Proof. By Proposition 4.4, there exists a split ξ-complete resolution $\mathbf{S} \xrightarrow{\mu} \mathbf{P} \xrightarrow{\pi} M$ of M such that μ_i is an isomorphism for each $i \geq m$. Thus $\mathbf{T}$ and $\mathbf{S}$ are homotopy equivalences by Lemma 4.5, and hence $H^n(\mathcal{C}(\mathbf{S}, N) \cong H^n(\mathcal{C}(\mathbf{T}, N)$ for any integer n. Next we show that $\widetilde{\xi \mathrm{xt}}^n_{\mathcal{P}}(M, N) \cong H^n(\mathcal{C}(\mathbf{S}, N))$ for any integer n.

Note that $\mathbf{S} \xrightarrow{\mu} \mathbf{P} \xrightarrow{\pi} M$ is a split ξ-complete resolution. Then we have a sequence $\mathbf{X} \xrightarrow{\lambda} \mathbf{S} \xrightarrow{\mu} \mathbf{P}$ of complexes such that $X_i \xrightarrow{\lambda_i} S_i \xrightarrow{\mu_i} P_i \overset{\delta_i}{\dashrightarrow}$ is a split $\mathbb{E}$-triangle for each $i \in \mathbb{Z}$. Let $\rho : \mathbf{Q} \longrightarrow N$ be a ξ-projective resolution of N. Applying the functor $\mathcal{C}(-, \mathbf{Q})$ to the sequence above, we have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \overline{\mathcal{C}}(\mathbf{P}, \mathbf{Q}) \arrow[r] \arrow[d] & \mathcal{C}(\mathbf{P}, \mathbf{Q}) \arrow[r] \arrow[d] & \widetilde{\mathcal{C}}(\mathbf{P}, \mathbf{Q}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overline{\mathcal{C}}(\mathbf{S}, \mathbf{Q}) \arrow[r] \arrow[d] & \mathcal{C}(\mathbf{S}, \mathbf{Q}) \arrow[r] \arrow[d] & \widetilde{\mathcal{C}}(\mathbf{S}, \mathbf{Q}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overline{\mathcal{C}}(\mathbf{X}, \mathbf{Q}) \arrow[r] \arrow[d] & \mathcal{C}(\mathbf{X}, \mathbf{Q}) \arrow[r] \arrow[d] & \widetilde{\mathcal{C}}(\mathbf{X}, \mathbf{Q}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

Since $\mathbf{X}$ is bounded above, $\overline{\mathcal{C}}(\mathbf{X}, \mathbf{Q}) = \mathcal{C}(\mathbf{X}, \mathbf{Q})$. For each integer $n \in \mathbb{Z}$, we have

$$H^n(\widetilde{\mathcal{C}}(\mathbf{P}, \mathbf{Q})) \cong H^n(\widetilde{\mathcal{C}}(\mathbf{S}, \mathbf{Q})).$$

Note that $\mathbf{S}$ is a ξ-exact complex such that $\mathcal{C}(\mathbf{S}, Q)$ is exact for all $Q \in \mathcal{P}(\xi)$. Then $\overline{\mathcal{C}}(\mathbf{S}, \mathbf{Q})$ is exact by [14, Proposition A.2(b)]. Thus

$$H^n(\mathcal{C}(\mathbf{S}, \mathbf{Q})) \cong H^n(\widetilde{\mathcal{C}}(\mathbf{S}, \mathbf{Q})), \forall n \in \mathbb{Z}.$$

Note that $\rho : \mathbf{Q} \longrightarrow N$ is a ξ-projective resolution of N. Then for any ξ-projective object P, $\mathcal{C}(P, \rho) : \mathcal{C}(P, \mathbf{Q}) \longrightarrow \mathcal{C}(P, N)$ is a quasi-isomorphism. It is not hard to check that the morphism $\mathcal{C}(\mathbf{S}, \rho) : \mathcal{C}(\mathbf{S}, \mathbf{Q}) \longrightarrow \mathcal{C}(\mathbf{S}, N)$ is a quasi-isomorphism and the proof is similar to that of Lemma 3.8(1). So

$$H^n(\mathcal{C}(\mathbf{S}, \mathbf{Q})) \cong H^n(\mathcal{C}(\mathbf{S}, N)), \forall n \in \mathbb{Z}.$$

Hence for each integer n, we have

$$H^n(\widetilde{\mathcal{C}}(\mathbf{P}, \mathbf{Q})) \cong H^n(\widetilde{\mathcal{C}}(\mathbf{S}, \mathbf{Q})) \cong H^n(\mathcal{C}(\mathbf{S}, \mathbf{Q})) \cong H^n(\mathcal{C}(\mathbf{S}, N)).$$

So $\widetilde{\xi xt}^n_{\mathcal{P}}(M, N) \cong H^n(\mathcal{C}(\mathbf{S}, N)) \cong H^n(\mathcal{C}(\mathbf{T}, N)$ *for all* $n \in \mathbb{Z}$. This completes the proof. □

Remark 4.7. *Assume that $\mathcal{C}$ is a triangulated category and ξ is a proper class of triangles which is closed under suspension (see [7, Section 2.2]). Let M be an object in $\mathcal{C}$ with finite ξ-$\mathcal{G}$projective dimension and N an object in $\mathcal{C}$. By Theorem 4.6, one can check that for any integer*

References

[1] L. Angeleri-Hügel, F.U. Coelho, Infinitely generated tilting modules of finite projective dimension, *Forum Math.* **13** (2001) 239-250.

[2] J. Asadollahi, Sh. Salarian, Gorenstein objects in triangulated categories, *J. Algebra* **281** (2004) 264-286.

[3] J. Asadollahi, Sh. Salarian, Tate cohomology and Gorensteinness for triangulated categories, *J. Algebra* **299** (2006) 480-502.

[4] J. Asadollahi, Sh. Salarian, Cohomology theories for complexes, *J. Pure Appl. Algebra* **210** (2007) 771-787.

[5] L.L. Avramov, H.-B. Foxby, S. Halperin, *Differential Graded Homological Algebra*, preprint, 2009.

[6] L.L. Avramov, A. Martsinkovsky, Absolute, relative, and Tate cohomology of modules of finite Gorenstein dimension, *Proc. London Math. Soc.* **85** (3) (2002) 393-440.

[7] A. Beligiannis, Relative homological algebra and purity in triangulated categories, *J. Algebra* **227** (1) (2000) 268-361.

[8] A. Beligiannis, H. Krause, Thick subcategories and virtually Gorenstein algebras, *Illinois J. Math.* **52** (2008) 551-562.

[9] A. Beligiannis, I. Reiten, *Homological and Homotopical Aspects of Torsion Theories*, Amer. Math. Soc., Rhode Island, 2007.

[10] D.J. Benson, J.F. Carlson, Products in negative cohomology, *J. Pure Appl. Algebra* **82** (1992) 107-130.

[11] D. Bravo, M.A. Pérez, Finiteness conditions and cotorsion pairs, *J. Pure Appl. Algebra* **221** (2017) 1249-1267.

[12] T. Bühler, Exact categories, *Expo. Math.* **28** (2010) 1-69.

[13] H. Cartan, S. Eilenberg, *Homological Algebra*, Princeton Univ. Press, Princeton, 1956.

[14] O. Celikbas, L. W. Christensen, L. Liang, G. Piepmeyer, Stable homology over associative rings, *Trans. Amer. Math. Soc.* **369** (2017) 8061-8086.

[15] X.W. Chen, Homotopy equivalences induced by balanced pairs, *J. Algebra* **324** (2010) 2718-2731.

[16] L.W. Christensen, A. Frankild, H. Holm, On Gorenstein projective, injective and flat dimensions- a functorial description with applications, *J. Algebra* **302** (2006) 231-279.

[17] R. Colpi, J. Trlifaj, Tilting modules and tilting torsion theories, *J. Algebra* **178** (1995) 614-634.

[18] E.E. Enochs, O.M.G. Jenda, *Relative Homological Algebra*, Walter de Gruyter, Berlin-New York, 2000.

[19] S. Estrada, M.A. Pérez, H.Y. Zhu, Balanced pairs, cotorsion triplets and quiver representations, *Proc. Edinb. Math. Soc.* **63** (2020) 67-90.

[20] F. Goichot, Homologie de Tate-Vogel équivariante, *J. Pure Appl. Algebra* **82** (1992) 39-64.

[21] R. Göbel, J. Trlifaj, *Approximations and Endomorphism Algebras of Modules*, Walter de Gruyter, Berlin-New York, 2006.

[22] H. Holm, Gorenstein homological dimensions, *J. Pure Appl. Algebra* **189** (2004) 167-193.

[23] J.S. Hu, D.D. Zhang, P.Y. Zhou, Proper classes and Gorensteinness in extriangulated categories, *J. Algebra* **551** (2020) 23-60.

[24] J.S. Hu, D.D. Zhang, P.Y. Zhou, Gorenstein homological dimensions for extriangulated categories (arXiv:1908.00931, 2019).

[25] Z.Y. Huang, Generalized tilting modules with finite injective dimension, *J. Algebra* **311** (2007) 619-634.

[26] H. Krause, Smashing subcategories and the telescope conjecture: An algebraic approach, *Invent. Math.* **139** (2000) 99-133.

[27] F. Mantese, I. Reiten, Wakamatsu tilting modules, *J. Algebra* **278** (2004) 532-552.

[28] G. Mislin, Tate cohomology for arbitrary groups via satellites, *Topology Appl.* **56** (1994) 293-300.

[29] H. Nakaoka, Y. Palu, Extriangulated categories, Hovey twin cotorsion pairs and model structures, *Cahiers de Topologie et Geometrie Differentielle Categoriques*, Volume LX-2 (2019) 117-193.

[30] W. Ren, Z.K. Liu, Balance of Tate cohomology in triangulated categories, *Appl. Categ. Struct.* **23** (6) (2015) 819-828.

[31] W. Ren, R.Y. Zhao, Z.K. Liu, Cohomology theoreies in triangulated categories, *Acta Math. Sin. Engl. Ser.* **32** (11) (2016) 1377-1390.

[32] O. Veliche, Gorenstein projective dimension for complexes, *Trans. Amer. Math. Soc.* **358** (2006) 1257-1283.

[33] T. Wakamatsu, Tilting modules and Auslander's Gorenstein property, *J. Algebra* **275** (2004) 3-39.

[34] J. Wang, Y.X Li, J.S. Hu, When the kernel of a complete hereditary cotorsion pair is the additive closure of a tilting module, *J. Algebra* **530** (2019) 94-113.

[35] J.F. Wang, H.H. Li, Z.Y. Huang, Applications of exact structures in abelian categories, *Publ. Math. Debrecen* **88** (2016) 269-286.

[36] J.Q. Wei, Auslander bounds and homological conjectures, *Rev. Mat. Iberoam.* **27** (2011) 871-884.

[37] F. Zareh-Khoshchehreh, M. Asgharzadeh, K. Divaani-Aazar, Gorenstein homology, relative pure homology and virtually Gorenstein rings, *J. Pure Appl. Algebra* **218** (12) (2018) 2356-2366.

[38] P.Y. Zhou, B. Zhu, Triangulated quotient categories revisited, *J. Algebra* **502** (2018) 196-232.

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