Dimension of a Vector Space

If a vector space V has a basis consisting of n vectors, then n is called the dimension of V, denoted by

$$\dim(V) = n.$$

If V consists of zero vector alone, the dimension of V is defined as zero.

1) $\dim(R^n) = n$ The standard basis has n vectors
2) $\dim(P_n) = n+1$ The standard basis has $n+1$ vectors
3) $\dim(M_{m,n}) = mn$ The standard basis has mn vectors.

Example 8 Determine the dimension of each subspace of R^3

(a) $H = \{(d, c-d, c) : c, d \in \mathbb{R}\}$

(b) $H = \{(2b, b, 0) : b \in \mathbb{R}\}$.

(a) $(d, c-d, c) = (0, c, c) + (d, -d, 0)$

$= c(0,1,1) + d(1,-1,0)$

H is spanned by the set $S = \{(0,1,1), (1,-1,0)\}$.

The set S is linearly independent (verify). So its a basis for H.

$\dim(H) = 2.$

(b) $(2b, b, 0) = b(2, 1, 0)$

H is spanned by the set $S = \{(2, 1, 0)\}$.

$$\dim(H) = 1$$

<u>Theorem</u> Let V be an n-dimensional vector space, and let S be a set in V with exactly n vectors. Then S is a basis for V iff S spans V or S is linearly independent.

<u>Example 9</u> (a) By inspection, explain why $\vec{v}_1 = (-3, 7)$ and $\vec{v}_2 = (5, 5)$ form a basis for R^2.

(b) By inspection, explain why $\vec{v}_1 = (2, 0, -1)$, $\vec{v}_2 = (4, 0, 7)$ and $\vec{v}_3 = (-1, 1, 4)$ form a basis for R^3.

(a) Since neither vector is a multiple of the other, the two vectors form a linearly independent S in R^2. Hence they form a basis.

(b) The vectors $\vec{v}_1$ and $\vec{v}_2$ form a linearly independent set in xz-plane. The vector $\vec{v}_3$ is outside of xz-plane. So the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent. Hence they form a basis for R^3.

<u>Theorem</u> Let S be a set of vectors in a finite dimensional vector space V.

(a) If S spans V but is not a basis for V, then S can be reduced to a basis by removing appropriate vectors from S.

(b) If S is a linearly independent set that is not already a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.

<u>Theorem</u> If H is a subspace of a finite dimensional vector space V, then

(a) H is finite dimensional

(b) $\dim(H) \leq \dim(V)$

(c) $H = V$ iff $\dim(H) = \dim(V)$.

<u>Example 10</u>. Find the dimension of the subspace H of R^4 spanned by

$$S = \{(-1, 2, 5, 0), (3, 0, 1, -2), (-5, 4, 9, 2)\}$$

Although H is spanned by the set S, S is not a basis for H because S is linearly dependent set.

$$\vec{v}_3 = 2\vec{v}_1 - \vec{v}_2$$

we discard $\vec{v}_3$ and we have set

$$S_1 = \{(-1, 2, 5, 0), (3, 0, 1, -2)\}$$

than spans H. Moreover S_1 is linearly independent because neither vector is a scalar multiple of the other.

$$\dim(H) = 2$$

Example (11) Find the dimension of the subspace

$$H = \{(a-3b+6c,\ 5a+4d,\ b-2c-d,\ 5d) : a,b,c,d \in \mathbb{R}\}$$

$$(a-3b+6c,\ 5a+4d,\ b-2c-d,\ 5d)$$

$$= (a, 5a, 0, 0) + (-3b, 0, b, 0) + (6c, 0, -2c, 0) + (0, 4d, -d, 5d)$$

$$= a(1,5,0,0) + b(-3,0,1,0) + c(6,0,-2,0) + d(0,4,-1,5)$$

H is spanned by the set S given by

$$S = \{(1,5,0,0), (-3,0,1,0), (6,0,-2,0), (0,4,-1,5)\}$$

The vector $\vec{v}_3$ is a multiple of $\vec{v}_2$. We may discard $\vec{v}_3$ and still have a set

$$S_1 = \{(1,5,0,0), (-3,0,1,0), (0,4,-1,5)\}$$

that spans H. The set S_1 is linearly independent.

$$\dim(H) = 3.$$

Example (12) Find a basis and dimension of the solution space of the homogeneous system.

$$2x_1 + 2x_2 - x_3 + x_5 = 0$$
$$-x_1 - x_2 + 2x_3 - 3x_4 + x_5 = 0$$
$$x_1 + x_2 - 2x_3 - x_5 = 0$$
$$x_3 + x_4 + x_5 = 0$$

$$\begin{bmatrix} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

The reduced-echelon form is

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 + x_2 + x_5 = 0$ $\qquad$ $x_2 = \text{free} = s$

$x_3 + x_5 = 0$ $\qquad$ $x_5 = \text{free} = t$

$x_4 = 0$

$x_1 = -s - t$ $\qquad$ $(-s-t, s, -t, 0, t)$

$x_2 = s$ $\qquad$ $= (-s, s, 0, 0, 0) + (-t, 0, -t, 0, t)$

$x_3 = -t$ $\qquad$ $= s(-1, 1, 0, 0, 0) + t(-1, 0, -1, 0, 1)$

$x_4 = 0$

$x_5 = t$ $\qquad$ The vectors $\vec{v_1} = (-1, 1, 0, 0, 0)$

$\vec{v_2} = (-1, 0, -1, 0, 1)$

spans solution space.

Since neither vector is a scalar multiple of other, they are linearly independent. Hence they form a basis for solution space.

dim = 2.

Example (13) Find a basis and the dimension of the solution space of the homogenous system.

$$x_1 + 3x_2 - 2x_3 + 2x_5 = 0$$
$$2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = 0$$
$$5x_3 + 10x_4 + 15x_6 = 0$$
$$2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 0$$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 5 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 + 3x_2 + 4x_4 = 0 \qquad x_2 = \text{free} = r$$
$$x_3 + 2x_4 = 0 \qquad x_4 = \text{free} = s$$
$$x_6 = 0 \qquad x_5 = \text{free} = t$$

$$x_1 = -3r - 4s - 2t$$
$$x_2 = r$$
$$x_3 = -2s$$
$$x_4 = s$$
$$x_5 = t$$
$$x_6 = 0$$

$(-3r+4s-2t, r, -2s, t, 0)$

$= (-3r, r, 0, 0, 0) + (-4s, 0, -2s, 0, 0)$

$+ (-2t, 0, 0, t, 0)$

$= r(-3, 1, 0, 0, 0) + s(-4, 0, -2, 0, 0)$

$+ t(-2, 0, 0, 1, 0)$

The vectors $\vec{v}_1 = (-3, 1, 0, 0, 0)$, $\vec{v}_2 = (-4, 0, -2, 0, 0)$

$\vec{v}_3 = (-2, 0, 0, 1, 0)$ span solution space. The vectors are linearly independent (verify).

dim = 3.

$\boxed{\text{Exercise 4.5 Q1-22}}$

Row Space, Column Space and Null Space

For an $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

the vectors $\vec{r_1} = [a_{11} \; a_{12} \; \cdots \; a_{1n}]$

$\vec{r_2} = [a_{21} \; a_{22} \; \cdots \; a_{2n}]$

$\vdots$

$\vec{r_m} = [a_{m1} \; a_{m2} \; \cdots \; a_{mn}]$

in R^n that are formed from the rows of A are called row vectors of A, and the vectors

$$\vec{c_1} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \vec{c_2} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \quad \cdots, \quad \vec{c_n} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

in R^m formed from columns of A are called the column vectors of A.

<u>Example</u>

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}$$

The row vectors of A: $\vec{r_1} = [2 \; 1 \; 0]$, $\vec{r_2} = [3 \; -1 \; 4]$

The column vectors of A: $\vec{c_1} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\vec{c_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\vec{c_3} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

Let A be an $m \times n$ matrix.

1) The subspace of R^n spanned by the row vectors of A is called the row space of A.

2) The subspace of R^m spanned by the column vectors of A is called the column space of A.

3) The solution space of the homogeneous system of equations

$$AX = 0$$

which is a subspace of R^n, is called null space of A.

<u>Theorem</u> If an $m \times n$ matrix A is row equivalent to an $m \times n$ matrix B, then the row space of A is equal to the row space of B.

<u>Theorem</u> If a matrix A is row equivalent to a matrix B in row echelon form, then

1) the row vectors with leading 1's (the non-zero vectors) form a basis for the row space of B.

2) the column ~~space~~ vectors with the leading 1's of the row vectors form a basis for the column space of B.

<u>Example</u> 1

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix}$$

The row echelon form of the matrix A is

$$B = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The vectors $\vec{r}_1 = [1 \quad 3 \quad 1 \quad 3]$, $\vec{r}_2 = [0 \quad 1 \quad 1 \quad 0]$

$\vec{r}_3 = [0 \quad 0 \quad 0 \quad 1]$ form a basis for row space of B

and hence form a basis for row space of A.

The vectors

$$\vec{c}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \vec{c}_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \vec{c}_4 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

form a basis for the column space of B

<u>Example ②</u> Find a basis for the row space of the matrix

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

The row echelon form of matrix A is

$$B = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The non-zero row vectors of B form a basis for the row space of B and hence form a basis for the row space of A.

The basis for vectors are.

$$\vec{r_1} = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \end{bmatrix}$$

$$\vec{r_2} = \begin{bmatrix} 0 & 0 & 1 & 3 & -2 & -6 \end{bmatrix}$$

$$\vec{r_3} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

<u>Example ③</u> Find a basis for the column space of the matrix

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The vectors

$$\vec{c}_1' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \vec{c}_3' = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad c_5' = \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for the column space of B. The corresponding vectors (column) of A are

$$\vec{c}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} \quad \vec{c}_3 = \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix} \quad \vec{c}_5 = \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix}$$

form a basis for the column space of A.

<u>Example 4</u> Find the basis for the row space of

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

Consisting entirely of row vectors from A.

Take transpose of A and reduced the matrix A^T in row-echelon form.

$$A^T = \begin{bmatrix} 1 & 2 & 0 & 2 \\ -2 & -5 & 5 & 6 \\ 0 & -3 & 15 & 18 \\ 0 & -2 & 10 & 8 \\ 3 & 6 & 0 & 6 \end{bmatrix}$$

The row-echelon form is

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -5 & -10 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The column vectors of A^T that form a basis for the column space of A^T are

$$\vec{c_1} = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 3 \end{bmatrix} \quad \vec{c_2} = \begin{bmatrix} 2 \\ -5 \\ -3 \\ -2 \\ 6 \end{bmatrix} \quad \vec{c_4} = \begin{bmatrix} 2 \\ 6 \\ 18 \\ 8 \\ 6 \end{bmatrix}$$

Transposing again, the basis vectors for the row space of A

$$\vec{r_1} = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \end{bmatrix}$$

$$\vec{r_2} = \begin{bmatrix} 2 & -5 & -3 & -2 & 6 \end{bmatrix}$$

$$\vec{r_4} = \begin{bmatrix} 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

Example 5 Find the basis for the row space of

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix}$$

consisting entirely of row vectors from A.

Take transpose of A and reduce the matrix A^T in row-echelon form.

$$A^T = \begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 3 & 1 & 0 & 4 & 0 \\ 1 & 1 & 6 & -2 & -4 \\ 3 & 0 & -1 & 1 & -2 \end{bmatrix}$$

The row-echelon form of A^T is

$$\begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 0 & 1 & 9 & -5 & -6 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\vec{c}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 3 \end{bmatrix} \quad \vec{c}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \vec{c}_3 = \begin{bmatrix} -3 \\ 0 \\ 6 \\ -1 \end{bmatrix}$$

form a basis for column space of A^T.

Transposing again, the basis vectors for row space of A are

$$\vec{r}_1 = \begin{bmatrix} 1 & 3 & 1 & 3 \end{bmatrix}, \quad \vec{r}_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\vec{r}_3 = \begin{bmatrix} -3 & 0 & 6 & 1 \end{bmatrix}$$

<u>Basis for a Vector Space</u> up to now, we have focused on methods for finding basis associated with matrices. These methods can be used to find a basis for the space spanned by a set of vectors in R^n.

<u>Example</u> 6 (a) Find a basis of R^3 spanned by

$$S = \{ (-1, 2, 5), (3, 0, 3), (5, 1, 8) \}$$

(b) Find a basis for the subspace of R^5 spanned by the vectors

$$\vec{v}_1 = (1, -2, 0, 0, 3)$$

$$\vec{v}_2 = (2, -5, -3, -2, 6)$$

$$\vec{v}_3 = (0, 5, 15, 10, 0)$$

$$\vec{v}_4 = (2, 6, 18, 8, 6)$$

(a) The vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ to form rows of a matrix A.

$$A = \begin{bmatrix} -1 & 2 & 5 \\ 3 & 0 & 3 \\ 5 & 1 & 8 \end{bmatrix}$$

The row-echelon form of A is

$$\begin{bmatrix} 1 & -2 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

The non-zero vectors

$$\vec{r}_1 = \begin{bmatrix} 1 & -2 & -5 \end{bmatrix} \qquad \vec{r}_2 = \begin{bmatrix} 0 & 1 & 3 \end{bmatrix}$$

form a basis for the row space of A. They form a basis for the subspace spanned by $S = \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$

(b) The ~~subspace~~ space spanned by these vectors is the row space of the matrix A

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

The row-echelon form of the matrix is

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The non-zero vectors in the matrix are

$$\vec{r_1} = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \end{bmatrix}$$

$$\vec{r_2} = \begin{bmatrix} 0 & 1 & 3 & 2 & 0 \end{bmatrix}$$

$$\vec{r_3} = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

These vectors form a basis for the row-space of A and consequently form a basis for the subspace of R^5 spanned by $\vec{v_1}, \vec{v_2}, \vec{v_3}$ and $\vec{v_4}$.

<u>Example 7</u> (a) Find a subset of the vectors

$\vec{v_1} = (1, -2, 0, 3)$, $\vec{v_2} = (2, -5, -3, 6)$

$\vec{v_3} = (0, 1, 3, 0)$ $\vec{v_4} = (2, -1, 4, -7)$

$\vec{v_5} = (5, -8, 1, 2)$

that form a basis for the space spanned by these vectors

(b) Express each vector, not in the basis, as a linear combination of the basis vectors.

(a) The matrix that has $\vec{v_1}, \vec{v_2}, \ldots \vec{v_5}$ as its column vectors.

$$A = \begin{array}{c} \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix} \\ \begin{matrix} v_1 & v_2 & v_3 & v_4 & v_5 \end{matrix} \end{array}$$

The row-echelon form of A is

$$B = \begin{array}{c} \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} c_1 & c_2 & c_3 & c_4 & c_5 \end{matrix} \end{array}$$

The leading 1's occure in column 1, 2 and 4. The vectors $\vec{c_1}$, $\vec{c_2}$ and $\vec{c_4}$ is a basis for the column space of B and consequently $\vec{v_1}$, $\vec{v_2}$ and $\vec{v_4}$ is a basis for the column space of A.

(b)

$$c_3 = 2c_1 - c_2$$

$$c_5 = c_1 + c_2 + c_4$$

$$\vec{v_1} = 2\vec{v_1} - \vec{v_2}$$

$$\vec{v_5} = \vec{v_1} + \vec{v_2} + \vec{v_4}.$$

Basis For Null Space of a Matrix

Consider a system of linear equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

$$\begin{array}{c} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \\ \vec{c}_1 \quad \vec{c}_2 \quad \quad \vec{c}_n \end{array} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$x_1 \vec{c}_1 + x_2 \vec{c}_2 + \cdots + x_n \vec{c}_n = \vec{b}$$

The system $A\vec{x} = \vec{b}$ is consistent if and only if $\vec{b}$ is expressible as a linear combination of the column vectors of A.

<u>Theorem</u> A system of linear equations $A\vec{x} = \vec{b}$ is consistent iff b is in the column space of A.

<u>Example 8</u> Let $A\vec{x} = \vec{b}$ be the linear system

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

Show that $\vec{b}$ is in the column space of A by expressing it as a linear combination of the column vectors of A.

Solving the system by Gaussian elimination

$$x_1 = 2 \qquad x_2 = -1, \quad x_3 = 3.$$

$$2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + (-1)\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

<u>Theorem</u> If $\vec{x}_0$ is any solution of a consistent linear system

$$A\vec{x} = \vec{b}$$

and if $S = \{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_r\}$ be a basis for the null space of A, then every solution $A\vec{x} = \vec{b}$ can be expressed in the form

$$\vec{x} = \vec{x}_0 + k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_r\vec{v}_r.$$

<u>Example (9)</u>

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 + 3x_2 + 4x_4 + 2x_5 = 0$$
$$x_3 + 2x_4 = 0$$
$$x_6 = 0$$

$x_2 = r$, $x_4 = s$, $x_5 = t$

$(-3r-4s-2t, r, -2s, s, t, 0)$

$x_1 = -3r - 4s - 2t$

$x_2 = r$

$x_3 = -2s$

$x_4 = s$

$x_5 = t$

$x_6 = 0$

$$= (-3r, r, 0, 0, 0, 0) + (-4s, 0, -2s, s, 0, 0) + (-2t, 0, 0, 0, t, 0)$$

$$= r(-3, 1, 0, 0, 0, 0) + s(-4, 0, -2, 1, 0, 0) + t(-2, 0, 0, 0, 1, 0)$$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \begin{bmatrix} x_4 \\ x_2 \\ x_3 \\ x_1 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 5 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 + 3x_2 + 4x_4 + 2x_5 = 0$$
$$x_3 + 2x_4 = 0$$
$$x_6 = 1/3$$

$x_2 = r, \quad x_4 = s, \quad x_5 = t$

$x_1 = -3r - 4s - 2t$

$x_2 = r$

$x_3 = -2s$

$x_4 = s$

$x_5 = t$

$x_6 = 1/3$

$$(-3r-4s-2t, r, -2s, s, t, 1/3)$$
$$= (0,0,0,0,0,1/3)$$
$$+ (-3r, r, 0, 0, 0, 0)$$
$$+ (-4s, 0, -2s, s, 0, 0)$$
$$+ (-2t, 0, 0, 0, t, 0)$$

$$= (0,0,0,0,0,1/3)$$
$$+ r(-3,1,0,0,0,0)$$
$$+ s(-4,0,-2,1,0,0)$$
$$+ t(-2,0,0,0,1,0)$$

$$\boxed{\text{Exercise 4.7 Q1-19}}$$

Theorem If A is an $m \times n$ matrix, Then the row space and column

Rank and Nullity

Rank of a Matrix The dimension of the row (or column) space of a matrix A is called rank of A and is denoted by

$$\text{rank}(A).$$

Example 1 Find the rank of the following matrices

(a)
$$A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ 0 & 1 & 3 & 5 \end{bmatrix}$$

(b)
$$A = \begin{bmatrix} 1 & +2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

(a) The row-echelon form of A is

$$\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\text{rank}(A) = 3.$$

Theorem If A is an $m \times n$ matrix, then the row space and column space of A have the same dimension.

(b) The reduced row-echelon form of the matrix A is

$$\begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A) = 2$

<u>Nullity of a Matrix</u> If A is an $m \times n$ matrix, then the set of all solutions of the homogeneous system of linear equations

$$A\vec{X} = \vec{0}$$

is a subspace of R^n, called the null space of A.
The dimension of the null space of A is called nullity of A and is denoted by

$$\text{nullity}(A).$$

<u>Example (2)</u> Find the nullity of following matrices

(a) $A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$

(a) The null space of A is the solution of the homogenous system $A\vec{X} = \vec{0}$

$$\begin{bmatrix} 1 & 2 & -2 & 1 & 0 \\ 3 & 6 & -5 & 4 & 0 \\ 1 & 2 & 0 & 3 & 0 \end{bmatrix}$$

The reduced row-echelon form of augmented matrix is

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 + 2x_2 + 3x_4 = 0$ $\qquad x_2 = s$

$x_3 + x_4 = 0$ $\qquad x_4 = t$

$x_1 = -2s - 3t, \quad x_2 = s, \quad x_3 = -t, \quad x_4 = t$

$(-2s-3t, s, -t, t)$

$= (-2s, s, 0, 0) + (-3t, 0, -t, t)$

$= s(-2, 1, 0, 0) + t(-3, 0, -1, 1)$

The basis for the null space of A is

$\{(-2, 1, 0, 0), (-3, 0, -1, 1)\}$

nullity $(A) = 2$.

(b).

$$\begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 & 0 \\ 3 & -7 & 2 & 0 & 1 & 4 & 0 \\ 2 & -5 & 2 & 4 & 6 & 1 & 0 \\ 4 & -9 & 2 & -4 & -4 & 7 & 0 \end{bmatrix}$$

The reduced row-echelon form of augmented matrix is

$$\begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 & 0 \\ 0 & 1 & -2 & -12 & -16 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - 4x_3 - 28x_4 - 37x_5 + 13x_6 = 0$$

$$x_2 - 2x_3 - 12x_4 - 16x_5 + 5x_6 = 0$$

$$x_3 = r, \quad x_4 = s, \quad x_5 = t, \quad x_6 = u$$

$$x_1 = 4r + 28s + 37t - 13u$$

$$x_2 = 2r + 12s + 16t - 5u$$

$$x_3 = r, \quad x_4 = s, \quad x_5 = t, \quad x_6 = u$$

$$(4r + 28s + 37t - 13u, 2r + 12s + 16t - 5u, r, s, t, u)$$

$$= r(4, 2, 1, 0, 0, 0) + s(28, 12, 0, 1, 0, 0)$$
$$+ t(37, 16, 0, 0, 1, 0) + u(-13, -5, 0, 0, 0, 1)$$

nullity $(A) = 4$.

Dimension of the Solution Space

Theorem If A is an $m \times n$ matrix, then

$$\text{rank}(A) + \text{nullity}(A) = n.$$

Example 3 Find the rank and nullity of matrix

$$A = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{bmatrix}$$

The reduced echelon form of A is

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$n = 5$ = no. of columns

$\text{rank}(A) = 3$

$\text{rank}(A) + \text{nullity}(A) = n$

$3 + \text{nullity}(A) = 5$

$\text{nullity}(A) = 5 - 3 = 2$

<u>Theorem</u> If A is an $m \times n$ matrix, then

(a) rank(A) = no. of leading variable in the general solution of $AX = 0$

(b) nullity(A) = no. of free variables (parameters) in the general solution of $AX = 0$

<u>Example 4</u> Find the number of parameters in the general solution of $AX = 0$ if A is a 5×7 matrix of rank 3.

$n = 7 \quad \text{rank}(A) = 3$

$$\text{rank}(A) + \text{nullity}(A) = n$$
$$3 + \text{nullity}(A) = 7$$
$$\text{nullity}(A) = 7 - 3 = 4$$

<u>Theorem</u> If $AX = b$ is a consistent linear system of m equations and n unknowns and A has rank r, then the general solution of the system consists of $n - r$ parameters.

<u>Example</u> what conditions must be satisfied by b_1, b_2, b_3 b_4 and b_5 for the overdetermined system

$$x_1 - 2x_2 = b_1$$
$$x_1 - x_2 = b_2$$
$$x_1 + x_2 = b_3$$
$$x_1 + 2x_2 = b_4$$
$$x_1 + 3x_2 = b_5$$ to be consistent.

$$\begin{bmatrix} 1 & -2 & b_1 \\ 1 & -1 & b_2 \\ 1 & 1 & b_3 \\ 1 & 2 & b_4 \\ 1 & 3 & b_5 \end{bmatrix}$$

The reduced row echelon form of the matrix is

$$\begin{bmatrix} 1 & 0 & 2b_2 - b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & b_3 - 3b_2 + 2b_1 \\ 0 & 0 & b_4 - 4b_2 + 3b_1 \\ 0 & 0 & b_5 - 5b_2 + 4b_1 \end{bmatrix}$$

The system is consistent iff $b_1, b_2, \ldots b_5$ satisfy the conditions

$$2b_1 - 3b_2 + b_3 = 0$$
$$3b_1 - 4b_2 + b_4 = 0$$
$$4b_1 - 5b_2 + b_5 = 0$$

$$\begin{bmatrix} 2 & -3 & 1 & 0 & 0 & 0 \\ 3 & -4 & 0 & 1 & 0 & 0 \\ 4 & -5 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -5 & 4 & 0 \\ 0 & 1 & 0 & -4 & 3 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{bmatrix}$$

$$b_1 - 5b_4 + 4b_5 = 0$$
$$b_2 - 4b_4 + 3b_5 = 0$$
$$b_3 - 2b_4 + b_5 = 0$$

$$b_1 = 5b_4 - 4b_5 \qquad b_4 = r$$
$$b_2 = 4b_4 - 3b_5 \qquad b_5 = s$$
$$b_3 = 2b_4 - b_5$$

$$b_1 = 5r - 4s$$
$$b_2 = 4r - 3s$$
$$b_3 = 2r - s$$
$$b_4 = r$$
$$b_5 = s.$$

<u>Theorem</u> If A is any matrix, then

$$\text{rank}(A) = \text{rank}(A^T)$$

$\boxed{\text{Exercise 4.8 Q1 – 20}}$