Discrete Mathematics:

Combinatorics.

• Euler totient fn :

→ $\phi(n) = \dfrac{n(P_1-1)(P_2-1)(P_3-1)\cdots}{P_1 P_2 P_3 \cdots}$

→ $\phi(P) = P-1$

→ $\phi(a \times b) = \phi(a) \times \phi(b)$

* when, a & b are coprimes.

• Inclusion & Exclusion :

→ $A \cup B = A + B - A \cap B$

→ #elements divisible by k in $[1, N]$ = $\text{floor}\left(\frac{n}{k}\right)$.

→ #elements divisible by k, R in $[1, N]$ = $\text{floor}\left(\frac{n}{\text{lcm}(k,R)}\right)$.

• Binomial Theorem :

→ $(a+b)^n = \Sigma \binom{n}{i} a^i b^{n-i}$

• Catalan's Number :

→ $C_n = \binom{2n}{n} - \binom{2n}{n-1}$

$= \dfrac{1}{n+1}\binom{2n}{n}$

• Generating fn :

→ $G(x) = a_0x^0 + a_1x^1 + \cdots a_ix^i + \cdots \infty$

→ $G(x) = \sum_{i=0}^{\infty} a_i x^i$

• Permutation & Combination

→ ${}^nP_r = \dfrac{n!}{(n-r)!}$ → ${}^nC_r = \dfrac{n!}{(n-r)!\,r!}$

→ $\binom{n}{r} = \binom{n}{n-r}$. also →

→ $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$: Pascals Triangle.

→ $\binom{-n}{k} = \binom{n+k-1}{k}(-1)^k$: extended Binomial Theorem.

• Dearrangement.

D_n = # perm. of n elements where, none are in the correct place.

→ $D_n = n!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \cdots \frac{(-1)^n}{n!}\right]$

→ $D_1 = 0$ $D_4 = 9$

$D_2 = 1$ $D_5 = 44$

$D_3 = 2$ $D_6 = 265$.

• Some important Series :

→ $\frac{1}{1-x} = 1 + x + x^2 + \cdots \infty$

→ $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \infty$

→ $\frac{1}{1-ax} = 1 + ax + (ax)^2 + \cdots \infty$

→ $\frac{1}{1+ax} = 1 - ax + (ax)^2 - \cdots \infty$

→ $\frac{1}{(1-x)^2} = \frac{d}{dx}\left[\frac{1}{1-x}\right]$

$= 1 + 2x + 3x^2 + \cdots \infty$

• Pigeon hole Principle :

If there are 'm' pigeons & 'n' holes, then there is atleast $\lceil \frac{m}{n} \rceil$ pigeons in atleast one hole.

→ Always consider worst case distribution.

Functions

• Basic functions

$f : A \to B$: mapping

→ All elements in A must be mapped

→ One element in A points to exactly one element at B

• No of function.

$= B^A$

• One - one function.

A fn $f : A \to B$ is one-one if :

→ $|A| \leq |B|$

→ $\forall a \forall b\, (f(a) = f(b) \to a = b)$.

• No of one - one fn :

$= {}^BP_A$

- **Onto function:**

A fn $f: A \to B$ is onto iff:

→ $|A| \geq |B|$

→ $\forall b \exists a$

- **Bijective function:**

→ $|A| = |B|$

→ one-one

→ onto

- **No of Bijective fn**

$= n! = A! = B!$

- **No of onto fn:**

$$= \sum_{i=0}^{B} (-1)^i \binom{B}{i} (B-i)^A .$$

- $\boxed{\text{NOTE}}$

→ $f(S \cup T) = f(S) \cup f(T)$

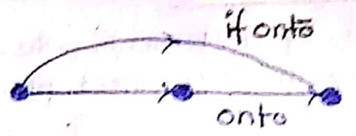
→ $f(S \cap T) \subseteq f(S) \cap f(T)$

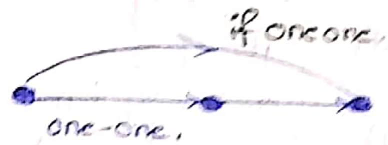
- **function composition:**

→ $g \circ f = g(f(x))$

→ $f \circ g = f(g(x))$.

- **Summary (fn composition):**

→ 

→ 

$\boxed{\text{Relations.}}$

→ **defn:** Subset of cross product.

If $|A| = n$ & $|B| = m$.

Then, $|A \times B| = n \times m$.

- **No of relations possible**

$= 2^{mn}$

$= 2^{n^2}$ <Same set>.

→ **REF**: $\forall a \in A, aRa$

→ **SYM**: $\forall a \forall b ((a,b) \to (b,a))$

→ **ANT**: $\forall a \forall b ((a,b) \wedge (b,a) \to a = b)$

→ **ASY**: $\forall a \forall b ((a,b) \to (b,a) \notin R)$.

→ **IRR**: $\forall a \in A, (a,a) \notin R$.

→ **TRA**: $\forall a \forall b \forall c ((a,b)(b,c) \to (a,c))$.

- **Closure properties**

KoR	∪	∩
REF	✓	✓
IRR	✓	✓
SYM	✓	✓
ANT	~	✓
ASY	~	✓
TRA	~	✓
EQ	~	✓
PAR	~	✓

- **No of possible relations** (w/ given types)

∩	IRR	SYM	ANT	ASY
REF	0	$1^n \times 2^{n(n-1)/2}$	$1^n \times 3^{(n^2-n)/2}$	0
IRR	–	$1^n \times 2^{n(n-1)/2}$	$1 \times 3^{(n^2-n)/2}$	$1 \times 3^{(n^2-n)/2}$
SYM	–	–	$2^n \times 1$	$1 \times 1 (= \phi)$
ANT	–	–	–	$1 \times 3^{n(n-1)/2}$

- **Equivalence Relation:** (RST)

→ (i) REF (ii) SYM (iii) TRA. then EQ Relation

→ No of EQ Relation $= B_n$ ← Bell number.

→ Smallest EQ Relation : Δ_A

→ Largest EQ Relation : $(A \times A)$

- **Partial order relation**

→ (i) REF (ii) ANT (iii) TRA.

→ Smallest PAR : Δ_A

→ Largest PAR : depends on set.

* How many relations, both PAR & EQ = Δ_A = 1 Relation.

- **EQ classes:**

Let R is a relation on A. R ≡ EQ Reln.

$\forall x \in A;\ [x] = \{ y \mid (x, y) \in R,\ y \in A \}$

⊛ iff union of all EQ class gives back original set A, then it's called a partition

- **Partition.**

Let $P_1 P_2 \cdots P_n \subseteq A$ ie all are subset of A. And it satisfies the following condition :

<i> $P_i \cap P_j = \phi \quad \forall i \neq j$ ... disjoint (exclusive.)

<ii> $\bigcup_{i=1}^{n} P_i = A$ ... mutually exhaustive.

then, $P = \{P_1 P_2 \cdots P_n\}$ is partition of A.

- **Set Refinement** ($\prec$)

$P_1 = \{a_1 a_2 \cdots a_n\}$

$P_2 = \{b_1 b_2 \cdots b_n\}$

P_1 is refinement of P_2 iff $\forall a_i \in P_1$, $\exists\, b_j \in P_2$ such that $A_i \subseteq B_j$

- $\boxed{\text{Any}}$ **closure**

→ Smallest $\boxed{\text{Any}}$ relation which contains original relation.

→ (Set, PAR Relation) : POSET

→ (Set, Total order) : TOSET.

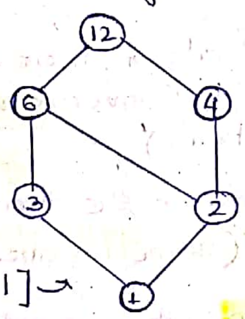
Lattice

- **Hasse diagram.**

→ No self Loops } implied.

→ No TRA Edge }

$[D_{12}, |]$ →



- **greatest element :**

→ all elements ∈ A ≤ x ← Always unique.

- **Least element :**

→ x ≤ all elements ∈ A

- **Greatest Lower bound - GLB** $(\cap, \wedge, \circ)$

→ LB's of B ≤ x ∈ A ≤ all elements ∈ B.

Lower point of convergence.

- **Least upper bound - LUB** $(\cup, \vee, +)$

→ all elements ∈ B ≤ x ∈ A ≤ UB's of B

Greatest point of convergence.

- **Lattice :** $[L, \wedge, \vee]$

GLB & LUB exist for each pair & they are unique.

→ $a \wedge b = b \wedge a$, $a \vee b = b \vee a$] Commutative Law.

→ $a \vee a = a$, $a \wedge a = a$] idempotent Law.

→ $a \wedge (a \vee b) = a$, $a \vee (a \wedge b) = a$] absorptive Law.

→ $a \vee (b \vee c) = (a \vee b) \vee c$, $a \wedge (b \wedge c) = (a \wedge b) \wedge c$] Associative Law.

• Bounded Lattice :

→ greatest & least element exists.

→ greatest element = 1

→ Least element = 0

• Complement Lattice.

→ Bounded Lattice

→ $a + b = 1$

$a \cdot b = 0$

→ All elements must have atleast one complement

• Sub Lattice

B is a sub-Lattice of A iff :

→ $B \subseteq A$

→ B itself is a Lattice.

• Distributive Lattice

Every pair must satisfy :

→ $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

→ $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

⇒ **theorem** : Lattice is distributive iff. every vertex has exactly one complement.

• Boolean Algebra.

Lattice + Complemented + distributive.

Groups

• Quasi Group	• Semi Group	• Monoid	• Group	• Abelian
→ closed	→ closed	→ closed	→ closed	→ closed
	→ ass	→ ass	→ ass	→ ass
		→ identity	→ identity	→ identity
			→ inverse	→ inverse
				→ Comm.

• Order of Group o(G)

→ The no of elements in Group.

→ $o(G) = n$ (finite Group)

→ $o(G) = 0$ or ∞ (inf. Gr.)

→ upto $o(G) = 5$, all Groups are Abelian group.

→ $o(G) \geq 6$: may or may not Abelian.

*Note : $G = \{1, \omega, \omega^2, \ldots \omega^{P-1}\}$ where $P \in$ Prime

→ $\{\omega^P = 1\}$ → Always an Abelian Gr. → $(\omega^r)^{-1} = \omega^{P-r}$ → $e = 1$

• Properties of Group :

→ e is unique

→ $\forall a \in G$, a^{-1} is unique.

→ $\forall a, b \in (G, \circ)$: $(a \circ b)^{-1} = b^{-1} \circ a^{-1}$

$(a^{-1})^{-1} = a$

→ $\forall a \in G$ if $a^{-1} = a$, then G is Abelian (Converse need not be true.)

→ $\forall a \in G$ if $a \circ a = e$ then G is Abelian (same as above)

• Order of an element :

$o(a) = n$, n = Least +ve integer s.t.

→ $a^n = e$.

* the min value of n is order.

→ $o(e) = 1$

→ $\forall a \in G$ $o(a) \mid o(G)$.

→ $\forall a \in G$ $o(a) = o(a^{-1})$

→ $\forall a \in G$, if $o(a) = n$ & $a^m = e$ then $n \mid m$.

• Sub Group : H is a sub Grp of G, iff :

→ $H \subseteq G$

→ H is a group

→ $[H = e$ & $H = G]$: Trivial

- Lagrange's theorem :

Let G be a finite Gr. if H is sub Gr. of G, then always :

→ $o(H) \mid o(G)$ → Hence if $o(G) =$ Prime → $o(H) = 1$, $o(H) = P$] Trivial only.

* Converse of this may be false.

- Properties of Sub-Gr :

H & K are 2 Sub Gr. of G

→ $(H \cup K)$ may not be sub-gr of G

→ ∃ $(H \cup K)$ is Sub Gr. of G if $H \subseteq K$ OR $K \subseteq H$.

→ $(H \cap K)$ is always sub gr.

- Cyclic Group :

∃ $a \in G$ s.t $G = \{a^n \mid n \in \mathbb{Z}^+\}$ then G → cyclic gr.

→ a is generator. ∴ Representation : $G = \langle a \rangle$.

→ if $o(G) > 1$ then $G \neq \langle e \rangle$.

→ if a is generator then a^{-1} also.

→ if G → cy. Gr. & $o(G) = n$, #generator $= \phi(n)$.

→ Every gr. of prime order is cyclic.

Graph Theory.

- Basic Concepts :

→ Graph : $G(V,E)$ $V > 0 ; E \geq 0$;

→ Theo : $\sum d(v_i) = 2e$.

→ Theo : #odd Deg vertex = EVEN.

→ Theo : In simple graph, atleast 2 vertex will have same degree.

→ max deg : $\Delta(G)$

min-deg : $\delta(G)$.

→ $\delta(G) \leq \frac{2e}{n} \leq \Delta(G)$.
↳ Avg. deg.

- Bipartite Graph :

$G(V,E)$ s.t. V can be divided into 2 sets V_1 & V_2 such that edges will be from one set to another but no in same.

→ Theo : Bipartite Graph cannot contain odd-Len cycle.

→ Theo : Every even-Len cy. can be converted to Bi-partite.

- Complete Bipartite graph : $(K_{m,n})$

$\|V_1\| = m$; $\|V_2\| = n$.

→ $\delta(K_{m,n}) = \min(m,n)$

$\Delta(K_{m,n}) = \max(m,n)$.

→ $V = m+n$
$E = m * n$.

→ $\max(e) = \left\lfloor \frac{n^2}{4} \right\rfloor$ ← when split b/w m & n is as close as possible.

- Complement of a Graph :

→ $K_n - G = \bar{G}$.

→ one of G or $\bar{G}$ will always be conn.

→ if G = discon. ; $\bar{G}$ = connected

if G = con. ; $\bar{G}$ = discon/con.

- Hyper Cube (Q_n).

→ $\deg(v_i) = n$

→ $\chi(Q_n) = 2$

→ Total $V = 2^n$

→ diameter $(Q_n) = n$.

Connected & disconnected :

→ Path exists b/w all pair of vertex → Conn.

→ $(n-1) \le e \le n(n-1)/2$] for conn. graph. ; $K=1$

→ $(n-k) \le e \le (n-k)(n-k+1)/2$] for disconn. $[K \ge 2]$

→ if config of disconn graph given : then calculate max edges manually.

→ **Theo** : $\boxed{\underset{\text{Vertex}}{K(G)} \le \underset{\text{Edge.}}{\lambda(G)} \le \delta(G) \le \frac{2e}{n} \le \Delta G \le n-1}$

→ **Theo** : if cut Edge exists then cut vertex also exists $(n \ge 3)$

Walk | Path | Trail :

	Repeat E	Repeat V
Walk →	✓	✓
Trail →	×	✓
path →	×	×

→ **Theo** : Euler Graph is possible iff deg of all vertices are EVEN.

→ **theo** : Euler Path possible only when exactly 2 odd deg. vertex. [only for conn. graph].

Euler Graph :

Closed trail cover all edges exactly one time.

Euler Line | Path :

opentrail cover all edges exactly once.

Hamiltonian Graph (H.G.)

Closed path that covers all vertices.

* **Hamil. Path** : ~~cto~~ open path that covers all vertices. (H.P.)

→ Every HG contains HP, but reverse need not be true.

→ **Theo** : $\forall n$ $W_n, C_n \cong$ H.G. → **Theo** : K_n $\forall (n \ge 3)$ is HG. $\forall (n \ge 2)$ is HP.

Graph Colouring. $\chi(G)$.

→ defn : Min no of col. to colour G.

→ $\chi(\text{Tree} \mid \text{Bipartite}) = 2$ → $\chi(\text{isolated}) = 1$

→ $\chi(W_n) = \begin{cases} 4 & \text{even} \\ 3 & \text{odd} \end{cases}$ → $\chi(C_n) = \begin{cases} 2 & \text{even} \\ 3 & \text{odd} \end{cases}$

Independence Set :

Set of non-adj vertices.

→ Maximal independence set (MIS)

→ $\beta(G)$ = # vertex in max MIS.

Dominating Set :

→ Set of vertices (D) Such that if we take away any $v \in G$, then either that vertex OR its adj vertex belongs to D

→ "Me or my friend".

→ Minimal Dominating set. (MDS).

→ $\alpha(G)$ = # vertices in min MDS.

Scanned with CamScanner

- **Matching Set :**
  - → defn : set of non adj edges.
  - → maximal matching set.
  - → NOTE : $m(K_n) = \lfloor \frac{n}{2} \rfloor$
  - → $m(G)$ = # edges in max maximal matching set.

- **Perfect Matching (P.M.) :**
  - → maximal matching s.t. induced deg of all vertices to 1
  - → Total $P.M.(K_{2n}) = \frac{(2n)!}{n! \times 2^n}$
  - → **Theo** : If PM exists then the no of vertex is always surely even.

- **Covering Set :**
  - → set of Edges s.t. all vertices are included (adj) to atleast 1 Edge of that set.
  - → minimal covering set.
  - → Covering no : #edges in minimum covering set.
  - → **Theo** : Every PM is minimal cover but the converse is not True.

- **Isomorphism :**
  H is isomorphic to G iff :
  - → $\alpha : V(G) \rightarrow V(H)$
  - → Bijective.
  - → $xy \in E(G) \Leftrightarrow \alpha(x)\alpha(y) \in E(H)$.
  - → Adjacency & non-adjacency is preserved.

- **Homomorphism**
  H is homomorphic to G iff :
  - → $\alpha : V(G) \rightarrow V(H)$
  - → need not bijective
  - → $xy \in E(G) \rightarrow \alpha(x)\alpha(y) \in E(H)$.

- **Kuratoski's theorem :**
  A graph is non planar iff it has a sub-graph homomorphic to $K_{3,3}$ & K_5.

- → **Euler formula** :
  - → $v - e + f = 2$ ... Con.
  - → $v - e + f = 1 + k$ ... discon

- → if a graph is planar : $\boxed{3f \leq 2e.}$
  $\therefore e \leq (3v - 6)$.
- → if graph is planar + Bipartite : $\boxed{4f \leq 2e.}$
  $\therefore e \leq (2v - 4)$.
- → **dirac** : $\forall v_i \in G ; d(v_i) \geq n/2$
- → **Ores** : $d(v) + d(u) \geq n \; \forall (u,v) \notin E$.

Logic

- → Connectives
  - $\wedge$ Conjunction
  - $\vee$ disjunction.
  - $\rightarrow$ ... if ...
  - $\leftrightarrow$ ... iff ...
- → $p \rightarrow q$
  - → Converse : $q \rightarrow p$
  - → Inverse : $\sim p \rightarrow \sim q$
  - → Contrapos : $\sim q \rightarrow \sim p$
- → $p \rightarrow q \equiv \sim q \rightarrow \sim p \equiv \sim p \vee q$.
- → $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$.
- → $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$.
- → $(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$

• Inference Rule:

→ $\dfrac{P \quad P \to q}{\therefore q}$ → $\dfrac{P \to q \quad \sim q}{\sim P}$ → $\dfrac{P \to q \quad q \to R}{\therefore P \to R}$ → $\dfrac{P \vee q \quad \sim P}{q}$ → $\dfrac{P \quad q}{P \wedge q}$

→ $\dfrac{P \wedge q}{\therefore P}$ → $\dfrac{P}{P \vee q}$ → $\dfrac{P \vee q \quad \sim P \vee r}{q \vee r}$

• Predicate Logic & Quantifiers:

→ $\forall x\, P(x) = P(x_1) \wedge P(x_2) \wedge \dots P(x_n)$. . . . Universal

→ $\exists x\, P(x) = P(x_1) \vee P(x_2) \vee \dots P(x_n)$ . . . Existential.

→ $\left.\begin{array}{l} \sim \forall x\, P(x) = \exists x \sim P(x) \\ \sim \exists x\, P(x) = \forall x \sim P(x) \end{array}\right]$ negation.

→ $\forall x\, P(x) \to \exists x\, P(x)$.

→ English to Predicate
ALL →
SOME ∧.

• Nested Quantifiers:

→ $\forall y \forall x \equiv \forall x \forall y$.

→ $\exists x \exists y \equiv \exists y \exists x$.

→ $\left.\begin{array}{l} \exists y \forall x \to \forall x \exists y. \\ \forall x \exists y \not\to \exists y \forall x. \end{array}\right]$ ∃ girl is gf of all the boys.

$\boxed{\text{Rec. Relation.}}$

• Method of char. Root: homogeneous.

→ Type 1: $a_n = \alpha a_{n-1}$
Roots $= \alpha$
Soln: $a_n = (\alpha)^n \cdot a_0$

→ $a_n = \alpha a_{n-1} + \beta a_{n-2}$
Roots $=$ A, B
Soln: $a_n = (A)^n \cdot c_1 + (B)^n \cdot c_2$

→ $a_n = \alpha a_{n-1} + \beta a_{n-2}$
Roots: A, A
Soln: $a_n = (A)^n c_1 + n(A^n) c_2$.

∮ Roots are same.

• Method of undetermined coeff:

→ $a_n = a_n^h + a_n^p$

→ Guess:

$f(n)$	a_n^p
const.	A
n	$A_1 n + A_0$
n^2	$A_2 n^2 + A_1 n + A_0$
n^3	$A_3 n^3 + A_2 n^2 + A_1 n + A_0$
r^n	$A r^n$

→ For same roots, multiply a_n^p with n, to avoid collision.