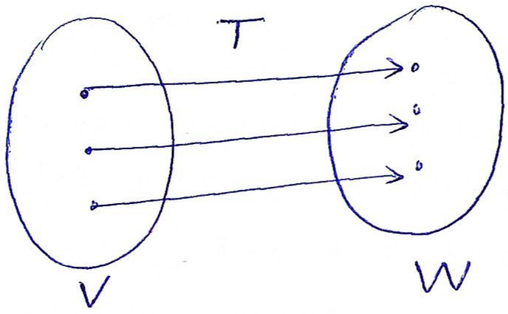
One-to-one and Onto Linear Transformations

One-to-one A function $T: V \longrightarrow W$ is called one-to-one if T maps distinct vectors in V into distinct vectors in W.



T is one-to-one if and only if for all $\vec{u}, \vec{v}$ in V

$$T(\vec{u}) = T(\vec{v}) \Rightarrow \vec{u} = \vec{v}.$$

Theorem Let $T: V \longrightarrow W$ be a linear transformation. Then T is one-to-one if and only if $\mathrm{Ker}(T) = \{\vec{0}\}$.

Example ① Determine whether the linear transformation is one-to-one. $T: R^2 \longrightarrow R^2$ defined by

$$T(x, y) = (x+y, x-y)$$

$$T(\vec{v}) = \vec{0}$$

$$T(x, y) = \vec{0}$$

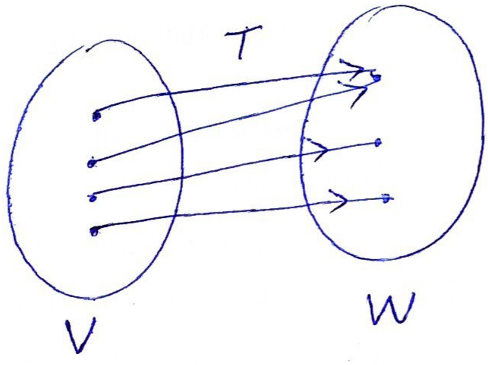
$$(x+y, x-y) = (0, 0)$$

$$\begin{array}{l} x + y = 0 \\ x - y = 0 \end{array} \Rightarrow \begin{array}{l} x = 0 \\ y = 0 \end{array}$$

$$\mathrm{Ker}(T) = \{(0,0)\} = \{\vec{0}\}.$$

T is one-to-one.

<u>onto.</u> A function $T : V \longrightarrow W$ is said onto if every element in W has a preimage in V.



T is onto when $W = \text{range}(T)$.

<u>Theorem</u> Let $T : V \longrightarrow W$ be a linear transformation, where W is finite dimensional. Then T is onto if and only if

$$\text{rank}(T) = \dim(W).$$

<u>Example (2)</u> The linear transformation $T : R^3 \longrightarrow R^3$ is represented by $T(\vec{X}) = A\vec{X}$. Find nullity (T) and rank (T) and determine whether T is one-to-one or onto

(a) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(a) The matrix is in echelon form.

$\text{rank}(T) = \dim(\text{range}) = 3$ $n = 3$

$\text{nullity}(T) = \dim(\text{Kernel}) = 0$

T is one-to-one.

rank $(T) = \dim(W) = 3$

T is onto.

(b) The matrix is in echelon form

rank $(T) = \dim(\text{range}) = 2$ $\qquad n = 3$

nullity $(T) = \dim(\text{Kernel}) = 1$

T is not one-to-one

rank $(T) = 2$ $\qquad \dim(W) = 3$

$\text{rank}(T) \neq \dim(W)$

T is not - ~~one to~~ onto.

Example 3 Find nullity (T) and rank (T) and determine whether T is one-to-one or onto.

(a) $T: R^2 \longrightarrow R^3$ defined by $T(\vec{x}) = A\vec{X}$

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(b) $T: R^3 \longrightarrow R^2$ defined by $T(\vec{x}) = A\vec{X}$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

(a) rank (T) = dim (range) = 2 $\qquad n = 2$

Nullity (T) = dim (Kernel) = 0

T is one-to-one

rank $(T) = 2 \quad \dim(W) = 3$

rank$(T) \neq \dim(W)$

T is not onto.

(b) rank (T) = dim(range) = 2 $\qquad n = 3$

nullity (T) = dim(Kernel) = 1

T is not one-to-one

rank $(T) = \dim(W) = 2$

T is onto.

Isomorphism A linear transformation $T: V \longrightarrow W$ that is both one-to-one and onto is called an isomorphism. Moreover, if V and W are vector spaces such that there exists ~~and~~ an isomorphism from V to W, then V and W are said to be isomorphic to each other.

Theore Two finite dimensional vector spaces V and W are said to be isomorphic if they are of the same dimension.

Example (4) The vector spaces are isomorphic to each other.

(a) R^4 = 4-space

(b) M_{41} = space of all 4×1 matrices

(c) M_{22} = space of all 2×2 matrices

(d) P_3 = space of all polynomials of degree 3 or less

(e) $V = \{ (x_1, x_2, x_3, x_4, 0) : x_i \in \mathbb{R} \}$ (subspace of R^5)

$$\boxed{\text{Exercise } 8{\cdot}2 \quad \text{Q1–10 (odd)}}$$

$$(T_2 \circ T_1)(x, y, z) = (2x + y,\ x + z,\ 0)$$

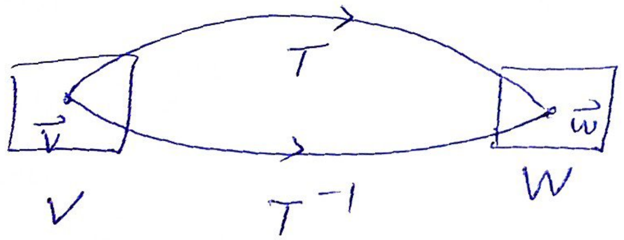
$$(T_1 \circ T_2)(\vec{v}) = (A_1 A_2)(\vec{v})$$

$$(T_1 \circ T_2)(x, y, z) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= (2x - 2y + z,\ 0,\ x)$$

Inverse of a Linear Transformation



If $T: V \longrightarrow W$, then $T^{-1}: W \longrightarrow V$ is called inverse linear transformation.

For $T: V \longrightarrow W$ defined by $T(\vec{v}) = A\vec{v}$

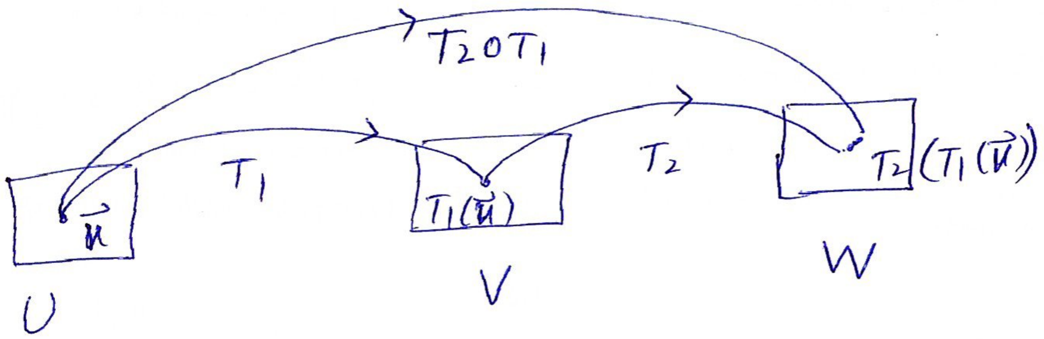
the inverse transformation is

$T^{-1}: W \longrightarrow V$ defined by $T^{-1}(\vec{v}) = A^{-1}\vec{v}$

Composition and Inverse Transformations

If $T_1 : U \longrightarrow V$ and $T_2 : V \longrightarrow W$ are linear transformations, then the composition of T_2 with T_1, denoted by $T_2 \circ T_1$ is a function defined by the formula.

$$(T_2 \circ T_1)(\vec{v}) = T_2(T_1(\vec{v}))$$



Theorem Let $T_1 : U \longrightarrow V$ and $T_2 : V \longrightarrow W$ be linear transformations with standard matrices A_1 and A_2. The composition $T_2 \circ T_1 : U \longrightarrow W$ defined by

$$(T_2 \circ T_1)(\vec{v}) = T_2(T_1(\vec{v}))$$

is a linear transformation.

Moreover, the standard matrix A of $T_2 \circ T_1$ is given by the matrix product

$$A = A_2 A_1$$

$T_1 : U \longrightarrow V \qquad T_1(\vec{v}) = A_1 \vec{v}$

$T_2 \; V \longrightarrow W \qquad T_2(\vec{v}) = A_2 \vec{v}$

$$(T_2 \circ T_1)(\vec{v}) = T_2(T_1(\vec{v})) = T_2(A_1\vec{v}) = A_2(A_1\vec{v}) = (A_2 A_1)\vec{v}$$

Example 5 Let T_1 and T_2 be two transformations from R^3 to R^3 such that

$$T_1(x,y,z) = (2x+y,\ 0,\ x+z)$$

$$T_2(x,y,z) = (x-y,\ z,\ y)$$

Find $T_2 \circ T_1$ and $T_1 \circ T_2$.

$$T_1(x,y,z) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$T_2(x,y,z) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$(T_2 \circ T_1)(\vec{v}) = (A_2 A_1)\vec{v}$$

$$(T_2 \circ T_1)(x,y,z) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$(T_2 \circ T_1)(x,y,z) = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Example (6) The linear transformation

$T: R^3 \longrightarrow R^3$ is defined by

$$T(x_1, x_2, x_3) = (2x_1 + 3x_2 + x_3,\ 3x_1 + 3x_2 + x_3,\ 2x_1 + 4x_2 + x_3)$$

Find inverse transformation.

The standard matrix for T is

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

$$T^{-1}(\vec{v}) = A^{-1}\vec{v}$$

$$T^{-1}(x_1, x_2, x_3) = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$T^{-1}(x_1, x_2, x_3) = (-x_1 + x_2,\ -x_1 + x_3,\ 6x_1 - 2x_2 - 3x_3)$$

Exercise 8.3

Q1–12 (odd)