Chapter 1 :

Solve linear system by matrix

Section 3

Matrix

Definition: Is a rectangular arrangement of numbers into rows and columns.

ترتيب العناصر فى شكل صفوف و أعمدة

✓ **Example:** Matrix ***A*** has two rows and three columns

$$A = \begin{bmatrix} -2 & 5 & 6 \\ 5 & 2 & 7 \end{bmatrix}_{2*3}$$

Size (Order) of the matrix

$$\boxed{\text{Size} = M * N}$$

(where M is number of rows, and N is number of columns)

✓ **Example:**

$$A = \begin{bmatrix} -8 & -4 \\ 23 & 12 \\ 18 & 10 \end{bmatrix}_{3*2} \qquad B = \begin{bmatrix} -2 & 9 \end{bmatrix}_{1*2}$$

Matrix Elements: Each element in a matrix is identified by naming the row and the column in which it appears.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3*3}$$

✓ **Example:**

$$B = \begin{bmatrix} 4 & 14 & -7 \\ 18 & 5 & 13 \\ -20 & 4 & 22 \end{bmatrix}_{3*3} \qquad \boxed{b_{21} = 18} \qquad \boxed{b_{13} = -7}$$

Equal Matrices

Matrix A = Matrix B............When?

1) Size of **A** = Size of **B**

2) $a_{ij} = b_{ij}$ العناصر المتناظرة متساوية

✓ **Example:**

$$A = \begin{bmatrix} -7 & \sqrt{16} & 13 \\ \frac{1}{2} & 2 & 8 \end{bmatrix}_{2*3} \qquad B = \begin{bmatrix} -7 & 4 & 13 \\ 0.5 & 2 & 8 \end{bmatrix}_{2*3}$$

Then A = B

✓ **Equal matrices can be used to solve for variables.**

If $\begin{bmatrix} 3 & 4 \\ -7 & 4 \end{bmatrix} = \begin{bmatrix} x & y+2 \\ -7 & 4 \end{bmatrix}$

Then $\boxed{x = 3}$ $\boxed{y = 2}$

Matrix operations

Addition and Subtraction

$\boxed{\text{Size(A)} = \text{Size(B)}}$ ⟸ الشرط

Let $A = \begin{bmatrix} 4 & 5 & 2 \\ -3 & 6 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 3 & -5 \\ 4 & -5 & 7 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix}$

Then $A + B = \begin{bmatrix} 4 & 8 & -3 \\ 1 & 1 & 7 \end{bmatrix} \qquad A - B = \begin{bmatrix} 4 & 2 & 7 \\ -7 & 11 & -7 \end{bmatrix}$

✓ **Notes:**

1) The Expressions $A + C$, $A - C$, $B - C$, $B + C$ are undefined. not the same size.
2) $A + B = B + A$ **but** $A - B \neq B - A$

Scalar multiples (Multiplying a Matrix by a Constant)

✓ We can multiply a matrix by some value by multiplying each element with that value.

✓ The value can be positive or negative

Let $A = \begin{bmatrix} 3 & 8 \\ 2 & -6 \\ 4 & 0 \end{bmatrix}$ **Then** $3A = \begin{bmatrix} 9 & 24 \\ 6 & -18 \\ 12 & 0 \end{bmatrix}$

✓ **Example: Consider the matrices**

$A=\begin{bmatrix} 2 & 3 & 4 \\ 1 & 3 & 1 \end{bmatrix}$ $B=\begin{bmatrix} 0 & 2 & 7 \\ -1 & 3 & -5 \end{bmatrix}$ $C=\begin{bmatrix} 9 & -6 & 3 \\ 3 & 0 & 12 \end{bmatrix}$

Find $2A - B + \frac{1}{3}C$

Solution

$2A = \begin{bmatrix} 4 & 6 & 8 \\ 2 & 6 & 2 \end{bmatrix}$ $(-1)B = \begin{bmatrix} 0 & -2 & -7 \\ 1 & -3 & 5 \end{bmatrix}$

$\frac{1}{3}C = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 4 \end{bmatrix}$

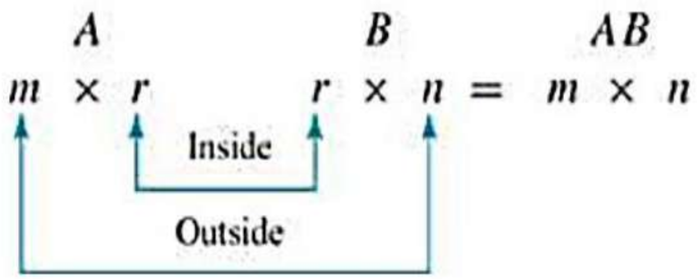
$$2A - B + \frac{1}{3}C = \begin{bmatrix} 4 & 6 & 8 \\ 2 & 6 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -2 & -7 \\ 1 & -3 & 5 \end{bmatrix} + \begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 2 & 2 \\ 4 & 3 & 11 \end{bmatrix}$$

Multiplying Matrices

We can multiply a matrix (A) by another matrix (B) if the number of columns in **A** is equal to the number of rows in **B**

الشرط ⟸ أن يكون عدد الأعمدة فى المصفوفة الأولى يساوى عدد الصفوف فى المصفوفة الثانية



✓ **Example: Consider the matrices, find AB**

$$A=\begin{bmatrix}1&2&4\\2&6&0\end{bmatrix},\qquad B=\begin{bmatrix}4&1&4&3\\0&-1&3&1\\2&7&5&2\end{bmatrix}$$

✓ **Solution**

Since **A** is 2 × 3 a matrix and **B** is a 3 × 4 matrix, the product **AB** Is a 2 × 4 matrix.

$$\begin{bmatrix}1&2&4\\2&6&0\end{bmatrix}\begin{bmatrix}4&1&4&3\\0&-1&3&1\\2&7&5&2\end{bmatrix}=\begin{bmatrix}\boxed{}&\boxed{}&\boxed{}&\boxed{}\\\boxed{}&\boxed{}&\boxed{26}&\boxed{}\end{bmatrix}$$

$$(2\cdot4)+(6\cdot3)+(0\cdot5)=26$$

The entry in row 1 and column 4 of is computed as follows:

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \boxed{} & \boxed{} & \boxed{} & \boxed{13} \\ \boxed{} & \boxed{} & \boxed{} & \boxed{} \end{bmatrix}$$

$$(1 \cdot 3) + (2 \cdot 1) + (4 \cdot 2) = 13$$

The computations for the remaining entries are:

$$\begin{aligned} (1 \cdot 4) + (2 \cdot 0) + (4 \cdot 2) &= 12 \\ (1 \cdot 1) - (2 \cdot 1) + (4 \cdot 7) &= 27 \\ (1 \cdot 4) + (2 \cdot 3) + (4 \cdot 5) &= 30 \\ (2 \cdot 4) + (6 \cdot 0) + (0 \cdot 2) &= 8 \\ (2 \cdot 1) - (6 \cdot 1) + (0 \cdot 7) &= -4 \\ (2 \cdot 3) + (6 \cdot 1) + (0 \cdot 2) &= 12 \end{aligned}$$

$$AB = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

✓ **Note:** **AB ≠ BA**

✓ **Exercise:** Suppose the following matrices along with their sizes

A(4*5) , B(4*5) , C(5*2) , D(4*2) , E(5*4)

Determine which of the following is defined

BA :

AC + D :

AE + B :

E(A + B) :

Special Matrices

1- square matrix (n*n) is a matrix with the same number of columns and rows (2 * 2, 3 * 3, etc.)

يتساوى فيها عدد الصفوف مع عدد الأعمدة

$$\mathbf{A} = \begin{bmatrix} 5 & 0 \\ 9 & -2 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

2- Diagonal matrix is a square matrix in which all entries that are not on the main diagonal are **zero.**

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$$\mathbf{E} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

However, the following matrices are not diagonal

$$\mathbf{H} = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{J} = \begin{bmatrix} 0 & 4 & 3 \\ -7 & 0 & 6 \\ 5 & -2 & 0 \end{bmatrix}$$

مصفوفة قطرية عناصر القطر الرئيسى لها كلها تساوى 1

3- <u>Identity matrix</u> is a diagonal matrix with all main diagonal entries equal to **1**

$$\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{I}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

✓ **<u>Notes:</u>**

1) كل مصفوفة وحدة تعتبر مصفوفة قطرية، <u>وليس العكس</u>

2) $AI = A$ and $IA = A$

4- <u>Vector matrix</u> مصفوفة مكونة من صف واحد او عمود واحد فقط

① Column vector

$$\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}_{3*1}$$

② Row vector

$$\begin{bmatrix} 1 & -9 & -3 \end{bmatrix}_{1*3}$$

5- <u>Zero(null) matrix</u> is a matrix with all zeros

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$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \end{bmatrix}$$

✓ **<u>Notes:</u>**

1) كل مصفوفة صفرية مربعة تعتبر مصفوفة قطرية

2) $A+O = O+A = A-O = A$

Matrix Transpose A^T

✓ If A is **m*n** matrix, then A^T is **n*m**

$$B=\begin{bmatrix}2 & 3\\1 & 4\\5 & 6\end{bmatrix},\qquad C=[1\ \ 3\ \ 5],\qquad D=[4]$$

$$B^T=\begin{bmatrix}2 & 1 & 5\\3 & 4 & 6\end{bmatrix},\qquad C^T=\begin{bmatrix}1\\3\\5\end{bmatrix},\qquad D^T=[4]$$

✓ Properties of transpose

1) $(A^T)^T$ = A
2) $(KA)^T$ = KA^T (Where K is a constant)
3) $(AB)^T$ = $B^T A^T$
4) $(A+B)^T$ = A^T + B^T and $(A-B)^T$ = A^T - B^T
5) if A is a square matrix (n*n) , then A is said to be Symmetric matrix when $\Rightarrow$ $\boxed{A = A^T}$

Let **A** = $\begin{bmatrix}4 & -3 & 0\\-3 & 5 & 2\\0 & 2 & 7\end{bmatrix}$ $\qquad$ A^T = $\begin{bmatrix}4 & -3 & 0\\-3 & 5 & 2\\0 & 2 & 7\end{bmatrix}$

A = A^T Then A is symmetric matrix

Matrix Inverse A^{-1}

- ✓ If **A** and **B** are square matrices of the same size and **AB = BA = I** , Then **A** is said to be invertible and **B** is its inverse.
- ✓ If no such matrix **B** can be found , Then **A** is said to be no inverse (Singular)

Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ **and its inverse** $B = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

Then $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

And $BA = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Finding Inverse (2*2 matrix)

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ **Then** $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

✓ **Note:**

if ad – bc ≠ 0 , A is invertible , if ad – bc = 0 , A is singular

Let $A = \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$ **Then** $A^{-1} = \frac{1}{-1}\begin{bmatrix} 7 & -4 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -7 & 4 \\ 2 & -1 \end{bmatrix}$

Let $A = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$

(3*4) – (6*2) = 0 **Then A is not invertible**

*Finding Inverse (n*n matrix)*

$$[A \,/\, I] \xrightarrow{\text{Row Operations}} [I \,/\, A^{-1}]$$

✓ *Example 1*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \quad , \quad \textit{Find } A^{-1}$$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-3\boldsymbol{R_1} + \boldsymbol{R_3} \rightarrow \boldsymbol{R_3}]{-2\boldsymbol{R_1} + \boldsymbol{R_2} \rightarrow \boldsymbol{R_2}} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\text{Interchange } \boldsymbol{R_2} \text{ with } \boldsymbol{R_3}} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[-\boldsymbol{R_3} \rightarrow \boldsymbol{R_3}]{-\boldsymbol{R_2} \rightarrow \boldsymbol{R_2}} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{-2\boldsymbol{R_2} + \boldsymbol{R_1} \rightarrow \boldsymbol{R_1}} \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & -5 & 0 & 2 \\ 0 & 1 & 3 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[-3\boldsymbol{R_3} + \boldsymbol{R_2} \rightarrow \boldsymbol{R_2}]{3\boldsymbol{R_3} + \boldsymbol{R_1} \rightarrow \boldsymbol{R_1}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3 & 2 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

✓ *Example 2: Show the following matrix is not invertible*

$$A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-R_1 + R_3 \to R_3]{-2R_1 + R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 / -8 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & 1 & 9/8 & 1/4 & -1/8 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-8R_2 + R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & 1 & 9/8 & 1/4 & -1/8 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

$\boxed{\text{A is not invertible}}$

✓ *Properties of inverse*

1) $(A^{-1})^{-1} = A$

2) $(KA)^{-1} = \frac{1}{k} A^{-1}$ (Where K is a constant)

3) $A . A^{-1} = A^{-1} . A = I$

4) If A and B are invertible matrices of the same size, then:
   1- AB is invertible 2- $(AB)^{-1} = B^{-1} . A^{-1}$

5) If A is invertible, Then A^T is invertible and

$$(A^{-1})^T = (A^T)^{-1}$$