Boolean Algebra and Logic Circuits

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Learning Objectives

- **In this chapter you will learn about:**

✓Boolean algebra

✓Fundamental concepts and basic laws of Boolean algebra

✓Boolean function and minimization

✓Logic gates

✓Logic circuits and Boolean expressions

✓Combinational circuits and design

Boolean Algebra

- An algebra that deals with **binary number system**
- **George Boole (1815-1864)**, an English mathematician, developed it for:
  - Simplifying representation
  - Manipulation of propositional logic
- In 1938, **Claude E. Shannon** proposed using **Boolean algebra** in design of **relay switching circuits**
- Provides **economical** and **straightforward** approach
- Used extensively in designing **electronic circuits** used in computers

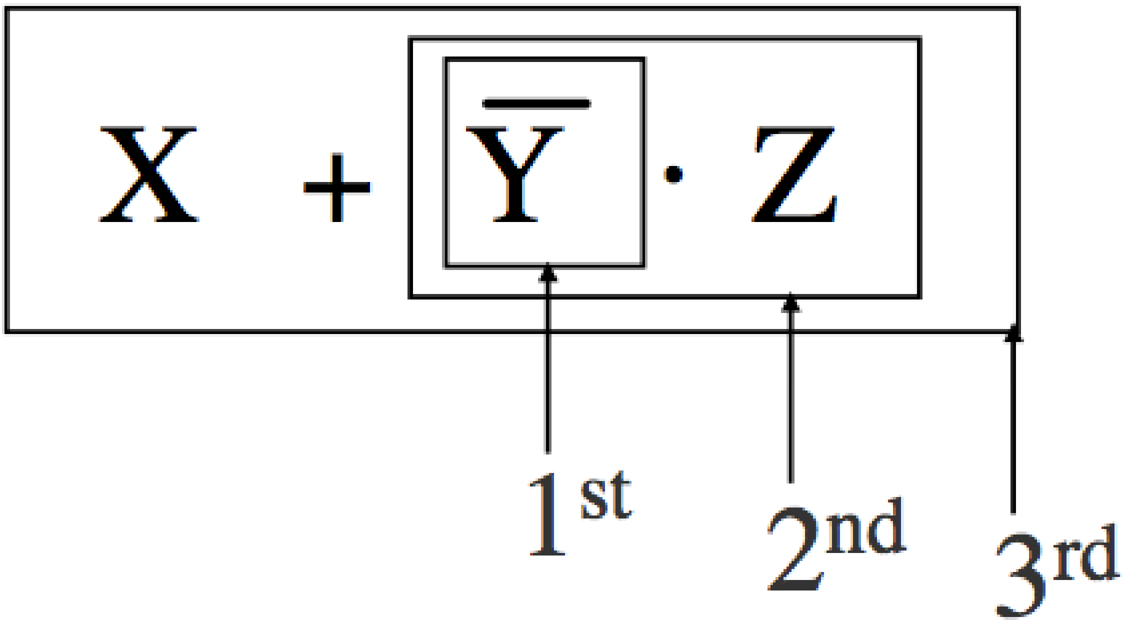
Fundamental Concepts of Boolean Algebra

- **Use of Binary Digit**
  - **Boolean equations** can have either of two possible values, 0 and 1
- Logical Addition
  - Symbol ‘**+**’, also known as ‘**OR**’ operator, used for **logical addition**.
  - Follows law of **binary addition**
- **Logical Multiplication**
  - Symbol ‘**.** ’, also known as ‘**AND**’ operator, used for logical multiplication.
  - Follows law of binary multiplication
- **Complementation**
  - Symbol ‘**-**’, also known as ‘**NOT**’ operator, used for complementation.
  - Follows law of binary compliment

Operator Precedence

- Each operator has a **precedence level**
- Higher the **operator's precedence level**, earlier it is evaluated
- Expression is scanned from **left to right**
- First, expressions enclosed within **parentheses are evaluated**
- Then, all **complement (NOT) operations** are performed
- Then, **all '·' (AND) operations** are performed
- Finally, **all '+' (OR) operations** are performed

Operator Precedence



Postulates of Boolean Algebra

- ***Postulate 1:***
  1. A = 0, if and only if, A is not equal to 1
  2. A = 1, if and only if, A is not equal to 0
- ***Postulate 2:***
  1. $x + 0 = x$
  2. $x \cdot 1 = x$
- ***Postulate 3: Commutative Law***
  1. $x + y = y + x$
  2. $x \cdot y = y \cdot x$

Postulates - are the basic structure from which lemmas and theorems are derived.
Theorem - a mathematical statement that is proved using rigorous mathematical reasoning.
Lemma - a minor result whose sole purpose is to help in proving a theorem. It is a stepping stone on the path to proving a theorem.

Postulates of Boolean Algebra

- ***Postulate 4: Associative Law***
  ① $x + (y + z) = (x + y) + z$
  ② $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
- ***Postulate 5: Distributive Law***
  ① $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$
  ② $x + (y \cdot z) = (x + y) \cdot (x + z)$
- ***Postulate 6:***
  ① $x + \overline{x} = 1$
  ② $x \cdot \overline{x} = 0$

The Principle of Duality

There is a precise duality between the operators . (AND) and + (OR), and the digits 0 and 1.

For example, in the table below, the second row is obtained from the first row and vice versa simply by interchanging '+' with '.' and '0' with '1'

	Column 1	Column 2	Column 3
Row 1	$1 + 1 = 1$	$1 + 0 = 0 + 1 = 1$	$0 + 0 = 0$
Row 2	$0 \cdot 0 = 0$	$0 \cdot 1 = 1 \cdot 0 = 0$	$1 \cdot 1 = 1$

Therefore, if a particular theorem is proved, its dual theorem automatically holds and need not be proved separately

Some Important Theorems of Boolean Algebra

Sr. No.	Theorems/ Identities	Dual Theorems/ Identities	Name (if any)
1	$x + x = x$	$x \cdot x = x$	Idempotent Law
2	$x + 1 = 1$	$x \cdot 0 = 0$	
3	$x + x \cdot y = x$	$x \cdot x + y = x$	Absorption Law
4	$\overline{\overline{x}} = x$		Involution Law
5	$x \cdot \overline{x} + y = y$	$x + \overline{x} \cdot y = x + y$	
6	$\overline{x+y} = \overline{x}\ \overline{y} \cdot$	$\overline{x \cdot y} = \overline{x}\ \overline{y} +$	De Morgan's Law

Methods of Proving Theorems

- **The theorems of Boolean algebra may be proved by using one of the following methods:**
  1. By using postulates to show that **L.H.S. = R.H.S**
  2. By ***Perfect Induction* or *Exhaustive Enumeration*** method where all possible combinations of variables involved in **L.H.S.** and **R.H.S.** are checked to **yield identical results**
  3. By the ***Principle of Duality*** where the dual of an already proved theorem is **derived from the proof of its corresponding pair**

Proving a Theorem by Using Postulates (Example)

- ***Theorem:*** $x + x \cdot y = x$
- ***Proof:***

L.H.S.	
$= x + x \cdot y$	
$= x \cdot 1 + x \cdot y$	by postulate 2(b)
$= x \cdot (1 + y)$	by postulate 5(a)
$= x \cdot (y + 1)$	by postulate 3(a)
$= x \cdot 1$	by theorem 2(a)
$= x$	by postulate 2(b)
$=$ R.H.S.	

Proving a Theorem by Perfect Induction (Example)

- **Theorem:** $x + x \cdot y = x$

x	y	$x \cdot y$	$x + x \cdot y$
0	0	0	0
0	1	0	0
1	0	0	1
1	1	1	1

Proving a Theorem by Perfect Induction (Example)

- **Theorem:** $x + x = x$

Proof:

L.H.S.

$= x + x$

$= (x + x) \cdot 1$ by postulate 2(b)

$= (x + x) \cdot (x + \overline{x})$ by postulate 6(a)

$= x + x \cdot \overline{x}$ by postulate 5(b)

$= x + 0$ by postulate 6(b)

$= x$ by postulate 2(a)

$=$ R.H.S.

Proving a Theorem by Perfect Induction(Example)

Dual Theorem:

$$x \cdot x = x$$

Proof:

L.H.S.		Notice that each step of the proof of the dual theorem is derived from the proof of its corresponding pair in the original theorem
$= x \cdot x$		
$= x \cdot x + 0$	by postulate 2(a)	
$= x \cdot x + x \cdot \overline{x}$	by postulate 6(b)	
$= x \cdot (x + \overline{x})$	by postulate 5(a)	
$= x \cdot 1$	by postulate 6(a)	
$= x$	by postulate 2(b)	
$=$ R.H.S.		

Boolean Functions

- A **Boolean function** is an expression formed with:
  - Binary variables
  - Operators (OR, AND, and NOT)
  - Parentheses, and equal sign
- The value of a **Boolean function** can be either 0 or 1
- A **Boolean function** may be represented as:
  - An algebraic expression, or
  - A truth table

Representation as an Algebraic Expression

$$W = X + \overline{Y} \cdot Z$$

Variable W is a function of X, Y, and Z, can also be written as W = f (X, Y, Z)

The RHS of the equation is called an ***expression***

The symbols X, Y, Z are the ***literals*** of the function

For a given Boolean function, there may be more than one algebraic expressions

Representation as a Truth Table

- The number of rows in the table is equal to 2^n, where n is the number of literals in the function

X	Y	Z	W
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

Minimization of Boolean Functions

- **Minimization of Boolean functions deals** with
  - Reduction in number of literals
  - Reduction in number of terms
- Minimization is achieved through manipulating expression to obtain equal and simpler expression(s) (having fewer literals and/or terms)

Minimization of Boolean Functions

$$F_1 = \overline{x} \cdot \overline{y} \cdot z + \overline{x} \cdot y \cdot z + x \cdot \overline{y}$$

F_1 has 3 literals (x, y, z) and 3 terms

$$F_2 = x \cdot \overline{y} + \overline{x} \cdot z$$

F_2 has 3 literals (x, y, z) and 2 terms

F_2 can be realized with fewer electronic components, resulting in a cheaper circuit

Minimization of Boolean Functions

x	y	z	F_1	F_2
0	0	0	0	0
0	0	1	1	1
0	1	0	0	0
0	1	1	1	1
1	0	0	1	1
1	0	1	1	1
1	1	0	0	0
1	1	1	0	0

- Both F_1 and F_2 produce the same result

Try out some Boolean Function Minimization

(a) $x + \overline{x} \cdot y$

(b) $x \cdot \left(\overline{x} + y \right)$

(c) $\overline{x} \cdot \overline{y} \cdot z + \overline{x} \cdot y \cdot z + x \cdot \overline{y}$

(d) $x \cdot y + \overline{x} \cdot z + y \cdot z$

(e) $\left(x + y \right) \cdot \left(\overline{x} + z \right) \cdot \left(y + z \right)$