Maximum and Minimum

Consider the function $y = f(x)$.

Critical Points: The points on the graph of the function $y = f(x)$ at which the slope of the tangent line is zero are called Critical Points. The Critical points are obtained by solving $\frac{dy}{dx} = 0$ $i.e.$ $f'(x) = 0$.

$\boldsymbol{x = c}$	**Test condition**	**Decision**
If $f'(c) = 0$	$f''(c) < 0$	$f(x)$ has maximum value at c
If $f'(c) = 0$	$f''(c) > 0$	$f(x)$ has minimum value at c
If $f'(c) = 0$	$f''(c) = 0$	Test is inconclusive

Q. Find the Maximum and Minimum values of the $\boldsymbol{f(x) = x^5 - 5x^4 + 5x^3 - 1}$.

Solution:

Given $f(x) = x^5 - 5x^4 + 5x^3 - 1$

$\therefore f'(x) = 5x^4 - 20x^3 + 15x^2$ and $f''(x) = 20x^3 - 60x^2 + 30x$

For maximum and minimum values (for critical point)

$f'(x) = 0$

$\Rightarrow 5x^4 - 20x^3 + 15x^2 = 0$

$\Rightarrow 5x^2(x^2 - 4x + 3) = 0$

$\Rightarrow 5x^2(x^2 - 3x - x + 3) = 0$

$\Rightarrow x = 0, 1, 3$

Now $f''(0) = 0$, Therefore $f(x)$ has no maximum or minimum values at $x = 0$.

Again at $x = 1$, $f''(1) = -10 < 0$. Therefore $f(x)$ has a maximum value at $x = 1$ which is $f(1) = 1 - 5 + 5 - 1 = 0$.

Again at $x = 3$, $f''(3) = 20 \times 3^3 - 50 \times 3^2 + 30 \times 3 = 20.27 > 0$. Therefore $f(x)$ has a minimum value at $x = 3$ which is $f(3) = 3^5 - 5.3^4 + 5.3^3 - 1 = -28$.

Therefore, the maximum value is 0 and the minimum value is -28.

Q. Find the Maximum and Minimum values of the $\boldsymbol{f(x) = 4x^3 - 9x^2 + 6x}$.

Solution:

Given $f(x) = 4x^3 - 9x^2 + 6x$

$\therefore f'(x) = 12x^2 - 18x + 6$ and $f''(x) = 24x - 18$

For maximum and minimum values: $f'(x) = 0 \Rightarrow 12x^2 - 18x + 6 = 0$

$$\Rightarrow 2x^2 - 3x + 1 = 0 \Rightarrow (2x - 1)(x - 1) = 0$$

$$\Rightarrow x = \frac{1}{2}\,,\ 1$$

At $x = \frac{1}{2}$ $f''\left(\frac{1}{2}\right) = 12 - 18 = -6 < 0$; Therefore $f(x)$ has a maximum value at $x = \frac{1}{2}$

Therefore $f\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 - 9\left(\frac{1}{2}\right)^2 + 6\left(\frac{1}{2}\right) = \frac{4}{8} - \frac{9}{4} + 3 = \frac{4-18+24}{8} = \frac{5}{4}$

Again at $x = 1$ $f''(1) = 24 - 18 = 6 > 0$

Therefore $f(x)$ has a minimum value at $x = 1$

Therefore $f(1) = 4(1)^3 - 9(1)^2 + 6(1) = 1$

So, the maximum value is $\frac{5}{4}$ and the minimum value is 1.

Q. Find the Maximum and Minimum values of the $f(x) = 2x^3 - 3x^2 - 12x$.

Solution:

Given $f(x) = 2x^3 - 3x^2 - 12x$

$\therefore f'(x) = 6x^2 - 6x - 12$ and $f''(x) = 12x - 6$

For maximum and minimum values: $f'(x) = 0$

$$\Rightarrow 6x^2 - 6x - 12 = 0$$

$$\Rightarrow x^2 - x - 2 = 0$$

$$\Rightarrow (x-2)(x+1) = 0$$

$$\Rightarrow x = 2, -1$$

At $x = -1$ $f''(2) = -12 - 6 = -18 < 0$

Therefore $f(x)$ has a maximum value at $x = -1$

Therefore $f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) = 7$

Again At $x = 2$ $f''(2) = 24 - 6 = 18 > 0$

Therefore $f(x)$ has a minimum value at $x = 2$

Therefore $f(2) = 2(2)^3 - 3(2)^2 - 12(2) = -20$

So, the maximum value is 7 and the minimum value is -20.

Exercise .

Find the Maximum and Minimum values of the following functions:

(a) $f(x) = 5x^6 - 18x^5 + 15x^4 - 10$

(b) $f(x) = 12x^5 - 5x^4 + 40x^3 + 6$