

<u>Course</u> > <u>Week 9</u>... > <u>Proble</u>... > Proble...

Problem Set 9

Problems 1-6 correspond to "Linear Projections"

Problem 1

1/1 point (graded)

In \mathbb{R}^2 , what is the unit vector corresponding to the x_1 -direction?

 $\bigcirc (0,0)$

 \bigcirc (1,0)

 $\bigcirc (0,1)$

 \bigcirc (1,1)



Submit

1 Answers are displayed within the problem

Problem 2

1/1 point (graded)

What is the unit vector in the same direction as (3, 2, 2, 2, 2)?

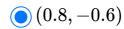
\bigcirc (1.5, 1, 1, 1, 1)	
\bigcirc (1, 0.67, 0.67, 0.67, 0.67)	
\bigcirc (0.6, 0.4, 0.4, 0.4, 0.4)	
$\bigcirc (0.5, 0.33, 0.33, 0.33, 0.33)$	
✓	
? Hint (1 of 1): To get a unit vector in the same direction as x , simply divide by $\ x\ $.	Next Hint
Submit	
Answers are displayed within the problem	
Problem 3	
1/1 point (graded) What is the projection of the vector $(3,5,-9)$ onto the direction $(0.6,-0.8,0)$	0)?
-2.2 ✓ Answer: -2.2	
-2.2	
? Hint (1 of 1): If u is a unit vector, then the projection of x onto direction u is simply $u \cdot x$.	Next Hint
Submit	

0	Answers are	displayed	within the	problem
•		J		p. c.c.c

Problem 4

1/1 point (graded)

What is the (unit) direction along which the projection of (4,-3) is largest?



- \bigcirc (-0.6, -0.8)
- \bigcirc (-0.8, 0.6)
- $\bigcirc \, (0.8,0.6)$



Explanation

The projection of x=(4,-3) is going to be largest in the direction of x itself.

Submit

1 Answers are displayed within the problem

Problem 5

1/1 point (graded)

What is the (unit) direction along which the projection of (4,-3) is smallest?

- \bigcirc (0.8, -0.6)
- $\bigcirc (-0.6, -0.8)$
- (-0.8, 0.6)

 (0.8, 0.6) ✓ Explanation The presidential of man (4 = 2) will be appelled in the direction appearing to the matrix that is the second of the control of the control
The projection of $x=(4,-3)$ will be smallest in the direction opposite to x , that is, the direction of $-x$.
Answers are displayed within the problem
Problem 6
1/1 point (graded) The projection of vector \boldsymbol{x} onto direction \boldsymbol{u} is exactly zero. Which of the following statements is necessarily true? Select all that apply.
ightharpoonup u is orthogonal to x .
$oxedsymbol{oxed}u$ is in the opposite direction to $x.$
ightharpoonup u is at right angles to x .
It is not possible to have a projection of zero.
Submit
Answers are displayed within the problem
Problems 7-8 correspond to "Principal component analysis I: one-dimensional projection"

Problem 7

4/4 points (graded)

A three-dimensional data set has covariance matrix

$$\Sigma = \left(egin{array}{ccc} 4 & 2 & -3 \ 2 & 9 & 0 \ -3 & 0 & 9 \end{array}
ight).$$

a) What is the variance of the data in the x_1 -direction?



b) What is the correlation between x_1 and x_3 ?



c) What is the variance in the direction (0,-1,0)?



d) What is the variance in the direction of (1,1,0)?



? Hint (1 of 3): For part (a): the diagonal entry Σ_{ii} is the variance of X_i .
Hint (2 of 3): For part (b): the entry Σ_{ij} is the <i>covariance</i>
between X_i and X_j . This is not the same as the <i>correlation</i> .
Do you remember how to get from one to the other?
Hint (3 of 3): For part (c,d): the variance in direction u , where u is a unit vector, is given by $u^T \Sigma u$.
Submit
Answers are displayed within the problem
Problem 8
1/1 point (graded) Which of the following covariance matrices has the property that the variance is the same in any direction? Select all that apply.
✓ The all-zeros matrix.
The all-ones matrix.
✓ The identity matrix.
Any diagonal matrix.

Explanation

Let u be any unit vector in d-dimensional space.

If A is the all-zeros matrix, then $u^TAu=0$, the same for all u. If B is the all-ones matrix, then $u^TBu=\sum_{ij}u_iu_j=(\sum_iu_i)^2$, which is not the same for $\mathsf{all}\ u.$

With the identity matrix: $u^T I u = u^T u = 1$, the same for all u.

Let D be the diagonal matrix where $D_{11}=1$ and all other diagonal entries are zero. Then $u^TDu=u_1^2$, not the same for all u.

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1 Answers are displayed within the problem

Problems 9-11 correspond to "Principal component analysis II: the top k directions"

Problem 9

8/8 points (graded)

Let $u_1,u_2\in\mathbb{R}^d$ be two vectors with $\|u_1\|=\|u_2\|=1$ and $u_1\cdot u_2=0$. Define U to be the matrix whose columns are u_1 and u_2 .

What are the dimensions of the following matrices?

a) ${\it U}$

of Rows =

d **✓ Answer:** d

of Columns =

2 **✓** Answer: 2

b) U^T

of Rows =

2 **✓ Answer:** 2

# of Columns	
d	✓ Answer: d
c) UU^T	
# of Rows =	
d	✓ Answer: d
# of Columns =	
d	✓ Answer: d
d) $u_1u_1^T$	
# of Rows =	
d	✓ Answer: d
# of Columns =	
d	✓ Answer: d
Submit	

1 Answers are displayed within the problem

Problem 10

1/1 point (graded)

Continuing from the previous problem, let $u_1,u_2\in\mathbb{R}^d$ be two vectors with $\|u_1\|=\|u_2\|=1$ and $u_1\cdot u_2=0$, and define U to be the matrix whose columns are u_1 and u_2 .

Which of the following linear transformations sends points $x\in\mathbb{R}^d$ to their (two-dimensional) projections onto directions u_1 and u_2 ? Select all that apply.

$$\checkmark x \mapsto (u_1 \cdot x, u_2 \cdot x)$$

$$igcup x \mapsto (u_1 \cdot x) \, u_1 + (u_2 \cdot x) \, u_2$$

$$\checkmark x \mapsto U^T x$$

$$\bigcap x\mapsto UU^Tx$$



Explanation

The first and third maps send 4-d to 2-d. The second and fourth maps send 4-d to 4-d.

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1 Answers are displayed within the problem

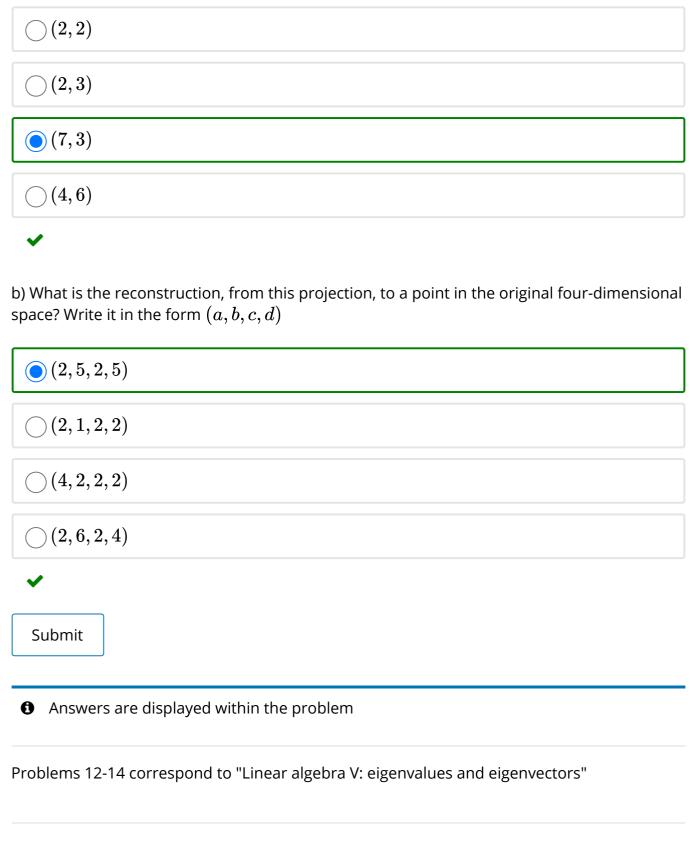
Problem 11

2/2 points (graded)

For a particular four-dimensional data set, the top two eigenvectors of the covariance matrix are:

$$\frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}.$$

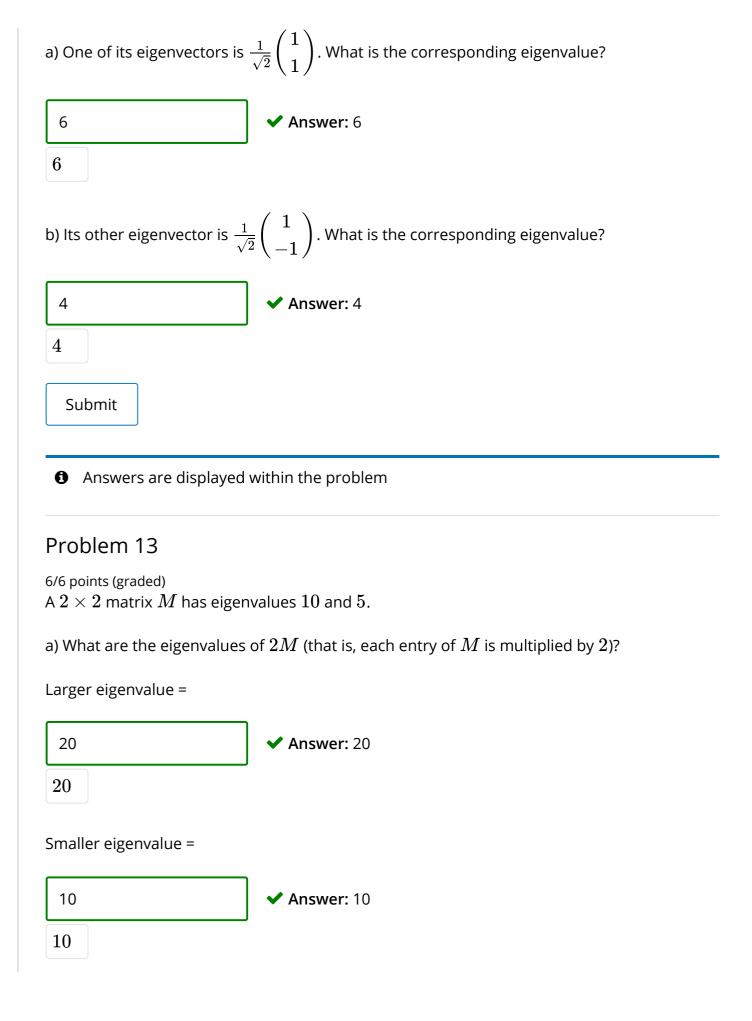
a) What is the PCA projection of point (2,4,2,6) into two dimensions? Write it in the form (a,b).



Problem 12

2/2 points (graded)

Consider the 2 imes 2 matrix $M = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}$.



b) What are the eigenvalues of M+3I, where I is the 2×2 identity matrix? Larger eigenvalue = 13 **✓ Answer:** 13 13Smaller eigenvalue = 8 ✓ Answer: 8 8 c) What are the eigenvalues of $M^2=MM$? Larger eigenvalue = **✓ Answer:** 100 100 100 Smaller eigenvalue = 25 **✓ Answer:** 25 25

Explanation

Suppose (u,λ) is an (eigenvector, eigenvalue) pair for M, that is, $Mu=\lambda u$.

Part (a): for any constant c, we have $(cM)\,u=M\,(cu)=c\lambda u$. Thus $(u,c\lambda)$ is an (eigenvector, eigenvalue) pair for cM.

Part (b): for any constant c, we have $(M+cI)\,u=Mu+cu=(\lambda+c)\,u$. Thus $(u,\lambda+c)$ is an (eigenvector, eigenvalue) pair for M+cI.

Part (c): For any positive integer c, we have $M^cu=\lambda^cu$, and thus (u,λ^c) is an (eigenvector, eigenvalue) pair for M^c .

? Hint (1 of 3): For part (a): if $Mu=\lambda u$, what do we know about $(2M)\,u=M\,(2u)$?

Next Hint

Hint (2 of 3): For part (b): Note that $(M+3I)\,u=Mu+3u$

Hint (3 of 3): For part (c): $M^{2}u=M\left(Mu
ight)$

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1 Answers are displayed within the problem

Problem 14

6/7 points (graded)

A certain three-dimensional data set has covariance matrix

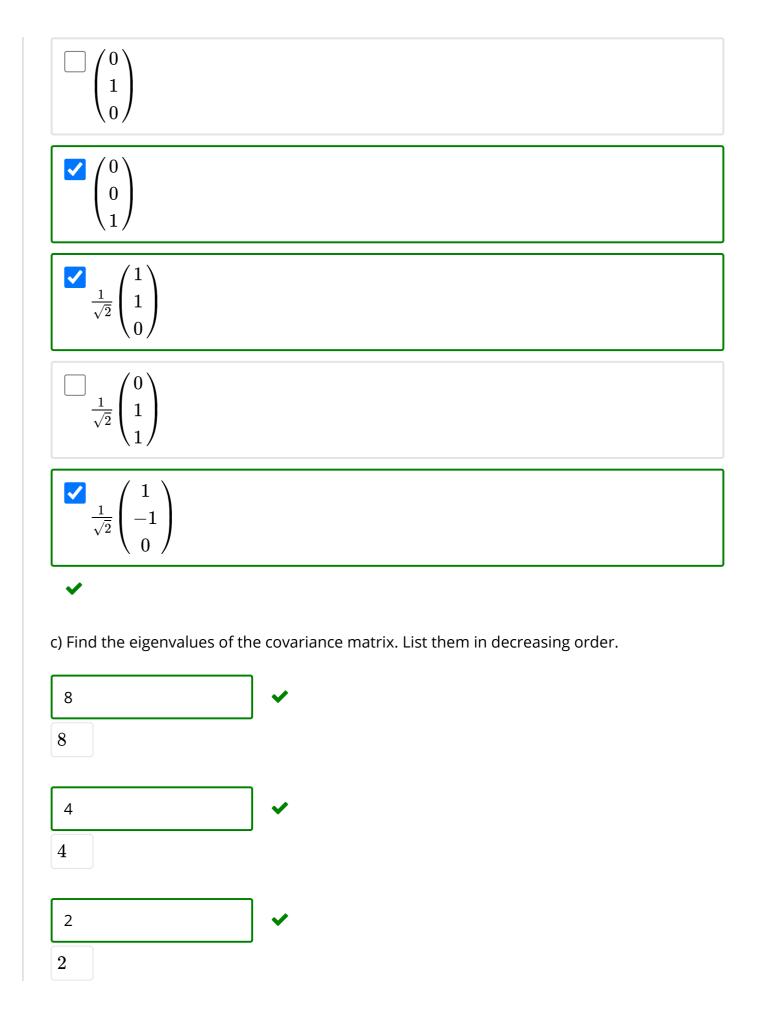
$$\begin{pmatrix}
5 & -3 & 0 \\
-3 & 5 & 0 \\
0 & 0 & 4
\end{pmatrix}$$

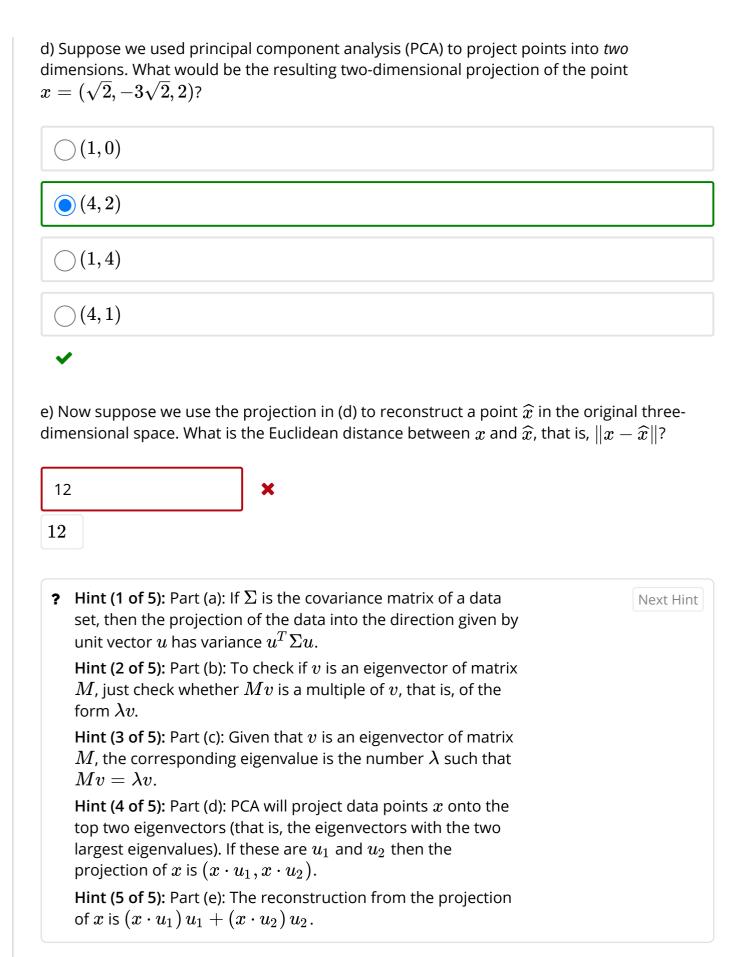
.

a) Consider the direction $u=\left(1,1,1\right)/\sqrt{3}$. What is variance of the projection of the data onto direction u?

8/3 **✓**

b) Which of the following are eigenvectors of the covariance matrix? Select all that apply.





★ Partially correct (6/7 points)

Problems 15-17 correspond to "Linear algebra VI: spectral decomposition"

Problem 15

1/1 point (graded)

M is a 2 imes 2 real-valued symmetric matrix with eigenvalues $\lambda_1 = 6, \lambda_2 = 1$ and corresponding eigenvectors

$$u_1=rac{1}{\sqrt{5}}inom{2}{1}, \ \ u_2=rac{1}{\sqrt{5}}inom{-1}{2}.$$

What is M?

- $\bigcirc \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$
- $igcup \left(egin{matrix} 4 & 2 \ 2 & 1 \end{matrix}
 ight)$
- $\bigcirc \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$
- **~**
- **?** Hint (1 of 1): This is a direct application of the spectral decomposition theorem. We went through an example just like this in lecture.

Submit		
1 Answer	s are displayed within the problem	
interesting p		
Each of	$^{\mathrm{f}}$ the data points has at most k nonzero coordinates.	
✓ The dat	ta can be perfectly reconstructed from their PCA projection onto k (dimensions.
Z Each da	ata point can be expressed as a linear combination of the top k eige	envectors.
It is pos	ssible to discard $d-k$ of the coordinates without losing any of the $lpha$	variance in
		variance in
the dat Hint (1 of subspace)		variance in
the dat Hint (1 of subspace)	of 1): Intuitively, the data lies in a k -dimensional ce, but this subspace need not be aligned with the	
the dat Hint (1 or subspace coordin Submit	of 1): Intuitively, the data lies in a k -dimensional ce, but this subspace need not be aligned with the	

1/1 point (graded) A data set in \mathbb{R}^d has a covariance matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$. Under which of the following conditions is PCA most likely to be effective as a form of dimensionality reduction? Select all that apply.
$oxedsymbol{oxed}$ When the λ_i are approximately equal.
$igwedge$ When most of the λ_i are close to zero.
$oxedsymbol{oxed}$ When most of the λ_i are close to 1.
$lacksquare$ When the sequence $\lambda_1,\lambda_2,\ldots$ is rapidly decreasing.
Explanation The overall variance in the data is $\lambda_1+\lambda_2+\cdots+\lambda_d$. When PCA is used to reduce the dimension to k , the amount of variance in the projected points is $\lambda_1+\lambda_2+\cdots+\lambda_k$. PCA is most effective when this second quantity is not too much smaller than the first, in other words, when the fraction of variance lost, $\left(\lambda_{k+1}+\cdots+\lambda_d\right)/\left(\lambda_1+\cdots+\lambda_d\right)$, is small.

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1 Answers are displayed within the problem

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