Spanning Sets If every vector in a vector space can be written as a linear combination of vectors in a set S, then S is a spanning set of the vector space.

Let $S = \{V_1, V_2, \ldots V_n\}$ be a subset of a vector space V. The set S is a spanning set of V if every vector in V can be written as a linear combination of vectors in S. In such cases it is said that S spans V. Span(S) or Span$\{V_1, V_2 \ldots V_n\}$

Example (14) (a) The set $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ spans R^3.

Every vector (u_1, u_2, u_3) in R^3 can be written as

$$(u_1, u_2, u_3) = u_1(1,0,0) + u_2(0,1,0) + u_3(0,0,1)$$

X (b) The set $S = \{1, x, x^2\}$ spans P_2.

Any polynomial of degree 2 $p(x) = a + bx + cx^2$ in P_2 can be written as

$$p(x) = a(1) + b(x) + c(x^2)$$

Example (15) Show that the set $S = \{(1,2,3), (0,1,2), (-2,0,1)\}$ spans R^3.

Let $\vec{u} = (u_1, u_2, u_3)$ be any vector in R^3.

$$(u_1, u_2, u_3) = k_1(1,2,3) + k_2(0,1,2) + k_3(-2,0,1)$$

$$= (k_1 - 2k_3,\ 2k_1 + k_2,\ 3k_1 + 2k_2 + k_3)$$

$$k_1 - 2k_3 = u_1$$
$$2k_1 + k_2 = u_2$$
$$3k_1 + 2k_2 + k_3 = u_3$$

The coefficient matrix of the system is

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}, \quad \det A = -1$$

The system has a unique solution. So any vector in R^3 can be written as a linear combination of the vectors in S. Therefore S pans R^3.

<u>Example</u> (16) Determine the vectors $\vec{v_1} = (1,1,2)$, $\vec{v_2} = (1,0,1)$ and $\vec{v_3} = (2,1,3)$ span the vector space R^3.

$$(u_1, u_2, u_3) = k_1(1,1,2) + k_2(1,0,1) + k_3(2,1,3)$$
$$= (k_1 + k_2 + 2k_3,\ k_1 + k_3,\ 2k_1 + k_2 + 3k_3)$$

$$k_1 + k_2 + k_3 = u_1$$
$$k_1 + k_3 = u_2$$
$$2k_1 + k_2 + 3k_3 = u_3$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

$\det A = 0$.

The vectors $\vec{v_1}$, $\vec{v_2}$ and $\vec{v_3}$ do not span R^3.

Theorem If $S = \{v_1, v_2, \ldots v_r\}$ is a set of vectors in a vector space V, then

$$\text{Span}(S) = \{k_1 v_1 + k_2 v_2 + \cdots + k_n v_n : k_1, k_2 \ldots k_n \text{ are real}\}$$

is a subspace of V.

Moreover, span(S) is the smallest subspace of V that contains S, in the sense that every other subspace of V that contains S must contain Span(S).

Solution Space of Homogeneous System

The solutions of a homogeneous linear system $AX = 0$ of m equations and n unknowns can be viewed as vectors in R^n.

Theorem The solution set of a homogenous linear system $AX = 0$ in n unknowns is a subspace of R^n.

Example (17) Consider the linear systems

(a)
$$x - 2y + 3z = 0$$
$$-3x + 7y - 8z = 0$$
$$4x + y + 2z = 0$$

(b)
$$x - 2y + 3z = 0$$
$$-3x + 7y - 8z = 0$$
$$-2x + 4y - 6z = 0$$

(c)

$$x - 2y + 3z = 0$$
$$2x - 4y + 6z = 0$$
$$3x - 6y + 9z = 0$$

(a)

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ -3 & 7 & -8 & 0 \\ 4 & 1 & 2 & 0 \end{bmatrix} \quad \begin{matrix} 3R_1 + R_2 \\ -4R_1 + R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 9 & -10 & 0 \end{bmatrix} \quad -9R_2 + R_3$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -19 & 0 \end{bmatrix} \quad -\frac{1}{19}R_3$$

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$x - 2y + 3z = 0 \Rightarrow x = 0$

$y + z = 0 \Rightarrow y = 0$

$z = 0$

The solution space is $\{0\}$.

(b)

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ -3 & 7 & -8 & 0 \\ -2 & 4 & -6 & 0 \end{bmatrix} \quad \begin{matrix} 3R_1 + R_2 \\ 2R_1 + R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x - 2y + 3z = 0 \qquad z = \text{free}$$
$$y + z = 0$$

let $z = t \quad y = -z = -t$

$$x = 2y - 3z = 2(-t) - 3t = -5t$$

$$x = -5t, \; y = -t, \; z = t$$

parametric equations for the line through origin and parallel to the vector $\vec{v} = (-5, -1, 1)$.

(c)

$$x - 2y + 3z = 0$$
$$2x - 4y + 6z = 0$$
$$3x - 6y + 9z = 0$$

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 2 & -4 & 6 & 0 \\ 3 & -6 & 9 & 0 \end{bmatrix} \quad \begin{matrix} -2R_1 + R_2 \\ -3R_1 + R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x - 2y + 3z = 0 \qquad y, z = \text{free}$

Let $y = s$, $z = t$ $\quad x = 2s - 3t$

This is the equation of plane through origin with normal vector $\vec{n} = (1, -2, 3)$.

$\boxed{\text{Exercise 4.2} \quad Q\ 1 - 20}$