} \times \vec{B})$ at $(1, 0, -2)$.

PROBLEM (25): Find a unit normal vector $\hat{n}$ to each of the following surface:

(i) $\vec{r}(u, v) = a\cos u\,\hat{i} + a\sin u\,\hat{j} + v\hat{k}$

(ii) $\vec{r}(u, v) = u\cos v\,\hat{i} + u\sin v\,\hat{j} + v\hat{k}$

(iii) $\vec{r}(u, v) = u\cos v\,\hat{i} + u\sin v\,\hat{j} + u^2\hat{k}$

(iv) $\vec{r}(u, v) = a\cos u\cos v\,\hat{i} + a\sin u\cos v\,\hat{j} + a\sin v\,\hat{k}$

PROBLEM (26): Find a unit normal vector $\hat{n}$ at $(0, 0, 0)$ for the surface S represented by $z = 3x^2 + 4y^2$.

PROBLEM (27): Find a unit normal vector $\hat{n}$ at $(1, 1, 1)$ for the surface S represented by

(i) $x^2 + y^2 - z - 1 = 0$ (ii) $x^3 + xyz + 2y^3 - z^3 - 5 = 0$

PROBLEM (28): If $\vec{R}(t) = (3t^2 - t)\hat{i} + (2 - 6t)\hat{j} - 4t\hat{k}$, find

(i) $\int \vec{R}(t)\,dt$ (ii) $\int_2^4 \vec{R}(t)\,dt$

PROBLEM (29): Find

(i) $\int [\cos t\,\hat{i} + (t\sec^2 t + \tan t)\hat{j} - \sin t\,\hat{k}]\,dt$

(ii) $\int_0^{\pi/2} (3\sin t\,\hat{i} + 2\cos t\,\hat{j})\,dt$

PROBLEM (30): If $\vec{A}(t) = t\hat{i} - t^2\hat{j} + (t-1)\hat{k}$ and $\vec{B}(t) = 2t^2\hat{i} + 6t\vec{k}$, evaluate

(i) $\int_0^2 \vec{A} \cdot \vec{B}\,dt$ (ii) $\int_0^2 \vec{A} \times \vec{B}\,dt$.

PROBLEM (31): Let $\vec{A} = t\hat{i} - 3\hat{j} + 2t\hat{k}$, $\vec{B} = \hat{i} - 2\hat{j} + 2\hat{k}$, $\vec{C} = 3\hat{i} + t\hat{j} - \hat{k}$, evaluate

(i) $\int_1^2 \vec{A} \cdot \vec{B} \times \vec{C}\,dt$ (ii) $\int_1^2 \vec{A} \times (\vec{B} \times \vec{C})\,dt$

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PROBLEM (32): Evaluate $\int_0^7 \vec{r} \cdot \frac{d\vec{r}}{dt}\, dt$ if $\vec{r}(0) = 5\hat{i} - 3\hat{j} + 2\hat{k}$ and $\vec{r}(7) = \hat{i} + 8\hat{j} + 9\hat{k}$.

PROBLEM (33): Solve the differential equation $\frac{d^2\vec{r}}{dt^2} = \vec{A}t + \vec{B}$, where $\vec{A}$ and $\vec{B}$ are constant vectors. It is given that both $\vec{r}$ and $\frac{d\vec{r}}{dt}$ are zero when $t = 0$.

PROBLEM (34): If $\frac{d^2\vec{A}}{dt^2} = 6t\hat{i} - 24t^2\hat{j} + 4\sin t\,\hat{k}$, find $\vec{A}(t)$ given that when $t = 0$, $\vec{A} = 2\hat{i} + \hat{j}$ and $\frac{d\vec{A}}{dt} = -\hat{i} - 3\hat{k}$.

PROBLEM (35): The acceleration $\vec{a}$ of a particle at any time $t \geq 0$ is given by
$\vec{a} = e^{-t}\hat{i} - 6(t+1)\hat{j} + 3\sin t\,\hat{k}$.
If the velocity $\vec{V}$ and displacement $\vec{r}$ are zero at $t = 0$, find $\vec{V}$ and $\vec{r}$ at any time $t > 0$.

PROBLEM (36): The acceleration $\vec{a}$ of an object at any time t is given by $\vec{a} = -g\hat{j}$, where g is a constant. At $t = 0$ the velocity is given by $\vec{V} = V_0\cos\theta_0\,\hat{i} + V_0\sin\theta_0\,\hat{j}$ and the displacement $\vec{r} = \vec{0}$. Find $\vec{V}$ and $\vec{r}$ at any time $t > 0$.

PROBLEM (37): If $\vec{r} = \vec{C}_1 e^{wt} + \vec{C}_2 e^{-wt}$, where $\vec{C}_1$ and $\vec{C}_2$ are constant vectors, then show that $\frac{d^2\vec{r}}{dt^2} - w^2\vec{r} = 0$.

PROBLEM (38): If $\vec{r} = \vec{A}\sin wt + \vec{B}\cos wt + \frac{\vec{C}t}{w^2}\sin wt$ where $\vec{A}$, $\vec{B}$, and $\vec{C}$ are constant vectors and w is a scalar constant, show that

$$\frac{d^2\vec{r}}{dt^2} + w^2\vec{r} = \frac{2\vec{C}}{w}\cos wt$$

PROBLEM (39): Solve the following differential equations:

(i) $\frac{d^2\vec{r}}{dt^2} - 4\frac{d\vec{r}}{dt} - 5\vec{r} = \vec{0}$

(ii) $\frac{d^2\vec{r}}{dt^2} + 2\frac{d\vec{r}}{dt} + \vec{r} = \vec{0}$

(iii) $\frac{d^2\vec{r}}{dt^2} + 4\vec{r} = \vec{0}$

(iv) $\frac{d^2\vec{r}}{dt^2} + 2\frac{d\vec{r}}{dt} + 5\vec{r} = \vec{0}$

PROBLEM (40): If $\vec{C}_1$ and $\vec{C}_2$ are constant vectors and λ is a constant scalar, show that $\vec{H} = e^{-\lambda x}(\vec{C}_1\sin\lambda y + \vec{C}_2\cos\lambda y)$ satisfies the partial differential equation

$$\frac{\partial^2\vec{H}}{\partial x^2} + \frac{\partial^2\vec{H}}{\partial y^2} = \vec{0}$$

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