# PARTIAL FRACTIONS

# LEARNING OBJECTIVES

- distinguish between proper and improper fractions
- identify the types of partial fractions
- express a given algebraic fraction as the sum of its partial fractions
- use the cover up rule
- express an improper algebraic fraction as a proper fraction before expressing it in partial fractions

# DEFINITIONS

## Rational Fraction

If $P(x)$ and $Q(x)$ are two polynomials in $x$ and $Q(x) \neq 0$ then the quotient $\frac{P(x)}{Q(x)}$ is called a rational fraction.

For example, $\frac{x^2+1}{x^3-2x+3}$ is a rational fraction.

## Proper Fraction

A rational fraction $\frac{P(x)}{Q(x)}$ in which the degree of numerator $P(x)$ is less than the degree of denominator $Q(x)$ is called a proper fraction.

For example, $\frac{6x+27}{3x^3-9x}$ is a proper fraction.

## Improper Fraction

A rational fraction $\frac{P(x)}{Q(x)}$ in which the degree of numerator $P(x)$ is greater than or equal to the degree of denominator $Q(x)$ is called an improper fraction.

For example, $\frac{x^2+1}{x^2-2x+3}$ and $\frac{x^3+1}{x^2-2x+3}$ are improper fractions.

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# PARTIAL FRACTIONS

A complex algebraic fraction can be expressed as the sum of 2 or more simpler fractions, known as partial fractions.

$$\frac{7x-25}{(x-3)(x-4)} = \frac{4}{x-3}+\frac{3}{x-4}$$

The complex algebraic fraction

Its partial fractions

# FORM OF PARTIAL FRACTIONS

The number and the form of partial fractions of an algebraic fraction $\frac{P(x)}{Q(x)}$ depends on the factors in $Q(x)$:

$$\text{Consider} \quad \frac{P(x)}{Q(x)} = \frac{2x+3}{(x+1)(x-2)^2(x^2+3)(x^2+1)^3}$$

In $Q(x)$, there is a:

- non-repeated linear factor of the form $ax+b$
- repeated linear factor, i.e. $(x-2)$ is repeated twice
- non-repeated quadratic factor of the form $ax^2+bx+c$
- repeated quadratic factor, i.e. $(x^2+1)$ is repeated thrice

# FORM OF PARTIAL FRACTIONS

For every non-repeated linear factor $ax + b$ in $Q(x)$, there will be a partial fraction of the form $\frac{A}{ax+b}$, where $A$ is a constant whose value is to be determined.

$$\frac{5x-2}{(2x-1)(x+5)} = \frac{A}{2x-1} + \frac{B}{x+5}$$

$$\frac{5x-2}{(2x-1)(x+5)(3x+2)} = \frac{A}{2x-1} + \frac{B}{x+5} + \frac{C}{3x+2}$$

# FORM OF PARTIAL FRACTIONS

For a repeated linear factor $(ax+b)^n$ in $Q(x)$, there will be a partial fraction of the form $\frac{A}{(ax+b)^n}+\frac{B}{(ax+b)^{n-1}}+\frac{C}{(ax+b)^{n-2}}+\ldots+\frac{D}{ax+b}$, where A, $B, C, \ldots, D$ are constants whose values are to be determined.

$$\frac{7x+4}{(x+1)^3}=\frac{A}{(x+1)^3}+\frac{B}{(x+1)^2}+\frac{C}{x+1}$$

$$\frac{7x+4}{(x+1)^3(3x+2)}=\frac{A}{(x+1)^3}+\frac{B}{(x+1)^2}+\frac{C}{x+1}+\frac{D}{3x+2}$$

# FORM OF PARTIAL FRACTIONS

For every quadratic factor $ax^2 + bx + c$ in $Q(x)$, there will be a partial fraction of the form $\frac{Ax+B}{ax^2+bx+c}$, where $A$ and $B$ are constants whose values are to be determined.

$$\frac{x}{(2x^2+1)(x+3)} = \frac{Ax+B}{2x^2+1} + \frac{C}{x+3}$$

$$\frac{1}{x^2(x^2-7)} = \frac{A}{x^2} + \frac{B}{x} + \frac{Cx+D}{x^2-7}$$

# FORM OF PARTIAL FRACTIONS

For a repeated quadratic factor $(ax^2+bx+c)^n$ in $Q(x)$, there will be a partial fraction of the form $\frac{Ax+B}{(ax^2+bx+c)^n}+\frac{Cx+D}{(ax^2+bx+c)^{n-1}}+\ldots+\frac{Ex+F}{ax^2+bx+c}$, where A, $B, C, D, \ldots, E, F$ are constants whose values are to be determined.

$$\frac{x^2+x+2}{(x^2+3)^2}=\frac{Ax+B}{(x^2+3)^2}+\frac{Cx+D}{x^2+3}$$

$$\frac{x^2+x+2}{(x^2+3)^2(x-1)}=\frac{Ax+B}{(x^2+3)^2}+\frac{Cx+D}{x^2+3}+\frac{E}{x-1}$$

# FINDING NUMERATOR CONSTANTS IN PARTIAL FRACTIONS

After writing out the form of the partial fractions, we need to find the constants in the numerators of these partial fractions

For example,

$$\frac{42-19x}{(x-4)(x^2+1)} = \frac{A}{x-4} + \frac{Bx+C}{x^2+1}$$

Our aim is to find the values of $A, B$ & $C$

We can determine these numerator constants by:

- using the cover-up rule
- comparing like terms in $x$ on both sides, or
- substituting suitable values of $x$

# Finding Numerator Constants in Partial Fractions

| **Cover-up Rule** | **Comparing** | **Substitution** |
|---|---|---|

The cover-up rule is used to find the numerator constant of a partial fraction corresponding to a non-repeated linear factor in $Q(x)$.

For example,

$$\frac{42-19x}{(x-4)(x^2+1)} = \frac{A}{x-4} + \frac{Bx+C}{x^2+1}$$

Non-repeated linear factor

We can use the cover-up rule to find $A$ as it is the numerator of the partial fraction corresponding to the non-repeated linear factor $x-4$.

# Finding Numerator Constants in Partial Fractions

| **Cover-up Rule** | Comparing | Substitution |
| --- | --- | --- |

$$\frac{42 - 19x}{(x-4)(x^2+1)} = \frac{A}{x-4} + \frac{Bx+C}{x^2+1}$$

- The root of $x - 4$ is $4$ (since if $x - 4 = 0$, then $x = 4$)
- We "cover-up" $x - 4$ in the given algebraic fraction
- Substitute the root $x = 4$ into given algebraic fraction *(excluding the cover up term)* to find $A$:

$$A = \frac{42 - 19(4)}{(4)^2 + 1} = \frac{42 - 76}{17} = -2$$

# Finding Numerator Constants in Partial Fractions

Cover-up Rule | **Comparing** | Substitution

$$\frac{42-19x}{(x-4)(x^2+1)} = \frac{-2}{x-4} + \frac{Bx+C}{x^2+1} \ldots (1)$$

Now to find $B$ and $C$, we either compare like terms in $x$, or substitute in suitable values of $x$. For comparing, multiplying both sides of (1) by the terms in denominator in the given algebraic fraction [here, $(x-4)(x^2+1)$]:

$$42-19x = -2(x^2+1) + (Bx+C)(x-4)$$

$$42-19x = -2x^2 - 2 + Bx^2 - 4Bx + Cx - 4C$$

$$42-19x = (B-2)x^2 + (C-4B)x + (-4C-2)$$

# Finding Numerator Constants in Partial Fractions

| Cover-up Rule | **Comparing** | Substitution |
|---|---|---|

We can now compare like terms in $x$ on both sides:

$$42 - 19x = (B-2)x^2 + (C-4B)x + (-4C-2)$$

Comparing the $x^2$ terms,

$$B - 2 = 0 \qquad 0x^2\ (\text{L.H.S.}) = (B-2)x^2\ (\text{R.H.S.})$$

$$\therefore B = 2$$

Comparing the $x$ terms, $C - 4B = -19 \qquad -19x\ (\text{L.H.S.}) = (C-4B)x\ (\text{R.H.S.})$

$$C = -19 + 4(2)$$

$$\therefore C = -11$$

OR,

Comparing the constant terms, $-4C - 2 = 42 \qquad 42\ (\text{L.H.S.}) = (-4c-2)x\ (\text{R.H.S.})$

$$-4C = 44$$

$$\therefore C = -11$$

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# Finding Numerator Constants in Partial Fractions

Cover-up Rule | Comparing | **Substitution**

$$42 - 19x = (B-2)x^2 + (C-4B)x + (-4C-2)$$

We can also substitute suitable values of $x$ to find $B$ and $C$.

Substitute $x = 0$:

$$42 - 19(0) = (B-2)(0)^2 + (C-4B)(0) + (-4C-2)$$

$$42 = -4C - 2$$

$$4C = -44$$

$$\therefore C = -11$$

Substitute $x = 1$:

$$42 - 19(1) = (B-2)(1)^2 + (C-4B)(1) + (-4C-2)$$

$$23 = B - 2 + (-11 - 4B)(1) + (44 - 2)$$

$$3B = 6$$

$$\therefore B = 2$$

# Finding Numerator Constants in Partial Fractions

| Cover-up Rule | Comparing | Substitution |
| --- | --- | --- |

For example,

$$\frac{42-19x}{(x-4)(x^2+1)} = \frac{A}{x-4} + \frac{Bx+C}{x^2+1}$$

Our aim is to find the values of $A, B$ & $C$

And finally, write out the partial fractions with the found values for the numerator constants:

$$\frac{42-19x}{(x-4)(x^2+1)} = \frac{-2}{x-4} + \frac{2x-11}{x^2+1}$$

***Note:***

The cover-up rule can also be used to find the numerator of the partial fraction which denominator is the highest power of a repeated linear factor in Q(x).