Eigenvalues and Eigenvectors

The eigen value problem is one of the most important problem in linear algebra.

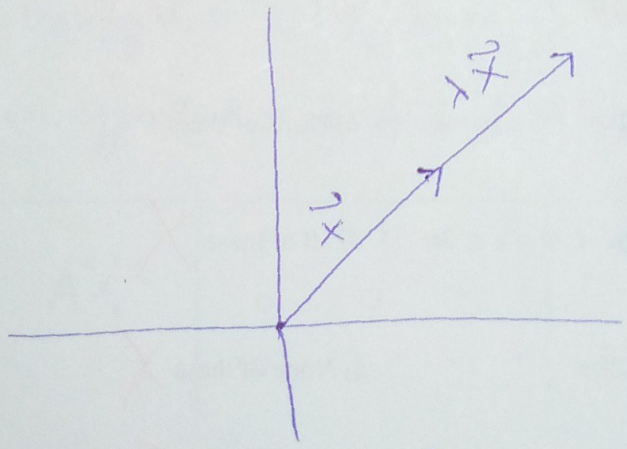
If A is an $n \times n$ matrix, do non zero vectors $\vec{x}$ in R^n exist such that $A\vec{x}$ is a scalar multiple of $\vec{x}$?

The scalar, denoted by the Greek letter λ, is called an eigenvalue of the matrix A and the non-zero vector $\vec{x}$ is called an eigenvector of A corresponding to λ.

<u>Definition</u> Let A be an $n \times n$ matrix. The scalar λ is called an eigenvalue of A if there is a non-zero vector $\vec{x}$ s.t

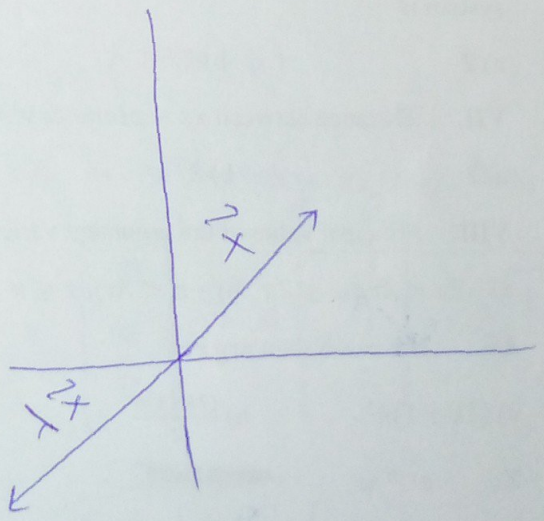
$$A\vec{x} = \lambda \vec{x}.$$

The vector $\vec{x}$ is called eigenvector of A corresponding to λ.



$A\vec{x} = \lambda \vec{x}$

$\lambda > 0$



$A\vec{x} = \lambda \vec{x}$

$\lambda < 0$

<u>Example 1</u> For the matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

verify that $\vec{X}_1 = (1, 0)$ is an eigenvector of A corresponding to eigen value $\lambda_1 = 2$ and that

$\vec{X}_2 = (0, 1)$ is an eigenvector of A corresponding to eigenvalue $\lambda_2 = -1$.

$$A\vec{X}_1 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda_1 \vec{X}_1$$

$$A\vec{X}_2 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -1\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \lambda_2 \vec{X}_2$$

<u>Example 2</u> For the matrix $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

verify that $\vec{X}_1 = (-3, -1, 1)$ and $\vec{X}_2 = (1, 0, 0)$ are eigenvectors of A and find their corresponding eigenvalues.

$$A\vec{X}_1 = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} = \lambda_1 \vec{X}_1$$

$\lambda_1 = 0$

$$A\vec{X}_2 = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \lambda_2 \vec{X}_2$$

$\lambda_2 = 1$

<u>Eigenspace</u> The set of all eigenvectors of a given value λ, together with zero vector, is a subspace of R^n. This subspace is called the eigenspace of λ.

<u>Theorem</u> If A is an $n \times n$ matrix with eigenvalue λ, then the set of all eigenvectors of λ, together with zero vector

$$\{\vec{0}\} \cup \{\vec{x} : \vec{x} \text{ is an eigenvector of } \lambda\},$$

is a subspace of R^n. This subspace is called the eigenspace of λ.

<u>Computing Eigenvalues and Eigenvectors</u>

To find the eigenvalues and eigenvectors of an $n \times n$ matrix A, let I be the $n \times n$ identity matrix.

$$A\vec{x} = \lambda\vec{x} \Rightarrow A\vec{x} = \lambda I\vec{x}$$

$$\Rightarrow (\lambda I - A)\vec{x} = \vec{0}.$$

This homogeneous system of equations has non-zero solutions if and only if the coefficient matrix $(\lambda I - A)$ is not invertible i.e, if and only if the determinant of $(\lambda I - A)$ is zero.

<u>Theorem</u> Let A be an $n \times n$ matrix.

1) An eigenvalue of A is a scalar λ such that

$$\det(\lambda I - A) = 0$$

This is called the characteristic equation of A.

2) The eigenvectors of A corresponding to λ are the non zero solutions of $(\lambda I - A)\vec{x} = \vec{0}$.

<u>Example 3</u> Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$$

The characteristic polynomial of A is

$$\det(\lambda I - A) = 0$$

$$\begin{vmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{vmatrix} = 0$$

$$(\lambda - 2)(\lambda + 5) - (-12) = 0$$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$(\lambda + 1)(\lambda + 2) = 0$$

$$\lambda_1 = -1 \qquad \lambda_2 = -2$$

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be eigenvector of the matrix A corresponding to eigenvalue λ.

$$(\lambda I - A)\vec{x} = \vec{0}$$

$$\begin{pmatrix} \lambda-2 & 12 \\ -1 & \lambda+5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For $\lambda_1 = -1$ $\begin{pmatrix} -3 & 12 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

The augmented matrix is

$$\begin{pmatrix} -3 & 12 & 0 \\ -1 & 4 & 0 \end{pmatrix}$$

The echelon form is

$$\begin{pmatrix} 1 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_1 - 4x_2 = 0, \quad x_2 = t$

$x_1 = 4t$

$\vec{x} = (4t, t) = t(4,1) = t\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \; t \neq 0.$

For $\lambda_2 = -2$ $\begin{pmatrix} -4 & 12 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

The augmented matrix is

$$\begin{pmatrix} -4 & 12 & 0 \\ -1 & 3 & 0 \end{pmatrix}$$

The echelon form is

$$\begin{pmatrix} 1 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_1 - 3x_2 = 0 \qquad x_2 = t$

$x_1 = 3t$

$\vec{x} = (3t, t) = t(3,1) = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \; t \neq 0$

<u>Example 4</u> Find the eigenvalues of

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$$

The characteristic polynomial of A is

$\det(\lambda I - A) = 0$

$$\begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda - 8 \end{vmatrix} = 0$$

$\lambda[\lambda(\lambda-8)+17] + 1(0-4) + 0 = 0$

$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0 \longrightarrow (1)$

The possible integer solutions of (1) are the divisors of -4

$\pm 1, \pm 2, \pm 3, \pm 4$

$\lambda = 4$ is a solution $\quad 4^3 - 8(4)^2 + 17(4) - 4 = 0$

$$(\lambda-4)(\lambda^2-4\lambda+1)=0$$

4	1	-8	17	-4
		4	-16	4
	1	-4	1	0

$$\lambda^2-4\lambda+1=0$$

$$\lambda=\frac{4\pm\sqrt{16-4(1)(1)}}{2(1)}$$

$$=\frac{4\pm\sqrt{12}}{2}=\frac{4\pm2\sqrt{3}}{2}=2\pm\sqrt{3}$$

$$\lambda_1=4,\ \lambda_2=2+\sqrt{3},\ \lambda_3=2-\sqrt{3}$$

<u>Example 5</u> Find the eigenvalues and corresponding eigenvectors of

$$A=\begin{bmatrix}2&1&0\\0&2&0\\0&0&2\end{bmatrix}$$

The characteristic polynomial is

$$\det(\lambda I-A)=0$$

$$\begin{vmatrix}\lambda-2&-1&0\\0&\lambda-2&0\\0&0&\lambda-2\end{vmatrix}=0$$

$$(\lambda-2)^3=0$$

$$\lambda=2$$

is the eigen value of the matrix A.

$$(\lambda I - A)\vec{X} = \vec{0}$$

$$\begin{pmatrix} \lambda-2 & -1 & 0 \\ 0 & \lambda-2 & 0 \\ 0 & 0 & \lambda-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

For $\lambda = 2$ $\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$x_1 = \text{free} = s$

$x_2 = 0$

$x_3 = \text{free} = t$

$$\vec{X} = (s, 0, t) = (s, 0, 0) + (0, 0, t)$$

$$= s(1, 0, 0) + t(0, 0, 1)$$

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are eigenvectors

of A corresponding to eigenvalue $\lambda = 2$.

Eigenvalues of an upper Triangular Matrix

The eigenvalues of the upper triangular matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

are $\lambda_1 = a_{11}$, $\lambda_2 = a_{22}$, $\lambda_3 = a_{33}$, $\lambda_4 = a_{44}$.

<u>Theorem</u> If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, diagonal), then the eigenvalues of A are the entries on the main diagonal of A.

<u>Example ①</u> Find the eigenvalues of the lower triangular matrix.

$$A = \begin{bmatrix} 1/2 & 0 & 0 \\ -1 & 2/3 & 0 \\ 5 & -8 & -1/4 \end{bmatrix}$$

$\lambda_1 = 1/2$, $\lambda_2 = 2/3$, $\lambda_3 = -1/4$.

<u>Finding Basis for Eigenspaces.</u> Let A be an $n \times n$ matrix. The eigenspace of A corresponding to the eigenvalue λ is the solution space of the homogeneous system

$$(\lambda I - A)\vec{x} = \vec{0}.$$

<u>Example 7</u> Find basis for the eigenspace of

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}.$$

The Characteristic equation is

$$\det(\lambda I - A) = 0$$

$$\begin{vmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{vmatrix} = 0$$

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$\lambda = 1$ is a root $\quad 1^3 - 5(1)^2 + 8(1) - 4 = 0$

1	1	-5	8	-4
		1	-4	4
	1	-4	4	0

$$(\lambda - 1)(\lambda^2 - 4\lambda + 4) = 0$$

$$(\lambda - 1)(\lambda - 2)^2 = 0$$

$$\lambda_1 = 1, \quad \lambda_2 = 2.$$

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigenvector of A corresponding to eigenvalue λ.

$$(\lambda I - A)\vec{x} = \vec{0}$$

$$\begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda_1 = 1$ $\begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Solving this system, $x_1 = -2s$, $x_2 = s$, $x_3 = s$

$$\vec{x} = \begin{bmatrix} -2s \\ s \\ s \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$B_1 = \{(-2, 1, 1)\}$ is a basis for eigenspace corresponding to $\lambda = 1$

For $\lambda_2 = 2$ $\begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Solving this system yields

$$x_1 = -s, \quad x_2 = t, \quad x_3 = s.$$

$$\vec{x} = \begin{bmatrix} -s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix} + \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix}$$

$$= s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$B_2 = \{(-1, 0, 1), (0, 1, 0)\}.$$

The vectors are linearly independent. These vectors form a basis for the eigenspace corresponding to $\lambda = 2$.

<u>Example 8</u> Find the eigenvalues of

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

and find a basis for each of the corresponding eigenspace.

$$\det(\lambda I - A) = 0$$

$$\begin{vmatrix} \lambda-1 & 0 & 0 & 0 \\ 0 & \lambda-1 & -5 & 10 \\ -1 & 0 & \lambda-2 & 0 \\ -1 & 0 & 0 & \lambda-3 \end{vmatrix} = 0$$

$$(\lambda-1)^2(\lambda-2)(\lambda-3)=0$$

$\lambda_1=1, \quad \lambda_2=2, \quad \lambda_3=3$

$$(\lambda I - A)\vec{x}=0.$$

$$\begin{bmatrix} \lambda-1 & 0 & 0 & 0 \\ 0 & \lambda-1 & -5 & 10 \\ -1 & 0 & \lambda-2 & 0 \\ -1 & 0 & 0 & \lambda-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda_1=1$ $\quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 10 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 10 & 0 \\ -1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1=-2t, \quad x_2=s, \quad x_3=2t, \quad x_4=t$

$$\vec{x} = \begin{bmatrix} -2t \\ s \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ s \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2t \\ 0 \\ 2t \\ t \end{bmatrix}$$

$$= s\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

$B_1 = \{(0,1,0,0), (-2,0,2,1)\}$ is a basis for the eigenspace corresponding to $\lambda_1 = 1$.

For $\lambda_2 = 2$ $\quad B_2 = \{(0,5,1,0)\}$ (verify)

For $\lambda_3 = 3$ $\quad B_3 = \{(0,-5,0,1)\}$. (verify)

Eigenvalues of powers of a Matrix

If λ is an eigenvalue of matrix A and $\vec{x}$ is a corresponding eigenvector, then

$$A^2\vec{x} = A(A\vec{x}) = A(\lambda\vec{x}) = \lambda(A\vec{x}) = \lambda(\lambda\vec{x})$$

$$A^2\vec{x} = \lambda^2\vec{x}.$$

This shows that λ^2 is an eigenvalue of A^2 and $\vec{x}$ is a corresponding eigenvectors.

<u>Theorem</u> If K is a positive integer, λ is an eigenvalue of a matrix A, and $\vec{x}$ is a corresponding eigenvector, then λ^K is an eigenvalue of A^K and $\vec{x}$ is a corresponding eigenvector.

<u>Example(9)</u> Given the matrix

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Find eigenvalues and eigenvectors of A^7.

$$\det(\lambda I - A) = 0$$

$$\begin{vmatrix} \lambda-1 & -3 & 0 \\ -3 & \lambda-1 & 0 \\ 0 & 0 & \lambda+2 \end{vmatrix} = 0$$

$$(\lambda-1)\left[(\lambda-1)(\lambda+2)\right] + 3\left[3(\lambda+2)\right] - 0 = 0$$

$$(\lambda+2)(\lambda-1)^2 - 9(\lambda+2) = 0$$

$$(\lambda+2)\left[(\lambda-1)^2 - 9\right] = 0$$

$\lambda_1 = -2$ $\quad$ $\lambda_2 = 4$ are eigenvalues of A

$(-2)^7$, $(4)^7$ are eigenvalues of A^7

$= -128$ $\quad$ $= 16384$

For $\lambda_1 = -2$, $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are

eigenvectors of A^T (verify).

For $\lambda_2 = 4$ $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector of A^T (verify)

Eigenvalues and Invertibility

A square matrix A is invertible iff $\lambda = 0$ is not an eigenvalue of A.

Example 10 The matrix $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

is invertible since it has $\lambda_1 = -2$, $\lambda_2 = 4$, neither of which is zero. (check $\det A \neq 0$).

$\boxed{\text{Exercise 5.1 Q1 – 29 odd}}$