Problem Statement:

To solve for temperature in a square domain of 1x1, whose boundaries are at 20^0C at all times, and the temperature at t=0 is given by T(x,y) = {40^0C, if $(x-0.5)^2 + (y-0.5)^2 = 0.2$; 20^0C otherwise}.

Solution:

The equation defining the 2-dimensional transient conduction without heat generation, also known as Fourier equation, is

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \dots\dots\dots\dots\dots\dots\dots\dots (1)$$

where α: Thermal Diffusivity

In order to program this equation, it must be converted to algebraic form. This is done by discretizing using Finite Difference Method (FDM). By implementing Forward Differencing for time derivatives and Central Differencing for special derivatives, we get

$$\left(\frac{T_{new(i,j)} - T_{old(i,j)}}{\Delta t}\right) = \alpha\left(\frac{T_{(i-1,j)} - 2 * T_{(i,j)} + T_{(i+1,j)}}{\Delta x^2} + \frac{T_{(i,j-1)} - 2 * T_{(i,j)} + T_{(i,j+1)}}{\Delta y^2}\right) \dots\dots (2)$$

Central Differencing is a 2nd order scheme whereas Forward Differencing is 1st order scheme. Hence this discretization is 2nd order accurate in space derivatives and 1st order accurate in time derivative.

Uniform grid sizes and time steps are used for easy computation. Hence constants K_1 and K2 can be defined as

$$K_1 = \alpha\left(\frac{\Delta t}{\Delta x^2}\right) \text{ and } K_2 = \alpha\left(\frac{\Delta t}{\Delta y^2}\right)$$

Solving implicitly for T_{new},

$$T_{new(i,j)} = \frac{T_{old(i,j)} + K_1 * \left(T_{new(i-1,j)} + T_{prime(i+1,j)}\right) + K_2 * \left(T_{new(i,j-1)} + T_{prime(i,j+1)}\right)}{(1 + 2 * K_1 + 2 * K_2)} \dots\dots (3)$$

where $T_{new(i,j)}$: New value of temperature at $(i,j)^{th}$ point and time = t
$T_{old(i,j)}$: Old value of temperature at $(i,j)^{th}$ point and time = t - Δt
$T_{prime(i,j)}$: New value of temperature at $(i,j)^{th}$ point, time = t, and previous iteration

This equation is solved iteratively to get T_{new} at every timestep. The iterations are continued till the convergence criterion i.e., maximum absolute value in the matrix [$T_{new} - T_{prime}$] is less than 1e-5. Once this is achieved, the matrix [T_{prime}] is updated with the values of [T_{new}] using over-relaxation factor (SOR). Successive over-relaxation helps in faster convergence and SOR of 1.5 was chosen. The other methods of updating the values includes Gauss-Seidel and Jacobi which are slower than over-relaxation.

Once the solution is converged, the solving is continued for the next time step. This is done by copying the values from [T_{new}] to [T_{old}]. The values for [T_{new}] is calculated iteratively as described above. This is continued till the time variable reaches the desired end time of the simulation. The data is then saved and plotted. The solution procedure is shown in Fig1.

Conclusion:

The program was solved for 7200s, and the solution data were analyzed. Temperature plots at various interval of time were plotted. The system has reached steady state at 7200s, with all heat lost through the boundaries. This was evident by the temperature values which were 20^0C at all points.

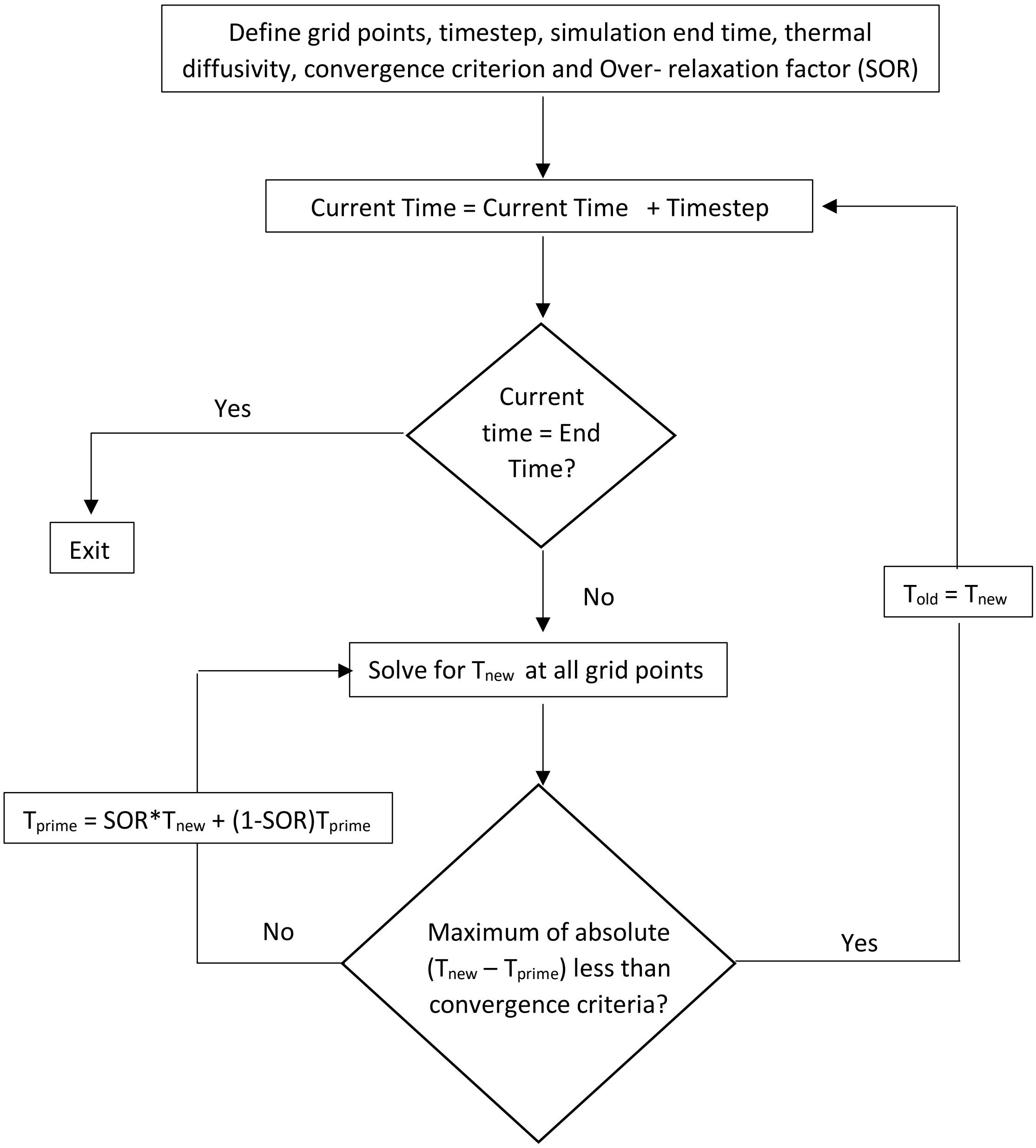

Fig1: Flowchart of the FDM algorithm used