Integrable Systems in Celestial Mechanics

UG1707

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Overview

Motivation: Celestial Mechanics

Celestial mechanics is the study of celestial bodies under gravitational forces, with a rich history:

- **Newton (1687)**: Formulated the laws of motion and universal gravitation.
- **Euler, Lagrange, Poincaré**: Developed mathematical frameworks for orbital dynamics.

The **two-body problem**, describing two bodies mutually attracting via an inverse-square law, is exactly solvable in Newtonian mechanics. A special case of this is the **Kepler problem**, which describes planetary motion around a central mass.

The Kepler Problem: A Fundamental Model

Physical Setup:

- A point mass moving under an inverse-square central force.
- Governed by Newton's second law:

$$m\ddot{\mathbf{x}} = -\nabla V(\mathbf{x}), \quad V(r) = -\frac{GM}{r}.$$

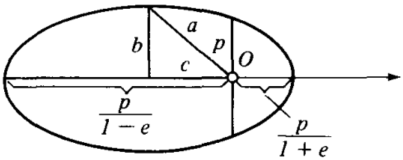


Figure: Planar Kepler orbit [Arnol'd, 2013]

Key Properties:

- The equation of the orbit:

$$r = \frac{p}{1 + e\cos\varphi}.$$

- The problem is **integrable**, meaning it admits conserved quantities.
- Bound orbits are either **ellipses** or **circles**.

The Euler Problem: A Step Beyond Kepler

Motivation: The three-body problem is **non-integrable**, but if two of the masses is much larger and fixed, the system simplifies and becomes **integrable**. This leads to the Euler problem.

Physical Setup:

- A small mass moving under the gravitational field of two fixed masses.
- The potential takes the form:

$$V(x,z) = -\frac{Gm_+}{r_+} - \frac{Gm_-}{r_-}.$$

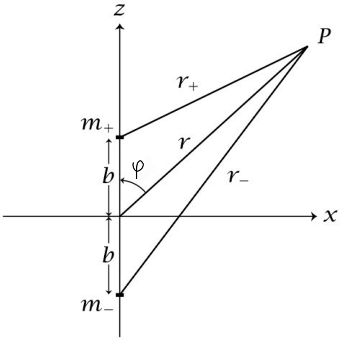


Figure: Schematic of the Euler problem

Why Study This?

- Models the Earth's gravitational influence on a satellite.
- Provides insight into classical equivalent to a quantum study of H^+ system

Lagrangian Mechanics

Definition

Mechanical systems consists of positions in a manifold Q called the **configuration space** of a system where the positions are parametrised by coordinates q_i

Definition

The **Lagrangian** is a function on a manifold P (**tangent bundle** of some **configuration space** Q) with coordinates q_i and $\dot{q}_i$ (velocities of the curves in Q) defined as the difference between kinetic and potential energy

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) = T - V$$

For our purposes we take $P = \mathbb{R}^{2n}$! so $\mathbf{q}, \dot{\mathbf{q}} \in \mathbb{R}^n$

Euler-Lagrange Equation

Definition

The **Action** functional for a given a Lagrangian is given by

$$S[\mathbf{q}] = \int_{t_0}^{t_1} \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}})\, dt$$

where $t_0, t_1 \in \mathbb{R}$ are initial and final times of the motion

Theorem [Euler-Lagrange Equations]

When the action functional is extremised (i.e. $\delta S = 0$) the Lagrangian satisfy the following equation

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i} \text{ for each } i \in \{1, \cdots, n\}$$

But why is this useful?

Example of Lagrangian Mechanics

Example

Consider a particle of mass m moving in $\mathbb{R}^3$ under a potential $V(\mathbf{x})$. The Lagrangian is given by:

$$\mathcal{L} = \frac{m}{2}\sum_{i=1}^{3} \dot{x}_i^2 - V(\mathbf{x}).$$

Applying the Euler-Lagrange equation to x_i:

$$m\ddot{x}_i = -\frac{\partial V}{\partial x_i},$$

which recovers Newton's second law!

Hamiltonian Mechanics

Definition

Given a Lagrangian function the **Hamiltonian function** is defined on the **phase space** (also a manifold M in fact the **cotangent bundle** of Q) with coordinates q_i and p_i (generalised momenta associated to q_i) of a system is defined as follows:

$$H(\mathbf{q}, \mathbf{p}) = \sum_{i=1}^{n} p_i \dot{q}_i - \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) \ ; \ p_i := \frac{\partial \mathcal{L}}{\partial \dot{q}_i}.$$

Now repeating the procedure above but replacing the Lagrangian in the action functional with the Hamiltonian and extremising the action we get:

Hamilton's equation

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \ , \ \frac{\partial H}{\partial q_i} = -\dot{p}_i.$$

Hamiltonian Example

Do we still get Newton's laws?

Example

Repeating the example above with a particle of mass m moving in $\mathbb{R}^3$ under a potential $V(\mathbf{x})$. The Hamiltonian is given by:

$$H = \frac{1}{2m}\sum_{i=1}^{n} p_i^2 + V(\mathbf{x})$$

Hamilton's equations give:

$$\frac{p_i}{m} = \dot{x}_i \ , \ \frac{\partial V}{\partial x_i} = -\dot{p}_i .$$

when put together we recover Newtons law again:

$$m\ddot{x}_i = -\frac{\partial V}{\partial x_i}.$$

Poisson Brackets

Definition

For functions $f, g \in C^\infty(M)$, the **Poisson bracket** is an $\mathbb{R}$-bilinear map which is defined as:

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right).$$

- The Poisson bracket describes how one function evolves under Hamilton's equations i.e.

$$\dot{q}_i = \{q_i, H\} \ , \ \dot{p}_i = \{p_i, H\}.$$

- If $\{f, H\} = 0$, then f is a **constant of motion** $\left(\frac{df}{dt} = 0 \right)$.

Liouville Integrability

Definition

A Hamiltonian system is **Liouville integrable** if there exist n independent functions $f_1, \ldots, f_n$ such that:

- They **Poisson commutes** (are in **involution**):

$$\{f_i, f_j\} = 0, \quad \forall i, j \in \{1, \cdots, n\}.$$

- They are functionally **independent** on the level sets

$$M_c := \{p \in M \mid f_i(p) = c_i, \quad i = 1, \ldots, n\}.$$

Liouville-Arnold Theorem

Theorem

If a Hamiltonian system is **Liouville integrable**, then:

- The level set

$$M_c := \{p \in M \mid f_i(p) = c_i, \quad i = 1, \dots, n\}$$

  is an n-dimensional submanifold invariant under time evolution of the Hamiltonian.
- If M_c is **compact and connected**, then M_c is diffeomorphic to an n-torus:

$$T^n = \{(\theta_1, \dots, \theta_n) \bmod 2\pi\}.$$

- in a neighbourhood of M_c there is a canonical change of coordinates $(p_i, q_i) \to (I_i, \theta_i \bmod 2\pi)$ (**action-angle coordinates**) such that in the new coordinates the Hamiltonian $H = H(\mathbf{I})$ and $I(\mathbf{c})$. The flow is linear in angle coordinates:

$$\theta_i = \omega_i(\mathbf{c})t + \theta_0 \ , \ \dot{\theta}_i = \frac{\partial H}{\partial I_i} \ ; \ \omega_i(\mathbf{c}) \equiv \dot{\theta}_i.$$

A visualisation of transformation in one dimension

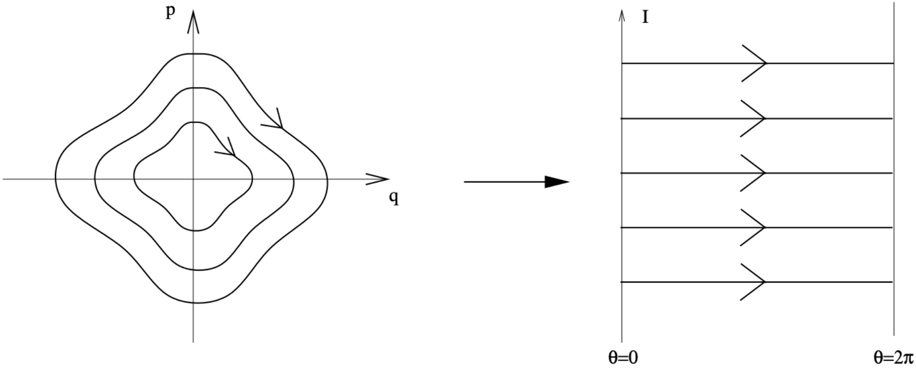


Figure: transformation from (q, p) to (θ, I) [Tong, 2015]

Action-Angle Coordinates

Definition

The **action variables** are defined as:

$$I_i = \frac{1}{2\pi} \int_{\gamma_i} \sum_j p_j dq_j.$$

where γ_i denote a basis of one dimensional cycles on the torus $M_{\mathbf{c}} \cong T^n$

- This is defined as preserving the form of Hamilton's equations
- These coordinates **simplify the equations of motion**.
- Motion on tori is **quasi-periodic**.
- They allow explicit integration of the system.

Kepler Problem Revisited

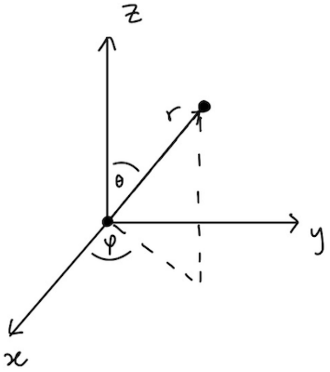


Figure: Setup of the Kepler problem

Kepler Problem: A point mass under the influence of a central potential governed by an inverse-square law.

The Hamiltonian of the system is given by:

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2\theta}\right) - \frac{\mu}{r}$$

where $\mu = Gm$ is a parameter related to the gravitational constant.

Conserved Quantities in the Kepler Problem

A functionally independent set of conserved quantities on the level set:

- **Hamiltonian (Total Energy):**

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2\theta}\right) - \frac{\mu}{r}$$

- **Angular Momentum in the z-direction:**

$$L_z = p_\varphi$$

- **Total Angular Momentum Squared:**

$$L^2 = p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta}$$

Explicitly, we consider the level set for bounded orbits:

$$H = -\alpha^2, \quad L_z = C, \quad L^2 = J^2, \quad \text{where } \alpha, C, J \in \mathbb{R}.$$

Checking Involution

Since H, L_z, and L^2 are conserved quantities (i.e., constants of motion), we have:

$$\{H, H\} = \{L_z, H\} = \{L^2, H\} = 0$$

The only non-trivial Poisson bracket:

$$\{L^2, L_z\} = 0$$

Conclusion: The Kepler problem is Liouville integrable! as $Q = \mathbb{R}^3 + 3$ independent commuting constants H, L_z, L^2.
How about the action-angle coordinates what does that tell us?

Kepler Problem in Action-Angle Coordinates

Action Variables:

- **Azimuthal action:**

$$I_\varphi = \frac{1}{2\pi}\int_{\gamma_\varphi} C \; d\varphi = p_\varphi$$

- **Polar action:**

$$I_\theta = \frac{1}{2\pi}\int_{\gamma_\theta} \sqrt{J^2 - \frac{p_\varphi^2}{\sin^2\theta}}\; d\theta = J - I_\varphi$$

- **Radial action:**

$$I_r = \frac{1}{2\pi}\int_{\gamma_r} \frac{\sqrt{-2\alpha^2 r^2 + 2\mu r - J^2}}{r}\; dr = -(I_\theta + I_\varphi) - \frac{\mu}{\sqrt{2\alpha^2}}$$

Hamiltonian in the new coordinates:

$$H = -\frac{\mu^2}{2(I_r + I_\theta + I_\varphi)^2}$$

Degeneracy in Frequencies

We observe a degeneracy in the fundamental frequencies ω_i:

$$\dot{\theta}_r = \dot{\theta}_\theta = \dot{\theta}_\varphi = \frac{\mu^2}{4(I_r + I_\theta + I_\varphi)^3}; \ \omega_i \equiv \dot{\theta}_i.$$

Implication:

- The trajectory in **complete phase space** is periodic.
- Both **position and velocity** return to their initial values after a finite time.
- This means orbits in **configuration space** are closed!

This expected as bounded orbits in the kelper problem is elliptical or circular!

Euler problem/Two Fixed Centre Problem Revisted

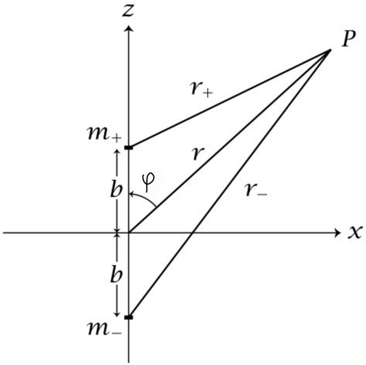


Figure: Setup of the Euler problem [Ó'Mathúna, 2008]

Euler Problem: A point mass under the influence of of two fixed centres governed by an inverse-square law.

The Hamiltonian of the system is given by:

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) - G\left(\frac{m_+}{r_+} + \frac{m_-}{r_-}\right)$$

where r_+ and r_- are given by:

$$r_+^2 = r^2 + b^2 - 2br\cos\varphi$$

$$r_-^2 = r^2 + b^2 + 2br\cos\varphi.$$

Coordinate Choice/Planar Prolate Coordinates

Hard to solve with current choice of coordinates. Maybe choose one that favours the symmetry of the setup?

$$r\sin\varphi = x := \pm\sqrt{R^2 - b^2}\sin\sigma \ , \ r\cos\varphi = z := R\cos\sigma \ , \ R > b \ \& \ \sigma \in [0, \pi]$$

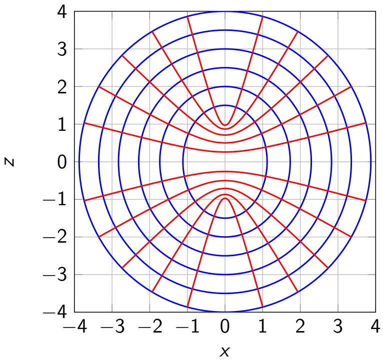


Figure: Blue curves represent constant R (ellipses), and red curves are constant σ (hyperbolae).

Conserved Quantities and Integrability

Functionally Independent Conserved Quantities:

- **Hamiltonian (Total Energy)**:

$$H = \frac{1}{2}\left(\frac{p_R^2(R^2 - b^2)}{R^2 - b^2\cos^2\sigma} + \frac{p_\sigma^2}{R^2 - b^2\cos^2\sigma}\right) - \mu\frac{R + \beta b\cos\sigma}{R^2 - b^2\cos^2\sigma}$$

  where

$$\beta = \frac{m_+ - m_-}{m_+ + m_-}, \quad \mu = G(m_+ + m_-).$$

  Turns out there's a hidden symmetry in this system...

Hidden Symmetry

The independent, non-trivial conserved quantity is

$$\tilde{L} = \frac{p_\sigma^2}{2} + Hb^2 \cos^2 \sigma + \mu\beta b \cos \sigma.$$

- In the limit $b \to 0$, this reduces to the Keplerian conserved quantity p_φ.
- Since $\tilde{L}$ is conserved, we have
$$\{\tilde{L}, H\} = 0,$$
implying the system is Liouville integrable as $Q = \mathbb{R}^2$ + 2 independent commuting constants $H, \tilde{L}$.

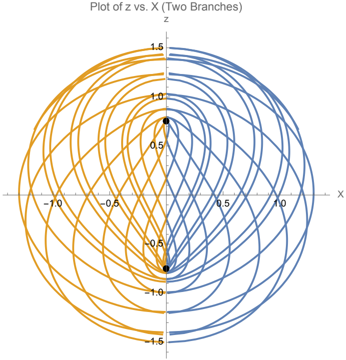
In fact on the level set

$$H = -\alpha^2, \quad \tilde{L} = \frac{C^2}{2}, \quad \text{where } \alpha, C \in \mathbb{R},$$

Hamilton's equations can be solved explicitly using **Jacobi elliptic functions**. Let's look at some orbit plots

Orbit solutions

(a) Rotating ellipse-like orbit



(b) Figure 8-like orbit

Figure: Examples of orbits generated via Mathematica.

Can we make a qualitative inference via action-angle coordinates?

Action-Angle Coordinates

The action angle variables are given by

$$I_R = -C + i\alpha \sum_{k=0}^{\infty} \frac{(1+2k)!!}{(2k)!2^k k!} b^{2k} \sum_{n=0}^{2k} \binom{2k}{n} \frac{(-1)^n \Gamma(2k-n-\frac{1}{2})\Gamma(n-\frac{1}{2})}{\pi} (-R_-)^{\frac{1}{2}-2k+n} (R_+)^{\frac{1}{2}-n}.$$

$$I_\sigma = \frac{\sqrt{2}\alpha}{2\pi} \sum_{k=0}^{\infty} \frac{b^{2k}\sqrt{\pi}\Gamma\left(2k+\frac{1}{2}\right)}{2(2k)!\Gamma\left(2k+1\right)} \sum_{n=0}^{2k} \binom{2k}{n} \frac{\Gamma\left(2k-n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right)}{\pi} (-x_-)^{\frac{1}{2}-2k+n} (-x_+)^{\frac{1}{2}-n}.$$

Looks horrible! maybe consider upto $\mathcal{O}(b^2)$?

$$I_R = -C + \frac{\mu}{\sqrt{2\alpha^2}} - \frac{3b^2}{16}\left(\frac{-2\alpha^2}{C} + \frac{\mu^2}{C^3}\right) + \mathcal{O}(b^4)$$

$$I_\sigma = C + \frac{3b^2}{64}\left(\frac{2\alpha^2}{C} - \frac{\mu^2\beta^2}{C^3}\right) + \mathcal{O}(b^4).$$

Looks like Kepler plus other $\mathcal{O}(b^2)$ stuff! What about the frequencies?

Breaking of Degeneracy

Considering the Hamiltonian to order $\mathcal{O}(b^4)$

$$H = -\frac{\mu^2}{2(I_R + I_\sigma)^2} - b^2 \frac{G_1}{2G_0^2} + \mathcal{O}(b^4)$$

where

$$G_0 = \frac{(I_R + I_\sigma)^2}{\mu^2}\,, \; G_1 = (I_R + I_\sigma)\frac{3}{32}\left(\frac{10}{I_\sigma(I_\sigma + I_R)^2} - \frac{(\beta^2 + 4)}{I_\sigma^3}\right)$$

We now see $\dot{\theta}_\sigma \neq \dot{\theta}_R$ as $\dot{\theta}_i = \frac{\partial H}{\partial I_i}$, thus no longer closed in configuration space and orbits precess! This was seen above.

Conclusion

Key Takeaways:

- The Kepler and Euler problems serve as fundamental examples of **integrable** systems in celestial mechanics.
- Liouville integrability provides a structured framework to analyse conserved quantities and dynamics.
- The transition from Kepler to Euler reveals how small perturbations can break degeneracy in frequencies and lead to orbit precession.

Broader Impact:

- Understanding integrable models aids in studying real-world orbital mechanics and perturbation theory.
- These ideas extend to modern physics, including quantum integrability and astrophysical modeling.

Future work

Areas that could be worked on:

- Finding a general expression for the hamiltonian for in action angle variables for Euler
- Finishing the plotter to allow for non symmetric plots and classifying orbits
- Work on the Vinti problem (modelling a satellite orbitting around the earth)

References

Arnol'd, V. I. (2013).
Mathematical methods of classical mechanics, volume 60.
Springer Science & Business Media.

Ó'Mathúna, D. (2008).
Integrable systems in celestial mechanics, volume 51.
Springer Science & Business Media.

Tong, D. (2015).
Lecture notes on classical dynamics, hamiltonian formulation.

Thank you!

Solution to the Euler Problem(Extra-Slide)

In terms of R and σ we get

$$\cos\sigma = \frac{k'_{S2}\mathrm{sn}[f+f_0 : k_{S2}] + \delta_S \mathrm{dn}[f+f_0 : k_{S2}]}{\mathrm{dn}[f+f_0 : k_{S2}] + \delta_S k'_{S2}\mathrm{sn}[f+f_0 : k_{S2}]}. \tag{1}$$

$$R = p\frac{\mathrm{dn}[f_v : k_v] + \delta_v \mathrm{cn}[f_v : k_v]}{(1+e\delta_v)\mathrm{dn}[f_v : k_v] + (e+\delta_v)\mathrm{cn}[f_v : k_v]}. \tag{2}$$

Jacobi elliptic functions are introduced via the *amplitude* function φ defined by

$$u = \int_0^{\varphi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}, \tag{3}$$

where u is the **elliptic argument**. The three primary Jacobi elliptic functions are :

$$\mathrm{sn}(u;k) = \sin\varphi, \tag{4}$$

$$\mathrm{cn}(u;k) = \cos\varphi, \tag{5}$$

$$\mathrm{dn}(u;k) = \sqrt{1-k^2\sin^2\varphi}. \tag{6}$$