

Integrable Systems in Celestial Mechanics

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Abstract

The Kepler and Euler problems are special cases of the two-body and three-body problem, respectively. These have been extensively studied by mathematicians such as Euler, Lagrange, Poincaré, and Liouville. In this report, we examine these problems within the broader framework of integrable systems, following Liouville's integrability criteria from the 18th century. We first introduce the Hamiltonian formulation of mechanics, which leads to the Liouville-Arnold theorem, providing a geometric perspective on phase space trajectories. We then verify the integrability of the Kepler and Euler problems and analyse their explicit solutions, drawing on the reference text *Integrable Systems in Celestial Mechanics* [8]. Finally, we construct the corresponding action-angle variables and compare key differences between the two problems.

Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(Abdullah Mohamed Fouzuddeen)

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Chapter 1

Introduction

In 1687, Newton solved the two-body problem in his *Philosophiæ Naturalis Principia Mathematica* using purely geometric arguments. In Proposition 66 of Book 1 [6], he took the first steps toward studying the motion of three bodies under mutual gravitational attraction. Since then, mathematicians and physicists have been captivated by the three-body problem because of its fundamental role in understanding celestial orbits. Despite centuries of study, the three-body problem remains only partially understood. A key difference between the two problems was revealed in the 19th century when Poincaré proved that the three-body problem, in general, is non-integrable. This naturally raises the question: what does it mean for a mechanical system to be integrable? At first glance, the answer is not obvious. However, Liouville provided a precise criterion by requiring the existence of a complete set of independent first integrals in involution¹, a condition that is deeply rooted in the geometry of phase space. In essence, the presence of these integrals implies that the phase space can be foliated into invariant manifolds (typically tori), on which the motion is regular and can be described using action-angle variables. This geometric perspective lies at the heart of our study of integrable systems in celestial mechanics.

Celestial mechanics offers only a few integrable systems. Among these are the Kepler problem (the two-body problem), the Euler problem (the two-fixed-centre problem), and the Vinti problem (a variation of the two-fixed-centre problem). We focus on the Kepler problem and the planar case of the Euler problem. The Kepler problem is a classic example in which two bodies interact through a central force obeying an inverse-square law; one body is fixed at the origin while its gravitational influence on the other is studied. In contrast, the Euler problem involves two fixed bodies whose gravitational influence on a free body forms a variation of the three-body problem. This problem is particularly motivated by its application in modelling satellite orbits around Earth [8]. In this model, the two fixed centres are positioned along an axis at imaginary points located at $\pm ib$ from the origin², leading to analytic formulas that closely approximate the oblate nature of Earth's gravitational field.

In this report, we investigate integrable systems in celestial mechanics with two primary objectives. First, we define what it means for a system to be in-

¹A more precise definition can be seen later see 14.

²This the aforementioned Vinti problem, more of this can be read in the reference text [8]

tegrable and examine its properties through the Liouville-Arnold Theorem, emphasizing the deep geometric structure underlying phase space. Second, we apply these concepts to classical problems, drawing guidance from [8] to construct explicit solutions.

- **Chapter 2** develops a foundation in Hamiltonian mechanics and introduces the essential tools and background necessary to appreciate the geometric nature of phase space.
- **Chapter 3** explores the Kepler problem, proving the integrability criteria, deriving explicit solutions, and constructing the corresponding action-angle variables via the Liouville-Arnold Theorem.
- **Chapter 4** examines the planar Euler problem, exploring its non-trivial geometric structure through Möbius transforms and Jacobi elliptic functions to solve for trajectories. We then construct action-angle variables for the Euler problem and compare them with the Kepler case.

This report assumes that the reader is comfortable with Lagrangian mechanics and familiar with differential geometry. For clarity, we also outline the key influences on this work and highlight our own contributions.

- **Chapter 2** draws from multiple sources, particularly [2], [5], [7] and [9].
- **Chapter 3** presents an original discussion on integrability criteria and Hamiltonian formulation, while the sections on trajectories follow [8]. The construction of action-angle variables was aided by [5].
- **Chapter 4** follows [8] for trajectory solutions. However, our discussion on coordinate choices, integrability, and action-angle variables is entirely original. To our knowledge, the final result is not explicitly available in existing literature. The plots and Mathematica code provided in this chapter were created by us (see Appendix D).

Chapter 2

Classical Mechanics and Symplectic Geometry

Before discussing integrable systems, it's worth reviewing the formalisms used. We'll start by briefly reminding ourselves of some key ingredients in the Lagrangian framework and why it's considered useful, followed by a brief introduction to the Hamiltonian formulation. It's also important to understand the methods used to approach these mechanical problems. We explore these methods and the geometric context at the end of the chapter. This chapter is based on various sources, particularly [2], [5], [7], and [9].

2.1 Hamiltonian Mechanics

Definition 1 (Lagrangian and Action). Consider a system with generalized coordinates $\mathbf{q} \in \mathbb{R}^n$ and velocities $\dot{\mathbf{q}} \in \mathbb{R}^n$, with kinetic energy T and potential energy V , and assume a time-independent Lagrangian¹. The Lagrangian is a smooth function $\mathcal{L} : \mathbb{R}^{2n} \rightarrow \mathbb{R}$, sending $(\mathbf{q}, \dot{\mathbf{q}}) \mapsto \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}})$. The action functional associated to \mathcal{L} is defined as

$$S[\mathbf{q}] = \int_{t_0}^{t_1} \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) dt,$$

where the Lagrangian function is defined as

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) = T - V \tag{2.1}$$

and $t_0, t_1 \in \mathbb{R}$ denote the initial and final times, respectively.

A fundamental result in the Lagrangian formalism is given by the Euler-Lagrange theorem.

Theorem 2.1.1 (Euler-Lagrange). *The extremals of the action functional in Definition 1 satisfy the Euler-Lagrange equations, given by Equation 2.2 for $i \in$*

¹We are only concerned with time-independent Lagrangians in this report and will make this simplification where possible.

$\{1, \dots, n\}$.

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) = \frac{\partial \mathcal{L}}{\partial q_i} \quad (2.2)$$

Having established the key terminology, we now illustrate these concepts with an example.

Example 1. Consider a conserved mechanical system with mass m and potential $V(\mathbf{x})$ in \mathbb{R}^3 . The Lagrangian function would be defined as follows,

$$\mathcal{L} = \frac{m}{2} \sum_{i=1}^3 \dot{x}_i^2 - V(\mathbf{x}) \quad (2.3)$$

Applying the Euler-Lagrange equations to the above Lagrangian for the x_i^{th} (the i^{th} component of \mathbf{x}) coordinate, we obtain

$$m\ddot{x}_i = -\frac{\partial V}{\partial x_i}$$

which is i^{th} component of Newton's 2nd law for the system. ($\mathbf{F} = m\ddot{\mathbf{x}}$ where, $\mathbf{F} = -\nabla V$).

This highlights the advantages of Lagrangian mechanics, as it provides an equivalent formulation to Newton's laws while allowing for a more convenient coordinate choice without the complexities of the vectorial approach used in Newtonian mechanics. Now that we're familiar with the formulation we can start discussing the Hamiltonian formalism.

Definition 2 (Hamiltonian). Given an explicit, time-independent Lagrangian function $\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}})$ we define the Hamiltonian function as

$$H(\mathbf{q}, \mathbf{p}) = \sum_i p_i \dot{q}_i - \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}). \quad (2.4)$$

This is a smooth function, $H : \mathbb{R}^{2n} \rightarrow \mathbb{R}$, where \mathbf{q} and \mathbf{p} denote the generalised coordinates and momenta of the system. The i^{th} generalised momentum p_i is defined via the Lagrangian as

$$p_i := \frac{\partial \mathcal{L}}{\partial \dot{q}_i}.$$

Hamilton's equations are defined as follows

$$\frac{\partial H}{\partial p_i} = \dot{q}_i, \quad \frac{\partial H}{\partial q_i} = -\dot{p}_i. \quad (2.5)$$

It's useful at this stage to consider an example of this formulation.

Example 2. The Hamiltonian for a conserved mechanical system with mass m and potential $V(\mathbf{x})$ in \mathbb{R}^3 reads as

$$H = \frac{1}{2m} \sum_{i=1}^n p_i^2 + V(\mathbf{x}). \quad (2.6)$$

Hamilton's equation now gives

$$\frac{p_i}{m} = \dot{x}_i, \quad \frac{\partial V}{\partial x_i} = -\dot{p}_i.$$

Putting this together we get

$$m\ddot{x}_i = -\frac{\partial V}{\partial x_i}.$$

This result recovers Newton's second law and is consistent with the Euler-Lagrange equations, as expected.

Theorem 2.1.2. *Suppose the system is mechanical and the kinetic energy is of quadratic with respect to $\dot{\mathbf{q}}$, then the Hamiltonian H is the total energy of the system*

Proof. Recall that the Hamiltonian is defined as Definition 2,

$$H(\mathbf{q}, \mathbf{p}) = \sum_i p_i \dot{q}_i - \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}})$$

since kinetic energy is quadratic with respect to $\dot{\mathbf{q}}$, T takes the form

$$T = \frac{1}{2} \sum_{i,j} a_{ij} \dot{q}_i \dot{q}_j,$$

where $a_{ij} = a_{ij}(\mathbf{q})$, now p_i have the form

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \sum_{i,k} a_{ik} \dot{q}_k,$$

thus the Hamiltonian now reads

$$H = \sum_{i,j} a_{ij} \dot{q}_j \dot{q}_i - \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) = 2T - \mathcal{L}$$

$$H = 2T - (T - V) = T + V.$$

□

This underlies the physical interpretation of the Hamiltonian function as the total energy of a mechanical system.

Definition 3. A coordinate q_1 is said to be cyclic if it does not appear explicitly in the Hamiltonian, i.e., i.e $\partial H / \partial q_1 = 0$.

An immediate consequence of a cyclic coordinate is that its corresponding conjugate momenta is a constant of motion this follows directly from Hamilton's equations.

While the Hamiltonian formalism reduces the order of the differential equations from n second-order ODEs to $2n$ first-order ODEs (where n is the number of degrees of freedom), it also increases the total number of equations. In practice,

while this does not make it easier to solve problems, we gain much more conceptually and structure-wise. One such example is the Poisson bracket structure.

2.1.1 Poisson Bracket

The Poisson bracket is an operation on smooth functions on phase space² defined as follows.

Definition 4. Let $f(\mathbf{q}, \mathbf{p})$ and $g(\mathbf{q}, \mathbf{p})$ be smooth functions on phase space. Then the Poisson bracket is given by

$$\{f, g\} := \sum_i \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i}.$$

Proposition 2.1.3. Let f, g, h be smooth functions on phase space and $\alpha, \beta \in \mathbb{R}$, then the Poisson bracket satisfies the following properties:

1. *Skew-symmetry:* $\{f, g\} = -\{g, f\}$
2. *Linear in its entries in \mathbb{R} :* $\{\alpha f + \beta g, h\} = \alpha \{f, h\} + \beta \{g, h\}$
3. *Leibniz rule:* $\{fg, h\} = f \{g, h\} + \{f, h\} g$
4. *Jacobi identity:* $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$

Proof. 1. $\{f, g\} = \sum_i \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} = \sum_i \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} = -\{g, f\}.$

2. This follows from linearity of the partial derivative.

3. This follows from the Leibniz rule for partial derivatives.

4. It follows from straightforward expansion of the expression.

□

The Poisson bracket plays a fundamental role in Hamiltonian mechanics, as seen in the following propositions.

Proposition 2.1.4. Let f be a smooth function on phase space then

$$\frac{df}{dt} = \{H, f\} + \frac{\partial f}{\partial t}.$$

Proof. f is a smooth function on phase space thus

$$\frac{df}{dt} = \sum_i \frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i + \frac{\partial f}{\partial t}.$$

²Phase space is given a mathematical definition later in terms of the cotangent bundle see 2.3. For now one can just assume its equivalent to \mathbb{R}^{2n} where it is the space of position and momenta.

Using Hamilton's equations we obtain

$$= \sum_i \frac{\partial f}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q_i} + \frac{\partial f}{\partial t}$$

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}.$$

□

An immediate consequence of this is the fact that Hamilton's equation can be rewritten in terms of the Poisson bracket as follows

$$\frac{dq_i}{dt} = \{q_i, H\} \quad , \quad \frac{dp_i}{dt} = \{p_i, H\}.$$

Proposition 2.1.5. *Let I be a smooth function on phase space that does not depend explicitly on time. I is a constant of motion (i.e. $\frac{dI}{dt} = 0$) if and only if*

$$\{I, H\} = 0.$$

Proof. Since I is a constant of motion with no explicit time dependence, it follows from Proposition 2.1.4 that

$$\frac{dI}{dt} = 0 \implies \{I, H\} + \frac{\partial I}{\partial t} = 0$$

since there is no explicit time dependence

$$\{I, H\} = 0.$$

the argument works both ways as required. □

The Poisson bracket provides a powerful tool for identifying conserved quantities and plays a crucial role in integrable systems, as we will explore later.

Remark 1. We may use the following terminology that if $\{I, H\} = 0$, then I and H are in involution.

Before proceeding, we establish some fundamental Poisson bracket identities that will be useful for later calculations.

Proposition 2.1.6. *Let p_i and q_i be the generalised momenta and coordinates respectively then the following identity holds with $\{.,.\}$*

$$\{q_i, q_j\} = 0 \quad , \quad \{p_i, p_j\} = 0 \quad , \quad \{p_i, q_j\} = \delta_{ij}$$

Proof. This follows from the definition of the Poisson brackets. □

2.2 Hamilton-Jacobi Theory and Action-Angle Variables

In this section, we introduce a key technique in Hamiltonian mechanics that allows us to analyse phase space trajectories by determining periodic motion frequencies. This is achieved through action-angle variables, which provide insight without requiring explicit trajectory solutions. In later chapters, we shall use these techniques to understand a few problems in detail. We begin by identifying transformations that preserve the structure of Hamilton's equations and justify a specific transformation that renders the Hamiltonian cyclic with respect to a new set of coordinates.

2.2.1 Canonical Transformations

A key feature of the Lagrangian formulation is that the equations of motion remain invariant under changes in generalised coordinates. In this section, we identify the transformations in Hamiltonian mechanics that preserve this property. To do so, we first introduce a method for simultaneously handling Hamilton's equations. Let $\mathbf{x} = (q_1, \dots, q_n, p_1, \dots, p_n)^T$, Hamilton's equations can then be rewritten as follows:

$$\dot{\mathbf{x}} = \Omega \frac{\partial H}{\partial \mathbf{x}} \quad \text{or} \quad \dot{x}_i = \Omega_{ij} \frac{\partial H}{\partial x_j}, \quad (2.7)$$

where Ω is a $2n \times 2n$ matrix of the form,

$$\Omega = \begin{bmatrix} \mathbf{0}_{n \times n} & I_{n \times n} \\ -I_{n \times n} & \mathbf{0}_{n \times n} \end{bmatrix}. \quad (2.8)$$

Now we seek a transformation $\mathbf{x} \rightarrow \mathbf{y}$ that keeps Hamilton's equations invariant, i.e.,

$$\dot{y}_i = \Omega_{ij} \frac{\partial H}{\partial y_j}.$$

The relation between \dot{y}_i and \dot{x}_l is given by

$$\dot{y}_i = \frac{\partial y_i}{\partial x_l} \dot{x}_l,$$

then using Equation 2.7, we obtain

$$\dot{y}_i = \frac{\partial y_i}{\partial x_l} \Omega_{lm} \frac{\partial H}{\partial x_m} = \frac{\partial y_i}{\partial x_l} \Omega_{lm} \frac{\partial H}{\partial y_j} \frac{\partial y_j}{\partial x_m}.$$

Thus, the transformation preserves Hamilton's equation if

$$\Omega_{ij} = \mathcal{J}_{il} \Omega_{lm} \mathcal{J}_{mj}^T,$$

where $\mathcal{J}_{il} = \frac{\partial y_i}{\partial x_l}$ is the Jacobian matrix. This motivates the following definition.

Definition 5 (Canonical transformations). A change of coordinates in phase

space $\mathbf{x} \rightarrow \mathbf{y}$ is called a canonical transformation if the Jacobian matrix \mathcal{J} obeys

$$\mathcal{J}\Omega\mathcal{J}^T = \Omega$$

and then the Jacobian matrix \mathcal{J} is called symplectic.

Proposition 2.2.1. *The Poisson bracket is invariant under canonical transformations.*

Proof. Consider the smooth functions $f(\mathbf{x}), g(\mathbf{x})$ on phase space then the Poisson bracket of f and g is given by

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right),$$

or using the notation above

$$\{f, g\} = \frac{\partial f}{\partial x_i} \Omega_{ij} \frac{\partial g}{\partial x_j}.$$

Under a transformation $\mathbf{x} \rightarrow \mathbf{y}(\mathbf{x})$ we obtain

$$\begin{aligned} \frac{\partial f}{\partial x_i} &= \frac{\partial f}{\partial y_k} \mathcal{J}_{ki}, \\ \{f, g\} &= \frac{\partial f}{\partial y_k} \mathcal{J}_{ki} \Omega_{ij} \mathcal{J}_{lj} \frac{\partial g}{\partial y_l} = \frac{\partial f}{\partial y_k} \Omega_{kl} \frac{\partial g}{\partial y_l} \end{aligned}$$

where the final equality is because the transformation is symplectic. Therefore

$$= \{f(\mathbf{y}), g(\mathbf{y})\}.$$

□

Proposition 2.2.2. *The converse statement is also true, if the Poisson bracket structure is conserved such that*

$$\{Q_i, Q_j\} = 0, \{P_i, P_j\} = 0, \{Q_i, P_j\} = \delta_{ij},$$

where P_i, Q_i are the new coordinates which were represented as \mathbf{y} . Then $\mathbf{x} \rightarrow \mathbf{y}$ is a canonical transformation, where \mathbf{x} is the old-coordinates.

Having established that symplectic transformations preserve the structure of Hamiltonian mechanics, we now introduce a systematic way to construct these transformations: generating functions. These functions allow us to transition between coordinate systems while ensuring that the new equations of motion remain Hamiltonian.

Let (q_i, p_i) be the original phase-space coordinates, and define the new coordinates (Q_i, P_i) via an invertible transformation:

$$Q_i := Q_i(q_i, p_i), \quad P_i := P_i(q_i, p_i).$$

If there exists a function $K(\mathbf{Q}, \mathbf{P})$ (the Hamiltonian for the new coordinate system) such that Hamilton's equations hold:

$$\dot{Q}_i = \frac{\partial K}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial K}{\partial Q_i},$$

then we say (Q_i, P_i) form a canonical coordinate system. Since we are interested in a canonical transformation from the old set of coordinates (q_i, p_i) to (Q_i, P_i) , we require that the equations of motion coincide we shall do this via the action ³. Recall Definition 2 of the Hamiltonian via the Lagrangian:

$$H(\mathbf{q}, \mathbf{p}, t) = \sum_i p_i \dot{q}_i - \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t),$$

which gives the action functional:

$$S = \int_{t_0}^{t_1} \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) dt = \int_{t_0}^{t_1} \left(\sum_i p_i \dot{q}_i - H(\mathbf{q}, \mathbf{p}, t) \right) dt.$$

The transformation is canonical if the new action

$$S' = \int_{t_0}^{t_1} \left(\sum_i P_i \dot{Q}_i - K(\mathbf{Q}, \mathbf{P}, t) \right) dt$$

leads to equivalent equations of motion, i.e.,

$$\delta S = \delta S' = 0.$$

To ensure that the new and old coordinate systems yield equivalent equations of motion, we introduce an appropriate generating function F that satisfies the condition:

$$\sum_i p_i \dot{q}_i - H(\mathbf{q}, \mathbf{p}, t) = \sum_i P_i \dot{Q}_i - K(\mathbf{Q}, \mathbf{P}, t) + \frac{dF}{dt} \quad (2.9)$$

where F is any function of the phase space coordinates as variation of the path are zero at the end points (i.e $\delta Q_i = \delta P_i = 0$ and $\delta p_i = \delta q_i = 0$). While there are 4 types of generating functions we focus on what's usually referred to as F_1 in the literature [5], where explicitly

$$F = F_1(\mathbf{q}, \mathbf{Q}, t).$$

We consider the case that q_i cannot be expressed as a function of Q_i and t ; if this is not the case, a different generating function should be used. From Equation 2.9 we get the relation

$$\sum_i p_i \dot{q}_i - H = \sum_i \left(P_i \dot{Q}_i - K + \frac{\partial F_1}{\partial q_i} \dot{q}_i + \frac{\partial F_1}{\partial Q_i} \dot{Q}_i + \frac{\partial F_1}{\partial t} \right)$$

³We've already shown by way of an example Hamilton's equations and the Euler-Lagrange equations are equivalent for conserved mechanical systems.

$$= \sum_i \left(-K + \frac{\partial F_1}{\partial q_i} \dot{q}_i + \frac{\partial F_1}{\partial Q_i} \dot{Q}_i + \frac{\partial F_1}{\partial t} + P_i \dot{Q}_i \right)$$

since the variations of \dot{q}_i and \dot{Q}_i are independent, the coefficients must match. Comparing the coefficients of \dot{q}_i and \dot{Q}_i gives

$$p_i = \frac{\partial F_1}{\partial q_i}, \quad P_i = -\frac{\partial F_1}{\partial Q_i}, \quad (2.10)$$

thus we get the defining relationship between the Hamiltonian functions K and H as

$$K = H + \frac{\partial F_1}{\partial t}. \quad (2.11)$$

In fact one can check these equations ensure that the new coordinates (Q_i, P_i) preserve Poisson bracket structure and thus are canonical.

2.2.2 Hamilton-Jacobi Equation and Action-Angle Variables

We now examine a specific class of canonical transformations using the generating function F_1 to map from (q_i, p_i) to (Q_i, P_i) . In particular, we seek a transformation where Hamilton's equations in the new coordinates take the form $K(\mathbf{Q}, \mathbf{P}, t) = 0$, i.e. Q_i and P_i are constants of motion throughout phase space leading to the equations

$$\frac{\partial K}{\partial Q_i} = \frac{\partial K}{\partial P_i} = 0 \implies Q_i = \alpha_i \in \mathbb{R}, \quad P_i = \beta_i \in \mathbb{R}. \quad (2.12a)$$

$$H \left(q_i, \frac{\partial F_1}{\partial q_i}, t \right) + \frac{\partial F_1}{\partial t} = 0. \quad (2.12b)$$

In [Equation 2.12b](#), we make use of the relationship established by the generating function [Equation 2.10](#). Due to historical conventions, this specific function $F_1(\mathbf{q}, \mathbf{Q}, t)$ is often referred to as "Hamilton's principle function", which we will denote as S . The [Equation 2.12b](#) is then recognised as the Hamilton-Jacobi equation. To illustrate its application, we consider an explicit example.

Example 3. Consider a one dimensional simple harmonic oscillator (SHO) with unit mass and frequency ω . The Hamiltonian is given by

$$H(q, p) = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2. \quad (2.13)$$

Using the relation from [Equation 2.10](#),

$$p = \frac{\partial S}{\partial q}. \quad (2.14)$$

The Hamilton-Jacobi equation takes the form

$$\frac{1}{2} \left(\frac{\partial S}{\partial q} \right)^2 + \frac{1}{2}\omega^2 q^2 + \frac{\partial S}{\partial t} = 0. \quad (2.15)$$

Then we assume the standard separability ansatz

$$S(q, \alpha, t) = S_q(q) + S_t(t), \quad (2.16)$$

where $Q = \alpha \in \mathbb{R}$, as defined in Equation 2.12a. Substituting this into the Hamilton-Jacobi equation, we obtain

$$\frac{1}{2} \left(\frac{\partial S_q}{\partial q} \right)^2 + \frac{1}{2} \omega^2 q^2 = -\frac{\partial S_t}{\partial t}. \quad (2.17)$$

Since both sides must equal a constant, we define this constant as α , leading to the two ordinary differential equations:

$$\frac{dS_t}{dt} = -\alpha, \quad (2.18)$$

$$\frac{1}{2} \left(\frac{\partial S_q}{\partial q} \right)^2 + \frac{1}{2} \omega^2 q^2 = \alpha. \quad (2.19)$$

Integrating the first equation gives

$$S_t = -\alpha t + C_0. \quad (2.20)$$

We leave S_q in its implicit form for convenience. Combining these results, the solution for S is

$$S(q, \alpha, t) = \int \sqrt{2 \left(\alpha - \frac{1}{2} \omega^2 q^2 \right)} dq - \alpha t + C_0. \quad (2.21)$$

Using Equation 2.10, the new generalised momentum P is given by

$$P = -\frac{\partial S}{\partial \alpha} = -\frac{1}{\sqrt{2}} \int \frac{dq}{\sqrt{\alpha - \frac{1}{2} \omega^2 q^2}} + t. \quad (2.22)$$

Evaluating the integral, we find

$$P = -\frac{1}{\omega} \arcsin \left(q \frac{\omega}{\sqrt{2\alpha}} \right) + t. \quad (2.23)$$

By definition, P is a constant, which we denote as $\beta \in \mathbb{R}$. Solving for $q(t)$, we have:

$$q(t) = \frac{\sqrt{2\alpha}}{\omega} \sin \omega(t - \beta), \quad (2.24)$$

which is the well-known solution for the simple harmonic oscillator. Notably, α corresponds to the energy of the system, since the Hamiltonian H is a conserved quantity. Similarly, solving for p using Equation 2.10, we obtain

$$p(t) = \sqrt{2\alpha} \cos \omega(t - \beta). \quad (2.25)$$

This example illustrates the key idea behind Hamilton-Jacobi theory it reformulates Hamilton's equations originally a set of ordinary differential equations (ODEs) into a single partial differential equation (PDE). While this is not always practically useful, it plays a crucial role in the derivation of action-angle variables, as we will see in the following chapters.

Before proceeding, we simplify the Hamilton-Jacobi equation for time-independent Hamiltonians.

Proposition 2.2.3. *If H is time independent Hamiltonian for a system and S is Hamilton's principle function. Then the Hamilton-Jacobi equation simplifies to,*

$$H\left(q, \frac{\partial W}{\partial q}\right) = E, \text{ where } S = W(Q, q) - tE$$

and E is the total energy of the system.

Proof. Since $\frac{\partial H}{\partial t} = 0$, we recall the identity from proposition 2.1.4

$$\frac{dH}{dt} = \{H, H\} + \frac{\partial H}{\partial t} = 0,$$

this implies that H is a conserved quantity, i.e., $H = E$. Substituting the ansatz $S = W(Q, q) + S_t$ into Hamilton-Jacobi equation, we obtain

$$E + \frac{\partial S_t}{\partial t} = 0.$$

Thus $S_t = -Et$ (upto to an integration constant). Therefore, we arrive at the result

$$S = W(Q, q) - Et,$$

where W is known as Hamilton's characteristic function, that satisfies the time independent Hamilton-Jacobi equation. \square

We now turn our attention to a specific type of canonical transformation central to this report, the action-angle variables. To motivate this transformation, we revisit the example of the simple harmonic oscillator (SHO) and consider the change of variables from $(q, p) \rightarrow (\theta, I)$. Recall the solution for the SHO:

$$q(t) = \frac{\sqrt{2\alpha}}{\omega} \sin(\omega(t - \beta)), \quad p(t) = \sqrt{2\alpha} \cos(\omega(t - \beta)).$$

We define the new angle variable $\theta = \omega(t - \beta)$ and the action variable $I = \alpha$. To verify that this transformation is canonical, we compute the Poisson bracket $\{q, p\}$:

$$\{q, p\} = \frac{\partial q}{\partial \theta} \frac{\partial p}{\partial I} - \frac{\partial q}{\partial I} \frac{\partial p}{\partial \theta} = \left\{ \frac{\sqrt{2I}}{\omega} \sin(\theta), \sqrt{2I\omega} \cos(\theta) \right\},$$

simplifying, we obtain

$$2\{\sqrt{I} \sin(\theta), \sqrt{I} \cos(\theta)\} = 1.$$

Since the Poisson bracket is preserved, this confirms that the transformation is canonical. Having established this, we now examine the Hamiltonian in terms of the new variables (I, θ) :

$$H(I, \theta) = \omega I.$$

This reveals the key insight, eliminating the dependence on the angle coordinate θ the Hamiltonian becomes cyclic in θ , effectively "straightening" the flow lines in phase space. A visual representation of this transformation is shown in Figure 2.1. A natural question arises, can such a transformation always be performed? In general, the answer is no. However, for a special class of systems known as integrable systems, such a transformation is possible. This will be discussed in more detail in the following section. For now let us define the action variable in one dimension.

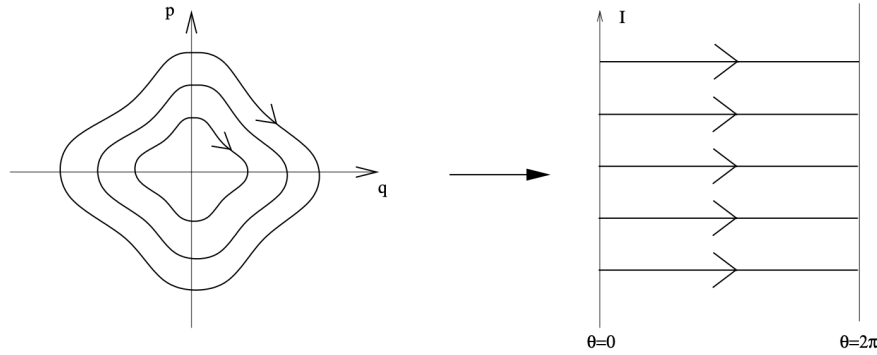


Figure 2.1: transformation from (q, p) to (θ, I) [9]

Definition 6. Let $H(q, p)$ be the Hamiltonian for a one dimensional conserved mechanical system. Then the action variable I is defined as

$$I = \frac{1}{2\pi} \int_{H=E} p dq,$$

where the integral is taken over the area enclosed by a periodic orbit in phase space, normalised by 2π . The frequency associated with the orbit is then given by

$$\omega := \dot{\theta} = \frac{\partial H(\mathbf{I})}{\partial I}.$$

A few important observations follow. First, the periodic orbit is uniquely determined by the energy E since $H = E$, the action variable I depends only on E and is therefore conserved. Second, The transformation to action-angle variables is canonical; however, we will not explicitly prove this in this report and will instead assume it as given.

The crucial statement in this construction was H was conserved, turns out this enough for a one dimensional system to be integrable. However, if a system

is integrable and there exists compact level set then we may be able to construct action-angle variables via the Liouville-Arnold theorem we shall now motivate this theorem in the following section.

2.3 Symplectic Geometry

In this section, we provide a brief introduction to symplectic geometry, highlighting its role in Hamiltonian mechanics and its relevance to the Liouville-Arnold theorem. Proofs are omitted for conciseness, but references for further details can be found in [2] and [7]. This report is largely self-contained, assuming a background in Lagrangian mechanics and some familiarity with ordinary differential equations (ODEs). However, for this section, we assume prior knowledge of differential geometry, particularly from the differential geometry course [4].

Hamiltonian mechanics describes the evolution of a mechanical system in phase space, which, for a system with n degrees of freedom, is a $2n$ -dimensional manifold. The coordinates of this manifold consist of n position variables and their conjugate momenta. The system's evolution is governed by a function on phase space known as the Hamiltonian.

To develop this framework, we first introduce key concepts from symplectic geometry, including Hamiltonian vector fields and their flows. Building on these foundations, we then present the Liouville-Arnold theorem, which formalises the notion of integrability in our context.

2.3.1 Symplectic Manifolds and Darboux's Theorem

Definition 7 (Symplectic Vector space). A symplectic vector space is a pair (V, ω) , where V is a $2n$ -dimensional real vector space and symplectic structure $\omega \in \Lambda^2 V$ is a skew-symmetric, non-degenerate bilinear form on V . Explicitly, $\omega : V \times V \rightarrow \mathbb{R}$ such that

$$\omega(\mathbf{u}, \mathbf{v}) = -\omega(\mathbf{v}, \mathbf{u}) \quad \forall \mathbf{u}, \mathbf{v} \in V \quad (\text{skew-symmetry})$$

$$\omega(\mathbf{u}, \mathbf{v}) = 0 \quad \forall \mathbf{u} \in V \implies \mathbf{v} = 0 \quad (\text{non-degenerate}).$$

Remark 2. The symplectic structure may already be familiar. Recall the matrix Ω from Equation 2.8. Using this, we obtain the canonical example of a symplectic vector space, $V = \mathbb{R}^{2n}$ and $\omega(\mathbf{u}, \mathbf{v}) = \mathbf{u}^T \Omega \mathbf{v}$.

Definition 8 (Symplectic Manifold). A symplectic manifold is a pair (M, ω) where M is a $2n$ -dimensional smooth manifold and $\omega \in \Omega^2(M)$ satisfying:

$$d\omega = 0 \quad (\omega \text{ is a closed 2-form})$$

$$\omega(X, Y) = 0 \quad \forall X \in \mathfrak{X}(M) \implies Y = 0 \quad (\text{non-degenerate})$$

The two-form ω is called a symplectic form. Importantly, at each point $p \in M$, the tangent space $T_p M$ inherits structure of a symplectic vector space.

Example 4. Let $M = \mathbb{R}^{2n}$ with coordinates (p_i, q_i) for $i \in \{1, \dots, n\}$ and

$$\omega = \sum_{i=1}^n dp_i \wedge dq_i.$$

Here ω is exact (since $\omega = d(p_i dq_i)$) and hence closed. In the coordinate basis $\{\frac{\partial}{\partial p_i}, \frac{\partial}{\partial q_i}\}$ we have

$$\omega \left(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j} \right) = 0, \quad \omega \left(\frac{\partial}{\partial q_i}, \frac{\partial}{\partial q_j} \right) = 0, \quad \omega \left(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial q_j} \right) = \delta_{ij}.$$

Thus, the matrix representation of ω is invertible at every point, making ω non-degenerate. The coordinate system (p_i, q_i) is then said to be a symplectic chart.

Definition 9 (Symplectic Chart). A *symplectic chart* on (M, ω) is a coordinate chart (p_i, q_i) , $i \in \{1, \dots, n\}$, on M such that

$$\omega = \sum_{i=1}^n dp_i \wedge dq_i.$$

This is equivalent to the coordinate basis $\{\frac{\partial}{\partial p_i}, \frac{\partial}{\partial q_i}\}$ being a symplectic basis.

We now state one of the central result of symplectic geometry.

Theorem 2.3.1 (Darboux). *Let (M, ω) be a symplectic manifold. Then every point $p \in M$ is contained in a symplectic chart.*

Remark 3. Darboux's theorem implies that all symplectic manifolds are locally equivalent to a symplectic vector space. Moreover, one can show that all symplectic vector spaces correspond to the one from Remark 2.

Before proceeding, we briefly discuss the significance of symplectic geometry in classical mechanics. Consider the following construction. Let Q be the configuration space which is an n -dimensional smooth manifold with a chart $\phi(x) = (q_i(x))$ containing the point $x \in Q$. Then the cotangent bundle $T^*Q = \{(p_x, x) | p_x \in T_x^*Q, x \in Q\}$ is the phase space of the system and a $2n$ -dimensional smooth manifold⁴ which has a chart $\Phi(p_x, x) = (p_i(x), q_i(x))$, where p_i are component functions of the 1-form $p_x = p_i(x)(dq_i)_x$.

Theorem 2.3.2. *T^*Q has a symplectic form ω which in the above chart is*

$$\omega = \sum_i dp_i \wedge dq_i.$$

Remark 4. This provides a precise mathematical definition of phase space in a geometric context. For the purposes of this report, unless explicitly stated otherwise, we assume phase space to be diffeomorphic to \mathbb{R}^{2n} , as our configuration spaces are taken to be \mathbb{R}^n .

⁴This has been proved in the differential geometry course, see [4].

2.3.2 Hamiltonian Vector Fields and the Poisson Bracket

We now derive Hamilton's equations from a geometric perspective.

Definition 10 (Hamiltonian Vector Field). Let X_H be a Hamiltonian vector field associated with the Hamiltonian function $H \in C^\infty(M)$. It is defined by

$$\iota_{X_H}\omega = -dH.$$

In local symplectic coordinates (p_i, q_i) , along an integral curve $\dot{\gamma} = X_H(\gamma(t))$, this definition leads to Hamilton's equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

Definition 11 (Poisson Bracket). For functions $f, g \in C^\infty(M)$, we define the Poisson bracket as follows

$$\{f, g\} := X_g(f) = -\omega(X_f, X_g).$$

In a symplectic chart (p_i, q_i) the Poisson bracket takes the familiar form from Definition 4

$$\{f, g\} = \sum_{i=1}^n \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right).$$

We now introduce the notion of a Hamiltonian system and its constants of motion in a geometric framework.

Definition 12. A Hamiltonian system $(M, \omega; H)$ is a symplectic manifold (M, ω) with a preferred function $H \in C^\infty(M)$ called the Hamiltonian. Motion, or 'time-evolution', of a Hamiltonian system is defined by the Hamiltonian flow of X_H (which is a Hamiltonian vector field.).

Definition 13 (Constant of Motion). A constant of motion (or first integral) of a Hamiltonian system $(M, \omega; H)$ is any function $f \in C^\infty(M)$ that is invariant under the flow of X_H . Thus, f is a constant of motion if and only if it commutes with H under the Poisson bracket:

$$\{f, H\} = 0.$$

Theorem 2.3.3. *The Hamiltonian H is invariant under the flow of X_f if and only if $f \in C^\infty(M)$ is a constant of motion.*

Proof. Recall that

$$X_f(H) = \{H, f\} = -\{f, H\}.$$

Therefore, $X_f(H) = 0$ if and only if $\{f, H\} = 0$, as required. \square

2.3.3 Liouville Integrability and the Liouville–Arnold Theorem

In Hamiltonian mechanics, a system is integrable in the Liouville sense if it satisfies the following conditions.

Definition 14. A $2n$ -dimensional Hamiltonian system $(M, \omega; H)$ is Liouville integrable if:

1. There exist functions $f_1 := H, f_2, \dots, f_n \in C^\infty(M)$ that are in involution, i.e.

$$\{f_i, f_j\} = 0, \quad \forall i, j = 1, \dots, n.$$

2. The functions f_i are independent on the level sets

$$M_c := \{p \in M \mid f_i(p) = c_i, i = 1, \dots, n\},$$

that is, the 1-forms df_i are linearly independent on M_c .

Remark 5. It is known (see [2]) that a $2n$ -dimensional symplectic manifold can admit at most n independent functions in involution.

The Liouville–Arnold theorem describes the structure of phase space for Liouville-integrable systems.

Theorem 2.3.4 (Liouville–Arnold Theorem). *Let $(M, \omega; H)$ be a $2n$ -dimensional Hamiltonian system that is Liouville integrable. Then:*

1. *The common level set*

$$M_c := \{p \in M \mid f_i(p) = c_i, \text{ for } i = 1, \dots, n\},$$

is a smooth n -dimensional submanifold that is invariant under the flow of X_H .

2. *If M_c is compact and connected, then it is diffeomorphic to an n -torus*

$$T^n = \{(\theta_1, \dots, \theta_n) \bmod 2\pi\}.$$

3. *The flow of X_H on a compact $M_c \cong T^n$ takes the form*

$$\dot{\theta}_i = \omega_i(\mathbf{c}),$$

where ω_i depend on the constant values \mathbf{c} .

4. *In a neighbourhood of $M_c \cong T^n$, there exist canonical action–angle coordinates (I_i, θ_i) such that the actions $I_i = I_i(\mathbf{c})$ are constants of motion.*

Remark 6. For $n = 1$, the 1-torus is a circle (S^1). For $n = 2$, T^2 is topologically equivalent to a torus (or $S^1 \times S^1$).

Geometrically, this result reveals that the phase space of an integrable system is foliated by an invariant tori, where each torus represents a compact, regular orbit on which the dynamics are quasi-periodic. This rich geometric structure highlights how integrability partitions the phase space into regions of regular motion. Having established the notion of integrability and the existence of action-angle variables via [Theorem 2.3.4](#), we conclude this chapter with their general definition. In the following chapters, we will explore examples of how to determine these variables using the Hamilton–Jacobi equation.

Definition 15. Let γ_i denote a basis of one dimensional cycles on the torus $M_{\mathbf{c}} \cong T^n$, where $i \in \{1, \dots, n\}$. The action variables are defined by

$$I_i(\mathbf{c}) = \frac{1}{2\pi} \int_{\gamma_i} \sum_j p_j dq_j. \quad (2.26)$$

Remark 7. A few remarks: First, when comparing with the $n = 1$ case from the previous section we see that the level set M_c is just the locus $H(p, q) = E$ and thus must be diffeomorphic to a circle. Therefore the two definitions are compatible. Second, notice that proving this change of variables is canonical amounts to proving that (I_i, θ_i) are symplectic coordinates as this yields Hamilton's equation. While we do not prove it in this report, the proof is available in [2].

Chapter 3

Kepler's Problem

3.1 Introduction

This chapter examines the Kepler problem in \mathbb{R}^3 . The Kepler problem is a special case of the two-body problem in classical mechanics, where two bodies interact through a central force that follows an inverse-square law. We will structure this chapter as follows: First, we set up the problem, then identify the constants of motion, and demonstrate that the system satisfies the integrability criteria established in Definition 14. Next, we analyse the explicit solution of the problem before finally using Theorem 2.3.4 to construct action-angle variables.

We begin by constructing the Lagrangian for the problem. Consider the position vector in \mathbb{R}^3 :

$$\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r \sin \theta \cos \varphi \\ r \sin \theta \sin \varphi \\ r \cos \theta \end{bmatrix}.$$

This leads to a kinetic and potential term:

$$T = \frac{M}{2} \langle \dot{\mathbf{X}}, \dot{\mathbf{X}} \rangle = \frac{M}{2} (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\varphi}^2),$$

$$V = \frac{-GMm}{r},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product in \mathbb{R}^3 . The mass of the fixed centre and of the moving body are m, M respectively, and G is the gravitational constant. For convenience we shall set the mass of the moving body to unity $M = 1$ and define the constant μ as, $\mu := Gm$.

Since the problem has a natural spherical symmetry, spherical coordinates provide the most convenient representation. The Lagrangian of the system now reads:

$$\mathcal{L}(q, \dot{q}, t) = T - V = \frac{1}{2} (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\varphi}^2) + \frac{\mu}{r}. \quad (3.1)$$

This leads to the following second-order ODEs via the Euler-Lagrange equations:

- For φ :

$$\frac{d}{dt} [r^2 \sin^2 \theta \dot{\varphi}] = 0. \quad (3.2)$$

- For θ :

$$\frac{d}{dt} [r^2 \dot{\theta}] = r^2 \sin \theta \cos \theta \dot{\varphi}^2. \quad (3.3)$$

- For r :

$$\frac{d}{dt} [\dot{r}] = r \dot{\theta}^2 + r \sin^2 \theta \dot{\varphi}^2 - \frac{\mu}{r^2}. \quad (3.4)$$

3.2 Constants of motion

We begin by identifying the constants of motion, starting with the conservation of the conjugate momentum associated with φ , denoted as p_φ . By noting that p_φ is a constant and using Equation 3.2, we get

$$\dot{\varphi} = \frac{p_\varphi}{r^2 \sin^2 \theta}, \quad (3.5)$$

which allows us to rewrite Equation 3.3 as

$$\frac{d}{dt} [r^2 \dot{\theta}] = \frac{\cos \theta p_\varphi^2}{r^2 \sin^3 \theta},$$

in which upon multiplying both sides by $2r^2 \dot{\theta}$ and rearranging terms, we find another conserved quantity

$$J^2 = (r^2 \dot{\theta})^2 + \frac{p_\varphi^2}{\sin^2 \theta}. \quad (3.6)$$

The constant J^2 corresponds to the conservation of the total angular momentum squared, implying confinement of the body to a plane. The third conserved quantity is the total energy of the system which may be derived by using Equation 3.5 and Equation 3.6, simplifying Equation 3.4 as

$$\frac{\dot{r}^2}{2} + \frac{J^2}{2r^2} - \frac{\mu}{r} = -\alpha^2. \quad (3.7)$$

This is expected since the Hamiltonian is time independent ¹ and here we have chosen the sign of the constant α^2 so that the orbit remains bounded. Since we have identified three conserved quantities, we proceed to find the Hamiltonian and verify that the Kepler problem in \mathbb{R}^3 is integrable in the Liouville sense.

3.2.1 Hamiltonian Formulation and Integrability

To demonstrate the integrability of the system, we first determine the constants of motion in coordinates (q_i, p_i) . By Definition 2 the conjugate momenta associated to (r, θ, φ) are given by

$$p_r = \frac{\partial L}{\partial \dot{r}} = \dot{r}, \quad (3.8)$$

¹As we shall see in the next section.

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = r^2 \dot{\theta}, \quad (3.9)$$

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = r^2 \sin^2 \theta \dot{\varphi}. \quad (3.10)$$

Since $M = 1$, we obtain the Hamiltonian by Definition 2, as follows

$$H = \sum_i p_i \dot{q}_i - \mathcal{L} = \frac{1}{2} \left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2 \theta} \right) - \frac{\mu}{r}.$$

The Hamiltonian is a conserved quantity so we take the level set as $H = -\alpha^2$ as in the above section 3.2. The remaining constant of motion, J^2 is explicitly given by

$$J^2 = p_\theta^2 + \frac{p_\varphi^2}{\sin^2 \theta}. \quad (3.11)$$

To ensure consistency, we define a function f_J as $f_J = p_\theta^2 + \frac{p_\varphi^2}{\sin^2 \theta} - J^2$, the level set is then given by $f_J = 0$. To check for involution, we recall the Poisson bracket relation, given by Proposition 2.1.4

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\}.$$

From the definition of constants of motion, we get

$$\{f_J, H\} = \frac{df_J}{dt} = 0,$$

$$\{H, H\} = \frac{dH}{dt} = 0,$$

$$\{p_\varphi, H\} = \frac{dp_\varphi}{dt} = 0,$$

thus the only thing remaining to check is $\{f_J, p_\varphi\}$. We have

$$\{f_J, p_\varphi\} = \left\{ p_\theta^2 + \frac{p_\varphi^2}{\sin^2 \theta} - J^2, p_\varphi \right\}$$

hence by bi-linearity of the Poisson bracket, we obtain

$$= \{p_\theta^2, p_\varphi\} + \left\{ \frac{p_\varphi^2}{\sin^2 \theta}, p_\varphi \right\} - \{J^2, p_\varphi\} = 0.$$

The first term is zero because of the identity from Theorem 2.1.6 and the other two terms evaluate to zero. Thus, the quantities H , f_J , and p_φ are in involution. Next, we verify the independence of H, f_J, p_φ on level sets by considering the following expression

$$H = -\alpha^2, \quad p_\varphi = C, \quad f_J = 0,$$

where $J, \alpha, C \in \mathbb{R}$. Now consider the differential of these functions on the above level sets

$$dH = \left(-\frac{p_\theta^2}{2r^3} - \frac{p_\varphi^2}{2r^3 \sin^2 \theta} + \frac{\mu}{r^2} \right) dr - \frac{p_\varphi^2 \cos \theta}{r^2 \sin^3 \theta} d\theta + p_r dp_r + \frac{p_\theta}{r^2} dp_\theta + \frac{p_\varphi}{r^2 \sin^2 \theta} dp_\varphi,$$

$$dp_\varphi = dp_\varphi,$$

$$f_J = d \left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2 \theta} - J^2 \right) = -\frac{2p_\varphi^2 \cos \theta}{\sin^3 \theta} d\theta + 2p_\theta dp_\theta + \frac{2p_\varphi}{\sin^2 \theta} dp_\varphi.$$

To prove linear independence consider the following linear combination

$$adH + bdf_J + cdp_\varphi = 0$$

where $a, b, c \in \mathbb{R}$. Examining the coefficient of dr and noting we're looking at it on the level set we get

$$\frac{a}{r^2} \left(\mu r - \frac{J^2}{2} \right) = 0.$$

However, we notice this has to be true for arbitrary r on the level set for fixed $J \in \mathbb{R}$ and because the 1-forms dq_i 's and dp_i 's forms a basis at the cotangent space at the level set, we conclude that $a = 0$. Now considering the coefficient of dp_φ which gives

$$b \frac{2p_\varphi}{\sin^2 \theta} + c = 0.$$

Since this holds for arbitrary θ in the level set and assuming θ is not degenerate we conclude $b = 0$. This means $a = b = c = 0$ which implies the constants of motion are linearly independent on the level sets hence the system is Liouville integrable by Definition 14.

3.3 Solving for Trajectories

Now that we have shown the system is integrable, we proceed to find explicit solutions for the trajectories. We begin by introducing a new independent variable f , which satisfies

$$J \frac{d}{df} = r^2 \frac{d}{dt}. \quad (3.12)$$

Substituting this into Equation 3.5, Equation 3.6, and Equation 3.7, we obtain

$$\sin^2 \theta \left(\frac{d\theta}{df} \right)^2 = \left(1 - \frac{p_\varphi^2}{J^2} \right) - \cos^2 \theta, \quad (3.13a)$$

$$\left(\frac{dr}{df} \right)^2 = -r^2 \left[1 - \frac{2\mu r}{J^2} + \frac{2\alpha^2 r^2}{J^2} \right], \quad (3.13b)$$

$$\frac{d\varphi}{df} = \frac{\frac{p_\varphi}{J}}{\sin^2 \theta}. \quad (3.13c)$$

Remark 8 (Notation). For convenience, we denote $\frac{d}{df}$ as $'$ i.e., $\frac{dR}{df}$ will be written as R'

We now solve for the trajectory of r in terms of the angle parameter f . Substituting

$$u = \frac{1}{r}, r = \frac{1}{u}, \text{ where } r' = -\frac{u'}{u^2},$$

we get

$$u'^2 = - \left[u^2 - \frac{2u\mu}{J^2} + \frac{2\alpha^2}{J^2} \right].$$

Completing the square on the right-hand side and defining

$$u - \frac{\mu}{J^2} = \sqrt{\frac{\mu^2}{J^4} - \frac{2\alpha^2}{J^2}} w,$$

we obtain

$$w'^2 = 1 - w^2,$$

giving the solution

$$w = \cos(f + \omega_0).$$

Here, ω_0 is an integration constant. Returning to r we get

$$r = \frac{\frac{J^2}{\mu}}{1 + \sqrt{1 - \frac{2J^2\alpha^2}{\mu^2}} \cos(f + \omega_0)}. \quad (3.14)$$

Remark 9. The quantity $\sqrt{1 - \frac{2J^2\alpha^2}{\mu^2}}$ is usually denoted as e , the eccentricity of the orbit [8]. In particular, $e \begin{smallmatrix} \leq \\ \geq \end{smallmatrix} 1$ corresponds to negative/positive energy solutions, representing elliptic/hyperbolic orbits. If $e = 0$, the orbit is circular.

Returning to solving Equation 3.13a for θ , we introduce the substitution

$$\cos \theta = \sqrt{1 - \frac{p_\varphi^2}{J^2}} S,$$

which leads to

$$S'^2 = 1 - S^2,$$

giving the solution

$$S = \sin(f + \omega_1).$$

Again, ω_1 is an integration constant, and we obtain

$$\cos \theta = \sqrt{1 - \frac{p_\varphi^2}{J^2} \sin^2 (f + \omega_1)}. \quad (3.15)$$

Finally, we solve Equation 3.13c for φ . Substituting the solution for $\cos \theta$ and simplifying, we get

$$\varphi' = \frac{\frac{p_\varphi}{J} \sec^2 (f + \omega_1)}{1 + \left(\frac{p_\varphi}{J}\right)^2 \tan^2 (f + \omega_1)}.$$

Defining $\tan \phi = \frac{p_\varphi}{J} \tan (f + \omega_1)$, and differentiating with respect to f , we get

$$\sec^2 (\phi) \phi' = \frac{p_\varphi}{J} \sec^2 (f + \omega_1),$$

where

$$\sec^2 \phi = 1 + \left(\frac{p_\varphi}{J}\right)^2 \tan^2 (f + \omega_1).$$

This implies $\phi' = \varphi'$, leading to

$$\phi = \varphi + \varphi_0,$$

where φ_0 is an integration constant. This results in the equation for φ

$$\tan (\varphi + \varphi_0) = \frac{p_\varphi}{J} \tan (f + \omega_1), \quad (3.16)$$

which completes the solution for the trajectories in terms of the variable f , which is often called the true anomaly [8]. For completeness, we solve for the time-angle relationship in Appendix A.

3.4 Action-Angle Variables for Kepler

In this section, we examine the implication of the Liouville-Arnold Theorem 2.3.4 and explicitly construct action-angle variables for this problem. We start by once again recalling the conserved quantities for the system:

$$\begin{aligned} -\alpha^2 &= \frac{1}{2} \left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2 \theta} \right) - \frac{\mu}{r}, \\ J^2 &= p_\theta^2 + \frac{p_\varphi^2}{\sin^2 \theta}, \\ C &= p_\varphi. \end{aligned}$$

Since the Hamiltonian is time-independent, we can apply Proposition 2.2.3 to obtain

$$H\left(q_i, \frac{\partial W}{\partial q_i}, t\right) = -\alpha^2; \quad p_i = \frac{\partial W}{\partial q_i}.$$

We note that the sign is $-\alpha^2$ is to ensure bounded orbits. Additionally, one can show that this choice is necessary for obtaining compact level sets. Substituting the ansatz

$$W = W_r + W_\theta + W_\varphi,$$

into the Hamilton-Jacobi equation and rearranging we obtain

$$-2\alpha^2 r^2 \sin^2 \theta + 2\mu r \sin^2 \theta - \left(\frac{\partial W_r}{\partial r}\right)^2 r^2 \sin^2 \theta - \left(\frac{\partial W_\theta}{\partial \theta}\right)^2 \sin^2 \theta = \left(\frac{\partial W_\varphi}{\partial \varphi}\right)^2.$$

Separating variables we arrive at

$$\frac{\partial W_\varphi}{\partial \varphi} = C, \quad (3.17)$$

$$\frac{\partial W_\theta}{\partial \theta} = \sqrt{J^2 - \frac{p_\varphi^2}{\sin^2 \theta}}, \quad (3.18)$$

$$\frac{\partial W_r}{\partial r} = \frac{\sqrt{-2\alpha^2 r^2 + 2\mu r - J^2}}{r}. \quad (3.19)$$

Finally recalling the action-angle coordinate are given by Definition 15, we obtain

$$I_\varphi = \frac{1}{2\pi} \int_{\gamma_\varphi} C \, d\varphi, \quad (3.20)$$

$$I_\theta = \frac{1}{2\pi} \int_{\gamma_\theta} \sqrt{J^2 - \frac{p_\varphi^2}{\sin^2 \theta}} \, d\theta, \quad (3.21)$$

$$I_r = \frac{1}{2\pi} \int_{\gamma_r} \frac{\sqrt{-2\alpha^2 r^2 + 2\mu r - J^2}}{r} \, dr. \quad (3.22)$$

Where the limits of the integrals are determined by closed curves in phase space. Since the Hamiltonian does not depend explicitly on φ , the coordinate is cyclic, and we obtain

$$I_\varphi = p_\varphi.$$

To evaluate I_θ , we identify the integration limits as the closed curve spanning from $-\theta^*$ to $+\theta^*$ in phase space. The shape of this curve is determined by the condition $p_\theta|_{\theta=\theta^*} = 0$ (see figure 3.1b), ensuring the local maxima/minima of the motion with respect to t . Since the motion is symmetric, we compute the integral from $\frac{\pi}{2}$ to θ^* and multiply by 4. Introducing $\frac{p_\varphi}{J} = \cos l$ (this is known as the polar angle [5] and is a unit less parameter) we get a convenient relationship of $\theta^* = \frac{\pi}{2} - l$.

Remark 10. A quick remark here, $\frac{p_\varphi}{J} = \cos l$ is a valid substitution as for I_θ to be a valid closed curve $\frac{p_\varphi}{J} \leq 1$.

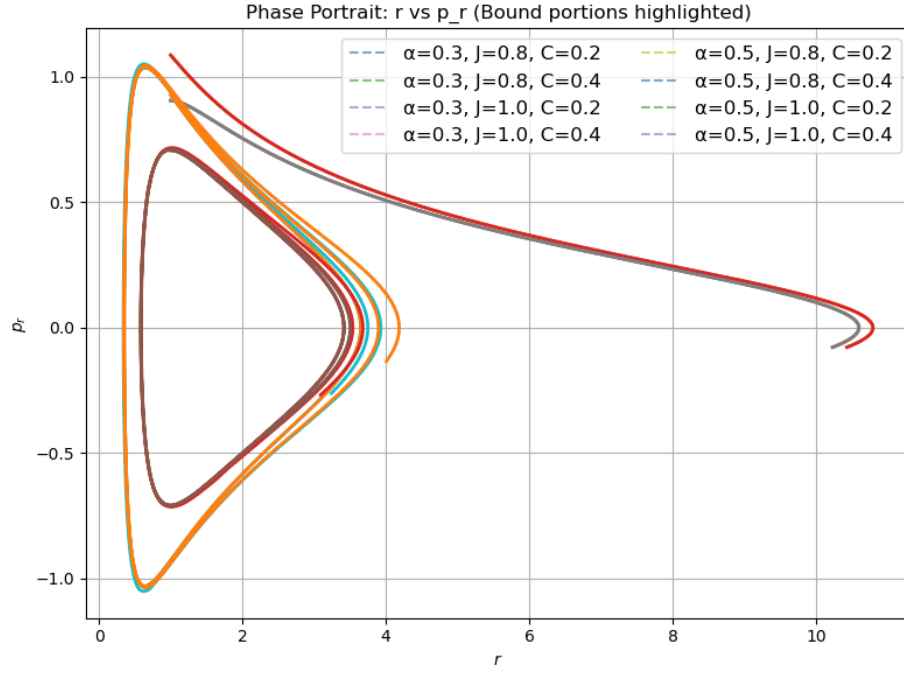
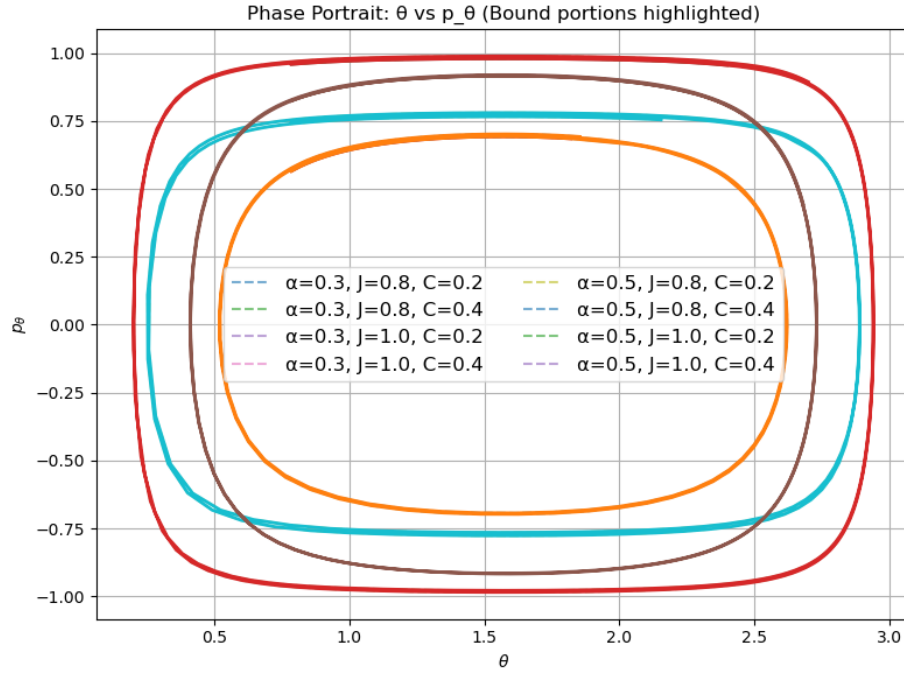
(a) Plot of p_r vs r (b) Plot of p_θ vs θ

Figure 3.1: Phase portraits illustrating the motion in $r - p_r$ and $\theta - p_\theta$ phase space. These were generated numerically for different values of J, C, α .

Thus, integral 3.21 may be rewritten as

$$I_\theta = \frac{2J}{\pi} \int_{\frac{\pi}{2}}^{\theta^*} \csc \theta \sqrt{\sin^2 \theta - \cos^2 l} d\theta = \frac{2J}{\pi} \int_{\frac{\pi}{2}}^{\theta^*} \csc \theta \sqrt{\sin^2 l - \cos^2 \theta} d\theta,$$

now using the substitution

$$\cos \theta = \sin l \sin \Phi$$

gives

$$I_\theta = \frac{2J \sin^2 l}{\pi} \int_0^{\frac{\pi}{2}} \frac{\cos^2 \Phi}{1 - \sin^2 l \sin^2 \Phi} d\Phi.$$

We make one more substitution

$$u = \tan \Phi,$$

to get

$$I_\theta = \frac{2J \sin^2 l}{\pi} \int_0^\infty \frac{1}{(1+u^2)(1+u^2 \cos^2 l)} du.$$

Which can be simplified via partial fractions and is a known integral that evaluates to

$$I_\theta = J \left(1 - \frac{p_\varphi}{J} \right) = J - p_\varphi. \quad (3.23)$$

We now evaluate the final integral for I_r which was given by

$$I_r = \frac{1}{2\pi} \int_{\gamma_r} dr \frac{\sqrt{-2\alpha^2 r^2 + \mu r - (I_\theta + I_\varphi)^2}}{r}.$$

The limits of integration for I_r are determined by the points where the expression under the square root vanishes, corresponding to the turning points of the radial motion (where $p_r = \dot{r} = 0$). Thus let r_+ and r_- be the roots for the quadratic above, the closed counter then goes from r_- to r_+ and then back to r_- thus the integral may be written as 2 times the integral from r_- to r_+ , i.e.

$$I_r = \frac{1}{\pi} \int_{r_-}^{r_+} \frac{\sqrt{-2\alpha^2 r^2 + 2\mu r - (I_\theta + I_\varphi)^2}}{r} dr.$$

Remark 11. Note that the quadratic must have real roots; otherwise, no closed curve can exist. As seen in Figure 3.1a, for certain values of $J, \alpha, C \in \mathbb{R}$, not all trajectories in phase space correspond to closed orbits.

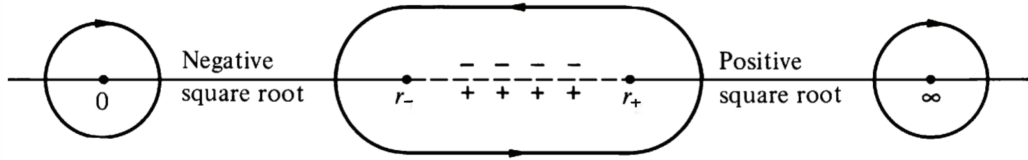


Figure 3.2: Complex plane r via in the neighbourhood of the real axis [5]

This integral can be evaluated via the use complex contour integration technique, identifying branch points at r_\pm and enclosing the branch cut from r_- to r_+ with a contour in the complex plane (as seen in 3.2). We split the contour into four parts, where two of them cancel. The remaining two pick up a phase contribution from the branch cut, leading to an integral that is twice the real

integral. Although we cannot directly apply the residue theorem to the given contour, we can instead consider its complement, which encloses the rest of the complex plane in the clockwise direction. This contour is analytic everywhere except at the poles at $r = 0$ & $r \rightarrow \infty$, allowing us to apply the residue theorem.

$$2I_r = \frac{2\pi i}{\pi} [\text{Res}(I_r, 0) + \text{Res}(I_r, r \rightarrow \infty)].$$

Using the residue formula for poles, at $r = 0$ we get

$$\begin{aligned} \text{Res}(I_r, 0) &= \frac{1}{(1-1)!} \lim_{r \rightarrow 0} \frac{d^0}{dr^0} \left(\frac{r \sqrt{-2\alpha^2 r^2 + 2\mu r - (I_\theta + I_\varphi)^2}}{r} \right) \\ &= i(I_\theta + I_\varphi) \end{aligned}$$

now at $r \rightarrow \infty$, We first start by changing variables of the integral by setting $r = \frac{1}{z}$ and then evaluate the residue at $z = 0$,

$$\begin{aligned} \text{Res}(I_r, r \rightarrow \infty) &= \frac{1}{(2-1)!} \lim_{z \rightarrow 0} \frac{d}{dz} \left(z^2 \frac{-1}{z^2} \sqrt{-2\alpha^2 + 2\mu z - (I_\theta + I_\varphi)^2 z^2} \right) \\ &= \frac{-\mu}{\sqrt{-2\alpha^2}}. \end{aligned}$$

Putting it all together we get

$$\begin{aligned} I_r &= -(I_\theta + I_\varphi) - \frac{\mu i}{\sqrt{-2\alpha^2}} \\ -\alpha^2 &= -\frac{\mu^2}{2(I_r + I_\theta + I_\varphi)^2}. \end{aligned}$$

Since the Hamiltonian is conserved and we define it such that $H = -\alpha^2$, thus in action-angle coordinates it is given by

$$H = -\frac{\mu^2}{2(I_r + I_\theta + I_\varphi)^2}. \quad (3.24)$$

Now, considering the angular frequencies via [Theorem 2.3.4](#), we have

$$\omega_i = \dot{\theta}_i,$$

which shows that the frequencies ω_i are degenerate, as

$$\dot{\theta}_r = \dot{\theta}_\theta = \dot{\theta}_\varphi = \frac{\mu^2}{4(I_r + I_\theta + I_\varphi)^3}.$$

This degeneracy implies that the frequencies ω_i are commensurate, meaning their ratios are rational. As a consequence, the trajectory in phase space is periodic, and the motion repeats after a finite time. Since a closed trajectory in phase

space means that both position and velocity return to their initial values, the orbit in configuration space must also be closed.

Thus, in the Kepler problem, bound orbits do not precess but remain elliptical. This is expected, as the solution to the orbital trajectories for $e \leq 1$ corresponds to an ellipse or a circle. In contrast, for more general gravitational systems, frequency degeneracies are broken, leading to precessing orbits, as we shall see in the following chapter.

Chapter 4

Euler Problem Planar Case

4.1 Introduction

The Planar Euler problem, also known as the two fixed centre problem, describes a system where three bodies interact in a plane. Two fixed masses are symmetrically situated on the z axis (at $z = \pm b$) where they induce a gravitational field on the free body P (for simplicity, we assume P is a point mass). We shall structure our discussion of this chapter in a similar fashion to [chapter 3](#). Starting by addressing the potential and the choice of coordinates. The gravitational potential

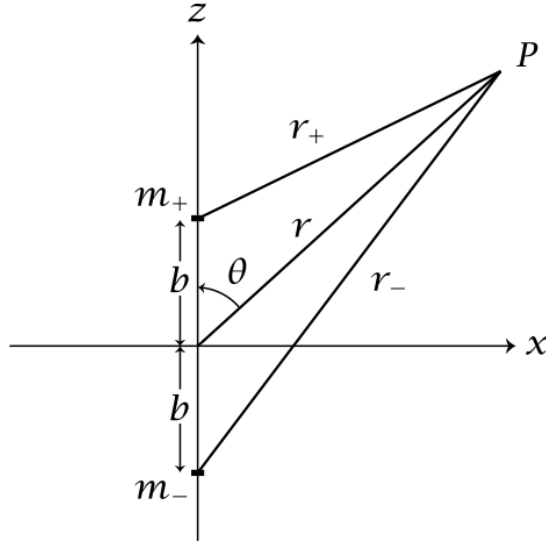


Figure 4.1: Set up of the two fixed centre problem [8]

per unit mass at P is given by

$$V = G \left(\frac{m_+}{r_+} + \frac{m_-}{r_-} \right). \quad (4.1)$$

Here, G is the gravitational constant and the potential is given in terms of polar coordinates (r, θ) , where θ is measured from positive z axis (as indicated in Figure

4.1), the distances r_+ and r_- are given by:

$$r_+^2 = r^2 + b^2 - 2br \cos \theta, \quad (4.2a)$$

$$r_-^2 = r^2 + b^2 + 2br \cos \theta. \quad (4.2b)$$

However, working directly with (r, θ) is cumbersome. A more natural choice, given the problem's symmetry, is planar prolate spheroidal coordinates (R, σ) which simplify the equations of motion. These are defined by

$$r \sin \theta = x := \pm \sqrt{R^2 - b^2} \sin \sigma, \quad r \cos \theta = z := R \cos \sigma. \quad (4.3)$$

Before proceeding, it is worth briefly discussing what we mean by a ‘natural’ choice of coordinates and why (R, σ) is particularly well suited for this problem. We first observe that the curves corresponding to constant R are ellipses. To see this, note that setting R to a constant in

$$\frac{x^2}{R^2 - b^2} + \frac{z^2}{R^2} = 1$$

yields an ellipse (since $R > b$). Similarly, the lines corresponding to constant σ are given by

$$x^2 = \left(\frac{z^2}{\cos^2 \sigma} - b^2 \right) \sin^2 \sigma.$$

which represents a hyperbolae. One can also see both of these constant lines from the figure 4.2 below.

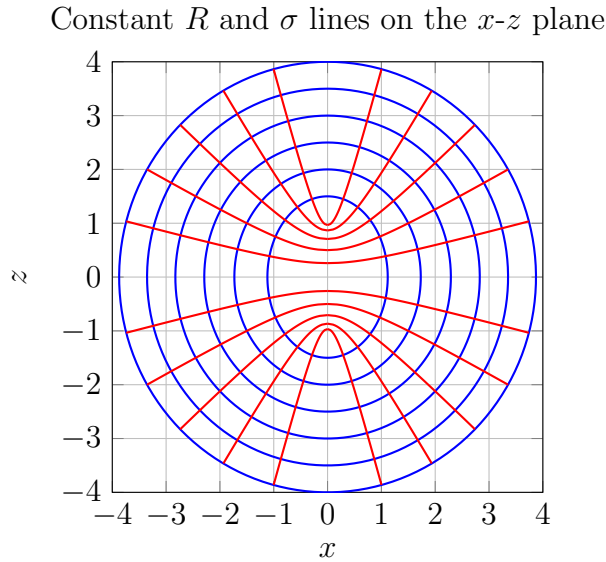


Figure 4.2: Blue curves represent constant R (ellipses), and red curves are constant σ (hyperbolae).

This choice is convenient because, for a bounded system, we expect the orbits of the free body to resemble elliptical shapes that are slightly extended along

the axis where the fixed centres are located. We shall also see that this is a great choice later on, as it turns out that the Hamilton–Jacobi equation becomes separable in these coordinates.

Rewriting the potential in Equation 4.1 in terms of the new coordinates yields

$$V = G \left(\frac{m_+}{(R - \cos \sigma)} + \frac{m_-}{(R + \cos \sigma)} \right) = G(m_+ + m_-) \left(\frac{R + b \left(\frac{m_+ - m_-}{m_+ + m_-} \right) \cos \sigma}{R^2 - \cos^2 \sigma} \right). \quad (4.4)$$

For ease of calculation, we define the following constants

$$\beta = \frac{m_+ - m_-}{m_+ + m_-}, \quad (4.5a)$$

$$\mu = G(m_+ + m_-). \quad (4.5b)$$

To compute the kinetic energy T in the new coordinates, consider the position vector,

$$\mathbf{X} = \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} \pm \sqrt{R^2 - b^2} \sin \sigma \\ R \cos \sigma \end{bmatrix}$$

T is then given by

$$T = \frac{1}{2} \langle \dot{\mathbf{X}}, \dot{\mathbf{X}} \rangle = \frac{1}{2} \left(\frac{R^2 - b^2 \cos^2 \sigma}{R^2 - b^2} \dot{R}^2 + (R^2 - b^2 \cos^2 \sigma) \dot{\sigma}^2 \right), \quad (4.6)$$

where the Lagrangian is given by

$$\begin{aligned} \mathcal{L} &= T - V \\ &= \frac{1}{2} \left[\frac{R^2 - b^2 \cos^2 \sigma}{R^2 - b^2} \dot{R}^2 + (R^2 - b^2 \cos^2 \sigma) \dot{\sigma}^2 \right] + \mu \frac{R + \beta b \cos \sigma}{R^2 - b^2 \cos^2 \sigma}. \end{aligned} \quad (4.7)$$

4.2 Constants of Motion

We start by making a final change of coordinates to aid in finding the constants of motion. In particular, we look for coordinates that are of ‘Liouville form’ [8]. We do so by defining

$$R = b \cosh \xi, \quad \dot{R} = b \sinh \xi \dot{\xi}, \quad R^2 - b^2 = b^2 \sinh^2 \xi, \quad (4.8)$$

thus leading to the Lagrangian being

$$\mathcal{L} = b^2 (\cosh^2 \xi - \cos^2 \sigma) \left[\frac{\dot{\xi}^2}{2} + \frac{\dot{\sigma}^2}{2} \right] + \mu \frac{b \cosh \xi + \beta b \cos \sigma}{b^2 \cosh^2 \xi - b^2 \cos^2 \sigma}. \quad (4.9)$$

Now considering Lagrange's equations:

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\xi}} \right) = \frac{\partial \mathcal{L}}{\partial \xi}, \quad (4.10a)$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\sigma}} \right) = \frac{\partial \mathcal{L}}{\partial \sigma}. \quad (4.10b)$$

We now derive the energy integral by multiplying Equation 4.10a by $\dot{\xi}$ and Equation 4.10b by $\dot{\sigma}$, and adding them together to obtain

$$\frac{d}{dt} \left[\dot{\xi} \frac{\partial \mathcal{L}}{\partial \dot{\xi}} + \dot{\sigma} \frac{\partial \mathcal{L}}{\partial \dot{\sigma}} \right] = \dot{\xi} \frac{\partial \mathcal{L}}{\partial \xi} + \dot{\xi} \frac{\partial \mathcal{L}}{\partial \dot{\xi}} + \dot{\sigma} \frac{\partial \mathcal{L}}{\partial \sigma} + \dot{\sigma} \frac{\partial \mathcal{L}}{\partial \dot{\sigma}} = \frac{d\mathcal{L}}{dt},$$

where the last equality is 0 as \mathcal{L} is not explicitly time dependent. Now integrating the above we get

$$\dot{\xi} \frac{\partial \mathcal{L}}{\partial \dot{\xi}} + \dot{\sigma} \frac{\partial \mathcal{L}}{\partial \dot{\sigma}} - \mathcal{L} = E, \quad (4.11)$$

where we recognise that the constant is the total energy of the system as this is the definition of the Hamiltonian.

Remark 12. This procedure can be done in general to any explicitly time-independent Lagrangian and that is to say that total energy is conserved.

Now coming back to Equation 4.10a which gives

$$\frac{d}{dt} \left[(b^2 \cosh^2 \xi - b^2 \cos^2 \sigma) \dot{\xi} \right] = \frac{d(b^2 \cosh^2 \xi)}{d\xi} \left[\frac{\dot{\xi}^2}{2} + \frac{\dot{\xi}^2}{2} \right] - \frac{\partial V}{\partial \sigma}. \quad (4.12)$$

For convenience, we first define $Q(\sigma, \xi) = (b^2 \cosh^2 \xi - b^2 \cos^2 \sigma) \dot{\xi}$, then we multiply Equation 4.12 by $Q\dot{\xi}$ which leads to

$$Q\dot{\xi} \frac{d}{dt} [Q\dot{\xi}] = \frac{\partial Q}{\partial \xi} T\dot{\xi} - \frac{\partial V}{\partial \sigma} Q\dot{\xi}.$$

We then rewrite T as $E - V$ and after rearranging get

$$\frac{d}{dt} \left[\frac{(Q\dot{\xi})^2}{2} \right] = \dot{\xi} \left[E \frac{d(b^2 \cosh^2 \xi)}{d\xi} - \frac{\partial(QV)}{\partial \xi} \right],$$

where we notice

$$QV = Q\mu \frac{b \cosh \xi + \beta b \cos \sigma}{Q} = \mu(b \cosh \xi + \beta b \cos \sigma),$$

thus after manipulation of $\dot{\xi}$ we get

$$\frac{d}{dt} \left[\frac{(Q\dot{\xi})^2}{2} - Eb^2 \cosh^2 \xi - \mu b \cosh \xi \right] = 0.$$

Upon integration we obtain

$$\frac{(Q\dot{\xi})^2}{2} - Eb^2 \cosh^2 \xi - \mu b \cosh \xi = C_1. \quad (4.13)$$

If we apply a similar procedure to Equation 4.10b (for σ) we find

$$\frac{(Q\dot{\sigma})^2}{2} + Eb^2 \cos^2 \sigma - \mu\beta b \cos \sigma = C_2. \quad (4.14)$$

Remark 13. Here we get two constant of motion however one must note if we add Equation 4.13 and Equation 4.14 we get

$$Q^2 \left[\frac{\dot{\xi}^2}{2} + \frac{\dot{\sigma}^2}{2} \right] - EQ + VQ = Q[T - (T + V) + V] = C_1 + C_2,$$

which implies

$$C_1 + C_2 = 0, \quad (4.15)$$

therefore the constants of motions are linearly dependent on each other.

Now that we have sufficient constants of motion to work with we first rewrite Equation 4.13 and Equation 4.14 in terms of prolate spheroidal coordinates and then proceed to discuss integrability. Rewriting them we get

$$\frac{1}{2} \left(\frac{\dot{R}^2}{R^2 - b^2} (R^2 - b^2 \cos^2 \sigma)^2 \right) = ER^2 + \mu R + C_1, \quad (4.16a)$$

$$\frac{1}{2} (\dot{\sigma}^2 (R^2 - b^2 \cos^2 \sigma)^2) = -Eb^2 \cos^2 \sigma + \mu\beta b \cos^2 \sigma + C_2. \quad (4.16b)$$

Since in this report we are interested in bounded orbits it is important to choose the sign and value of constants E, C_1 and C_2 carefully. By inspecting Equation 4.16b, we observe as $b \rightarrow 0$, the condition $C_2 > 0$ holds as $\dot{\sigma}^2 (R^2 - b^2 \cos^2 \sigma)^2 > 0$ which in turn implies $C_1 < 0$. Now similar to the Kepler problem since we look at bounded orbits we let $E < 0$. Thus we choose the level sets as

$$E := -\alpha^2, \quad C_1 := \frac{-C^2}{2}, \quad C_2 := \frac{C^2}{2}. \quad (4.17)$$

The first integrals are then given by

$$\frac{1}{2} \left(\frac{\dot{R}^2}{R^2 - b^2} (R^2 - b^2 \cos^2 \sigma)^2 \right) = -\alpha^2 R^2 + \mu R - \frac{C^2}{2}, \quad (4.18a)$$

$$\frac{1}{2} (\dot{\sigma}^2 (R^2 - b^2 \cos^2 \sigma)^2) = \alpha^2 b^2 \cos^2 \sigma + \mu\beta b \cos^2 \sigma + \frac{C^2}{2}. \quad (4.18b)$$

4.2.1 Hamiltonian Formulation and Integrability

We begin by finding the conjugate momenta associated to R , σ which are

$$p_R = \frac{\partial L}{\partial \dot{R}} = \left[\frac{R^2 - b^2 \cos^2 \sigma}{R^2 - b^2} \right] \dot{R}, \quad (4.19a)$$

$$p_\sigma = [R^2 - b^2 \cos^2 \sigma] \dot{\sigma}. \quad (4.19b)$$

The Hamiltonian is then given by

$$H = \sum_i p_i \dot{q}_i - \mathcal{L} = \frac{1}{2} \left(\frac{p_R^2 (R^2 - b^2)}{R^2 - b^2 \cos^2 \sigma} + \frac{p_\sigma^2}{R^2 - b^2 \cos^2 \sigma} \right) - \mu \frac{R + \beta b \cos \sigma}{R^2 - b^2 \cos^2 \sigma},$$

where the first integrals in p_i 's and q_i 's are

$$f_R = \frac{p_R^2}{2} (R^2 - b^2) + \alpha^2 R^2 - \mu R + \frac{C^2}{2},$$

$$f_\sigma = \frac{p_\sigma^2}{2} - \alpha^2 b^2 \cos^2 \sigma + \mu \beta b \cos \sigma - \frac{C^2}{2}.$$

Remark 14. An interesting limiting case occurs as $b \rightarrow 0$, p_σ becomes a conserved quantity and coincides with p_φ from the Kelper case as $f_\sigma \rightarrow \frac{p_\sigma^2 - C^2}{2}$ (i.e. $p_\sigma = p_\varphi$).

At this stage, we make a few observations: First, f_R and f_σ are not independent on the level set $f_\sigma = f_R = 0$, as demonstrated in [Equation 4.15](#). Secondly, the Hamiltonian is time-independent and thus a conserved quantity. Therefore, H and f_σ are in involution due to the Poisson bracket relation given by [Proposition 2.1.4](#). To check linear independence, consider the following equation

$$a dH + b df_\sigma = 0$$

where $a, b \in \mathbb{R}$. The coefficient of the 1-form dp_R gives

$$a \left(\frac{p_R (R^2 - b^2)}{R^2 - b^2 \cos^2 \sigma} \right) = 0.$$

Since $R \geq b$ for all R and $p_R \neq 0$ for arbitrary R in the level set, it follows that a must be zero. Consequently, this implies $a = b = 0$, proving that dH and df_σ are linearly independent on the level set.

4.3 Solving for Trajectories

4.3.1 Reduction of First Integrals

We now aim to reduce the first integrals to facilitate the use of Jacobi elliptic functions to solve for trajectories. To do so, we start by introducing an independent

variable f given by

$$\frac{R^2 - b^2 \cos^2 \sigma}{\Lambda} \frac{d}{dt} = \frac{d}{df}$$

where Λ is a free parameter set according to appropriate cases, with units of angular momentum (this makes $\frac{\Lambda}{C}$ dimensionless).

Remark 15. Recall that we used the notation R' to denote $\frac{dR}{df}$.

Thus the first integrals now read:

$$\frac{\Lambda^2}{C^2} R'^2 = -(R^2 - b^2) \left[1 - \frac{2\mu R}{C^2} + \frac{2\alpha^2 R^2}{C^2} \right], \quad (4.20a)$$

$$\frac{\Lambda^2}{C^2} \sigma'^2 = 1 + 2\beta \frac{b\mu}{C^2} \cos \sigma + \frac{2b^2 \alpha^2}{C^2} \cos^2 \sigma. \quad (4.20b)$$

To maintain consistency with the reference text [8], we introduce some length scales. Let

$$a = \frac{\mu}{2\alpha^2}, \quad p = \frac{C^2}{\mu} \text{ hence, } ap = \frac{C^2}{2\alpha^2}. \quad (4.21)$$

The first integrals are then given by

$$\frac{\Lambda^2}{C^2} R'^2 = -(R^2 - b^2) \left[1 - \frac{2}{p} R + \frac{1}{ap} R^2 \right], \quad (4.22a)$$

$$\frac{\Lambda^2}{C^2} \sigma'^2 = 1 + 2\beta \frac{b}{p} \cos \sigma + \frac{b^2}{ap} \cos^2 \sigma. \quad (4.22b)$$

We now introduce a change of variable u into Equation 4.22a by letting

$$u = \frac{1}{R}, \text{ hence } R' = -\frac{u'}{u^2},$$

substituting this into Equation 4.22a and completing the square, we obtain

$$\frac{\Lambda^2}{C^2} u'^2 = -(1 - b^2 u^2) \left[\left(u - \frac{1}{p} \right)^2 - \frac{1}{p^2} \left(1 - \frac{p}{a} \right) \right]. \quad (4.23)$$

Following the pattern of the Kepler case, we define the eccentricity e as

$$e = \sqrt{1 - \frac{p}{a}} = \sqrt{1 - \frac{2\alpha^2 C^2}{\mu^2}},$$

which leaves Equation 4.23 as

$$\frac{\Lambda^2}{C^2} u'^2 = (1 - b^2 u^2) \left[\frac{e^2}{p^2} - \left(u - \frac{1}{p} \right)^2 \right].$$

Next, we simplify by introducing a substitution

$$\left(u - \frac{1}{p}\right) = \frac{e}{p}v, \text{ where } u' = \frac{e}{p}v'$$

which gives the form

$$\frac{\Lambda^2}{C^2}v'^2 = (1 - v^2) \left[1 - \frac{b^2}{p^2}(1 + ev)^2\right].$$

Where once again we introduce a dimensionless parameter η , which is a ratio between separation of the two fixed centres b and characteristic length scale p

$$\eta^2 = \frac{b^2}{p^2} = \frac{b^2}{a^2(1 - e^2)^2} = \frac{b^2}{a^2} \cdot \frac{1}{l^2},$$

where, for ease of calculation we let $(1 - e^2) = l$. Hence the equation takes the form

$$\begin{aligned} \frac{\Lambda^2}{C^2}v'^2 &= (1 - v^2) [1 - \eta^2(1 + ev)^2] \\ &= (1 - v^2) [(1 - \eta)^2 - 2\eta^2ev - \eta^2e^2v^2]. \end{aligned} \quad (4.24)$$

This now takes the form of an elliptic integral, although it is slightly awkward since one must perform a Möbius transformation to recast it into a standard form, as we shall see in the next section. Now we focus our attention on the constant of motion associated to σ . Multiplying it through with $\sin^2 \sigma$, we obtain

$$\frac{\Lambda^2}{C^2}\sigma'^2 \sin^2 \sigma = \sin^2 \sigma + 2\beta \frac{b}{p} \sin^2 \sigma \cos \sigma + \frac{b^2}{ap} \sin^2 \sigma \cos^2 \sigma.$$

This suggests a substitution of $S = \cos \sigma$, where $S' = \sigma' \sin \sigma$ which leads to the form

$$\frac{\Lambda^2}{C^2}S'^2 = (1 - S^2) [1 + 2\eta\beta S + \eta^2 l S^2]. \quad (4.25)$$

We now observe that both the constants of motion are of a generic form that looks like

$$\frac{\Lambda^2}{C^2}y' = (1 - y^2) [(1 - d^2) + 2sy + qy^2], \quad (4.26)$$

which can be transformed into a form amenable to solution using Jacobi elliptic functions.

Remark 16. It's worth remarking if $e = 0$ (and hence $p = a$), then u becomes constant; in particular, $u = \frac{1}{p}$, giving solution of the form

$$\frac{x^2}{p^2(1 - \eta^2)} + \frac{z^2}{p^2} = 1.$$

Moreover, if η^1 is specified by:

¹ $\eta^2 > 1$ violates the chosen coordinate system.

- $\eta = 0$, we get circular motion,
- $\eta^2 = 1$, the orbit is on the z axis and is in a collision orbit (we will not discuss this case as we are interested in bounded orbits),
- $\eta^2 < 1$, the orbit is a closed elliptic orbit.

Now that we have completed the reduction of the equations for S and R we can proceed with analysis of solutions for [Equation 4.26](#).

4.3.2 Generic Analysis via Möbius Transformation

In this section, we outline the procedure for transforming the generic ODE of the form given in [Equation 4.26](#) into the standard elliptic form:

$$\frac{\Lambda^2}{C^2} Y'^2 = (1 - Y^2)(A + BY^2), \quad (4.27)$$

which allows solutions to be expressed in terms of the Jacobi elliptic functions sn and cn (see [appendix B](#) for Definitions of Jacobi elliptic functions). To achieve this transformation, we apply a Möbius transformation. The steps for constructing this transformation are as follows:

1. **Factorizing $(1 - y^2)$:** Rewrite the term $(1 - y^2)$ in [Equation 4.26](#) as

$$(1 - y^2) = J^2 \left[(1 - \delta y)^2 - (y - \delta)^2 \right],$$

where the parameters J and δ must satisfy the relation

$$J^2(1 - \delta^2) = 1. \quad (4.28)$$

2. **Rewriting the quadratic term:** Express the quadratic factor in [Equation 4.26](#) as

$$(1 - d^2) + 2sy + qy^2 = J^2 \left[A(1 - \delta y)^2 + B(y - \delta)^2 \right].$$

Equating coefficients yields relations between the parameters (see equations in [Appendix C](#)).

3. **Solving for transformation parameters:** By appropriately combining these relations and introducing the intermediary parameter

$$\rho^2 := \left(\frac{1 - \delta}{1 + \delta} \right)^2,$$

we obtain explicit expressions for δ , A , B , and J (see [Appendix C](#) for details).

4. **Applying the Möbius transformation:** The transformation is given by

$$Y = \frac{y - \delta}{1 - \delta y}, \quad \Longleftrightarrow \quad y = \frac{Y + \delta}{1 + \delta Y}. \quad (4.29)$$

A straightforward calculation shows that

$$Y'^2 = \frac{y'}{J^2(1 - \delta y)^2},$$

which transforms the equation into the required form of [Equation 4.27](#). The intermediary parameters provide a means to express the transformed ODE's coefficients A and B .

To be self-contained, we explicitly state the relations between these parameters:

$$h = \frac{1}{2} \left[1 - \sqrt{1 - \frac{4s^2}{[(1 - d^2) + q]^2}} \right], \quad (4.30a)$$

$$A + B = [(1 - d^2) + q][1 - 2h], \quad A - B = [(1 - d^2) - q], \quad (4.30b)$$

$$\delta = \frac{1 - \rho}{1 + \rho} = \frac{1 - \rho^2}{(1 + \rho^2)^2} = -\frac{s}{[(1 - d^2) + q][1 - h]}. \quad (4.30c)$$

Remark 17. The parameter h is introduced to simplify calculations, while the remaining parameters have been briefly outlined in the steps above.

The detailed algebraic steps for equating coefficients and solving for the parameters are omitted here; see [Appendix C](#) or the reference text [\[8\]](#) for a complete derivation.

4.3.3 Explicit Parameter Ranges for S

Having done the generic analysis for ODEs of the form of [Equation 4.26](#), we now solve the ODEs for $S(\sigma)$ and $v(R)$. Starting with S ; recall [Equation 4.25](#)

$$\frac{\Lambda^2}{C^2} S'^2 = (1 - S^2) [1 + 2\eta\beta S + \eta^2 l S^2], \quad (4.31)$$

here comparing with [Equation 4.26](#) we have

$$d^2 = 0, \quad s = \eta\beta, \quad q = \eta^2(1 - e^2). \quad (4.32)$$

Thus h for this now reads

$$h_S = \frac{1}{2} \left[1 - \sqrt{1 - \frac{4\eta^2\beta^2}{[1 + \eta^2(1 - e^2)]^2}} \right], \quad (4.33)$$

where we introduce the notation h_S to specify that this is the parameter h used in [Section 4.3.2](#) for the particular ODE governing S . This notation will be con-

sistently applied to all relevant parameters introduced in section 4.3.2. Applying the procedure outlined there, we obtain

$$1 - h_S = \frac{1}{2} \left[1 + \sqrt{1 - \frac{4\eta^2\beta^2}{[1 + \eta^2(1 - e^2)]^2}} \right], \quad (4.34)$$

$$\delta_S = -\frac{2\eta\beta}{[1 + \eta^2(1 - e^2)] + \sqrt{[1 + \eta^2(1 - e^2)]^2 - 4\eta^2\beta^2}}. \quad (4.35)$$

The parameters A_S and B_S are now given by

$$A_S + B_S = \sqrt{[1 + \eta^2(1 - e^2)]^2 - 4\beta^2\eta^2}, \quad (4.36)$$

$$A_S - B_S = 1 - \eta^2(1 - e^2). \quad (4.37)$$

Solving for A_S and B_S , we find

$$A_S = \frac{1}{2}[1 - \eta^2(1 - e^2)] + \frac{1}{2}\sqrt{[1 + \eta^2(1 - e^2)]^2 - 4\eta^2\beta^2}, \quad (4.38)$$

$$B_S = -\frac{1}{2}[1 - \eta^2(1 - e^2)] + \frac{1}{2}\sqrt{[1 + \eta^2(1 - e^2)]^2 - 4\eta^2\beta^2}. \quad (4.39)$$

Setting a new variable via a Möbius transform

$$\zeta = \frac{S - \delta_S}{1 - \delta_S S}, \quad S = \frac{\zeta + \delta_S}{1 + \delta_S \zeta}, \quad (4.40)$$

we obtain,

$$\frac{\Lambda^2}{C^2} \zeta'^2 = (1 - \zeta^2)[A_S + B_S \zeta^2]. \quad (4.41)$$

As pointed out in Section 4.3.2, this has solutions in the form of Jacobi elliptic functions, pending the sign of B_S . Considering $B_S \lesseqgtr 0$, we derive the condition

$$\beta^2 \lesseqgtr 1 - e^2, \quad (4.42)$$

thus we only need to consider two cases:

$$B_S \lesseqgtr 0 \quad \text{as} \quad \beta^2 + e^2 \lesseqgtr 1. \quad (4.43)$$

Remark 18. A subtlety arises in ensuring that the condition

$$[1 + \eta^2(1 - e^2)]^2 \geq 4\eta^2\beta^2 \quad (4.44)$$

is satisfied. While we do not address this case in detail due to space constraints, the key idea is to define $\lambda = \frac{b}{a}/(\eta(1 - e^2))$ and analyse the corresponding quadratic inequality. By examining the roots, we can determine the parameter regions where our solutions remain valid. For a more comprehensive discussion, see [8].

Case I: $\beta^2 + e^2 \geq 1$, this is the case where B_S is negative. Now recall we

have a free parameter Λ in Equation 4.25, we will set it as

$$\Lambda_1^2 := \Lambda^2 = C^2 A_S, \quad (4.45)$$

we then define a parameter k_{S1} as follows

$$k_{S1}^2 = -\frac{B_S}{A_S}. \quad (4.46)$$

Now Equation 4.41 becomes

$$\zeta'^2 = (1 - \zeta^2)[1 - k_{S1}^2 \zeta^2], \quad (4.47)$$

which has the solution in terms of Jacobi elliptic functions sn of modulus k_{S1} (see appendix B.4), where explicitly it is given by

$$\zeta = \text{sn}[f + f_0 : k_{S1}].$$

Here, f_0 is the constant of integration. Now via the Möbius transformation, we get S

$$S = \cos \sigma = \frac{\text{sn}[f + f_0 : k_{S1}] + \delta_S}{1 + \delta_S \text{sn}[f + f_0 : k_{S1}]}$$

If we let $-\omega$ denote the value of f at which orbit makes its first crossing of the x -axis, then

$$z = 0, \quad S = 0 \quad \text{for } f = -\omega,$$

which gives the relation to determine the constant of integration

$$\text{sn}[f_0 - \omega : k_{S1}] + \delta_S = 0.$$

Case II: $\beta^2 + e^2 \leq 1$, in this case, B_S is positive, thus allowing Equation 4.41 to be integrated. We write the equation in the form

$$\frac{\Lambda^2}{C^2} \zeta'^2 = (1 - \zeta^2) [A_S + B_S - B_S(1 - \zeta^2)] \quad (4.48)$$

$$= (A_S + B_S)(1 - \zeta^2) \left[1 - \frac{B_S}{A_S + B_S} (1 - \zeta^2) \right]. \quad (4.49)$$

Once again, we choose to set the parameter Λ and k_{S2} as convenient by setting

$$\Lambda_2^2 := \Lambda = C^2 (A_S + B_S),$$

$$k_{S2}^2 = \frac{B_S}{A_S + B_S},$$

where we have used the subscript 2 to keep track of the case for Λ . Now, Equation 4.41 reads

$$\zeta'^2 = (1 - \zeta^2)[1 - k_{S2}^2(1 - \zeta^2)]$$

which gives rise to the solution

$$\zeta = \text{cn}[f + f_0 - K_{S_2} : k_{S_2}]. \quad (4.50)$$

K_{S_2} above is the quarter period of the Jacobi elliptic function of modulus k_{S_2} . Inverting the Möbius transform for S we obtain

$$S = \frac{\text{cn}[f + f_0 - K_{S_2} : k_{S_2}] + \delta_S}{1 + \delta_S \text{cn}[f + f_0 - K_{S_2} : k_{S_2}]},$$

using the relation given by Equation B.7a we get

$$\text{cn}[f + f_0 - K_{S_2} : k_{S_2}] = k'_{S_2} \frac{\text{sn}[f + f_0 : k_{S_2}]}{\text{dn}[f + f_0 : k_{S_2}]}$$

where $k'^2_{S_2} = (1 - k^2_{S_2})$. Finally we get the form

$$S = \cos \sigma = \frac{k'_{S_2} \text{sn}[f + f_0 : k_{S_2}] + \delta_S \text{dn}[f + f_0 : k_{S_2}]}{\text{dn}[f + f_0 : k_{S_2}] + \delta_S k'_{S_2} \text{sn}[f + f_0 : k_{S_2}]} \quad (4.51)$$

If we impose the same initial condition as case I ($z = 0$, $S = 0$ when $f = -\omega$), f_0 can be determined via

$$k'_{S_2} \text{sn}[f_0 - \omega : k_{S_2}] + \delta_S \text{dn}[f_0 - \omega : k_{S_2}] = 0.$$

This completes the solution for S , we may now move on to Equation 4.24 for $v(R)$.

4.3.4 Explicit Parameter Ranges for R and Plots

Recall Equation 4.24 and the procedure from Section 4.3.2 for solving these types of equations in general. The parameters for v are given by

$$d^2 = \eta^2, \quad s = -\eta^2 e, \quad q = -\eta^2 e^2.$$

Using the subscript v for parameters defined in Equation 4.24, we obtain

$$h_v = \frac{1}{2} \left[1 - \sqrt{1 - \frac{4\eta^4 e^2}{[1 - \eta^2(1 + e^2)]^2}} \right],$$

$$1 - h_v = \frac{1}{2} \left[1 + \sqrt{1 - \frac{4\eta^4 e^2}{[1 - \eta^2(1 + e^2)]^2}} \right].$$

From this, we define δ_v as

$$\delta_v = \frac{2\eta^2 e}{[1 - \eta^2(1 + e^2)] + \sqrt{[1 - \eta^2(1 + e^2)]^2 - 4\eta^4 e^2}},$$

which leads to

$$\begin{aligned} A_v + B_v &= \sqrt{[1 - \eta^2(1 - e^2)]^2 - 4\eta^4 e^2}, \\ A_v - B_v &= 1 - \eta^2(1 - e^2). \end{aligned}$$

Solving for A_v and B_v , we obtain

$$\begin{aligned} A_v &= \frac{1}{2}[1 - \eta^2(1 - e^2)] + \frac{1}{2}\sqrt{[1 - \eta^2(1 - e^2)]^2 - 4\eta^4 e^2}, \\ B_v &= -\frac{1}{2}[1 - \eta^2(1 - e^2)] + \frac{1}{2}\sqrt{[1 - \eta^2(1 - e^2)]^2 - 4\eta^4 e^2}. \end{aligned}$$

Here, we note that A_v is always positive, whereas B_v is always negative or zero.

Applying the Möbius transformation with a new variable w , Equation 4.24 transforms into

$$\frac{\Lambda^2}{C^2} w'^2 = (1 - w^2)[A_v + B_v w^2], \quad \text{where} \quad w = \frac{v - \delta_v}{1 - \delta_v v}. \quad (4.52)$$

Rewriting it in a more favourable form,

$$\frac{\Lambda^2}{A_v^2 C^2} w'^2 = (1 - w^2) \left[1 + \frac{B_v}{A_v} w^2 \right].$$

Unlike in the case of S , we cannot set Λ freely, so we must consider the two cases for S separately:

Case I ($e^2 + \beta^2 \geq 1$):

$$\frac{1}{j_{v1}^2} := \frac{\Lambda_1^2}{C^2 A_v^2}.$$

Case II ($e^2 + \beta^2 \leq 1$):

$$\frac{1}{j_{v2}^2} := \frac{\Lambda_2^2}{C^2 A_v^2}.$$

For convenience, we define parameters j_{v1} and j_{v2} and introduce another parameter k_v as

$$k_v^2 = -\frac{B_v}{A_v}.$$

Since both cases affect the solution of R only through the constants j_{v1} and j_{v2} , we define a general parameter j_v covering both cases:

$$j_v = \begin{cases} j_{v1}, & \text{if } e^2 + \beta^2 \geq 1, \\ j_{v2}, & \text{if } e^2 + \beta^2 \leq 1. \end{cases} \quad (4.53)$$

Rewriting Equation 4.52 in terms of j_v

$$\frac{w'^2}{j_v^2} = (1 - w^2)[1 - k_v^2 w^2], \quad (4.54)$$

and defining a rescaled angle $f_v = j_v f$, we obtain

$$\left(\frac{dw}{df_v}\right)^2 = (1 - w^2)[1 - k_v^2 w^2].$$

This has a known solution in terms of the Jacobi elliptic function sn with modulus k_v given by

$$w = \text{sn}[f_v + f_{v0} : k_v].$$

Applying the inverse Möbius transform, we find the solution for v

$$v = \frac{\text{sn}[f_v + f_{v0} : k_v] + \delta_v}{1 + \delta_v \text{sn}[f_v + f_{v0} : k_v]}.$$

Differentiating gives

$$v' = j_v^2 (1 - \delta_v^2) \frac{\text{cn}[f_v + f_{v0} : k_v] \text{dn}[f_v + f_{v0} : k_v]}{[1 + \delta_v \text{sn}[f_v + f_{v0} : k_v]]^2},$$

where applying the initial condition $f = 0$ ($f_v = 0$), we set

$$\text{cn}[f_{v0} : k_v] = 0.$$

Using [Equation B.7a](#) we may be able to deduce

$$f_{v0} = K_v,$$

where K_v is a quarter-period of the Jacobi elliptic function. This simplifies the solution

$$w = \frac{\text{cn}[f_v : k_v]}{\text{dn}[f_v : k_v]}.$$

Thus, the solution for v is

$$v = \frac{\text{cn}[f_v : k_v] + \delta_v \text{dn}[f_v : k_v]}{\text{dn}[f_v : k_v] + \delta_v \text{cn}[f_v : k_v]},$$

transforming from v to u , we obtain

$$u = \frac{1}{p} \left[\frac{(1 + e\delta_v) \text{dn}[f_v : k_v] + (e + \delta_v) \text{cn}[f_v : k_v]}{\text{dn}[f_v : k_v] + \delta_v \text{cn}[f_v : k_v]} \right].$$

Thus, the explicit solution for R is

$$R = p \frac{\text{dn}[f_v : k_v] + \delta_v \text{cn}[f_v : k_v]}{(1 + e\delta_v) \text{dn}[f_v : k_v] + (e + \delta_v) \text{cn}[f_v : k_v]}.$$

This Completes the solution for R . Before we move on, it's worth briefly looking at a few examples of Case I orbits and Case II orbits in the (x, z) plane. For case I orbits we tend to see for small β the orbits looks like rotating ellipses as seen

in figures 4.3. Whilst for case II orbits we see almost overlapping nearby ellipses and in a special case when $\beta = 0$ we see figure 8-like orbits as seen in figures 4.4.

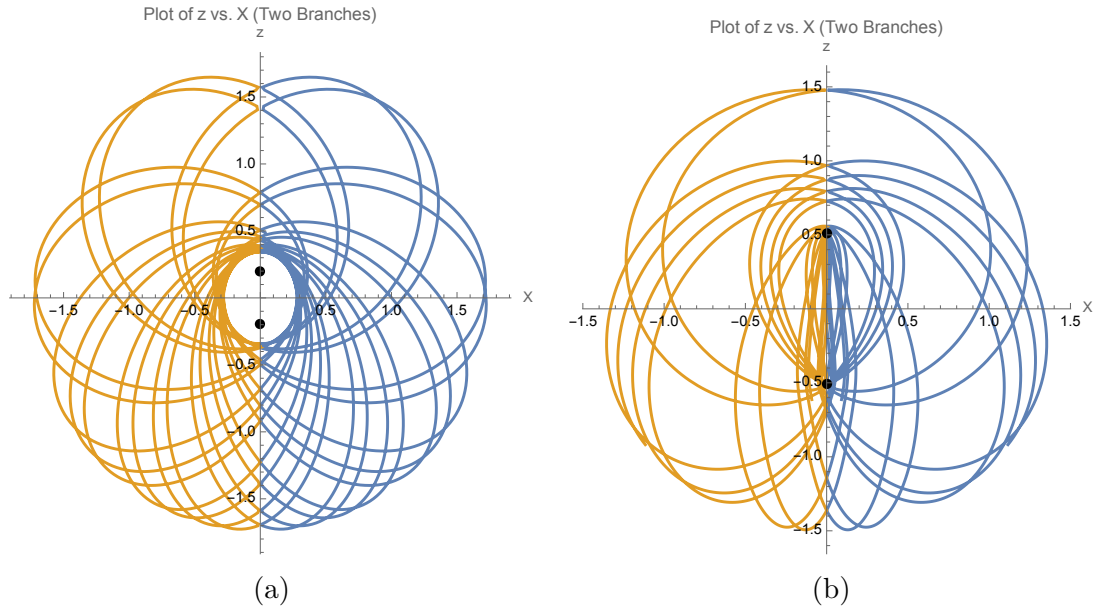


Figure 4.3: Examples of Case I orbits via Mathematica code appendix D.

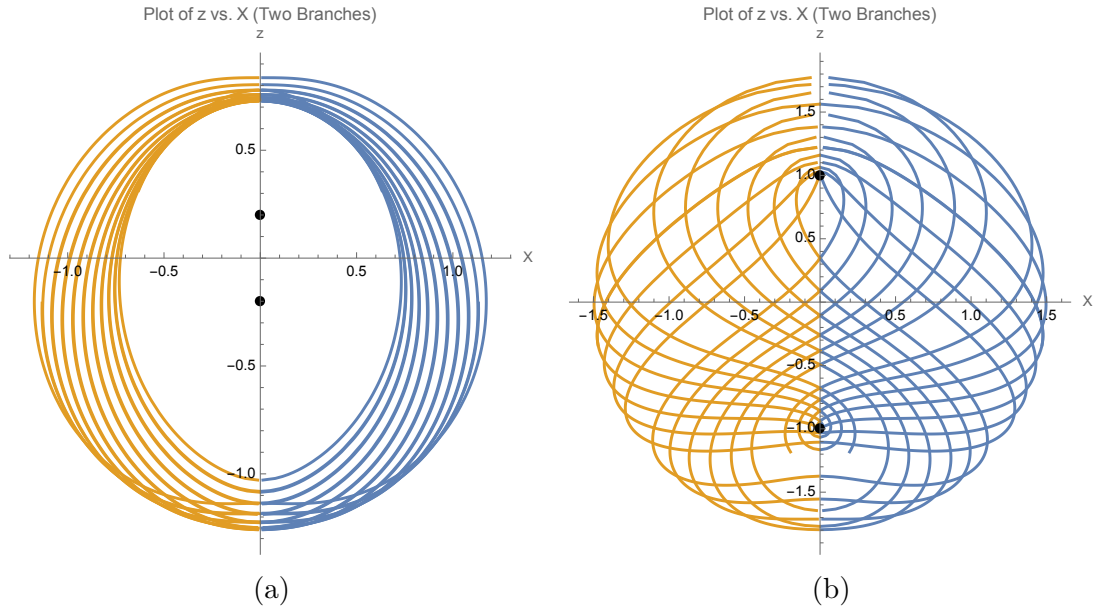


Figure 4.4: Examples of Case II orbits via Mathematica code appendix D.

Remark 19. A few quick remarks here, First recall that the coordinate system was defined as

$$r \sin \theta = x := \pm \sqrt{R^2 - b^2} \sin \sigma, \quad r \cos \theta = z := R \cos \sigma.$$

Thus, for each set of σ and S , we obtain $(\pm x, z)$, hence the orange and blue colours in the plots in figures 4.3 & 4.4. This leads to us being limited to looking at symmetric orbit trajectories this is of course not the only possible orbits that

could physically occur but the code created was limited to it. Second while it would be appropriate to include the time angle analysis for this problem we opted not to for the sake of space and because the primary interest was the solutions of trajectories.

4.4 Action-Angle Variables for Euler

Having solved for the trajectories we now try determining the action-angle variables for this problem. First recall the two constants of motion: the Hamiltonian and the constant associated with p_σ . Following the approach in Section 4.2.1, we focus on compact level sets, thus we get

$$\frac{1}{2} \left(\frac{p_R^2(R^2 - b^2)}{R^2 - b^2 \cos^2 \sigma} + \frac{p_\sigma^2}{R^2 - b^2 \cos^2 \sigma} \right) - \mu \frac{R + \beta b \cos \sigma}{R^2 - b^2 \cos^2 \sigma} = -\alpha^2, \quad (4.55)$$

$$\frac{p_\sigma^2}{2} - \alpha^2 b^2 \cos^2 \sigma + \mu \beta b \cos \sigma = \frac{C^2}{2}. \quad (4.56)$$

Recognising that the Hamiltonian is time-independent, we apply Proposition 2.2.3 and use an ansatz of the form $W(R, \sigma) = W_R(R) + W_\sigma(\sigma)$ in the Hamilton-Jacobi equation

$$H \left(q_i, \frac{\partial W}{\partial q_i}, t \right) = -\alpha^2, \quad \text{where } p_i = \frac{\partial W}{\partial q_i}. \quad (4.57)$$

Rearranging, we obtain

$$-\left(\frac{\partial W_R}{\partial R} \right)^2 (R^2 - b^2) + 2\mu R - 2\alpha^2 R^2 = \left(\frac{\partial W_\sigma}{\partial \sigma} \right)^2 - 2\mu \beta b \cos \sigma - 2\alpha^2 b^2 \cos^2 \sigma. \quad (4.58)$$

Separating variables yields

$$\frac{\partial W_R}{\partial R} = \sqrt{\frac{-2\alpha^2 R^2 + 2\mu R - C^2}{R^2 - b^2}}, \quad (4.59a)$$

$$\frac{\partial W_\sigma}{\partial \sigma} = \sqrt{2\alpha^2 b^2 \cos^2 \sigma + 2\mu \beta b \cos \sigma + C^2}. \quad (4.59b)$$

Thus, the action coordinates I_i are given by

$$I_R = \frac{1}{2\pi} \int_{\gamma_R} p_R dR = \frac{1}{2\pi} \int_{\gamma_R} \sqrt{\frac{-2\alpha^2 R^2 + 2\mu R - C^2}{R^2 - b^2}} dR, \quad (4.60a)$$

$$I_\sigma = \frac{1}{2\pi} \int_{\gamma_\sigma} p_\sigma d\sigma = \frac{1}{2\pi} \int_{\gamma_\sigma} \sqrt{2\alpha^2 b^2 \cos^2 \sigma + 2\mu \beta b \cos \sigma + C^2} d\sigma. \quad (4.60b)$$

To evaluate these integrals, it is necessary to understand the integration contours in phase space. For I_R , similar to I_r in the Kepler problem (see Section 3.4), we require that R have real roots in the numerator under the square root. Therefore, the contour is defined by integrating from R_- to R_+ and multiplying by 2, where R_{\pm} are the roots of the quadratic. Note that, by our choice of coordinates, $R > b$ so the denominator poses no issue. For I_{σ} , the integral runs from 0 to π because $\sigma \in [0, \pi]$.

Remark 20. Explicit solutions for these coordinates exist and can be found via [3] (equation 256.19). Although this evaluates to a combination of elliptic functions, we found it less useful for discussion in this report since it proved difficult to obtain the Hamiltonian via that approach.

We shall instead aim to look close to the Kepler limit (i.e. b is small) and seek a solution in terms of a series expansion via a Taylor series. To proceed, note that by our choice of coordinates $R > b$. Therefore, since $\frac{b}{R} < 1$ for small b , we may expand the variable $\frac{b^2}{R^2}$ in a Taylor series:

$$\begin{aligned} \frac{1}{\sqrt{R^2 - b^2}} &= R^{-1} \left(1 - \frac{b^2}{R^2} \right)^{-\frac{1}{2}}, \\ &= R^{-1} \sum_{k=0}^{\infty} \left((-1)^{2k} \frac{(1+2k)!!}{2^k k!} \left(\frac{b^{2k}}{R^{2k}} \right) \right). \end{aligned} \quad (4.61)$$

Substituting this series into the expression for I_R yields

$$I_R = \frac{\alpha}{\pi} \int_{R_-}^{R_+} \sqrt{-2R^2 + \frac{2\mu}{\alpha^2}R - \frac{C^2}{\alpha^2}} \left(\sum_{k=0}^{\infty} \frac{(1+2k)!!}{2^k k!} \frac{b^{2k}}{R^{2k+1}} \right) dR.$$

Similar to the Kepler case, one may solve a complex contour integral that encloses the complement of the branch cut region from R_- to R_+ (i.e., perform a residue calculation at $R = 0$ and as $R \rightarrow \infty$). However, we observe that pole as $R \rightarrow \infty$ only contributes to leading order specifically, the $\frac{1}{R}$ term in the expansion. In fact, the leading order term is the Kepler term, as it is independent of b and evaluates, as before, to

$$\frac{\alpha}{\pi} \int_{R_-}^{R_+} \frac{\sqrt{-2R^2 + 2\frac{\mu}{\alpha^2}R - \frac{C^2}{\alpha^2}}}{R} dR = -C + \frac{\mu}{\sqrt{2\alpha^2}}.$$

It is now sufficient to evaluate the residue of the k^{th} order term at $R = 0$ for a solution for I_R . To that end, we employ the residue formula:

$$\text{Res}(I_R, 0) = \frac{1}{(2k)!} \lim_{R \rightarrow 0} \left(\frac{(1+2k)!! b^{2k}}{2^k k!} \frac{d^{2k}}{dR^{2k}} \left[R^{2k+1} \frac{\sqrt{-2R^2 + 2\frac{\mu}{\alpha^2}R - \frac{C^2}{\alpha^2}}}{R^{2k+1}} \right] \right).$$

To determine the $2k^{\text{th}}$ derivative, we use the general Leibniz rule:

$$\frac{d^k}{dx^k}[f(x)g(x)] = \sum_{n=0}^k \binom{k}{n} f^{(k-n)}(x)g^{(n)}(x),$$

for our purposes, we set

$$f(R) = \sqrt{R - R_-} \quad \text{and} \quad g(R) = \sqrt{R_+ - R}.$$

The n^{th} derivative of f is then given by

$$f^{(n)} = \frac{1}{2} \left(\frac{1}{2} - 1 \right) \cdots \left(\frac{1}{2} - (n-1) \right) (R - R_-)^{\frac{1}{2}-n} = \frac{(-1)^{n-1} (2n-3)!!}{2^n} (R - R_-)^{\frac{1}{2}-n}.$$

Since this expression is only valid for $n > 1$, we may alternatively use the Gamma function as an analytic continuation of the factorial (see appendix B.4). In general, the relationship between the double factorial and the Gamma function is

$$z!! = \sqrt{\frac{2}{\pi}} 2^{\frac{z}{2}} \Gamma\left(\frac{z}{2} + 1\right). \quad (4.62)$$

Thus $f^{(n)}$ and $g^{(n)}$ can be expressed as

$$f^{(n)} = \frac{(-1)^{n-1} \Gamma\left(n - \frac{1}{2}\right)}{\sqrt{\pi}} (R - R_-)^{\frac{1}{2}-n},$$

$$g^{(n)} = \frac{-\Gamma\left(n - \frac{1}{2}\right)}{\sqrt{\pi}} (R_+ - R)^{\frac{1}{2}-n}.$$

Returning to the residue calculation, we have

$$\begin{aligned} \text{Res}(I_R, 0) &= \frac{1}{(2k)!} \frac{(1+2k)!! b^{2k}}{2^k k!} \lim_{R \rightarrow 0} \left(\frac{d^{2k}}{dR^{2k}} \left[\sqrt{-R^2 + 2\frac{\mu}{\alpha^2} R - \frac{C^2}{\alpha^2}} \right] \right) \\ &= \frac{1}{(2k)!} \frac{(1+2k)!! b^{2k}}{2^k k!} \sum_{n=0}^{2k} \binom{2k}{n} \frac{(-1)^n \Gamma\left(2k - n - \frac{1}{2}\right) \Gamma\left(n - \frac{1}{2}\right)}{\pi} (-R_-)^{\frac{1}{2}-2k+n} (R_+)^{\frac{1}{2}-n}, \end{aligned}$$

This gives the general expansion for I_R :

$$\begin{aligned} I_R &= -C + i\alpha \sum_{k=0}^{\infty} \frac{1}{(2k)!} \frac{(1+2k)!! b^{2k}}{2^k k!} \sum_{n=0}^{2k} \binom{2k}{n} \\ &\quad \times \frac{(-1)^n \Gamma\left(2k - n - \frac{1}{2}\right) \Gamma\left(n - \frac{1}{2}\right)}{\pi} \\ &\quad \times (-R_-)^{\frac{1}{2}-2k+n} (R_+)^{\frac{1}{2}-n}. \end{aligned} \quad (4.63)$$

Here, we do not worry about the factor i since we have the term $(-R_-)^{\frac{1}{2}-2k+n}$ and we define the branch cut from R_- to R_+ .

Next, we shift our focus to the I_σ integrand. Making the substitution $x = \cos \sigma$

yields

$$I_\sigma = \frac{\sqrt{2}\alpha}{2\pi} \int_{-1}^1 \frac{\sqrt{x^2 b^2 + \frac{\mu\beta}{\alpha^2}bx + \frac{C^2}{2\alpha^2}}}{\sqrt{1-x^2}} dx.$$

Since $\sigma \in [0, \pi]$ implies $|x| < 1$ and near the Kepler limit b is small so that $xb \ll 1$, we may expand in a Taylor series about 0. Using a procedure similar to that for I_R and letting x_\pm be the roots of the quadratic in the I_σ integrand, we have

$$\begin{aligned} \sqrt{x^2 b^2 + \frac{\mu\beta}{\alpha^2}bx + \frac{C^2}{2\alpha^2}} &= \sqrt{(xb - x_-)(xb - x_+)} \\ &= \sum_{k=0}^{\infty} \frac{(xb)^k}{k!} \sum_{n=0}^k \binom{k}{n} \frac{(-1)^k \Gamma(k - n - \frac{1}{2}) \Gamma(n - \frac{1}{2})}{\pi} (-x_-)^{\frac{1}{2}-k+n} (-x_+)^{\frac{1}{2}-n}. \end{aligned}$$

Putting everything together, we obtain

$$I_\sigma = \int_{-1}^1 \sum_{k=0}^{\infty} \frac{(xb)^k}{k! \sqrt{1-x^2}} \sum_{n=0}^k \binom{k}{n} \frac{(-1)^k \Gamma(k - n - \frac{1}{2}) \Gamma(n - \frac{1}{2})}{\pi} (-x_-)^{\frac{1}{2}-k+n} (-x_+)^{\frac{1}{2}-n} dx.$$

This is now aided by the following identity

$$\int_{-1}^1 \frac{x^n}{\sqrt{1-x^2}} = \begin{cases} \frac{\Gamma(n + \frac{1}{2})\sqrt{\pi}}{2\Gamma(n+1)}, & \text{if } n \text{ is even.} \\ 0, & \text{if } n \text{ is odd.} \end{cases} \quad (4.64)$$

Since only even powers contribute, we may replace k by $2k$, leading to the explicit series

$$I_\sigma = \frac{\sqrt{2}\alpha}{2\pi} \sum_{k=0}^{\infty} \frac{b^{2k} \sqrt{\pi} \Gamma(2k + \frac{1}{2})}{2(2k)! \Gamma(2k+1)} \sum_{n=0}^{2k} \binom{2k}{n} \frac{\Gamma(2k - n - \frac{1}{2}) \Gamma(n - \frac{1}{2})}{\pi} (-x_-)^{\frac{1}{2}-2k+n} (-x_+)^{\frac{1}{2}-n}.$$

A quick sanity check shows that, at leading order (i.e., $k = 0$) the value is C , which is indeed the expected Kepler limit. In general, we have not been able to find the Hamiltonian as directly as in the Kepler case. However, we may extract H in a perturbative manner by considering terms up to $\mathcal{O}(b^2)$. The action coordinates are then given by

$$I_R = -C + \frac{\mu}{\sqrt{2}\alpha^2} - \frac{3b^2}{16} \left(\frac{-2\alpha^2}{C} + \frac{\mu^2}{C^3} \right) + \mathcal{O}(b^4), \quad (4.65a)$$

$$I_\sigma = C + \frac{3b^2}{64} \left(\frac{2\alpha^2}{C} - \frac{\mu^2\beta^2}{C^3} \right) + \mathcal{O}(b^4). \quad (4.65b)$$

At this stage, one may proceed either via implicit inversion or by applying Lagrange's inversion theorem². In the implicit method, ignoring higher-order terms

²See section 5.1 of chapter of [10] for the general procedure, we do not include it in this report for the sake of space.

(i.e., $\mathcal{O}(b^4)$) and rearranging for α^2 yields

$$\alpha^2 = \frac{C}{2} \left(\frac{64}{3b^2}(I_\sigma - C) + \frac{\mu^2 \beta^2}{C^3} \right).$$

For convenience, we define

$$X = \sqrt{2\alpha^2} = \sqrt{C \left(\frac{64}{3b^2}(I_\sigma - C) + \frac{\mu^2 \beta^2}{C^3} \right)},$$

rearranging the expression for I_R in (4.65a) gives

$$X = \frac{\mu}{I_R + C - \frac{3b^2}{16C}X^2 + \frac{3b^2}{16C^3}\mu^2},$$

and since $X = \sqrt{2\alpha^2}$, we finally obtain

$$\alpha = \frac{1}{\sqrt{2}} \frac{\mu}{I_R + C - \frac{3b^2}{16C}X^2 + \frac{3b^2}{16C^3}\mu^2}. \quad (4.66)$$

Recalling the level set $H = -\alpha^2$, we then have

$$H = -\frac{\mu^2}{2(I_R + C - \frac{3b^2}{16C}X^2 + \frac{3b^2}{16C^3}\mu^2)^2}.$$

Remark 21. At this stage, note that to obtain the Hamiltonian in terms of I_R and I_σ only, one must eliminate the C dependence. However, it is clear that in the limit $b \rightarrow 0$ (where $I_\sigma \rightarrow C$), the Kepler Hamiltonian is recovered.

To find an explicit relation between H and the action variables I_i , we first invert the relation for I_σ to express C as a function of I_σ . Using the Lagrange inversion theorem, suppose

$$I_\sigma = f(C) = C + b^2 g(C),$$

with b^2 as a small parameter and

$$g(C) = \frac{3}{64} \left(\frac{2\alpha^2}{C} - \frac{\mu^2 \beta^2}{C^3} \right),$$

then for $b = 0$ we have the base solution $C = I_\sigma$. Assuming an expansion

$$C = I_\sigma + c_1 b^2 + \mathcal{O}(b^4),$$

we substitute back into the definition of I_σ and match the coefficients order by order. At order b^2 , one finds

$$c_1 = -g(I_\sigma).$$

Thus, the inverted series is

$$C = I_\sigma - \frac{3b^2}{64} \left(\frac{2\alpha^2}{I_\sigma} - \frac{\mu^2 \beta^2}{I_\sigma^3} \right) + \mathcal{O}(b^4).$$

We may use the same idea once again by substituting C in terms of I_σ into [Equation 4.65a](#) and inverting for $\frac{\mu}{\sqrt{2\alpha^2}}$, we get

$$\frac{\mu}{\sqrt{2\alpha^2}} = (I_R + I_\sigma) - \frac{3b^2}{64} \left(\frac{2\mu^2}{I_\sigma(I_\sigma + I_R)^2} - \frac{\mu^2 \beta^2}{I_\sigma^3} \right) + \frac{3b^2}{16} \left(\frac{-2\mu^2}{I_\sigma(I_\sigma + I_R)^2} + \frac{\mu^2}{I_\sigma^3} \right) + \mathcal{O}(b^4).$$

Now squaring and rearranging for $\frac{1}{\alpha^2}$ we obtain

$$\frac{1}{2\alpha^2} = \frac{(I_R + I_\sigma)^2}{\mu^2} - 2(I_R + I_\sigma) \frac{3b^2}{64} \left(\frac{10}{I_\sigma(I_\sigma + I_R)^2} - \frac{(\beta^2 + 4)}{I_\sigma^3} \right) + \mathcal{O}(b^4),$$

for convenience we define the functions

$$G_0 = \frac{(I_R + I_\sigma)^2}{\mu^2},$$

$$G_1 = (I_R + I_\sigma) \frac{3}{32} \left(\frac{10}{I_\sigma(I_\sigma + I_R)^2} - \frac{(\beta^2 + 4)}{I_\sigma^3} \right) + \mathcal{O}(b^4).$$

Now ignoring higher order corrections and rearranging for $-\alpha^2$ we get

$$-\alpha^2 \approx \frac{-1}{2G_0} \left(1 - b^2 \frac{G_1}{G_0} \right)^{-1}.$$

Using a binomial expansion for small b^2 , we obtain

$$H := -\alpha^2 \approx -\frac{\mu^2}{2(I_R + I_\sigma)^2} - b^2 \frac{G_1}{2G_0^2} + \mathcal{O}(b^4).$$

Where once again we've gotten the Kepler Hamiltonian in terms of some corrections (as $b \rightarrow 0$, $I_\sigma \rightarrow I_\varphi$). While the implicit expression in some sense is cleaner to obtain, the explicit one makes it clear that the action variables for the Euler Problem could be seen as a perturbation on the Kepler problem. In fact, one sees that just the $\mathcal{O}(b^2)$ corrections were enough to lift the degeneracy of the orbital frequencies ω_i 's displaying the complexity in contrast to the Kepler case. This was expected as we no longer in general had an ellipse as an orbit but rather had rotating ellipses and more complicated orbits as seen in [figure 4.3](#).

Chapter 5

Conclusion

In this report, motivated by the study of integrable systems in celestial mechanics [8], we examined special cases of the two-body and three-body problems: the Kepler problem and the planar Euler problem. Beginning with an introduction to Hamiltonian mechanics and symplectic geometry, we established the framework for integrability through the Liouville-Arnold theorem.

We first analysed the Kepler problem, solving it using Lagrangian mechanics and ODE methods while verifying its integrability. This led naturally to the construction of action-angle variables, drawing on Classical Mechanics by Goldstein [5]. Our analysis confirmed that Keplerian orbits remain closed in configuration space due to degeneracies in orbital frequencies.

We then turned to the Planar Euler problem, where the system's unique geometry justified the use of prolate spheroidal coordinates. By applying Möbius transformations and Jacobi elliptic functions, we derived explicit solutions and utilised Mathematica code to visualise symmetric orbits. The solutions exhibited diverse behaviours, including rotating ellipses and figure-eight-like trajectories, consistent with findings in [8]. However, a deeper exploration of these behaviours proved difficult due to the limitations of the code.

Finally, we constructed action variables for the Planar Euler problem, revealing how its Hamiltonian in action variables could be understood as a perturbation of the Keplerian case. The addition of the correction terms broke the degeneracy in the frequencies leading to precessing orbits, one of the defining differences between the two problems.

While our study confirmed the integrability of both systems, it also highlighted the richness of the Euler problem, where even first-order corrections to the Kepler limit introduced significant qualitative changes. Given more time, we would have pursued an exact solution for the Hamiltonian in action-angle variables, investigated the Vinti problem, and refined our orbit plotters to capture non-symmetric trajectories. These extensions would further illuminate the interplay between two problems deepening our understanding of celestial mechanics.

Appendix A

Time Angle Relation Kelper

This appendix is dedicated to solving the time angle relation in the Kelper problem. Recall the time angle relation (Equation 3.12) and the solution for r (Equation 3.14):

$$\frac{dt}{df} = \frac{r^2}{J}, \quad r = \frac{J^2/\mu}{1 + \sqrt{1 - \frac{2J^2\alpha^2}{\mu^2}} \cos(f + \omega_0)}.$$

For consistency with [8], we introduce the following constants:

$$e := \sqrt{1 - \frac{2J^2\alpha^2}{\mu^2}}, \quad (\text{A.1})$$

$$a := \frac{J^2}{\mu(1 - e^2)}, \quad (\text{A.2})$$

$$n := \sqrt{\frac{\mu}{a^3}}. \quad (\text{A.3})$$

With these definitions, Equation 3.12 becomes

$$n \frac{dt}{df} = \frac{(1 - e^2)^{3/2}}{(1 + e \cos f)^2}. \quad (\text{A.4})$$

To integrate Equation A.4, note that

$$\frac{d}{df} \left[\frac{e \sin f}{1 + e \cos f} \right] = \frac{e^2 + e \cos f}{(1 + e \cos f)^2} = \frac{1}{1 + e \cos f} - \frac{1 - e^2}{(1 + e \cos f)^2}.$$

Multiplying both sides by $\sqrt{1 - e^2}$ gives

$$\frac{(1 - e^2)^{3/2}}{(1 + e \cos f)^2} = \frac{\sqrt{1 - e^2}}{1 + e \cos f} - \frac{d}{df} \left[\frac{e \sqrt{1 - e^2} \sin f}{1 + e \cos f} \right]. \quad (\text{A.5})$$

To integrate the first term on the right, define the auxiliary variable χ by

$$\tan \chi = \frac{\sqrt{1 - e^2} \sin f}{e + \cos f}. \quad (\text{A.6})$$

One may verify that

$$\cos \chi = \frac{e + \cos f}{1 + e \cos f}, \quad \sin \chi = \frac{\sqrt{1 - e^2} \sin f}{1 + e \cos f}.$$

Differentiating Equation A.6 with respect to f yields

$$\chi' = \frac{\sqrt{1 - e^2}}{1 + e \cos f}. \quad (\text{A.7})$$

Thus,

$$\int \frac{\sqrt{1 - e^2}}{1 + e \cos f} df = \chi = \arctan \left[\frac{\sqrt{1 - e^2} \sin f}{e + \cos f} \right]. \quad (\text{A.8})$$

Combining Equation A.4, Equation A.5, and Equation A.8, we obtain the integrated form:

$$M := n(t - t_0) = \arctan \left[\frac{\sqrt{1 - e^2} \sin f}{e + \cos f} \right] - \frac{e\sqrt{1 - e^2} \sin f}{1 + e \cos f}, \quad (\text{A.9})$$

where t_0 is an integration constant and M is known as the mean anomaly [8].

Appendix B

Special Functions

This appendix is dedicated to a concise overview of the special functions used throughout this report. Rather than providing a comprehensive introduction, we highlight key definitions, properties, and identities that play a central role in our analysis.

B.1 Jacobi Elliptic Functions

This section is primarily based on Chapter 16 of [1]. The Jacobi elliptic functions form a class of doubly periodic meromorphic functions that generalize the trigonometric functions to the setting of elliptic curves. They depend on the elliptic modulus k and its complementary modulus k' , where

$$k'^2 = 1 - k^2,$$

or equivalently,

$$k^2 + k'^2 = 1.$$

Remark 22. In order to maintain consistency with [8], we have modified the notation originally used in [1].

The complete elliptic integrals of the first kind are defined by

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}, \quad (\text{B.1})$$

$$iK'(k) = i \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k'^2 \sin^2 \theta}}. \quad (\text{B.2})$$

These integrals represent the real and imaginary quarter-periods, respectively, and they establish a rectangular lattice structure in the complex plane.

B.1.1 Definition and Fundamental Identities

The Jacobi elliptic functions are introduced via the *amplitude* function φ defined by

$$u = \int_0^\varphi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}, \quad (\text{B.3})$$

where u is the *elliptic argument*. The three primary Jacobi elliptic functions are then defined as follows:

$$\text{sn}(u; k) = \sin \varphi, \quad (\text{B.4})$$

$$\text{cn}(u; k) = \cos \varphi, \quad (\text{B.5})$$

$$\text{dn}(u; k) = \sqrt{1 - k^2 \sin^2 \varphi}. \quad (\text{B.6})$$

Among the many identities these functions satisfy, the following are particularly useful:

$$\text{cn}[u \pm K(k); k] = \mp k' \frac{\text{sn}(u; k)}{\text{dn}(u; k)}, \quad (\text{B.7a})$$

$$\text{sn}[u \pm K(k); k] = \pm \frac{\text{cn}(u; k)}{\text{dn}(u; k)}. \quad (\text{B.7b})$$

B.2 Gamma Function

The Gamma function provides an analytic continuation of the factorial function to the complex plane. For non-negative integers n , the factorial is defined by

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 1, \quad \text{with } 0! = 1.$$

A related integral representation is given by

$$I(n) = \int_0^\infty e^{-t} t^n dt.$$

To extend this definition to complex numbers z (with $\text{Re}(z) > 0$ for convergence), we define the Gamma function by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad \text{Re}(z) > 0.$$

This definition satisfies the recurrence relation

$$\Gamma(z+1) = z \Gamma(z),$$

so that for natural numbers, $\Gamma(n+1) = n!$.

Some key identities involving the Gamma function are:

- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi},$

- The double factorial for z can be expressed as

$$z!! = \sqrt{\frac{2}{\pi}} 2^{\frac{z}{2}} \Gamma\left(\frac{z}{2} + 1\right).$$

Appendix C

Explicit Procedure

This appendix is to work out the intermediary steps for the Möbius transformation in the Euler problem. We begin by considering equations of the form

$$\frac{\Lambda^2}{C^2} y'^2 = (1 - y^2) \left[(1 - d^2) + 2sy + qy^2 \right],$$

for the variables S and v (with R related to v as defined above). It is convenient to transform the above equation (see [Equation 4.26](#)) into one of the form

$$\frac{\Lambda^2}{C^2} Y'^2 = (1 - Y^2) (A + BY^2), \quad (\text{C.1})$$

since we have a free parameter Λ (recall that Λ has units of angular momentum) which can be chosen so that the solutions are expressed in terms of elliptic functions. For example, one obtains

$$Y'^2 = (1 - Y^2)(1 - k_1^2 Y^2) \implies Y = \text{sn}[f + f_0 : k_1], \quad (\text{C.2a})$$

$$Y'^2 = (1 - Y^2) \left(1 - k_2^2 (1 - Y^2) \right) \implies Y = \text{cn}[f + f_0 - K_2 : k_2], \quad (\text{C.2b})$$

where K_2 is the quarter period of the Jacobi elliptic function of modulus k_2 .

Remark 23. We do not provide a rigorous definition of these elliptic functions here; see [Equation B.4](#) for further details and useful relationships.

It turns out that a Möbius transformation achieves the desired form. To show this explicitly, note that the generic equation ([Equation 4.26](#)) contains two quadratic factors on the right-hand side. Our goal is to recast these so that both quadratic terms become linear. We proceed as follows.

First, consider the factor $1 - y^2$. Write it as

$$(1 - y^2) = J^2 \left[(1 - \delta y)^2 - (y - \delta)^2 \right] = J^2 (1 - \delta^2) (1 - y^2),$$

which requires the arbitrary constants J and δ to satisfy

$$J^2 (1 - \delta^2) = 1. \quad (\text{C.3})$$

Next, for the second factor in Equation 4.26, set

$$(1 - d^2) + 2sy + qy^2 = J^2 \left[A(1 - \delta y)^2 + B(y - \delta)^2 \right],$$

which expands to

$$J^2 \left[(A + B\delta^2) - 2\delta(A + B)y + (A\delta^2 + B)y^2 \right].$$

To obtain the form in Equation C.1, we impose the following restrictions:

$$J^2(A + B\delta^2) = 1 - d^2, \quad (\text{C.4a})$$

$$J^2\delta(A + B) = -s, \quad (\text{C.4b})$$

$$J^2(A\delta^2 + B) = q. \quad (\text{C.4c})$$

Adding Equation C.4a and Equation C.4c yields

$$J^2(1 + \delta^2)(A + B) = (1 - d^2) + q, \quad (\text{C.5})$$

while subtracting Equation C.4c from Equation C.4a gives

$$J^2(1 - \delta^2)(A - B) = (1 - d^2) - q. \quad (\text{C.6})$$

Multiplying Equation C.4b by 2 and adding to Equation C.5 leads to

$$J^2(1 + \delta)^2(A + B) = (1 - d^2) - 2s + q, \quad (\text{C.7})$$

and subtracting the same from Equation C.5 results in

$$J^2(1 - \delta^2)(A + B) = (1 - d^2) + 2s + q. \quad (\text{C.8})$$

Dividing Equation C.8 by Equation C.7 and defining a new parameter ρ for convenience, we set

$$\rho^2 := \left(\frac{1 - \delta}{1 + \delta} \right)^2 = \frac{(1 - d^2) + 2s + q}{(1 - d^2) - 2s + q}. \quad (\text{C.9})$$

Thus,

$$\rho = \sqrt{\frac{(1 - d^2) + 2s + q}{(1 - d^2) - 2s + q}} \implies \frac{1 - \delta}{1 + \delta} = \pm \rho. \quad (\text{C.10})$$

Requiring $\rho \rightarrow 1$ as $\delta \rightarrow 0$ selects the positive sign, so that

$$\delta = \frac{1 - \rho}{1 + \rho}. \quad (\text{C.11})$$

Then,

$$1 - \delta^2 = \frac{4\rho}{(1 + \rho)^2}, \quad J^2 = \frac{1 + \rho^2}{4\rho}. \quad (\text{C.12})$$

It follows from Equation C.9 that

$$1 - \rho^2 = -\frac{4s}{(1 - d^2) - 2s + q}. \quad (\text{C.13})$$

Using Equation C.9 and Equation C.10, we write

$$(1 + \rho)^2 = 1 + \rho^2 + 2\rho,$$

and further,

$$(1 + \rho)^2 = 2 \left[\frac{(1 - d^2) + q + \sqrt{[(1 - d^2) - 2s + q][(1 - d^2) + 2s + q]}}{(1 - d^2) - 2s + q} \right]. \quad (\text{C.14})$$

Next, observe that

$$[(1 - d^2) - 2s + q][(1 - d^2) + 2s + q] = [(1 - d^2) + q]^2 \left[1 - \frac{4s^2}{[(1 - d^2) + q]^2} \right]. \quad (\text{C.15})$$

Define a parameter h by

$$(1 - 2h)^2 = 1 - \frac{4s^2}{[(1 - d^2) + q]^2}, \quad (\text{C.16})$$

so that

$$h = \frac{1}{2} \left[1 - \sqrt{1 - \frac{4s^2}{[(1 - d^2) + q]^2}} \right], \quad (\text{C.17a})$$

$$1 - h = \frac{1}{2} \left[1 + \sqrt{1 - \frac{4s^2}{[(1 - d^2) + q]^2}} \right]. \quad (\text{C.17b})$$

Then, from Equation C.15,

$$\sqrt{[(1 - d^2) - 2s + q][(1 - d^2) + 2s + q]} = [(1 - d^2) + q](1 - 2h). \quad (\text{C.18})$$

Recall Equation C.14, which can now be written in terms of h as

$$(1 + \rho)^2 = \frac{4(1 - h)[(1 - d^2) + q]}{(1 - d^2) - 2s + q}. \quad (\text{C.19})$$

Returning to Equation C.11 and using Equation C.13 and Equation C.19, we obtain

$$\delta = \frac{1 - \rho}{1 + \rho} = \frac{1 - \rho^2}{(1 + \rho^2)^2} = -\frac{s}{[(1 - d^2) + q](1 - h)}. \quad (\text{C.20})$$

From Equation C.7, introducing J^2 from Equation C.3, we have

$$(1 - d^2) - 2s + q = \frac{(1 + \delta)^2}{1 - \delta^2}(A + B) = \frac{A + B}{\rho},$$

so that

$$A + B = \rho \left[(1 - d^2) - 2s + q \right] = \sqrt{[(1 - d^2) - 2s + q][(1 - d^2) + 2s + q]}.$$

Using Equation C.18, this gives

$$A + B = [(1 - d^2) + q](1 - 2h).$$

Similarly, from Equation C.6,

$$A - B = (1 - d^2) - q.$$

Solving these yields

$$A = (1 - d^2)(1 - h) - hq, \quad (\text{C.21a})$$

$$B = -h(1 - d^2) + q(1 - h). \quad (\text{C.21b})$$

With h determined in terms of d , s , and q from Equation C.17a, and with δ , A , and B given by Equation C.20 and Equation C.21a respectively (and J^2 determined by Equation C.3), all the quantities are now fully determined algebraically.

Finally, putting all factors together, we have

$$\frac{\Lambda^2}{C^2} y'^2 = J^4 \left[(1 - \delta y)^2 - (y - \delta)^2 \right] \left[A(1 - \delta y)^2 + B(y - \delta)^2 \right].$$

This can be rewritten as

$$\frac{\Lambda^2}{C^2} \left[\frac{y'}{J^2(1 - \delta y)^2} \right]^2 = \left[1 - \left(\frac{y - \delta}{1 - \delta y} \right)^2 \right] \left[A + B \left(\frac{y - \delta}{1 - \delta y} \right)^2 \right].$$

We now identify the Möbius transformation

$$Y = \frac{y - \delta}{1 - \delta y} \quad \text{or equivalently} \quad y = \frac{Y + \delta}{1 + \delta Y}, \quad (\text{C.22})$$

so that

$$Y'^2 = \frac{y'}{J^2(1 - \delta y)^2}.$$

In terms of Y , the equation becomes

$$\frac{\Lambda^2}{C^2} Y'^2 = (1 - Y^2) [A + B Y^2],$$

which is exactly the desired form given in Equation C.1.

Appendix D

Mathematica Code

The following Mathematica code was used to plot orbits for the planar Euler problem.

```
(* Case I *)
(* Variables set up *)
B = 0.75;
e = 1.4;
b = 2.2;

(* Relations to main parameters *)
g = Sqrt[e^2 + B^2 - 1];

eeta[b_, e_] := b/(1 - e^2);
p = (1 - e^2);

f = eeta[b, e];

d0[f_, e_] :=
  2/((1 - f^2*(1 - e^2)) + Sqrt[(1 - f^2*(1 + e^2))^2 - 4*f^4*e^2]);
deltaV = f^2*d0[f, e];
eStar = e*((1 + f^2*d0[f, e])/(1 + f^2*d0[f, e]*e^2));

kS1 = Sqrt[((1 - f^2*(1 - e^2) -
  Sqrt[(1 + f^2*(1 - e^2))^2 - 4*f^2*B^2]))/(1 - f^2*(1 - e^2) +
  Sqrt[(1 + f^2*(1 - e^2))^2 - 4*f^2*B^2])];
deltaS = (-2*f*B)/(1 + f^2*(1 - e^2) +
  Sqrt[(1 + f^2*(1 - e^2))^2 - 4*f^2*B^2]);
kv = Sqrt[f^2*e^2*(1 + 2*f^2 + f^4*(2 + e^2))];
jv = Sqrt[(1 - f^2*(1 - e^2) +
  Sqrt[(1 - f^2*(1 - e^2))^2 - 4*f^2*e^2])/(2*
  Sqrt[(1 + f^2*(1 - e^2))^2 - 4*f^2*e^2])];
kS0 = ((2*Sqrt[b*g] - Sqrt[(b + g)^2 - B^2])/(2*Sqrt[b*g] +
  Sqrt[(b + g)^2 - B^2]));
deltaSS = (((-b - g)*(b + B - g))/((b + g)^2 + B*(b - g) +
```

```

2*Sqrt[b*g*((b + g)^2 - B^2)]));

deltaR = -(((b + e)*(1 - (b - e)))/((b + e)^2 - (b - e) +
2*Sqrt[b*e]*Sqrt[(b + e)^2 - 1]));
kR = ((2*Sqrt[b*e] - Sqrt[(b + e)^2 - 1])/(2*Sqrt[b*e] +
Sqrt[(b + e)^2 - 1]));

(*for 1-e<=b<=B-g*)
jr1 = Sqrt[(2*Sqrt[b*e] +
Sqrt[(b + e)^2 - 1])^2/(2*((1 - e^2 + b^2) +
Sqrt[(1 - e^2 + b^2)^2 - 4*b^2*B^2]))];
(*for B-g<=b<=1+e*)
jr0 = (2*Sqrt[b*e] + Sqrt[(b + e)^2 - 1])/(2*Sqrt[b*g] +
Sqrt[(b + g)^2 - B^2]);

(* Case A *)
(* Define R as a function of f_v using dn and cn elliptic functions *)
R[f_, p_, deltaV_, eStar_, kv_, jv_] :=
p*(JacobiDN[f*jv, kv] +
deltaV*JacobiCN[f*jv, kv])/(JacobiDN[f*jv, kv] +
eStar*JacobiCN[f*jv, kv]);

R2[f_, p_, deltaR_, b_, e_, kR_, jr_] :=
0.5*(1 - b +
e)*((JacobiCN[f*jr, kR] +
deltaR*JacobiDN[f*jr, kR])/(JacobiDN[f*jr, kR] +
deltaR*JacobiCN[f*jr, kR])) + 0.5*(1 + b + e);

(* Define conditional R function *)
Rconditional[f_, p_, deltaV_, eStar_, kv_, jv_, deltaR_, b_, e_, kR_,
jr1_, jr0_] :=
Which[0 <= b <= 1 - e, R[f, p, deltaV, eStar, kv, jv],
1 - e <= b <= B - g, R2[f, p, deltaR, b, e, kR, jr1],
B - g <= b <= 1 + e, R2[f, p, deltaR, b, e, kR, jr0]];

(* Define \[Sigma] as a function of f using sn elliptic function *)
sigma[f_, fS0_, kS1_, deltaS_] :=
ArcCos[(JacobiSN[f + fS0, kS1] + deltaS)/(1 +
deltaS*JacobiSN[f + fS0, kS1])];

sigma2[f_, fS0_, kS0_, deltaSS_] :=
ArcCos[(((1 + deltaSS)*b + (1 - deltaSS)*(B - g))*
JacobiSN[f + fS0,
kS0] + ((1 + deltaSS)*b - (1 - deltaSS)*(B - g)))/(1 +
deltaSS*JacobiSN[f + fS0, kS0])];

(* Define conditional sigma function *)

```

```

sigmaconditional[f_, fS0_, kS1_, deltaS_, kS0_, deltaSS_] :=
  Which[0 <= b <= B - g, sigma[f, fS0, kS1, deltaS], b >= B - g,
    sigma2[f, fS0, kS0, deltaSS]];

(* Set parameters for \[Sigma] *)
fS0 = 0.5; (* Offset for elliptic function *)

(* Define X and z in terms of R and \[Sigma], including both branches for X *)
XPlus[f_] :=
  Sqrt[Rconditional[f, p, deltaV, eStar, kv, jv, deltaR, b, e, kR,
    jr1, jr0]^2 - b^2]*
  Sin[sigmaconditional[f, fS0, kS1, deltaS, kS0, deltaSS]];
XMinus[f_] := -Sqrt[
  Rconditional[f, p, deltaV, eStar, kv, jv, deltaR, b, e, kR, jr1,
    jr0]^2 - b^2]*
  Sin[sigmaconditional[f, fS0, kS1, deltaS, kS0, deltaSS]];
z[f_] :=
  Rconditional[f, p, deltaV, eStar, kv, jv, deltaR, b, e, kR, jr1,
    jr0]*Cos[sigmaconditional[f, fS0, kS1, deltaS, kS0, deltaSS]];

(* Define specific points to be included *)
specificPoints = {{0, b}, {0, -b}};

(* Plot z vs. X for both branches of X as a parametric plot by varying f *)
Show[ParametricPlot[{{XPlus[f], z[f]}, {XMinus[f], z[f]}}, {f, 0,
  25 Pi}, PlotRange -> All,
  PlotLabel -> "Plot of z vs. X (Two Branches)",
  AxesLabel -> {"X", "z"}, AspectRatio -> 1,
  PlotStyle -> {LightOrangeOrange, LightBlueBlue}],
  ListPlot[specificPoints, PlotStyle -> {Black, PointSize[Large]}]]

(*Case II parameters*)B = 0;
e = 0.7005;
b = 0.9;

(*Relations to main parameters*)
g = Sqrt[e^2 + B^2 - 1];

eeta[b_, e_] := b/(1 - e^2);
p = (1 - e^2);

f = eeta[b, e];
ps = p[e];
d0[f_, e_] :=
  2/((1 - f^2*(1 - e^2)) + Sqrt[(1 - f^2*(1 + e^2))^2 - 4*f^4*e^2]);
deltaV = f^2*d0[f, e];
eStar = e*((1 + f^2*d0[f, e])/(1 + f^2*d0[f, e]*e^2));

```

```

deltaR = -(((b + e)*(1 - (b - e)))/((b + e)^2 - (b - e) +
      2 Sqrt[b*e]*Sqrt[(b + e)^2 - 1]));
kR = ((2*Sqrt[b*e] - Sqrt[(b + e)^2 - 1])/(2*Sqrt[b*e] +
      Sqrt[(b + e)^2 - 1]));

kS2 = 0.5 -
      0.5*((1 - f^2*(1 - e^2))/(Sqrt[(1 + f^2*(1 - e^2))^2 - 4*f^2*B^2]));
deltaS = (-2*f*B)/(1 + f^2 (1 - e^2) +
      Sqrt[(1 + f^2 (1 - e^2))^2 - 4*f^2*B^2]);
kv = Sqrt[f^2*e^2 (1 + 2*f^2 + f^4*(2 + e^2))];
jv = Sqrt[(1 - f^2 (1 - e^2) +
      Sqrt[(1 - f^2 (1 - e^2))^2 - 4*f^2*e^2])/(1 - f^2 (1 - e^2) +
      Sqrt[(1 + f^2 (1 - e^2))^2 - 4*f^2*e^2])];
jr2 = 0.5*
      Sqrt[(2*Sqrt[b*e] +
      Sqrt[(b + e)^2 - 1])^2/(Sqrt[(1 - e^2 + b^2)^2 - 4*b^2*B^2])];

(*Case B*)
(*Define R as a function of f_v using dn and cn elliptic functions*)
(*for 0<=b<=1-e*)
R[f_, p_, deltaV_, eStar_, kv_, jv_] :=
  p*(JacobiDN[f*jv, kv] +
    deltaV*JacobiCN[f*jv, kv])/(JacobiDN[f*jv, kv] +
    eStar*JacobiCN[f*jv, kv]);

(*for 1-e<=b<=1+e*)
R2[f_, p_, deltaR_, b_, e_, kR_, jr2_] :=
  0.5*(1 - b +
    e)*((JacobiCN[f*jr2, kR] +
    deltaR*JacobiDN[f*jr2, kR])/(JacobiDN[f*jr2, kR] +
    deltaR*JacobiCN[f*jr2, kR])) + 0.5*(1 + b + e);

(*Define conditional R function*)
Rconditional[f_, p_, deltaV_, eStar_, kv_, jv_, deltaR_, b_, e_, kR_,
  jr2_] :=
  Which[0 <= b <= 1 - e, R[f, p, deltaV, eStar, kv, jv],
    1 - e <= b <= 1 + e, R2[f, p, deltaR, b, e, kR, jr2]];

(*Define \[Sigma] as a function of f using sn elliptic function*)
sigma[f_, fS0_, kS2_, deltaS_] :=
  ArcCos[((Sqrt[1 - kS2^2])*JacobiSN[f + fS0, kS2] +
    deltaS*JacobiDN[f + fS0, kS2])/(JacobiDN[f + fS0,
    kS2] + (Sqrt[1 - kS2^2])*deltaS*JacobiSN[f + fS0, kS2])];

(*Set parameters for \[Sigma]*)
fS0 = 0; (*Offset for elliptic function*)

```



```

(*Define X and z in terms of R and \[Sigma], including both branches \
for X*)
XPlus[f_] :=
  Sqrt[Rconditional[f, p, deltaV, eStar, kv, jv, deltaR, b, e, kR,
    jr2]^2 - b^2]*Sin[sigma[f, fS0, kS2, deltaS]];
XMinus[f_] := -Sqrt[
  Rconditional[f, p, deltaV, eStar, kv, jv, deltaR, b, e, kR,
    jr2]^2 - b^2]*Sin[sigma[f, fS0, kS2, deltaS]];
z[f_] :=
  Rconditional[f, p, deltaV, eStar, kv, jv, deltaR, b, e, kR, jr2]*
  Cos[sigma[f, fS0, kS2, deltaS]];

(*Define specific points to be included*)
specificPoints = {{0, b}, {0, -b}};

(*Plot z vs.X for both branches of X as a parametric plot by varying \
f*)
Show[ParametricPlot[{{XPlus[f], z[f]}, {XMinus[f], z[f]}}, {f, 0,
  25 Pi}, PlotRange -> All,
  PlotLabel -> "Plot of z vs. X (Two Branches)",
  AxesLabel -> {"X", "z"}, AspectRatio -> 1,
  PlotStyle -> {LightOrangeOrange, LightBlueBlue}],
  ListPlot[specificPoints, PlotStyle -> {Black, PointSize[Large]}]]

```

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