Week 10

PROBABILITY THEORY

Permutation

A **permutation** is an arrangement of all or part of a set of objects.

The number of ways of arranging *n* objects taken *r* at a time is denoted by $^{n}P_{r}$ and is defined as:

$$^{n}P_r = \frac{n!}{(n-r)!}.$$

Example: How many distinct four-digit numbers can be formed from the following integers 1, 2, 3, 4, 5, 6 if each integer is used only once?

$$^{n}P_r = \frac{n!}{(n-r)!}.$$

Here, $n = 6$ and $r = 4$

$$^{6}P_4 = \frac{6!}{(6-4)!}$$

$$= \frac{6!}{2!}$$

$$= 360$$

Example Evaluate: (i) $^{5}P_{3}$ (ii) $^{10}P_{6}$

Combinations

When a selection of objects is made without paying regard to the order of selection, it is called combination. The number of combinations of n things taken r at·a time is denoted by $^n\mathbf{C}_r$ or by $\binom{n}{r}$ and is defined by :

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Example: Evaluate: i) $^5\mathbf{C}_3$ (ii) $^6\mathbf{C}_{r2}$ (iii) $^8\mathbf{C}_5$

1.15 Multiplication Theorem of Probability for Independent Events

If A and B are the two events with positive probabilities $\{P(A)\neq 0, P(B)\neq 0\}$ then A and B are independent if and only if $P(A\cap B) = P(A).P(B)$

Proof

If an event happen in $n_1$ways of which a_1 are successful and the event B can happen in n_2 ways of which a_2 are successful, and to combine each successful event in the first with each successful event in the second case. Thus the total number of possible cases in $a_1 x\ a_2$. Similarly, the total number of possible cases is $n_1 x\ n_2$. By the definition the

probability occurrence of both events, $\frac{a_1 X\ a_2}{n_1 X\ n_2} = \frac{a_1}{n_1} X \frac{a_2}{n_2}$

$$\text{But } P(A) = \frac{a_1}{n_1} \text{ and } P(B) = \frac{a_2}{n_2}$$

$$P(A\cap B) = P(A).P(B)$$

Example

a)

Let A and B be events with $P(A \cup B) = \frac{3}{4}$, $P(\bar{A}) = \frac{2}{3}$ and $P(A \cap B) = \frac{1}{4}$, find (i) $P(A)$, (ii) $P(B)$, (iii) $P(A \cap \bar{B})$.

Solution:

$P(A \cup B) = \frac{3}{4}$, $P(\bar{A}) = \frac{2}{3}$ and $P(A \cap B) = \frac{1}{4}$ (given)

$\therefore$ (i) $P(A) = 1 - P(\bar{A}) = 1 - \frac{2}{3} = \frac{1}{3}$

(ii) Now $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\therefore$ $P(B) = P(A \cup B) - P(A) + P(A \cap B)$

$$= \frac{3}{4} - \frac{1}{3} + \frac{1}{4} = \frac{9 - 4 + 3}{12} = \frac{8}{12} = \frac{2}{3}.$$

(iii) $P(A \cap \bar{B}) = P(A) - P(A \cap B)$

$$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

b)

If $P(A) = \frac{1}{2}$, $P(A \cup B) = \frac{3}{4}$ and $P(\bar{B}) = \frac{5}{8}$, then find (i) $P(A \cap B)$, (ii) $P(\bar{A} \cap \bar{B})$ (iii) $P(\bar{A} \cup \bar{B})$ and (iv) $P(B \cap \bar{A})$.

Solution:

Given: $P(A) = \frac{1}{2}$, $P(A \cup B) = \frac{3}{4}$ and $P(\bar{B}) = \frac{5}{8}$.

Then $\quad P(\bar{A}) = 1 - P(A) = \frac{1}{2}$, and $P(B) = 1 - P(\bar{B}) = \frac{3}{8}$.

$\therefore$ (i) $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

or $\quad P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$$= \frac{1}{2} + \frac{3}{8} - \frac{3}{4} = \frac{4 + 3 - 6}{8} = \frac{1}{8}$$

(ii) Using De Morgan's Law $(\overline{A \cup B}) = \bar{A} \cap \bar{B}$, we have

$$P(\bar{A} \cap \bar{B}) = P[(\overline{A \cup B})]$$

$$= 1 - P(A \cup B) = 1 - \frac{3}{4} = \frac{1}{4}.$$

(iii) Using De Morgan's Law $\overline{(A \cap B)} = \bar{A} \cup \bar{B}$, we have

$$P(\bar{A} \cup \bar{B}) = P[\overline{(A \cap B)}]$$

$$= 1 - P(A \cap B) = 1 - \frac{1}{8} = \frac{7}{8}.$$

(iv) $$P(B \cap \bar{A}) = P(B - A \cap B) = P(B) - P(B \cap A)$$

$$= \frac{3}{8} - \frac{1}{8} = \frac{1}{4}.$$

Example

Two dice are thrown. Let A be the event that the sum of the upper face numbers is odd, and B the event of at least one ace? Assuming a sample space of 36 points, list the sample points which belong to the events $A \cap B$, $A \cup B$ and $A \cap \bar{B}$. Find the probabilities of these events, assuming equally likely events.

Solution:

S consists of 36 sample points.

Let A = (sum of the upper face numbers is odd), and

B = (at least one ace). Then

$$A = \begin{Bmatrix} (1,2) & (2,1) & (3,2) & (4,1) & (5,2) & (6,1) \\ (1,4) & (2,3) & (3,4) & (4,3) & (5,4) & (6,3) \\ (1,6) & (2,5) & (3,6) & (4,5) & (5,6) & (6,5) \end{Bmatrix}$$

$$B = \{(1,1),\ (1,2),\ (1,3),\ (1,4),\ (1,5),\ (1,6),\ (2,1),\ (3,1),\ (4,1),\ (5,1),\ (6,1)\}$$

$$\bar{B} = \begin{Bmatrix} (2,2) & (3,2) & (4,2) & (5,2) & (6,2) \\ (2,3) & (3,3) & (4,3) & (5,3) & (6,3) \\ (2,4) & (3,4) & (4,4) & (5,4) & (6,4) \\ (2,5) & (3,5) & (4,5) & (5,5) & (6,5) \\ (2,6) & (3,6) & (4,6) & (5,6) & (6,6) \end{Bmatrix}$$

Now $A \cap B$ = The intersection of A and B consists of those points which belong to both A and B.

$\therefore$ $A \cap B = \{(1,2), (1,4), (1,6), (2,1), (4,1), (6,1)\}$

It thus contains 6 sample points.

And $A \cup B$ = The unon of A and B consists of those points which belong to A or B or both.

$$\therefore \quad A \cup B = \begin{Bmatrix} (1,1) & (1,5) & (2,5) & (4,1) & (5,1) & (6,1) \\ (1,2) & (1,6) & (3,1) & (4,3) & (5,2) & (6,3) \\ (1,3) & (2,1) & (3,2) & (4,5) & (5,4) & (6,5) \\ (1,4) & (2,3) & (3,4) & (3,6) & (5,6) & \end{Bmatrix}$$

i.e. $A \cup B$ contains 23 sample points.

Again $A \cap \bar{B}$ = The intersection of A and $\bar{B}$ consists of points which belong to both A and $\bar{B}$.

Thus $A \cap \bar{B} = \begin{Bmatrix} (2,3), (2,5), (3,2), (3,4), (3,6), (4,3) \\ (4,5), (5,2), (5,4), (5,6), (6,3), (6,5) \end{Bmatrix}$

i.e. $A \cap \bar{B}$ consists of 12 sample points.

Hence $P(A \cap B) = \dfrac{6}{36} = \dfrac{1}{6}$, $P(A \cup B) = \dfrac{23}{36}$, and

$$P(A \cup \bar{B}) = \frac{12}{36} = \frac{1}{3}.$$

Example

Two good dice are rolled simultaneously. Let A denote the event "the sum shown is 8" and B the event "the two show the same number." Find $P(A)$, $P(B)$, $P(A \cap B)$, and $P(A \cup B)$.

Solution:

Let S = sample space, A = the sum shown is 8, and

B = the two show the same number. Then

S consists of 36 elements.

$$A = \{(6,2), (5,3), (4,4), (3,5), (2,6)\}$$

$$B = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$$

nce $P(A) = \frac{5}{36}$; $\quad P(B) = \frac{6}{36} = \frac{1}{6}$;

$$P(A \cap B) = \frac{1}{36};\ \text{as } A \cap B = \{(4,4)\}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{5}{36} + \frac{6}{36} - \frac{1}{36} = \frac{10}{36} = \frac{5}{18};$$

1.15 CONDITIONAL PROBABILITY

The multiplication theorem is not applicable in the case of dependent events. If A and B are the two events are said to be dependent, when B can occur only when A is known to have occurred. The probability attached to such an event is called conditional probability and its denoted by $P(B/A)$

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

The general terms of multiplication in its modified form in terms of conditional probability becomes

$$P(A \cap B) = P(B)\,P(A/B)$$

$$P(A \cap B) = P(A)\,P(B/A)$$

BAYE's Theorem

Statement

If $E_1, E_2, E_3, \ldots, E_n$ are mutually disjoint events with $P(E_i) \neq 0, (i = 1,2,3,\ldots,n)$, then for any arbitrary event A which is a subset of $\bigcup_{i=1}^{n} E_i$ such that $P(A) > 0$, we have

$$P(E_i \mid A) = \frac{P(E_i)P(A \mid E_i)}{\sum_{i=1}^{n} P(E_i)P(A \mid E_i)} = \frac{P(E_i)P(A \mid E_i)}{P(A)}$$

Proof :

Let $E_1, E_2, E_3, \ldots, E_n$ are and mutually disjoint events and A be any another event on the sample space, then $(A \cap E_i) = E_i$ $(i = 1,2,3,\ldots n)$ are mutually disjoint events

$$A = (A \cap E_1) \cup (A \cap E_2) \cup (A \cap E_3) \cup \ldots \cup (A \cap E_n)$$

$\Rightarrow$ $P(A) = P[(A \cap E_1) \cup (A \cap E_2) \cup (A \cap E_3) \cup \ldots \cup (A \cap E_n)]$ by using axioms of (iii)

$\Rightarrow$ $P(A) = P(A \cap E_1) + P(A \cap E_2) + P(A \cap E_3) + \ldots + (A \cap E_n)$

$$P(A) = \sum_{i=1}^{n} P(A \cap E_i)$$

Multiplication theorem of probability

$$P(A \cap B) = P(A \mid B)P(B)$$

$$P(A)=\sum_{i=1}^{n} P(E_i)P(A|E_i)$$

$$P(E_i|A)=\frac{P(A\cap E_i)}{P(A)}$$

$$P(E_i|A)=\frac{P(E_i)P(A|E_i)}{P(A)}=\frac{P(E_i)P(A|E_i)}{\sum_{i=1}^{n} P(E_i)P(A|E_i)}$$

$$\therefore\ P(E|A)=\frac{P(E)P(A|E)}{\sum P(E)P(A|E)}=\frac{P(E)P(A|E)}{P(A)}$$

Example 6.17 Two coins are tossed. What is the conditional probability that two heads given that there is at least one head?

The sample space S for this experiment is

$$S=\{HH, HT, TH, TT\}.$$

Let A represent the event that two heads appear, and B, the event that there is at least one Then we need $P(A/B)$.

Since $A=\{HH\}$, $B=\{HH, HT, TH\}$ and $A\cap B=\{HH\}$.

$$\therefore\quad P(A)=\frac{1}{4},\ P(B)=\frac{3}{4} \text{ and } P(A\cap B)=\frac{1}{4}$$

Hence $P(A/B)=\dfrac{P(A\cap B)}{P(B)}=\dfrac{1/4}{3/4}=\dfrac{1}{3}.$

Example: A man tosses two fair dice. What is the probability that the sum of two dice will be 7, given that (i) the sum is odd (ii) the sum is greater than 6 (iii) the two dice have same outcome?

Solution:

Let $A = \{\text{the sum is } 7\}$, $B = \{\text{the sum is odd}\}$.

$C = \{\text{the sum is greater than } 6\}$, and

$D = \{\text{the two dice had the same outcomes}\}$. Then

$A = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$,

$B = \{(1, 2), (1, 4), (1, 6), (2, 1), (2, 3), (2, 5), (3, 4), \ldots, (6, 5)\}$,

$C = \{(1, 6), (2, 5), (2, 6); (3, 4), (3, 5), (3, 6), (4, 3), (4, 4), \ldots, (6, 6)\}$,

$D = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$,

$A \cap B = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$,

$A \cap C = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$,

$A \cap D = \phi$

$$\therefore \quad P(A)=\frac{6}{36},\ P(B)=\frac{18}{36},\ P(C)=\frac{21}{36},\ P(D)=\frac{6}{36}$$

$$P(A\cap B)=\frac{6}{36},\ P(A\cap C)=\frac{6}{36} \text{ and } P(A\cap D)=0.$$

using the definition of conditional probability, we get

$$P(A/B)=\frac{P(A\cap B)}{P(B)}=\frac{6}{36}\times\frac{36}{18}=\frac{1}{3}$$

$$P(A/C)=\frac{P(A\cap C)}{P(C)}=\frac{6}{36}\times\frac{36}{21}=\frac{2}{7}$$

$$P(A/D)=\frac{P(A\cap D)}{P(D)}=0\times\frac{36}{6}=0.$$

Example 6.19 What is the probability that a randomly selected poker hand, contains exactly 3 ... given that it contains *at least* 2 aces?

Let A represent the event that exactly 3 aces are selected and B, the event that *at least* 2 aces are ... Then we need $P(A/B)$.

Since a poker hand consists of 5 cards, therefore the sample space S contains $\binom{52}{5}=2{,}598{,}960$ ...

$$n(A)=\binom{4}{3}\binom{48}{2} \text{ outcomes;}$$

$$n(B)=\binom{4}{2}\binom{48}{3}+\binom{4}{3}\binom{48}{2}+\binom{4}{4}\binom{48}{1}$$

... at *least* 2 aces means 2 or 3 or 4 aces; and

$$n(A \cap B) = \binom{4}{3}\binom{48}{2} \text{ as } A \subset B.$$

Thus $$P(A \cap B) = \frac{\binom{4}{3}\binom{48}{2}}{\binom{52}{5}}, \text{ and}$$

$$P(B) = \frac{\binom{4}{2}\binom{48}{3} + \binom{4}{3}\binom{48}{2} + \binom{4}{4}\binom{48}{1}}{\binom{52}{5}},$$

Hence $$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{\binom{4}{3}\binom{48}{2}}{\binom{4}{2}\binom{48}{3} + \binom{4}{3}\binom{48}{2} + \binom{4}{4}\binom{48}{1}}.$$

$$= \frac{4{,}512}{108{,}336} = 0.0416.$$

Example 6.22 Box A contains 5 green and 7 red balls. Box B contains 3 green, 3 red and 6 yel[low] balls. A box is selected at random and a ball is drawn at random from it. What is the probability tha[t the] ball drawn is green?

Let E represent the event that the green ball is drawn. Then E can occur in one of the followin[g] mutually exclusive ways:

i) Box A is selected and a green ball is drawn, i.e. $A \cap E$, or

ii) Box B is selected and a green ball is drawn, i.e. $B \cap E$.

Therefore P(green ball) = P(box A and green ball) + P(box B and green ball)

= P(box A) P(green ball/box A) + P(box B) x P(green ball/box B)

In symbols,

$$P(E) = P(A \cap E) + P(B \cap E)$$

$$= P(A)\,P(E / A) + P(B)\,P(E / B)$$

$$= \frac{1}{2}.\frac{5}{12} + \frac{1}{2}.\frac{3}{12} = \frac{1}{3}$$

Example 6.28 Let A be the event that a family has children of both sexes and B be the event that a [fam]ily has at most one boy. If a family is known to have (i) three children, then show that A and B are [in]dependent events, (ii) four children, then show that A and B are dependent events.

Let b denote a boy and g a girl. Then

i) the equiprobable sample space S would be

$$S = \{bbb, bbg, bgb, gbb, bgg, gbg, ggb, ggg\}$$

The two events are

$$A = \{\text{children of both sexes}\},$$
$$= \{\text{bbg, bgb, gbb, bgg, gbg, ggb}\}, \text{ and}$$
$$B = \{\text{at most one body}\},$$
$$= \{\text{bgg, gbg, bgg, ggb, ggg}\}$$

The event $A \cap B$ is

$$A \cap B = \{\text{bgg, gbg, ggb}\}$$

Thus their respective probabilities are

$$P(A) = \frac{6}{8} = \frac{3}{4},\ P(B) = \frac{4}{8} = \frac{1}{2}, \text{ and } P(A \cap B) = \frac{3}{8}$$

$$P(A)\ P(B) = \frac{3}{4} \times \frac{1}{2} = \frac{3}{8} = P(A \cap B).$$

Hence A and B are independent.

ii) the sample space S may be represented by the following 16 equally likely outcomes:

S = {bbbb, bbbg, bbgb, bgbb, gbbb, bbgg, bgbg, gbbg, gbgb, bggb, ggbb, bggg, gbgg, ggbg, gggb, gggg}

The events are:

A = {bbbg, bbgb, bgbb, gbbb, bbgg, bgbg, gbbg, gbgb, bggb, ggbb, bggg, gbgg, ggbg, gggb}

B = {bggg, gbgg, ggbg, gggb, gggg}, and

$A \cap B$ = {bggg, gbgg, ggbg, gggb}

Their probabilities are

$P(A)=\frac{14}{16}=\frac{7}{8}$, $P(B)=\frac{5}{16}$ and $P(A\cap B)=\frac{4}{16}=\frac{1}{4}$.

$P(A)\,P(B)=\frac{7}{8}.\frac{5}{16}\neq P(A\cap B)$.

Hence A and B are dependent events.