Differential equation of simple harmonic motion:

If "F" be the force acting on a particle executing simple harmonic motion and "Y" it's displacement from its mean position, then

$$F \propto -Y$$

$$\therefore\ F = -KY \;\ldots\ldots\ldots\ldots(1)$$

According to newton's second law of motion, we have

$$F = ma \;\ldots\ldots\ldots\ldots(2)$$

From equation (1) and (2)

$$ma = -ky$$

$$\Rightarrow m\frac{d^2y}{dt^2} = -ky$$

$$\Rightarrow \frac{d^2y}{dt^2} = -\frac{k}{m}y$$

$$\Rightarrow \frac{d^2y}{dt^2} + \frac{k}{m}y = 0 \;\ldots\ldots\ldots\ldots\ (3)$$

When , $\omega = \sqrt{\frac{k}{m}}$ is the angular frequency. This equation is called the differential equation of motion of a body execating simple harmonic motion.

To obtain a general solution of the differential equation of simple harmonic motion , let us multiply both sides of equation (3) by $2\frac{dy}{dt}$, we get

$$2\frac{dy}{dt}\cdot\frac{d^2y}{dt^2} = \omega^2 y\,.2\frac{dy}{dt}$$

$$\Rightarrow\ 2\frac{dy}{dt}\cdot\frac{d^2y}{dt^2} = -2\omega^2 .\, y\frac{dy}{dt}$$

Integrating with respect to time, we have

$$\left(\frac{dy}{dt}\right)^2 = -\omega^2 y^2 + c \ \ldots\ldots\ldots\ldots(4)$$

Where, "C" is a constant of integration.

At maximum displacement, the velocity is zero.

$$\frac{dy}{dt} = 0 \text{ when } Y = a$$

From equation (4), we have, $0 = -\omega^2 y^2 + c$

$$\therefore c = \omega^2 a^2$$

Substituting the value of "c" in equation (4)

$$\left(\frac{dy}{dt}\right)^2 = -\omega^2 y^2 + \omega^2 a^2$$

$$\Rightarrow \left(\frac{dy}{dt}\right)^2 = \omega^2(a^2 - y^2)$$

$$\Rightarrow \frac{dy}{dt} = \omega\sqrt{a^2 - y^2}$$

$$\Rightarrow \frac{dy}{\sqrt{a^2-y^2}} = \omega dt$$

$$\Rightarrow \int \frac{dy}{\sqrt{a^2-y^2}} = \int \omega dt$$

$$\Rightarrow \sin^{-1}\frac{y}{a} = \omega t + \varphi$$

$$\Rightarrow \frac{y}{a} = \sin(\omega t + \varphi)$$

$$\therefore y = a\sin(\omega t + \varphi)$$

This is the general solution of the differential equation of simple harmonic motion.

Average Values of Energies of Harmonic Oscillator:

The potential energy of the particle at a displacement y is given by.

$$PE = \frac{1}{2} m \omega^2 y^2 \qquad [y = a\sin(\omega t + \phi)]$$

$$= \frac{1}{2} m \omega^2 a^2 \sin^2(\omega t + \phi),$$

So the average potential energy of a particle over a complete cycle

$$= \frac{1}{T}\int_0^T \frac{1}{2} m\omega^2 a^2 \sin^2(\omega t + \phi)\, dt$$

$$= \frac{1}{T}\cdot\frac{m\omega^2 a^2}{4}\int_0^T 2\sin^2(\omega t + \phi)\, dt$$

$$= \frac{m\omega^2 a^2}{4T}\int_0^T [1 - \cos 2(\omega t + \phi)]\, dt$$

$$= \frac{m\omega^2 a^2}{4T}\left\{\int_0^T dt - \int_0^T \cos 2(\omega t + \phi)\, dt\right.$$

The average value of both a sine and a cosine function for a complete cycle is zero.

Therefore,

$$P.E. = \frac{1}{4T} m\omega^2 a^2 [t]_0^T - 0$$

$$= \frac{1}{4} m\omega^2 a^2$$

$$= \frac{1}{4} k a^2 \qquad \left[\omega^2 = \frac{k}{m}\right] \text{ ——— } (1)$$

The kinetic energy of the particle at displacement y is given by

$$KE = \frac{1}{2} m \left(\frac{dy}{dt}\right)^2$$

$$KE = \frac{1}{2} m \left[\frac{d}{dt} a\sin(\omega t + \phi)\right]^2$$

$$KE = \frac{1}{2} m \left[\frac{d}{dt} a \sin(\omega t + \varphi)\right]^2$$

$$= \frac{1}{2} m \omega^2 a^2 \cos^2(\omega t + \varphi)$$

The average KE of the particle for a complete cycle is

$$= \frac{1}{T}\int_0^T \frac{1}{2} m \omega^2 a^2 \cos^2(\omega t + \varphi)\, dt$$

$$= \frac{1}{T} \frac{m\omega^2 a^2}{4} \int_0^T \left[1 + \cos 2(\omega t + \varphi) dt\right]$$

$$= \frac{m\omega^2 a^2}{4T}\left[\int_0^T dt + \int_0^T \cos 2(\omega t + \varphi)\right.$$

The average value of both sine and cosine function over a complete cycle is zero.

$$\therefore \text{Average K.E.} = \frac{m\omega^2 a^2}{4T}\left[t\right]_0^T = \frac{1}{4} m\omega^2 a^2 = \frac{1}{4} k a^2$$

Average PE = average KE $= \frac{1}{4} k a^2 =$ half the total energy.

Composition of two simple harmonic vibrations in a straight line:

Let two simple harmonic vibrations be represented by the equations.

$$y_1 = a_1 \sin(\omega t + \varphi_1)$$

$$y_2 = a_2 \sin(\omega t + \varphi_2)$$

Where, y_1 and y_2 are the displacement of the particle due to the individual vibrations of amplitudes a_1 and a_2 respectively.

$$\therefore y = y_1 + y_2$$

$$= a_1 \sin(\omega t + \varphi_1) + a_2 \sin(\omega t + \varphi_2)$$

$$= a_1(\sin\omega t \cos\varphi_1 + \cos\omega t \sin\varphi_1) + a_2(\sin\omega t \cos\varphi_2 + \cos\omega t \sin\varphi_2)$$

$$= (a_1 \cos\varphi_1 + a_2 \cos\varphi_2) \sin\omega t + (a_1 \sin\varphi_1 + a_2 \sin\varphi_2) \cos\omega t$$

The amplitudes a_1 and a_2 are constant .Hence putting,

$$a_1 \cos\varphi_1 + a_2 \cos\varphi_2 = A \cos\varphi$$

$$a_1 sin\varphi_1 + a_2 sin\varphi_2 = A \sin\varphi$$

∴The resultant amplitude can be written as,

$$Y = A \cos\varphi \sin \omega t + A \sin\varphi \cos \omega t .$$

$$= \text{A} \sin (\omega t + \varphi).$$

<u>Composition of two simple harmonic vibrations at right angles to each other having equal frequencies but differing in phase and amplitude.</u>

Let two simple harmonic motion of the same frequency but of amplitude "a" and "b" and having their vibrations mutually perpendicular to one another.

If "φ" is the phase difference between the two vibrations, then

$x = \text{a} \sin(\omega t + \varphi)$ ....(1)

$y = \text{b} \sin \omega t$ .... (2)

From equation (1), $\frac{x}{a} = \sin(\omega t + \varphi) = \sin \omega t \cos\varphi + \cos\omega t \sin\varphi$

$= \sin \omega t \cos\varphi + \sqrt{1 - sin^2 \omega t} \sin\varphi$ .... (3)

From equation (2), $\sin \omega t = \frac{y}{b}$

In equation (3) $\frac{x}{a} = \frac{y}{b} \cos\varphi + \sqrt{1 - \frac{y}{b^2}^2} \sin\varphi.$

$$\Rightarrow (\frac{x}{a} - \frac{y}{b} \cos\varphi)^2 = (1 - \frac{y}{b^2}^2) sin^2\varphi$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} cos^2\varphi - \frac{2xy}{ab} \cos\varphi = (1 - \frac{y}{b^2}^2) sin^2\varphi$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} (cos^2\varphi + sin^2\varphi) - \frac{2xy}{ab} \cos\varphi = sin^2\varphi$$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} \cos\varphi = sin^2\varphi.$$

Expression ~~of~~ for a plane progressive wave :

A progressive wave is one which travels onward through the medium in a given direction without attenuation i.e., with its amplitude constant.

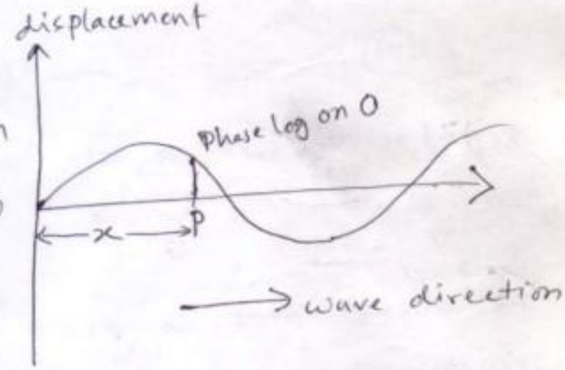

Fig → progressive wave

A typical wave form is shown in fig. Let a wave is originating at O, travel to the right along the x-axis. The equation of motion of this particle at O is obviously

$$Y = a \sin \omega t \qquad \text{——— (1)}$$

where y is the displacement of the particle at time t, a its amplitude and ω its angular velocity.

For a particle at P which is at a distance x away from O, let this phase difference be ϕ. Hence the equation of motion of the particle at P is

$$Y = a\sin(\omega t - \phi) \quad \text{——— (ii)}$$

For a difference in path of λ, the difference in phase is 2π. Hence for a distance x, the corresponding phase difference is $\frac{2\pi}{\lambda}.x$. Substituting this value in eqn (ii) we get

$$Y = a\sin\left(\omega t - \frac{2\pi}{\lambda}x\right)$$

$$\text{or, } Y = a\sin(\omega t - kx) \quad \text{——— (iii)}$$

Where $k = \frac{2\lambda}{\lambda}$ is referred to as the propagative constant.

Now $\omega = \frac{2\pi}{T}$, where T is a time period for a complete ~~co~~ oscillation. n is the frequency, therefr. $v = n\lambda$, or $\frac{1}{T} = n = \frac{v}{\lambda}$, Then eqn (iii) becomes.

$$Y = a\sin\left(\frac{2\pi v}{\lambda}t - \frac{2\pi}{\lambda}x\right)$$

$$\text{a } Y = a\sin\frac{2\pi}{\lambda}(vt - x)$$

The most commonly used equation of ~~wave~~ a progressive wave.

Differential equation of wave motion (from general equ^n of a progressive wave)

=> The general equation of a progressive simple harmonic wave is

$$Y = a \sin \frac{2\pi}{\lambda} (vt - x) \quad \text{---} (I)$$

Differentiating equ^n (1) with respect to time

$$\frac{dy}{dt} = a \frac{2\pi v}{\lambda} \cos \frac{2\pi}{\lambda} (vt - x) \quad \text{---} (II)$$

Again differentiating equ^n (II) w r t time

$$\frac{d^2y}{dt^2} = - a \cdot \frac{2\pi v}{\lambda} \cdot \frac{2\pi v}{\lambda} \sin(vt - x)$$

$$= - a \frac{4\pi^2 v^2}{\lambda^2} \sin(vt - x) \quad \text{---} (III)$$

To find the value of compression, differentiate equ^n (1) with respect to x.

$$\frac{dy}{dx} = - a \frac{2\pi}{\lambda} \cos \frac{2\pi}{\lambda} (vt - x) \quad \text{---} (IV)$$

differentiate equ^n (IV) with respect to x

$$\frac{d^2y}{dx^2} = - \frac{4\pi^2 a}{\lambda^2} \sin \frac{2\pi}{\lambda} (vt - x) \quad \text{---} (V)$$

From equ's (II) and (IV) we get

$$\frac{dy}{dt} = - v \frac{dy}{dx} \quad \text{---} (VI)$$

from eqn's (III) and (V) we get

$$\frac{d^2y}{dt^2} = v^2 \frac{d^2y}{dx^2} \quad \text{——————} \quad (VII)$$

Equation (VII) represents the differential equation of wave motion.

The general differential equation of wave motion can be written as

$$\frac{d^2y}{dt^2} = K \frac{d^2y}{dx^2} \quad \text{——————} \quad (VIII)$$

where, $K = v^2$

and $v = \sqrt{K}$

Thus knowing the value of K, the value of v can be calculated