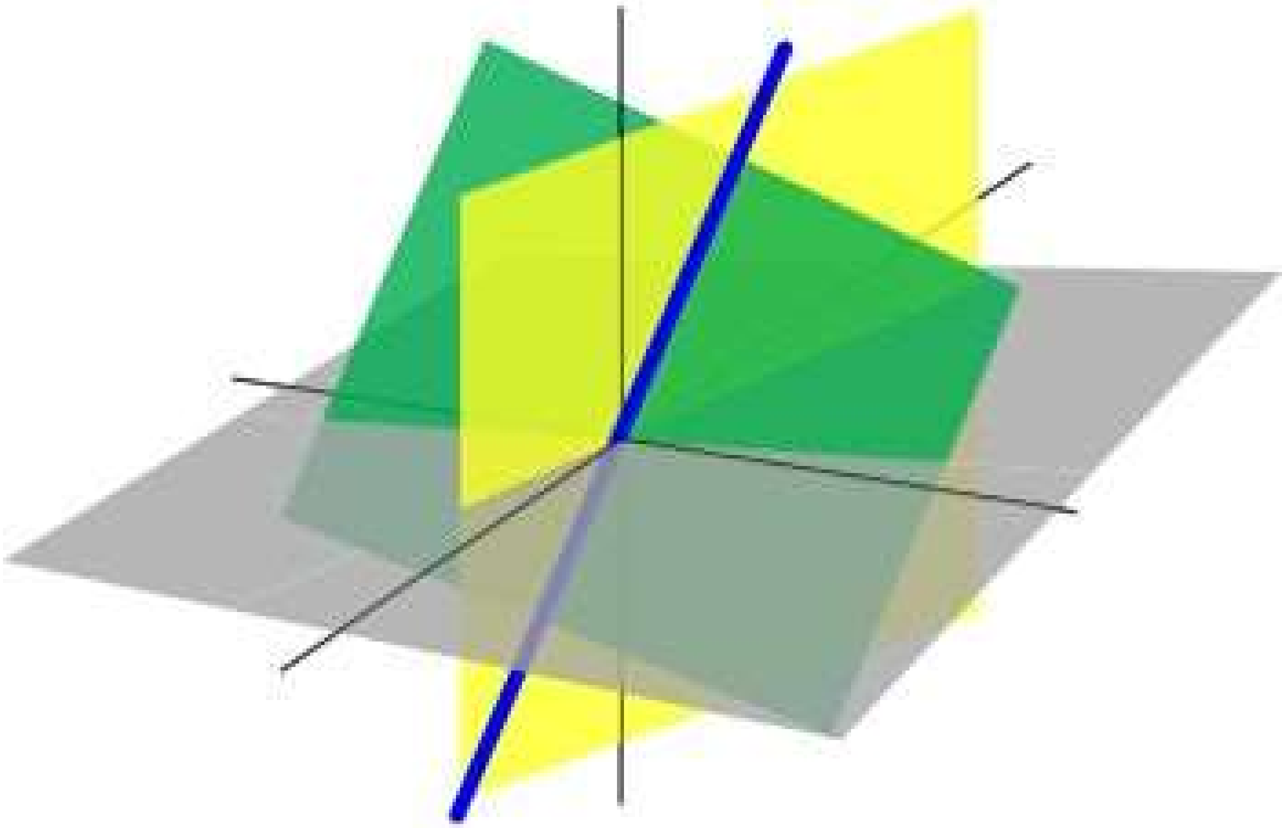
Linear Algebra Summary



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1 Introduction

Linear equations:

- $x + 3y - z = 0$
- $3x - 5y + 7z = 20$

*** examples of Non-linear equations:**

- $x - 3y + \sqrt{z} = 1$ $(\sqrt{z})$
- $\frac{1}{x} + y + z = 8$ $\left(\frac{1}{x}\right)$
- $xy + 3x + z = 2$ (xy)
- $x^2 + y + 3z = 19$ (x^2)
- $\sin x + y + \ln(z) = e^y$ $(\sin x)$ $(\ln(z))$ (e^x)

* System of linear equations :-

$$a_{11} X_1 + a_{12} X_2 + a_{13} X_3 = b_1$$
$$a_{21} X_1 + a_{22} X_2 + a_{23} X_3 = b_2$$
$$a_{31} X_1 + a_{32} X_2 + a_{33} X_3 = b_3$$

$$\boxed{AX = B}$$

$A \Rightarrow$ Coefficient Matrix

$X \Rightarrow$ Variables Matrix (unknowns)

$B \Rightarrow$ Constants Matrix

$$\underset{(A)}{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}} \underset{(X)}{\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}} = \underset{(B)}{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}$$

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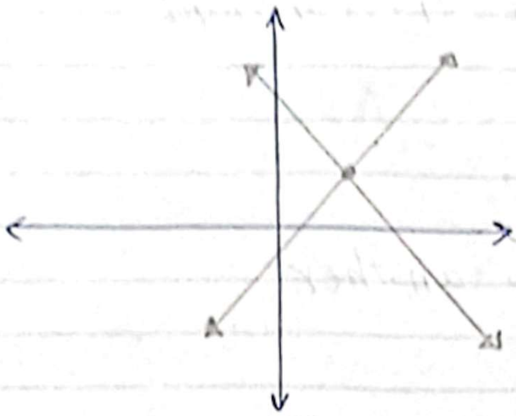
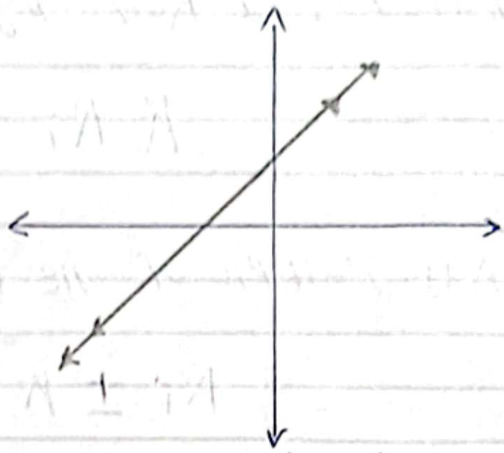
x system is :

↳ A) Consistent

① Unique Solution or ② Infinite solutions



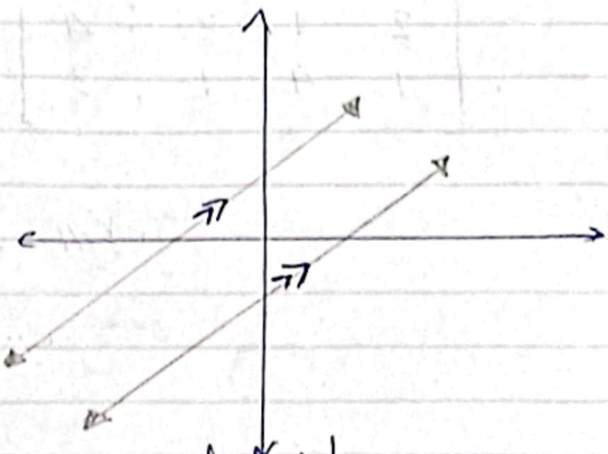
" متقاطعان في نقطة "



" متطابقان "

↳ B) InConsistent

↳ ③ No solution



" متوازيات "

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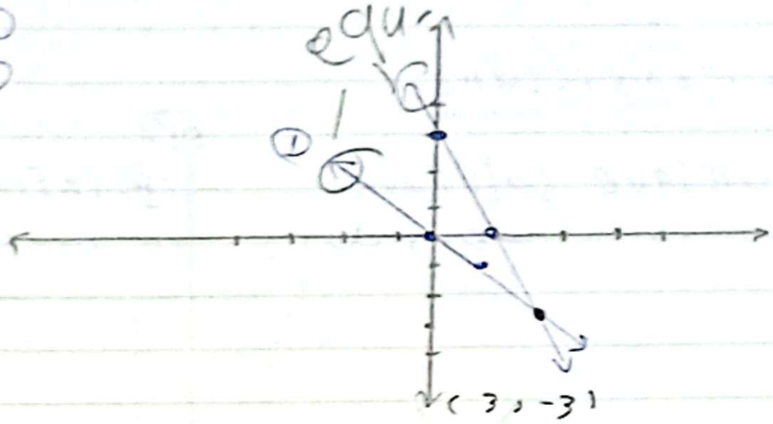
Q4: solve (2x2) system Geometricely

$x + y = 0 \longrightarrow ①$

$2x + y = 3 \longrightarrow ②$



$x = 0 \quad \therefore y = 0$

$x = 1 \quad y = -1$

$x = 0 \quad y = 3$

$x = 1, \; y = 1$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$ $\quad x = 3$, $\quad y = -3$

*Important note:- (Imp.)

IF two linear system have the same solution they are called $\Rightarrow$ equivalent system

ex:- 1) $x - y = 1$

$x + y = 3$

solve

$2x = 4$

$x = 2 \quad y = 1$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

2) $x - y = 1$

$y = 1$

solve

$x = 1 + 1 = 2 \quad x = 2$

$y = 1$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

* solve the given system by back substitutions.

① $x - y + z = 0$
$2y - z = 1$
$3z = -1$

⇒ back substitution solve

$\boxed{3z = -1}$ ∴ $\cancel{z = -1/3}$ $\boxed{z = \frac{-1}{3}}$

$2y = 1 + -\frac{1}{3} = \frac{2}{3}$ $\boxed{y = \frac{1}{3}}$

$\boxed{x} = y - z = \frac{1}{3} + \frac{1}{3} = \boxed{\frac{2}{3}}$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

* Find the augmentes matrix of linear systems.

$2x_1 + 3x_2 - x_3 = 1$

$x_1 + x_3 = 0$

$-x_1 + 2x_2 - 2x_3 = 0$

$$\rightarrow \left[\begin{array}{ccc|c} 2 & 3 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ -1 & 2 & -2 & 0 \end{array}\right]$$

* Elementry Row operations:

① Interchange Two rows:

$$\boxed{R_i \longleftrightarrow R_J}$$

② multiply a Row by Non-Zero Constant

$$\boxed{K\, R_i}$$

③ Add Multiple of one Row to another

$$\boxed{R_i \pm K\, R_J}$$

Note

change

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 2 & 1 & 3 & 4 \\ 1 & 4 & 1 & 3 \end{array}\right] \xrightarrow{R_1 + 2R_2} \left[\begin{array}{ccc|c} 5 & 2 & 7 & 13 \\ 2 & 1 & 3 & 4 \\ 1 & 4 & 1 & 3 \end{array}\right]$$

* التغير بيحصل كل أول Term

1 Row Echelon Form (REF)

Pivot

$$\begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2 Reduced Row Echelon Form (RREF)

ex:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

* $\boxed{\text{Pivot} = 1}$

ex:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

* use elementry row operation to reduce the given ~~matt~~ matrices to: → row echelon form
→ reduced row echelon form.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xleftrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 + -R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xleftrightarrow{R_2 - R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note ① row echelon form

1- $\begin{bmatrix} 0 & 5 & 6 & 0 & 0 \\ 0 & 0 & 2 & -4 & -5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$

2- $\begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

3- $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

4- $\begin{bmatrix} 1 & 5 & 6 & 1 & -1 \\ 0 & 0 & 2 & -4 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

5- $\begin{bmatrix} 1 & 5 & 6 & 0 & 0 \\ 0 & 0 & 2 & -4 & -5 \\ 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

② Reduced row echelon form

1- $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$

2- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3- $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

4- $\begin{bmatrix} 1 & 5 & 0 & 1 & -1 \\ 0 & 0 & 1 & -4 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

5- $\begin{bmatrix} 1 & \cancel{5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

× Methods of Solution :-

[1] Guassian elemination method

[2] Guass - Jordan elemination method

[3] Inverse Matrix method

[4] Cramer's method

1 " Guassian G (REF) "

① م حول system ⟵ ** "augmented matrix"

② هستخدم طريقة ال "row echelon form"

③ هرجع ال "Augmented matrix" ⟵ الى "system"

$$[A | b] \xrightarrow[\text{oper.}]{\text{row}} [U | c]$$

↳ upper matrix

ex:- Using Guass Elimination, solve:-

$$x_1 + x_2 + x_3 = 3$$
$$2x_1 + 3x_2 + 3x_3 = 7$$
$$4x_1 + 3x_2 - 2x_3 = 3$$

Solve

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 3 & 3 & 7 \\ 4 & -3 & -2 & 3 \end{array}\right]$$

$$\xrightarrow[R_3 + -4R_1]{R_2 + -2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & -7 & -6 & -9 \end{array}\right]$$

$$\xrightarrow{R_3 + 7R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$x_1 + x_2 + x_3 = 3$

$x_2 + x_3 = 1$

$\boxed{x_3 = -2}$ $\quad x_2 = 1 - -2 = 3$

$\boxed{x_2 = 3}$, $x_1 = 3 - 1$ $\boxed{x_1 = 2}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$$

~~e x2) : $x_1 + 2x_2 +$~~

e x_2 :

$-2x_1 + x_2 - x_3 = 4$

$x_1 + 2x_2 + 3x_3 = 13$

$3x_1 + x_3 = -1$

Solve

$$\left[\begin{array}{ccc|c} -2 & 1 & -1 & 4 \\ 1 & 2 & 3 & 13 \\ 3 & 0 & 1 & -1 \end{array}\right] \xrightarrow{R_1 \Leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 13 \\ -2 & 1 & -1 & 4 \\ 3 & 0 & 1 & -1 \end{array}\right]$$

$$\begin{array}{l} R_2 + 2R_1 \longrightarrow \\ R_3 + -3R_1 \longrightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 13 \\ 0 & 5 & 5 & 30 \\ 0 & -6 & -8 & -40 \end{array}\right] \xrightarrow{\frac{1}{5}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & -6 & -8 & -40 \end{array}\right]$$

$$R_3 + 6R_2 \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & -2 & -4 \end{array}\right] \xrightarrow{-\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & -1 & -2 \end{array}\right]$$

$x_1 + 2x_2 + 3x_3 = 13$

$x_2 + x_3 = 6$

$-x_3 = -2$

$\therefore \boxed{x_3 = 2}$ $\boxed{x_2 = 4}$ $x_1 =$

$x_1 = 3 - (2 \times 4 + 3 \cdot 2) = -11$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$$

2] ((Gauss - Jordan)) method :⟶ (RREF)

① حول ال System ⟵ "augmented matrix"

② حولها الى ⟵ "Reduced Row echelon method" (RREF)

③ حول ال ((augmented matrix)) ال ⟵ System

$$\boxed{[A \mid b] \xrightarrow[\text{oper}]{\text{row}} [I \mid x]}$$

ex:

$$2x_2 + x_3 = -8$$
$$x_1 - 2x_2 - 3x_3 = 0$$
$$-x_1 + x_2 + 2x_3 = 3$$

"Solve"

$$\left[\begin{array}{ccc|c} 0 & 2 & 1 & -8 \\ 1 & -2 & -3 & 0 \\ -1 & 1 & 2 & 3 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 2 & 1 & -8 \\ -1 & 1 & 2 & 3 \end{array}\right]$$

$$R_3 + R_1 \longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 2 & 1 & -8 \\ 0 & -1 & -1 & 3 \end{array}\right] \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 1 & \frac{1}{2} & -4 \\ 0 & -1 & -1 & 3 \end{array}\right]$$

$$R_1 + 2R_2 \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & -8 \\ 0 & 1 & \frac{1}{2} & -4 \\ 0 & -1 & -1 & 3 \end{array}\right] \xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & \frac{1}{2} & -4 \\ 0 & 0 & -\frac{1}{2} & -1 \end{array}\right]$$

$$\xrightarrow{-2R_3} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & \frac{1}{2} & -4 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow[R_2 + -\frac{1}{2}R_3]{R_1 + 2R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

النواتج تطلع على طول

$\boxed{x_1 = -4}$ $\boxed{x_2 = -5}$ $\boxed{x_3 = 2}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -5 \\ 2 \end{bmatrix}$$

note:- matrices are ⇒ row equivalent

(يكافئ)

1] If there is a sequence of "elementary row operation" that converts $A \longrightarrow B$

$$[A] \xrightarrow{\text{oper.}} \longrightarrow \longrightarrow [B]$$

∴ A, B are row equivalent

2] If they can be reduced to the same row echelon form.

note: zero matrix :-

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

row echelon form ← (and) → Reduced row echelon form

is

✓ ✓

Note :- (1) (imp) IF two matrices can be reduced to the same row echelon form.

they are $\Longrightarrow$ row equivalent

ex :- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}$,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{R_2 - 3R_1} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$$

$$\xrightarrow{R_1 + R_2} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \xrightarrow{-\frac{1}{2} R_2} \begin{bmatrix} 1 & 0 \\ 0 & +1 \end{bmatrix} \checkmark$$

$$B = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} \xrightarrow[-R_2]{R_1 \Leftrightarrow R_2} \begin{bmatrix} 1 & 0 \\ 3 & +1 \end{bmatrix}$$

$$\xrightarrow{R_2 + -3R_1} \begin{bmatrix} 1 & 0 \\ 0 & +1 \end{bmatrix} \checkmark$$

ex(1) :-

$$2x - 3y + z = 1$$
$$x - 2y + z = 3$$

n=3
m=2

((solve)

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & 1 \\ 1 & -2 & 1 & 3 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 2 & -3 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 + -2R_1} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 1 & -1 & -5 \end{array}\right]$$

$$\xrightarrow{R_1 + 2R_2} \left[\begin{array}{ccc|c} \boxed{1} & 0 & -1 & -7 \\ 0 & \boxed{1} & -1 & -5 \end{array}\right]$$

$$\cancel{x_1 - x_3 = -7} \qquad \boxed{\begin{array}{l} x - z = -7 \\ y - z = -5 \end{array}}$$
$$\cancel{x_2 - x_3 = -5}$$

free

"أفرض أول المتغير الموجود في كل المعادلات"

$$\therefore \boxed{z = t} \qquad \therefore \boxed{x = -7 + t} \qquad \therefore \boxed{y = -5 + t}$$

(n > m)

* عدد المعادلات = 2 ، عدد المجاهيل = 3

$\Rightarrow$ "Infinitely many solutions" $\therefore$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t - 7 \\ t - 5 \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} + \begin{bmatrix} -7 \\ -5 \\ 0 \end{bmatrix}$$

$$= t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -7 \\ -5 \\ 0 \end{bmatrix}$$

ex 2 :-

$$x + 3y = 7$$
$$2x - y = 4$$
$$3x + 2y = 11$$

$n = 2$
$m = 3$

" solve "

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 2 & -1 & 9 \\ 3 & 2 & 11 \end{array}\right] \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & -7 & -10 \\ 0 & -7 & -10 \end{array}\right]$$

$$R_3 - R_2 \longrightarrow \left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & -7 & -10 \\ 0 & 0 & 0 \end{array}\right]$$ ⟸ (مش موجود)

$$x + 3y = 7$$
$$-7y = -10 \qquad \boxed{y = \frac{10}{7}}$$
$$x = 7 - 3y = 7 - 3 * \frac{10}{7} = \boxed{\frac{19}{7}} \#$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{19}{7} \\ \frac{10}{7} \end{bmatrix} \Rightarrow 1/2 \ (n = m) = \frac{1}{7}\begin{bmatrix} 19 \\ 10 \end{bmatrix}$$

$\therefore \Longrightarrow$ " unique solution "

Ex3 :- $x_1 - 2x_2 - x_3 + 3x_4 = 0$

$-2x_1 + 4x_2 + 5x_3 - 5x_4 = 3$

$3x_1 - 6x_2 - 6x_3 + 8x_4 = 2$

"solve"

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ -2 & 4 & 5 & -5 & 3 \\ 3 & -6 & -6 & 8 & 2 \end{array}\right]$$

$$\xrightarrow[R_3 - 3R_1]{R_2 + 2R_1} \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & -3 & -1 & 2 \end{array}\right]$$

$$\xrightarrow{R_3 + R_2} \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{array}\right]$$

$(n < m)$

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & b \end{array}\right]$$

هذا الشكل ⇐ يعني أن لا يوجد حل

$\Longrightarrow$ no solution

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* special matrices:

1 square matrix:

size = $m \times n$ $n = m$

$$\begin{bmatrix} 5 & 3 \\ 2 & 5 \end{bmatrix}_{2\times 2} , \quad \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 1 \\ 7 & 7 & 9 \end{bmatrix}_{3\times 3}$$

2 row vector

size = $1 \times n$

$$\begin{bmatrix} 1 & 3 & 7 & 8 \end{bmatrix}$$

3 column vector

size = $m \times 1$

$$\begin{bmatrix} 1 \\ 3 \\ 7 \\ 2 \end{bmatrix}$$

4 Diagonal matrix :

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \rightarrow \text{Main Diagonal}$$

5 scalar matrix :

$$M_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} , \quad M_2 = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$$M = K \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underline{\underline{K}}\, I_n$$

$K \rightarrow$ Constant $\neq 0$

6 Identity Matrix

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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7 Upper-triangle Matrix

$$U=\begin{bmatrix} 5 & 3 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

8 Lower-Triangle Matrix

$$L=\begin{bmatrix} 5 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 3 & 5 \end{bmatrix}$$

9 REF

$$\begin{bmatrix} 7 & 0 & 1 & 0 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

10 RREF

$$\begin{bmatrix} 1 & 3 & 0 & \cancel{7} \\ 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note Upper-Matriax is REF

11 zero Matrix: ($\theta_{n \times m}$)

$$\theta_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note θ = zero-Matrix is → REF & → RREF

7] zero matrix $\Rightarrow$ ($\theta_{n \times m}$)

$\Rightarrow$ A matrix all of whose entries are $\Rightarrow$ zero (0)

ex:- $\theta_{2\times3} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\theta_{2\times1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

* matrix operations :-

, let matrix A , matrix B

$A = B$ if :
- $\Rightarrow$ $\boxed{\text{the same size}}$ (لهم نفس الحجم)
- $\Rightarrow$ their corresponding Entries are equal (عناصرهم المتناظرة متساوية)

ex:- 1] (A ± B) If $\Rightarrow$ the same size

ex:

$$\begin{bmatrix} 1 & 7 & 3 \\ 3 & 5 & 5 \\ 5 & 2 & 7 \end{bmatrix} + \begin{bmatrix} 1 & -3 & 2 \\ 9 & 5 & 1 \\ -1 & 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & 5 \\ 12 & 10 & 6 \\ 4 & 5 & 12 \end{bmatrix}$$

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2) $rA \rightarrow$ multiplication By scalar ((الرابعة))

«الضرب في جميع عناصر المصفوفة»

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\therefore$ $3A = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$

Properties (خواص)

* $A + B = B + A$ ((الابدال))

* $(A + B) + C = A + (B + C)$ ((الدمج))

* $r(A + B) = rA + rB$ (التوزيع)

* $A + \theta = \theta + A = A$

* $A - A = \theta$, $IA = A$

3] matrix multiplication :-

$$\underbrace{\underset{m \times \boxed{P}}{A} \rightarrow \; = \; \leftarrow \underset{\boxed{P} \times n}{B}}_{\text{(size) المصفوفة الجديدة}} \times \; = \; \underset{m \times n}{C}$$

الشرط ⇐ عدد أعمدة الأول = عدد صفوف الثاني

$\Rightarrow$ the dot product

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ex: $\begin{bmatrix} 2 & 3 & 4 \end{bmatrix}_{1\times(3)} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}_{(3)\times 1} = \begin{bmatrix} 38 \end{bmatrix}_{1\times 1}$

ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times(2)} \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}_{(2)\times 2} = \begin{bmatrix} 0 & -1 \\ 4 & -5 \end{bmatrix}$

$BA = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -5 & -4 \\ 1 & 0 \end{bmatrix}$

Note:

$AB \neq BA$ is not commutative

but:

If (($AB = BA$)) then A and B are said to ⇒ commute matrices

Note

① $AD = 0_{m\times n}$ $\xrightarrow{or}$ $A \neq 0$, $D \neq 0$ (ليس شرطاً)

② $AB = AC \longrightarrow B \neq C$ (ليس شرطاً)

ex $A = \begin{bmatrix} 1 & -3 & 0 & 4 \\ -2 & 5 & -8 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 5 & 3 \\ -3 & 10 & 2 \\ 2 & 0 & -4 \\ -1 & -7 & 5 \end{bmatrix}$

2×4 $\quad\quad$ 4×3

Sol

$$AB = \begin{bmatrix} 13 & -53 & 17 \\ -56 & -23 & 81 \end{bmatrix}$$

$BA \Rightarrow$ $\boxed{\text{We can't do it.}}$

Note

يمكن كتابة المصفوفة كالآتي :-

$A = [a_1 \ a_2 \ \cdots \ a_n]$

$\bar{a}_i = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix} \longrightarrow$ column

باختصار كل الـ (Colums) ب $(\bar{a})$

or

$A = \begin{bmatrix} \cancel{b_1} \\ \cancel{b_2} \\ \vdots \\ \cancel{b_n} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

$\bar{b}_J = [b_1 \ b_2 \ b_3 \cdots b_n]$

باختصار كل الـ ((rows)) ب $(\bar{b})$

* matrix row $\boxed{\text{representation}}$:- إيجاد صف ما مباشرة من حاصل ضرب مصفوفتين

$$A = \begin{bmatrix} 1 & -3 & 0 & 4 \\ -2 & 5 & -8 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} 8 & 5 & 3 \\ -3 & 10 & 2 \\ 2 & 0 & -4 \\ -1 & -7 & 5 \end{bmatrix}$$

① To Find: 2^{nd} row of matix(AB)

$\boxed{\Rightarrow (2^{nd} \text{ row of matrix } A) \times B}$

ex:- $$\begin{bmatrix} -2 & 5 & -8 & 9 \end{bmatrix} \begin{bmatrix} 8 & 5 & 3 \\ -3 & 10 & 2 \\ 2 & 0 & -4 \\ -1 & -7 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -56 & -23 & 81 \end{bmatrix}$$

To Find 1^{st} row of (AB)

$\boxed{\Rightarrow (1^{st} \text{ row of } (A)) \times B}$

$$= \begin{bmatrix} 1 & -3 & 0 & 4 \end{bmatrix} \begin{bmatrix} 8 & 5 & 3 \\ -3 & 10 & 2 \\ 2 & 0 & -4 \\ -1 & -7 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & -53 & 17 \end{bmatrix}$$

* matrix column representation:　　إيجاد عمود ما مباشرةً من حاصل ضرب مصفوفتين

i) To find 2^{nd} column of (AB)

$$\Rightarrow \boxed{A \times (2^{nd} \text{ column of } (B))}$$

$$= \underset{2 \times 4}{\begin{bmatrix} 1 & -3 & 0 & 4 \\ -2 & 5 & -2 & 9 \end{bmatrix}} \underset{4 \times 1}{\begin{bmatrix} 5 \\ 10 \\ 0 \\ -7 \end{bmatrix}} = \underset{2 \times 1}{\begin{bmatrix} -53 \\ -23 \end{bmatrix}}$$

ii) To find 3^{rd} column of (AB):-

$$\Rightarrow \boxed{A \times (3^{rd} \text{ column of } (B))}$$

$$= \begin{bmatrix} 1 & -3 & 0 & 4 \\ -2 & 5 & -2 & 9 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -4 \\ 5 \end{bmatrix} = \begin{bmatrix} 17 \\ 81 \end{bmatrix}$$

لو عايز ← صف ← من ⇐ أخده من المصفوفة الأولى (A)

(AB)

← عمود من ⇐ أخده من المصفوفة الثانية (B)

Block multiplication :-

تقسيم كل مصفوفة إلى Block ونجري عملية الضرب على الـ (Blocks) وكأنها عناصر (entries)

$$B = \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 5 & 6 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad A = \left[\begin{array}{ccc} 6 & 5 & 4 \\ 3 & 2 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

$3 \times 5 \qquad 5 \times 3$

$$= \begin{bmatrix} I_2 & A_{2\times3} \\ \Theta_{1\times2} & \Theta_{1\times3} \end{bmatrix} \begin{bmatrix} B_{2\times3} \\ I_3 \end{bmatrix} = \begin{bmatrix} I_2 B_{2\times3} + A_{2\times3} I_3 \\ \Theta_{1\times2} B_{2\times3} + \Theta_{1\times3} I_{3\times3} \end{bmatrix}$$

$\cancel{2} \times 2 \qquad 2 \times \cancel{1} \qquad 2 \times 1$

$$= \begin{bmatrix} B_{2\times3} + A_{2\times3} \\ \Theta_{1\times3} + \Theta_{1\times3} \end{bmatrix} = \begin{bmatrix} B_{2\times3} + A_{2\times3} \\ \Theta_{1\times3} \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 7 & 7 \\ 7 & 7 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

أول ما تحاول تقسمها خليها الـ Block ←

$\boxed{I_n}$

$\boxed{\Theta_{n\times n}}$

To find : 1^{st} column :-

$$A e_1 = \underset{2\times3}{\begin{bmatrix}4 & 2 & 1\\ 0 & 5 & -1\end{bmatrix}} \underset{3\times1}{\begin{bmatrix}1\\0\\0\end{bmatrix}} = \begin{bmatrix}4\\0\end{bmatrix} \rightarrow$$

To find 3^{rd} column :-

$$A e_3 = \begin{bmatrix}4 & 2 & 1\\ 0 & 5 & -1\end{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}$$

Properities of matrix multiplication :-

1] $(AB)C = A(BC)$ لنفس الترتيب (الدمج)

2] $rAB = (rA)B = A(rB)$

3] $A(B+C) = AB + AC$ $\quad * \;(AB \neq BA)$

4] $(B+C)A = BA + CA$

5] $I_m A_{m\times n} = A_{m\times n} I_n = A$

6] $\Theta_{1\times2}\, A_{2\times3} = \Theta_{1\times3}$

7] $A_{2\times3}\, \Theta_{3\times2} = \Theta_{2\times2}$

* matrix power :- $\longrightarrow$ (square matrix)

A $\Rightarrow$ matrix then. $A^k = \underbrace{A\,A\,A \cdots A}_{(k)}$

$A^3 = AAA$

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $A^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$A^2 = \begin{bmatrix} 7 & 9 \\ 15 & 22 \end{bmatrix}$

If (A) is (square matrix) and ~~(r, s) nonnegative integer~~

|| r, s $\Rightarrow$ non negative Integer :- || ($r, s = 0, 1, 2, \dots$)

then, || Properties of Power of matrix ||

1] ~~$A^r A^s = A$~~

1] $A^r A^s = A^{\boxed{r+s}}$

2] $(A^r)^s = A^{rs} = (A^s)^r$

3] $A^1 = A$

4] $A^0_{n \times n} = I_n$ $\hookrightarrow$ "Identity matrix"

موضوع الدرس : اليـــوم : التاريخ : / /

* the transpose of a matrix :

" تقلب الصفوف أعمدة و الأعمدة صفوف "

$\Rightarrow A^T$ الرمز

$\underset{n \times m}{A} \longrightarrow \underset{m \times n}{A^T}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Note IF A is $\boxed{\text{square matrix}}$:-

then, A and A^T $\Rightarrow$ has (the same Diagonal entries)

" square لازم تكون بس "

ex :-

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

ex :-

$$B = \begin{bmatrix} 2 & -1 & 5 \\ 0 & 1 & 3 \\ 0 & 6 & 4 \end{bmatrix} \qquad B^T = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 6 \\ 5 & 3 & 4 \end{bmatrix}$$

Properties of transpose

① $(A^T)^T = A$

② $(A \pm B)^T = A^T \pm B^T$

③ $(kA)^T = kA^T$

④ $(AB)^T = B^T A^T$

⑤ $(ABC)^T = C^T B^T A^T$

* If A is a square matrix

① Symmetric matrix

$$\boxed{A^T = A}$$

e.g.

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & 6 \\ 3 & 6 & 7 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & 6 \\ 3 & 6 & 7 \end{bmatrix}$$

$(A = A^T)$

② Skew-symmetric matrix

$$\boxed{A^T = -A}$$

$$A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix}$$

$A = -A^T$

theorems:-

If (A) is $\boxed{\text{square matrix}}$:-

$$\boxed{(A + A^T) \Rightarrow \text{symmetric matrix}}$$

If for any matrix (A),

$$\boxed{(AA^T) \text{ and } (A^TA) \Rightarrow \text{symmetric matrix}}$$

Proof

$$(A + A^T)^T = A^T + (A^T)^T = A^T + A$$

L.H.S R.H.S

$$\therefore \text{L.H.S} = \text{R.H.S}$$

$$(AA^T)^T = (A^T)^T A^T = AA^T$$

L.H.S R.H.S

$$\therefore \boxed{\text{L.H.S} = \text{R.H.S}}$$

$$(A^TA)^T = (\cancel{A^T)^T A^T = A^TA}$$

$$= A^T (A^T)^T = A^T A$$

$$(\text{L.H.S} = \text{R.H.S})$$

Note

$$\boxed{A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)}$$

* what is the benefit of:
  - 1) upper triangle Matrix
  - 2) REF
  - 3) RREF

↳ To solve system of linear equations and then using of Back substitution

ex:

$$\begin{bmatrix} 5 & 2 & 3 \\ 6 & 1 & 5 \\ 1 & 2 & 13 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 12 \\ 100 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$3x + 2y + z = 1 \quad \cancel{x}$$

$$2y + 6z = 2$$

$$5z = 3$$

Note:

We can write linear system as a linear combination of column vectors:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

if and only if:

$\boxed{AX = B} \xRightarrow{\text{is}}$ Consistent

ex:

$$\begin{bmatrix} 2 & 3 & -1 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 18 \end{bmatrix}$$

$$\equiv x_1 \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 18 \end{bmatrix}$$

التاريخ : الموضوع :

* Rank of Matrix:

⇒ is a No. of (Non-dependent vectors)

⇒ is a No. of (Non-zero rows) in → REF
↳ RREF

$$M_1 = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \qquad r(M_1) = 3$$

$$M_2 = \left[\begin{array}{ccc|c} 1 & -5 & 3 & 1 \\ 0 & 7 & 0 & 2 \end{array}\right] \qquad r(M_2) = 2$$

$$M_3 = \left[\begin{array}{ccc|c} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad r(M_3) = 2$$

Note

$$r(\Theta) = 0$$

* "the Rank theorem"

$[A \vdots b]$

the augmented matrix of linear system of → m "equations" → معادلات
→ n "variables" → مجاهيل

(≠)

① $r(A) < r(A \vdots b)$ → no solution
[Augmented]

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 8 \end{array}\right] \rightarrow$$

$r(A) = 2$ $r(A \vdots b) = 3$

"no solution"

② $\boxed{r(A) = r(A|b)} = n$ ⇒ a unique solution

$$\begin{array}{c} \begin{array}{cccc} x & y & z & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \end{array}\right] \end{array} \rightarrow$$

$r(A) = 3$ $r(A|b) = 3$

$(n = 3)$

"unique solution"

③ $\boxed{r(A) = r(A|b)} < n$ ⇒ Infinitely many solutions

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & -4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow$$

$r(A) = 2$ $r(A|b) = 2$

$(n = 3)$

"Infinitly many solutions"

IF → (a) "over determined" $(m > n)$

⇒ It's "Inconsistent"

↳ "no solution"

$\boxed{AX = b}$

→ (b) "under determined" $(m < n)$

⇒ It's "Consistent"

↳ "Infinitely many solution" (only)

ex:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 3 & 5 \end{array}\right] \rightarrow \quad m = 3 \quad n = 2$$

$(m > n)$

∴ (over-determined) ⇒ no solution

ex2:

$$\left[\begin{array}{ccc|c} 1 & 5 & 7 & 3 \\ 0 & 3 & 2 & 4 \end{array}\right] \quad m = 2 \quad n = 3$$

→ $(m < n)$

under determined ⇒ "Infinitely many solutions"

Note

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$$ "الموزون دي نَ آ ز صف"

c① $\boxed{(m = n)}$ → (unique solution)

or

c② → $(m < n)$ → Infinitely many solution ex1-

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & -4 \\ 2 & 5 & -1 & 7 \\ 4 & 5 & -3 & -5 \\ 4 & 9 & -5 & -1 \end{array}\right]$$

Sol:

$$\xrightarrow[R_4 - R_3]{R_2 - 2R_1,\ R_3 - 4R_1} \left[\begin{array}{ccc|c} 1 & 2 & -2 & -4 \\ 0 & 1 & 3 & 15 \\ 0 & -3 & 5 & 11 \\ 0 & 4 & -2 & 4 \end{array}\right]$$

$$\xrightarrow{R} \left[\begin{array}{ccc|c} 1 & 0 & -8 & -34 \\ 0 & 1 & 3 & 15 \\ 0 & 0 & 14 & 56 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow m = 3 \ \{ \ n = 3$$

unique solution

or

$r(A) = 3 = r(A|b) = 3$

$n = 3$

ex1

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 5 & 5 & 8 \end{array}\right] \qquad (m < n)$$

$$\xrightarrow{R} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow m = 2 \ \{ \ n = 3$$

Infinitely many solution

* A linear system is :

① homogeneous → $[A \mid 0]$, $[b] = 0$

$[b] = \theta_{n \times 1}$

ex: $3x_1 + 2x_2 + 5x_3 = 0$
$2x_1 - x_2 + x_3 = 0$

② non homogeneous → $[A \mid b]$, $(b \neq 0)$

ex: $3x_1 + 2x_2 + 5x_3 = 2$
$2x_1 - x_2 + x_3 = 5$ $\neq 0$

* the homogeneous system

→ Every homogeneous system is consistent

c① ~~m < n~~

c① $m < n$ " under determined "

⇒ Infinitely many solution including zero solution

c② $m = n$

⇒ unique solution / the zero solution / travial solution

الحل الصفري

✗ Inverse matrix ✗

→ ((square matrix.))

express:-

$A_{n\times n}$ and $B_{n\times n}$

where:.

$$\boxed{AB = BA = I_n}$$

لشرط :- $\boxed{A}$ و $\boxed{B}$ ⇐ ((square matrices))

في هذه الحالة :

أن (A) ⇐ لها inverse ← $\boxed{A^{-1}}$

$$AA^{-1} = A^{-1}A = I_n$$

$$((AB = BA = I_n))$$

IF:

→ $\cancel{A = B = A^{-1}}$

IF:

① → $\boxed{B = A^{-1}}$ → A لها معكوس → Invertible

② → $\boxed{B \neq A^{-1}}$ → A ليس لها معكوس → singular

* Properties of (Inverse matrix)

Let: A is the invertible matrix → A^{-1}

① $(A^{-1})^{-1} = A$

② $(cA)^{-1} = \frac{1}{c}A^{-1}$ * " $c \longrightarrow$ "scalar" " (−1)

③ $(AB)^{-1} = B^{-1}A^{-1}$

④ $(ABC\cdots)^{-1} = \cdots C^{-1}B^{-1}A^{-1}$

$\boxed{I^{-1} = I}$

⑤ $(A^T)^{-1} = (A^{-1})^T$

⑥ $(A^n)^{-1} = (A^{-1})^n$ " ~~n ⇒ positive in~~

↳ n ⇒ "nonnegative integer"

عدد صحيح غير سالب = (0, 1, 2, ...)

Definition

$$A^{-n} = (A^{-1})^n = (A^n)^{-1}$$

* Elementary matrix:

(E) ⇐ أي مصفوفة جاية من (I_n) ولكن مُطبق عليها (one row operation)

" عملية واحدة فقط " //

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\boxed{1}]{R_1 \leftrightarrow R_2} \underline{E_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\boxed{2}]{5 R_3} \underline{E_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\boxed{3}]{R_1 + 2R_3} E_3 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note ① the inverse of ((Elementary matrix)) is $\boxed{\text{Elementary matrix}}$

الاتنين زي بعض

② The transpose of Elementary matrix is $\boxed{\text{Elementary matrix}}$

③ the product of two elementary matrices isn't $\boxed{\text{Elementary matrix}}$

موضوع الدرس: اليوم: التاريخ: / /

Note: ال Row operation يكافئ ضرب المصفوفة من الشمال في $\boxed{\text{Elementary matrix (E)}}$ = مطبق عليها نفس ال operation

ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$

$\therefore I \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1$

$\therefore E_1 A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$

ex (2):

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow{3R_2} \begin{bmatrix} 1 & 2 & 3 \\ 12 & 15 & 18 \\ 7 & 8 & 9 \end{bmatrix}$

$\therefore I \xrightarrow{3R_2} = E_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore E_2 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 12 & 15 & 18 \\ 7 & 8 & 9 \end{bmatrix}$

* the inverse of Elementary matrix:-

للحصول على (E^{-1}) :-

+ نطبق عكس العملية على الـ (I_n)

ex. 1) $R_1 \longleftrightarrow R_2 \Longrightarrow R_2 \longleftrightarrow R_1$ هو هو (الإبدال)

2) $nR_1 \Longrightarrow \frac{1}{n} R_1$ (المعكوس الضربي)

3) $R_1 + 2R_3 \longrightarrow R_1 - 2R_3$ (المعكوس الجمعي)

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \therefore \quad I_3 \xrightarrow{R_1 \leftrightarrow R_2} E_1$$

$$\therefore \quad E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1^{-1}$$

Note في عملية الابدال المرفوق ⟸ ($E_1 = E_1^{-1}$)

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad \therefore \quad I_3 \xrightarrow{4R_3} E_2$$

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{1}{4} R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} = E_2^{-1}$$

ex: $E_3 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\because I_3 \xrightarrow{R_1 + 2R_3} E_3$

$\therefore E_3^{-1} = I_3 \xrightarrow{R_1 - 2R_3} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_3^{-1}$

* matrices $\underline{A}$ and $\underline{B}$ are row equivalent.

لو قدرت أوصل ؟

$A \xrightarrow[\text{operation}]{\text{row}} B$

$$\boxed{E_K E_{K-1} \cdots E_2 E_1 A = B}$$

هذا الضرب من الشمال
يكافئ الـ row operation اللازمة
لتحويل : $(A \xrightarrow{} B)$

$A \longrightarrow B$
or
$B \longrightarrow A$

ex: $(A \longrightarrow B)$

$A \longrightarrow R_1 \longleftrightarrow R_2$ E_1
$3R_3$ E_2
$R_1 + 2R_3$ E_3
$R_2 \longleftrightarrow R_3$ E_4
$\longrightarrow B$

$$\therefore \boxed{B = E_4 E_3 E_2 E_1 A}$$

موضوع الدرس : اليـــوم : التاريخ : / /

* Invertible matrix theorem:-

→ If a matrix A is square matrix $(n \times n)$ then the following are equivalent:-

(I) A is invertible.

(II) $AX = b$ → the system has a "unique solution" "له حل وحيد"

(III) $AX = 0$ → the system has only "trivial solution" "له الحل الصفري"

(IV) A is Row equivalent to I_n $(A_{n \times n} \longrightarrow I_n)$

(V) A is a product of elementary matrices.

Note: If: A is "Invertible matrix"

then, (A, I_n) are row equivalent

$$\therefore \underline{A} \xrightarrow[\text{oper.}]{\text{row}} \underline{I_n}$$

$$\boxed{\underbrace{E_k E_{k-1} \cdots E_2 E_1}_{} A = I_n}$$

((row operations)) شوية

" Inversion of matrices "

* Gauss - Jordan $\boxed{\text{inversion}}$:-

Let :- a square matrix (A) is Invertible :-

step ① super - Augmented matrix :-

$$\left[A_{n \times n} \,\middle|\, I_n \right]$$

step ② Row operation on $[A \mid I]$

معك (Row oper.) على ال $[A \mid I]$

لحد أما أوصل ال $[A]$ إلى ((Reduced row echlon form)) $[I_n \mid A^{-1}]$

IF :- $[\cancel{A}]$ (rrEF) $\longrightarrow$ $= I_n \Rightarrow$ (Invertible)

ليها معكوس

$\longrightarrow \neq I_n \Rightarrow$ (signulare)

مالها معكوس

$$[A \mid I] \xrightarrow[\text{operations}]{\text{row}} [I \mid A^{-1}]$$

Note

$$A = E_1^{-1} \, E_2^{-1} \, E_3^{-1} \cdots$$

$$A^{-1} = E_3 \, E_2 \, E_1$$

ex:

Find inverse of (A) $A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$

$$[A \mid I_2] = \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 0 & 3 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 0 & 1 & 1/3 & -2/3 \end{array}\right] \xrightarrow{\cancel{R_2}}$$

$$\xrightarrow{R_1 + R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1/3 & 1/3 \\ 0 & 1 & 1/3 & -2/3 \end{array}\right]$$

$$\therefore [I \mid A^{-1}] = \left[\begin{array}{cc|cc} 1 & 0 & 1/3 & 1/3 \\ 0 & 1 & 1/3 & -2/3 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{bmatrix} = 1/3 \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

Note: طريقة ثانية للمعكوس المصفوفة الـ $\boxed{2 \times 2}$ فقط

IF A is a square matrix (2 × 2).

$$A = \begin{bmatrix} a & \pm b \\ \pm c & d \end{bmatrix}$$

((أ بدل عناصر القطر الرئيسي))
((وأ غير إشارات القطر الآخر))

$$\boxed{A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

$$\det(A) = \boxed{ad - cb} \Rightarrow \text{المحدد المصفوفة}$$

ex:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\det(A) = 2 * -1 - 1 * 1 = -3$$

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix} = -1/3 \begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= 1/3 \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

Note: solving the linear system using ⇒ (Gauss - Jordan inversion)

ex:

$$\begin{aligned} x + 2z &= 3 \\ -x + y - 2z &= -3 \\ 2x + 2y + z &= 6 \end{aligned}$$

(Sol)

$$A^{-1} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & -2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \overset{A^{-1}}{=} \begin{bmatrix} 3 \\ -3 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & -2 \\ 2 & 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ -3 \\ 6 \end{bmatrix}$$

#

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -1 & 1 & -2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 - 2R_1]{R_2 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 2 & -3 & -2 & 0 & 1 \end{array}\right]$$

$$\cancel{\xrightarrow{R_3 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{array}\right]}$$

$$\xrightarrow{-\frac{1}{3} R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & -2/3 & 1 & \frac{2}{3} & 0 & -1/3 \end{array}\right]$$

$$\xrightarrow{R_3 + \frac{2}{3} R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 4/3 & 0 & -1/3 \end{array}\right]$$

$$\xrightarrow{R_1 - 2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5/3 & 0 & 2/3 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 4/3 & 0 & -1/3 \end{array}\right] = [I \mid A^{-1}]$$

$$\therefore A^{-1} = \begin{bmatrix} -5/3 & 0 & 2/3 \\ 1 & 1 & 0 \\ 4/3 & 0 & -1/3 \end{bmatrix}$$

$$X = A^{-1} b = \underset{3 \times 3}{\begin{bmatrix} -5/3 & 0 & 2/3 \\ 1 & 1 & 0 \\ 4/3 & 0 & -1/3 \end{bmatrix}} \underset{3 \times 1}{\begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}}$$

$$X = \underset{3 \times 1}{\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$$

$x = -1$

$y = 0$

$\cancel{z} = z = 3$

Note solving the system using :-
(Gauss - Jordan Elimination)

$$* \quad \boxed{A\,X = b} \longrightarrow \boxed{X = A^{-1}b}$$

Augumented matrix $\longrightarrow [A \mid b]$

$$\boxed{[A \mid b] \xrightarrow[\text{oper.}]{\text{row}} [I \mid \underbrace{A^{-1}b}_{\text{Solution}}]}$$

$$\boxed{X = A^{-1}b}$$

$$\boxed{\det(I_n) = 1}$$

Ex :-

$$\left[\begin{array}{cc|ccc} 1 & 2 & 3 & -1 & 2 \\ 2 & 6 & 5 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|ccc} 1 & 2 & 3 & -1 & 2 \\ 0 & 2 & -1 & 4 & -4 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{cc|ccc} 1 & 0 & 4 & -5 & 6 \\ 0 & 2 & -1 & 4 & -4 \end{array}\right]$$

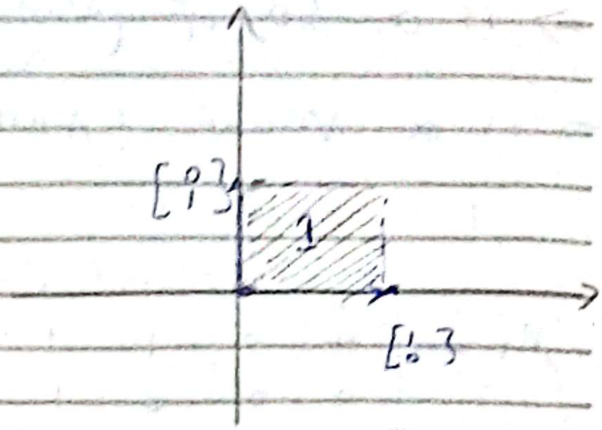
$$\xrightarrow{1/2\,R_2} \left[\begin{array}{cc|ccc} 1 & 0 & 4 & -5 & 6 \\ 0 & 1 & -1/2 & 2 & -2 \end{array}\right]$$

(Solution)

* Matrix Determinant *

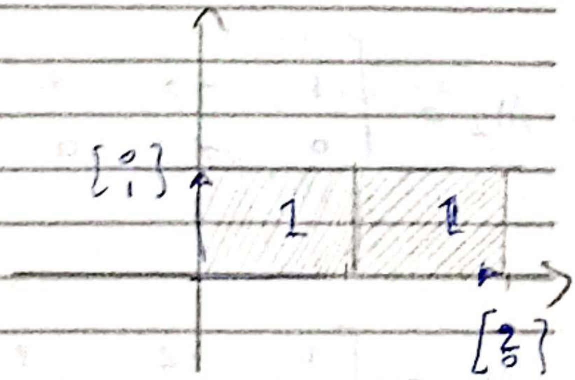
① in (2D):

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



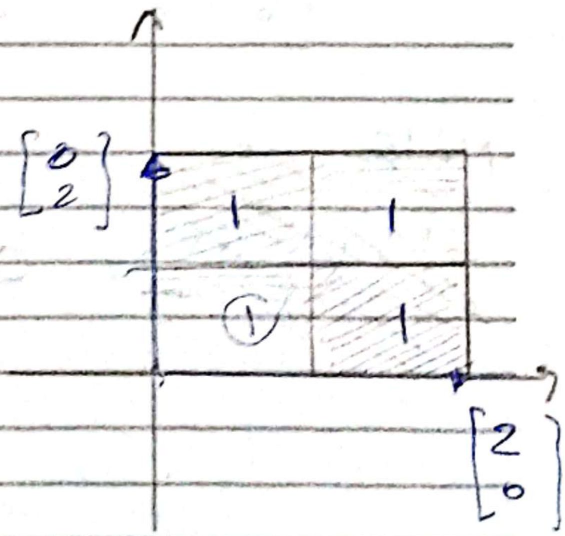
$\det(A) = 1 - 0 = 1$

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$



$\det(A) = 2 - 0 = 2$

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$



$\det(A) = 4$

التاريخ :   الموضوع :

Note

if ① det(A) > 0 (+ve)

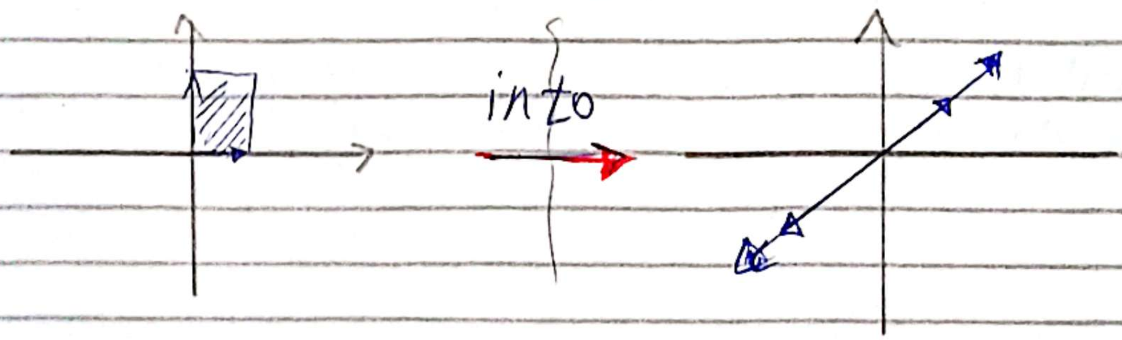
→ the shape keeps its orientation
and
→ scalling by |det(A)| value

② det(A) < 0 (-ve)

→ the shape is flipped
→ the Basis vector switches its positions "
⇐ علاقة ال axes ببعضهن انعكست

③ det(A) = 0

↳ the shape is collapsed to line
" Area = 0 ⇒ det(A) = 0 "
↳ the Matrix (A) Not Invertable



الموضوع: التاريخ:

② in (3D):

$$\therefore A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

det (A) = Volume of the cube

* when:

det(A) = 0 ⟹ 3D shape is squashed flat
"مسطح مضغوط"

Singular Matrix ⟵ (No Volume) ⟵

⟶ Not Invertible

★ General Interpretation (n x n) matrices

$$A_{n\times n} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & \vdots \\ \vdots & & \\ a_{n1} & & a_{nn} \end{bmatrix}$$

$\boxed{\det(A_{n\times n})} \Rightarrow$ Transform (unit cube) or (shape)

→ tells how the transformation will:

1. Scale n-dim Volume
2. preserves or reverse orientation
   (+) (−)
3. whether the transformation is → invertible
   or
   → Not

Note

(2D)	(3D)
det(A) = The Area of shape	det(A) ⇒ the Volume of shape

موضوع الدرس : اليـــوم : التاريخ : / /

* Properties of determinants :-

① If A and B are square matrices of the same size

$$\boxed{\det(AB) = \det(A)\ \det(B)}$$

② suppose A is invertible matrix :-

$$\boxed{\det(A^{-1}) = \frac{1}{\det(A)}}$$

but :- $\boxed{(\det(A) \neq 0)}$

Proof

$$A A^{-1} = I_n$$

$\xrightarrow[\text{للطرفين}]{\det}$ $\therefore \det(A A^{-1}) = \det(I_n)$

$$\det(A)\ \det(A^{-1}) = 1$$

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Note ~~det(I) =~~

$$\boxed{\det(I_n) = 1}$$

③ For any square matrix :-

$$\boxed{\det(A) = \det(A^T) = \text{نفس الناتج}}$$

⇐ لهم نفس الناتج .

④ matrix A is (Invertible) if and only if (($\boxed{\det(A) \neq 0}$))

matrix A is signular if and only if (($\det(A) = 0$))

⑤ I) upper-triangle matrix

If all of the entries ($\boxed{\text{below تحت}}$) the main Diagonal are zero.

$$\begin{bmatrix} a_{11} & \cdots & & \\ 0 & a_{22} & & \\ 0 & 0 & a_{33} & \\ 0 & 0 & 0 & \ddots \end{bmatrix}$$

" upper triagnle matrix "

II) lower triagnle matrix

If all of the entries ($\boxed{\text{above فوق}}$) the main Diagonal are zero.

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ & a_{22} & 0 & 0 \\ & & a_{33} & 0 \\ & & & \ddots \end{bmatrix}$$

" lower triagnle matrix

Note) (upper or lower triegnle) قيمة المحدد للمصفوفة

$$\det(A) = a_{11}\, a_{22}\, a_{33} \cdots a_{nn}$$

⇐ ($\boxed{\text{يساوي حاصل ضرب عناصر القطر الرئيسي}}$)

موضوع الدرس: اليوم: التاريخ: / /

6) IF:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = K$$

① ممكن آخذ من أي (صف / عمود) عامل مشترك

ex:
$$\begin{vmatrix} 2a & 2b \\ c & d \end{vmatrix} = 2\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 2K$$

ex:
$$\begin{vmatrix} a & 2b \\ c & 2d \end{vmatrix} = 2\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 2K$$

Note: IF A is n×n matrix

then,

$$\det(KA) = K^n \ast \det(A)$$

(ثابت)

② ممكن الضرب

$$\begin{vmatrix} a & b \\ c & 3d \end{vmatrix} = 3K$$

$$5\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 5a & b \\ 5c & d \end{vmatrix} = \begin{vmatrix} a & 5b \\ c & 5d \end{vmatrix} = 5K$$

③ ممكن أبدل أي (صفين) أو (عمودين) ← و أضرب في (-1)

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} c & d \\ a & b \end{vmatrix} = -K$$

موضوع الدرس :   اليوم :   التاريخ :   /   /

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} b & a \\ d & c \end{vmatrix} = -K$$

④ $(R_j + \alpha R_i)$ ⇐ ضرب (صف أو عمود) في ثابت
(ثابت) وجمعه على (صف أو عمود) آخر
⇐ لا يغير من قيمة المحدد

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \xrightarrow[=]{R_1 + 2R_2} \begin{vmatrix} a+2c & b+2d \\ c & d \end{vmatrix} = K$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \xrightarrow[=]{C_1 - 3C_2} = \begin{vmatrix} a-3b & b \\ c-3d & d \end{vmatrix}$$

$$= K$$

Note : Elementary matrix → (E)   det(E) :

- تبديل (صفين / عمودين) في (I_n) ← -1
- لو ضربت I_n في scalar (α) ← α
- لو طبقت عملية ($R_i + \alpha R_j$) ← 1

$$\det(E) = \begin{cases} -1 \\ \alpha \\ 1 \end{cases}$$

موضوع الدرس : اليــــوم : التاريخ : / /

(7) لو المحدد (A) فيه ← (صفين متساويين - عمودين متساويين)

$$\det(A) = zero$$

(8) لو المحدد فيه صف واحد أو عمود واحد كله أصفار

$$\det(A) = zero$$

ex
$$A = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 5 & 5 & 6 \end{vmatrix} = zero \quad \text{و} \quad B = \begin{vmatrix} 3 & 4 & 2 \\ 0 & 3 & 1 \\ 3 & 4 & 2 \end{vmatrix} = 0$$

(على الأقل)

(9) IF $\boxed{\text{atleast}}$ one ((Diagonal entry)) in {upper or lower} triangular matrix is (zero)

$\Rightarrow$ ($\det(A) = 0$)

لو في عنصر واحد على الأقل في المصفوفة (upper or lower triangular) "من القطر"
~~لـ عناصر القطر الرئيسي~~ "
يساوي صفر
← قيمة المحدد تكون صفر.
← ((عنصر = 0) من عنا من القطر))

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix} = zero$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 5 & 0 & 0 \\ 3 & 1 & 0 \end{bmatrix} = zero$$

(9) If

$$\boxed{\det(A+B) \neq \det(A) + \det(B)}$$

Note: If All three matrices differ only in the 2nd row and the 2nd row of C can be found By Adding the corresponding entries from 2nd row of A, B.

then,

$$\boxed{\det(C)} = \det(A) + \det(B)$$

ex:-

$$A = \begin{bmatrix} 4 & 2 & -1 \\ 6 & 1 & 7 \\ -1 & -3 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 2 & -1 \\ -2 & -5 & 3 \\ -1 & -3 & 9 \end{bmatrix}$$

$$C = \begin{bmatrix} 4 & 2 & -1 \\ 4 & -4 & 10 \\ -1 & -3 & 9 \end{bmatrix}$$

[A] ~~+ [B]~~ ⟹ 2nd row (A) + 2nd row of (B) = 2nd row of (C)

$$\Rightarrow \begin{vmatrix} 4 & 2 & -1 \\ 4 & -4 & 10 \\ -1 & -3 & 9 \end{vmatrix} = \begin{vmatrix} 4 & 2 & -1 \\ 6 & 1 & 7 \\ -1 & -3 & 9 \end{vmatrix} + \begin{vmatrix} 4 & 2 & -1 \\ -2 & -5 & 3 \\ -1 & -3 & 9 \end{vmatrix}$$

[ii] Cramer's Rule :-

IF: $\boxed{AX = b}$

then, the unique solution is:

$$x_i = \frac{\det(A_i(b))}{\det(A)}$$

where:-

$A_i(b)$ ⇒ matrix obtained by → replacing column $\boxed{i}$ by matrix(b)

ex:-

$$A = \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$A_1(b) = \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix},$$

$$A_2(b) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

#

ex:- solve using Cramer :- $\rightsquigarrow$ ($n \times n$ matrices)

$$3x_1 - 2x_2 = 6$$
$$-5x_1 + 4x_2 = 8$$

(sol)

$$A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 6 \\ 8 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_1 = \frac{\det(A_1(b))}{\det A} \Rightarrow A_1(b) = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix}$$

$$\det(A_1(b)) = 24 - -16 = 40$$

$$\det(A) = 12 - 10 = 2$$

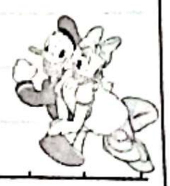
$$x_1 = \frac{40}{2} = 20$$

$$x_2 = \frac{\det(A_2(b))}{\det(A)} \qquad A_2(b) = \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix}$$

$$= 24 + 30 = 54$$

$$x_2 = \frac{54}{2} = 27$$

so, $x_1 = 20$, $x_2 = 27$ $\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 27 \end{bmatrix}$



* Inverse Formula *

$$A^{-1} = \frac{1}{\det(A)} \cdot \underline{adj(A)}$$

adj(A) :⇒ the transpose of the matrix of cofactors of A
(منقول مصفوفة العوامل)

↳ (Adjoint or Adjugate)

* Cofactor of A * ⟶ C_{ij}

$$C_{ij} = \underbrace{(-1)^{i+j}}_{} \det(A_{ij})$$

↳ (الإشارة للعنصر نفسه)
(وليس العنصر نفسه)

$$A = \begin{bmatrix} \overset{+}{2} & \overset{-}{3} & \overset{+}{1} \\ \overset{-}{0} & \overset{+}{5} & \overset{-}{1} \\ \overset{+}{3} & \overset{-}{4} & \overset{+}{9} \end{bmatrix}$$

$$C_{11} = + \begin{vmatrix} 5 & 1 \\ 4 & 9 \end{vmatrix} = +(45 - 4) = 41$$

$$C_{23} = (-1)^5 \times \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = -(8 - 9) = --1 = 1$$

$$C_{31} = + \begin{vmatrix} 3 & 1 \\ 5 & 9 \end{vmatrix} = 27 - 5 = 22$$

ex- Find the inverse of A using inverse formula:-

$$A = \begin{bmatrix} 1 & 3 & -1 \\ -2 & -6 & 0 \\ 1 & 4 & -3 \end{bmatrix}$$

(sol)

$$\boxed{A^{-1} = \frac{1}{\det(A)} \cdot adj(A)}$$

$adj(A) = (\text{co-factors matrix})^T = (C)^T$

(co-factors matrix) $\Rightarrow C$

$C_{11} = -18$ $C_{12} = -6$ $C_{13} = \cancel{-14}\ -2$

~~$C_{21} = \ldots$~~

$C_{21} = 5$ $C_{22} = -2$ $C_{23} = -1$

$C_{31} = -6$ $C_{32} = +2$ $C_{33} = 0$

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$C = \begin{bmatrix} -18 & -6 & -2 \\ 5 & -2 & -1 \\ -6 & 2 & 0 \end{bmatrix}$$

$$C^T = \begin{bmatrix} -18 & 5 & -6 \\ 6 & -2 & 2 \\ -2 & -1 & 0 \end{bmatrix}$$

$$\therefore \quad \cancel{A}\ \mathrm{adj}(A) = C^T = \begin{bmatrix} -18 & 5 & -6 \\ 6 & -2 & 2 \\ -2 & -1 & 0 \end{bmatrix}$$

$$\cancel{\det(A)} = \quad \det(A) = 2$$

$$A^{-1} = \frac{1}{\det(A)} \cdot \mathrm{adj}(A)$$

$$A^{-1} = \frac{1}{2} * \begin{bmatrix} -18 & 5 & -6 \\ 6 & -2 & 2 \\ -2 & -1 & 0 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -9 & 5/2 & -3 \\ 3 & -1 & 1 \\ -1 & -1/2 & 0 \end{bmatrix}$$

* If: $A^T = A^{-1}$ and $\det(A) > 0$

$\therefore (A \Rightarrow I_n)$

((Properties of determinant))

① $\det(AB) = \det(A)\det(B)$

② $\det(A^T) = \det(A)$

③ $\det(A^n) = (\det(A))^n$

④ $\det(A^{-1}) = \dfrac{1}{\det(A)}$

⑤ $\det(kA) = (k^n)\det(A)$ مهم

⑥ $\det(A^{-n}) = \dfrac{1}{(\det(A))^n}$

Note

① $D = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ → (D) diagonal matrix

② $U = \begin{bmatrix} a & d & e \\ 0 & b & F \\ 0 & 0 & c \end{bmatrix}$ → (U) upper triangular matrix

③ $L = \begin{bmatrix} a & 0 & 0 \\ d & b & 0 \\ e & f & c \end{bmatrix}$ → (L) Lower triangular matrix

* $\det(D) = \det(U) = \det(L) =$

$= abc$

Note

$A_{3\times 3} = [a_1 \quad a_2 \quad a_3]$

$B_{3\times 3} = [a_1 \quad a_2 \quad a_4]$

$C_{3\times 3} = [a_1 \quad a_2 \quad a_3 + a_4]$

$\det(C) = \det(A) + \det(B)$

* The linear system:

① Non-homogenous system:

$$AX = b \Rightarrow [A|b], \quad b \neq \theta$$

if ① $\boxed{\det(A) \neq 0}$ → Invertible
→ has unique solution

② $\boxed{\det(A) = 0}$ → Not Invertible
→ has → No solution
or
→ infinite solu

VE

② homogenous system

$$AX = \Theta \rightarrow [A \mid \theta]$$

① if $\boxed{\det(A) \neq 0}$ → is Invertible
→ has zero-solution "trivial solution"

لم كل الخطوط تمر بنقطة الأصل

② if $\boxed{\det(A) = 0}$ → is Not Intervible "singular"
→ has infinite solutions (including zero-solution)

* Tensor *

① 1D Tensor / vector :

9
8
7
3
-4
3

② 2D Tensor / matrix

-9	4	2	7	30
3	0	3	8	1
1	23	1	1	2
5	1	5	20	5

③ 3D Tensor / cube

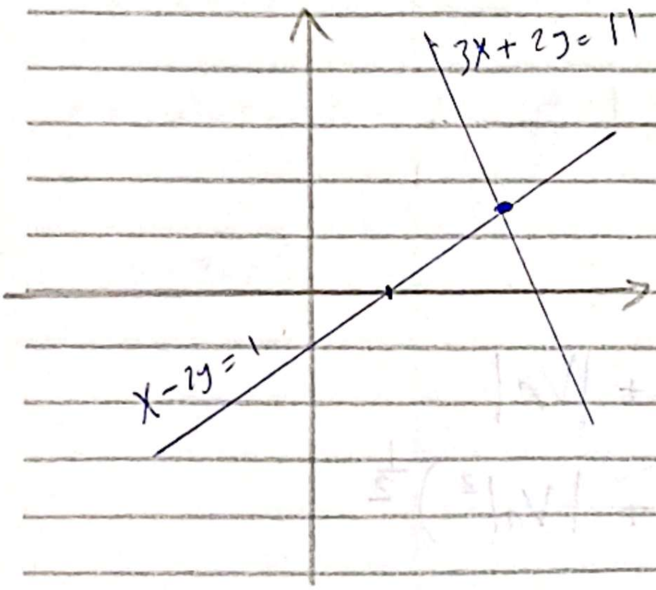
-9	4	2	3
1	5	6	7
3	1	2	3
1	2	3	4
-3	1	2	4

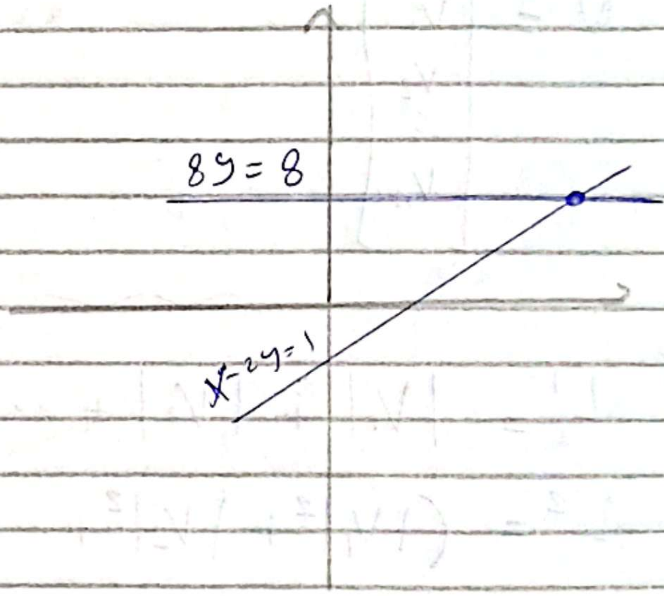
Note

when apply Gaussian elimination

→ We convet Transforming matrix to another matrix (RREF) to get the same solution

$$\begin{bmatrix}1&-2\\3&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\11\end{bmatrix} \xrightarrow{RREF} \begin{bmatrix}1&-2\\0&8\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\8\end{bmatrix}$$





* Vector Norm:

$\|V\| \rightarrow$ is a distance measure.

$\hookrightarrow$ L^1 norm
L^2 norm
L^3 norm
$\vdots$
L^∞ norm

$$V = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix}$$

* $L^1 = |V_1| + |V_2| + \cdots + |V_n|$

$L^2 = (|V_1|^2 + |V_2|^2 + \cdots + |V_n|^2)^{\frac{1}{2}}$

$L_3 = (|V_1|^3 + |V_2|^3 + \cdots + |V_n|^3)^{\frac{1}{3}}$

$\vdots$

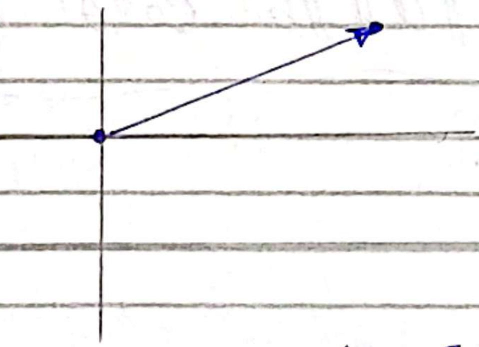
$$\boxed{L^P = (|V_1|^P + |V_2|^P + \cdots + |V_n|^P)^{\frac{1}{P}}}$$

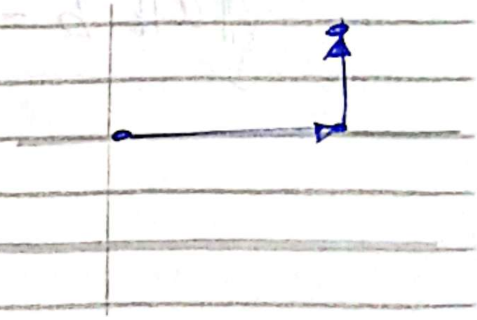
example :

(actual value) y | (predicted value) $\hat{y}$

$$\boxed{\hat{e} = \hat{y}_i - y}$$

	L_2 norm	L_1 norm
Distance measure	Euclidean distance	Manhattan distance
Regularization	Ridge	LASSO
optimization	MSE	MAE





$$MSE = \frac{1}{m} \| \hat{e} \|^2$$

$$\boxed{MAE = \frac{1}{m} \sum_{i=1}^{m} |\hat{y} - y|}$$

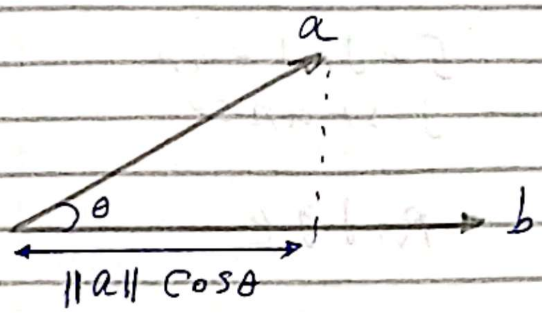
الموضوع: التاريخ:

* Dot Product:

↳ It tells us something about how much two vectors point in the same direction.

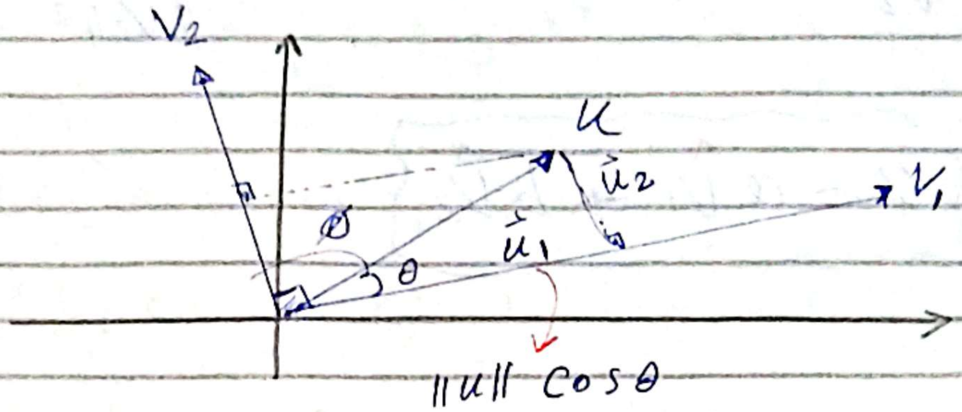
⇒ Measure of how closely two vectors align in terms of the direction they point.

⇒ it is an "equivalent" projection



$$\boxed{a \cdot b = \|a\| \|b\| \cos(\theta)}$$

* Application:



$$u \cdot V_1 = \|u\| \, \|V_1\| \cos(\theta)$$

$$\therefore \|u\| \cos(\theta) = \frac{u \cdot V_1}{\|V_1\|}$$

$$\times \frac{1}{\|V_1\|}$$

$$\frac{\|u\| \cos(\theta)}{\|V_1\|} = \frac{u \cdot V_1}{\|V_1\|^2} \to a$$

$$u_{V_1} = \frac{u \cdot V_1}{\|V_1\|^2} \vec{V}_1 = a \hat{V}_1$$

$\hat{V}_1 , \hat{V}_2 \to$ orthogonal

ممكن أعمل نفس الكلام مع مركبة الـ u في اتجاه $\hat{V}_2$

$$u_{V_2} = b\hat{V}_2 \longrightarrow b = \frac{u\cdot V_2}{\|V_2\|^2}$$

$$\therefore \boxed{u = a\hat{V}_1 + b\hat{V}_2}$$

$$\therefore u = \frac{u\cdot\hat{V}_1}{\|V_1\|^2}\hat{V}_1 + \frac{u\cdot\hat{V}_2}{\|V_2\|^2}\hat{V}_2$$

$$= \frac{u\cdot\hat{V}_1}{V_1\cdot V_1}\hat{V}_1 + \frac{u\cdot\hat{V}_2}{\hat{V}_2\cdot V_2}\hat{V}_2$$

summary

Scalar projection	Vector projection
$u_{V_1} = \frac{u\cdot V_1}{\|V_1\|}$	$\vec{u}_{V_1} = \frac{u\cdot V_1}{\|V_1\|}\frac{V_1}{\|V_1\|}$ → unit vector

* Vector space

لـ عندي space أقدر أمثل فيها مجموعة (vectors)

* كل (vector) في الـ vector space هو عبارة عن (linear combination) لـ من الـ (basis vectors)

* يعني باستخدام: ١) lineal Addition
٢) scalar multiplication

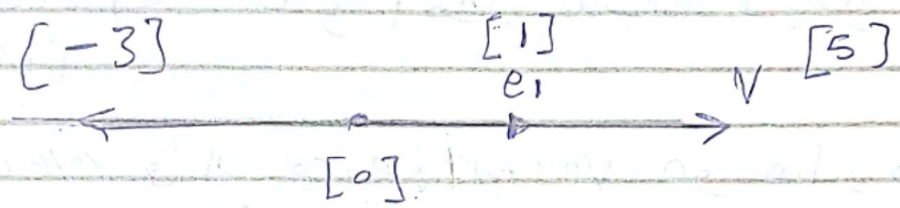
أقدر أوصل لأي (vector) في الـ vector space

* بنوع من الـ vector space ⟵ (R^n)

- R^1
- R^2
- R^3
- ⋮
- R^n

Note: zero-vector $(\vec{\theta}) \Rightarrow$ is Not a scalar

(1) R^1

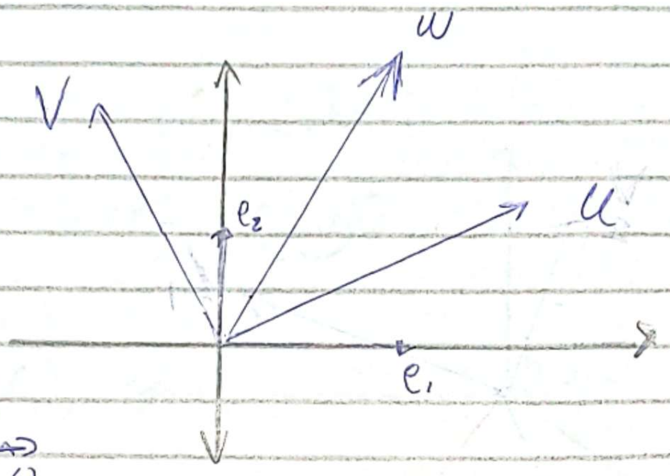


لم أحتاج إلا vector واحد فقط لتمثيل
كل الـ vectors في الـ (R^1)؛

$V = a\, e_1$

$U = b\, e_2$

② R^2 :



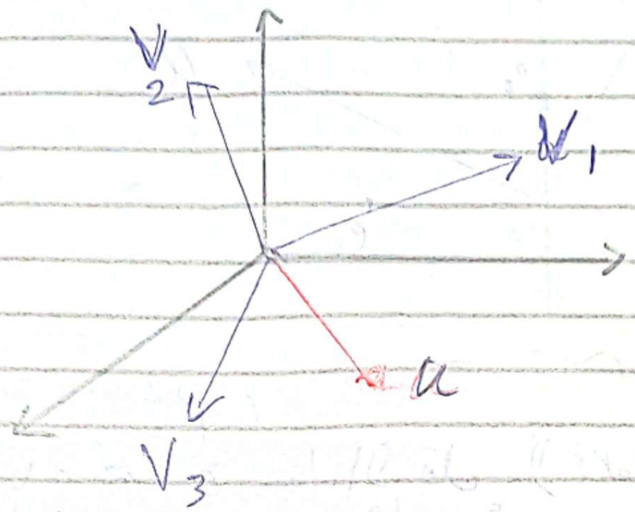
$(\vec{u})$, $(\vec{V})$

* في الـ (R^2)
أنا محتاج كل الأقل ((2 independent vectors))
علشان أقدر أمثل أي vector في الـ R^2
كـ (linear combination) من الـ u, V

$$w = a\vec{u} + b\vec{V}$$

* $u, V \Rightarrow$ linear independent

③ R^3 :



ال R^3 :

← محتاج على الأقل (Three independent Vectors)

علشان أقدر أمثل أي vector في ال R^3

$$u = a\vec{V_1} + b\vec{V_2} + c\vec{V_3}$$

$V_1, V_2, V_3 \rightarrow$ linear independent

* R^n :

← هحتاج على الأقل ← (n- vectors)

لشرط إنها تكون (independent)

↳ $V_1, V_2, V_3, \ldots\ldots, V_n$

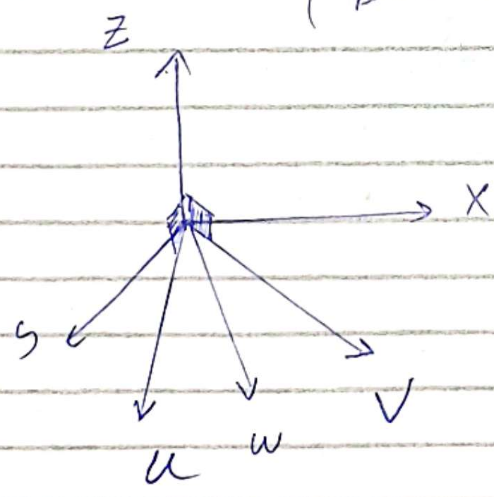
← (هقدرش أوصل لأي vector فيهم بدلالة الباقي)

* intuition

لازم يكون معايا عدد (n) من ال independent vectors علشان أقدر أمثل أي vector في ال (R^n)

لأنه ← كل (independent vector) هيدي معلومة عن ال (Basis vectors)

مثلاً :



هنا ال $\vec{u}$, $\vec{v}$, $\vec{w}$ واقعين في ال (X y- Plan)

وبالتالي مفيش أي إسقاط على محور ($\hat{z}$) .

ف ال w يعتبر linear combination من $\vec{u}, \vec{v}$

Axioms $\Rightarrow (\vec{u}, \vec{v} \in V)$

(1) → (addition) الجمع

① $(\vec{u} + \vec{v}) \in V \Rightarrow$ (closed under addition)

② $\vec{u} + \vec{v} = \vec{v} + \vec{u} \in V$

③ $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

④ $\vec{u} + \vec{o} = \vec{o} + \vec{u} = \vec{u}$, $(\vec{o} \in V)$

⑤ $\vec{u} + (-\vec{u}) = (-\vec{u}) + u = \vec{o}$, $(-\vec{u} \in V)$

(2) → (scalar multiplication) الضرب في ثابت

$\Rightarrow a \in R$, $b \in R$
ثوابت

⑥ $a\vec{u} \in V \Rightarrow$ (closed under multiplication)

⑦ $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$

⑧ $(a+b)\vec{u} = a\vec{u} + b\vec{u}$

⑨ $a(b\vec{u}) = (ab)\vec{u}$

⑩ $1.\vec{u} = \vec{u}$

① لو حققت العشر فرضيات (10 axioms) ⇐ (Vector space)
لو ال (set)
② لو محققتش فرضية واحدة مش X (Vector space) ⟵ $\boxed{(set)}$

موضوع الدرس : اليـــوم : التاريخ: / /

* Null space *

let matrix ($A_{m\times n}$) :-

* the null space of $A \Rightarrow \boxed{N(A)}$

where :- ((مجموعة الحل))

$N(A) \Rightarrow$ is the set of all solutions of $\boxed{(AX=0)}$

$\{AX=0\} \Rightarrow$ (Homogenous system)

$\Rightarrow$ consistent

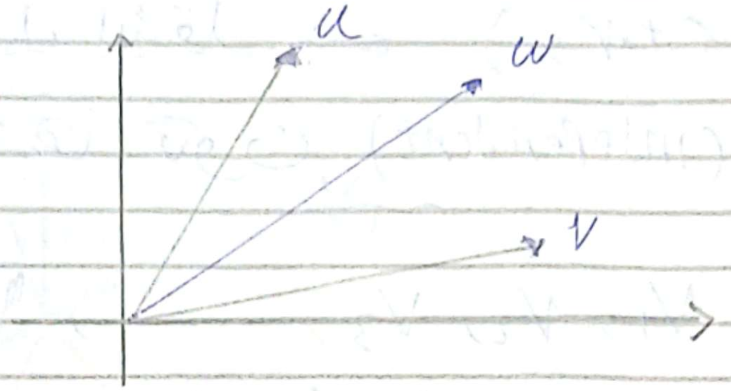
theorem :-

IF A is an $m\times n$ matrix

then, the null space of $A \Rightarrow N(A)$ is the subspace of (R^n)

(($N(A) \xrightarrow{\text{subspace of}} R^n$))

× linear combination:



$w \rightarrow$ is a linear combination of u, v

$$\boxed{w = av + bu}$$

ex:

$w = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ is a linear combination from $\left\{ \begin{matrix} u \\ v \end{matrix} \right.$??

$u = \begin{bmatrix} -0.1 \\ -4 \end{bmatrix}$, $v = \begin{bmatrix} 1.5 \\ 0.3 \end{bmatrix}$

$$\begin{bmatrix} 3 \\ 5 \end{bmatrix} = a \begin{bmatrix} 1.5 \\ 0.3 \end{bmatrix} + b \begin{bmatrix} -0.1 \\ -4 \end{bmatrix}$$

solution

*1) augmented matrix:

$$\left[\begin{array}{cc|c} 1.5 & -0.1 & 3 \\ 0.3 & -4 & 5 \end{array}\right]$$

* $R_1 * \frac{2}{3} \longrightarrow R_1$

$$\left[\begin{array}{cc|c} 1 & -\frac{1}{15} & 2 \\ 0.3 & -4 & 5 \end{array}\right]$$

⋮

لحد ما أوصل لـ (RREF)

$$\left[\begin{array}{cc|c} 1 & 0 & 1150/697 \\ 0 & 1 & -220/199 \end{array}\right]$$

$$\therefore \quad a = \frac{1150}{697} \quad , \quad b = \frac{-220}{199}$$

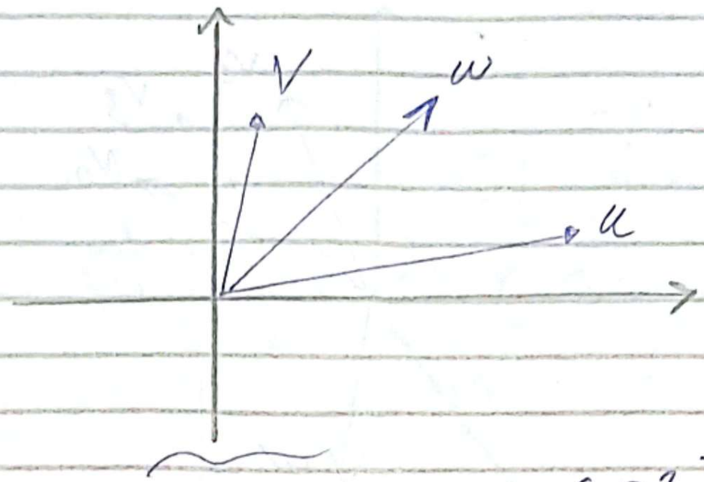
then $w = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ is a linear combination from u, v

solve using Inverse :

if

- has inverse $\Rightarrow$ unique solution
- hasn't inverse $\begin{cases} \text{No solution} \\ \text{infinite solution} \end{cases}$

Note



في الـ (R^2)

هل الـ (w) عبارة عن linear comb. من u , v ؟؟

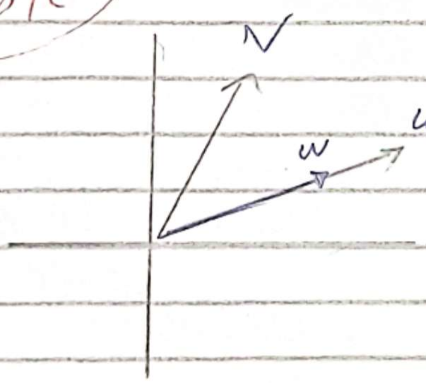
⇐ الـ (w) هتكون linear combination من u, v لو كانت واقعة في نفس المستوى اللي بيجمعهم (R^2)

" أي واقعة في نفس الـ vector space "

لكن

لو الـ $(\vec{w})$ عامل زاوية (ماتاع) مع المستوى اللي بيجمع $u, v \in$ فمقدرش أقول إنه هيكون (linear comb.) منهم

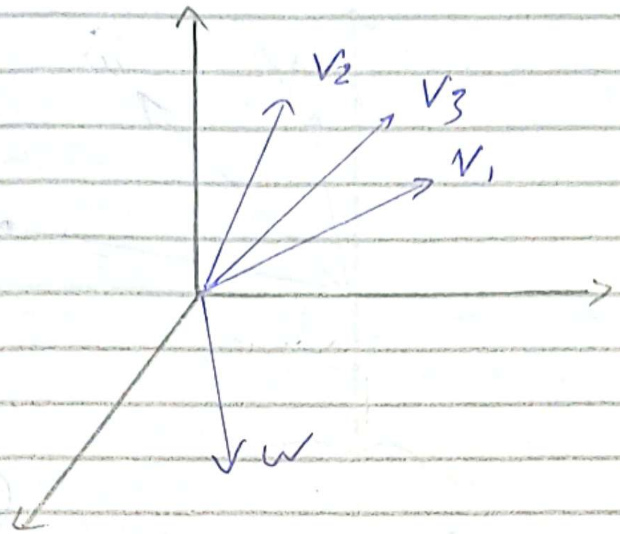
Note



لو الـ $(\vec{w})$ هتنطبق على إحدى الـ vectors

هيكون عندي (infinite solutions)

Note



لا يمكن كتابة ($\vec{w}$) بدلالة V_1, V_2, V_3 فال $\boxed{R^3}$

لأن ال (V_3) عبارة عن linear combinations من $\vec{V_1}$, $\vec{V_2}$

$$\vec{V_3} = a\vec{V_1} + b\vec{V_2}$$

وبالتالي:

$\vec{V_3} \leftarrow$ هو هنا فتش أي معلومة جديدة للبعد الثالث

موضوع الدرس :   اليـــوم :   التاريخ :   /   /

Dimension

⇒ let V be a vector space If V has a basis consiting of ($\boxed{n}$ vectors). we say that V has dimension (n)

~~and write as~~

$$\Rightarrow \dim V = n$$

n = (Bases) الـ هو عدد ∈ $\dim V$

Note

① Vector space $\{0\}$ ⇒ $\boxed{\dim\{0\} = 0}$

② $\dim R^n = n$

③ $\dim M_{mn} = m * n$ ⇝ (matrices)

④ $\dim P_n = n + 1$ ⇝ (Polynomial)

Note ⇒ ((Geometric Interpretation)) of subspaces of R^3 :-

(i) $\dim\{0\} = 0$ → (only the zero subspace)

(ii) $\dim(V) = 1$ → any subspace spanned by all lines through the origin $(0,0)$

theorem

If (V) is a vector space with ($\dim V = n$) then:-

(i) any set consisting of $\underline{n}$ L.I vectors

$\rightarrow$ is span V.

(ii) any $\underline{n}$ vectors that span V

$\rightarrow$ are (L.I)

* الشرطين بيحققوا بعض :

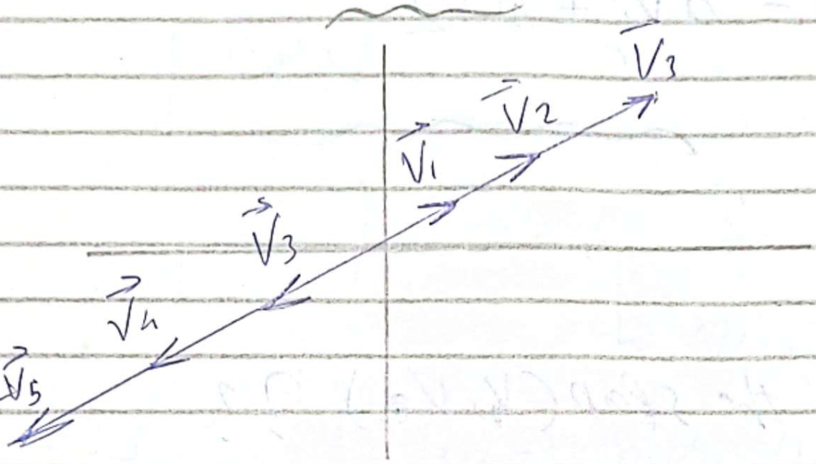
ليه :- لأن : ← ($\dim V = n$)

← يبقى المفروض عندي ($\underline{\underline{n}}$ Bases)

وأكيد : الشرطين متحققين .

X span :

لـ $span(V_1)$ ⇐ هو كل الـ (vectors) اللي أقدر أكتبها / أمثلها
كـ linear combination من ($\vec{V_1}$)



الـ $span(V_1)$ هو كل الـ (vectors) اللي على نفس
الـ (line).

$\vec{V_2} = a\vec{V_1}$

$\vec{V_3} = b\vec{V_1}$

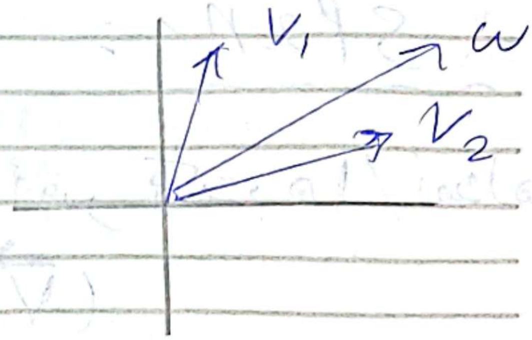
$\vdots \qquad \vdots$

الموضوع: التاريخ:

*

$w \in$ span(V_1, V_2)

$\therefore$ $w = aV_1 + bV_2$



example

is w in the span(V_1, V_2) ??

$w = \begin{bmatrix} 2 \\ 5 \\ 3 \\ 2 \end{bmatrix}$, $V_1 = \begin{bmatrix} 11 \\ 5 \\ -7 \\ 0 \end{bmatrix}$, $V_2 = \begin{bmatrix} 2 \\ 13 \\ 0 \\ -7 \end{bmatrix}$

$$\begin{bmatrix} 2 \\ 5 \\ 3 \\ 2 \end{bmatrix} = a \begin{bmatrix} 11 \\ 5 \\ -7 \\ 0 \end{bmatrix} + b \begin{bmatrix} 2 \\ 13 \\ 0 \\ -7 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 11 & 2 & 2 \\ 5 & 13 & 5 \\ -7 & 0 & 3 \\ 0 & -7 & 2 \end{array}\right]$$

$$\Big\downarrow \text{RREF}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & , \\ 0 & 1 & 0 & , \\ 0 & 0 & 1 & \cdot \\ 0 & 0 & 0 & \cdot \end{array}\right]$$

$a = 0, \quad b = 0$

↳ trivial solution

w is not linear comb. of V_1, V_2

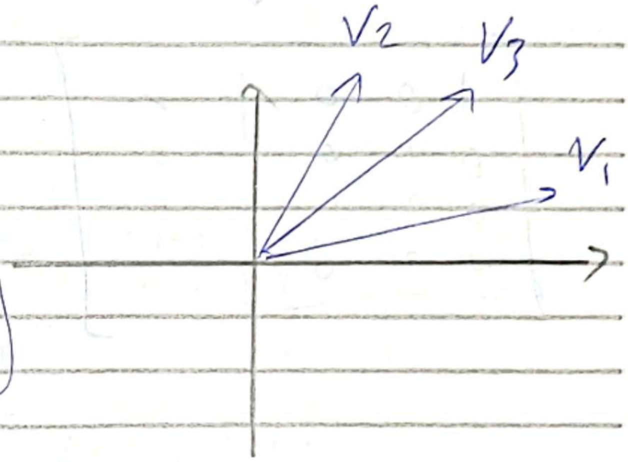
So, $w \notin \text{span}(V_1, V_2)$

* Vector linear independence

=> A vector is linearly dependent on other vectors if it can be expressed as a linear combination of other vectors.

ex: $V_1 = 5V_2 + 3V_3$

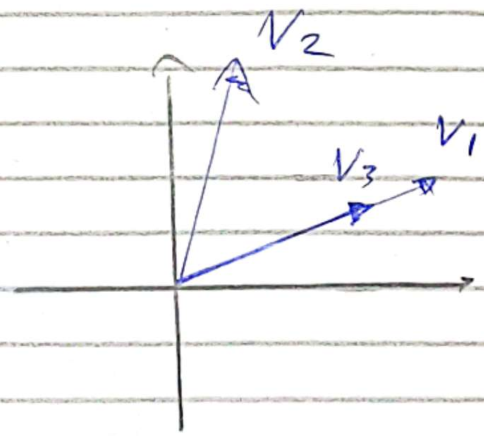
$V_3 = aV_1 + bV_2$



$\therefore \boxed{aV_1 + bV_2 + cV_3 = 0}$

dependent

$aV_1 + bV_2 + cV_3 = 0$



so in R^n :

if :

$$C_1V_1 + C_2V_2 + \cdots + C_nV_n = 0$$

then $\boxed{V_1, V_2, \ldots, V_n}$

→ dependent if $C_1, C_2, \ldots C_n$ have a non-zero value

else

→ independent if

$$(C_1 = C_2 = C_3 = \cdots = C_n = 0)$$

Note

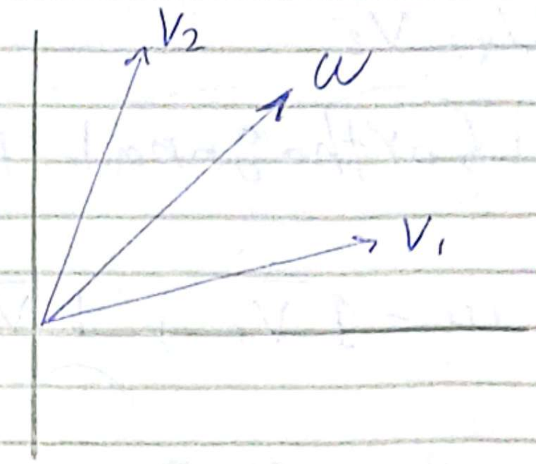
2 independent vectors in R^3

$\rightarrow$ give me a plane in R^3

* if I need to span (R^3)

$\Rightarrow$ I need third vector ((independent))

* Basis vectors

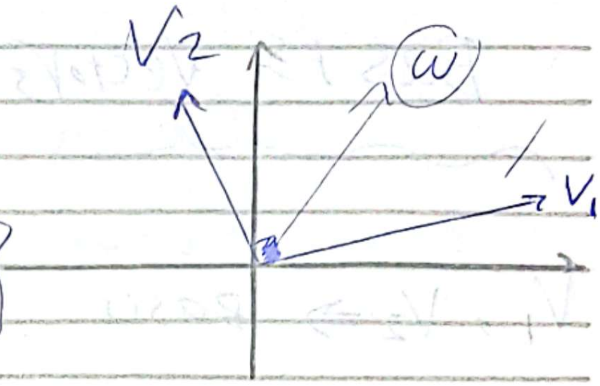


$V_1, V_2 \Rightarrow$ Basis
must be independent
// refrence vectors \\

V_1, V_2 are Basis, if
I can write any vector in vector space
using them as a linear combination

* $V_1, V_2 \rightarrow$ must be independent
  $\rightarrow$ span$(V_1, V_2) \Rightarrow$ all vectors in vector space

*



v_1, v_2

↳ orthogonal Basis

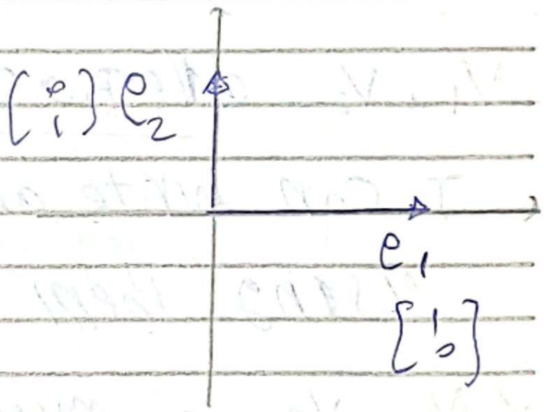
$w = 1\vec{V_1} + 1\vec{V_2}$

* special Basis :

orthonormal Basis

↓

"standard Basis"



$\|\hat{e}_1\| = 1$

$\|\hat{e}_2\| = 1$

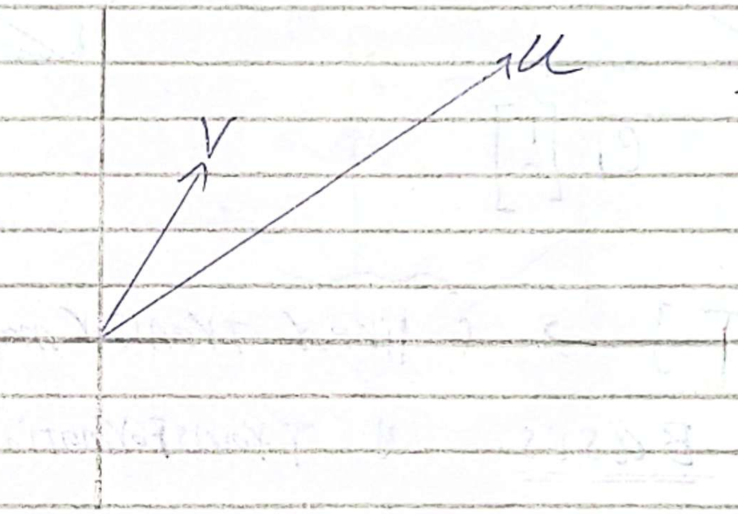
Linear Transformation:

$$\begin{array}{ccc} T & V & u \\ \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} & \begin{bmatrix} 1 \\ 2 \end{bmatrix} = & \begin{bmatrix} 9 \\ 11 \end{bmatrix} \end{array}$$

↓

Transformation matrix

↳ tell us where the basis vectors go

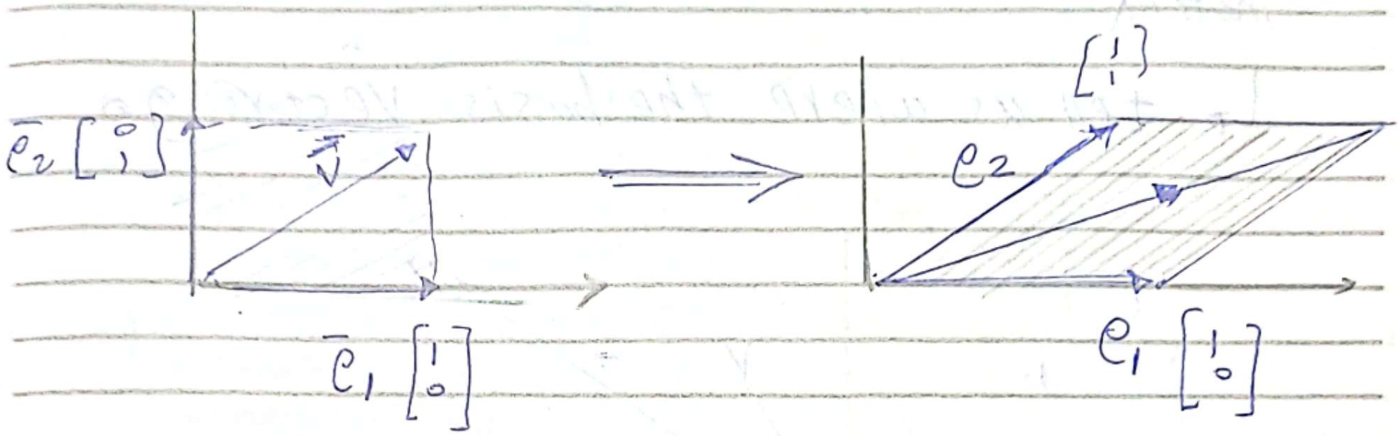


⇐ لما أثرت على $V = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ بـ $T = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$

هيتحول لـ $u = \begin{bmatrix} 9 \\ 11 \end{bmatrix}$

* intuition

$$\underset{T}{\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}}\underset{\vec{V}}{\begin{bmatrix}\text{٣}\\ \text{٢}\end{bmatrix}} = \underset{\vec{w}}{\begin{bmatrix}\text{٥}\\ \text{٥}\end{bmatrix}}$$



* لما بنعمل (linear Transformation) بـ [T]

⇐ بيحصل الـ Transformation للـ Basis

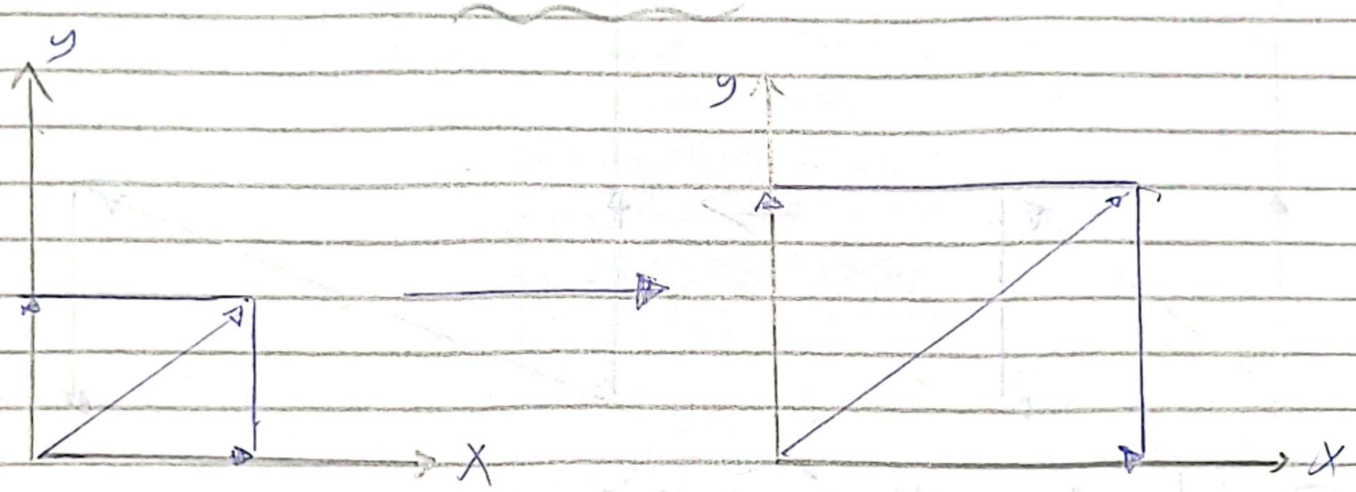
وبالتالي هيحصل (Transformation) لكل الـ Vectors

في الـ (Vector space)

* Types of linear Transformation

① Scaling

$$\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{new}} = \underbrace{\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}}_{\substack{T \\ \text{"scalar matrix"}}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{old}}$$



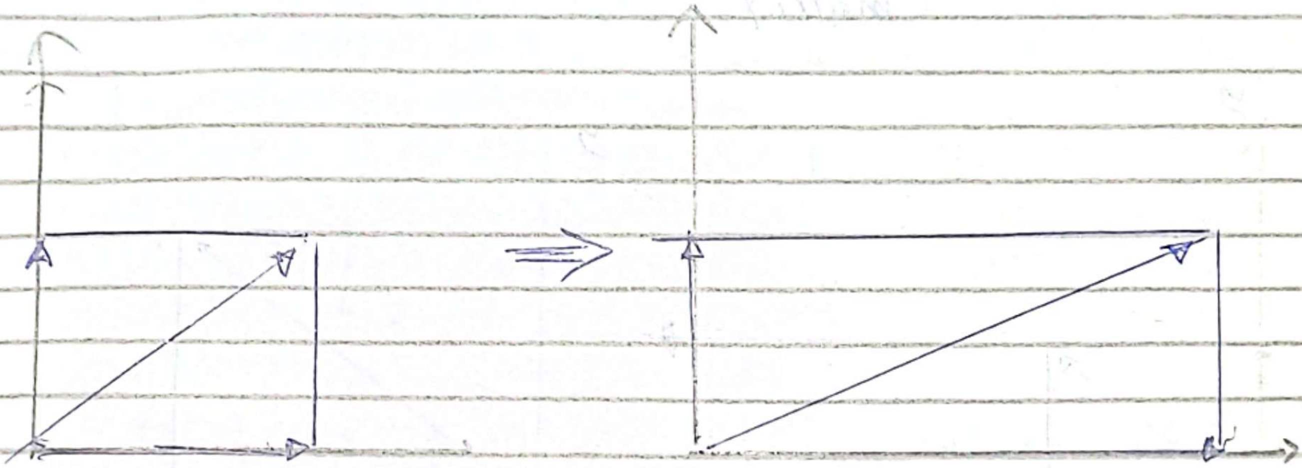
* الزيادة في e_1 و e_2 بنفس المقدار

* تحصل scaling فقط للـ vectors ولكن مش هتغير إتجاهها

② shearing

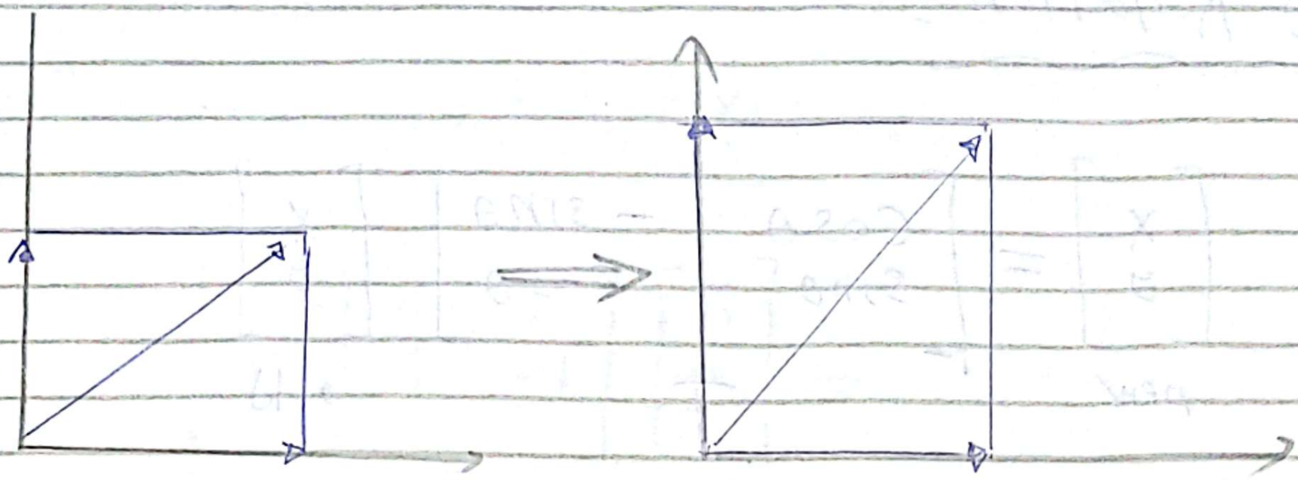
ⓐ shearing in X axis

$$\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{new} = \underbrace{\begin{bmatrix} 1 & sh_x \\ 0 & 1 \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{old}$$



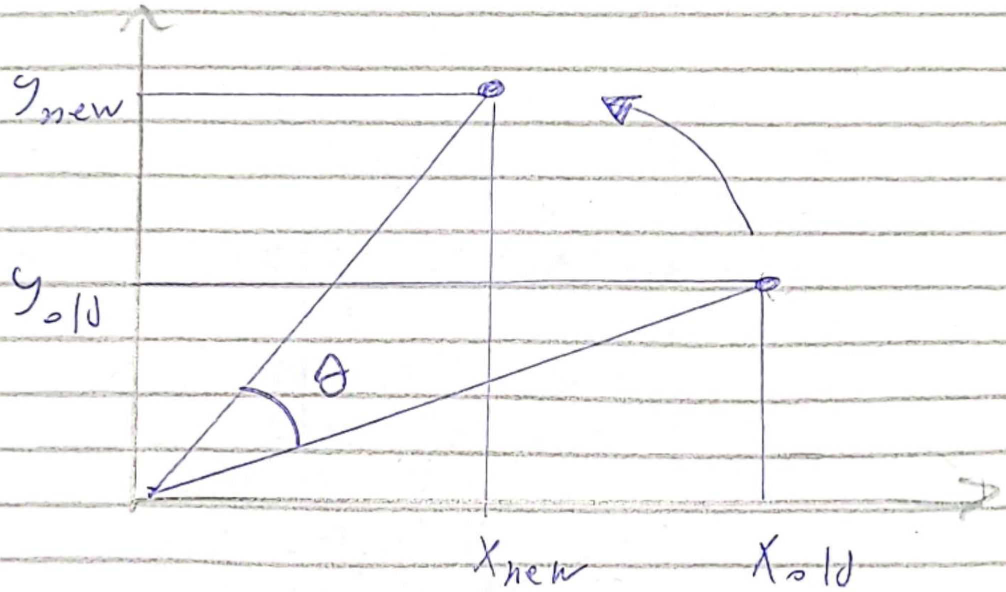
ⓑ shearing in y-axis

$$\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{new} = \underbrace{\begin{bmatrix} 1 & 0 \\ sh_y & 1 \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{old}$$



③ Rotation :

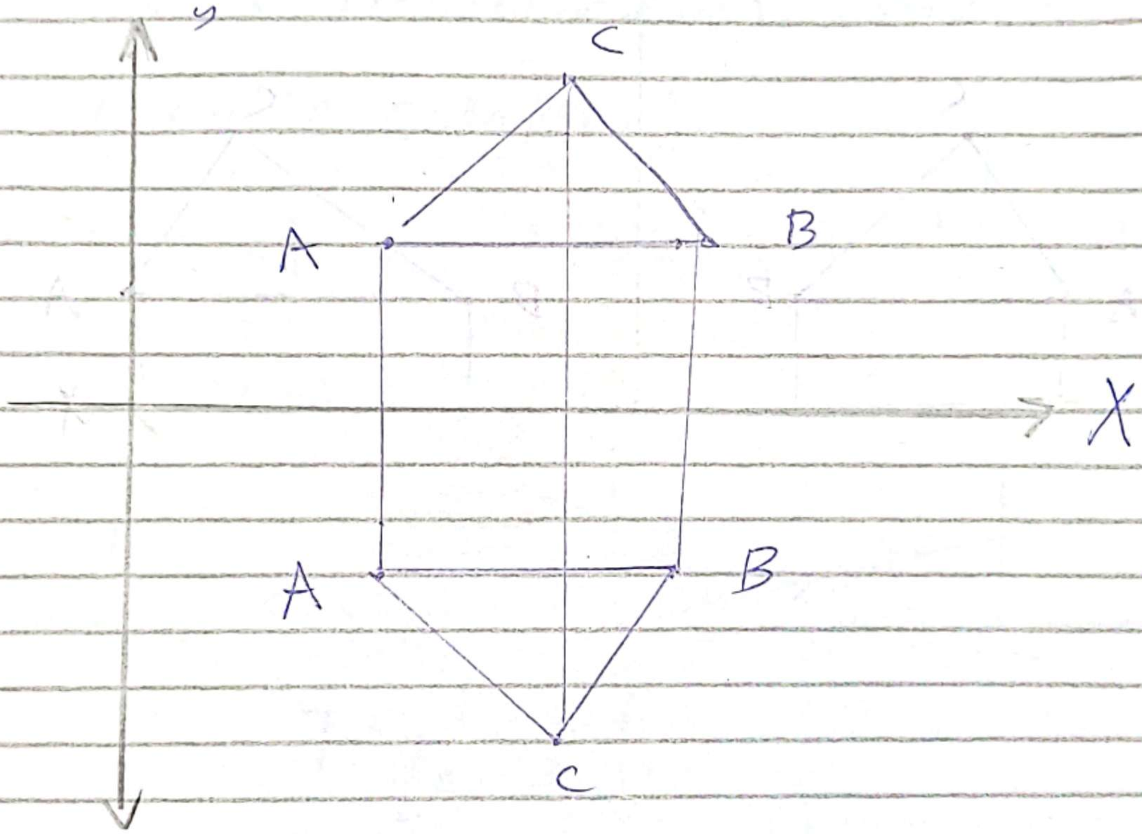
$$\begin{bmatrix} x \\ y \end{bmatrix}_{new} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{T} \begin{bmatrix} x \\ y \end{bmatrix}_{old}$$



4 Reflection

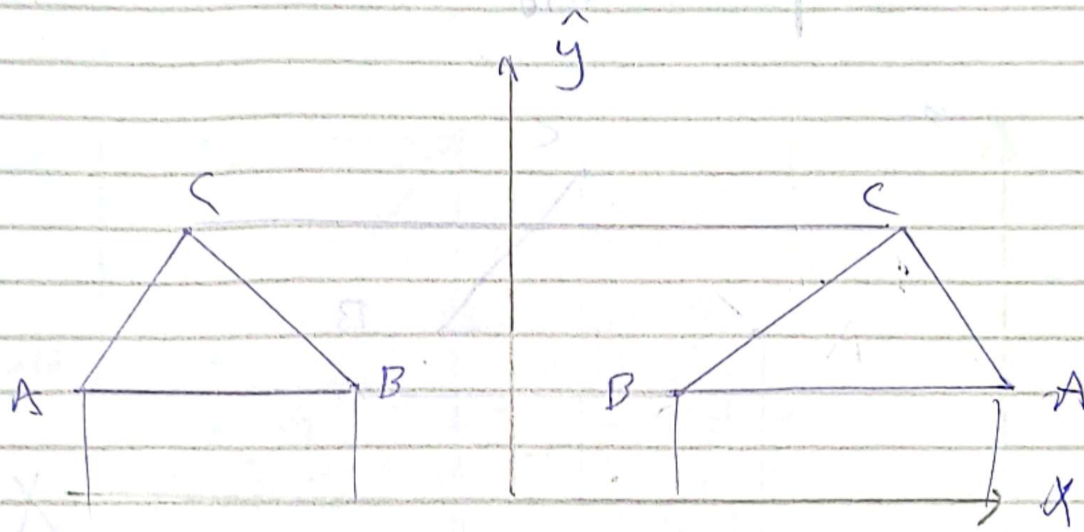
a Reflection along X-axis :

$$\underbrace{\begin{bmatrix} X \\ y \end{bmatrix}}_{new} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} X \\ y \end{bmatrix}}_{old}$$



(b) Reflection along y-axis :

$$\begin{bmatrix} x \\ y \end{bmatrix}_{\text{new}} = \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_{T} \begin{bmatrix} x \\ y \end{bmatrix}_{\text{old}}$$



Note

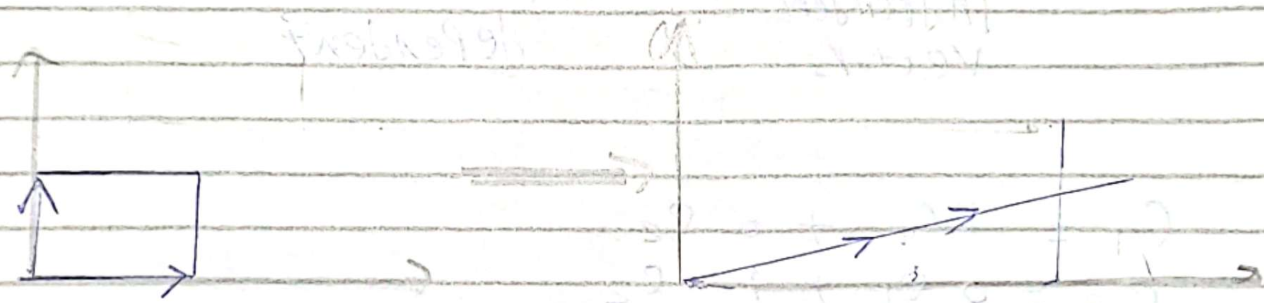
if $\det(A) = 0 \Rightarrow$ A is Not Invertible

then, transformation can't be $\boxed{\text{restored}}$

ملاحظة

لو ممكن أعمل Transformation ويدمل
↳ (space collapsed)

$\boxed{\det(T) = 0}$ ← طب أعرف منين ؟



$$T = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

* collapsed space cannot be restored

* Matrix Factorization
  // decomposition

$$\boxed{A \longrightarrow CF}$$

$$\begin{bmatrix} 1 & 1 & 2 & 4 & 2 \\ 2 & 1 & 3 & 5 & 4 \\ 1 & 1 & 2 & 4 & 2 \\ 0 & 1 & 1 & 3 & 0 \end{bmatrix}$$

independent vectors

dependent

$C_1 = 1\, C_1 + 0\, C_2$

$C_2 = 0\, C_1 + 1\, C_2$

$C_3 = 1\, C_1 + 1\, C_2$

$C_4 = 1\, C_1 + 3\, C_2$

$C_5 = 2\, C_1 + 0\, C_2$

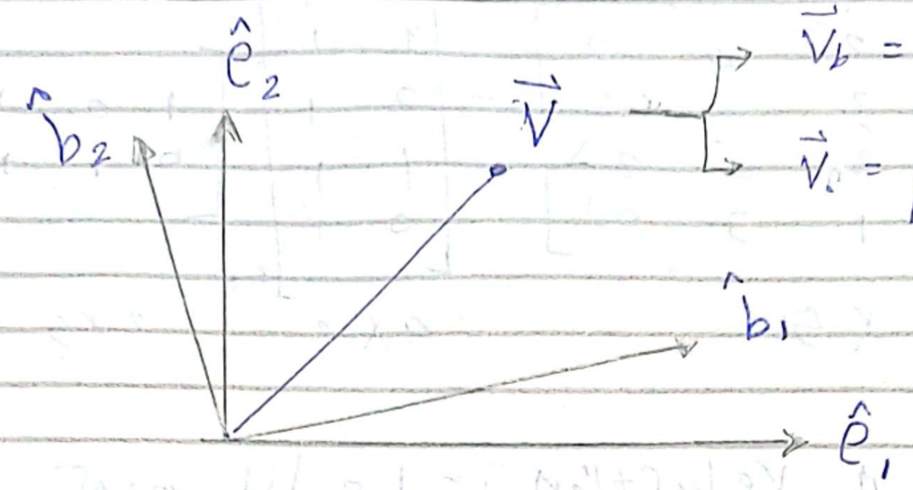
$$\begin{bmatrix} 1 & 1 & 2 & 4 & 2 \\ 2 & 1 & 3 & 5 & 4 \\ 1 & 1 & 2 & 4 & 2 \\ 0 & 1 & 1 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 & 0 \end{bmatrix}$$

$$4 \times 5 \qquad\qquad 4 \times 2 \qquad\qquad 2 \times 5$$

← كده انا عملت: reduction لل size

وبالتالي اقدر أعمل (restoration) بسهولة

* Changing Basis *



$\vec{V}_b = \begin{bmatrix} 2 \\ 0.5 \end{bmatrix}$

$\vec{V}_e = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

⇐ الـ basis vectors اللى بتكون منها الـ vector ($\vec{V}$) هى اللى بتتغير، لكن $\boxed{\text{Position}}$ بتاع الـ data

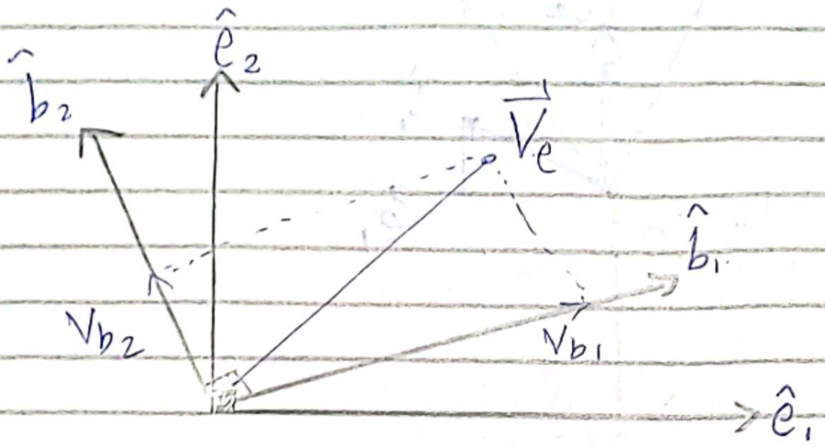
مش بيتغير

$\vec{V}$ → w.r.t $\vec{e}$ Basis → $\vec{V}_e = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$\vec{V}$ → w.r.t $\vec{b}$ Basis → $\vec{V}_b = \begin{bmatrix} 2 \\ 0.5 \end{bmatrix}$

$\boxed{i}$ orthogonal Basis

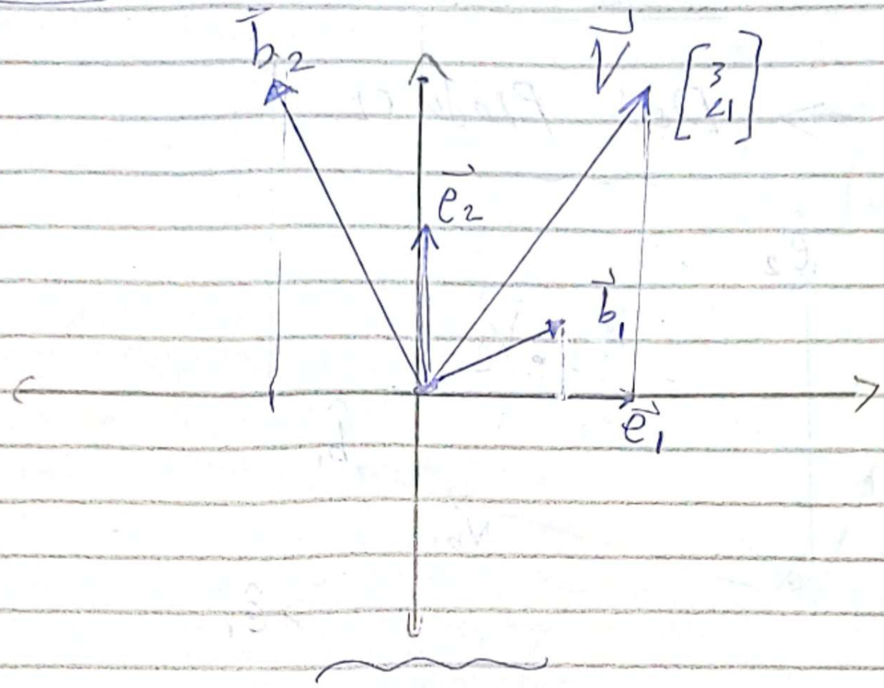
↳ using ⇒ Dot Product



$$\boxed{V_{b_1} = \frac{V_e \cdot b_1}{b_1 \cdot b_1} \, b_1}$$

$$\boxed{V_{b_2} = \frac{V_e \cdot b_2}{b_2 \cdot b_2} \, b_2}$$

example

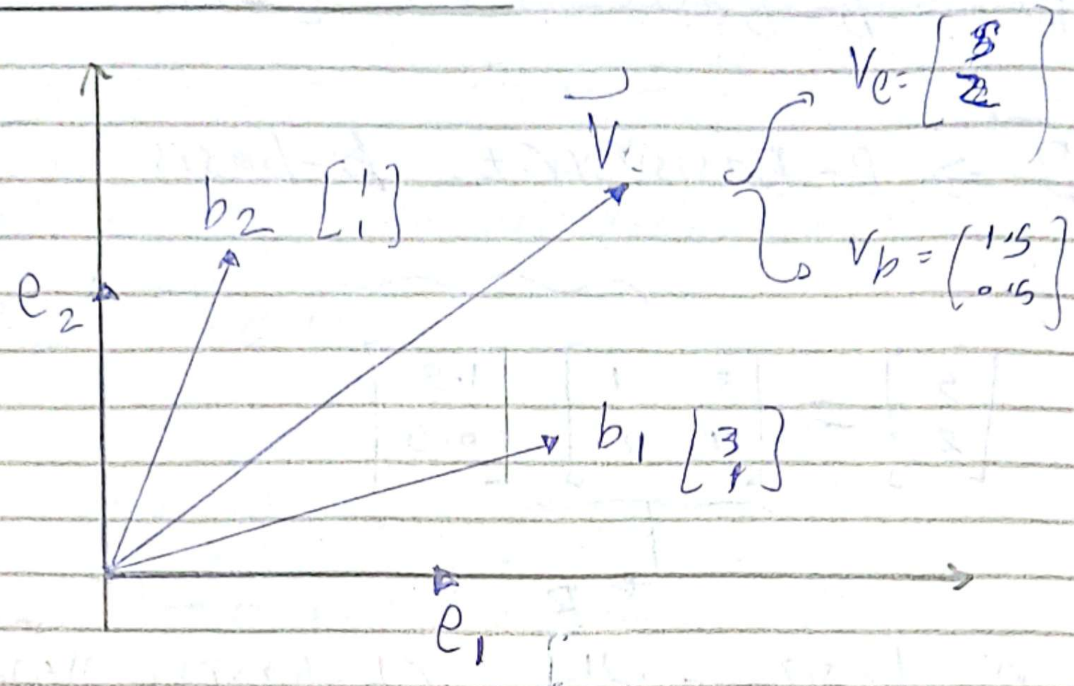


$$V_{b_1} = \frac{\begin{bmatrix}3\\4\end{bmatrix} \cdot \begin{bmatrix}2\\1\end{bmatrix}}{\begin{bmatrix}2\\1\end{bmatrix} \cdot \begin{bmatrix}2\\1\end{bmatrix}} b_1 = 2$$

$$V_{b_2} = \frac{\begin{bmatrix}3\\4\end{bmatrix} \cdot \begin{bmatrix}-2\\4\end{bmatrix}}{\begin{bmatrix}-2\\4\end{bmatrix} \cdot \begin{bmatrix}-2\\4\end{bmatrix}} \cdot b_2 = 0.5$$

$$V_b = \begin{bmatrix}2\\0.5\end{bmatrix} \qquad V_e = \begin{bmatrix}3\\4\end{bmatrix}$$

ii) Non-orthogonal Basis :



$e_{1_b} = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}$ $\qquad$ $e_{2b} = \begin{bmatrix} -0.5 \\ 1.5 \end{bmatrix}$

$b_{1e} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ $\qquad$ $b_{2e} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\boxed{\begin{aligned} u_e &= B\, u_b \\ u_b &= B^{-1} u_e \end{aligned}}$$

$B \Rightarrow$ b-Basis w.r.t e-basis

$B^{-1} \Rightarrow$ e-Basis w.r.t. b-basis

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}}_{B} \begin{bmatrix} 1.5 \\ 0.5 \end{bmatrix}$$

هـ- حولنا من ال (b-basis) الى e-basis

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix} \leftarrow \begin{bmatrix} 1.5 \\ 0.5 \end{bmatrix}$$

Note

$\boxed{B = E^{-1}}$, $\boxed{E = B^{-1}}$

$$B = \begin{bmatrix} | & | \\ b'_{1e} & b'_{2e} \\ | & | \end{bmatrix} \qquad E = \begin{bmatrix} | & | \\ e'_{1b} & e'_{2b} \\ | & | \end{bmatrix}$$

الموضوع: التاريخ:

* Transformation in non-orthogonal space

steps

① convert the vector to orthogonal space

$B \rightarrow \boxed{B\vec{V}}$

② perform transformation using T

$T \rightarrow \boxed{TB\vec{V}}$

③ convert the transformed vector back to non-orthogonal space

$B^{-1} \rightarrow \boxed{B^{-1}TB\vec{V}_{b}}$

$\boxed{B^{-1}TB\vec{V}_{b}} \longrightarrow \vec{V}_{b}$

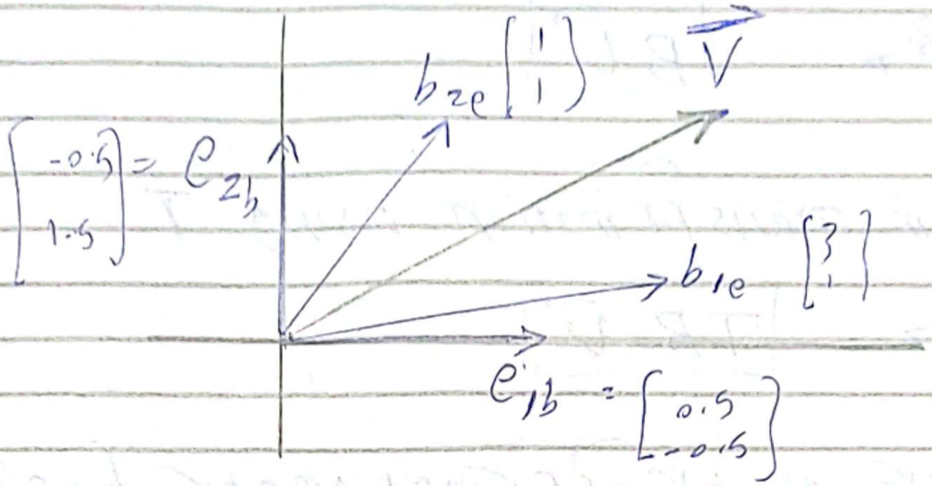
$B^{-1} \Rightarrow$ e-basis w.r.t. b-basis

$T \Rightarrow$ Transformation matrix

$B \Rightarrow$ b-basis w.r.t e-basis

*example:

→ you need to transform (reflect) vector $\vec{V}$ in B-space



$$\vec{V}_t = \underbrace{\boxed{B^{-1} T B}}_{T_B} \vec{V}$$

t → transformed

$$T_B = B^{-1} T B$$

$$B = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$B^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$$

$$\overline{V_t} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} V_{1b} \\ V_{2b} \end{bmatrix}$$

$$= \boxed{\frac{1}{2}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} V_{1b} \\ V_{2b} \end{bmatrix}}$$

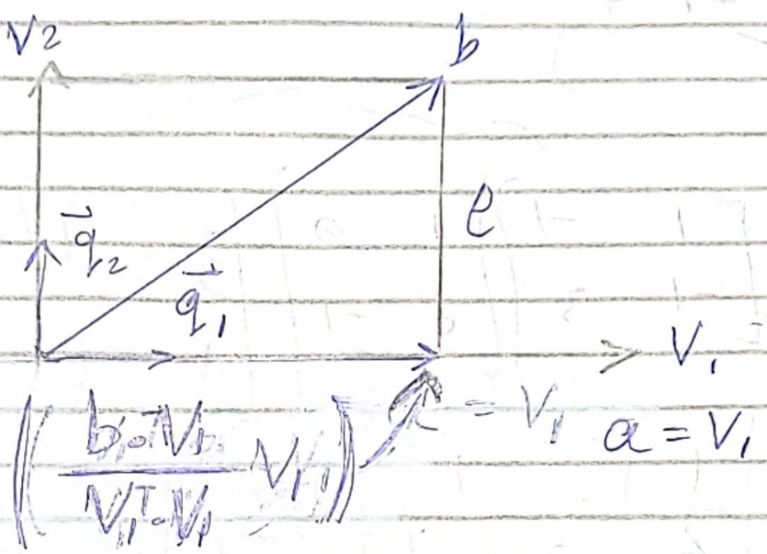
//

((orthogonalization))

↳ Gram - schmidt process

⇒ method for orthonormalizing a set of linearly independent vectors

① 2D



$a = V_1$

$$b = V_2 + \frac{b \cdot V_1}{V_1 \cdot V_1} V_1$$

$$\therefore V_2 = b - \frac{b \cdot V_1}{V_1 \cdot V_1} V_1$$

$$\vec{q}_1 = \frac{V_1}{\|V_1\|} \qquad \vec{q}_2 = \frac{V_2}{\|V_2\|}$$

② 3D

a, b, c → three linearly independent vectors

$$\left.\begin{aligned} V_1 &= a \\ V_2 &= b - \frac{b \cdot V_1}{V_1 \cdot V_1} V_1 \end{aligned}\right\} 2D$$

* $V_3 \Rightarrow$ is $\perp$ V_1, V_2

$$C = V_3 + \frac{C \cdot V_1}{V_1 \cdot V_1} V_1 + \frac{C \cdot V_2}{V_2 \cdot V_2} V_2$$

$$V_3 = C - \frac{C \cdot V_1}{V_1 \cdot V_1} V_1 - \frac{C \cdot V_2}{V_2 \cdot V_2} V_2$$

$$q_1 = \frac{V_1}{\| V_1 \|}$$

$$q_2 = \frac{V_2}{\| V_2 \|}$$

$$q_3 = \frac{V_3}{\| V_3 \|}$$

* General Case :

$\cdot\ V_1 = a$

$\cdot\ V_2 = b - \frac{b \cdot V_1}{V_1 \cdot V_1} V_1$

$\cdot\ V_3 = c - \frac{c \cdot V_1}{V_1 \cdot V_1} V_1 - \frac{c \cdot V_2}{V_2 \cdot V_2} V_2$

$\cdot\ V_4 = d - \frac{d \cdot V_1}{V_1 \cdot V_1} V_1 - \frac{d \cdot V_2}{V_2 \cdot V_2} V_2 - \frac{d \cdot V_3}{V_3 \cdot V_3} V_3$

$$\boxed{V_n = V - \sum_{j=1}^{n-1} \frac{V \cdot V_j}{V_j \cdot V_j} V_j}$$

$q_1 = \frac{V_1}{\| V_1 \|}$

$q_2 = \frac{V_2}{\| V_2 \|}$

$\vdots$

$\boxed{q_n = \frac{V_n}{\| V_n \|}}$

→ unit vectors

* Gram - Schmidt process

$u_1 = v_1$ | $e_1 = \frac{u_1}{\|u_1\|}$

$u_2 = v_2 - proj_{u_1}(v_2)$ | $e_2 = \frac{u_2}{\|u_2\|}$

$u_3 = v_3 - proj_{u_1}(v_3) - proj_{u_2}(v_3)$ | $e_3 = \frac{u_3}{\|u_3\|}$

$\vdots$

$u_n = v_n - \sum_{j=1}^{n-1} proj_{u_j}(v_n)$ | $e_n = \frac{u_n}{\|u_n\|}$

ex : ③

consider the following set of vectors in R^2

$$S = \left\{ V_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, V_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$$

$$u_1 = V_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$u_2 = V_2 - proj_{u_1}(V_2)$$

$$= \begin{bmatrix} 2 \\ 2 \end{bmatrix} - proj_{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \frac{8}{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2/5 \\ 6/5 \end{bmatrix}$$

* check the orthogonality of u_1, u_2:

$$u_1 \cdot u_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -2/5 \\ 6/5 \end{bmatrix} =$$

$$= -\frac{6}{5} + \frac{6}{5} = 0$$

then $u_1 \perp u_2$

الموضوع: التاريخ:

ex:

Consider following set in R^3:

$$V_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, V_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, V_3 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

$$u_1 = V_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \boxed{e_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

$$u_2 = \cancel{u}_2 - proj_{u_1}(V_2) =$$

$$= \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\boxed{e_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}$$

$$u_3 = V_3 - proj_{u_1}(V_3) - proj_{u_2}(V_3)$$

$$= \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} - \frac{\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \text{------} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$e_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

Care

التاريخ :   الموضوع :

* orthogonal Matrix :

E ⇒ is orthogonal matrix :

characteristics :-

① The Inverse is easy to compute

② Transformation is reversible because

it doesn't collapse space

~~3~~

③ The projection is just ⇒ dot product

محمد و أميرة

* QR Decomposition "Factorization"

⇒ let A: real square matrix

A → can be decomposed as $\boxed{QU}$

$\boxed{Q \equiv E}$ ⇒ From Gram-shmidt

usage:

$$\Rightarrow \quad AX = b$$

$$(QU)X = b$$

$$UX = Q^{-1} b$$

$$\boxed{UX = Q^T b}$$

upper-triangle matrix

Back substitution لحل باستخدام

* Eigen Vectors & Eigen Values

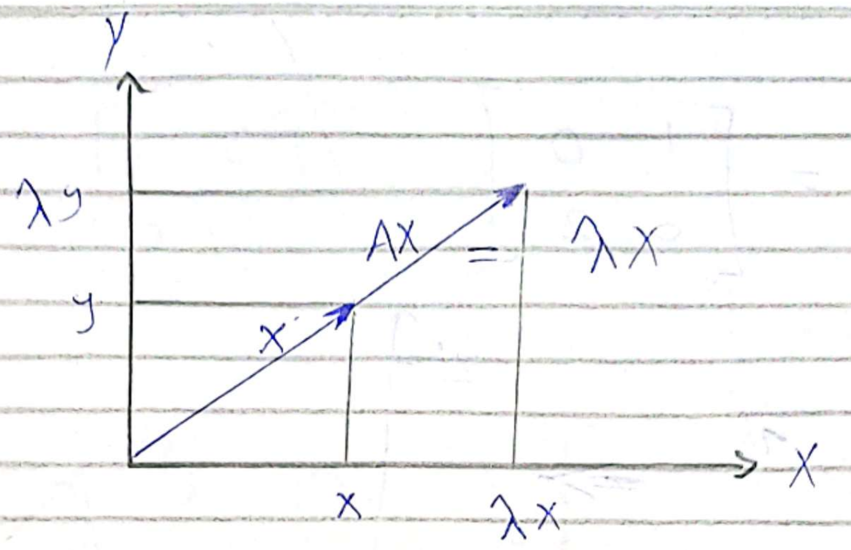
① Eigen Vector

↳ is a vector that doesn't change its $\boxed{\text{span}}$ after Transformation.

and considered the characteristic of transformation

② Eigen Value

↳ the values associated (of scaling) with eigenvectors after Transformation



$$\boxed{AX = \lambda X}$$

$$\boxed{(A - \lambda I)X = 0}$$

أي (Transformation matrix) وليكن (A) بيكون ليها (eigen vectors)

لـ هي vectors ليها خصائص مميزة

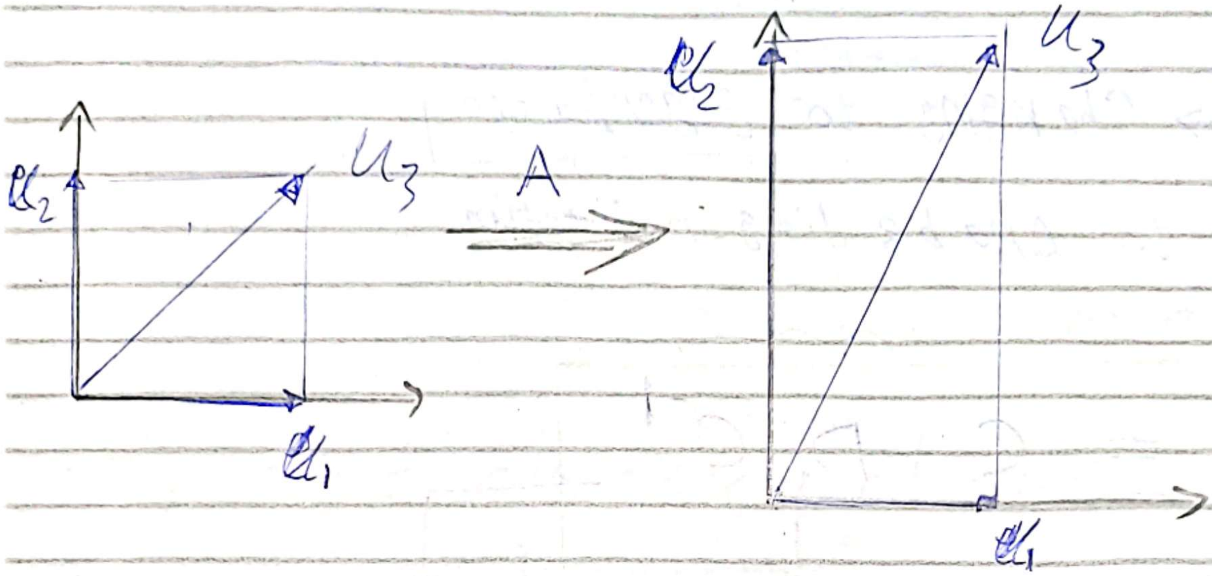
بعد الـ Transformation ① واقعة في نفس الـ (span)

② حصلها scaling فقط

ومقدار الـ scaling الي حصل هو (eigen value) λ

ex:

let $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$



eigen vectors $\Rightarrow$ $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\boxed{\lambda = 1}$

$e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\boxed{\lambda = 2}$

Given:- square matrix ($A_{n\times n}$)

$\because AX = \lambda X$, $\therefore AX - \lambda X = 0$, $\cancel{X}(A - \lambda I)\cancel{X} = 0$

$\Rightarrow$ $\boxed{(A - \lambda I)X = 0}$ ((homogeneous system))

✗	✓
" unique solution "	" Infinity many solution "
حل وحيد "الحل الصفري"	⇐ عدد لا نهائي من الحلول
$\Rightarrow$ $((\lvert det \rvert \neq 0))$	$\Rightarrow$ $((\lvert det \rvert = 0))$
$\Rightarrow$ $\det(A - \lambda I) \neq 0$	$\Rightarrow$ $\det(A - \lambda I) = 0$
$\rightarrow$ (trivial solution)	$\Rightarrow$ $\boxed{\text{non-trivial solution}}$
$\therefore$ $(\vec{X} = 0)$ ✗ ⇐ هنا مرفوض	$\therefore$ matrix $(A - \lambda I)$ $\Rightarrow$ (signular)
⇐ لأن الـ (eigen vetor) لـ $\boxed{\text{is a non-zero vector}}$	⇐ في هذه الحالة هيكون عندي عدد لا نهائي من الحلول لـ X و أقدر أجيب الـ (eigen vectors)

موضوع الدرس:　　　　اليوم:　　　　التاريخ:　/　/

steps

① نجيب $(A - \lambda I)$ (مصفوفة) ← ①

⇐ أطرح λ من عناصر القطر للمصفوفة (A) ← (λ)

② نساوي محدد المصفوفة $(A - \lambda I)$ بالصفر

$$\Rightarrow \det(A - \lambda I) = 0 \longrightarrow ②$$

③ قيمة المحدد
⇐ تكون معادلة في (λ) تسمى "characteristic eq."

Note ⇐ لو كان المصفوفة $(n \times n)$
⇐ تكون المعادلة من الدرجة (n) ⇐ أعلى قوة (λ^n)

④ أجيب قيم (λ)
التي تحقق المعادلة $[\det(A - \lambda I) = 0]$.

⑤ أعوض بقيم (λ) وأحل ⇐ $((A - \lambda I)X = 0)$ (مصفوفة الوحدة) ← I
وأجيب قيمة $(\vec{X})$ عند كل قيمة (λ).

$\lambda \Rightarrow$ eigen value

$\vec{X} \Rightarrow$ (eigenvector) corresponding to (λ) eigen value.

أي: لكل ← (λ) فيه ← X (eigenvector)

موضوع الدرس : اليــــوم : التاريخ: / /

* Definition *

⇒ eigenspace ⇒ $N(A-\lambda I)$

⇒ is $\boxed{\text{Null space}}$ of $(A-\lambda I)$

لأنه السيستم   مجموعة حل لهذا السيستم :-

homogeneous ⇒

$$((A-\lambda I)x=0)$$

$(\lambda)_K$ ← ((eigenspaces))

ex:-

$$\boxed{N(A-\lambda I)} = \left\{ t\begin{bmatrix}1\\2\\3\end{bmatrix} + s\begin{bmatrix}-1\\5\\-7\end{bmatrix} ;\ t, s \in R \right\}$$

((eigenvectors)) (λ_K)

نرمز :

so,

eigenvectors is dimensions of eigenspace.

Amira

ex(1): Find eigenvalues and eigenvector of A:

$$A = \begin{bmatrix} 5 & 7 & -5 \\ 0 & 4 & -1 \\ 2 & 8 & -3 \end{bmatrix}$$

(sol)

① $(A - \lambda I) = \begin{bmatrix} 5-\lambda & 7 & -5 \\ 0 & 4-\lambda & -1 \\ 2 & 8 & -3-\lambda \end{bmatrix}$

② $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 5-\lambda & 7 & -5 \\ 0 & 4-\lambda & -1 \\ 2 & 8 & -3-\lambda \end{vmatrix} = 0$$

$= (5-\lambda)\left((4-\lambda)(-3-\lambda)+8\right) + 2\left((4-\lambda)(-3-\lambda)+8\lambda\right)$

$= (5-\lambda)(-12-4\lambda+3\lambda+\lambda^2+8) + 2(-12-4\lambda+3\lambda+\lambda^2 + 8)$

$= -60 - 20\lambda + 15\lambda + 5\lambda^2 + 40 + 12\lambda + 4\lambda^2 - 3\lambda^2 - \lambda^3 + 8\lambda$

$\cancel{+8\lambda} - 24 - 8\lambda + 6\lambda + 2\lambda^2$

$= -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$

$= \boxed{\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0}$

(Characteristic equation)

نبدأ حلول ← $\forall$ (بالتالة) *

$\lambda_1 = 1 \quad , \quad \lambda_2 = 2 \quad , \quad \lambda_3 = 3$

$\therefore (\lambda_1 = 1), (\lambda_2 = 2) (\lambda_3 = 3) \Rightarrow$ (the eigen values)

at $(\lambda_1 = 1)$:-

solve: $(A - \lambda I)X = 0$ at $\boxed{\lambda_1 = 1}$

$(A - I)X = 0$

$$[\cancel{A - I \mid 0}] \Rightarrow \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 4 & 7 & -5 & 0 \\ 0 & 3 & -1 & 0 \\ 2 & 8 & -4 & 0 \end{array}\right] \end{array} \Rightarrow [A - I \mid 0]$$

$$\xrightarrow[\text{oper.}]{\text{row}} (\text{R.E.F}) \left[\begin{array}{ccc|c} 1 & 4 & -2 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} r(A - I) = 2 \\ r(A - I \mid 0) = 2 \\ n = m \quad \boxed{n = 3} \end{array}$$

(Infinity many sol.) $\Leftarrow$ ($\boxed{r(A) = r(A \mid b)} < n$) $\Leftarrow$

⇐ عدد لانهائي من الحلول ، ولازم يكون محددة لأنه

أننا هنجيب قيم ال (λ) اللي تخليه كده :

Free variables $\Rightarrow 3 - 2 = 1 \quad (\boxed{x_3 = t})$

$x_1 + 4x_2 - 2t = 0$

$3x_2 - t = 0 \qquad \boxed{x_2 = \frac{1}{3}t}$

$x_1 + \frac{4}{3}t - 2t = 0 \qquad \boxed{x_1 = +\frac{2}{3}t}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}t \\ \frac{1}{3}t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}.$$

$$\therefore N(A - I) = \left\{ t \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix} ;\ t \in R \right\}$$

eigenspace

⇒ eigenspace corresponding to $(\lambda = 1)$:-

sol:-

⇒ eigenvalue:- → $\lambda = 1$

⇒ eigenvectors ⇒ $X = \boxed{t \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}}$

⇐ any non zero multiple of $\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}$, $(t \neq 0)$

⇒ $AX = \lambda X$ ⇒ A

$\therefore AX = X \rightsquigarrow (\lambda = 1)$

⇐ يعني لو ضربت المصفوفة (A) في الـ (Vector) X

هيطلع هو نفسه اللي ضربت فيه لأن الـ $(\lambda = 1)$:

موضوع الدرس :   اليـــوم :   التاريخ :   /   /

~~ex:-(2)~~ (Note)

؛ عند ما أطلع القيم الـ (λ) من الـ (characteristic eq))

علشان أتأكد من الحل :-

⇒ (to check))

⇒ $\boxed{\sum \lambda = \sum(\text{عناصر القطر}) \; \cancel{= \det(A)}}$

$\lambda_1 \times \lambda_2 \times \lambda_3 = \det(A)$

ex:-

$$A = \begin{bmatrix} 5 & 7 & -5 \\ 0 & 4 & -1 \\ 2 & 8 & -3 \end{bmatrix}$$

~~by (λ~~ by: $\det(A - \lambda I) = 0$

↳ $\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 3 \quad \to ①$

$\sum \lambda = 1 + 2 + 3 = 6$

② $\sum$ عناصر القطر $= 5 + 4 - 3 = 6$

③ $\det(A) = 6$

* Diagonalization :-

⇒ eigendecomposition

↳ Changing to $\boxed{\text{eigenbasis}}$ to enable diagonalization

$$\boxed{T = C \; D \; C^{-1}}$$

$$\begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 \\ | & | & | \\ | & | & | \end{bmatrix}^{-1}$$

T ⇒ Transformation (Diagonaziable Matrix)

C ⇒ matrix of eigenvectors (w.r.t my basis)

D → diagonal matrix that contain eigenvalues

C^{-1} → matrix of my basis (w.r.t eigenbasis)

* Application *

⇒ makes the process more $\boxed{\text{computationally effective}}$

if we need matrix multiplication $\boxed{n}$ times

$$T = C D C^{-1}$$

$$T T = C D \cancel{C^{-1}} \cancel{C} D C^{-1} = C D^2 C^{-1}$$

$$T^4 = C D \cancel{C^{-1}} \cancel{C} D \cancel{C^{-1}} \cancel{C} D \cancel{C^{-1}} \cancel{C} D C^{-1} = C D^4 C^{-1}$$

$$\boxed{T^n = C D^n C^{-1}}$$

intuition

when we transforming to eigen basis

⇒ it makes any transformation is Just a Scaling by eigen values

{ Dimensionality Reduction }

Intro

⇒ Many ML problems involve thousands or millions of features for each training instance.

⇒ make the training very slow

⇒ make it much harder to find a good solution.

* this problem is the curse of Dimensionality

Dimensionality Reduction

↳ Find a low-dimensional representation of the data that retrains as much information as possible.

Used For

1. Data compression
   a) save data
   b) speed up learning algorithm
   c) Avoide model overfitting

2. Data Visualization
   ⇒ reduce high dimension data to 2D / 3D

Note

لأن فضل إني أعمل (training) بال (original Data) قبل ال Dim. reduction

لأن ال (Dimensionality reduction) قد يسبب :
→ ① lose some information
→ ② make a pipelines bit more complex and harder to maintain

So, make your system perform slightly worse.

* in some cases:
it may filter out some noise and unnecessary details

* Covariance

⇒ is a measure of the joint probability of two random variables.

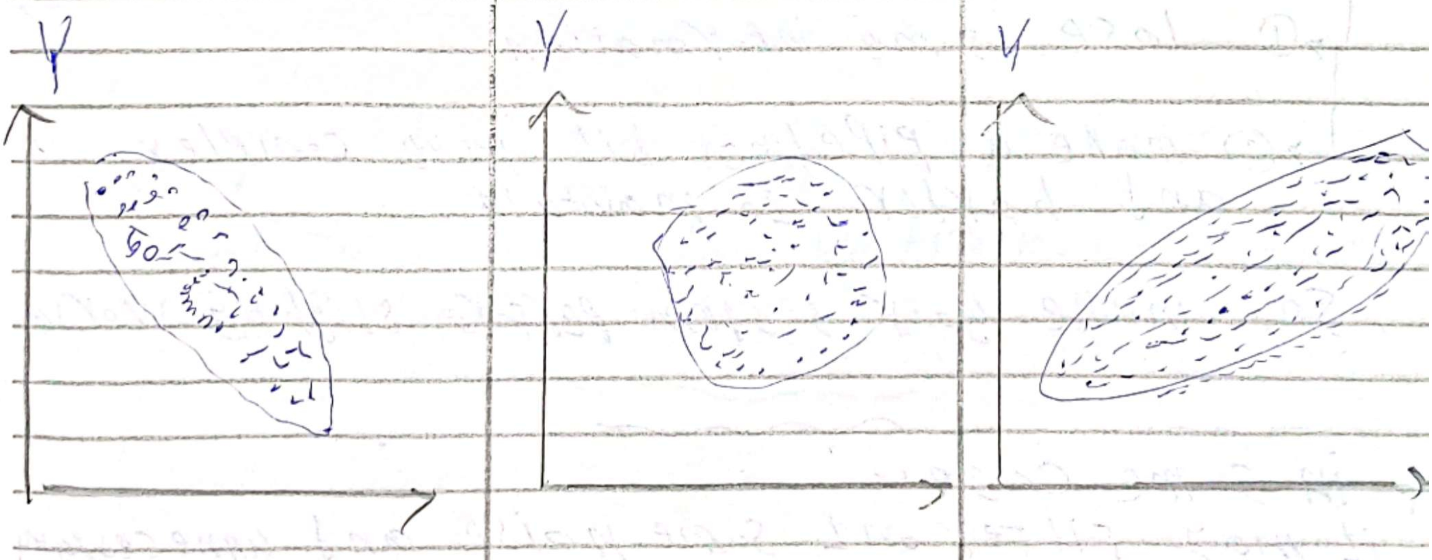
→ describe: → how far the variables are spread out
and
→ the nature of their relationship

~~~

Cov(X, Y) ـ بيشرح لما المتغير X بيزيد ـ الـ Y ←
- بيزيد
- بيقل
- يتغير عشوائياً

~~~

$Cov(X,Y) < 0$	$Cov(X,Y) = 0$	$Cov(X,Y) > 0$



* Relationship of Covariance

$$\boxed{Cov(X, Y) = \frac{\sum_{i=1}^{n} (x_i - \mu_x)(y_i - \mu_y)}{(n-1)}}$$

Note) ✓

Variance is a special case of Covariance when two variables are $\boxed{\text{identical}}$

$$Cov(X, X) = Var(X) = \frac{\sum_{i=1}^{n} (x_i - \mu_x)^2}{n-1}$$

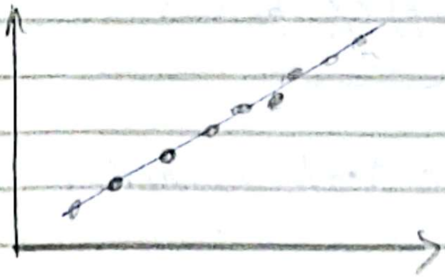
* Correlation :

$$\boxed{\rho_{X,Y} = \frac{cov(X,Y)}{\sigma_X \sigma_Y}}$$

↳ is the Normalized Version of the Covariance

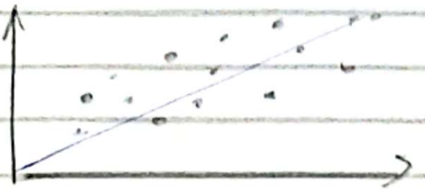
⇒ measure Both < strength / Direction > of the linear relationship Between the Two variables.

⇒ measure in standard scale : $\boxed{-1 \rightarrow +1}$



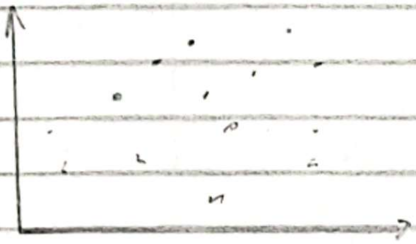
* strong +ve correlation

correlation $= +1$



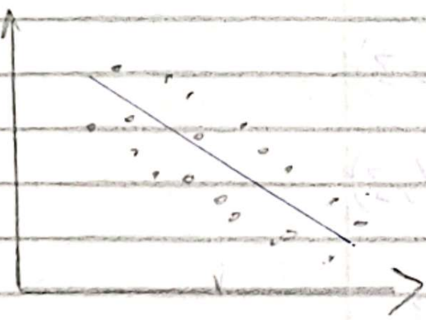
* +ve correlation

$0 <$ correlation < 1



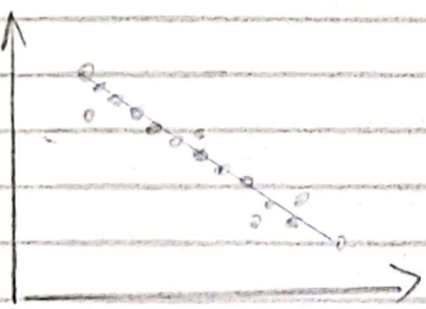
* No correlation

correlation $= 0$



* -ve correlation

$-1 >$ correlation > 0



* strong negative correlation

correlation $= -1$

الموضوع: التاريخ:

* Covariance & Correlation Matrix :

↳ square Matrix describes the realationship Between two variables.

⇛ symmetric Matrix

* Covariance Matrix

	X	Y	Z
X	Var_X	$Cov(X,Y)$	$Cov(X,Z)$
Y	$Cov(X,Y)$	Var_Y	$Cov(Y,Z)$
Z	$Cov(Z,X)$	$Cov(Z,Y)$	Var_Z

الموضوع: التاريخ:

* Correlation Matrix :

→ the Diagonal always = $\boxed{1}$

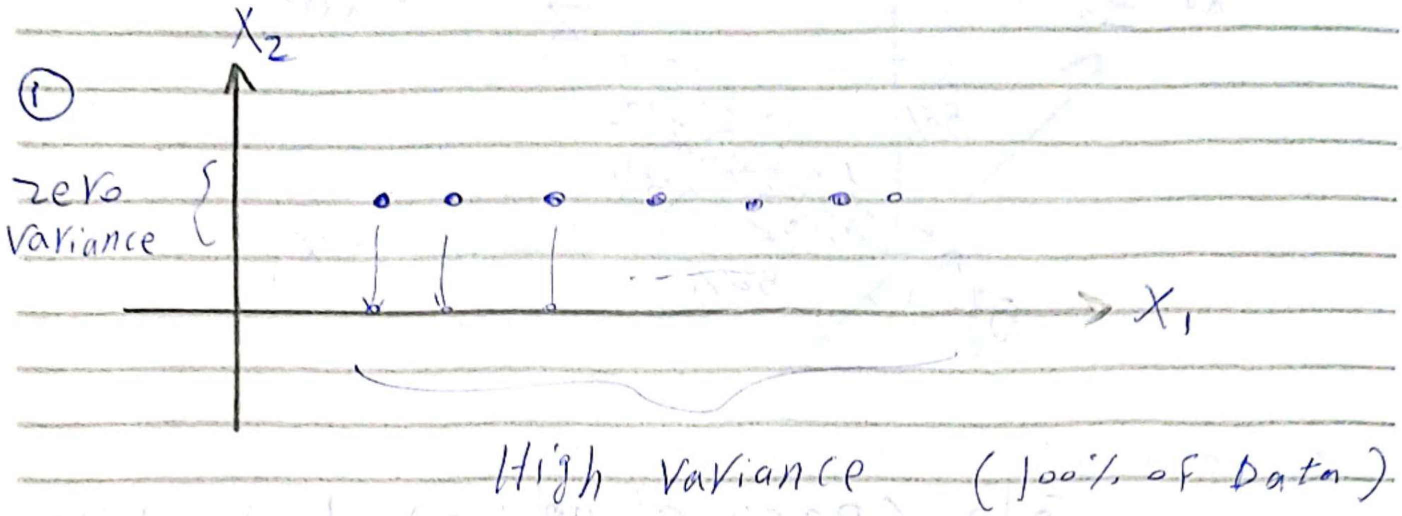
	X	Y	Z		
X	1	-0.567	0.732	-	
Y	-0.567	1	-0.153	-	
Z	0.732	-0.153	1	-	
⋮	⋮	⋮	⋮	1	
					1

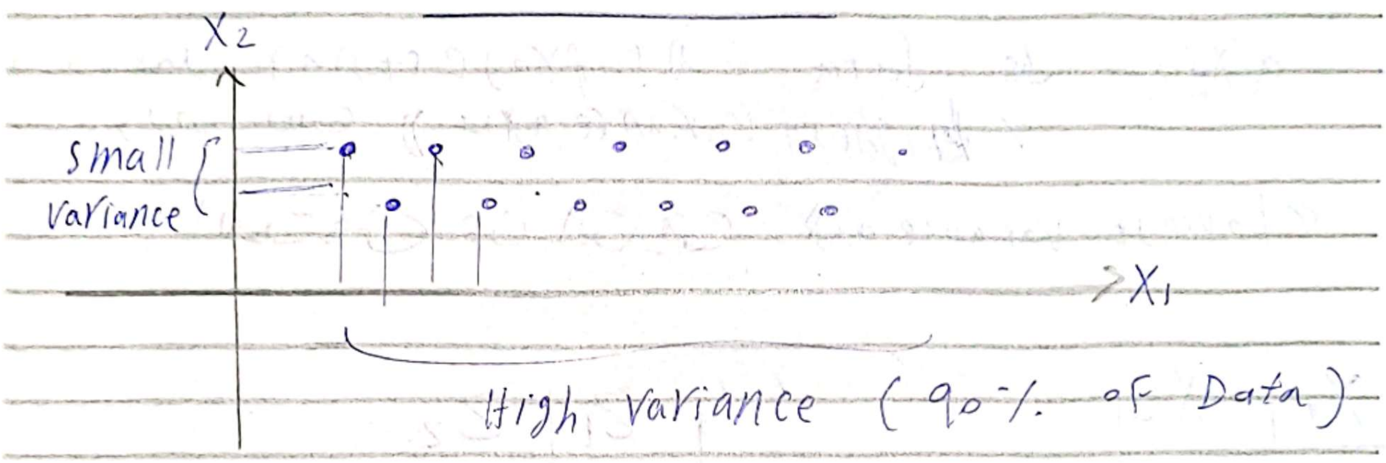
* Dimensionality Reduction Techniques :-

1. principal Components Analysis (PCA)
   - ➔ eigen decomposition

2. Singular Value decomposition (SVD)

*introduction :-

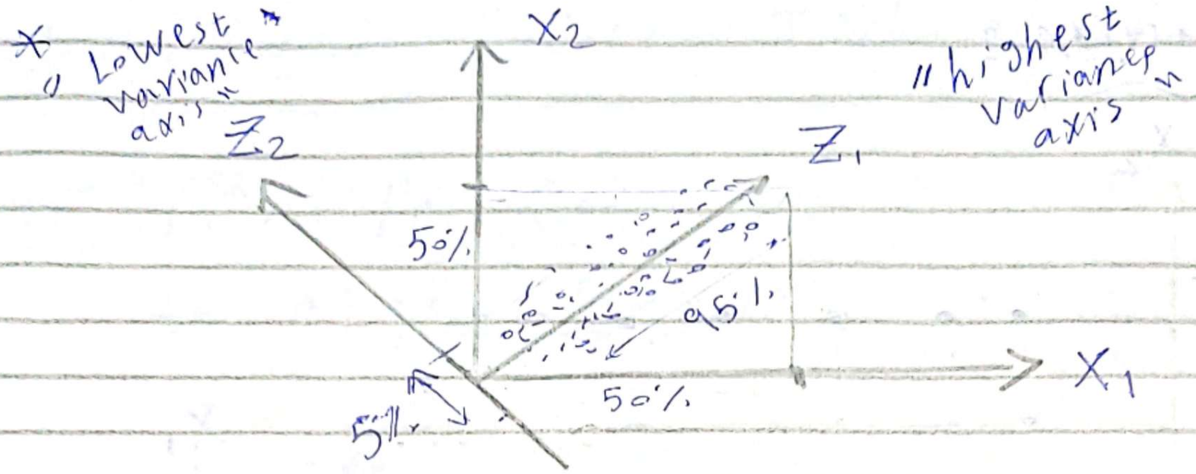




* أقدر أعمل projection على X_1 فقط

و استغنى عن X_2 لأنها (low variance)

⇐ و كده قللت عدد ال (Dimensions)

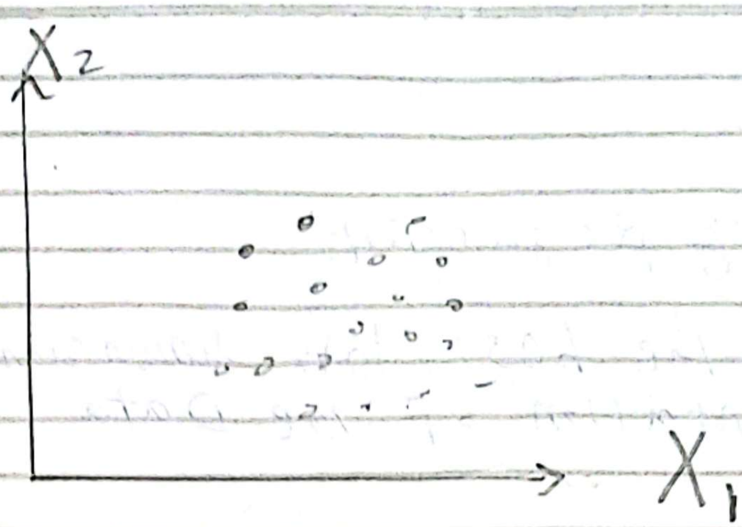


هنا لما أعمل (Basis Changing) هاقدر

أعمل (projection) للـ data كل axis

واحد بيبص ((highest variance axis))

والاستغنى عن التاني ((lowest variance axis))



← هنا صعب أعمل Dimensionality Reduction
في الـ (2 D).

لذلك ⇐ الـ projection هيكون على $\boxed{\text{Planes}}$
في الـ 3 D

* summary :

Dimensionality Reduction:

↳ Finding the best low dimensional Representation of the Data

Note the Error is kept to a minimum.

* Finding (PCA) :

I) Eginende composition of Variance matrix

II) singular value decomposition (SVD)

I) EigenDecomposition of variance matrix :

$$\boxed{A = Q D Q^{-1}}$$

A ⇒ data matrix

D ⇒ matrix of eigen values (Diagonal matrix)

Q ⇒ matrix of full orthonormal eigen vectors

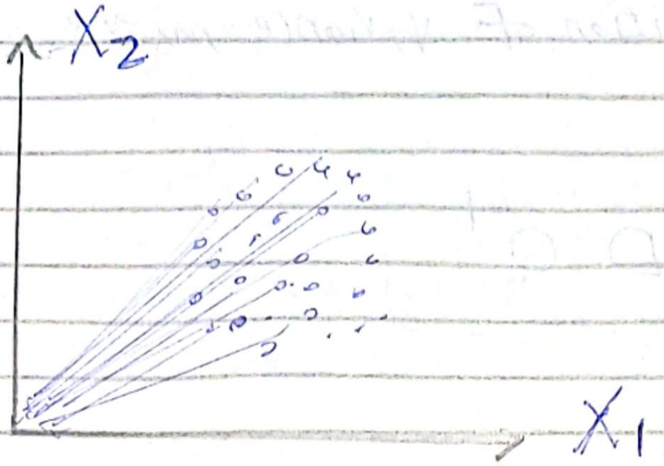
Note $\boxed{Q^{-1} = Q^T}$

* Q → is a semi-difinite matrix

$$\boxed{\lambda \geq 0}$$

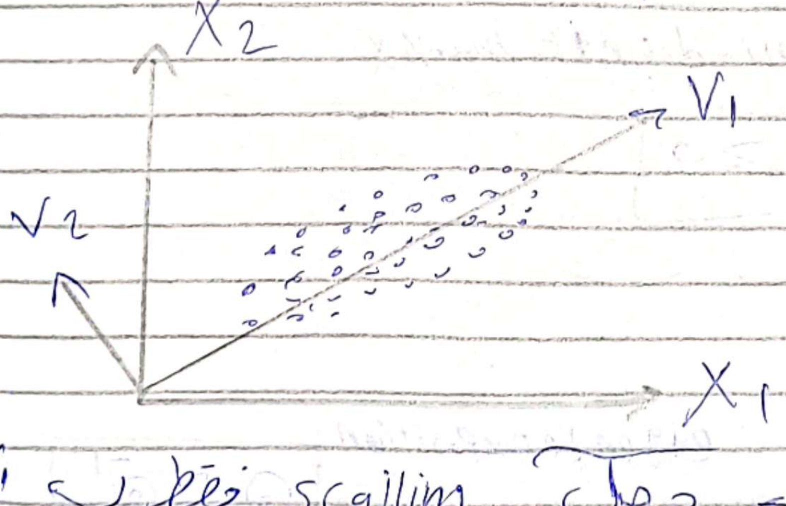
$$A \xrightarrow[\text{matrix}]{\text{covariance}} C \xrightarrow{\text{eigendecomposition}} \boxed{Q D Q^{-1}}$$

data matrix



$cov(X_1, X_2) = 0$

لما أعمل transformation باستخدام (covariance matrix)



$cov(X_1, X_2) > 0$

بعد ال transformation ← حصل scaling فقط لـ V_1, V_2

$V_1 \rightarrow$ axis of highest variance

$V_2 \rightarrow$ // // lowest //

$\lambda_1 \gg \lambda_2$

* Steps of Eigen Decomposition

Step ①: Standardize the dataset

$$\boxed{X_{new} = \frac{X - \mu}{\sigma}}$$

Step ②: Calculate the covariance matrix for the features in the dataset

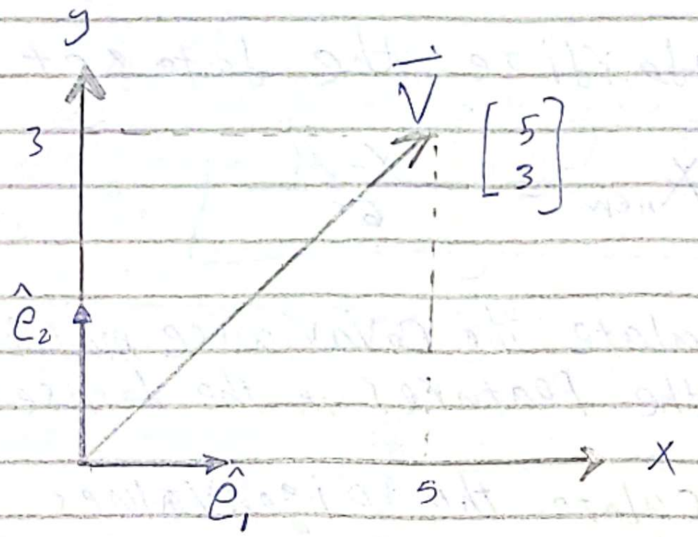
Step ③: Calculate the eigenvalues and eigenvectors for the covariance matrix.

Step ④: Sort eigenvalues and their corresponding eigenvectors

Step ⑤: Pick largest $\underline{\underline{K}}$ eigenvalues and then form the matrix of these $\underline{\underline{K}}$ vectors

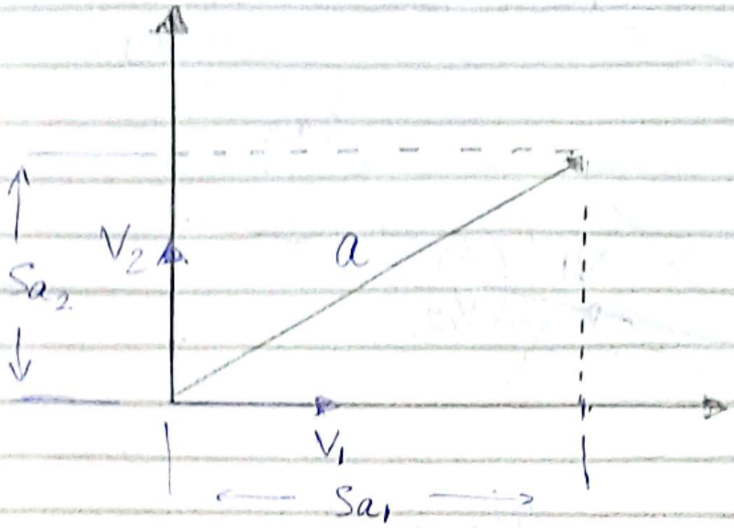
Step ⑥: Transform the original matrix by projecting the data onto the ~~K~~ picked $\underline{\underline{K}}$ $\boxed{\text{eigenvectors}}$

II) Singular Value Decomposition (SVD)



$$\vec{V} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} = 5\hat{e}_1 + 3\hat{e}_2$$

* SVD $\Rightarrow$ is decomposing vectors onto orthogonal axes.



$Sa_1 \rightarrow$ project of $\vec{a}$ on $\vec{V_1}$
$Sa_2 \rightarrow$ " " " " $\vec{V_2}$

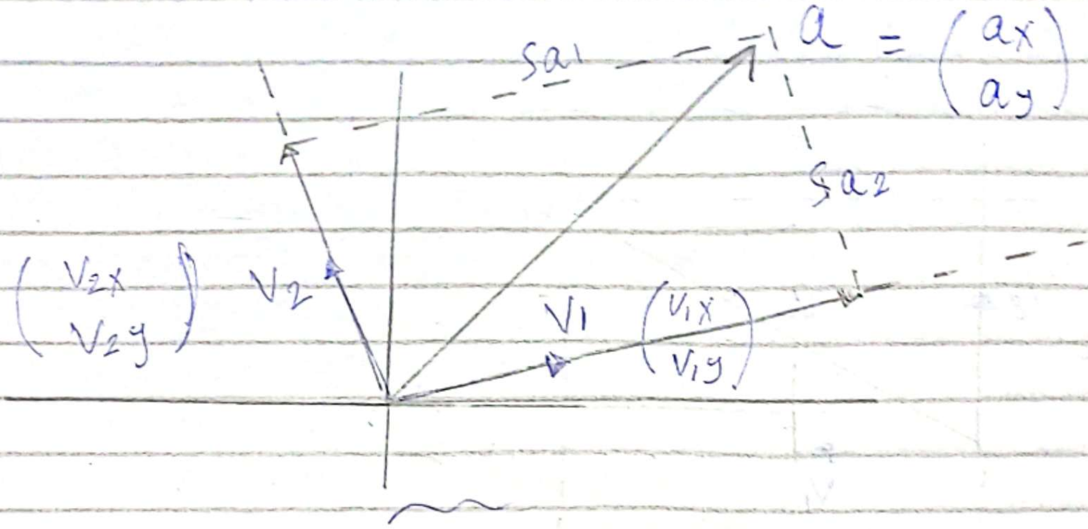
$\therefore$ $\boxed{a = Sa_1 V_1 + Sa_2 V_2}$

Rule

x Any vector can be expressed in terms of :

1) projection Directions unit vectors $(V_1, V_2, \ldots)$ orthonormal

2) the length of projections onto them $\rightarrow$ Sa_1, Sa_2, $\vdots$

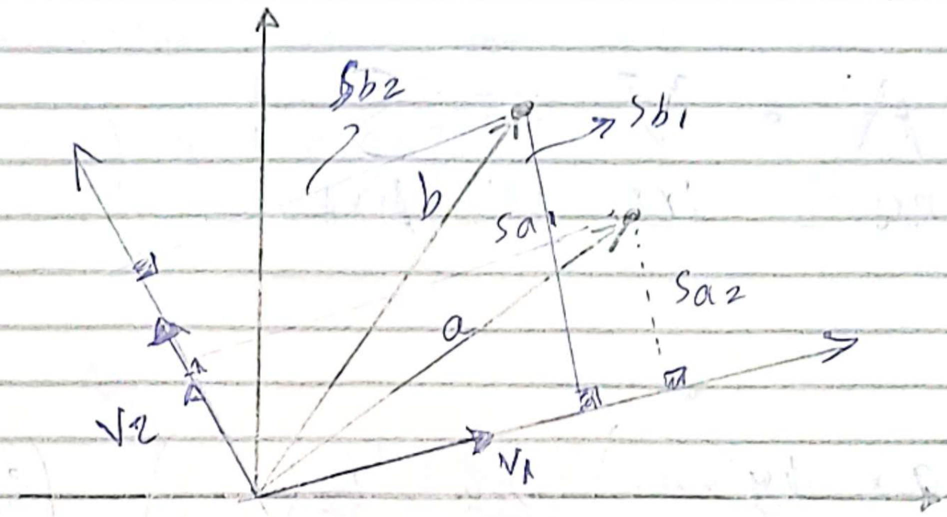


$$a^T \cdot V_1 = (a_x \ a_y) \cdot \begin{pmatrix} V_{1x} \\ V_{1y} \end{pmatrix} = S_{a1}$$

$$a^T \cdot V_2 = (a_x , a_y) \cdot \begin{pmatrix} V_{2x} \\ V_{2y} \end{pmatrix} = S_{a2}$$

$$\therefore a^T \cdot V = (a_x \ a_y) \begin{pmatrix} V_{1x} & V_{2x} \\ V_{1y} & V_{2y} \end{pmatrix} = (S_{a1} \ S_{a2})$$

* For more than one point:



$$\begin{pmatrix} a_x & a_y \\ b_x & b_y \end{pmatrix} \cdot \begin{pmatrix} V_{1x} & V_{2x} \\ V_{1y} & V_{2y} \end{pmatrix} = \begin{pmatrix} Sa_1 & Sa_2 \\ Sb_1 & Sb_2 \end{pmatrix}$$

$$\boxed{A \cdot V = S}$$

* The general case:

$$\boxed{\underset{n\times d}{A} \cdot \underset{d\times d}{V} = \underset{n\times d}{S}}$$

$$\begin{pmatrix} a_x & a_y & \cdots \\ b_x & b_y & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} V_{1x} & V_{2x} & \cdots \\ V_{1y} & V_{2y} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} S_{a1} & S_{a2} & \cdots \\ S_{b1} & S_{b2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

$$n \times d \qquad\qquad d \times d \qquad\qquad n \times d$$

$n \Rightarrow$ Nomber of points

$d \Rightarrow$ No. of Dimensions

A → matrix containing points

V → matrix containing the dimension axes

S → matrix containing lengths of projection

Note:

V : is orthogonal matrix

then, $\boxed{V^{-1} = V^{T}}$

$$A \cdot V = S$$

$$A = S\,V^{-1} = \boxed{S\,V^{T}}$$

$$A = S\,\underline{V^{T}}$$

all Vectors (PC)

Normalizing the column vectors in S

$$S = \begin{pmatrix} \frac{S_{a1}}{\sigma_1} & \frac{S_{a2}}{\sigma_2} \\ \frac{S_{b1}}{\sigma_1} & \frac{S_{b2}}{\sigma_2} \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$$

$$= \begin{pmatrix} u_{a1} & u_{a2} \\ u_{b1} & u_{b2} \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$$

$$= U \quad \Sigma$$

Note

$\sigma_1 = \sqrt{(S_{a1})^2 + (S_{b1})^2}$ ⟹ Magnitude of 1st col

$\sigma_2 = \sqrt{(S_{a2})^2 + (S_{b2})^2}$ ⟹ " " 2nd col

↳ singular values

Note

$$A = S V^T$$

$$A = U \Sigma V^T$$

* PCA with SVD :-

$$\boxed{A = U \Sigma V^T}$$

$$\underbrace{\begin{pmatrix} ax & ay & \cdots \\ bx & by & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}}_{\text{Data set}} = \begin{pmatrix} Sa_1 & Sa_2 & \cdots \\ Sb_1 & Sb_2 & \cdots \\ \vdots & \vdots & \end{pmatrix} \begin{pmatrix} V_{1x} & V_{2x} & \cdots \\ V_{1y} & V_{2y} & \\ \vdots & \vdots & \end{pmatrix}$$

$Sa_1, Sb_1, \ldots$	$Sa_2, Sb_2, \ldots$	$V_{1x}, V_{1y}, \ldots$	$V_{2x}, V_{2y}, \ldots$
the proj. length of the dataset (on) 1st PC	" " " " " " on 2nd PC	1st principle component	2nd principle component

* Pricipal components

↳ are vectors selected from ($V_1, V_2, \ldots, V_n$) that has the largest value (σ) ↑

* PCA summary :

① eigen Decomposition

1. standardize datamatrix (A) → $\boxed{\frac{X-\mu}{\sigma}}$

2. get covariance matrix $\boxed{C}$

3. decomposition of covariance matrix (C) $\boxed{Q D Q^T}$

4. select PCs from eigenvectors have the largest (eigenvalues)

② singular value decomposition (SVD)

1. ✓

2. $= \boxed{U \Sigma V^T}$

3. ✓