Inner Product spaces

Length and Dot product in R^n

The length, or magnitude, or norm of a vector $\vec{V}$ given by

$$\vec{V} = (V_1, V_2, --- V_n) \text{ in } R^n$$

is $$\|\vec{V}\| = \sqrt{V_1^2 + V_2^2 + V_3^2 + --- + V_n^2}$$

If $\|\vec{V}\| = 1$, the vector is called unit vector.

Example (1) $\vec{V} = (2, -2, 3)$ in R^3

$$\|\vec{V}\| = \sqrt{2^2 + (-2)^2 + 3^2} = \sqrt{17}$$

Unit Vector If $\vec{V}$ is a non zero vector in R^n, then the vector

$$\vec{u} = \frac{\vec{V}}{\|\vec{V}\|}$$

has length 1 and has the same direction as $\vec{V}$. This vector $\vec{u}$ is called the unit vector in the direction of $\vec{V}$.

Example (2) $\vec{V} = (3, -1, 2)$

$$\frac{\vec{V}}{\|\vec{V}\|} = \frac{(3, -1, 2)}{\sqrt{3^2 + (-1)^2 + 2^2}} = \left(\frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{2}{\sqrt{14}}\right)$$

Dot product The dot product of $\vec{u} = (u_1, u_2, -- u_n)$ and $\vec{v} = (v_1, v_2, -- v_n)$ is the scalar quantity

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + -- + u_n v_n$$

Example 3 $\vec{u} = (1, 2, 0, -3)$, $\vec{v} = (3, -2, 4, 2)$

$$\vec{u} \cdot \vec{v} = (1)(3) + (2)(-2) + (0)(4) + (-3)(2) = -7$$

Angle Between Two Vectors The angle θ $(0 \leq \theta \leq \pi)$ between two non-zero vectors $\vec{u}$ and $\vec{v}$ is

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}$$

Example 4 $\vec{u} = (-4, 0, 2, -2)$ $\vec{v} = (2, 0, -1, 1)$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|} = \frac{-12}{\sqrt{24}\sqrt{6}} = -1$$

$$\theta = \pi.$$

Orthogonal vectors Two vectors $\vec{u}$ and $\vec{v}$ in R^n are orthogonal if $\vec{u} \cdot \vec{v} = 0$

Example $\vec{u} = (3, 2, -1, 4)$ $\quad \vec{v} = (1, -1, 1, 0)$

$$\vec{u} \cdot \vec{v} = (3)(1) + (2)(-1) + (-1)(1) + (4)(0)$$
$$= 0$$

Orthogonal and Orthonormal Set

A set S of vectors is called orthogonal if every pair of vectors in S is orthogonal.

In addition, each vector in the set is a unit vector, then S is called orthonormal.

For $S = \{ \vec{v}_1, \vec{v}_2, \dots \vec{v}_n \}$

1) $\vec{v}_i \cdot \vec{v}_j = 0, \; i \neq j$ $\quad$ orthogonal

2) $\left.\begin{array}{ll} \vec{v}_i \cdot \vec{v}_j = 0 & i \neq j \\ \|\vec{v}_i\| = 1 & i = 1, 2, \dots n \end{array}\right\}$ orthonormal.

Example 5 $\vec{v}_1 = (0, 1, 0)$ $\quad \vec{v}_2 = (1, 0, 1)$, $\vec{v}_3 = (1, 0, -1)$

$$\vec{v}_1 \cdot \vec{v}_2 = 0 + 0 + 0 = 0$$
$$\vec{v}_1 \cdot \vec{v}_3 = 0 + 0 + 0 = 0$$
$$\vec{v}_2 \cdot \vec{v}_3 = 0 + 0 + 0 = 0$$

$\vec{v}_1, \vec{v}_2, \vec{v}_3$ are orthogonal.

Example 6 Show that the set is an orthonormal set

$$S = \left\{ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{-\sqrt{2}}{6}, \frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3}\right), \left(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}\right) \right\}.$$

$$\vec{v}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$\vec{v}_2 = \left(\frac{-\sqrt{2}}{6}, \frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3}\right)$$

$$\vec{v}_3 = \left(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}\right)$$

$$\vec{v}_1 \cdot \vec{v}_2 = -\frac{1}{6} + \frac{1}{6} + 0 = 0$$

$$\vec{v}_1 \cdot \vec{v}_3 = \frac{2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}} + 0 = 0$$

$$\vec{v}_2 \cdot \vec{v}_3 = -\frac{\sqrt{2}}{9} - \frac{\sqrt{2}}{9} + \frac{2\sqrt{2}}{9} = 0$$

$$\|\vec{v}_1\| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + 0^2} = 1$$

$$\|\vec{v}_2\| = \sqrt{\left(\frac{-\sqrt{2}}{6}\right)^2 + \left(\frac{\sqrt{2}}{6}\right)^2 + \left(\frac{2\sqrt{2}}{3}\right)^2} = 1$$

$$\|\vec{v}_3\| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2}$$

$$= \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{1}{9}} = 1$$

Theorem: If $S = \{ \vec{v}_1, \vec{v}_2, \cdots \vec{v}_n \}$ is an orthogonal set of nonzero vectors, then S is linearly independent.

Theorem: If V is an ~~inner product~~ vector space of dimension n, then any orthogonal set of n vectors form a basis for V.

1) A basis consisting of orthogonal vectors is called an orthogonal basis

2) A basis consisting of orthonormal vectors is called an orthonormal basis.

Example 7: Show that the following set is an orthogonal basis for R^4.

$$S = \{ (2,3,2,-2), (1,0,0,1), (-1,0,2,1), (-1,2,-1,1) \}$$

$\vec{v}_1 = (2,3,2,-2)$

$\vec{v}_2 = (1,0,0,1)$

$\vec{v}_3 = (-1,0,2,1)$

$\vec{v}_4 = (-1,2,-1,1)$

$\vec{v}_1 \cdot \vec{v}_2 = 0$

$\vec{v}_1 \cdot \vec{v}_3 = 0$

$\vec{v}_1 \cdot \vec{v}_4 = 0$

$\vec{v}_2 \cdot \vec{v}_3 = 0$

$\vec{v}_2 \cdot \vec{v}_4 = 0$

$\vec{v}_3 \cdot \vec{v}_4 = 0$

The set S form an orthogonal basis for R^4.

Example 8 show that the set

$$S = \{ (0,1,0), (\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}), (\tfrac{1}{\sqrt{2}}, 0, \tfrac{-1}{\sqrt{2}}) \}$$

form an orthonormal basis for R^3.

$\vec{v}_1 = (0,1,0)$ $\qquad$ $\vec{v}_1 \cdot \vec{v}_2 = 0$

$\vec{v}_2 = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$ $\qquad$ $\vec{v}_1 \cdot \vec{v}_3 = 0$

$\vec{v}_3 = (\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})$ $\qquad$ $\vec{v}_2 \cdot \vec{v}_3 = 0$

$\|\vec{v}_1\| = 1$, $\|\vec{v}_2\| = 1$ $\quad$ $\|\vec{v}_3\| = 1$

Theorem Every nonzero finite dimensional vector space has an orthonormal basis.

Gram-Schmidt orthonormalization process

1) To convert a basis $\{\vec{v}_1, \vec{v}_2, \cdots \vec{v}_n\}$ into an orthogonal basis $\{\vec{w}_1, \vec{w}_2, \cdots \vec{w}_n\}$, perform the following computations.

$$\vec{w}_1 = \vec{v}_1$$

$$\vec{w}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{w}_1}{\|\vec{w}_1\|^2} \vec{w}_1$$

$$\vec{w}_3 = \vec{v}_3 - \frac{\vec{v}_3 \cdot \vec{w}_1}{\|\vec{w}_1\|^2} \vec{w}_1 - \frac{\vec{v}_3 \cdot \vec{w}_2}{\|\vec{w}_2\|^2} \vec{w}_2$$

$$\vdots$$

$$\vec{w}_n = \vec{v}_n - \frac{\vec{v}_n \cdot \vec{w}_1}{\|\vec{w}_1\|^2} \vec{w}_1 - \frac{\vec{v}_n \cdot \vec{w}_2}{\|\vec{w}_2\|^2} \vec{w}_2 - \cdots \frac{\vec{v}_n \cdot \vec{w}_{n-1}}{\|\vec{w}_{n-1}\|^2} \vec{w}_{n-1}$$

2) To convert $\{\vec{w}_1, \vec{w}_2, \dots \vec{w}_n\}$ to orthonormal basis

$$\vec{p}_i = \frac{\vec{w}_i}{\|\vec{w}_i\|} \quad i = 1, 2, \dots n.$$

<u>Example 9</u> Apply Gram-Schmidt process to the basis for R^2 $\{(1,1), (0,1)\}$

$\vec{v}_1 = (1,1) \qquad \vec{v}_2 = (0,1)$

$$\vec{w}_1 = \vec{v}_1 = (1,1)$$

$$\vec{w}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{w}_1}{\|\vec{w}_1\|^2}\vec{w}_1$$

$$= (0,1) - \frac{0+1}{(\sqrt{2})^2}(1,1)$$

$$= (0,1) - \frac{1}{2}(1,1) = \left(-\frac{1}{2}, \frac{1}{2}\right)$$

$$\vec{p}_1 = \frac{\vec{w}_1}{\|\vec{w}_1\|} = \frac{(1,1)}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\vec{p}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|} = \frac{\left(-\frac{1}{2}, \frac{1}{2}\right)}{\frac{1}{\sqrt{2}}} = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

Example 10 Apply Gram-Schmidt process to the basis for R^3

$$\{(1,1,0),(1,2,0),(0,1,2)\}$$

$\vec{v}_1=(1,1,0)$, $\vec{v}_2=(1,2,0)$, $\vec{v}_3=(0,1,2)$

$$\vec{w}_1=\vec{v}_1=(1,1,0)$$

$$\vec{w}_2=\vec{v}_2-\frac{\vec{v}_2\cdot\vec{w}_1}{\|\vec{w}_1\|^2}\vec{w}_1=(1,2,0)-\frac{1+2+0}{(\sqrt{2})^2}(1,1,0)$$

$$=(1,2,0)-\frac{3}{2}(1,1,0)=\left(-\frac{1}{2},\frac{1}{2},0\right)$$

$$\vec{w}_3=\vec{v}_3-\frac{\vec{v}_3\cdot\vec{w}_1}{\|\vec{w}_1\|^2}\vec{w}_1-\frac{\vec{v}_3\cdot\vec{w}_2}{\|\vec{w}_2\|^2}\vec{w}_2$$

$$=(0,1,2)-\frac{0+1+0}{(\sqrt{2})^2}(1,1,0)-\frac{0+\frac{1}{2}+0}{\left(\frac{1}{\sqrt{2}}\right)^2}\left(-\frac{1}{2},\frac{1}{2},0\right)$$

$$=(0,1,2)-\frac{1}{2}(1,1,0)-\frac{1/2}{1/2}\left(-\frac{1}{2},\frac{1}{2},0\right)$$

$$=(0,1,2)-\left(\frac{1}{2},\frac{1}{2},0\right)-\left(-\frac{1}{2},\frac{1}{2},0\right)$$

$$=(0,0,2)$$

$$\vec{p}_1=\frac{\vec{w}_1}{\|\vec{w}_1\|}=\frac{(1,1,0)}{\sqrt{2}}=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right).$$

$$\vec{p}_2=\frac{\vec{w}_2}{\|\vec{w}_2\|}=\frac{(-1/2,1/2,0)}{1/\sqrt{2}}=\left(\frac{-\sqrt{2}}{2},\frac{\sqrt{2}}{2},0\right)$$

$$\vec{p}_2=\frac{\vec{w}_3}{\|\vec{w}_3\|}=\frac{(0,0,2)}{2}=(0,0,1).$$