Basis for a Vector Space

A set of vectors $S = \{v_1, v_2, \ldots v_r\}$ in a vector space V is a basis for V when the following conditions are true.

1) S spans V 2) S is linearly independent.

Example 1 Show that $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ is a basis for R^3.

1) S spans V. Let $(u_1, u_2, u_3) \in R^3$.

$$(u_1, u_2, u_3) = u_1(1,0,0) + u_2(0,1,0) + u_3(0,0,1)$$

2) S is linearly independent

$$k_1(1,0,0) + k_2(0,1,0) + k_3(0,0,1) = (0,0,0)$$

$$(k_1, k_2, k_3) = (0,0,0)$$

$$\Rightarrow k_1 = 0, \quad k_2 = 0, \quad k_3 = 0$$

S is a basis for R^3.

The basis $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ is the standard basis for R^3. The vectors

$$\hat{e}_1 = (1, 0, 0, \ldots, 0)$$
$$\hat{e}_2 = (0, 1, 0, \ldots, 0)$$
$$\vdots$$
$$\hat{e}_n = (0, 0, 0, \ldots, 1)$$

form the standard basis for R^n.

<u>Example 2</u> Show that the set $S = \{(1,1), (1,-1)\}$ is a basis for R^2.

1) <u>S spans V</u> Let $(u_1, u_2) \in R^2$.

$$(u_1, u_2) = k_1(1,1) + k_2(1,-1)$$
$$= (k_1 + k_2, k_1 - k_2)$$

$$k_1 + k_2 = u_1 \qquad A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
$$k_1 - k_2 = u_2$$

$\det A = -2$. The system has a unique solution.

2) <u>S is linearly independent</u>

$$k_1(1,1) + k_2(1,-1) = (0,0)$$

$$k_1 + k_2 = 0$$
$$k_1 - k_2 = 0 \quad \Rightarrow k_1 = 0, \; k_2 = 0$$

S is linearly independent. Therefore S is a basis.

<u>Example 3</u> $S = \{(1,2,3), (0,1,2), (-2,0,1)\}$

S spans R^3 and is linearly independent. So S is a basis for R^3.

Example (4) Show that the vector space P_3 has the basis.

$$S = \{1, x, x^2, x^3\}$$

1) S spans V

$$P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

2) S is linearly independent

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 = 0$$

$$a_0 = 0,\ a_1 = 0,\ a_2 = 0,\ a_3 = 0.$$

S is linearly independent. Therefore S is a basis for P_3.

$S = \{1, x, x^2, \ldots x^n\}$ is a basis for the vector space P_n of polynomials of degree n or less.

Example (5) The set

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

for a basis for the vector space $M_{2,2}$ of 2×2 matrices.

1) S spans V Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2,2}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

2) <u>S is linearly independent</u>

$$k_1\begin{bmatrix}1&0\\0&0\end{bmatrix}+k_2\begin{bmatrix}0&1\\0&0\end{bmatrix}+k_3\begin{bmatrix}0&0\\1&0\end{bmatrix}+k_4\begin{bmatrix}0&0\\0&1\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$\begin{bmatrix}k_1&k_2\\k_3&k_4\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$k_1=0,\ k_2=0,\ k_3=0,\ k_4=0.$

The set S is linearly independent. Therefore from the basis for $M_{2,2}$.

<u>Example 8</u> Show that the vectors $\vec{v_1}=(1,2,1)$, $\vec{v_2}=(2,9,0)$ $\vec{v_3}=(3,3,4)$ form basis for R^3.

1) <u>S spans V</u> Let $(u_1,u_2,u_3)\in R^3$.

$$(u_1,u_2,u_3)=k_1(1,2,1)+k_2(2,9,0)+k_3(3,3,4)$$
$$=(k_1+2k_2+3k_3,\ 2k_1+9k_2,\ 3k_3,\ k_1+4k_3)$$

$$\begin{aligned}k_1+2k_2+3k_3&=u_1\\2k_1+9k_2+3k_3&=u_2\\k_1\quad\quad+4k_3&=u_3\end{aligned}\qquad A=\begin{bmatrix}1&2&3\\2&9&3\\1&0&4\end{bmatrix}$$

$\det A=-1$. The system has unique solution.

2) <u>S is linearly independent</u>

$$k_1 + 2k_2 + 3k_3 = 0$$
$$2k_1 + 9k_2 + 3k_3 = 0$$
$$k_1 \quad + 4k_3 = 0$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 9 & 3 & 0 \\ 1 & 0 & 4 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$k_1 = 0, \quad k_2 = 0, \quad k_3 = 0.$

The vectors are linearly independent. Therefore the vectors $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ form a basis for R^3.

<u>Theorem</u> uniqueness of Basis Representation

If $S = \{v_1, v_2, \ldots v_r\}$ is a basis for a vector space V, then every vector in V can be written as a linearly combination of vectors in S in one and only one way.

<u>Coordinates of a Vector Relative to Basis</u>

If $S = \{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ is a basis for a vector space V and $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n$ is the expression for a vector $\vec{v}$ in terms of S, then the scalars $c_1, c_2 \ldots c_n$ are called coordinates of $\vec{v}$ relative to the basis S.

$$(\vec{v})_S = (c_1, c_2, \ldots c_n).$$

Example (7) (a) The vectors $\vec{v_1} = (1,2,1)$, $\vec{v_2} = (2,9,0)$, $\vec{v_3} = (3,3,4)$ form a basis for R^3. Find the coordinates of $\vec{v} = (5,-1,9)$ relative to the basis $S = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$.

(b) Find the vector $\vec{v}$ in R^3 whose coordinate vector relative to S is $(\vec{v})_S = (-1,3,2)$.

(a) To find $(\vec{v})_S$, we express $\vec{v}$ as a linear combination of the vectors in S.

$$\vec{v} = c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3}$$

$$(5,-1,9) = c_1(1,2,1) + c_2(2,9,0) + c_3(3,3,4)$$

$$= (c_1 + 2c_2 + 3c_3,\ 2c_1 + 9c_2 + 3c_3,\ c_1 + 4c_3)$$

$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= 5 \\ 2c_1 + 9c_2 + 3c_3 &= -1 \\ c_1 \qquad + 4c_3 &= 9 \end{aligned}$$

Solving this system, $c_1 = 1$, $c_2 = -1$, $c_3 = 2$.

$$(\vec{v})_S = (1,-1,2)$$

(b)

$$\vec{v} = c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3}$$

$$= (-1)(1,2,1) + 3(2,9,0) + 2(3,3,4)$$

$$= (11, 31, 7)$$

$\boxed{\text{Exercise 4.4 Q1-19}}$

Dimension of a Vector Space

If a vector space V has a basis consisting of n vectors, then n is called the dimension of V, denoted by

$$\dim(V) = n.$$

If V consists of zero vector alone, the dimension of V is defined as zero.

1) $\dim(R^n) = n$ The standard basis has n vectors
2) $\dim(P_n) = n+1$ The standard basis has $n+1$ vectors
3) $\dim(M_{m,n}) = mn$ The standard basis has mn vectors.

Example 8 Determine the dimension of each subspace of R^3

(a) $H = \{(d, c-d, c) : c, d \in \mathbb{R}\}$

(b) $H = \{(2b, b, 0) : b \in \mathbb{R}\}$.

(a) $(d, c-d, c) = (0, c, c) + (d, -d, 0)$

$$= c(0,1,1) + d(1,-1,0)$$

H is spanned by the set $S = \{(0,1,1), (1,-1,0)\}$.

The set S is linearly independent (verify). So its a basis for H.

$$\dim(H) = 2.$$

(b) $(2b, b, 0) = b(2, 1, 0)$

H is spanned by the set $S = \{(2, 1, 0)\}$.

$$\dim(H) = 1$$

<u>Theorem</u> Let V be an n-dimensional vector space, and let S be a set in V with exactly n vectors. Then S is a basis for V iff S spans V or S is linearly independent.

<u>Example 9</u> (a) By inspection, explain why $\vec{v_1} = (-3, 7)$ and $\vec{v_2} = (5, 5)$ form a basis for R^2.

(b) By inspection, explain why $\vec{v_1} = (2, 0, -1)$, $\vec{v_2} = (4, 0, 7)$ and $\vec{v_3} = (-1, 1, 4)$ form a basis for R^3.

(a) Since neither vector is a multiple of the other, the two vectors form a linearly independent S in R^2. Hence they form a basis.

(b) The vectors $\vec{v_1}$ and $\vec{v_2}$ form a linearly independent set in xz-plane. The vector $\vec{v_3}$ is outside of xz-plane. So the set $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is linearly independent. Hence they form a basis for R^3.

<u>Theorem</u> Let S be a set of vectors in a finite dimensional vector space V.

(a) If S spans V but is not a basis for V, then S can be reduced to a basis by removing appropriate vectors from S.

(b) If S is a linearly independent set that is not already a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.

<u>Theorem</u> If H is a subspace of a finite dimensional vector space V, then

(a) H is finite dimensional

(b) $\dim(H) \leq \dim(V)$

(c) $H = V$ iff $\dim(H) = \dim(V)$.

<u>Example 10</u> Find the dimension of the subspace H of R^4 spanned by

$$S = \{(-1, 2, 5, 0), (3, 0, 1, -2), (-5, 4, 9, 2)\}$$

Although H is spanned by the set S, S is not a basis for H because S is linearly dependent set.

$$\vec{v}_3 = 2\vec{v}_1 - \vec{v}_2$$

we discard $\vec{v}_3$ and we have set

$$S_1 = \{(-1, 2, 5, 0), (3, 0, 1, -2)\}$$

than spans H. Moreover S_1 is linearly independent because neither vector is a scalar multiple of the other.

$$\dim(H) = 2$$

Example (11) Find the dimension of the subspace

$$H = \{(a-3b+6c,\ 5a+4d,\ b-2c-d,\ 5d) : a,b,c,d \in \mathbb{R}\}$$

$$(a-3b+6c,\ 5a+4d,\ b-2c-d,\ 5d)$$

$$= (a, 5a, 0, 0) + (-3b, 0, b, 0) + (6c, 0, -2c, 0) + (0, 4d, -d, 5d)$$

$$= a(1,5,0,0) + b(-3,0,1,0) + c(6,0,-2,0) + d(0,4,-1,5)$$

H is spanned by the set S given by

$$S = \{(1,5,0,0), (-3,0,1,0), (6,0,-2,0), (0,4,-1,5)\}$$

The vector $\vec{v}_3$ is a multiple of $\vec{v}_2$. We may discard $\vec{v}_3$ and still have a set

$$S_1 = \{(1,5,0,0), (-3,0,1,0), (0,4,-1,5)\}$$

that spans H. The set S_1 is linearly independent.

$$\dim(H) = 3.$$

Example (12) Find a basis and dimension of the solution space of the homogeneous system.

$$2x_1 + 2x_2 - x_3 + x_5 = 0$$

$$-x_1 - x_2 + 2x_3 - 3x_4 + x_5 = 0$$

$$x_1 + x_2 - 2x_3 - x_5 = 0$$

$$x_3 + x_4 + x_5 = 0$$

$$\begin{bmatrix} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

The reduced-echelon form is

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 + x_2 + x_5 = 0$

$x_3 + x_5 = 0$

$x_4 = 0$

$x_2 = \text{free} = s$

$x_5 = \text{free} = t$

$x_1 = -s - t$

$x_2 = s$

$x_3 = -t$

$x_4 = 0$

$x_5 = t$

$(-s-t, s, -t, 0, t)$

$= (-s, s, 0, 0, 0) + (-t, 0, -t, 0, t)$

$= s(-1, 1, 0, 0, 0) + t(-1, 0, -1, 0, 1)$

The vectors $\vec{v_1} = (-1, 1, 0, 0, 0)$

$\vec{v_2} = (-1, 0, -1, 0, 1)$

spans solution space.

Since neither vector is a scalar multiple of other, they are linearly independent. Hence they form a basis for solution space.

dim = 2.

form the standard ...

Example 13 Find a basis and the dimension of the solution space of the homogenous system.

$$x_1 + 3x_2 - 2x_3 + 2x_5 = 0$$
$$2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = 0$$
$$5x_3 + 10x_4 + 15x_6 = 0$$
$$2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 0$$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 5 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 + 3x_2 + 4x_4 = 0$$
$$x_3 + 2x_4 = 0$$
$$x_6 = 0$$

$x_2 = \text{free} = r$

$x_4 = \text{free} = s$

$x_5 = \text{free} = t$

$$x_1 = -3r - 4s - 2t$$
$$x_2 = r$$
$$x_3 = -2s$$
$$x_4 = s$$
$$x_5 = t$$
$$x_6 = 0$$

$(-3r+4s-2t,\ r,\ -2s,\ t,\ 0)$

$= (-3r, r, 0, 0, 0) + (-4s, 0, -2s, 0, 0)$

$+ (-2t, 0, 0, t, 0)$

$= r(-3, 1, 0, 0, 0) + s(-4, 0, -2, 0, 0)$

$+ t(-2, 0, 0, 1, 0)$

The vectors $\vec{v}_1 = (-3, 1, 0, 0, 0)$, $\vec{v}_2 = (-4, 0, -2, 0, 0)$

$\vec{v}_3 = (-2, 0, 0, 1, 0)$ span solution space. The vectors are linearly independent (verify).

dim $= 3$.

Exercise 4.5
Q1-22