So. $\frac{dL}{dt} = I\alpha = \tau$ [according to Newton's second law of angular motion]

Now, $\tau = 0$, i.e., if torque is not active on the body.

$$\frac{dL}{dt} = 0 \quad \therefore \quad L = \text{constant}$$

Hence, **if the resultant torque acting on a body is zero, there will not be any change of angular momentum. This is the conservation principle of angular momentum.**

Example : You have seen **trapeze** game in the circus. There players demonstrate many exercises in air. While jumping from a swinging pad the hands and feet of the player remain stretched. At this time his angular velocity is minimum. Now when hands and feet are folded and brought near to the chest, the angular velocity increases; so, it becomes easier for him to somersault in air successively. Due to folding of hands and feet moment of inertia (I) of the player becomes less, but as his angular momentum $L = I\omega$ remains constant so his angular velocity increases [Fig. 4·27].

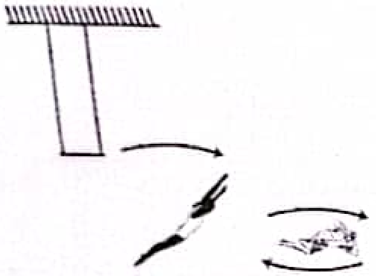

Fig. 4·27

Verify : Explain rotation of the body while jumping from the diving board or while skating on ice feat showing rotation on toes many times.

4·24 Moment of inertia and radius of gyration

4·24·1 Moment of inertia

When a rigid body is confined in a fixed axis, then if force is applied on that body, it cannot move in straight line due to confinement. The body rotates around the axis and there is angular displacement of each of the particles of the body. This type of motion of a body with respect to an axis is called rotational motion. The axis can either be inside or outside the body.

Definition : If a rigid body rotates around an axis, then moment of inertia of that body with respect to that axis means the summation of the product of square of distance from the axis and mass of each of the particles of that body.

Explanation : Let B be a rigid body which is rotating around a fixed axis XY with a uniform angular velocity ω [Fig. 4·28]. If the body is the summation of innumerable particles of masses m_1, m_2, m_3 .... m_n and the particles are respectively at distances r_1, r_2, r_3 .... r_n from the axis of rotation, then according to the definition with respect to that axis,

The moment of inertia of first particle $= m_1r_1^2$

The moment of inertia of second particle $= m_2r_2^2$

The moment of inertia of third particle $= m_3r_3^2$ and

The moment of inertia of n-th particle $= m_nr_n^2$

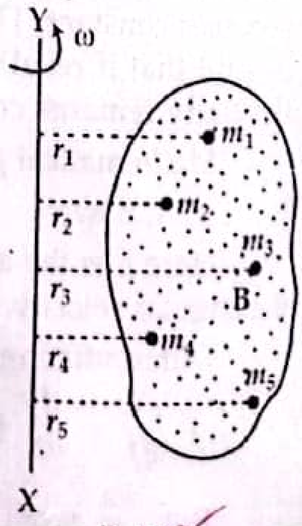

Fig. 4·28

So, according to definition the moment of inertia for the whole body with respect of that axis is,

$$I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \dots\dots + m_nr_n^2$$

$$\therefore\ I = \sum_{i=1}^{n} m_ir_i^2 \qquad \dots \qquad \dots \qquad \dots \qquad (4{\cdot}32)$$

$\left[\sum_{i=1}^{n}\right.$ sign indicates summation of all the quantities $\left.\right]$

By integration the moment of inertia can be expressed in the following way :

$$I = \int r^2 dm \qquad \dots \qquad \dots \qquad \dots \qquad (4{\cdot}33)$$

where dm is the mass of infinitesimally small element of the body and r is the distance of that small part from rotational axis.

Moment of inertia does not depend on angular velocity of particles, but depends on distribution of particles about the axis of rotation.

Unit and dimension of moment of inertia :

In M.K.S. and S.I. system unit of moment of inertia is kilogram-metre (kg-m^2) and dimension is [I] = [mass × (distance)^2] = [ML^2]

To know : Moment of inertia depends—

I. on the position of the rotational axis

II. on the shape of the rigid body

III. on the distribution of mass of the rotating body around the rotational axis.

4·24·2 Radius of gyration

Definition : If the total mass of a rigid body is assumed to be concentrated at a point and if the moment of inertia of that point mass with respect to a rotational axis is equal to the moment of inertia of that whole body, then the distance of that point from the axis is called the radius of gyration. It is denoted by K.

Explanation : Let B be a rigid body which is rotating with respect to an axis XY. The rigid body is composed of innumerable particles of masses $m_1, m_2, m_3 \dots\dots m_n$ and the particles are respectively at distances $r_1, r_2, r_3 \dots\dots r_n$ from the axis of rotation.

Now, suppose there is a point mass M at a distance of K from the rotational axis [Fig. 4·29].

$$M = \sum m_i = (m_1 + m_2 + m_3 + \dots\dots + m_n)$$

Clearly, moment of inertia in both cases will be same.

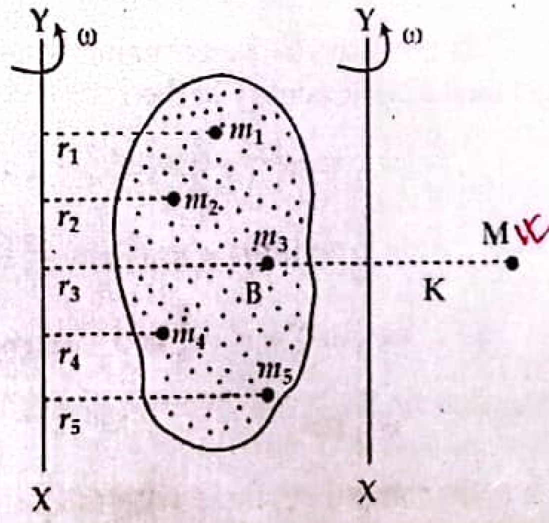

Fig. 4·29

That means,

$$MK^2 = \sum m_ir_i^2 = (m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \dots\dots + m_nr_n^2) \qquad \dots \qquad (4.34)$$

$$\therefore \quad K = \sqrt{\frac{m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \dots\dots\dots + m_nr_n^2}{M}}$$

$$= \sqrt{\frac{I}{M}} \qquad \dots \qquad \dots \qquad \dots \qquad (4.35)$$

Radius of gyration of a body with respect to a fixed axis is 0·2 m means that if moment of inertia is determined considering that total mass of the body is concentrated at a distance 0·2 m from that axis, then total moment of inertia is found out.

Example : Moment of inertia of solid sphere with respect to diameter is, $I = \frac{2}{5}MR^2$.

So, radius of gyration with respect to the diameter is,

$$K = \sqrt{\frac{I}{M}} = \sqrt{\frac{\frac{2}{5}MR^2}{M}} = \sqrt{\frac{2}{5}}\,R.$$

4·25 Rotational kinetic energy

Suppose a rigid body B is rotating in a circular orbit with uniform angular velocity of ω around XY axis [Fig. 4·29]. For this rotation the body contains some kinetic energy. **This energy is called rotational kinetic energy.**

Let, linear velocity of particle m_1 be v_1, so $v_1 = \omega r_1$

linear velocity of particle m_2 be v_2, so $v_2 = \omega r_2$

linear velocity of particle m_3 be v_3, so $v_3 = \omega r_3$

So, kinetic energy of the particle $m_1 = \frac{1}{2}m_1v_1^2 = \frac{1}{2}m_1\omega^2r_1^2$

kinetic energy of the particle $m_2 = \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_2\omega^2r_2^2$

kinetic energy of the particle $m_3 = \frac{1}{2}m_3v_3^2 = \frac{1}{2}m_3\omega^2r_3^2$

In this way, by determining kinetic energies of all the particles and adding them we get total kinetic energy of the body. So, rotational kinetic energy of the body,

$$= \frac{1}{2}m_1\omega^2r_1^2 + \frac{1}{2}m_2\omega^2r_2^2 + \frac{1}{2}m_3\omega^2r_3^2 + \dots\dots\dots\dots +$$

$$= \frac{1}{2}\omega^2\,[m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \dots]$$

$$= \frac{1}{2}\omega^2\Sigma m_ir_i^2 = \frac{1}{2}\omega^2.I \quad \text{[according to equation 4·32]}$$

$$= \frac{1}{2}I\omega^2 \qquad \dots \qquad \dots \qquad \dots \qquad (4·36)$$

By comparing force with torque we find that **the role played by a mass in linear motion, inertia plays the same role in rotational motion.**

Now if $\omega = 1$, i.e., moment of inertia for a rotating body with unit angular velocity. $I = 2E$ or twice the kinetic energy. So, it can be said that **moment of inertia of a body rotating with unit angular velocity is two times the kinetic energy.**

Proof : Let the lamina is composed of particle of masses m_1, m_2, m_3 etc. Distances from the axis CD of the particles are respectively x_1, x_2, x_3 etc. then moment of inertia of the particle of mass m_1 with respect to the axis AB

$$= m_1(x_1 + h)^2 = m_1x_1^2 + m_1h^2 + 2m_1x_1h$$

Similarly, moment of inertia of the particle of mass m_2 with respect to the axis AB

$$= m_2x_2^2 + m_2h^2 + 2m_2x_2h$$

moment of inertia of the particle of mass m_3

$$= m_3x_3^2 + m_3h^2 + 2m_3x_3h \text{ etc.}$$

$\therefore$ If I is moment of inertia of the whole lamina with respect to the axis AB, then it will be equal to the summation of the above moments of inertia.

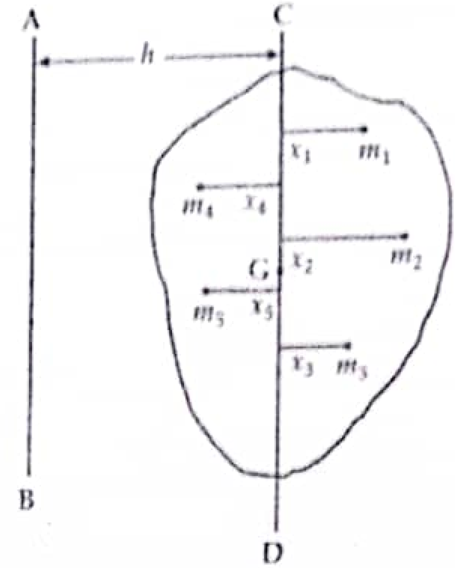

Fig. 4.32

$$\therefore I = m_1x_1^2 + m_1h^2 + 2m_1x_1h + m_2x_2^2 + m_2h^2 + 2m_2x_2h + m_3x_3^2 + m_3h^2 + 2m_3x_3h + \dots \dots$$

$$= \Sigma mx^2 + h^2\Sigma m + 2h\Sigma mx$$

Here, Σmx = total moment of the mass of the whole lamina about CD. But the weight of the lamina is acting downward along the line CD through the point G, so moment of the mass of the lamina about the axis CD.

$$\Sigma mx = 0$$

Again, $\Sigma m = M$ and $I_G = \Sigma mx^2$

$$\therefore I = I_G + Mh^2 \quad \dots \quad \dots \quad \dots \quad (4.42)$$

4·29 Determination of moment of inertia and radius of gyration for some special cases

(1) Moment of inertia and radius of gyration of a thin uniform rod about an axis through its centre and perpendicular to its length

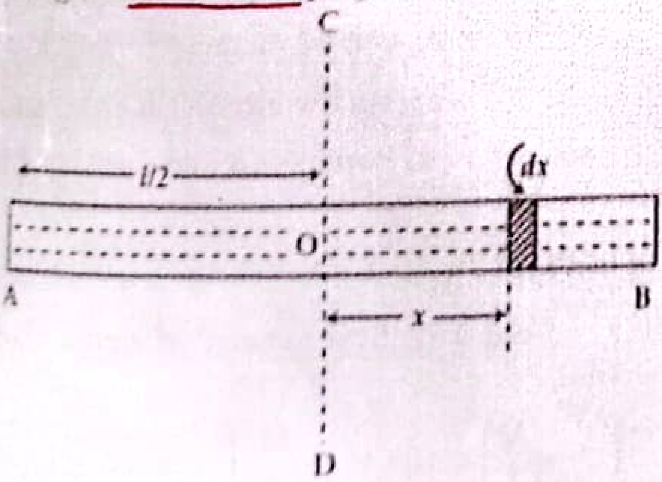

Fig. 4.33

Let AB be a thin uniform rod of length l and mass M, free to rotate about the axis CD which is passing through the centre O and perpendicular to the length of the rod [Fig. 4.33]. The moment of inertia about the axis CD and radius of gyration are to be found out.

Since the rod is uniform, the mass per unit length $= \frac{M}{l}$. So, at a distance x from the axis CD let dx be a small length whose mass is dM, then $dM = \frac{M}{l}dx$. As dx is very small, we can consider that all the particles in dx are at a distance x from CD. So, moment of inertia of dx about axis CD $= dM \times x^2 = \frac{M}{l} \times dx \times x^2$.

Now integrating the above equation within limits $x = l/2$ and $x = -l/2$ we get moment of inertia for the entire rod.

$\therefore$ Moment of inertia of the rod about the axis CD is,

$$I = \int_{-l/2}^{l/2} \left(\frac{M}{l}\right) x^2 dx = \frac{M}{l} \int_{-l/2}^{l/2} x^2 dx$$

$$= \frac{M}{l}\left[\frac{x^3}{3}\right]_{-l/2}^{l/2} = \frac{M}{l}\left[\frac{l^3}{3 \times 8} + \frac{l^3}{3 \times 8}\right]$$

$$= \frac{M}{l} \times \frac{2l^3}{24} = \frac{Ml^2}{12}$$

$$\therefore \quad I = \frac{M}{12} l^2 \qquad \ldots \qquad \ldots \qquad \ldots \qquad (4.43)$$

Let K be the radius of gyration

$$\therefore \quad MK^2 = I = \frac{M}{12} l^2$$

$$\text{or,} \quad K = \frac{l}{\sqrt{12}} = \frac{l}{2\sqrt{3}} \qquad \ldots \qquad \ldots \qquad \ldots \qquad (4.44)$$

(2) Moment of inertia and radius of gyration of a thin uniform rod about an axis passing through one end and perpendicular to its length

Let us consider a thin and uniform rod. Its mass is M and length is l. The rod is rotating around CD, one of its end A and perpendicular to its length [Fig. 4'34]. The moment of inertia and the radius of gyration of the rod around this CD are to be determined.

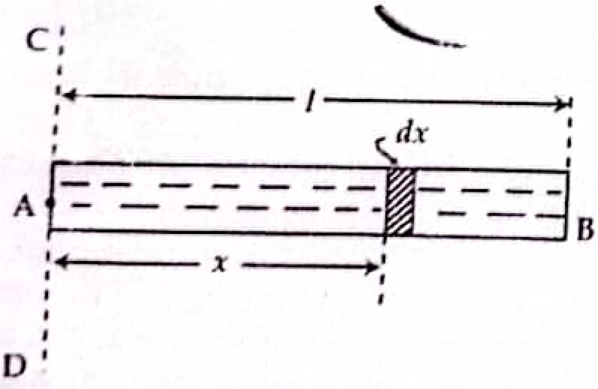

Fig. 4 34

According to the description, as the rod is uniform its mass per unit length is $\frac{M}{l}$. So, mass of a small element of length dx. $dM = \frac{M}{l} \times dx$. As the element is very small, so each of its particles can be considered to be at a distance of r from the axis CD.

$\therefore$ In respect of CD axis the moment of inertia of the small part this rod $= \frac{M}{l} \times dx \times x^2$

Now, if it is integrated within limits $x = 0$ and $x = l$, then moment of inertia of the whole rod with respect to CD will be found out

$$\therefore \text{ required moment of inertia, } I = \int_{x=0}^{x=l} \left(\frac{M}{l}\right) \times dx \times x^2 = \frac{M}{l} \int_{x=0}^{x=l} x^2 dx$$

$$= \frac{M}{l}\left[\frac{x^3}{3}\right]_0^l = \frac{M}{3l} \times l^3$$

$\therefore\ I = \frac{1}{3} Ml^2$ ... ... (4.45)

Now, if K is the radius of gyration, then $MK^2 = \frac{1}{3} Ml^2$

$\therefore\ K = \frac{l}{\sqrt{3}}$ ... ... (4.46)

(3) Moment of inertia and radius of gyration of a solid cylinder rotating about its own axis

Let mass of a uniform solid cylinder C be M, length l and radius r [Fig. 4·35]. It is rotating around its axis PQ. Moment of inertia and radius of gyration with respect to PQ are to be determined. According to the description, volume of the cylinder $= \pi r^2 \times l$

density of the material of the cylinder $= \frac{\text{mass}}{\text{volume}} = \frac{M}{\pi r^2 l}$

Let a hollow coaxial thin cylinder of radius r and width dx be considered around PQ.

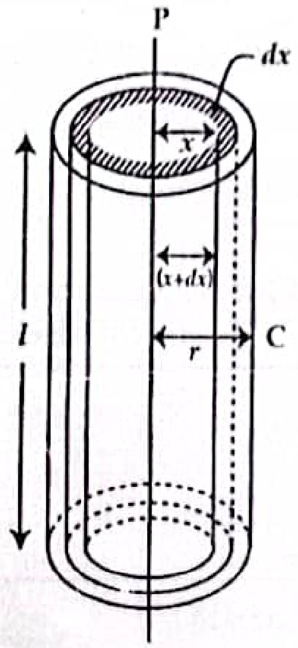

Fig. 4·35

Area of this thin cylinder $= 2\pi x dx$, volume $= 2\pi x \times dx \times l$

Now, mass = volume × density

$$= 2\pi x \times dx \times l \times \frac{M}{\pi r^2 l}$$

$$= \frac{2Mxdx}{r^2}$$

Since the cylinder of width dx is very thin hence its each particle can be considered to be at equal distance x from PQ. So, moment of inertia of this thin cylinder with respect to PQ

$$= \frac{2Mxdx}{r^2} \times x^2 = \frac{2M}{r^2} x^3 dx$$

The whole cylinder be considered to be formed by many similar thin cylinders.

So, within limits $x = 0$ and $x = r$ if the moment of inertia of the above hollow thin cylinder is integrated then we can find the moment of inertia I of the whole cylinder.

$$\therefore\quad I = \int_0^r \frac{2M}{r^2} x^3 dx = \frac{2M}{r^2}\int_0^r x^3 dx = \frac{2M}{r^2}\left[\frac{x^4}{4}\right]_0^r$$

$$= \frac{2M}{4r^2}\left[r^4 - 0\right]$$

$\therefore\quad I = \frac{1}{2} Mr^2$ ... ... (4.47)

In this case if radius of gyration is K, then

$MK^2 = \frac{1}{2} Mr^2$

$\therefore\ K = \frac{r}{\sqrt{2}}$ ... ... (4.48)