Diagonalization

In this section, we study another classical problem in linear algebra called the matrix diagonalization problem.

For a square matrix A, does there exist an invertible matrix P such that $P^{-1}AP$ is diagonal?

Similar Matrices Two square matrices A and B are called similar if there exists an invertible matrix P such that

$$B = P^{-1}AP$$

1) The matrices $A = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$

are similar. There exists an invertible matrix

$P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ such that $B = P^{-1}AP$

$$P^{-1}AP = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} = B.$$

2) The matrices

$$A = \begin{bmatrix} -2 & 7 \\ -3 & 7 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

are similar. There exists an invertible matrix P, where

$$P = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} \quad \text{such that } B = P^{-1}AP$$

<u>Diagonalizable Matrices</u> Matrices that are similar to diagonal matrices are called diagonalizable.

An $n \times n$ matrix A is diagonalizable if A is similar to a diagonal matrix. That is, A is diagonalizable if there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

The matrices $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & -2 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

are similar (diagonalizable). There exists an invertible matrix P, where

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{such that } P^{-1}AP = D$$

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -2 & -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D$$

<u>Theorem</u> If A and B are similar matrices, then they have the same eigenvalues.

The matrices $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & -2 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

are similar.

Eigenvalues of D are $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$

Eigenvalues of A:- The characteristic equation is

$$\det(\lambda I - A) = 0$$

$$\begin{vmatrix} \lambda-1 & 0 & 0 \\ 1 & \lambda-1 & -1 \\ 1 & 2 & \lambda-4 \end{vmatrix} = 0$$

$$(\lambda-1)\left[(\lambda-1)(\lambda-4)+2\right] - 0 + 0 = 0$$

$$(\lambda-1)\left[\lambda^2 - 5\lambda + 4 + 2\right] = 0$$

$$(\lambda-1)(\lambda^2 - 5\lambda + 6) = 0$$

$$(\lambda-1)(\lambda-2)(\lambda-3)=0$$

$$\lambda=1,\ \lambda=2,\ \lambda=3.$$

<u>Condition for Diagonalization</u>

<u>Theorem</u> An $n\times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.

<u>Example ①</u> check whether the following matrices are diagonalizable.

(a) $A=\begin{bmatrix}1&3&0\\3&1&0\\0&0&-2\end{bmatrix}$ (b) $A=\begin{bmatrix}1&0&0\\1&2&0\\-3&5&2\end{bmatrix}$

(a) The characteristic polynomial is

$$\det(\lambda I-A)=0$$

$$\begin{vmatrix}\lambda-1&-3&0\\-3&\lambda-1&0\\0&0&\lambda+2\end{vmatrix}=0$$

$$(\lambda-1)\left[(\lambda-1)(\lambda+2)\right]+3\left[-3(\lambda+2)\right]=0$$

$$(\lambda+2)\left[(\lambda-1)^2-9\right]=0$$

$\lambda=-2$ $\quad (\lambda-1)^2=9 \Rightarrow \lambda-1=\pm 3$

$$\lambda=4,\ \lambda=-2$$

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigenvector corresponding to eigenvalue λ. $(\lambda I - A)\vec{x} = \vec{0}$

$$\begin{bmatrix} \lambda-1 & -3 & 0 \\ -3 & \lambda-1 & 0 \\ 0 & 0 & \lambda+2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda = -2$ $\begin{bmatrix} -3 & -3 & 0 \\ -3 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The row echelon form of the matrix is

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 + x_2 = 0$ $\qquad$ $x_2 = s$

$x_1 = -s$ $\qquad$ $x_3 = t$

$$\vec{x} = \begin{bmatrix} -s \\ s \\ t \end{bmatrix} = \begin{bmatrix} -s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}$$

$$= s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda = -2$, $\vec{P_1} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{P_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

For $\lambda = 4$ $\begin{bmatrix} 3 & -3 & 0 \\ -3 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

The row echelon form of the matrix is

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 - x_2 = 0$ $\quad x_3 = 0$ $\quad x_2 = t$

$x_1 = t$

$$\vec{x} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\lambda_1 = -2$ $\quad \vec{p}_1 = (-1, 1, 0)$

$\lambda_2 = -2$ $\quad \vec{p}_2 = (0, 0, 1)$

$\lambda_3 = 4$ $\quad \vec{p}_3 = (1, 1, 0)$.

The matrix A is diagonalizable.

$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$

(b) The characteristic polynomial is

$$\det(\lambda I - A) = 0$$

$$\begin{vmatrix} \lambda-1 & 0 & 0 \\ -1 & \lambda-2 & 0 \\ 3 & -5 & \lambda-2 \end{vmatrix} = 0$$

$$(\lambda-1)\left[(\lambda-2)^2 + 0\right] - 0 + 0 = 0$$

$$(\lambda-1)(\lambda-2)^2 = 0$$

$$\lambda = 1, \quad \lambda = 2$$

Let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigenvector corresponding to λ.

$$(\lambda I - A)\vec{x} = \vec{0}$$

$$\begin{pmatrix} \lambda-1 & 0 & 0 \\ -1 & \lambda-2 & 0 \\ 3 & -5 & \lambda-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

For $\lambda = 1$

$$\begin{pmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 3 & -5 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The echelon form of the matrix is

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1/8 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 + x_2 = 0 \qquad x_3 = t$$
$$x_2 + \frac{1}{8}x_3 = 0$$

$$x_2 = -\frac{1}{8}t \qquad x_1 = \frac{1}{8}t.$$

$$\vec{x} = \begin{bmatrix} \frac{1}{8}t \\ -\frac{1}{8}t \\ t \end{bmatrix} = t\begin{bmatrix} \frac{1}{8} \\ -\frac{1}{8} \\ 1 \end{bmatrix}$$

For $\lambda = 2$.
$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 3 & -5 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The row echelon form is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= t \end{aligned}$$

$$\vec{x} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda_1 = 1 \qquad \vec{p}_1 = (\frac{1}{8}, -\frac{1}{8}, 1)$

$\lambda_2 = 2 \qquad \vec{p}_2 = (0, 0, 1)$

Since matrix is of order 3×3, there must be 3 linearly independent eigenvectors. the matrix A is not diagonalizable

<u>Theorem</u> If an $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.

<u>Example 2</u> Show that the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & -2 & 4 \end{bmatrix}$$

is diagonalizable.

The characteristic polynomial is

$$\det(\lambda I - A) = 0$$

$$\begin{vmatrix} \lambda-1 & 0 & 0 \\ 1 & \lambda-1 & -1 \\ 1 & 2 & \lambda-4 \end{vmatrix} = 0$$

$$(\lambda-1)(\lambda-2)(\lambda-3) = 0$$

$$\lambda = 1, \quad \lambda = 2, \quad \lambda = 3.$$

Since the matrix A has 3 distinct eigenvalues, A is diagonalizable.

Procedure for Diagonalizing a Matrix

Step I Confirm that the matrix is actually diagonalizable by finding n linearly independent eigen vectors. $\vec{P_1}, \vec{P_2}, \ldots \vec{P_n}$.

Step II Form the matrix $P = [\vec{P_1} \ \vec{P_2} \ \ldots \ \vec{P_n}]$ that has the columns consisting of eigenvectors.

Step III The matrix $P^{-1}AP$ will be diagonal and have the eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$ corresponding to the eigenvectors $\vec{P_1}, \vec{P_2} \ldots \vec{P_n}$ as its successive diagonal entries.

Example ③ Show that the matrix A is diagonalizable.

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

Then find a matrix P such that $P^{-1}AP$ is diagonal.

The characteristic polynomial is

$$\det(\lambda I - A) = 0$$

$$\begin{vmatrix} \lambda-1 & 1 & 1 \\ -1 & \lambda-3 & -1 \\ 3 & -1 & \lambda+1 \end{vmatrix} = 0$$

$$(\lambda-2)(\lambda+2)(\lambda-3) = 0.$$

$$\lambda_1 = 2, \quad \lambda_2 = -2, \quad \lambda_3 = 3.$$

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector corresponding to λ.

$$(\lambda I - A)\vec{x} = \vec{0}$$

$$\begin{bmatrix} \lambda-1 & 1 & 1 \\ -1 & \lambda-3 & -1 \\ 3 & -1 & \lambda+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda_1 = 2$ $\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 3 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The row echelon form is

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 + x_3 &= 0 \\ x_2 &= 0 \\ x_1 &= -t \end{aligned} \qquad x_3 = t$$

$$\vec{x} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \vec{p}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

For $\lambda_2 = -2$ $\begin{bmatrix} -3 & 1 & 1 \\ -1 & -5 & -1 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The row echelon form is

$$\begin{bmatrix} 1 & 0 & -1/4 \\ 0 & 1 & 1/4 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 - \tfrac{1}{4}x_3 &= 0 \\ x_2 + \tfrac{1}{4}x_3 &= 0 \end{aligned} \qquad x_3 = t$$

$$x_1 = \frac{1}{4}t, \quad x_2 = -\frac{1}{4}t$$

$$\vec{X} = \begin{bmatrix} \frac{1}{4}t \\ -\frac{1}{4}t \\ t \end{bmatrix} = \frac{1}{4}t \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \qquad \vec{P_2} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

For $\lambda_3 = 3$

$$\begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 3 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The row echelon form is

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} x_1 + x_3 = 0 \\ x_2 - x_3 = 0 \end{matrix} \qquad x_3 = t$$

$$x_1 = -t \qquad x_2 = +t \qquad x_3 = t$$

$$\vec{X} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{P_3} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

The matrix $P = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & 1 \end{bmatrix}$

$$P^{-1} = \begin{bmatrix} -1 & -1 & 0 \\ 1/5 & 0 & 1/5 \\ 1/5 & 1 & 1/5 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

<u>Example 4</u> Show that the matrix A is diagonalizable.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

Then find a matrix P such that $P^{-1}AP$ is diagonal.

From Example in Eigenvalues Section.

$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 3.$

$$\lambda_1 = 1 \quad \vec{P_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{P_2} = \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 2 \quad \vec{P_3} = \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix} \qquad \lambda_3 = 3, \quad \vec{P_4} = \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & -2 & 0 & 0 \\ 1 & 0 & 5 & -5 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} -5/2 & 1 & -5 & 5 \\ -1/2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Computing Power of a Matrix

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} = D$$

Squaring both sides

$$(P^{-1}AP)^2 = \begin{bmatrix} \lambda_1^2 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n^2 \end{bmatrix} = D^2$$

~~Start~~ $(P^{-1}AP)^2 = P^{-1}A\underline{PP^{-1}}AP = P^{-1}AIAP = P^{-1}A^2P$

$$P^{-1}A^2P = \begin{bmatrix} \lambda_1^2 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & \lambda_n^2 \end{bmatrix} = D^2$$

$$P^{-1}A^{k}P = D^{k} = \begin{bmatrix} \lambda_1^k & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^k & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n^k \end{bmatrix}$$

$$A^{k} = P D^{k} P^{-1} = P \begin{bmatrix} \lambda_1^k & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^k & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & \lambda_n^k \end{bmatrix} P^{-1}$$

<u>Example ⑤</u> Find A^{13}, where

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

The characteristic polynomial is

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$(\lambda - 1)(\lambda - 2)^2 = 0$$

$\lambda = 2$ $\quad \vec{P}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{P}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda = 1$ $\quad \vec{P}_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{pmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = D$$

$$D = P^{-1}AP \Rightarrow A = PDP^{-1}$$

$$A^{13} = PD^{13}P^{-1}$$

$$= \begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2^{13} & 0 & 0 \\ 0 & 2^{13} & 0 \\ 0 & 0 & 1^{13} \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -8190 & 0 & -16382 \\ 8191 & 8192 & 8191 \\ 8191 & 0 & 16383 \end{bmatrix}$$

Exercise 5.2

Q 1-34 odd