Orthogonal Diagonalization

Orthogonal Matrices A square matrix A is said to be orthogonal if

$$A^{-1} = A^T$$

or equivalently $AA^T = A^TA = I$

Example 1) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is orthogonal

$$A^{-1} = A^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

2) $A = \begin{bmatrix} 3/5 & 0 & -4/5 \\ 0 & 1 & 0 \\ 4/5 & 0 & 3/5 \end{bmatrix}$ is orthogonal

$$A^{-1} = A^T = \begin{bmatrix} 3/5 & 0 & 4/5 \\ 0 & 1 & 0 \\ -4/5 & 0 & 3/5 \end{bmatrix}$$

Theorem An $n \times n$ matrix A is orthogonal if and only if its column vectors form an orthonormal set.

Example ① show that

$$A = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ -2/3\sqrt{5} & -4/3\sqrt{5} & 5/3\sqrt{5} \end{bmatrix}$$

is orthogonal. Prove that column vectors of A form an orthonormal set.

$$AA^T = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ -2/3\sqrt{5} & -4/3\sqrt{5} & 5/3\sqrt{5} \end{bmatrix} \begin{bmatrix} 1/3 & -2/\sqrt{5} & \frac{-2}{3\sqrt{5}} \\ 2/3 & 1/\sqrt{5} & -4/3\sqrt{5} \\ 2/3 & 0 & 5/3\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = I$$

$$\Rightarrow \quad A^{-1} = A^T.$$

Let $\vec{P_1} = \begin{bmatrix} 1/3 \\ -2/\sqrt{5} \\ -2/3\sqrt{5} \end{bmatrix}$, $\vec{P_2} = \begin{bmatrix} 2/3 \\ 1/\sqrt{5} \\ -4/3\sqrt{5} \end{bmatrix}$, $\vec{P_3} = \begin{bmatrix} 2/3 \\ 0 \\ 5/3\sqrt{5} \end{bmatrix}$

$$\vec{P_1} \cdot \vec{P_2} = \vec{P_1} \cdot \vec{P_3} = \vec{P_2} \cdot \vec{P_3} = 0$$

$$\|\vec{P_1}\| = \|\vec{P_2}\| = \|\vec{P_3}\| = 1$$

So $\{\vec{P_1}, \vec{P_2}, \vec{P_3}\}$ is an orthonormal set.

<u>Theorem</u> (a) The inverse of an orthogonal matrix is orthogonal

(b) A product of orthogonal matrices is orthogonal

(c) If A is orthogonal, then $\det(A) = \pm 1$

Example 2 show that the matrix

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$ is orthogonal.

Prove that $\det(A) = 1$

$$AA^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\Rightarrow A^{-1} = A^T$$

$$\det(A) = \begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix}$$

$$= \frac{1}{2} + \frac{1}{2}$$

$$= 1$$

$$\boxed{\begin{array}{c} \text{EXERCISE } 7.1 \\ Q1, 3, 13, 17 \end{array}}$$

<u>Symmetric Matrices</u> A square matrix A is symmetric if

$$A = A^T$$

<u>Example</u> 1) $A = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix}$ is symmetric.

$$A^T = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix} = A$$

2) $A = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 3 & 0 \\ -2 & 0 & 5 \end{bmatrix}$ is symmetric

$$A^T = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 3 & 0 \\ -2 & 0 & 5 \end{bmatrix} = A$$

<u>Theorem</u> If A is a symmetric matrix, then

(a) the eigenvalues of A are all real numbers

(b) eigenvectors from different eigenspaces are orthogonal.

<u>Example 3</u> Show that any two eigenvectors of

$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ corresponding to distinct

eigenvalues are orthogonal.

The characteristic polynomial of A is

$$\det(\lambda I - A) = 0$$

$$\begin{vmatrix} \lambda-3 & -1 \\ -1 & \lambda-3 \end{vmatrix} = 0$$

$$(\lambda-2)(\lambda-4) = 0$$

$$\lambda_1 = 2 \qquad \lambda_2 = 4.$$

The eigenvectors corresponding to $\lambda=2$ and $\lambda=4$ are

$$\vec{x}_1 = \begin{bmatrix} s \\ -s \end{bmatrix} \qquad \vec{x}_2 = \begin{bmatrix} t \\ t \end{bmatrix}$$

$$\vec{x}_1 \cdot \vec{x}_2 = \begin{bmatrix} s \\ -s \end{bmatrix} \cdot \begin{bmatrix} t \\ t \end{bmatrix} = st - st = 0.$$

<u>The orthogonal Diagonalization problem</u>

We know that two square matrices A and B are similar if there is an invertible matrix P such that

$$P^{-1}AP = B$$

Is it possible to find an orthogonal matrix P for which this relationship holds?

Orthogonally Similar If A and B are square matrices, then we say that A and B are orthogonally similar if there is an orthogonal matrix P such that

$$P^{-1}AP = B \qquad P^{-1} = P^T$$

$$P^T AP = B.$$

Orthogonally Diagonalizable Matrices A matrix A is orthogonally diagonalizable if there exists an orthogonal matrix P such that $P^T AP = D$ is diagonal.

We say that P orthogonally diagonalizes A.

Fundamental Theorem of Symmetric Matrices

Let A be an $n \times n$ matrix. Then A is orthogonally diagonalizable and has real eigenvalues if and only if A is symmetric.

Example 4 Which matrices are orthogonally diagonalizable.

$$A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 1 & 8 \\ -1 & 8 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$$

$\boxed{\begin{matrix} A_1 \\ A_4 \end{matrix}}$

Orthogonal Diagonalization of a Symmetric Matrix

Let A be an $n \times n$ symmetric matrix.

Step I Find a basis for each eigenspace of A.

Step II Apply the Gram-Schmidt process to each of these bases to obtain an orthonormal basis for each eigenspace.

Step III Form the matrix P whose columns are the vectors constructed in step II. This matrix will orthogonally diagonalizable A and the eigenvalues on the diagonal of $P^T A P = D$ will be in the same order as their corresponding eigenvectors in P.

Example 5 Find an orthogonal matrix P that orthogonally diagonalizes

$$A = \begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}$$

Step I The characteristic polynomial is

$$\det(\lambda I - A) = 0$$

$$\begin{vmatrix} \lambda + 2 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = 0$$

$$(\lambda + 3)(\lambda - 2) = 0$$

$\lambda_1 = -3$ $\qquad$ $\lambda_2 = 2$

The eigenvectors corresponding to λ are

$$(\lambda I - A)\vec{x} = \vec{0}$$

$$\begin{bmatrix} \lambda+2 & -2 \\ -2 & \lambda-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For $\lambda_1 = -3$ $\quad \begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

The echelon form of the matrix is

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \qquad x_1 + 2x_2 = 0 \qquad x_2 = t \qquad \vec{x} = \begin{bmatrix} -2t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$x_1 = -2t$$

$$\vec{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

For $\lambda_2 = 2$ $\quad \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

The echelon form of matrix is

$$\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} \qquad x_1 - \frac{1}{2}x_2 = 0 \qquad x_2 = t \qquad \vec{x} = \begin{bmatrix} \frac{1}{2}t \\ t \end{bmatrix}$$

$$x_1 = \frac{1}{2}t \qquad = \frac{1}{2}t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

<u>step II</u> $\quad \vec{p}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|}$

$$= \frac{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}{\sqrt{5}} = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$\vec{p}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \frac{\begin{bmatrix}1\\2\end{bmatrix}}{\sqrt{5}} = \begin{bmatrix}1/\sqrt{5}\\2/\sqrt{5}\end{bmatrix}$$

The vectors $\vec{p}_1$ and $\vec{p}_2$ are orthonormal.

Step ①

$$P = \begin{bmatrix}-2/\sqrt{5} & 1/\sqrt{5}\\1/\sqrt{5} & 2/\sqrt{5}\end{bmatrix}$$

$$P^{-1}AP = P^{T}AP = \begin{bmatrix}-2/\sqrt{5} & 1/\sqrt{5}\\1/\sqrt{5} & 2/\sqrt{5}\end{bmatrix}\begin{bmatrix}-2 & 2\\2 & 1\end{bmatrix}\begin{bmatrix}2/\sqrt{5} & 1/\sqrt{5}\\1/\sqrt{5} & 2/\sqrt{5}\end{bmatrix}$$

$$= \begin{bmatrix}-3 & 0\\0 & 2\end{bmatrix}$$

Example ⑥ Find an orthogonal matrix P that diagonalizes

$$A = \begin{bmatrix}2 & 2 & -2\\2 & -1 & 4\\-2 & 4 & -1\end{bmatrix}$$

Step 1 The characteristic polynomial is

$$\det(\lambda I - A) = 0$$

$$\begin{vmatrix}\lambda-2 & -2 & 2\\-2 & \lambda+1 & -4\\2 & -4 & \lambda+1\end{vmatrix} = 0$$

$$(\lambda-3)^2(\lambda+6) = 0$$

$$\lambda_1 = 3 \qquad \lambda_2 = -6$$

The eigenvectors corresponding to λ are

$$(\lambda I - A)\vec{x} = \vec{0}$$

$$\begin{bmatrix} \lambda-2 & -2 & 2 \\ -2 & \lambda+1 & -4 \\ 2 & -4 & \lambda+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda_1 = 3$ $\begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 2 & -4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The echelon form is $\begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 - 2x_2 + 2x_3 = 0 \qquad x_2 = t \qquad x_3 = s$

$x_1 = 2t - 2s$

$$\vec{x} = \begin{bmatrix} 2t-2s \\ t \\ s \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -2s \\ 0 \\ s \end{bmatrix}$$

$$= t\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{v_1} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \qquad \vec{v_2} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

for $\lambda_2 = -6$

$$\begin{bmatrix} -8 & -2 & 2 \\ -2 & -5 & -4 \\ 2 & -4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The echelon form is $\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 - \frac{1}{2} x_3 = 0 \qquad x_3 = t$

$x_2 + x_3 = 0$

$x_1 = \frac{1}{2} t \qquad x_2 = -t \qquad x_3 = t$

$$\vec{x} = \begin{bmatrix} \frac{1}{2}t \\ -t \\ t \end{bmatrix} = \frac{1}{2} t \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

$$\vec{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

<u>Step II</u> The orthonormalization of eigenvectors using Gram-Schmidt process

$\vec{v}_1 = (2, 1, 0) \qquad \vec{v}_2 = (-2, 0, 1)$

$\vec{w}_1 = \vec{v}_1 = (2, 1, 0)$

$$\vec{w}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{w}_1}{\|\vec{w}_1\|^2} \vec{w}_1 = \left(-\frac{2}{5}, \frac{4}{5}, 1 \right)$$

~~2 = pick~~

The vectors normalizes to

$$\vec{p}_1 = \frac{\vec{w}_1}{\|\vec{w}_1\|} = \frac{(2, 1, 0)}{\sqrt{5}} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$$

$$\vec{p}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|} = \frac{(-2/5, 4/5, 1)}{3/\sqrt{5}} = \begin{bmatrix} -2/3\sqrt{5} \\ 4/3\sqrt{5} \\ 1/3\sqrt{5} \end{bmatrix}$$

The vector $\vec{v}_3$ is normalized by

$$\vec{v}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|} = \frac{(1, -2, 2)}{3} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

<u>Step III</u> $P = \begin{bmatrix} 2/\sqrt{5} & -2/3\sqrt{5} & 1/3 \\ 1/\sqrt{5} & 4/3\sqrt{5} & -2/3 \\ 0 & 5/3\sqrt{5} & 2/3 \end{bmatrix}$

$$P^{-1}AP = P^{T}AP = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\boxed{\begin{array}{c} \text{Exercise } 7.2 \\ \text{Q1} - 15 \text{ (odd)} \end{array}}$$