Wave and oscillation- part-2

Equilibrium Position

An equilibrium position is a point where an oscillating object experiences zero (0) resultant forces.

Amplitude

Amplitude is the maximum displacement on both sides of an object from its equilibrium position. The SI unit for amplitude is meter (m).

There are two types of amplitude (i) Linear amplitude denoted by **A** (ii) Angular amplitude denoted by θ.

Phase

The state of motion of a vibrating particle at any instant is called its phase at that instant.

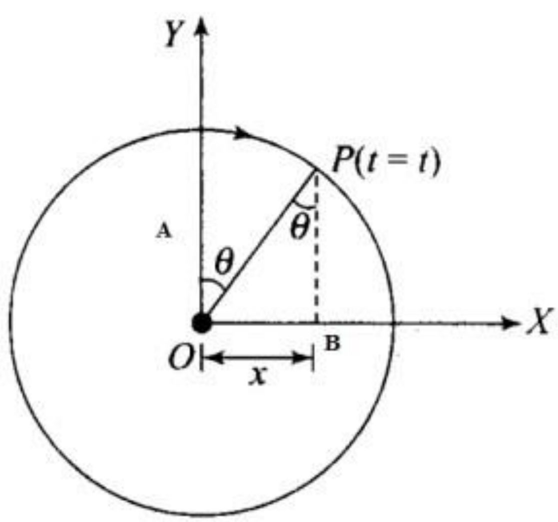
State of motion of a vibrating particle at any instant is determined by its displacement, velocity

and acceleration at that instant.

Particle Executing Simple Harmonic Motion

Displacement

Let angular velocity of the particle executing circular motion around a circle having radius A is ω and when it is at point P makes an angle θ as shown in the diagram.



From the figure

$$\sin\theta = \frac{OB}{OP}$$

$$\sin\theta = \frac{x}{A} \dots\dots..(1)$$

Here, x is the displacement from the origin O. And OP = A is the radius of the circle.

Equation number (1) can be written as

$$x = A\sin\theta$$

We know that, angular displacement , $\theta = \omega t$

So, the equation becomes

$$x = A\sin\omega t \dots\dots\dots\dots(2)$$

In another form

$$x = A\sin 2\pi nt \qquad (\text{Since}, \omega = \sin 2\pi nt\ )$$

Velocity:

The rate of change of displacement is called velocity. It is denoted by v . Mathematically, we can write

$$v = \frac{dx}{dt}$$

$$v = \frac{d}{dt}(x)$$

$$v = \frac{d}{dt}(A\sin\omega t) \quad [\text{using equation number 2}]$$

$$v = A\omega\cos\omega t \quad \dots\dots..(3)$$

From equation number (2)

$$x = A\sin\omega t$$

$$\sin\omega t = \frac{x}{A}$$

And

$$\cos\omega t = \sqrt{1-\sin^2\omega t} \qquad [\text{Since}, \sin^2\theta + \cos^2\theta = 1]$$

From equation (3)

$$v = A\omega\cos\omega t$$

$$= A\,\omega\sqrt{1-\ \sin^2\omega t}$$

$$= A\,\omega\ \sqrt{1-\frac{x^2}{A^2}}$$

$$= \omega\sqrt{A^2\ -\ x^2}\ \ \ldots\ldots\ldots\ldots\ldots.(4)$$

When (i) $x\ =\ A$ (The maximum displacement from the equilibrium position) then $v = 0$ (ii) when $x = 0$ then $v\ =\ A\,\omega$

Acceleration:

The rate of change of velocity is called acceleration. It is denoted by a . Mathematically, we can write

$$a = \frac{dv}{dt}$$

$$a = \frac{d}{dt}\left(\omega\sqrt{A^2\ -\ x^2}\right) \qquad \text{[Using equation number (4)]}$$

$$a =\ -A\,\omega^2 \sin\omega\,t$$

$$a =\ -\omega^2 A \sin\omega\,t$$

$$a =\ -\omega^2\,x \qquad\qquad [\text{Since,}\quad A\sin\omega\,t = x\ ]$$

When (i) $x\ =\ 0,\ a\ =\ 0$ (ii) when, $x\ = A,\ a\ = -\,\omega^2 A$

Solution of the Differential Equation of Simple Harmonic Motion

Differential equation of a particle executing simple harmonic motion is

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

In order to solve this equation, let us multiply both sides by $2\frac{dx}{dt}$, then

$$2\frac{dx}{dt}.\frac{d^2x}{dt^2} + 2\frac{dx}{dt}.\omega^2 x = 0$$

Integrating the above equation, we get

$$\left(\frac{dx}{dt}\right)^2 + \omega^2 x^2 = C \text{ .... } (1)$$

Here, C is a constant of integration. We need to find out the value of this.

Now, when

$x = A$, then velocity $\frac{dx}{dt} = 0$

Putting this value in equation (1)

$$C = \omega^2 A^2$$

From equation (1)

$$\left(\frac{dx}{dt}\right)^2 + \omega^2 x^2 = \omega^2 A^2$$

$$\left(\frac{dx}{dt}\right)^2 = \omega^2 A^2 - \omega^2 x^2$$

$$\frac{dx}{dt} = \omega \sqrt{A^2 - x^2}$$

$$\frac{dx}{\sqrt{A^2 - x^2}} = \omega \, dt$$

Integrating the above equation

$$\sin^{-1}\frac{x}{A} = \omega t + \delta \qquad \left[\text{Since, } \int \frac{dx}{\sqrt{A^2 - x^2}} = \sin^{-1}\frac{x}{A} \right]$$

Where, δ is the integrational constant.

Finally,

$$x = A\sin(\omega t + \delta)$$

This is the general solution of differential equation of simple harmonic oscillation.

Particle Executing Simple Harmonic Motion

Kinetic Energy:

Kinetic energy can be written as

$$K.E = \frac{1}{2}\, m\, v^2 \ldots\ldots\ldots\ldots (1)$$

Instantaneous velocity can be written as

$$v = \frac{d\,x}{d\,t}$$

$$= \frac{d}{d\,t}\{A \sin(\omega\ t + \delta)\}$$

$$= A\omega \cos(\omega\ t + \delta) \ldots\ldots\ldots\ldots (*)$$

$$= A\omega \sqrt{1 - \sin^2(\omega t + \delta)}$$

$$= A\,\omega \sqrt{1 - (\frac{x}{A})^2} \qquad [\text{Since}, x = A \sin(\omega t + \delta)]$$

$$= \omega \sqrt{A^2 - x^2} \ldots\ldots\ldots\ldots (2)$$

From equation (1)

$$K.E = \frac{1}{2}\, m\, v^2$$

$$= \frac{1}{2}\, m\, (\frac{dx}{dt})^2$$

$$= \frac{1}{2}\, m\, \omega^2 A^2 \cos^2(\omega t + \delta) \quad [\text{Using equation}(*)]$$

or

$$= \frac{1}{2}\, m\, \omega^2\, (A^2 - x^2) \qquad [\text{Using equation (2)}]$$

Potential Energy:

Now, the amount of work done for displacement x against the restoring force will remain stored as potential energy in the object.

Stored potential energy for the such kind of harmonic oscillator can be written as

$$P.E = \frac{1}{2}\ k\,x^2$$

$$= \frac{1}{2}\ k\,(\,A\sin(\omega t + \delta)\ )^2 \qquad [\,\text{Since}, x\ =\ A\sin(\omega t + \delta)\,]$$

$$= \frac{1}{2}\,k\,A^2\sin^2(\omega t + \delta)$$

$$= \frac{1}{2}\ m\omega^2 A^2 \sin^2(\omega t + \delta) \qquad [\,\text{Angular velocity}, \omega = \sqrt{\frac{k}{m}}\ ]$$

Total Energy:

Total Energy (E) = Kinetic energy (K.E) + Potential Energy (P.E)

$$E = K.E + P.E$$

$$= \frac{1}{2}\ m\,\omega^2 A^2\cos^2(\omega t + \delta) + \frac{1}{2}\ m\omega^2 A^2\sin^2(\omega t + \delta)$$

$$= \frac{1}{2} m\,A^2\,\omega^2$$