WEEK 12

Expectations For Random Variables

Definition:
The ***expected value*** of a random variable is the mean value of the variable X in the sample space, or population, of possible outcomes. *Expected value*, denoted by E(X), can also be interpreted as the mean value that would be obtained from an infinite number of observations on the random variable.

$$E(X) = \begin{cases} \sum_{i=1}^{n} x_i \, P(X = x_i) & \text{for discrete random variable} \\ \int_{-\infty}^{\infty} x \, f(x) \, dx & \text{for continuous random variable} \end{cases}$$

Expected Values of functions of r.v.

If $h(X)$ is a function of X, then

$$\mu_{h(X)} = E[h(X)] = \int_{-\infty}^{\infty} h(x) \cdot f(x) \, dx$$

$$E[h(X)] = E[aX + b] = aE[X] + b$$

Properties of Expectation

Property 1. Addition Theorem of Expectation

If X and Y are random variables then $E(X + Y) = E(X) + E(Y)$, provided all the expectation exists.

Property 2: Multiplication theorem of Expectation

If X and Y are independent random variables, then $E(XY) = E(X).E(Y)$

Property 3: If X is a random variable and 'a' is constant.

(i) $E[a\, H(X)] = a\, E[H(X)]$

(ii) $E[H(X) + a] = E[H(X)] + a$

Where H(X) is a function of X, is a r.v and all the expectation are exists.

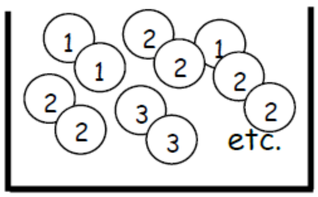
4.2 Variance

The variance of a random variable X is defines as

$$Var(X) = E(X^2) - (E(X))^2$$

Example:

Consider a population consisting of 100 families in a community. Suppose that 30 families have just 1 child, 50 families have 2 children, and 20 families have 3 children. What is the mean or average number of children per family for this population?



Population of 100 families

Mean = **(sum of all values)/100**
= [1(30) + 2(50) + 3(20)]/100
= **1**(30/100) + **2**(50/100) + **3**(20/100)
= 1(0.30) + 2(0.50) + 3(0.20)
= 1.9 children per family

Mean = **Sum of (value** x **probability of that value)**

Example: 4.1 Find the expectation and variance of the number on a die when thrown

Solution

Let X be a random variable representing the number on a die when thrown. Then X can take any one of the values 1,2,3,4,5,6 each with equal probability 1/6

x	1	2	3	4	5	6
P(X=x)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$E(X) = \sum_{i=1}^{6} x_i P(X = x_i) = 1\frac{1}{6} + 2\frac{1}{6} + 3\frac{1}{6} + 4\frac{1}{6} + 5\frac{1}{6} + 6\frac{1}{6}$$

$$= \frac{1+2+3+4+5+6}{6}$$

$$E(X) = \frac{21}{6}$$

Example 4.2 If a pair of fair dice is tossed and X denotes the sum of the numbers on them, find the expectation of X.

Solution: Clearly X may be at least 2 and at most 12

X	2	3	4	5	6	7	8	9	10	11	12
P(X)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$E(X)=\sum_{i=2}^{12} x_i P(X=x_i)=2\frac{1}{36}+3\frac{2}{36}+4\frac{3}{36}+5\frac{4}{36}+6\frac{5}{36}+7\frac{6}{36}+8\frac{5}{36}$$

$$+9\frac{4}{36}+10\frac{3}{36}+11\frac{2}{36}+12\frac{1}{36}$$

$$=\frac{1}{36}[2+6+12+20+30+42+48+36+30+22+12]$$

$$E(X)=\frac{252}{36}=7$$

Example 4.3 If X be a random variable with the following probability distribution

X	-3	6	9
P(x)	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$

Find E(X),E(X^2) and E$(2X+1)^2$

Solution

$$E(X)=\sum x_i P(X=x_i)=-3\frac{1}{6}+6\frac{1}{2}+x\frac{1}{3}=\frac{-3+18+18}{6}=\frac{33}{6}=\frac{11}{2}$$

$$E(X)=\frac{11}{2}$$

$$E(X^2)=\sum x_i^2 P(X=x_i)=(-3)^2\frac{1}{6}+6^2\frac{1}{2}+9^2\frac{1}{3}=\frac{93}{2}$$

$$E(X^2)=\frac{93}{2}$$

$$\mathrm{E}(2\mathrm{X}+1)^2=\mathrm{E}\left[4\mathrm{X}^2+4\mathrm{X}+1\right]=\mathrm{E}\left[4\mathrm{X}^2\right]+\mathrm{E}[4\mathrm{X}]+\mathrm{E}[1]$$

$$=4\mathrm{E}\left[\mathrm{X}^2\right]+4\mathrm{E}[\mathrm{X}]+1$$

$$=4.\frac{93}{2}+4.\frac{11}{2}+1=209$$

$$E(2X+1)^2=209$$

Example: 4.4 In a continuous distribution the probability density function of X is $f(x)=\begin{cases}\frac{3}{4}x(2-x) & ,0<x<2\\ 0 & ,otherwise\end{cases}$ Find the expectation of the distribution.

Solution.

$$E(X)=\int_0^2 x\,f(x)\,dx=\int_0^2 x.\frac{3}{4}x(2-x)\,dx$$

$$=\frac{3}{4}\int_0^2 x^2(2-x)\,dx=\frac{3}{4}\int_0^2 2x^2-x^3\,dx$$

$$=\frac{3}{4}\left[2\frac{x^3}{3}-\frac{x^4}{4}\right]_0^2=\frac{3}{4}\left[\left(2\frac{2^3}{3}-\frac{2^4}{4}\right)-\left(2\frac{0^3}{3}-\frac{0^4}{4}\right)\right]$$

$$=\frac{3}{4}\left[2\frac{8}{3}-\frac{16}{4}\right]=\frac{3}{4}\left[\frac{16}{3}-\frac{16}{4}\right]$$

$$=\frac{3}{4}\left[\frac{16}{3}-4\right]=\frac{3}{4}\left[\frac{16-12}{3}\right]=\frac{3}{4}\left[\frac{4}{3}\right]=1$$

$$E(X)=1$$

Example 7.10 **(a)** What is the mathematical expectation of the number of heads when 3 coins are tossed?

(b) What is the expectation of the number of failures preceding the first success in an infinite series of independent trials with constant probability of success? (P.U., [illegible])

a) Let the r.v. X represent the number of heads when three fair coins are tossed. Then [illegible] following p.d.

x	0	1	2	3
$f(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Hence the mathematical expectation of X is

$$E(X) = 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8} = \frac{12}{8} = 1.5$$

It should be noted that $E(X)$ is 1.5, which is not an integer and not any value X could actually have. It simply means that, if the experiment of tossing 3 coins is repeated a very large number of times, we would expect *on the average* to get 1.5 heads.

(ii) Let the r.v. X denote the number of failures preceding the first success. Then as X takes the value 0, 1, 2, 3, …, the respective probabilities are $p, qp, q^2p, q^3p, \ldots$ where $q = 1 - p$.

Hence $E(X) = x_1 f(x_1) + x_2 f(x_2) + x_3 f(x_3) + \ldots$

$$= 0.p + 1.qp + 2.q^2p + 3.q^3p + \ldots$$

$$= qp\,(1 + 2q + 3q^2 + \ldots)$$

$$= qp(1-q)^{-2} = qp(p)^{-2} = \frac{q}{p}$$

Example 7.12 Find the expected value of the r.v. X having the p.d.f.

$$f(x) = 2\,(1-x), \quad 0 < x < 1$$

$$= 0, \quad \text{elsewhere.}$$

Now $E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$

$$= 2\int_0^1 x(1-x)\, dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 2\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{1}{3}$$

Definitions:

Mean and standard deviation of a discrete random variable

Suppose that X is a discrete random variable with possible values $x_1, x_2, x_3, \ldots$ occurring with probabilities $p_1, p_2, p_3, \ldots$, then

the expected value (or mean) of X is given by $\mu = E(X) = \sum x_i p_i$

the *variance* of X is given by $V(X) = \sigma^2 = \sum (x_i - \mu)^2 p_i$

and so the *standard deviation* of X is given by $\sigma = \sqrt{\sum (x_i - \mu)^2 p_i}$

The sums are taken over all possible values of the random variable X.

Variance

The **variance** of a continuous random variable X with pdf $f(x)$ and mean value μ is

$$\begin{aligned}\sigma_X^2 = V(X) &= \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x)\, dx \\ &= E[(X - E(X))^2] \\ &= E(X^2) - E(X)^2\end{aligned}$$

The **standard deviation** (SD) of X is

$$\sigma_X = \sqrt{V(X)}$$

When $h(X) = aX + b$,

$$V[h(X)] = V[aX + b] = a^2 \cdot \sigma_X^2 \text{ and } \sigma_{aX+b} = |a| \cdot \sigma_X$$

We call this expected value the variance and denote it by Var(X) or σ^2. That is $\sigma^2 = E(X-\mu)^2 = E(X^2) - [E(X)]^2$. The positive square root of the variance, as before, is called the *standard deviation*.

It is useful to note the following important results about variance.

i) Var(X) cannot be negative.

ii) Var(a) = 0, where a is a constant.

iii) Var(aX) = a^2 Var(X), where a is a constant.

iv) Var($aX + b$) = a^2 Var(X), where a and b are constants.

Example 7.13 Let X have the following probability distribution:

x_i	1	2	3	4	5
$f(x_i)$	0.2	0.3	0.2	0.2	0.1

Find the probability functions of $3X-1$, X^2 and X^2+2; and find $E(3X-1)$, $E(X^2)$ and $E(X^2+2)$.

The probability distribution of the r.v. $H(X) = 3X-1$, is

Values of X, x_i	1	2	3	4	5
Probabilities, $f(x_i)$	0.2	0.3	0.2	0.2	0.1
Values of $3X-1$, $(3x_i-1)$	2	5	8	11	14

$$E(3X-1) = \Sigma H(x_i) f(x_i) = \Sigma(3x_i - 1) f(x_i)$$

$$= 2 \times 0.2 + 5 \times 0.3 + 8 \times 0.2 + 11 \times 0.2 + 14 \times 0.1$$

$$= 0.4 + 1.5 + 1.6 + 2.2 + 1.4 = 7.1$$

The p.d. of $H(X) = X^2$ is

x_i	1	2	3	4	5
$f(x_i)$	0.2	0.3	0.2	0.2	0.1
x_i^2	1	4	9	16	25

and $\quad E(X^2) = \sum x_i^2 f(x_i)$

$$= 1 \times 0.2 + 4 \times 0.3 + 9 \times 0.2 + 16 \times 0.2 + 25 \times 0.1$$

$$= 0.2 + 1.2 + 1.8 + 3.2 + 2.5 = 8.9$$

Similarly, $\quad E(X^2 + 2) = \sum (x_i^2 + 2) f(x_i)$

$$= 3 \times 0.2 + 6 \times 0.3 + 11 \times 0.2 + 18 \times 0.2 + 27 \times 0.1$$

$$= 0.6 + 1.8 + 2.2 + 3.6 + 2.7 = 10.9$$

Example 7.14 Let X be a r.v. with p.d.f.

$$f(x) = 2(x-1), \qquad 1 < x < 2$$

$$= 0, \qquad \text{elsewhere.}$$

Find the expected values of $H(X) = 2X - 1$ and $H(X) = X^2$.

Now $\quad E(2X-1) = \int_{-\infty}^{\infty} (2x-1) f(x)\,dx = 2\int_1^2 (2x-1)(x-1)\,dx$

$$= 2\int_1^2 (2x^2 - 3x + 1)\,dx = 2\left[\frac{2x^3}{3} - \frac{3x^2}{2} + x\right]_1^2$$

$$= 2\left[\left(\frac{16}{3} - 6 + 2\right) - \left(\frac{2}{3} - \frac{3}{2} + 1\right)\right]$$

$$= 2\left[\frac{4}{3} - \frac{1}{6}\right] = \frac{7}{3}$$

and $$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= 2\int_1^2 x^2(x-1)\,dx = 2\left[\frac{x^4}{4} - \frac{x^3}{3}\right]_1^2$$

$$= 2\left[\left(4 - \frac{8}{3}\right) - \left(\frac{1}{4} - \frac{1}{3}\right)\right]$$

$$= 2\left[\frac{4}{3} + \frac{1}{12}\right] = \frac{17}{6}$$

Example 7.15 If the continuous r.v. X has p.d.f.

$$f(x) = \frac{3}{4}(3-x)(x-5), \quad 3 \le x \le 5$$
$$= 0, \qquad \text{elsewhere.}$$

...ate the arithmetic mean, variance and standard deviation of X.

Now $$E(X) = \int_{-\infty}^{\infty} x\, f(x)\,dx$$

$$= \frac{3}{4}\int_3^5 x.(3-x)(x-5)\,dx$$

$$= \frac{3}{4}\int_3^5 (-x^3 + 8x^2 - 15x)\,dx = \frac{3}{4}\left[-\frac{x^4}{4} + \frac{8x^3}{3} - \frac{15x^2}{2}\right]_3^5$$

$$= \frac{3}{4}\left[\left(-\frac{625}{4} + \frac{1000}{3} - \frac{375}{2}\right) - \left(-\frac{81}{4} + \frac{216}{3} - \frac{135}{2}\right)\right]$$

$$= \frac{3}{4}\left[\left(-\frac{125}{12} + \frac{63}{4}\right)\right] = \frac{3}{4}\left(\frac{64}{12}\right) = 4$$

Again $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

$$= \frac{3}{4}\int_3^5 x^2 (3-x)(x-5)\,dx$$

$$= \frac{3}{4}\int_3^5 (-x^4 + 8x^3 - 15x^2)\,dx = \frac{3}{4}\left[-\frac{x^5}{5} + \frac{8x^4}{4} - \frac{15x^3}{3}\right]_3^5$$

$$= \frac{3}{4}\left[\frac{-1}{5}(3125) + 2(625) - 5(125) - \left\{\frac{-1}{5}(243) + 2(81) - 5(27)\right\}\right]$$

$$= \frac{3}{4}\left[0 + \frac{243}{5} - 162 + 135\right] = \frac{3}{4}\left[\frac{108}{5}\right] = \frac{81}{5}$$

$\therefore \quad \text{Var}(X) = E(X^2) - [E(X)]^2$

$$= \frac{81}{5} - (4)^2 = \frac{1}{5} = 0.2, \text{ and}$$

$$S.D.(X) = \sqrt{0.2} = 0.447$$

the mean = 4, variance = 0.2 an standard deviation = 0.447.

Example 7.16 The continuous *r.v.* X has p.d.f. $f(x)$, where $f(x) = \frac{3}{4}(1+x^2)$ for $0 \le x \le 1$. $E(X) = \mu$ and $Var(X) = \sigma^2$, find $P(|X-\mu| < \sigma)$.

Now $E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$

$$= \frac{3}{4}\int_0^1 x(1+x^2)\, dx = \frac{3}{4}\int_0^1 (x+x^3)\, dx$$

$$= \frac{3}{4}\left[\frac{x^2}{2}+\frac{x^4}{4}\right]_0^1 = \frac{3}{4}\left[\frac{1}{2}+\frac{1}{4}\right] = \frac{9}{16} = 0.5625$$

And $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$

$$= \frac{3}{4}\int_0^1 x^2(1+x^2)\, dx = \frac{3}{4}\int_0^1 (x^2+x^4)\, dx$$

$$= \frac{3}{4}\left[\frac{x^3}{3}+\frac{x^5}{5}\right]_0^1 = \frac{3}{4}\left[\frac{1}{3}+\frac{1}{5}\right]$$

$$= \frac{3}{4}\left(\frac{8}{15}\right) = \frac{2}{5} = 0.4$$

$$\therefore\ Var(X) = E(X^2) - [E(X)]^2 = \frac{2}{5} - \left(\frac{9}{16}\right)^2$$

$$= \frac{107}{1280} = 0.0836,\ \text{so that}$$

S.D. (X) or $\sigma = \sqrt{0.0836} = 0.289$

Now $P(|X-\mu|<\sigma) = P(-\sigma < X-\mu < \sigma)$

$$= P(\mu-\sigma < X < \mu+\sigma)$$

$$= P(0.5625-0.289 < X < 0.5625+0.289)$$

$$= P(0.2735 < X < 0.8515), \text{ and}$$

$$P(0.2735 < X < 0.8515) = \frac{3}{4}\int_{0.2735}^{0.8515}(1+x^2)\,dx$$

$$= \frac{3}{4}\left[x+\frac{x^3}{3}\right]_{0.2735}^{0.8515}$$

$$= \frac{3}{4}\left[0.8515+\frac{(0.8515)^3}{3}-(0.2735+\frac{(0.2735)^3}{3})\right]$$

$$= \frac{3}{4}[1.05729-0.28032] = 0.8527$$

Hence $P(|X-\mu|<\sigma) = 0.8527$.