Problem Set 7

Problems 1-4 correspond to "Kernels I: basis expansion"

Problem 1

1/1 point (graded)

Which of the following are quadratic functions of $x = (x_1, x_2, x_3)$? Select all that apply.

- [ ] $x_1^2 + x_2^2 + x_3^2 + x_1 x_2 x_3$
- [x] $x_1 x_2 + x_1 x_3 + x_2 x_3$
- [ ] $(x_1 + x_2 + x_1 x_2)^2$
- [x] $1 + x_1 + x_2 + x_3 + 10 x_2^2$

✔

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Problem 2

1/1 point (graded)

A decision boundary in $\mathbb{R}^2$ is given by the equation

$$x_1 + 3x_1x_2 = 6x_2^2 + 8.$$

This can be written in form $w \cdot \Phi(x) + b = 0$, where:

$x = (x_1, x_2)$

$\Phi(x) = (x_1, x_2, x_1^2, x_2^2, x_1x_2)$ is a basis expansion

$b = -8$

What is w?

- ◯ $w = (1, 3, -6, 0, 8)$
- ◯ $w = (1, 0, 0, 3, 0)$
- ◉ $w = (1, 0, 0, -6, 3)$
- ◯ $w = (1, -6, 0, 0, 3)$

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Problem 3

2/2 points (graded)

Data vectors $x = (x_1, \ldots, x_4)$ are augmented to give expanded features

$$\Phi(x) = (x_1, \ldots, x_4, x_1^2, \ldots, x_4^2, x_1x_2, \ldots, x_3x_4).$$

a) What is the dimension of $\Phi(x)$?

(2*4)+4*(4-1)/2 ✔ **Answer:** 14

$(2 \cdot 4) + \frac{4 \cdot (4-1)}{2}$

b) The Perceptron algorithm is run using the basis expansion $\Phi(x)$ and returns w, b. What is the dimension of w?

(2*4)+4*(4-1)/2 ✔ **Answer:** 14

$(2 \cdot 4) + \frac{4 \cdot (4-1)}{2}$

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Problem 4

1/1 point (graded)

We have a data set, of d-dimensional points and their labels, that is linearly separable. However, we use a basis expansion

$$\Phi(x) = (x_1, \ldots, x_d, x_1^2, \ldots, x_d^2, x_1 x_2, \ldots, x_{d-1} x_d),$$

and run the Perceptron algorithm with these expanded features. Which of the following outcomes necessarily occur? Select all that apply.

- [x] The algorithm converges.
- [ ] The algorithm does not converge.
- [ ] The algorithm returns a vector w in which the entries corresponding to quadratic terms in $\Phi(x)$ (such as x_1^2) are zero.

✔

Explanation

If the two classes can be perfectly separated by a linear function then they can also be separated by a quadratic function, and thus the Perceptron is guaranteed to converge. The boundary it finds need not be linear, though: it might find a non-linear quadratic boundary between the two classes.

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Problems 5-7 correspond to "Kernels II: the kernel trick"

Problem 5

1/1 point (graded)

For d-dimensional vectors x, let $\Phi(x)$ denote the basis expansion used to produce quadratic decision boundaries. If vectors $x, z \in \mathbb{R}^d$ are orthogonal to each other, what is $\Phi(x) \cdot \Phi(z)$?

1

✔ **Answer:** 1

1

Hint (1 of 1): Recall that $\Phi(x) \cdot \Phi(z)$ can be calculated directly from $x \cdot z$.

Next Hint

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Problem 6

3/3 points (graded)

For d-dimensional vectors x, let $\Phi(x)$ denote the basis expansion used to produce quadratic decision boundaries. In what follows, suppose we have a data set of n labeled points.

a) What is the dimension of $\Phi(x)$? Pick the best option below.

- ○ $O(d)$
- ◉ $O(d^2)$
- ○ $O(n)$

✔

b) Suppose we learn a quadratic decision boundary by using the Perceptron algorithm with the kernel trick. The algorithm performs k updates and then converges, returning a vector α. What is the dimension of α? Pick the best option below.

- ○ $O(d)$
- ○ $O(d^2)$
- ◉ $O(n)$

✔

c) Continuing from part (b), what is the time complexity of classifying a new data point using α? Pick the best option below.

- ○ $O(k)$
- ◉ $O(kd)$
- ○ $O(kd^2)$

○ $O(n)$

✔

Explanation
Part(c): Although α is n-dimensional, we are told that convergence occurs after just k updates: hence α contains at most k non-zero entries. As a result, classifying a new point x involves computing $\Phi(x) \cdot \Phi(x^{(i)})$ for at most k data points $x^{(i)}$. As we have seen, each such high-dimensional dot product can be computed directly from $x \cdot x^{(i)}$, and thus takes $O(d)$ time. Thus the total time to classify x is $O(kd)$.

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Problem 7

1/1 point (graded)
A data set consists of just four points in $\mathbb{R}^2$:

Label 1: points $x^{(1)} = (2, 3)$ and $x^{(2)} = (3, 6)$

Label -1: points $x^{(3)} = (1, 2)$ and $x^{(4)} = (0, 7)$

The kernel Perceptron algorithm (in dual form) is run on this data set, with a basis expansion $\Phi(\cdot)$ that produces a quadratic boundary. The algorithm converges after just five update steps: two updates on point $x^{(2)}$, two updates on point $x^{(3)}$ and one update on point $x^{(4)}$.

What vector α and value of b are returned?

◉ $\alpha = (0, 2, 2, 1),\ b = -1$

○ $\alpha = (0, 1, 1, 1),\ b = 2$

○ $\alpha = (0, 2, 2, 1),\ b = 1$

○ $\alpha = (0, 1, 1, 1)\,,\ b = 0$

✔

Explanation
Recall that α has one coordinate for each data point, and its value at that coordinate is simply the number of times an update was performed for that data point. Thus the final value of α is $(0, 2, 2, 1)$. Also note that b starts off at zero and, on each update, is incremented if the label is positive and decremented if it is negative. Therefore, the final value of b is -1.

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Problems 8-11 correspond to "Kernels III: kernel SVM"

Problem 8

1/1 point (graded)
To use the kernel trick with support vector machines, which version of the SVM optimization problem do we use?

○ The primal form, which returns w and b

◉ The dual form, which returns α and b

✔

Explanation
This is directly from lecture. But here's an interesting footnote: in some large-scale applications, the kernel SVM is used in its *primal* form, in conjunction with some clever methods for dimensionality reduction.

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Problem 9

1/1 point (graded)

In order to solve the kernel SVM optimization problem, what information about the data set $\{(x^{(i)}, y^{(i)})\}$ do we need to provide to the optimization procedure?

- ○ The labels $y^{(i)}$ and the basis expansions $\Phi\left(x^{(i)}\right)$
- ○ The labels $y^{(i)}$ and the squared norms $\Phi\left(x^{(i)}\right) \cdot \Phi\left(x^{(i)}\right)$
- ◉ The labels $y^{(i)}$ and the pairwise dot products $\Phi\left(x^{(i)}\right) \cdot \Phi\left(x^{(j)}\right)$

✔

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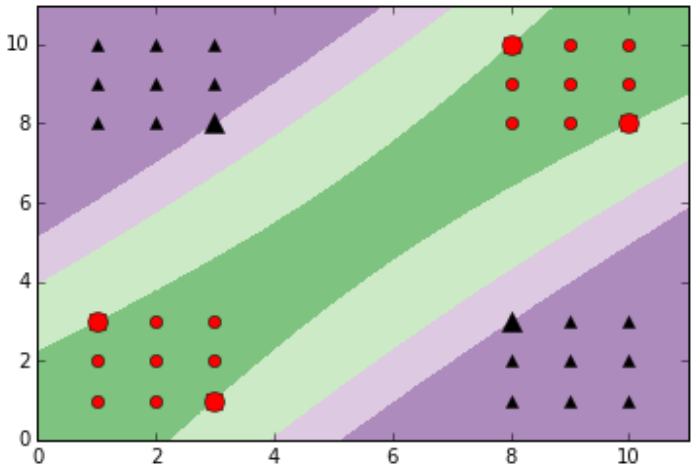
Problem 10

3/3 points (graded)

Consider the following example from lecture, which shows the decision boundary resulting from applying kernel SVM (with quadratic kernel) to a small two-dimensional data set.



a) What is the dimension of α in this case?

36

✔ **Answer:** 36

36

b) How many entries in α are > 0?

6

✔ **Answer:** 6

6

c) How many entries in α are < 0?

0

✔ **Answer:** 0

0

d) For you to think about: why does the margin on the green side appear to be larger than the margin on the purple side?

? **Hint (1 of 1):** Recall that α has one coordinate per data point; and its value on that coordinate is the number of times an update was performed on that data point.

Next Hint

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Answers are displayed within the problem

Problem 11

1/1 point (graded)

We have a data set of n labeled points and we wish to use kernel SVM to fit a quadratic boundary to it. To do this, we need to supply the labels of the points as well as the $n \times n$ matrix of pairwise dot products between the expanded feature vectors. Call this matrix K.

Now suppose that we change our mind and instead want a boundary that is a polynomial of degree 4. Again, we supply the labels of the data points and a matrix L. How does L compare to K? Select all that apply.

- [ ] K is $n \times n$ while L is $n^2 \times n^2$
- [x] L is also $n \times n$.
- [ ] $L = K^2$
- [x] Each entry of L is the square of the corresponding entry of K

✔

Explanation

K is an $n \times n$ matrix whose (i, j) entry is $(1 + x^{(i)} \cdot x^{(j)})^2$. Meanwhile, L is an $n \times n$ matrix whose (i, j) entry is $(1 + x^{(i)} \cdot x^{(j)})^4$.

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Problems 12-15 correspond to "Kernels IV: the kernel function"

Problem 12

1/1 point (graded)

In order to get a nonlinear decision boundary, we can use a basis expansion $\Phi(x)$. The corresponding kernel function is then $k(x, z) = \Phi(x) \cdot \Phi(z)$. Do we need knowledge of both $k(\cdot)$ and $\Phi(\cdot)$ to learn a classifier and subsequently use it, or is it enough to know just $k(\cdot)$?

- ○ In order to learn a classifier, we need to know $\Phi(\cdot)$ in addition to $k(\cdot)$. In order to use the classifier, it is sufficient to just know $k(\cdot)$.
- ○ In order to learn a classifier, it is sufficient to just know $k(\cdot)$. In order to use the classifier, we need to know $\Phi(\cdot)$ in addition to $k(\cdot)$.
- ◉ It is sufficient to just know $k(\cdot)$ for both learning a classifier and subsequently using it.

✔

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Problem 13

1/1 point (graded)

When using the RBF kernel, the final classifier assigns a label to a new point x by taking a weighted vote over the labels of the training points $x^{(i)}$. What do these weights depend upon? Select all that apply.

- ☑ The α_i found by the learning algorithm: larger α_i means a larger weight for $x^{(i)}$.
- ☑ The distance between x and $x^{(i)}$: closer points $x^{(i)}$ get a larger weight.
- ☐ The distance between x and $x^{(i)}$: farther points $x^{(i)}$ get a larger weight.

- [ ] The dot product $x \cdot x^{(i)}$: larger dot product implies a larger weight for $x^{(i)}$
- [ ] The dot product $x \cdot x^{(i)}$: larger dot product implies a smaller weight for $x^{(i)}$

✔

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Problem 14

1/1 point (graded)

We decide to define $k(x, z) = x \cdot z$. What can be said about this?

- ○ This is not a valid kernel function.
- ○ This is a valid kernel function and yields a quadratic decision boundary.
- ◉ This is a valid kernel function and yields a linear decision boundary.

✔

Explanation

This is the kernel function that corresponds to the embedding $\Phi(x) = x$; that is, the original data space. It finds a linear function in this space.

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Answers are displayed within the problem

Problem 15

1/1 point (graded)

We decide to define $k(x, z)$ to be 1 if $x = z$ and 0 otherwise. What can be said about this? Select all that apply.

- [ ] This is not a valid kernel function.
- [x] This is a valid kernel function.
- [x] Assuming the training points are distinct, we can achieve zero training error using a classifier based on $k(\cdot)$.
- [x] A classifier based on $k(\cdot)$ is unlikely to generalize well beyond the training set.

✔

Explanation
For any set of points, the kernel matrix (the matrix of similiarities between those points) is just the identity matrix, which is positive semidefinite. Hence this is a valid kernel function.

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Problems 16-19 correspond to "Decision Trees"

Problem 16

1/1 point (graded)
Decision trees have great expressive power: they can represent any decision boundary, by carving up the space to a fine enough granularity. Which of the other methods we've studied are similarly expressive? Select all that apply.

- [ ] Linear classifiers.
- [ ] Support vector machines with a quadratic kernel.

- [x] Nearest neighbor classifiers.
- [ ] Classifiers based on Gaussian generative models.

✔

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Answers are displayed within the problem

Problem 17

1/1 point (graded)

Let's say our training set consists of n data points in d-dimensional space. At the top node of the tree, we have to pick a split, which involves choosing (i) a feature and (ii) a split value along that feature. How many possibilities do we need to try out, roughly?

- ○ n
- ○ d
- ○ Infinitely many
- ◉ nd

✔

Explanation

For any given feature, we need only try $n - 1$ split values: imagine sorting the data points according to their values on this feature, and this gives us the set of distinct split possibilities.

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Problem 18

1/1 point (graded)

In order to grow a decision tree, we need a measure of how pure or impure (in terms of labels) a node is. A popular measure of this is the "Gini impurity index". If there are two labels, and a node has p fraction of one label and $1-p$ fraction of the other, the Gini impurity index is $2p\,(1-p)$. Notice that this is maximized when $p = 1/2$.

What is the Gini impurity index for a node in which 20% of the points have one label while 80% have the other label?

- ○ 0.2
- ○ 0.16
- ◉ 0.32
- ○ 0.8

✔

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Problem 19

1/1 point (graded)

We have talked about a measure of impurity in the case of binary classification. If there are k possible classes, and a node contains these in proportions $p_1, p_2, \ldots, p_k$ (summing to 1), which of the following is a suitable measure of impurity? That is, the higher the value, the more impure the node. Select all that apply.

- ☐ The maximum of $p_1, p_2, \ldots, p_k$
- ☐ $p_1^2 + p_2^2 + \cdots + p_k^2$

- [ ] The minimum of $p_1, p_2, \ldots, p_k$
- [x] $1 - (p_1^2 + p_2^2 \cdots + p_k^2)$

✔

Explanation
Choice 1: The maximum p_i is not a good measure: e.g. it is high when $p_1 = 1$ while $p_2 = \cdots = p_k = 0$.
Choice 2: Has a similar bad case.
Choice 3: What if $p_1 = 0$ while $p_2 = \cdots = p_k = 1/(k-1)$? Highly impure, but the minimum value is zero.

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Problems 20-21 correspond to "Boosting"

Problem 20

1/1 point (graded)
Boosting requires the weak learning algorithm to work with training data in which each point $(x^{(i)}, y^{(i)})$ has a positive weight $w_i > 0$. Intuitively, a weight of two would be equivalent to having two copies of that data point.

Here's something for you to think about: how would you incorporate weights into the following learning algorithms without explicitly making copies of data points?

- Decision trees.
- Gaussian generative models.
- Support vector machines.

Now select one of the following (you will get marked correct either way).

- ○ I have figured this out. ✔
- ◉ I am having some trouble with this.

✔

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Problem 21

1/1 point (graded)

Suppose we run boosting using weak classifiers from some class H (such as decision stumps), and suppose that on each round, the weak learner always returns a weak classifier whose error rate on the weighted data (for that round) is at most $1/2 - \epsilon$, for some $\epsilon > 0$. Which of the following is true? Select all that apply.

- ☐ Boosting converges to a classifier with zero test error.
- ☑ Boosting converges to a classifier with zero training error.
- ☐ Boosting converges to the classifier that has the smallest possible test error out of any classifier in H.

✔

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Problems 22-23 correspond to "Random forests"

Problem 22

1/1 point (graded)

Which of the following is an obvious benefit of random forests over boosted decision trees? Select all that apply.

- [x] The trees can be trained in parallel.
- [ ] Each individual tree is more highly optimized.
- [ ] Each individual tree has better accuracy.

✔

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Problem 23

1/1 point (graded)

When learning a tree in a random forest, the split at each node is chosen from only a subset of the full feature set. What is the main reason for this?

- ( ) It speeds up the training process.
- (•) It creates trees with greater variability.
- ( ) It improves the accuracy of each tree.
- ( ) It ignores irrelevant features.

✔

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- Answers are displayed within the problem