Week 11

Random Variables and Probability Distributions

Random Variables

Suppose that to each point of a sample space we assign a number. We then have a *function* defined on the sample space. This function is called a *random variable* (or *stochastic variable*) or more precisely a *random function* (*stochastic function*). It is usually denoted by a capital letter such as X or Y. In general, a random variable has some specified physical, geometrical, or other significance.

EXAMPLE 2.1

Suppose that a coin is tossed twice so that the sample space is

$$S = \{HH, HT, TH, TT\}.$$

Let X represent the number of heads that can come up. With each sample point we can associate a number for X as shown in Table 2-1. Thus, for example, in the case of HH (i.e., 2 heads), $X = 2$ while for TH (1 head), $X = 1$. It follows that X is a random variable.

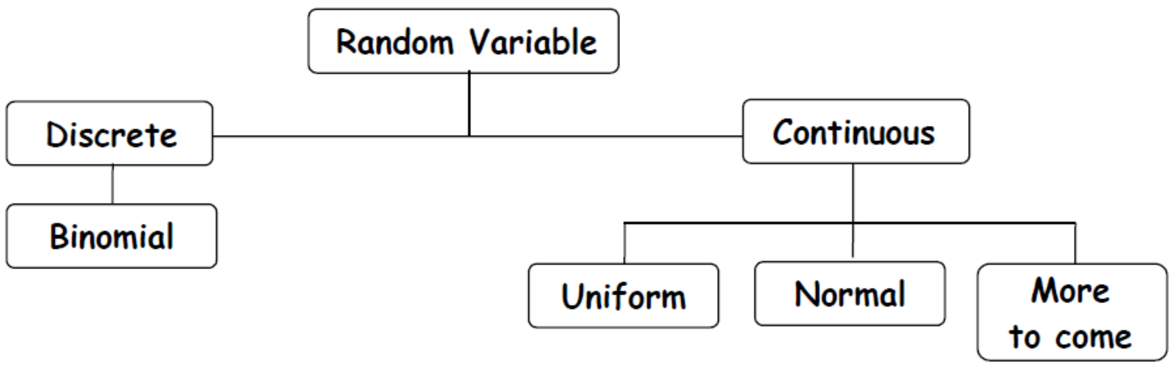
Table 2-1

Sample Point	HH	HT	TH	TT
X	2	1	1	0

It should be noted that many other random variables could also be defined on this sample space, for example, the square of the number of heads or the number of heads minus the number of tails. A random variable that takes on a finite or countably infinite number of values is called a ***discrete random variable*** while one which takes on a noncountably infinite number of values is called a *nondiscrete random variable*.

Definitions:
A ***discrete random variable*** can take one of a countable list of distinct values.
A ***continuous random variable*** can take any value in an interval or collection of intervals.

Types of Random Variables



General Discrete Random Variables

A **discrete** random variable, X, is a random variable with a finite or countable number of possible outcomes. The probability notation your text uses for a Discrete Random Variable is given next:

> ***Discrete Random Variable:***
> X = the random variable.
> k = a number that the discrete random variable could assume.
> $P(X = k)$ is the probability that the random variable X equals k.

The **probability distribution function (pdf) for a discrete random variable *X*** is a table or rule that assigns probabilities to the possible values of the X.
One way to show the distribution is through a table that lists the possible values and their corresponding probabilities:

Value of *X*	x_1	x_2	x_3	...
Probability	p_1	p_2	p_3	...

- **Two conditions** that must apply to the probabilities for a discrete random variable are:
  Condition 1: The sum of all of the individual probabilities must equal 1.
  Condition 2: The individual probabilities must be between 0 and 1.
- A ***probability histogram*** or better yet, a ***probability stick graph***, can be used to display the distribution for a discrete random variable.
  The x-axis represents the values or outcomes.
  The y-axis represents the probabilities of the values or outcomes.
- The **cumulative distribution function (cdf) for a discrete random variable *X*** is a table or rule that provides the probabilities $P(X \le k)$ for any real number k. Generally, the term cumulative probability refers to the probability that X is ***less than or equal to*** a particular value.

Probability Distribution or Probability Function

> The set of ordered pairs $(x, f(x))$ is a **probability function, probability mass function,** or **probability distribution** of the discrete random variable X if, for each possible outcome x,
>
> 1. $f(x) \geq 0$,
> 2. $\sum_x f(x) = 1$,
> 3. $P(X = x) = f(x)$.

Example

A shipment of 20 similar laptop computers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives.

***Solution* :** Let *X* be a random variable whose values *x* are the possible numbers of defective computers purchased by the school. Then *x* can only take the numbers 0, 1, and 2

Now,

$$f(0) = P(X = 0) = \frac{\binom{3}{0}\binom{17}{2}}{\binom{20}{2}} = \frac{68}{95}, \quad f(1) = P(X = 1) = \frac{\binom{3}{1}\binom{17}{1}}{\binom{20}{2}} = \frac{51}{190},$$

$$f(2) = P(X = 2) = \frac{\binom{3}{2}\binom{17}{0}}{\binom{20}{2}} = \frac{3}{190}.$$

Thus, the probability distribution of X is

x	0	1	2
$f(x)$	$\frac{68}{95}$	$\frac{51}{190}$	$\frac{3}{190}$

Example: Find the probability distribution and distribution function for the number of heads when 3 balanced coins are tossed.

Solution: The sample space for this experiment is

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}.$$

Let X be the r.v. that denotes the number of heads. Then the values of x are 0, 1, 2 and 3, and the probabilities are:

$$f(0) = P(X = 0) = P[\{TTT\}] = \frac{1}{8},$$

$$f(1) = P(X = 1) = P[\{HTT, THT, TTH\}] = \frac{3}{8}$$

$$f(2) = P(X = 2) = P[\{HHT, HTH, THH\}] = \frac{3}{8}$$

$$f(3) = P(X = 3) = P[\{HHH\}] = \frac{1}{8}$$

Putting this information in the tabular form, we obtain the desired probability distribution of the

Number of heads (x_i)	0	1	2	3
Probability $f(x_i)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

To obtain a formula, we need expressions for x heads to be selected out of 3 heads and the denominators for all values. Now, x heads can be selected in $\binom{3}{x}$ ways and the total number of outcomes is 8. Therefore the desired probability distribution in the form of an equation is

$$f(x) = P(X = x) = \frac{\binom{3}{x}}{8} = \binom{3}{x}\left(\frac{1}{2}\right)^3 \quad for \ x = 0, 1, 2, 3.$$

For distribution function, we compute the probabilities as below:

If $x < 0$, we have $P(X < x) = 0$

If $0 \le x < 1$, we have $P(X < x) = P(X = 0) = \frac{1}{8}$.

For $1 \le x < 2$, we have

$$P(X < x) = P(X = 0) + P(X = 1) = \frac{1}{8} + \frac{3}{8} = \frac{4}{8}$$

Similarly, for $2 \le x < 3$, we have

$$P(X < x) = P(X = 0) + P(X = 1) + P(X = 2) = \frac{4}{8} + \frac{3}{8} = \frac{7}{8}$$

Finally for $x \ge 3$, we have

$$P(X < x) = \sum_{i=0}^{3} P(X = i) = 1.$$

Hence the desired distribution function is

$$F(x)=\begin{cases} 0\ , & for\ x<0 \\ \frac{1}{8}\ , & for\ 0\le x<1 \\ \frac{4}{8}\ , & for\ 1\le x<2 \\ \frac{7}{8}\ , & 2\le x<3 \\ 1\ , & for\ x\ge 3 \end{cases}$$

Examples 7.2 (a) Find the probability distribution of the sum of the dots when two fair dice are thrown.

(b) Use the probability distribution to find the probabilities of obtaining (i) a sum of 8 or 11, (ii) a sum that is greater than 8, (iii) a sum that is greater than 5 but less than or equal to 10.

(a) The sample space S for the experiment of throwing two dice contains 36 sample points, which are equally likely, i.e. each point has probability $\frac{1}{36}$.

Let X be the random variable representing the sum of dots which appear on the dice. The values of the r.v. are 2, 3, 4, ..., 12. The probabilities of these values are computed as below;

$f(2)=P(X=2)=P[\{1,1\}]=\frac{1}{36}$, as there is only one point resulting in a sum of 2,

$$f(3)=P(X=3)=P[\{(1,2),(2,1)\}]=\frac{2}{36},$$

$$f(4)=P(X=4)=P[\{(1,3),(2,2),(3,1)\}]=\frac{3}{36},$$

Similarly, $f(5)=\frac{4}{36}$, $f(6)=\frac{5}{36}$, $f(7)=\frac{6}{36}$, $f(8)=\frac{5}{36}$, $f(9)=\frac{4}{36}$,

$f(10)=\frac{3}{36}$, $f(11)=\frac{2}{36}$ and $f(12)=\frac{1}{36}$.

Therefore the desired probability distribution of the r.v. X is

x_i	2	3	4	5	6	7	8	9	10	11	12
$f(x_i)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

(b) Using the probability distribution, we get the required probabilities as follows:

i) $P(\text{a sum of 8 or 11}) = P[(X=8) \text{ or } (X=11)]$

$$= P(X=8) + P(X=11)$$

$$= f(8) + f(11) = \frac{5}{36} + \frac{2}{36} = \frac{7}{36}$$

ii) $P(\text{a sum that is greater than 8})$

$$= P(X > 8)$$

$$= P(X=9) + P(X=10) + P(X=11) + P(X=12)$$

$$= f(9) + f(10) + f(11) + f(12)$$

$$= \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{10}{36}$$

iii) $P(\text{a sum that is greater than 5 but less than or equal to 10})$

$$= P(5 < X \leq 10)$$

$$= P(X=6) + P(X=7) + P(X=8) + P(X=9) + P(X=10)$$

$$= f(6) + f(7) + f(8) + f(9) + f(10)$$

$$= \frac{5}{36} + \frac{6}{36} + \frac{5}{36} + \frac{4}{36} + \frac{3}{36} = \frac{23}{36}$$

Question

Three balls are drawn from a bag containing 5 white and 3 black balls. If X de
number of white balls drawn from the bag, then find the p.d. of X.

Solution:

S consists of $\binom{8}{3} = 56$ sample points.

Let X be the r.v. that denotes the number of white balls, drawn from the bag. Then the values of x are 0, 1, 2, 3 and their probabilities are:

$$f(0) = P(X=0) = \binom{5}{0}\binom{3}{3} \div \binom{8}{3} = \frac{1}{56}$$

$$f(1) = P(X=1) = \binom{5}{1}\binom{3}{2} \div \binom{8}{3} = \frac{15}{56}$$

$$f(2) = P(X=2) = \binom{5}{2}\binom{3}{1} \div \binom{8}{3} = \frac{30}{56}$$

$$f(3) = P(X=3) = \binom{5}{3}\binom{3}{0} \div \binom{8}{3} = \frac{10}{56}$$

Putting this information in a tabular form, we obtain the desired probability distribution of X as

No. of white balls: x_i	0	1	2	3	Total
Probability: $f(x_i)$	$\frac{1}{56}$	$\frac{15}{56}$	$\frac{30}{56}$	$\frac{10}{56}$	1

Question

A large store places its last 15 clock radios in a clearance sale. Unknown to any one ... radios are defective. If a customer tests 3 different clock radios selected at rand... the p.d. of X = number of defective radios in the sample?

Solution:

S consists of $\binom{15}{3} = 455$ sample points.

Let X be the r.v. that denotes the number of defective clock radios in the sample. Then the values of x are 0, 1, 2, 3 and their probabilities are

$$f(0) = P(X=0) = \binom{5}{0}\binom{10}{3} \div \binom{15}{3} = \frac{120}{455}$$

$$f(1) = P(X=1) = \binom{5}{1}\binom{10}{2} \div \binom{15}{3} = \frac{225}{455}$$

$$f(2) = P(X=2) = \binom{5}{2}\binom{10}{1} \div \binom{15}{3} = \frac{100}{455}$$

$$f(3) = P(X=3) = \binom{5}{3}\binom{10}{0} \div \binom{15}{3} = \frac{10}{455}$$

Hence the desired probability distribution of X is

No. of defectives: x_i	0	1	2	3	Total
Probability: $f(x_i)$	$\frac{120}{455}$	$\frac{225}{455}$	$\frac{100}{455}$	$\frac{10}{455}$	1

Question:

Suppose X has a p.d. given by

x	-1	0	1
$f(x)$	$3c$	$3c$	$6c$

(i) Determine c. (ii) What is the p.d. of $Y=2X+1$?

Solution:

The probability distribution of $Y = 2X + 1$ is found as below:

x_i	-1	0	1
$f(x_i)$	$3c = \frac{3}{12}$	$\frac{3}{12}$	$\frac{6}{12}$
$y_i = 2x_i + 1$	-1	1	3

Hence the desired *p.d.* of $Y = 2X + 1$ is

y_i	-1	1	3
$f(y_i)$	$3/12$	$3/12$	$6/12$

Continuous Random Variables

A **continuous random variable** is a random variable where the data can take infinitely many values. For example, a random variable measuring the time taken for something to be done is continuous since there are an infinite number of possible times that can be taken. The height of a person, the temperature, the amount of rainfall etc. are examples of **continuous random variable**

Probability Density Function(PDF)

For **any** continuous random variable with probability density function f(x), we have that:

(i) $f(x) \geq 0 \quad \forall\, x$

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Question:

A continuous r.v. X has the p.d.f. as follows:

$$f(x) = \begin{cases} x/2 & \text{for } 0 < x \leq 1 \\ \frac{1}{4}(3-x) & \text{for } 1 < x \leq 2 \\ \frac{1}{4} & \text{for } 2 < x \leq 3 \\ \frac{1}{4}(4-x) & \text{for } 3 < x \leq 4 \\ 0, \text{ elsewhere} & \end{cases}$$

Compute $P(X \geq 3)$, $P(X = 2)$, $P(|X| < 1.5)$ and $P(1 < X < 3)$.

Solution:

Clearly $f(x) \geq 0$, and

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_0^1 \frac{x}{2}\, dx + \frac{1}{4}\int_1^2 (3-x)dx + \frac{1}{4}\int_2^3 dx + \frac{1}{4}\int_3^4 (4-x)\, dx$$

$$= \left[\frac{x^2}{4}\right]_0^1 + \frac{1}{4}\left[3x - \frac{x^2}{2}\right]_1^2 + \frac{1}{4}\left[x\right]_2^3 + \frac{1}{4}\left[4x - \frac{x^2}{2}\right]_3^4$$

$$= \frac{1}{4} + \frac{1}{4}\left[(6-2) - \left(3 - \frac{1}{2}\right)\right] + \frac{1}{4}[3-2]$$

$$+ \frac{1}{4}\left[\left(16 - \frac{16}{2}\right) - \left(12 - \frac{9}{2}\right)\right]$$

$$= \frac{1}{4} + \frac{1}{4}\left(\frac{3}{2}\right) + \frac{1}{4} + \frac{1}{4}\left(\frac{1}{2}\right)$$

$$= \frac{1}{4} + \frac{3}{2} + \frac{1}{4} + \frac{1}{3} = 1.$$

As $f(x)$ is a density function, therefore

$$P(X \geq 3) = \frac{1}{4}\int_3^4 (4-x)\, dx = \frac{1}{4}\left[4x - \frac{x^2}{2}\right]_3^4$$

$$= \frac{1}{4}\left[\left(16 - \frac{16}{2}\right) - \left(12 - \frac{9}{2}\right)\right] = \frac{1}{8};$$

$P(X=2) = 0$, because for a continuous r.v., the probability at any particular value is equal to zero.

$$P(|X| < 1.5) = P(-1.5 < X < 1.5)$$

$$= \int_{-1.5}^{0} 0.\, dx + \int_{0}^{1} \frac{x}{2}\, dx + \frac{1}{4} \int_{1}^{1.5} (3-x)\, dx$$

$$= \left[\frac{x^2}{4}\right]_0^1 + \frac{1}{4}\left[3x - \frac{x^2}{2}\right]_1^{3/2} = \frac{1}{4} + \frac{1}{4}\left[\left(\frac{9}{2} - \frac{9}{8}\right) - \left(3 - \frac{1}{2}\right)\right]$$

$$= \frac{1}{4} + \frac{1}{4}\left[\frac{27}{8} - \frac{5}{2}\right] = \frac{1}{4} + \frac{7}{32} = \frac{15}{32}, \text{ and}$$

$$P(1 < X < 3) = \frac{1}{4}\int_{1}^{2} (3-x)\, dx + \frac{1}{4}\int_{2}^{3} dx$$

$$= \frac{1}{4}\left[3x - \frac{x^2}{2}\right]_1^2 + \frac{1}{4}\left[x\right]_2^3$$

$$= \frac{1}{4}\left[\left(6 - \frac{4}{2}\right) - \left(3 - \frac{1}{2}\right)\right] + \frac{1}{4}[3 - 2]$$

$$= \frac{1}{4}\left(4 - \frac{5}{2}\right) + \frac{1}{4} = \frac{3}{8} + \frac{1}{4} = \frac{5}{8}.$$

Question:

A continuous r.v. X has the $p.d.f.$

$$f(x) = A(2-x)(2+x), \quad 0 \le x \le 2$$
$$= 0, \qquad \text{elsewhere.}$$

Find (i) the value of A, (ii) $P(X = \frac{1}{2})$, (iii) $P(X \le 1)$, (iv) $P(X \ge 2)$, (v) $P(1 \le x \le 2)$.

Solution:

Given $f(x) = A(2-x)(2+x)$, if $0 \le x \le 2$

$= 0$, elsewhere

(i) To find the value of A, we must have that

$$\int_0^2 A(2-x)(2+x)\,dx = 1$$

Solving, $A\int_0^2 (4-x^2)\,dx = 1$ or $A\left[4x - \frac{x^3}{3}\right]_1^2 = 1$ which gives $A = \frac{3}{16}$.

(ii) $P\left(X = \frac{1}{2}\right) = 0$, as probability at a particular value is zero.

(iii) $P(X \le 1) = \frac{3}{16}\int_0^1 (4-x^2)\,dx = \frac{3}{16}\left[4x - \frac{x^3}{3}\right]_0^1$

$$= \frac{3}{16}\left[\left(4 - \frac{1}{3}\right) - 0\right] = \frac{11}{16}.$$

(iv) $P(X \ge 2) = 0$, as outside the range 0 to 2, $f(x) = 0$.

(v) $P(1 \le X \le 2) = \frac{3}{16}\int_1^2 (4-x^2)\,dx = \frac{3}{16}\left[4x - \frac{x^3}{3}\right]_1^2$

$$= \frac{3}{16}\left[\left(8 - \frac{8}{3}\right) - \left(4 - \frac{1}{3}\right)\right]$$

$$= \frac{3}{16}\left[\frac{16}{3} - \frac{11}{3}\right] = \frac{3}{16} \times \frac{5}{3} = \frac{5}{16}.$$

Let X be a continuous r.v. with p.d.f.

$$f(x) = 6x\,(1-x), \; 0 \le x \le 1.$$
$$= 0, \qquad \text{otherwise.}$$

(i) Check that $f(x)$ is a p.d.f.

(ii) Obtain an expression for the distribution function of X.

(iii) Compute $P\left(\frac{1}{3} < X < \frac{2}{3}\right)$ and $P\left(X \le \frac{1}{2} \mid \frac{1}{3} < X < \frac{2}{3}\right)$.

7.8. (i) The function f(x) will be a density function, if

$$f(x) \ge 0 \text{ for every } x, \text{ and } \int_{-\infty}^{\infty} f(x)\,dx = 1.$$

Now the first condition is obvious and the second condition will be satisfied if

$$\int_0^1 6x\,(1-x)\,dx = 1,$$

i.e. $\quad 6\int_0^1 (x - x^2)\,dx = 1 \quad$ *i.e.* $\quad 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1$

i.e. $\quad 6\left[\frac{1}{2} - \frac{1}{3}\right] = 1$, which is true.

Therefore $f(x)$ is a legitimate *p.d.f.*

(ii) The cumulative distribution function (*c.d.f.*) is obtained as

$$F(x) = \int_{-\infty}^{x} f(x)\, dx = 6 \int_{0}^{x} x(1-x)\, dx$$

$$= 6 \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^x = 3x^2 - 2x^3$$

(iii) Now the probability in the interval $\frac{1}{3}$ to $\frac{2}{3}$ is

$$P\left(\frac{1}{3} < X < \frac{2}{3}\right) = \int_{1/3}^{2/3} f(x)\, dx$$

$$= 6 \int_{1/3}^{2/3} (x - x^2)\, dx = 6 \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{1/3}^{2/3}$$

$$= 6 \left[\left(\frac{1}{2} \cdot \frac{4}{9} - \frac{1}{3} \cdot \frac{8}{27}\right) - \left(\frac{1}{2} \cdot \frac{1}{9} - \frac{1}{3} \cdot \frac{1}{27}\right)\right]$$

$$= 6\left[\frac{10}{81} - \frac{7}{162}\right] = 6 \times \frac{13}{162} = \frac{13}{27}\text{; and}$$

$$P(X \le \tfrac{1}{2} / \tfrac{1}{3} \le X \le \tfrac{2}{3}) = \frac{P(\frac{1}{3} \le X \le \frac{1}{2})}{P(\frac{1}{3} \le X \le \frac{2}{3})}\text{, where}$$

$$P(\tfrac{1}{3} \le X \le \tfrac{1}{2}) = \int_{1/3}^{1/2} 6(x - x^2)\, dx = 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{1/3}^{1/2}$$

$$= 6\left[\left(\frac{1}{2}\cdot\frac{1}{4} - \frac{1}{3}\cdot\frac{1}{8}\right) - \left(\frac{1}{2}\cdot\frac{1}{9} - \frac{1}{3}\cdot\frac{1}{27}\right)\right]$$

$$= 6\left[\frac{1}{12} - \frac{7}{162}\right] = 6 \times \frac{13}{324} = \frac{13}{54},$$

$$\therefore\ P(X \le \tfrac{1}{2} / \tfrac{1}{3} \le X \le \tfrac{2}{3}) = \left(\frac{13}{54}\right) / \left(\frac{13}{27}\right) = \frac{13}{54} \times \frac{27}{13} = \frac{1}{2}.$$

Cumulative Distribution Function

The **cumulative distribution function (cdf) for a discrete random variable *X*** is a table or rule that provides the probabilities *P(X ≤ k)* for any real number *k*. Generally, the term cumulative probability refers to the probability that *X* is ***less than or equal to*** a particular value.

Cumulative Distribution Function Example

Question:

The random variable with PDF is given by:

$$f(x) = \begin{bmatrix} k(x^2 + x); if\ 0 \leq x \leq 1 \\ 0; else \end{bmatrix}$$

Find the cumulative distribution function(CDF)

Solution: The random variable with Probability Distribution Function is given to us. Let is find the CDF no

We know that, $\int_{-\infty}^{\infty} f(x)dx = 1$

$k \int_0^1 (x^2 + x)dx = 1$

$k\left(\frac{x^3}{3} + \frac{x^2}{2}\right)_0^1 = 1$

$k\left(\frac{5}{6}\right) = 1$

Therefore , k = $\frac{6}{5}$

The CDF F(X),is the function of PDF and it can be integrated within the interval $(-\infty, x)$

If x is in the interval $(-\infty, x)$, then

$F(x) = \int_{-\infty}^{x} f(x)dx\ F(x) = \int_{-\infty}^{x} 0dx\ F(x) = 0$

If x is in the interval [0,1], then

$F(x) = \int_{-\infty}^{x} f(x)dx\ F(x) = \int_{-\infty}^{0} f(x)dx + \int_0^x f(x)dx\ F(x) = 0 + \frac{6}{5}\left(\frac{x^3}{3} + \frac{x^2}{2}\right)$

If x is in the interval $(1, \infty)$,then

$F(x) = \int_{-\infty}^{x} f(x)dx\ F(x) = \int_{-\infty}^{0} f(x)dx + \int_0^1 f(x)dx + \int_1^x f(x)dx$

$F(x) = 0 + \frac{6}{5}\left(\frac{x^3}{3} + \frac{x^2}{2}\right)_0^1 + 0\ F(x) = \frac{6}{5} \times \frac{5}{6}\ F(x) = 1$

Therefore the cumulative distribution function CDF is given by,

$$F(x) = \begin{cases} 0\ ;\quad if\ x < 0 \\ \frac{6}{5}\left(\frac{x^3}{3} + \frac{x^2}{2}\right); if\ 0 \leq x \leq 1 \\ 1\ ;\ if\ x > 1 \end{cases}$$

Example 7.4 A r.v. X is of continuous type with p.d.f.

$$f(x) = 2x, \quad 0 < x < 1,$$
$$= 0, \quad \text{elsewhere.}$$

Find (i) $P\left(X = \frac{1}{2}\right)$, (ii) $P\left(X \le \frac{1}{2}\right)$, (iii) $P\left(X > \frac{1}{4}\right)$, iv) $P\left(\frac{1}{4} \le X < \frac{1}{2}\right)$

(v) $P\left(X \le \frac{1}{2} \mid \frac{1}{3} \le X \le \frac{2}{3}\right)$.

Clearly $f(x) \ge 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 2x\,dx = 1$.

i) Since $f(x)$ is a continuous probability function, therefore

$$P\left(X = \frac{1}{2}\right) = 0.$$

ii) $P\left(X \le \frac{1}{2}\right) = \int_{-\infty}^{0} 0\,dx + \int_0^{1/2} 2x\,dx = 0 + [x^2]_0^{1/2} = \frac{1}{4}$

iii) $P\left(X > \frac{1}{4}\right) = \int_{1/4}^{1} 2x\,dx + \int_1^{\infty} 0\,dx = [x^2]_{1/4}^{1} + 0 = \frac{15}{16}$

iv) $P\left(\frac{1}{4} \le X < \frac{1}{2}\right) = \int_{1/4}^{1/2} 2x\, dx = [x^2]_{1/4}^{1/2} = \frac{3}{16}$

v) Applying the definition of conditional probability, we get

$$P(X \le \frac{1}{2} | \frac{1}{3} \le X \le \frac{2}{3}) = \frac{P\left(\frac{1}{3} \le X \le \frac{1}{2}\right)}{P\left(\frac{1}{3} \le X \le \frac{2}{3}\right)} = \frac{\int_{1/3}^{1/2} 2x\, dx}{\int_{1/3}^{2/3} 2x\, dx}$$

$$= [x^2]_{1/3}^{1/2} \div [x^2]_{1/3}^{2/3}$$

$$= \frac{5}{36} \times \frac{9}{3} = \frac{5}{12}.$$

Example 7.5 A continuous r.v. X has the *d.f.* $F(x)$ as follows:

$$F(X) = 0, \qquad \text{for } x < 0,$$

$$= \frac{2x^2}{5}, \qquad \text{for } 0 < x \le 1,$$

$$= -\frac{3}{5} + \frac{2}{5}\left(3x - \frac{x^2}{2}\right), \qquad \text{for } 1 < x \le 2,$$

$$= 1 \qquad \text{for } x > 2.$$

Find the p.d. and $P(|X| < 1.5)$.

By definition, we have $f(x) = \frac{d}{dx}F(x)$.

Therefore

$$f(x) = \frac{4x}{5} \quad \text{for } 0 < x \le 1$$

$$= \frac{2}{5}(3-x) \quad \text{for } 1 < x \le 2$$

$$= 0 \quad \text{elsewhere.}$$

$$P(|X| < 1.5) = P(-1.5 < X < 1.5)$$

$$= \int_{-\infty}^{-1.5} 0\,dx + \int_{-1.5}^{0} 0\,dx + \int_{0}^{1} \frac{4x}{5}\,dx + \int_{1}^{1.5} \frac{2(3-x)}{5}\,dx$$

$$= 0 + 0 + \left[\frac{2x^2}{5}\right]_0^1 + \left[\frac{2}{5}\left(3x - \frac{x^2}{2}\right)\right]_1^{1.5}$$

$$= \frac{2}{5} + \frac{2}{5}\left[\left(4.5 - \frac{2.25}{2}\right) - \left(3 - \frac{1}{2}\right)\right]$$

$$= 0.40 + 0.35 = 0.75.$$