Linear Transformations

In this chapter, we learn about functions that map a vector space V into a vector space W. This type of functions is denoted by

$$T : V \longrightarrow W$$
$$T(\vec{v}) = \vec{w}$$

V = domain of T , W = Codomain of T

$\vec{w}$ = image of v under T.

Range = Set of all images of vectors in V.

Preimage = Set of all $\vec{v}$ in V such that $T(\vec{v}) = \vec{w}$.

<u>Example ①</u> For any vector $\vec{v} = (x_1, x_2)$ in R^2, let

$T : R^2 \longrightarrow R^2$ defined by

$$T(x_1, x_2) = (x_1 - x_2 , x_1 + 2x_2)$$

(a) Find the image of $\vec{v} = (-1, 2)$

(b) Find the preimage of $\vec{w} = (-1, 11)$

(a) $T(x_1, x_2) = (x_1 - x_2 , x_1 + 2x_2)$

$T(-1, 2) = (-1-2, -1 + 2(2)) = (-3, 3)$

(b) $T(x_1, x_2) = (-1, 11)$

$(x_1 - x_2 , x_1 + 2x_2) = (-1, 11)$

$x_1 - x_2 = -1$
$x_1 + 2x_2 = 11$ $\Rightarrow$ $x_1 = 3$, $x_2 = 4$

$\vec{v} = (3, 4)$

<u>Linear Transformation</u> Let V and W be vector spaces. The function

$$T : V \longrightarrow W$$

is called a linear transformation of V into W if the following two properties are true for all $\vec{u}$ and $\vec{v}$ in V and for any scalar k

1) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$
2) $T(k\vec{u}) = kT(\vec{u})$

<u>Example</u> ② show that the linear transformation

$$T : R^2 \longrightarrow R^2 \text{ defined by}$$

$$T(x_1, x_2) = (x_1 - x_2, x_1 + 2x_2)$$

is linear.

Let $\vec{u} = (x_1, x_2)$ $\quad \vec{v} = (y_1 + y_2)$

$$\vec{u} + \vec{v} = (x_1 + y_1, x_2 + y_2)$$

$$k\vec{u} = (kx_1, ky_1)$$

1) $T(\vec{u}+\vec{v}) = T(x_1+y_1, x_2+y_2)$

$$= (x_1 + y_1 - (x_2 + y_2), x_1 + y_1 + 2(x_2 + y_2))$$

$$= (x_1 + y_1 - x_2 - y_2, x_1 + y_1 + 2x_2 + 2y_2)$$

$$= (x_1 - x_2 + (y_1 - y_2), x_1 + 2x_2, y_1 + 2y_2)$$

$$= (x_1 - x_2, x_1 + 2x_2) + (y_1 - y_2, y_1 + 2y_2)$$

$$= T(\vec{u}) + T(\vec{v})$$

2) $T(k\vec{u}) = T(kx_1, kx_2)$

$= (kx_1 - kx_2, \; kx_1 + 2(kx_2))$

$= (k(x_1 - x_2), \; k(x_1 + 2x_2))$

$= k(x_1 - x_2, \; x_1 + 2x_2)$

$= kT(\vec{u})$

Properties of Linear Transformations

Let T be a linear transformation from V into W, where $\vec{u}$ and $\vec{v}$ are in V. Then the following properties are true.

1) $T(\vec{0}) = \vec{0}$

2) $T(-\vec{v}) = -T(\vec{v})$

3) $T(\vec{u} - \vec{v}) = T(\vec{u}) - T(\vec{v})$

4) If $\vec{v} = k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_n\vec{v}_n$

then $T(\vec{v}) = k_1T(\vec{v}_1) + k_2T(\vec{v}_2) + \cdots + k_nT(\vec{v}_n)$

Example ③ Consider the basis $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ for R^3, where $\vec{v}_1 = (1,1,1)$, $\vec{v}_2 = (1,1,0)$, $\vec{v}_3 = (1,0,0)$.

Let $T: R^3 \longrightarrow R^2$ be the linear transformation for which

$T(\vec{v}_1) = (1,0) \quad T(\vec{v}_2) = (2,-1) \quad T(\vec{v}_3) = (4,3)$.

Find a formula for $T(x_1, x_2, x_3)$ and then use that formula to compute $T(2,-3,5)$.

Let $\vec{v} = (x_1, x_2, x_3)$. We express $\vec{v}$ as a linear combination of $\vec{v_1}, \vec{v_2}, \vec{v_3}$.

$$\vec{v} = k_1\vec{v_1} + k_2\vec{v_2} + k_3\vec{v_3}$$

$$(x_1, x_2, x_3) = k_1(1,1,1) + k_2(1,1,0) + k_3(1,0,0)$$

$$x_1 = k_1 + k_2 + k_3$$

$$x_2 = k_1 + k_2$$

$$x_3 = k_1 \Rightarrow \boxed{k_1 = x_3}$$

$$x_2 = x_3 + k_2 \Rightarrow \boxed{k_2 = x_2 - x_3}$$

$$k_3 = \boxed{x_1 - x_2}$$

$$(x_1, x_2, x_3) = x_3(1,1,1) + (x_2 - x_3)(1,1,0) + (x_1 - x_2)(1,0,0)$$

$$\vec{v} = x_3\vec{v_1} + (x_2 - x_3)\vec{v_2} + (x_1 - x_2)\vec{v_3}$$

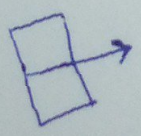
$$T(\vec{v}) = x_3 T(\vec{v_1}) + (x_2 - x_3)T(\vec{v_2}) + (x_1 - x_2)T(\vec{v_3})$$

$$T(x_1, x_2, x_3) = x_3(1,0) + (x_2 - x_3)(2,-1) + (x_1 - x_2)(4,3)$$

$$= (x_3, 0) + (2x_2 - 2x_3, -x_2 + x_3) + (4x_1 - 4x_2, 3x_1 - 3x_2)$$

$$= (4x_1 - 2x_2 - x_3, 3x_1 - 4x_2 + x_3)$$

$$T(2,-3,5) = (9, 23)$$



<u>Example 4</u> The function $T: R^2 \longrightarrow R^3$ is defined as follows

$$T(\vec{v}) = A\vec{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(a) Find $T(\vec{v})$ where $\vec{v} = (2, -1)$

(b) Show that T is a linear transformation from R^2 to R^3.

(a)
$$T(2,-1) = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 6+0 \\ 4-1 \\ -2+2 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$$

$$T(2,-1) = (6, 3, 0)$$

(b)

1) $T(\vec{u}+\vec{v}) = A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$

$= T(\vec{u}) + T(\vec{v})$

2) $T(k\vec{u}) = A(k\vec{u}) = kT(\vec{u})$.

The Kernel and Range of Linear Transformation

The Kernel of a Linear Transformation

Let $T: V \longrightarrow W$ be a linear transformation. Then the set of all vectors $\vec{v}$ in V that satisfy

$$T(\vec{v}) = \vec{0}$$

is called the Kernel of T and is denoted by $Ker(T)$.

$$Ker(T) = \{ \vec{v} : T(\vec{v}) = \vec{0}, \vec{v} \text{ in } V \}.$$

Example ⑤ Find the Kernel of the linear transformation

$T: R^2 \longrightarrow R^3$ represented by

$$T(x_1, x_2) = (x_1 - 2x_2, 0, -x_1)$$

To find $Ker(T)$, we need to find $\vec{v} = (x_1, x_2)$ such that $T(\vec{v}) = \vec{0}$

$$T(x_1, x_2) = (0, 0, 0)$$

$$(x_1 - 2x_2, 0, -x_1) = (0, 0, 0)$$

$$x_1 - 2x_2 = 0$$

$$0 = 0 \qquad \Rightarrow x_1 = 0$$

$$-x_1 = 0 \qquad x_2 = 0$$

$$Ker(T) = \{(0, 0)\}$$

Example 7

Find the kernel of the linear transformation $T: R^3 \longrightarrow R^2$ defined by

$$T(\vec{x}) = A\vec{x}$$

where

$$A = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix}$$

The kernel of T is set of all $\vec{v} = (x_1, x_2, x_3)$ such that

$$T(\vec{v}) = \vec{0}$$

$$T(x_1, x_2, x_3) = \vec{0}$$

$$\begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 - x_2 - 2x_3 = 0$$
$$-x_1 + 2x_2 + 3x_3 = 0$$

$$\begin{bmatrix} 1 & -1 & -2 & 0 \\ -1 & 2 & 3 & 0 \end{bmatrix}$$

The reduced echelon form is

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$x_1 = x_3$

$x_2 = -x_3$

$x_1 = t$

$x_2 = -t$

$x_3 = t$

$$\vec{v} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$Ker(T) = \{ t(1, -1, 1) : t \in \mathbb{R} \}.$$

<u>Theorem</u> The Kernel of a linear transformation

$$T : V \longrightarrow W$$

is a subspace of the domain V.

<u>Example 8</u> Let $T : R^5 \longrightarrow R^4$ be defined by

$$T(\vec{x}) = A(\vec{x})$$

where

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix}$$

Find a basis for $Ker(T)$ as a subspace of R^5.

$$T(\vec{v}) = \vec{0}$$

$$T(x_1, x_2, x_3, x_4, x_5) = \vec{0}$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 2 & 0 & 1 & -1 & 0 \\ 2 & 1 & 3 & 1 & 0 & 0 \\ -1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 8 & 0 \end{bmatrix}$$

The echelon form is

$$\begin{bmatrix} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -2x_3 + x_5$ $\qquad$ $x_3 = s$

$x_2 = x_3 + 2x_5$ $\qquad$ $x_5 = t$

$x_4 = -4x_5$

$$\vec{X} = \begin{bmatrix} -2s + t \\ s + 2s \\ s \\ -4t \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

A basis for Ker(T) is

$$S = \{(-2, 1, 1, 0, 0), (1, 2, 0, -4, 1)\}.$$

<u>Theorem</u> Let $T: R^n \longrightarrow R^m$ be the linear transformation given by $T(\vec{x}) = A\vec{x}$. Then $\ker(T)$ is equal to the solution space of $A\vec{x} = \vec{0}$.

The Range of a Linear Transformation

Let $T: V \longrightarrow W$ be a linear transformation. Then the set of all vectors $\vec{w}$ in W that are images of vectors in V is called range of T and is denoted by range (T)

$$\text{range}(T) = \{ T(\vec{v}) : \vec{v} \text{ is in } V \}.$$

<u>Theorem</u> The range of a linear transformation

$$T: V \longrightarrow W$$

is a subspace of W.

<u>Theorem</u> Let $T: R^n \longrightarrow R^m$ be the linear transformation given by $T(\vec{x}) = A\vec{x}$. Then the column space of A is equal to the range of T.

<u>Example (9)</u> Let $T: R^5 \longrightarrow R^4$ be defined by

$$T(\vec{x}) = A\vec{x}$$

where
$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix}$$

Find a basis for range (T) as a subspace of R^4.

$$T(\vec{x}) = A\vec{x}$$

$$T(x_1, x_2, x_3, x_4, x_5) = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

The echelon form of A is

$$\begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis for range (T) is

$$S = \{ (1, 2, -1, 0), (2, 1, 0, 0), (1, 1, 0, 2) \}.$$

<u>The Rank and Nullity of Linear Transformations</u>

Let $T: V \longrightarrow W$ be a linear transformation.

1) The dimension of the Kernel (T) is called the nullity of T and is denoted by nullity (T)

$$\text{nullity}(T) = \dim(\text{Ker}(T))$$

2) The dimension of the range of T is called the rank (T) and is denoted by rank (T).

$$\text{rank}(T) = \dim(\text{range}(T)).$$

Sum of Rank and Nullity

Let $T: V \longrightarrow W$ be a linear transformation from an n- dimensional vector space V into a vector space W.

Then

$$\text{rank}(T) + \text{nullity}(T) = n$$

or

$$\dim(\text{range}) + \dim(\text{Kernel}) = \dim(\text{domain})$$

Example ① Find the rank and nullity of the linear transformation $T: R^3 \longrightarrow R^3$ defined by the matrix

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ -1 & 0 & 2 \end{bmatrix}$$

The row echelon form of A is

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

rank $= 2$

$\text{rank}(T) = \dim(\text{range}(T)) = 2$

$\text{rank}(T) + \text{nullity}(T) = n$

$2 + \text{nullity}(T) = 3$

$\text{nullity}(T) = 3 - 2 = 1$

Example ① Let $T: R^5 \longrightarrow R^7$ be a linear transformation.

(a) Find the dimension of Ker (T) if dimension of range $(T) = 2$

(b) Find rank of T if nullity of T is 4

(c) Find rank of T if Ker $(T) = \{\vec{0}\}$.

(a) $n = 5$

$$\dim(\text{Kernel}) + \dim(\text{range}) = \dim(\text{domain})$$

$$\dim(\text{Kernel}) + 2 = 5$$

$$\dim(\text{Kernel}) = 3.$$

(b) $$\text{rank}(T) + \text{nullity}(T) = n$$

$$\text{rank}(T) + 4 = 5$$

$$\text{rank}(T) = 1$$

(c) $$\text{nullity}(T) = 0$$

$$\text{rank}(T) + \text{nullity}(T) = n$$

$$\text{rank}(T) + 0 = 5$$

$$\text{rank}(T) = 5$$

Exercise 8.1
Q1 — 31 (odd)