Projectile Motion

Projectile: A body or object which is projected with an initial velocity at an angle θ with horizontal in the vertical plane of the earth is called projectile. The path traversed by the projectile is called its trajectory.

Characteristics of Projectile motion:

1. Projectile motion is a fine example of curvilinear motion which is two dimensional.
2. The position and velocity change continuously with time.
3. The acceleration is constant and always directed vertically downward. It has no horizontal acceleration.

We show the trajectory by considering that,

- The only force that acting on projectile is gravitational force i.e. the air has no effect on it.
- The maximum height of projectile is negligible with compared to the radius of the earth. So the value of gravitational acceleration is constant.

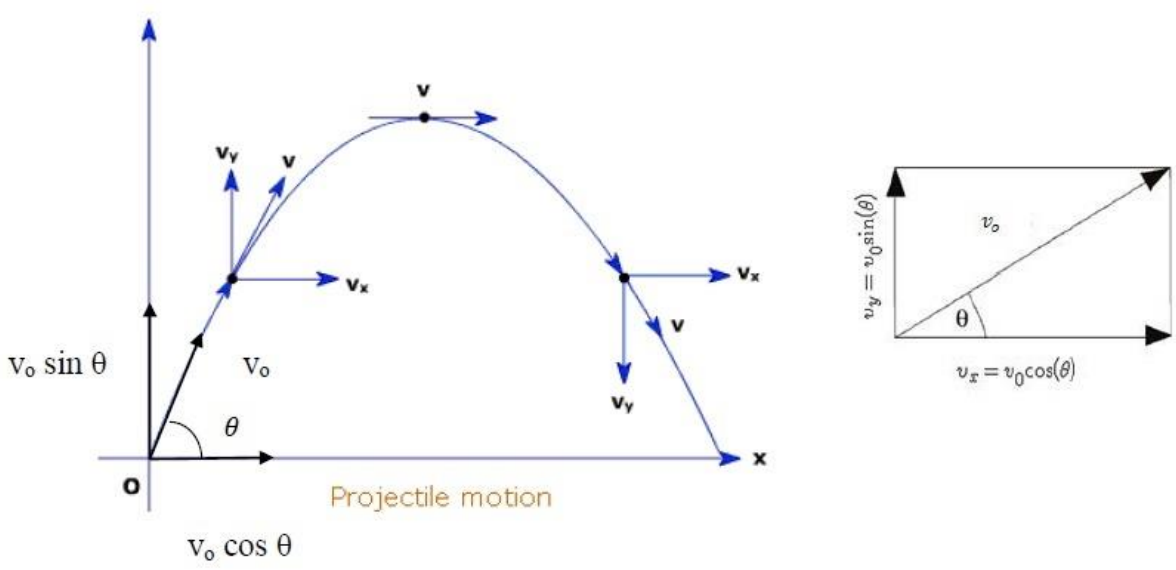
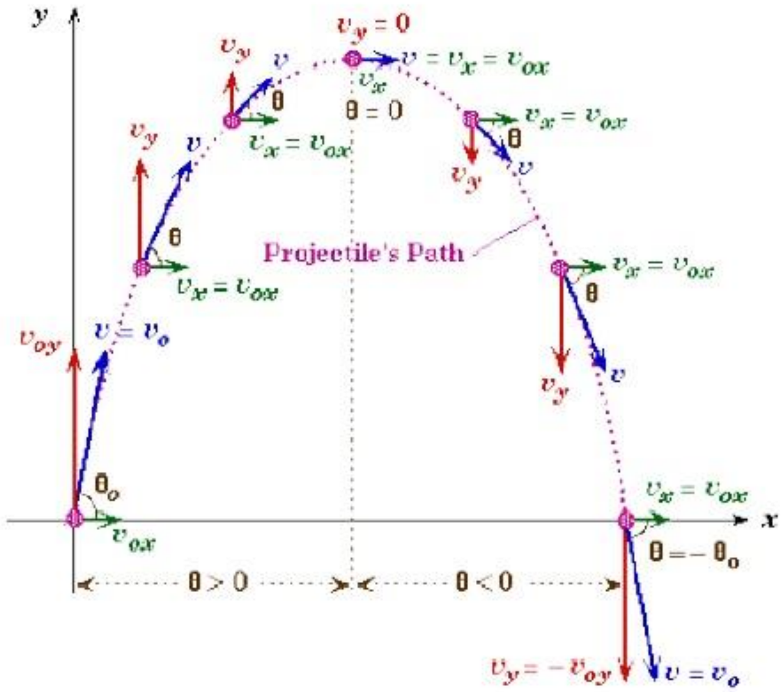


Fig.1. Projectile Motion



<u>The equation of Projectile path or trajectory</u>:

Let us consider a projectile begins its flight with an initial velocity v_o and making an angle θ with the horizontal direction from point O as shown in Fig. 1.

The initial velocity has two components i.e.,

horizontal component, $v_{ox} = v_o \cos\theta$, along OX-direction

and, vertical component, $v_{oy} = v_o \sin\theta$, along OY-direction

The accelerations components are

$a_x = 0$, along horizontal direction

$a_y = -g$, along vertical direction

At any time t, the horizontal and vertical components of displacement of a projectile are given by,

$$x = v_{ox}\, t + \frac{1}{2} a_x t^2$$

or, $x = v_o \cos\theta\, t$; (since $a_x = 0$) ----------------------------(1)

and, $y = v_{oy} t + \frac{1}{2} a_y t^2$

or, $y = v_o \sin\theta\, t - \frac{1}{2} g t^2$; (since, $a_y = -g$) -------------------- (2)

At any time t, the magnitude of the position of a projectile is given by

$$r = \sqrt{x^2 + y^2}$$

Now, at time t, the horizontal and vertical components of velocity of a projectile are given by,

$$v_x = v_{ox} + a_x t$$

or, $v_x = v_o \cos\theta$ --------------------------------------(3)

and, $v_y = v_{oy} + a_y t$

or, $v_y = v_o \sin\theta - g t$ ----------------------------------(4)

So, at time t, the magnitude of velocity of a projectile is

$$|\bar{v}| = \sqrt{v_X^2 + v_Y^2}$$

Let v makes an angle θ with the x axis, so

$$\tan\theta = \frac{v_y}{v_x}$$

or, $\theta = tan^{-1} (\frac{v_y}{v_x})$

which represent the direction of the projectile.

*** The horizontal velocity component of a projectile remain unchanged over time because there is no time dependent term as shown in equ. (3). On the other hand, from equ. (4) we can see the vertical velocity component of a projectile changes with time.

The equation of trajectory of motion of a body at any instant is the equation relating the co-ordinates of the body at that instant.

From equ.(1) and equ.(2) we find the co-ordinate as a function of t.

From equ.(1) , $x = v_o \cos\theta \, t$

$$\text{or,} \quad t = \frac{x}{v_o \cos\theta} \text{ ------------------------(5)}$$

Putting the value of t in equ.(2)

$$y = v_o \sin\theta \frac{x}{v_o \cos\theta} - \frac{1}{2} g \left(\frac{x}{v_o \cos\theta}\right)^2$$

$$\text{or,} \quad y = x \tan\theta - \frac{1}{2} \frac{gx^2}{V_0^2 cos^2\theta}$$

$$\text{or,} \quad y = ax - bx^2 \text{ -------------------------- (6)}$$

Here, a and b are constant. This is the equation of a parabola. So the path of a projectile is parabolic.

Time to reach maximum height:

At maximum height, the vertical component of velocity, $V_y = 0$

From the equation of motion, $v_y = v_o \sin\theta - g\,t$

$$v_o \sin\theta - g\,t = 0$$

$$\text{Or,} \quad t = \frac{v_0 \, sin\theta}{g} \text{ --------------------------------(7)}$$

By knowing the value of v_o, θ and g we can calculate t.

Time of flight: The time required for a projectile to come-back to the ground after it has been thrown is the time of flight.

Let the time of flight be T.

When the object come back to the ground then y = 0,

$$y = v_o \sin\theta\, T - \frac{1}{2} g\, T^2 = 0$$

$$\text{or, } (v_0 \sin\theta - \frac{1}{2} gT)T = 0$$

$$\text{Therefore, } T = 0 \quad \text{or, } v_0 \sin\theta - \frac{1}{2} g$$

Since, T = 0 indicates the time when the object has been thrown, so

$$T = \frac{2v_0 \sin\theta}{g} \text{ ------------------------------ (8)}$$

This is the equation of the time of flight of a projectile.

Maximum height:

From the equation of motion, $y = v_o \sin\theta\, t - \frac{1}{2} g\, t^2$

Let the height reached by the projectile be H where final verticle velocity $v_y = 0$ and the time to reach the maximum height, $t = \frac{v_0 \sin\theta}{g}$. So the equation becomes,

$$H = v_o \sin\theta \frac{v_0 \sin\theta}{g} - \frac{1}{2} \frac{g\, v_0^2 \sin^2\theta}{g^2}$$

$$\text{or, } H = \frac{v_0^2 \sin^2\theta}{2g} \text{ ------------------------------(9)}$$

When $\theta = 90^0$, The height will be maximum. Then equ.(8) will be,

$$H = \frac{v_0^2}{2g} \text{ ------------------------------- (10)}$$

Horizontal Range:

The linear distance from the point of projection to the end of flight is called the horizontal range. Alternatively, the distance travelled along the horizontal direction in the time of flight is called the horizontal range. This is denoted by R.

R = horizontal component of the initial velocity × time of flight

$$R = v_o \cos\theta \times T$$

$$R = v_o \cos\theta \times \frac{2v_0 \sin\theta}{g}$$

$$R = \frac{v_0^2 2\sin\theta\cos\theta}{g}$$

$$R = \frac{v_0^2 \sin 2\theta}{g} \quad \text{--------------------------} \ (11)$$

Knowing the value of v_o, θ and g, we can find the value of R.

Maximum Horizontal Range:

The equation of horizontal range, $R = \frac{v_0^2 \sin 2\theta}{g}$ ---------------------------- (12)

It is evident that R will be maximum, when $\sin 2\theta = 1$

$$\sin 2\theta = 90^0$$
$$2\theta = 90^0$$
$$\theta = 45^0$$

In that case, $R_{maximum} = \frac{v_0^2}{g}$ ------------------------- (13)

If an object is thrown at an angle 45^0 with the horizontal direction, the horizontal range will be maximum.

Time of ascend is equal to time of descend

Time taken by the projectile to reach the maximum height is called time of ascend.

At maximum height the final vertical velocity, $V_y = 0$

From the equation of motion, $v_y = v_o \sin\theta - g\,t$

$$v_o \sin\theta - g\,t = 0$$

$$\text{Or, } t = \frac{v_0 \sin\theta}{g}$$

$$\text{Or, Time of ascend} = \frac{v_0 \sin\theta}{g} \text{ ------------------------------ (1)}$$

By knowing the value of v_o, θ and g we can calculate t.

Time of flight

Time taken by the projectile to come back to the ground after it has been thrown is called the time of flight. When the projectile to come back to the ground then,

y = 0. So from the equation of motion,

$$y = v_o \sin\theta\, t - \frac{1}{2} g\, t^2; \quad (\text{since, } a_y = -g) \text{ -------------------- (2)}$$

$$\text{or, } 0 = v_o \sin\theta\, T - \frac{1}{2} g\, T^2$$

$$\text{or, } \frac{1}{2} g\, T^2 = v_o \sin\theta\, T$$

$$T = \frac{2v_0 \sin\theta}{g} \text{ ---------------------------- (3)}$$

Time of descend

Time taken by the projectile to reach the ground from maximum height is called time of descend.

Time of flight = Time of ascend + Time of descend

Or, Time of descend = Time of flight - Time of ascend

$$\text{Or, Time of descend} = \frac{2v_0 \sin\theta}{g} - \frac{v_0 \sin\theta}{g}$$

Or, Time of descend = $\frac{v_0 \sin\theta}{g}$

Or, Time of descend = Time of ascend

So, the time is same for projectile to reach the maximum height and come back from maximum height.

Problems:

1. A soccer player kicks a ball at an angle of 37^0 from the horizontal with an initial speed of 20 m/s. Assume that the ball moves in a vertical plane

a) Find the time at which the ball reaches the highest point of its trajectory.

b) How high does the ball go?

c) What is the horizontal range of the ball and how long is it in the air?

d) What is the velocity of the ball as it strikes ground?

2. A projectile is thrown at a velocity 30 ms^{-1} making an angle 30 deg with the ground. Calculate the magnitude of the velocity of the football after one second.

3. Suppose a football player kick a ball with an angle 30 degree with horizontal with a velocity 25 ms^{-1}. The distance between the player and goalkeeper is 80m. The keeper run with a uniform velocity of 10 ms^{-1} to catch the ball. Can the keeper catch the ball? [g=9.8 ms^{-2}]

4. A body is thrown at 20 ms_{-1} velocity making an angle 50 deg with horizontal direction.

a) Calculate horizontal range

b) Maximum height

c) Time to ascend the maximum height

d) Time of flight.

5. Suppose a football player kick a ball with an angle 30 degree with horizontal with a velocity 25 ms^{-1}. A keeper is on the line of the ball at 72 m from the ball striking would like to catch the ball. What is the velocity of the keeper?

Solution 5:

Here given that, θ = 30^0

V_0 = 25 ms^{-1}

We know, Time of flight, T = $\frac{2v_0\, sin\theta}{g}$

Or, T = $\frac{2\ x\ 25\ x\ sin30^0}{9.8}$

Or, T = 2.55 s

The horizontal range of the ball, R = $\frac{v_0^2 sin2\theta}{g}$

Or, R = $\frac{(25)^2\sin(2x\ 30^0)}{9.8}$

Or, R = 55.23 m

To take the catch the keeper has to cover (72 – 55.23) = 16.77 meter in 2.55 second.

The velocity of the keeper should be, V = $\frac{16.77}{2.55}$ ms^{-1} = 6.58 ms^{-1} (Ans.)

6. A Cricket player strike a ball at an initial speed 40 m/s with an initial angle 25 deg. A fielder is on the line of the ball at 72 m from the ball striking would like to catch the ball. Find the velocity of the fielder. Ans. (15.38 m/s)

Solution 6:

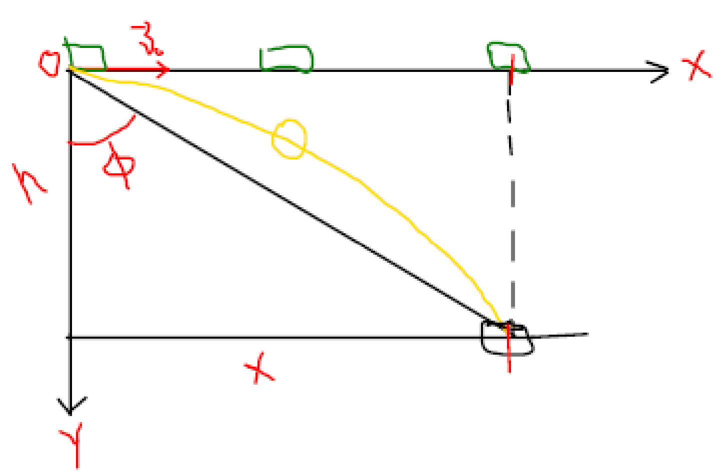
Given, v_0 = 155 km/h = $\frac{155\times1000}{3600}$ ms^{-1} = 43.01 ms^{-1}

h = 225 m

θ = 0^0

∅ = ?

From figure, tan∅ = $\frac{x}{h}$



We know,

$X = v_0 \cos\theta \, t$

$Y = v_0 \sin\theta \, t + \frac{1}{2} g t^2$

Or, $225 = 43.01 \sin 0^0 \times t + \frac{1}{2} \times 9.8 \, t^2$

Or, $225 = 43.01 \times 0 \times t + \frac{1}{2} \times 9.8 \, t^2$

Or, $225 = \frac{1}{2} \times 9.8 \, t^2$

Or, $\sqrt{\frac{225x2}{9.8}} = t$

Or, $t = 6.78$ s

So, $X = v_0 \cos\theta \, t$

Or, $X = 43.01 \cos 0^0 \times 6.78$

Or, $X = 291.6$ m

$\tan\emptyset = \frac{x}{h} = \frac{291.6}{225} = 1.296$

or, $\emptyset = \tan^{-1}(1.296)$

or, $\emptyset = 52.35 \, deg$ (Ans)

7. In a contest to drop a package on a target, one contest's plane is flying at a constant horizontal velocity of 155 Km/h at an elevation of 225 m toward a point directly above the target. At what angle of sight should the package be released to strike the target?