WEEK 13

Example 2.2 A random variable X has the following probability function

x	0	1	2	3	4	5	6	7
$P(x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

(i) Find k, (ii) Evaluate $P(X<6)$, $P(X\geq 6)$ and $P(o<X<5)$ (iii) Determine the distribution function of X and (iv) $P(X\leq a)>1/2$ find the minimum value of a,

Solution:

$$\sum_{x=0}^{7} P(x_i)=1$$

$$k+2k+2k+2k+3k+k^2+2k^2+7k^2+k=1$$

$$\Rightarrow 10k^2+9k-1=0 \quad \Rightarrow (10k-1)(k+1)=0 \quad \Rightarrow k=\frac{1}{10} \text{ or } k=-1(\text{negative})$$

$$Hence \;\; k=\frac{1}{10}$$

(ii) $P(X<6) = P(X=0)+ P(X=1)+ P(X=2)+ P(X=3)+ P(X=4)+ P(X=5)$

$$= k + 2k + 2k + 2k + 3k + k2$$

$$P(X < 6) = \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} + \frac{1}{100} = \frac{81}{100}$$

$$P(X \geq 6) = 1 - P(X<6) = 1 - \frac{81}{100} = \frac{19}{100}$$

$$P(X \geq 6) = \frac{19}{100}$$

$$P(0<X<5) = P(X=1)+ P(X=2)+ P(X=3)+ P(X=4)$$

$$= k+2k+2k+2k+3k = 8k = \frac{8}{10}$$

$$P(0<X<5) = \frac{8}{10}$$

(iii) *Distribution function of X*

$$F(x) = P(X \leq x)$$

x	$F(x) = P(X \le x)$
0	0
1	$k = \frac{1}{10}$
2	$k + 2k = 3k = \frac{3}{10}$
3	$k + 2k + 2k = 5k = \frac{5}{10}$
4	$k + 2k + 2k + 3k = 8k = \frac{8}{10}$
5	$k + 2k + 2k + 3k + k^2 = 8k + k^2 = \frac{8}{10} + \frac{1}{100} = \frac{81}{100}$
6	$k + 2k + 2k + 3k + k^2 + 2k^2 = 8k + 3k^2 = \frac{8}{10} + \frac{3}{100} = \frac{83}{100}$
7	$k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 9k + 10k^2 = \frac{9}{10} + \frac{10}{100} = 1$

(iv) *$P(X \le a) > 1/2$ find the minimum value of a*

From the distribution function $P(X \le 4) = \frac{8}{10} = \frac{4}{5} > \frac{1}{2}$

$a = 4$

Example 2.3 A discrete random variable X has the following probability distribution

x :	0	1	2	3	4	5	6	7	8
p(x):	a	3a	5a	7a	9a	11a	13a	15a	17a

(i) Find the value of 'a'

(ii) $P(0 < X < 3)$

(iii) $P(X \geq 3)$

(iv) Find the distribution function of X

Solution: We have $\sum_{i=1}^{n} P(X = x) = 1$

a+3a+ 5a+7a+9a+11a+13a+ 15a+17a =1

$\therefore\ 81a = 1 \Rightarrow a = \frac{1}{81}$

$\therefore$ The actual probability distribution is

x	0	1	2	3	4	5	6	7	8
P(X=x)	$\frac{1}{81}$	$\frac{3}{81}$	$\frac{5}{81}$	$\frac{7}{81}$	$\frac{9}{81}$	$\frac{11}{81}$	$\frac{13}{81}$	$\frac{15}{81}$	$\frac{17}{81}$

$$P(0 < X < 3) = P(X = 1) + P(X = 2) = \frac{3}{81} + \frac{5}{81} = \frac{8}{81}$$

$$P(0 < X < 3) = \frac{8}{81}$$

$$P(X \geq 3) = 1 - P(X<3) = 1 - \left\{\frac{1}{81} + \frac{3}{81} + \frac{5}{81}\right\} = \frac{72}{81}$$

The distribution function of X is

x	0	1	2	3	4	5	6	7	8
F(x)	0	$1/81$	$4/81$	$9/81$	$16/81$	$25/81$	$36/81$	$49/81$	1

Example 2.4 For the following density function, f(x) = $ae^{-|x|}$, $-\infty < x < \infty$, find the value of 'a'

Solution:

Given f(x) is a pdf.

$$\therefore \int_{-\infty}^{\infty} f(x)dx = 1$$

$$a \int_{-\infty}^{\infty} e^{-|x|} dx = 1$$

$$2a \int_{0}^{\infty} e^{-x} dx = 1$$

$$2a\left(\frac{e^{-x}}{-1}\right)_{0}^{\infty} = 1$$

$$2a\left(\frac{e^{-\infty}}{-1} - \frac{e^{-0}}{-1}\right) = 1$$

$$2a = 1 \Rightarrow a = \frac{1}{2}$$

Example 2.5 The diameter of an electric cable, say X, is assumed to be a continuous random variable with *p.d.f : f(x) = 6x(1-x) , 0≤x≤1.*

(i)Determine a number b such that P(X<b)=P(X>b).

(ii) Compute $P(X \le \frac{1}{2} / \frac{1}{3} \le X \le \frac{2}{3})$

Solution (i)

$$P(X < b) = P(X > b)$$

$$\Rightarrow \int_{0}^{b} f(x)dx = \int_{b}^{1} f(x)\, dx$$

$$\Rightarrow \int_{0}^{b} 6x(1-x)dx = \int_{b}^{1} 6x(1-x)dx$$

$$\Rightarrow 6\int_0^b (x - x^2)dx = 6\int_b^1 (x - x^2)dx$$

$$\Rightarrow \left(\frac{x^2}{2} + \frac{x^3}{3}\right)_0^b = \left(\frac{x^2}{2} + \frac{x^3}{3}\right)_b^1$$

$$\Rightarrow \left[\left(\frac{b^2}{2} + \frac{b^3}{3}\right) - \left(\frac{0^2}{2} + \frac{0^3}{3}\right)\right] = \left[\left(\frac{1^2}{2} + \frac{1^3}{3}\right) - \left(\frac{b^2}{2} + \frac{b^3}{3}\right)\right]$$

$$\Rightarrow 3b^2 - 2b^3 = (1 - 3b^2 + 2b^3)$$

$$\Rightarrow 4b^3 - 6b^2 + 1 = 0$$

$$\Rightarrow (2b - 1)(2b^2 - 2b - 1) = 0$$

$$\therefore 2b - 1 = 0 \quad \Rightarrow \quad b = \frac{1}{2} or$$

$$2b^2 - 2b - 1 = 0 \Rightarrow b = \frac{2 \pm \sqrt{4+8}}{4} = \frac{1 \pm \sqrt{3}}{2}$$

Hence $b = \frac{1}{2}$, is the only real value lying between 0 and 1

$$(ii)\ P(X \le \frac{1}{2} / \frac{1}{3} \le X \le \frac{2}{3}) = \frac{P\left(X \le \frac{1}{2} \cap \frac{1}{3} \le X \le \frac{2}{3} \right)}{P\left(\frac{1}{3} \le X \le \frac{2}{3} \right)}$$

$$= \frac{P\left(\frac{1}{3} \le X \le \frac{1}{2} \right)}{P\left(\frac{1}{3} \le X \le \frac{2}{3} \right)} = \frac{\int_{1/3}^{1/2} 6x(1-x)dx}{\int_{1/3}^{2/3} 6x(1-x)dx}$$

$$= \frac{13/54}{13/27} = \frac{11}{26}$$

$$P(X \le \frac{1}{2} / \frac{1}{3} \le X \le \frac{2}{3}) = \frac{11}{26}$$

Example 2.6 Let X be a continuous random variable with *p.d.f* given by

$$f(x) = \begin{cases} kx & ,0 \le x < 1 \\ k & ,1 \le x < 2 \\ -kx + 3k & ,2 \le x < 3 \\ 0 & ,otherwise \end{cases}$$

(i) find the value of k (ii)Determine the c.d.f

Solution:

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_{0}^{1} kx\,dx + \int_{1}^{2} k\,dx + \int_{2}^{3} (-kx+3k)\,dx = 1$$

$$k\left(\frac{x^2}{2}\right)_0^1 + k(x)_1^2 + \left(-k\frac{x^2}{2}+3kx\right)_2^3 = 1$$

$$k\left(\frac{1^2}{2}-\frac{0^2}{2}\right)+k(2-1)+\left(\left(-k\frac{3^2}{2}+3k3\right)-\left(-k\frac{2^2}{2}+3k2\right)\right)=1$$

$$k\left(\frac{1}{2}\right)+k+\left(\left(-k\frac{9}{2}+9k\right)-\left(-k\frac{4}{2}+6k\right)\right)=1$$

$$\frac{k}{2}+k+\left((k)\left(-\frac{9}{2}+9\right)-(k)(-2+6)\right)=1$$

$$\frac{k}{2}+k+\left((k)\left(\frac{-9+18}{2}-4\right)\right)=1$$

$$\frac{k}{2}+k+\left((k)\left(\frac{-9+18-8}{2}\right)\right)=1$$

$$\frac{k}{2}+k+\frac{k}{2}=1$$

$$\Rightarrow \frac{k+2k+k}{2}=1$$

$$\Rightarrow \frac{4k}{2}=1 \quad \Rightarrow 2k=1 \quad k=\frac{1}{2}$$

(ii) The $c.d.f$

For any x, such that $-\infty<x<0$;

$$F(x)=\int_{-\infty}^{x} f(x)dx = 0$$

For any x, where $0\leq x<1$;

$$F(x)=\int_{-\infty}^{0} 0\,dx+\int_{0}^{x} kx\,dx = k\int_{0}^{x} x\,dx = \frac{1}{2}\left(\frac{x^2}{2}\right)_0^x = \frac{1}{2}\left(\frac{x^2}{2}-\frac{0}{2}\right)=\frac{x^2}{4}$$

For any x, where $1 \le x < 2$;

$$F(x) = \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} kx\,dx + \int_{1}^{x} k\,dx = k\int_{0}^{1} x\,dx + k\int_{1}^{x} dx$$

$$= \frac{1}{2}\int_{0}^{1} x\,dx + \frac{1}{2}\int_{1}^{x} dx = \frac{1}{2}\left(\frac{x^2}{2}\right)_0^1 + \frac{1}{2}(x)_1^x = \frac{1}{2}\left(\frac{1^2}{2} - \frac{0^2}{2}\right) + \frac{1}{2}(x-1) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \frac{1}{2}(x-1)$$

$$= \frac{1}{4} + \frac{x-1}{2} = \frac{1+2(x-1)}{4} = \frac{1+2x-2}{4}$$

$$F(x) = \frac{2x-1}{4}$$

For any x, where $2 \le x < 3$;

$$F(x) = \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} kx\,dx + \int_{1}^{2} k\,dx + \int_{2}^{x} -kx + 3k\,dx = k\int_{0}^{1} x\,dx + k\int_{1}^{2} dx + k\int_{2}^{x} -x + 3\,dx$$

$$=\frac{1}{2}\int_{0}^{1} x\,dx+\frac{1}{2}\int_{1}^{2} dx+\frac{1}{2}\int_{2}^{x} -x+3\,dx$$

$$=\frac{1}{2}\left(\frac{x^2}{2}\right)_0^1+\frac{1}{2}(x)_1^2+\frac{1}{2}\left(-\frac{x^2}{2}+3x\right)_2^x$$

$$=\frac{1}{2}\left(\frac{1^2}{2}-\frac{0^2}{2}\right)+\frac{1}{2}(2-1)+\frac{1}{2}\left(\left(-\frac{x^2}{2}+3x\right)-\left(-\frac{2^2}{2}+3(2)\right)\right)$$

$$=\frac{1}{2}\left(\frac{1}{2}\right)+\frac{1}{2}(1)+\frac{1}{2}\left(\left(-\frac{x^2}{2}+3x\right)-(-2+6)\right)$$

$$=\frac{1}{4}+\frac{1}{2}+\frac{1}{2}\left(-\frac{x^2}{2}+3x-4\right)=\frac{1}{4}+\frac{1}{2}+\left(-\frac{x^2}{4}+\frac{3}{2}x-\frac{4}{2}\right)=\frac{1+2-x^2+6x-8}{4}$$

$$F(x)=\frac{-x^2+6x-5}{4}$$

For any *x*, $x \geq 3$;

$$F(x)=\int_{-\infty}^{0} 0\,dx+\int_{0}^{1} kx\,dx+\int_{1}^{2} k\,dx+\int_{2}^{1} -kx+3k\,dx+\int_{3}^{x} 0\,dx=k\int_{0}^{1} x\,dx+k\int_{1}^{2} dx+k\int_{2}^{3} -x+3\,dx$$

$$=\frac{1}{2}\int_{0}^{1} x\,dx+\frac{1}{2}\int_{1}^{2} dx+\frac{1}{2}\int_{2}^{3} -x+3\,dx$$

$$=\frac{1}{2}\left(\frac{x^2}{2}\right)_0^1+\frac{1}{2}(x)_1^2+\frac{1}{2}\left(-\frac{x^2}{2}+3x\right)_2^3$$

$$=\frac{1}{2}\left(\frac{1^2}{2}-\frac{0^2}{2}\right)+\frac{1}{2}(2-1)+\frac{1}{2}\left(\left(-\frac{3^2}{2}+3(3)\right)-\left(-\frac{2^2}{2}+3(2)\right)\right)$$

$$=\frac{1}{2}\left(\frac{1}{2}\right)+\frac{1}{2}(1)+\frac{1}{2}\left(\left(-\frac{9}{2}+9\right)-(-2+6)\right)$$

$$=\frac{1}{4}+\frac{1}{2}+\frac{1}{2}\left(-\frac{9}{2}+9-4\right)=\frac{1}{4}+\frac{1}{2}+\frac{1}{2}\left(\frac{-9}{2}+5\right)=\frac{1}{4}+\frac{1}{2}+\frac{1}{2}\left(\frac{-9+10}{2}\right)=\frac{1}{4}+\frac{1}{2}+\frac{1}{4}=1$$

$$F(x)=1$$

Hence the distribution function F(x) is given by

$$F(x) = \begin{cases} 0 & \text{for } -\infty \le x < 0 \\ \dfrac{x^2}{4} & \text{for } 0 \le x < 1 \\ \dfrac{2x-1}{4} & \text{for } 1 \le x < 2 \\ \dfrac{-x^2+6x-5}{4} & \text{for } 2 \le x < 3 \\ 1 & \text{for } 3 \le x < \infty \end{cases}$$

Example 2.7 The cumulative distribution of continuous random variable X is given by

$$F(x) = \begin{cases} 0, & x < 0 \\ x^2, & 0 \le x < \frac{1}{2} \\ 1 - \frac{3}{25}(3-x), & \frac{1}{2} \le x < 3 \\ 0, & x \ge 0 \end{cases}$$

Find (i) Probability density function of X (ii) $P(|X| \le 1)$ and (iii) $P(\frac{1}{3} \le X < 4)$

Solution:

We know that $f(x) = \dfrac{d}{dx} F(x)$

The points x = 0, ½ , 3 are points of continuity

$$\therefore f(x) = \begin{cases} 0, & x < 0 \\ 2x, & 0 \le x < \frac{1}{2} \\ \frac{6}{25}(3-x), & \frac{1}{2} \le x < 3 \\ 0, & x \ge 3 \end{cases}$$

$$P(|X| \le 1) = P(-1 \le X \le 1) = \mathrm{F}(1) - \mathrm{F}(-1) = \frac{3}{25}$$

$$\mathrm{P}(\tfrac{1}{3} \le \mathrm{X} < 4) = \mathrm{F}(4) - \mathrm{F}(\tfrac{1}{3}) = 1 - \frac{1}{9} = \frac{8}{9}$$

Discrete distributions

i) Binomial ii) Poisson

BINOMIAL DISTRIBUTION:

Binomial distribution is also known as Bernoulli distribution after the Swiss mathematician James Bernoulli (1654-1705) who discovered it in 1700 and was first published in 1712, eight years of his death. The distribution can be used in the following conditions

(i) The outcome of any trial can only take on two possible values, say success and Failure.
(ii) There is a constant probability p of success on each trial;
(iii) The experiment is repeated n times (i.e. n trials are conducted);

(iv) The trials are statistically independent (i.e. the outcome of past trials does not Affect subsequent trials);

Definition : A random variable X is said to be follow a binomial distribution if its probability function is given by

$$P(X=x) = {}^{n}c_{x}\, p^{x} q^{n-x},\quad x = 0,1,2,\ldots\ldots,n$$

$$\text{and } p + q = 1$$

Where n, p is called parameters of the binomial distribution. Mean and variance of the Binomial distribution is np and npq

Example 6.32 Five coins are tossed once (or one coin is tossed five times). What is the probability of getting precisely 3 heads?

We know that the probability of getting a head is $\frac{1}{2}$,

$$\therefore\quad P(\text{3 heads on a single toss of 5 coins}) = \binom{5}{3}\left(\frac{1}{2}\right)^{3}\left(\frac{1}{2}\right)^{5-3}$$

$$= 10 \times \left(\frac{1}{2}\right)^{5} = \frac{5}{16}.$$

Question:

Let X have a binomial distribution with $n=3$ and $p=0.4$
$P\left(X=\frac{3}{2}\right)$, $P(X=2)$, $P(X\le 2)$, $P(X=-2)$ and $P(X\ge 2)$.

Solution:

8.2. (b) The binomial probability distribution with n = 3 and p = 0.4 is

$$f(x) = \binom{3}{x}(0.4)^x(0.6)^{3-x}, \quad \text{for } x = 0, 1, 2, 3.$$

Now $P\left(X=\frac{3}{2}\right) = f\left(\frac{3}{2}\right) = 0$; because a random variable X with a binomial distribution takes only one of the integer values $0, 1, 2, \ldots, n$.

$$P(X=2) = \binom{3}{2}(0.4)^2(0.6)^{3-2} = 0.288;$$

$$\begin{aligned} P(X\le 2) &= \sum_{x=0}^{2}\binom{3}{x}(0.4)^x(0.6)^{3-x} \\ &= \binom{3}{0}(0.4)^0(0.6)^3 + \binom{3}{1}(0.4)^1(0.6)^2 + \binom{3}{2}(0.4)^2(0.6)^1 \\ &= 0.216 + 0.432 + 0.288 = 0.936; \end{aligned}$$

$P(X=-2) = f(-2) = 0$; because a random variable X with a binomial distribution takes only one of the non-negative integer values $0, 1, 2, 3, \ldots, n$.

$$\begin{aligned} P(X\ge 2) &= \sum_{x=2}^{3}\binom{3}{x}(0.4)^x(0.6)^{3-x} \\ &= \binom{3}{2}(0.4)^2(0.6)^1 + \binom{3}{3}(0.4)^3(0.6)^0 \\ &= 0.288 + 0.064 = 0.352. \end{aligned}$$

Example 6.33 If 60 percent of the voters in the City of Lahore prefer candidate X, what is the probability that in a sample of 12 voters exactly 7 will prefer X?

Here: $n = 12,\ k = 7,\ p = 0.60$ and $q = 0.40$

Therefore: $$P(7 \text{ out of } 12 \text{ prefer } X) = \binom{12}{7}(0.60)^7 (0.40)^{12-7}$$

$$= (792)(0.02799)(0.01024) = 0.227$$

Question:

a) The p.d. of a discrete random variable X is

$$f(x) = \binom{3}{x}\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{3-x}, \quad x = 0, 1, 2, 3.$$

Find $E(X)$ and $E(X^2)$.

b) Let X be a random variable with probability distribution

X	-1	0	1	2	3
$f(x)$	0.125	0.50	0.20	0.05	0.125

i) Find $E(X)$ and $\mathrm{Var}(X)$.

ii) Find the p.d. of the r.v. $Y = 2X + 1$. Using the p.d. of Y determine $E(Y)$ and $\mathrm{Var}(Y)$

iii) How are $E(X)$ and $E(Y)$, and $\mathrm{Var}(X)$ and $\mathrm{Var}(Y)$ related?

Solution:

7.22 (a) Computation of the expected values E(X) and E(X^2).

x_i	$f(x_i)$	$x_i f(x_i)$	$x_i^2 f(x_i)$
0	$\binom{3}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^3 = \frac{27}{64}$	0	0
1	$\binom{3}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^2 = \frac{27}{64}$	$\frac{27}{64}$	$\frac{27}{64}$
2	$\binom{3}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^1 = \frac{9}{64}$	$\frac{18}{64}$	$\frac{36}{64}$
3	$\binom{3}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^0 = \frac{1}{64}$	$\frac{3}{64}$	$\frac{9}{64}$
Σ	1	$\frac{48}{64}$	$\frac{72}{64}$

Hence $E(X) = \sum x_i f(x_i) = \frac{48}{64} = \frac{3}{4}$, and

$$E(X^2) = \sum x_i^2 f(x_i) = \frac{72}{64} = 1\frac{1}{8}.$$

(b) (i) Computation of E(X) and Var(X).

x_i	$f(x_i)$	$x_i f(x_i)$	$x_i^2 f(x)$
-1	0.125	-0.125	0.125
0	0.500	0.000	0.000
1	0.200	0.200	0.200
2	0.050	0.100	0.200
3	0.125	0.375	1.125
Total	1.000	0.550	1.650

Thus $\quad E(X) = \sum x f(x) = 0.55$; and

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = 1.650 - (0.55)^2 = 1.650 - 0.3025$$

$$= 1.3475 = 1.35$$

(ii) Computation of the p.d. of $Y = 2X + 1$, $E(Y)$ and $\text{Var}(Y)$.

$y = (2x+1)$	$f(y)$	$yf(y)$	$y^2f(y)$
-1	0.125	-0.125	0.125
1	0.500	0.500	0.500
3	0.200	0.600	1.800
5	0.050	0.250	1.250
7	0.125	0.875	6.125
Total	1.000	2.100	9.800

$\therefore \quad E(Y) = \sum yf(y) = 2.1$, and

$$\text{Var}(Y) = E(Y^2) - [E(Y)]^2 = 9.80 - (2.1)^2 = 9.80 - 4.41 = 5.39.$$

(iii) For the relationship between $E(X)$ and $E(Y)$ where $Y = 2X + 1$, we have

$$E(Y) = 2.1$$
$$= 2(0.55) + 1 = 2E(X) + 1.$$

For the relationship between $\text{Var}(X)$ and $\text{Var}(Y)$, we have

$$\text{Var}(Y) = 5.39$$
$$= 4(1.3475) = (2)^2 \text{Var}(X).$$

Question:

A multiple-choice quiz has 15 questions, each with 4 possible answers of which only 1 is the correct answer. What is the probability that sheer guess work yields from 5 to 10 correct answers?

Solution:

8.8. (a) The probability of a correct answer is $\frac{1}{4}$ = 0.25, and n = 15.

Let X denote the number of correct answers. Then

$$P(5 \le X \le 10) = \sum_{x=5}^{10} \binom{15}{x} (0.25)^x (0.75)^{15-x} = \sum_{x=5}^{10} b(x;\ 15, 0.25)$$

$$= \sum_{x=0}^{10} b(x;\ 15, 0.25) - \sum_{x=0}^{4} b(x;\ 15, 0.25)$$

$$= 0.9999 - 0.6865 \quad \text{(From binomial tables)},$$

$$= 0.3134$$

Example 2.8 Find the binomial distribution for which the mean is 4 and variance is 3

Solution

Mean =np Variance =npq

Given Mean= np=4

Variance =npq =3

$$\frac{npq}{np} = \frac{3}{4}$$

$$q = \frac{3}{4}$$

$$p = 1 - q = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\therefore np = 4$$

$$n\frac{1}{4} = 4$$

$$n = 16$$

The required binomial distribution is $P(X = x) = 16C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{n-x}$

Example 2.9 Find p for a binomial random variable X if n = 6 and if 9P[X=4] = P[X=2]

Solution:

Let X~ B(6,p)

$$\therefore P[X = x] = 6c_x p^x q^{n-x}, n = 0,1,2 --- 6$$

Given that 9P[X=4] = P[X=2]

$$9 \times 6c_4 p^4 q^2 = 6c_2 p^2 q^4$$

$$9p^2 = q^2$$

$$9p2 = (1-p)^2$$

ie., $8p^2+2p-1 = 0$

ie, $p = \frac{1}{4}$ or $-\frac{1}{2}$

But $p = -\frac{1}{2}$ is impossible

Hence $p = \frac{1}{4}$

Example 2.10 In a binomial distribution consisting of 5 independent trails, probabilities of 1 and 2

Successes are 0.4096 and 0.2048. Find the parameter of 'P' of the distribution

Soln : Let X ~ B(n,p) the probability mass function is

$$P(X=x) = \left({}^{n}C_{x}\right)P^{x}q^{(n-x)} \;\; \mathbf{;\ p+q=1\ ;\ x=0,1,2,3,\ldots\ldots}$$

$$P(X=1) = \binom{5}{1}pq^{4} = 0.4096 \text{ ---------------}\mathbf{(1)}$$

$$P(X=2) = \binom{5}{2}p^{2}q^{3} = 0.2048 \text{ ---------------}\mathbf{(2)}$$

$$= \frac{\binom{5}{1}pq^{4} = 0.4096}{\binom{5}{2}p^{2}q^{3} = 0.2048} = \frac{5q = 0.4096}{10p = 0.2048} = \frac{5(1-p)}{10p} = 2$$

$$= 5(1-p) = 2x10p$$

$$5 - 5p = 20p$$

$$5 = 20p + 5p$$

$$5 = 25p$$

$$==> 25p = 5$$

$$p = \frac{5}{25} = \frac{1}{5}$$

Example 2.11 Ten coins thrown simultaneously. Find the probability of getting at least 7 heads

Solution: Given n = 10, probability of getting a head = p = ½ , q = 1-p = ½

Probability mass function of binomial distribution is

$P(X = x) = nC_x p^x q^{n-x}$, $x = 0, 1, 2, \ldots\ldots 10$

ie., $P(X = x) = 10C_x (1/2)^x (1/2)^{10-x} = 10C_x (1/2)^{10}$

$P(X \geq 7) = P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10)$

$= 10C_7(1/2)^{10} + 10C_8(1/2)^{10} + 10C_9(1/2)^{10} + 10C_{10}(1/2)^{10}$

$P(X \geq 7) = 0.172$

2.4.2 POISSON DISTRIBUTION:

Poisson distribution is a discrete probability distribution, which is the limiting case of the binomial distribution under certain conditions.

1. When n is very indefinitely very large
2. Probability of success is very small.
3. $np = \lambda$ is finite

Definition : A discrete random variable X is said to be follow a Poisson distribution if the probability mass function is given by

$$p(X = x) = P(x; \lambda) = \frac{e^{-\lambda} \lambda^x}{x!}, x = 0,1,2,3.........\infty$$

Where e = 2.7183 and $\lambda > 0$

Here λ is called the parameter of the Poisson distribution.

Example 2.12 The probability of an item to be defective is 0.01. Find the probability that a sample
of 100 items randomly selected will contain not more than one defective item.

Solution: Given p = 0.01, n= 100, mean $\lambda = np = 0.01\times 100 = 1$,

Probability mass function of Poisson distribution is $P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$, x=0,1,2....

$$P(X \le 1) = P(X=0) + P(X=1) = e^{-1} + \frac{e^{-1}.1}{1!} = 2e^{-1}$$

Example 2.13 It is known from the past experience that in a certain plant there are on the average
4 industrial accidents .Find the probability that in a given year there will be less than 4 accidents.

Solution:

Let X denote the number of accidents in a year.

Given $\lambda = 4$

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!},\ x = 0,1,2,3,\ldots\ldots.$$

P(less than 4 accidents) = P(X < 4) = P(X =0)+ P(X =1) + P(X = 2)+ P(X = 3)

$$= \left(\frac{4^0 e^{-4}}{0!} + \frac{4^1 e^{-4}}{1!} + \frac{4^2 e^{-4}}{2!} + \frac{4^3 e^{-4}}{3!}\right) = 0.4335$$

$$= e^{-4}\left(\frac{4^0}{0!} + \frac{4^1}{1!} + \frac{4^2}{2!} + \frac{4^3}{3!}\right) = 0.4335$$

Question:

If X is a Poisson random variable with $\mu = 1.6$, find $P(X=0)$, $P(X=1)$, $P(X=2)$ and $P(X>2)$

Solution:

(c) Since X is a Poisson r.v. with μ = 1.6, therefore

$$P(X=0) = e^{-1.6}\frac{(1.6)^0}{0!} = 0.2019, \qquad (\because e^{-1.6} = 0.2019)$$

$$P(X=1) = \frac{e^{-1.6}(1.6)}{1!} = 0.3230,$$

$$P(X=2) = \frac{e^{-1.6}(1.6)^2}{2!} = 0.2584, \text{ and}$$

$$\begin{aligned} P(X>2) &= 1 - P(X \leq 2) \\ &= 1 - [P(X=0) + P(X=1) + P(X=2)] \\ &= 1 - 0.7833 = 0.2167. \end{aligned}$$

Question:

Suppose that the number of insurance claims closely approximates a Poisson distribution with $\mu = 0.05$. Find the probability of (i) no claim and (ii) 1 or fewer claims.

Solution:

8.36. (a) Let X denote the number of claims. Then the Poisson distribution with $\mu = 0.05$, is

$$p(x;\, 0.05) = \frac{e^{-0.05}\,(0.05)^x}{x!}, \quad x = 0, 1, 2, 3, \ldots$$

Therefore (i) $P(X=0) = e^{-0.05} = 0.9513$

(ii) $P(1 \text{ or fewer claims}) = P(X \leq 1) = e^{-0.05} + e^{-0.05}\,(0.05)$

$$= 0.9513 + (0.9513)(0.05)$$

$$= 0.9513 + 0.0476 = 0.9989.$$

Question:

A secretary makes 2 errors per page on the average. What is the probability that on the page she makes (i) 4 or more errors? (ii) no error?

Solution:

8.37. (a) Let X be the r.v. the number of errors per page. Then X has a Poisson distribution with parameter $\mu = 2$.

So $P(X=x) = \dfrac{e^{-2}(2)^x}{x!}$.

Now (i) $P(X \geq 4) = 1 - \sum_{x=0}^{3} \dfrac{e^{-2}\,(2)^x}{x!}$, where $e^{-2} = 0.1353$.

$$= 1 - e^{-2}\,[1 + 2 + 2 + 4/3]$$

$$= 1(0.1353)(6.3333) = 1 - 0.8569 = 0.1431, \text{ and}$$

(ii) $P(X=0) = e^{-2} = 0.1353$.

Question:

Given that X has a Poisson distribution with variance 1, calculate $P(X=2)$.

Solution:

8.39. (b) As the mean and variance of the Poisson distribution are equal, therefore $\sigma^2 = \mu = 1$; and the Poisson distribution will be

$$P(X=x) = \frac{e^{-1}(1)^x}{x!}.$$

Hence the desired probability is

$$P(X=2) = \frac{e^{-1}(1)^2}{2!} = \frac{e^{-1}}{2!} = \frac{1}{2e} = \frac{1}{2(2.7183)}$$

$$= \frac{1}{5.4366} = 0.1839.$$

Question:

a) Criticise the following statement:

"The mean of a Poisson distribution is 5 while its standard deviation is 4."

b) In a Poisson distribution the first two frequencies were 250 and 160. Find the frequencies of the next two values of the variable.

Solution:

8.40. (a) Let μ be the parameter of the Poisson distribution.

Then mean $= \mu$ and $\sigma = \sqrt{\mu}$, *i.e.* $\sigma = \sqrt{\text{mean}}$

In the given statement, mean $= 5$ and $\sigma = 4$

$$\therefore \qquad \sqrt{\text{mean}}, \quad i.e. \ \sqrt{5} \neq \sigma$$

Hence the given statement is wrong.

(b) Let μ be the parameter of the Poisson distribution and N be the total number of observations. Then the Poisson distribution is

$$\frac{N \,.\, e^{-\mu} \,.\, \mu^x}{x!}, \quad x = 0, 1, 2, \ldots, \infty$$

It is given that

$$N \,.\, e^{-\mu} = 250, \text{ and } N \,.\, e^{-\mu} \,.\, \mu = 160.$$

Dividing the second equation by the first, we get

$$\mu = \frac{16}{25} = 0.64$$

$$\therefore \qquad e^{-0.64} = 1 - (0.64) + \frac{(0.64)^2}{2} - \frac{(0.64)^3}{3!} + \frac{(0.64)^4}{4!} - \ldots$$

$$= 0.5273$$

The frequencies of the next two values are

$$N \,.\, e^{-\mu} \,.\, \frac{\mu^2}{2!} \text{ and } Ne^{-\mu} \,.\, \frac{\mu^3}{3!}$$

Now $\quad N \,.\, e^{-\mu} \,.\, \dfrac{\mu^2}{2!} = (Ne^{-\mu} \,.\, \mu)\left(\dfrac{\mu}{2}\right) = 160 \times \dfrac{0.64}{2} = 51$, and

$$Ne^{-\mu} \,.\, \frac{\mu^3}{3!} = (Ne^{-\mu} \,.\, \frac{\mu^2}{2!})\left(\frac{\mu}{3}\right) = 51 \times \frac{0.64}{3} = 11.$$

Continuous Probability Distributions

- Continuous Uniform Distribution.
- Normal Distribution.

2.4.3 Normal Distribution:

A continuous random variable X is said to follow a Normal distribution with parameter mean μ and variance σ^2 if its probability density is given by

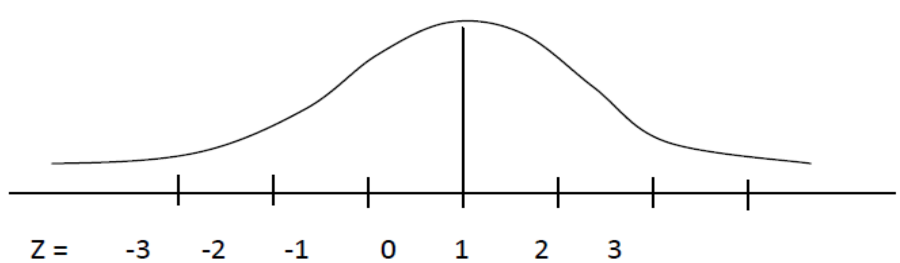
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Characteristics of a normal probability distribution

1. The normal curve is bell-shaped and has a single peak at the exact center of the distribution.
2. The arithmetic mean, median, and mode of the distribution are equal and located at the peak.
3. Half the area under the curve is above and half is below this center point (peak).
4. The normal probability distribution is symmetrical about its mean.
5. It is asymptotic - the curve gets closer and closer to the x-axis but never actually touches it.

The standard normal probability distribution is a normal distribution with a mean of 0 and a standard deviation of 1 is called the standard normal distribution.

$$\mathbf{z} = \frac{(\mathbf{X} - \boldsymbol{\mu})}{\boldsymbol{\sigma}}$$



Example 2.14 X is a normal variate with mean 30 and S.D 5. Find the probabilities that (i) $26 \le X \le 40$ (ii) $X \ge 4\ 5$ (iii) $|X - 30| > 5$

Solution Here mean $\mu = 30$ and $\sigma = 5$

(i)When X = 26 and $Z = \frac{x-\mu}{\sigma} = \frac{26-30}{5} = -0.8$

When X = 40 and $Z = \frac{x-\mu}{\sigma} = \frac{40-30}{5} = 2$

$$
\begin{aligned}
P(26 \le X \le 40) &= P(-0.8 \le X \le 2) \\
&= P(-0.8 \le X \le 0) + P(0 \le X \le 2) \\
&= P(0 \le X \le 0.8) + P(0 \le X \le 2) \text{ (symmetry)} \\
&= 0.2881 + 0.4772 = 0.7653
\end{aligned}
$$

$P(26 \le X \le 40) = 0.7653$

(ii) When X =45 $Z = \frac{x-\mu}{\sigma} = \frac{45-30}{5} = 3$

$P(X. \ge 45) = P(Z \ge 3) = 0.5 - P(0 \le X \le 3) = 0.5 - 0.4986 = 0.0014$

(iii) $P(|X - 30| \le 5) = P(25 \le X \le 35) = P(-1 \le Z \le 1) = 2\,P(0 \le Z \le 1) = 2 \times 0.3413 = 0.6826$

$P(|X - 30| > 5) = 1 - P(|X - 30| \le 5) = 1 - 0.6826 = 0.3174$

Example 2.15 The weight of adult cocker spaniel are normally distributed with a mean $\mu = 25$ lb and a standard deviations $\sigma = 3$ lb. find the probability that a) cocker's weight is less than 23 lb b) weight is between 20 lb and 27 lb c) weight is more than 29 lb

Solution

a) Find the probability that the cocker's weight is less than 23 lb.

$$P(x < 23) = P\left(z < \frac{23-25}{2}\right) = P(z < -.67) = .2514$$

b) Find the probability that the weight is between 20 lb and 27 lb.

$$P(20 < x < 27) = P\left(\frac{20-25}{3} < z < \frac{27-25}{3}\right) =$$

$$P(-1.67 < z < .67) = .7486 - .0475 = .7011$$

c) Find the probability that the weight is more than 29 lb.

$$P(x > 29) = P\left(z > \frac{29-25}{3}\right) = P(z > 1.33)$$

$$= 1 - .9082 = .0918$$

Example 2.16 In a distribution exactly normal, 10.03% of the items are under 25 kilogram weight and 8.97% of the items are under 70 kilogram weight. What are the mean and variance of the distribution?

Solution

Let x denote the weight (in kilograms) of the item.

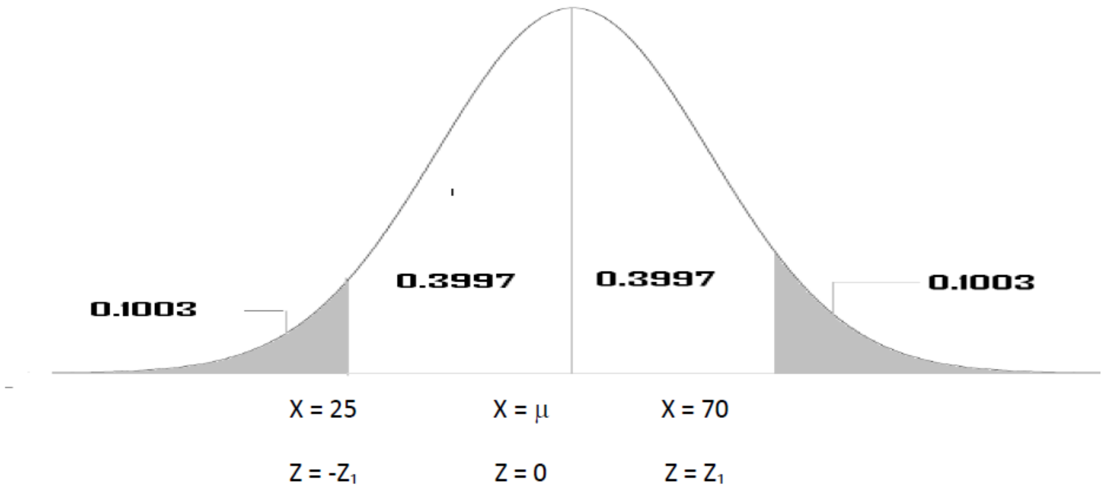
If $X \sim N(\mu, \sigma^2)$ then given are

$P(X < 25) = 0.1003$ and $P(X < 70) = 0.8997$

The points x =25 and x = 70 are located as shown below

When X = 25 , $Z = \frac{25-\mu}{\sigma} = -Z_1(\text{say})$ -----------------(1)

When X = 70 $Z = \frac{70-\mu}{\sigma} = Z_2(\text{say})$ ----------------------(2)



From the diagram

$P(Z < -Z_1) = 0.1003$

$P(Z < Z_2) = 0.8997$ and now $P(0<Z<Z_2) = 0.3997 \Rightarrow Z_2 = 1.28$ (from normal table)

$P(Z < -Z_1) = 0.1003 \Rightarrow P(Z>Z_1) = 0.1003$

$P(0<Z<Z_1) = 0.5 - 0.1003 = 0.3997 \Rightarrow Z_1 = 1.28$ (from normal table)

Substuting the values of Z_1 and Z_2 in (1) and (2)

$$\frac{25-\mu}{\sigma} = -1.28$$

$$\Rightarrow 25 - \mu = -1.28\sigma \quad \text{---------------(3)}$$

$$\frac{70-\mu}{\sigma} = 1.28 \qquad \text{--------------------(4)}$$

$$\Rightarrow 70 - \mu = 1.28\sigma$$

Solving the equation 3 and 4 we get $\mu = 47.5$ and $\sigma = 17.578$