Differentiation

Introduction: The derivative is a mathematical operator, which measures the rate of change of a quantity relative to another quantity. The process of finding a derivative is called differentiation. There are many phenomena related changing quantities such as speed of a particle, inflation of currency, intensity of an earthquake and voltage of an electrical signal etc. in the world. In this chapter we will discuss about various techniques of derivative.

Outcomes: After successful completion of the chapter, the students will be able to:

1. determine the speed, velocity and acceleration of a particle with respect to time.
2. calculate the rate at which the number of bacteria , the population changes with time.
3. measurethe rate at which the length of a metal rod changes with temperature.
4. find out the rate at which production cost changes with the quantity of a product .

Derivatives of elementary functions:

1. $\frac{d}{dx}(c) = 0,$ *where c is a constant.*

2. $\frac{d}{dx}(x) = 1.$

3. $\frac{d}{dx}(x^n) = nx^{n-1}.$

4. $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}.$

5. $\frac{d}{dx}(e^x) = e^x.$

6. $\frac{d}{dx}(a^x) = a^x \ln a.$

7. $\frac{d}{dx}(\ln x) = \frac{1}{x}.$

8. $\frac{d}{dx}(\sin x) = \cos x.$

9. $\frac{d}{dx}(\cos x) = -\sin x.$

10. $\frac{d}{dx}(\tan x) = \sec^2 x.$

11. $\frac{d}{dx}(\sec x) = \sec x \tan x.$

12. $\frac{d}{dx}(\cot x) = -\cos ec^2 x.$

13. $\frac{d}{dx}(\cos ecx) = -\cos ecx \cot x.$

14. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}.$

15. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}.$

16. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}.$

17. $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}.$

18. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}.$

19. $\frac{d}{dx}(\cos ec^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}.$

20. $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$

21. $\frac{d}{dx}(u^v) = u^v \frac{d}{dx}(v \ln u).$

22. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

where u and v are functions of x.

- ***Find the differential coefficient ($\frac{dy}{dx}$)of the following functions with respect to x.***

1. $y = 5x^8$

$\textbf{Sol}: Given\ that,\ y = 5x^8$

$Differentiating\ with\ respect\ to\ x\ we\ get,$

$$\frac{dy}{dx} = \frac{d}{dx}\left(5x^8\right)$$
$$= 5\frac{d}{dx}\left(x^8\right)$$
$$= 5\times 8x^{8-1}$$
$$= 40x^7 \quad (Ans.)$$

2. $y = 3x^7 + 2x + 1$

$\textbf{Sol}: Given\ that,\ y = 3x^7 + 2x + 1$

$Differentiating\ with\ respect\ to\ x\ we\ get,$

$$\frac{dy}{dx} = \frac{d}{dx}\left(3x^7 + 2x + 1\ \right)$$
$$= 3\frac{d}{dx}\left(x^7\right) + 2\frac{d}{dx}(x) + \frac{d}{dx}(1)$$
$$= 21x^6 + 2 + 0$$
$$= 21x^6 + 2 \quad (Ans.)$$

3. $y = 4\sin x - \cos x$

$\textbf{Sol}: Given\ that,\ y = 4\sin x - \cos x$

$Differentiating\ with\ respect\ to\ x\ we\ get,$

$$\frac{dy}{dx} = \frac{d}{dx}\left(4\sin x - \cos x\ \right)$$
$$= 4\frac{d}{dx}(\sin x) - \frac{d}{dx}(\cos x)$$
$$= 4\cos x - (-\sin x)$$
$$= 4\cos x + \sin x \quad (Ans.)$$

4. $y = \sec^2 x - \tan^2 x$

$\textbf{Sol}: Given\ that,\ y = \sec^2 x - \tan^2 x$

$Differentiating\ with\ respect\ to\ x\ we\ get,$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sec^2 x - \tan^2 x\right)$$
$$= \frac{d}{dx}\left(\sec^2 x\right) - \frac{d}{dx}\left(\tan^2 x\right)$$
$$= 2\sec x\frac{d}{dx}(\sec x) - 2\tan x\frac{d}{dx}(\tan x)$$
$$= 2\sec x(\sec x\tan x) - 2\tan x\left(\sec^2 x\right)$$
$$= 2\sec^2 x\tan x - 2\sec^2 x\tan x$$
$$= 0 \quad (Ans.)$$

5. $y = \ln\left(x + \sqrt{x^2 + a^2}\right)$

$\textbf{Sol}: Given\ that,\ y = \ln\left(x + \sqrt{x^2 + a^2}\right)$

$Differentiating\ with\ respect\ to\ x\ we\ get,$

$$\frac{dy}{dx} = \frac{d}{dx}\left\{\ln\left(x + \sqrt{x^2 + a^2}\right)\right\}$$
$$= \frac{1}{x + \sqrt{x^2 + a^2}}.\frac{d}{dx}\left(x + \sqrt{x^2 + a^2}\right)$$
$$= \frac{1}{x + \sqrt{x^2 + a^2}}.\left\{1 + \frac{d}{dx}\left(\sqrt{x^2 + a^2}\right)\right\}$$
$$= \frac{1}{x + \sqrt{x^2 + a^2}}.\left\{1 + \frac{1}{2\sqrt{x^2 + a^2}}.\frac{d}{dx}\left(x^2 + a^2\right)\right\}$$
$$= \frac{1}{x + \sqrt{x^2 + a^2}}.\left\{1 + \frac{2x}{2\sqrt{x^2 + a^2}}\right\}$$
$$= \frac{1}{x + \sqrt{x^2 + a^2}}.\left\{1 + \frac{x}{\sqrt{x^2 + a^2}}\right\}$$
$$= \frac{1}{x + \sqrt{x^2 + a^2}}.\frac{x + \sqrt{x^2 + a^2}}{\sqrt{x^2 + a^2}}$$
$$= \frac{1}{\sqrt{x^2 + a^2}} \quad (Ans.)$$

6. $y = \ln(\sec x + \tan x)$

$\textbf{Sol}: Given\ that,\ y = \ln(\sec x + \tan x)$

$Differentiating\ with\ respect\ to\ x\ we\ get,$

$$\frac{dy}{dx} = \frac{d}{dx}\{\ln(\sec x + \tan x)\}$$
$$= \frac{1}{\sec x + \tan x}.\frac{d}{dx}(\sec x + \tan x)$$
$$= \frac{\left(\sec x\tan x + \sec^2 x\right)}{\sec x + \tan x}$$
$$= \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x}$$
$$= \sec x$$
$$(Ans.)$$

7. $y = e^{ax^2+bx+c}$

Sol: *Given that*, $y = e^{ax^2+bx+c}$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{ax^2+bx+c}\right)$$
$$= e^{ax^2+bx+c}.\frac{d}{dx}\left(ax^2+bx+c\right)$$
$$= e^{ax^2+bx+c}\left(2ax+b+0\right)$$
$$= \left(2ax+b\right)e^{ax^2+bx+c}$$

(*Ans.*)

8. $y = e^{\sqrt{\cot x}}$

Sol: *Given that*, $y = e^{\sqrt{\cot x}}$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\sqrt{\cot x}}\right)$$
$$= e^{\sqrt{\cot x}}.\frac{d}{dx}\left(\sqrt{\cot x}\right)$$
$$= e^{\sqrt{\cot x}}.\frac{1}{2\sqrt{\cot x}}.\frac{d}{dx}\left(\cot x\right)$$
$$= \frac{e^{\sqrt{\cot x}}}{2\sqrt{\cot x}}.\left(-\cos ec^2 x\right)$$
$$= -\frac{e^{\sqrt{\cot x}}\cos ec^2 x}{2\sqrt{\cot x}}$$

(*Ans.*)

9. $y = \sqrt{x^3-2x+5}$

Sol: *Given that*, $y = \sqrt{x^3-2x+5}$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{x^3-2x+5}\right)$$
$$= \frac{1}{2\sqrt{x^3-2x+5}}.\frac{d}{dx}\left(x^3-2x+5\right)$$
$$= \frac{1}{2\sqrt{x^3-2x+5}}.\left(3x^2-2+0\right)$$
$$= \frac{3x^2-2}{2\sqrt{x^3-2x+5}}$$

(*Ans.*)

10. $y = \tan\ln\sin\left(e^{x^2}\right)$

Sol: *Given that*, $y = \tan\left(\ln\sin e^{x^2}\right)$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left\{\tan\left(\ln\sin e^{x^2}\right)\right\}$$
$$= \sec^2\left(\ln\sin e^{x^2}\right).\frac{d}{dx}\left\{\ln\left(\sin e^{x^2}\right)\right\}$$
$$= \sec^2\left(\ln\sin e^{x^2}\right).\frac{1}{\sin\left(e^{x^2}\right)}.\frac{d}{dx}\left\{\sin\left(e^{x^2}\right)\right\}$$
$$= \sec^2\left(\ln\sin e^{x^2}\right).\frac{1}{\sin\left(e^{x^2}\right)}.\cos\left(e^{x^2}\right).\frac{d}{dx}\left(e^{x^2}\right)$$
$$= \cot\left(e^{x^2}\right)\sec^2\left(\ln\sin e^{x^2}\right).e^{x^2}\frac{d}{dx}\left(x^2\right)$$
$$= 2xe^{x^2}\cot\left(e^{x^2}\right)\sec^2\left(\ln\sin e^{x^2}\right)$$

(*Ans.*)

11. $y = \cos^{-1}\left(e^{\cot^{-1}x}\right)$

Sol: *Given that*, $y = \cos^{-1}\left(e^{\cot^{-1}x}\right)$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left\{\cos^{-1}\left(e^{\cot^{-1}x}\right)\right\}$$
$$= -\frac{1}{\sqrt{1-e^{2\cot^{-1}x}}}.\frac{d}{dx}\left(e^{\cot^{-1}x}\right)$$
$$= -\frac{e^{\cot^{-1}x}}{\sqrt{1-e^{2\cot^{-1}x}}}.\frac{d}{dx}\left(\cot^{-1}x\right)$$
$$= -\frac{e^{\cot^{-1}x}}{\sqrt{1-e^{2\cot^{-1}x}}}\left(-\frac{1}{1+x^2}\right)$$
$$= \frac{e^{\cot^{-1}x}}{\left(1+x^2\right)\sqrt{1-e^{2\cot^{-1}x}}}$$

(*Ans.*)

12. $y = e^{\sin^{-1}x} + \tan^{-1}x$

Sol: *Given that*, $y = e^{\sin^{-1}x} + \tan^{-1}x$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\sin^{-1}x} + \tan^{-1}x\right)$$
$$= \frac{d}{dx}\left(e^{\sin^{-1}x}\right) + \frac{d}{dx}\left(\tan^{-1}x\right)$$
$$= e^{\sin^{-1}x}.\frac{d}{dx}\left(\sin^{-1}x\right) + \frac{1}{1+x^2}$$
$$= \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}} + \frac{1}{1+x^2}$$

(*Ans.*)

13. $y = x^2 \cot^{-1} x$

Sol : *Given that*, $y = x^2 \cot^{-1} x$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2 \cot^{-1} x\right)$$

$$= x^2 \frac{d}{dx}\left(\cot^{-1} x\right) + \cot^{-1} x \frac{d}{dx}\left(x^2\right)$$

$$= x^2 \left(\frac{-1}{1+x^2}\right) + \cot^{-1} x (2x)$$

$$= 2x \cot^{-1} x - \frac{x^2}{1+x^2}$$

$(Ans.)$

14. $y = x^3 \ln x$

Sol : *Given that*, $y = x^3 \ln x$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^3 \ln x\right)$$

$$= x^3 \frac{d}{dx}(\ln x) + \ln x \frac{d}{dx}\left(x^3\right)$$

$$= x^3 . \frac{1}{x} + \ln x \left(2x^2\right)$$

$$= x^2 + 2x^2 \ln x$$

$(Ans.)$

15. $y = xe^x \sin x$

Sol : *Given that*, $y = xe^x \sin x$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(xe^x \sin x\right)$$

$$= xe^x \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}\left(xe^x\right)$$

$$= xe^x \cos x + \sin x \left\{ x \frac{d}{dx}\left(e^x\right) + e^x \frac{d}{dx}(x) \right\}$$

$$= xe^x \cos x + \sin x \left(xe^x + e^x\right)$$

$$= xe^x \cos x + xe^x \sin x + e^x \sin x$$

$(Ans.)$

16. $y = e^{ax} \cos bx$

Sol : *Given that*, $y = e^{ax} \cos bx$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{ax} \cos bx\right)$$

$$= e^{ax} \frac{d}{dx}(\cos bx) + \cos bx \frac{d}{dx}\left(e^{ax}\right)$$

$$= e^{ax}(-b \sin bx) + \cos bx \left(ae^{ax}\right)$$

$$= ae^{ax} \cos bx - be^{ax} \sin bx$$

$(Ans.)$

17. $y = \left(x^2+1\right) \sin^{-1} x + e^{\sqrt{1+x^2}}$

Sol : *Given that*, $y = \left(x^2+1\right) \sin^{-1} x + e^{\sqrt{1+x^2}}$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left\{\left(x^2+1\right) \sin^{-1} x + e^{\sqrt{1+x^2}}\right\}$$

$$= \frac{d}{dx}\left\{\left(x^2+1\right) \sin^{-1} x\right\} + \frac{d}{dx}\left(e^{\sqrt{1+x^2}}\right)$$

$$= \left(x^2+1\right) . \frac{1}{\sqrt{1-x^2}} + \sin^{-1} x . (2x) + e^{\sqrt{1+x^2}} . \frac{1}{2\sqrt{1+x^2}} . 2x$$

$$= \frac{x^2+1}{\sqrt{1-x^2}} + 2x \sin^{-1} x + \frac{xe^{\sqrt{1+x^2}}}{\sqrt{1+x^2}}$$

$(Ans.)$

18. $y = e^{\sin x} \sin\left(a^x\right)$

Sol : *Given that*, $y = e^{\sin x} \sin\left(a^x\right)$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left\{e^{\sin x} \sin\left(a^x\right)\right\}$$

$$= e^{\sin x} \frac{d}{dx}\left\{\sin\left(a^x\right)\right\} + \sin\left(a^x\right) \frac{d}{dx}\left(e^{\sin x}\right)$$

$$= e^{\sin x} . \cos\left(a^x\right) . \frac{d}{dx}\left(a^x\right) + \sin\left(a^x\right) . e^{\sin x} . \cos x$$

$$= e^{\sin x} . \cos\left(a^x\right) . a^x \ln a + \sin\left(a^x\right) . e^{\sin x} . \cos x$$

$(Ans.)$

19. $y=\frac{\cos x}{1+\sin x}$

Sol : *Given that,* $y=\frac{\cos x}{1+\sin x}$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{\cos x}{1+\sin x}\right)$$

$$=\frac{(1+\sin x)\frac{d}{dx}(\cos x)-\cos x\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$$

$$=\frac{-\sin x-\sin^2 x-\cos^2 x}{(1+\sin x)^2}$$

$$=\frac{-\sin x-\left(\sin^2 x+\cos^2 x\right)}{(1+\sin x)^2}$$

$$=\frac{-\sin x-1}{(1+\sin x)^2}$$

$$=-\frac{1}{1+\sin x}$$

$(Ans.)$

20. $y=\frac{\cos x-\sin x}{\cos x+\sin x}$

Sol : *Given that,* $y=\frac{\cos x-\sin x}{\cos x+\sin x}$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)$$

$$=\frac{(\cos x+\sin x)\frac{d}{dx}(\cos x-\sin x)-(\cos x-\sin x)\frac{d}{dx}(\cos x+\sin x)}{(\cos x+\sin x)^2}$$

$$=\frac{(\cos x+\sin x)(-\sin x-\cos x)-(\cos x-\sin x)(-\sin x+\cos x)}{\cos^2 x+\sin^2 x+2\sin x\cos x}$$

$$=\frac{-(\cos x+\sin x)^2-(\cos x-\sin x)^2}{1+\sin 2x}$$

$$=\frac{-(1+\sin 2x)-(1-\sin 2x)}{1+\sin 2x}$$

$$=-\frac{2}{1+\sin 2x}$$

$(Ans.)$

21. $y=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

Sol : *Given that,* $y=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

put, $x=\sin\theta \quad \therefore\ \theta=\sin^{-1}x$

Now, $y=\tan^{-1}\left(\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right)$

$$=\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos^2\theta}}\right)$$

$$=\tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$$

$$=\tan^{-1}.\tan\theta$$

$$=\theta$$

$$=\sin^{-1}x$$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sin^{-1}x\right)$$

$$=\frac{1}{\sqrt{1-x^2}}$$

$(Ans.)$

22. $y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Sol : *Given that,* $y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

put, $x=\tan\theta \quad \therefore\ \theta=\tan^{-1}x$

Now, $y=\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$$=\cos^{-1}.\cos 2\theta$$

$$=2\theta$$

$$=2\tan^{-1}x$$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(2\tan^{-1}x\right)$$

$$=\frac{2}{1+x^2}$$

$(Ans.)$

23. $y=\tan^{-1}\left(\dfrac{\sqrt{(1+x^2)}-1}{x}\right)$

Sol : *Given that*, $y=\tan^{-1}\left(\dfrac{\sqrt{(1+x^2)}-1}{x}\right)$

$put,\ x=\tan\theta \quad \therefore\ \theta=\tan^{-1}x$

$$Now,\ y=\tan^{-1}\left(\frac{\sqrt{(1+\tan^2\theta)}-1}{\tan\theta}\right)$$

$$=\tan^{-1}\left(\frac{\sqrt{\sec^2\theta}-1}{\tan\theta}\right)$$

$$=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$$

$$=\tan^{-1}\left(\frac{1-\cos\theta}{\cos\theta}.\frac{\cos\theta}{\sin\theta}\right)$$

$$=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$=\tan^{-1}\left(\frac{2\sin^2\theta/2}{2\sin\theta/2\cos\theta/2}\right)$$

$$=\tan^{-1}\left(\frac{\sin\theta/2}{\cos\theta/2}\right)$$

$$=\tan^{-1}.\tan\theta/2$$

$$=\theta/2$$

$$=1/2\tan^{-1}x$$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(1/2\tan^{-1}x\right)$$

$$=\frac{1}{2(1+x^2)}$$

$(Ans.)$

Homework:-Find $\frac{dy}{dx}$ of the following functions:

1. $y=\ln\left(\sqrt{x-a}+\sqrt{x-b}\right)$ Ans: $\frac{1}{2\sqrt{(x-a)(x-b)}}$
2. $y=\cos(\ln x)+\ln(\tan x)$ Ans: $2\cos ec2x-\frac{\sin(\ln x)}{x}$
3. $y=sin^{-1}(e^{tan^{-1}x})$
4. $y=e^{ax}\sin^m rx$ Ans: $e^{ax}\sin^m rx(a+mr\cot rx)$
5. $y=\sin^{-1}x^2-xe^{x^2}$ Ans: $\frac{2x}{\sqrt{1-x^4}}-(2x^2+1)e^{x^2}$
6. $y=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ Ans: $\frac{1}{\sqrt{1-x^2}}$
7. $y=\tan^{-1}\left(\frac{4\sqrt{x}}{1-4x}\right)$ Ans: $\frac{2}{\sqrt{x}(1+4x)}$
8. $y=\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$ Ans: $-\frac{1}{2}$

Logarithmic differentiation: If we have functions that are composed of products, quotients and powers, to differentiate such functions it would be convenient first to take logarithm of the function and then differentiate. Such a technique is called the logarithmic differentiation.

1. $y=(\sin x)^{\ln x}$

Sol: *Given that*, $y=(\sin x)^{\ln x}$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left\{(\sin x)^{\ln x}\right\}$$
$$=(\sin x)^{\ln x}\frac{d}{dx}\{\ln x\ln(\sin x)\}$$
$$=(\sin x)^{\ln x}\left[\ln x.\frac{d}{dx}\{\ln(\sin x)\}+\ln(\sin x).\frac{d}{dx}(\ln x)\right]$$
$$=(\sin x)^{\ln x}\left[\ln x.\frac{1}{\sin x}.\cos x+\ln(\sin x).\frac{1}{x}\right]$$
$$=(\sin x)^{\ln x}\left[\cot x\ln x+\frac{\ln(\sin x)}{x}\right]$$

$(Ans.)$

2. $y=x^{1+x+x^2}$

Sol: *Given that*, $y=x^{1+x+x^2}$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^{1+x+x^2}\right)$$
$$=x^{1+x+x^2}\frac{d}{dx}\left\{(1+x+x^2)\ln x\right\}$$
$$=x^{1+x+x^2}\left[\ln x.\frac{d}{dx}(1+x+x^2)+(1+x+x^2).\frac{d}{dx}(\ln x)\right]$$
$$=x^{1+x+x^2}\left[\ln x.(0+1+2x)+(1+x+x^2).\frac{1}{x}\right]$$
$$=x^{1+x+x^2}\left[(2x+1)\ln x+\frac{(1+x+x^2)}{x}\right]$$

$(Ans.)$

3. $y=\left(\tan^{-1}x\right)^{\sin x+\cos x}$

Sol : *Given that*, $y=\left(\tan^{-1}x\right)^{\sin x+\cos x}$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left\{\left(\tan^{-1}x\right)^{\sin x+\cos x}\right\}$$

$$=\left(\tan^{-1}x\right)^{\sin x+\cos x}\frac{d}{dx}\left\{(\sin x+\cos x).\ln\left(\tan^{-1}x\right)\right\}$$

$$=\left(\tan^{-1}x\right)^{\sin x+\cos x}\left[(\sin x+\cos x)\frac{d}{dx}\left\{\ln\left(\tan^{-1}x\right)\right\}+\ln\left(\tan^{-1}x\right).\frac{d}{dx}(\sin x+\cos x)\right]$$

$$=\left(\tan^{-1}x\right)^{\sin x+\cos x}\left[\frac{(\sin x+\cos x)}{\tan^{-1}x}.\frac{1}{\left(1+x^2\right)}+\ln\left(\tan^{-1}x\right).(\cos x-\sin x)\right]$$

$(Ans.)$

4. $y=x^x+(\sin x)^{\ln x}$

Sol : *Given that*, $y=x^x+(\sin x)^{\ln x}$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left\{x^x+(\sin x)^{\ln x}\right\}$$

$$=\frac{d}{dx}\left(x^x\right)+\frac{d}{dx}\left\{(\sin x)^{\ln x}\right\}$$

$$=x^x\frac{d}{dx}(x\ln x)+(\sin x)^{\ln x}\frac{d}{dx}\left\{\ln x\ln(\sin x)\right\}$$

$$=x^x\left(x.\frac{1}{x}+\ln x\right)+(\sin x)^{\ln x}\left\{\ln x.\frac{1}{\sin x}.\cos x+\frac{\ln(\sin x)}{x}\right\}$$

$$=x^x(1+\ln x)+(\sin x)^{\ln x}\left\{\ln x.\cot x+\frac{\ln(\sin x)}{x}\right\}\quad Ans.$$

5. $y=(\sin x)^{\cos x}+(\cos x)^{\sin x}$

Sol : *Given that*, $y=(\sin x)^{\cos x}+(\cos x)^{\sin x}$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left\{(\sin x)^{\cos x}+(\cos x)^{\sin x}\right\}$$

$$=\frac{d}{dx}\left\{(\sin x)^{\cos x}\right\}+\frac{d}{dx}\left\{(\cos x)^{\sin x}\right\}$$

$$=(\sin x)^{\cos x}\frac{d}{dx}\left\{\cos x\ln(\sin x)\right\}+(\cos x)^{\sin x}\frac{d}{dx}\left\{\sin x\ln(\cos x)\right\}$$

$$=(\sin x)^{\cos x}\left[\cos x.\frac{1}{\sin x}.\cos x-\sin x\ln(\sin x)\right]+(\cos x)^{\sin x}\left[\sin x.\frac{1}{\cos x}.(-\sin x)+\cos x\ln(\cos x)\right]$$

$$=(\sin x)^{\cos x}\left[\cos x.\cot x-\sin x\ln(\sin x)\right]+(\cos x)^{\sin x}\left[\cos x\ln(\cos x)-\sin x.\tan x\right]\quad Ans.$$

6. $y = x^{\cos^{-1}x} - \sin x \ln x$

Sol : *Given that*, $y = x^{\cos^{-1}x} - \sin x \ln x$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{\cos^{-1}x} - \sin x \ln x\right)$$

$$= \frac{d}{dx}\left(x^{\cos^{-1}x}\right) - \frac{d}{dx}\left(\sin x \ln x\right)$$

$$= x^{\cos^{-1}x}\frac{d}{dx}\left(\cos^{-1}x \ln x\right) - \left(\frac{\sin x}{x} + \cos x \ln x\right)$$

$$= x^{\cos^{-1}x}\left[\frac{\cos^{-1}x}{x} - \frac{\ln x}{\sqrt{1-x^2}}\right] - \left(\frac{\sin x}{x} + \cos x \ln x\right) \quad Ans.$$

7. $y = \left(1+x^2\right)^{\tan x} + \left(2-\sin x\right)^{\ln x}$

Sol : *Given that*, $y = \left(1+x^2\right)^{\tan x} + \left(2-\sin x\right)^{\ln x}$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left\{\left(1+x^2\right)^{\tan x} + \left(2-\sin x\right)^{\ln x}\right\}$$

$$= \frac{d}{dx}\left\{\left(1+x^2\right)^{\tan x}\right\} + \frac{d}{dx}\left\{\left(2-\sin x\right)^{\ln x}\right\}$$

$$= \left(1+x^2\right)^{\tan x}\frac{d}{dx}\left\{\tan x \ln\left(1+x^2\right)\right\} + \left(2-\sin x\right)^{\ln x}\frac{d}{dx}\left\{\ln x \ln\left(2-\sin x\right)\right\}$$

$$= \left(1+x^2\right)^{\tan x}\left[\frac{2x\tan x}{1+x^2} + \sec^2 x \ln\left(1+x^2\right)\right] + \left(2-\sin x\right)^{\ln x}\left[\frac{\ln\left(2-\sin x\right)}{x} - \frac{\cos x \ln x}{\left(2-\sin x\right)}\right] \; Ans.$$

9. $(cosx)^y = (siny)^x$

Sol: Taking ln in both sides we get

$ylncosx = xlnsiny$

Differentiating w.r.to x we get

$$(lncosx)\frac{dy}{dx} + y\frac{1}{cosx}(-sinx) = 1.lnsiny + x\frac{1}{siny}cosy\frac{dy}{dx}$$

$$\Longrightarrow (lncosx - xcoty)\frac{dy}{dx} = (lnsiny + ytanx)$$

or $\frac{dy}{dx} = \frac{(\boldsymbol{lnsiny+ytanx})}{(\boldsymbol{lncosx-xcoty})}$ **Ans**

Homework:-Find $\frac{dy}{dx}$ of the following functions:

1. $y = x^{\sin^{-1}x}$ Ans: $x^{\sin^{-1}x}\left[\frac{\sin^{-1}x}{x} + \frac{\ln x}{\sqrt{1-x^2}}\right]$

2. $y = \left(\sin x\right)^{\cos^{-1}x}$ Ans: $\left(\sin x\right)^{\cos^{-1}x}\left[\cot x \cos^{-1}x - \frac{\ln \sin x}{\sqrt{1-x^2}}\right]$

3. $y = x^{x^x}$ Ans: $x^{x^x} x^x \left[1 + \ln x + \frac{1}{x}\right]$

4. $y = x^{\cos^{-1} x} + (\sin x)^{\ln x}$ Ans: $x^{\cos^{-1} x}\left[\frac{\cos^{-1} x}{x} - \frac{\ln x}{\sqrt{1-x^2}}\right] + (\sin x)^{\ln x}\left[\ln x \cot x + \frac{\ln \sin x}{x}\right]$

5. $y = (\tan x)^{\cot x} + (\cot x)^{\tan x}$ Ans: $(tanx)^{\cot x} \cos ec^2 x(1 - \ln \tan x) + (\cot x)^{\tan x} sec^2 x(\ln \cot x - 1)$

6. $x + y = \tan(xy)$

7. $\sqrt{\frac{y}{x}} + \sqrt{\frac{x}{y}} = 6$

Parametric Equation: If in the equation of a curve $y = f(x)$, x and y are expressed in terms of a third variable known as parameter i.e, $x = \varphi(t)$, $y = \psi(t)$ then the equations are called a parametric equation.

1. $x = a(t + \sin t)$, $y = a(1 - \cos t)$

sol : *Given that,*

$$x = a(t + \sin t) \cdots \cdots (1)$$

$$and \quad y = a(1 - \cos t) \cdots \cdots (2)$$

Differentiating (1) *and* (2) *with respect to* t *we get,*

$$\frac{dx}{dt} = a(1 + \cos t)$$

$$and \quad \frac{dy}{dt} = a \sin t$$

$$Now, \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{a \sin t}{a(1 + \cos t)}$$

$$= \frac{2 \sin \frac{t}{2} \cos \frac{t}{2}}{2 \cos^2 \frac{t}{2}}$$

$$= \tan \frac{t}{2} \quad (Ans.)$$

2. $x = a(\cos t + t \sin t)$, $y = a(\sin t - t \cos t)$

sol : *Given that,*

$$x = a(\cos t + t \sin t) \cdots \cdots (1)$$

$$and \quad y = a(\sin t - t \cos t) \cdots \cdots (2)$$

Differentiating (1) *and* (2) *with respect to* t *we get,*

$$\frac{dx}{dt} = a(-\sin t + t \cos t + \sin t)$$

$$= at \cos t$$

$$and \quad \frac{dy}{dt} = a(\cos t + t \sin t - \cos t)$$

$$= at \sin t$$

$$Now, \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{at \sin t}{at \cos t}$$

$$= \tan t \quad (Ans.)$$

3. $x = t - \sqrt{1-t^2}$, $y = e^{\sin^{-1} t}$

sol : *Given that*,

$$x = t - \sqrt{1-t^2} \cdots\cdots (1)$$

$$\text{and} \quad y = e^{\sin^{-1} t} \cdots\cdots (2)$$

Differentiating (1) *and* (2) *with respect to t we get*,

$$\frac{dx}{dt} = 1 - \frac{1}{2\sqrt{1-t^2}}.(-2t)$$

$$= 1 + \frac{t}{\sqrt{1-t^2}}$$

$$= \frac{t + \sqrt{1-t^2}}{\sqrt{1-t^2}}$$

$$\text{and} \quad \frac{dy}{dt} = e^{\sin^{-1} t}.\frac{1}{\sqrt{1-t^2}}$$

$$= \frac{e^{\sin^{-1} t}}{\sqrt{1-t^2}}$$

$$\text{Now,} \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{e^{\sin^{-1} t}}{\sqrt{1-t^2}}.\frac{\sqrt{1-t^2}}{t + \sqrt{1-t^2}}$$

$$= \frac{e^{\sin^{-1} t}}{t + \sqrt{1-t^2}} \quad (Ans.)$$

4. *Differentiate* $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ *with respect to* $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

sol : *Let*, $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$$= \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) \; ; \begin{bmatrix} putting \;\; x = \tan\theta \\ \therefore \;\; \theta = \tan^{-1} x \end{bmatrix}$$

$$= \tan^{-1}.\tan 2\theta$$

$$= 2\theta$$

$$= 2\tan^{-1} x \cdots\cdots (1)$$

$$\text{and} \quad z = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$= \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) \; ; \begin{bmatrix} putting \;\; x = \tan\theta \\ \therefore \;\; \theta = \tan^{-1} x \end{bmatrix}$$

$$= \sin^{-1}.\sin 2\theta$$

$$= 2\theta$$

$$= 2\tan^{-1} x \cdots\cdots (2)$$

Differentiating (1) *and* (2) *with respect to x we get*,

$$\frac{dy}{dx} = \frac{2}{1+x^2} \qquad \text{and} \quad \frac{dz}{dx} = \frac{2}{1+x^2}$$

$$\text{Now,} \quad \frac{dy}{dz} = \frac{dy/dx}{dz/dx}$$

$$= \frac{\frac{2}{1+x^2}}{\frac{2}{1+x^2}}$$

$$= 1 \quad (Ans.)$$

6. *Differentiate* $x^{\sin^{-1}x}$ *with respect to* $\sin^{-1}x$.

$sol: Let,\ y = x^{\sin^{-1}x} \ \cdots\cdots(1)$

$\quad and \quad z = \sin^{-1}x \cdots\cdots (2)$

Differentiating (1) *and* (2) *with respect to x we get,*

$$\frac{dy}{dx} = x^{\sin^{-1}x}\frac{d}{dx}\left(\sin^{-1}x\ln x\right) \ ;\left[\because \frac{d}{dx}\left(u^v\right) = u^v\frac{d}{dx}\left(v\ln u\right)\right]$$

$$= x^{\sin^{-1}x}\left(\frac{\sin^{-1}x}{x} + \frac{\ln x}{\sqrt{1-x^2}}\right)$$

$$and \ \frac{dz}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$Now,\ \frac{dy}{dz} = \frac{dy/dx}{dz/dx}$$

$$= \frac{x^{\sin^{-1}x}\left(\frac{\sin^{-1}x}{x} + \frac{\ln x}{\sqrt{1-x^2}}\right)}{\frac{1}{\sqrt{1-x^2}}}$$

$$= x^{\sin^{-1}x}\left(\frac{\sqrt{1-x^2}.\sin^{-1}x}{x} + \ln x\right) \quad (Ans.)$$

7. *Differentiate* $\tan^{-1}\left(\frac{\sqrt{1-x^2}-1}{x}\right)$ *with respect to* $\tan^{-1}x$.

$$sol: Let,\ y = \tan^{-1}\left(\frac{\sqrt{1-x^2}-1}{x}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{1-\sin^2\theta}-1}{\sin\theta}\right) \ ;\left[\begin{array}{r} putting \quad x = \sin\theta \\ \therefore \ \theta = \sin^{-1}x \end{array}\right]$$

$$= \tan^{-1}\left(\frac{\sqrt{\cos^2\theta}-1}{\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{\cos\theta-1}{\sin\theta}\right)$$

$$= \tan^{-1}\left\{-\frac{(1-\cos\theta)}{\sin\theta}\right\}$$

$$= \tan^{-1}\left\{-\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right\}$$

$$= \tan^{-1}\left\{-\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right\}$$

$$= \tan^{-1}\left\{-\tan\frac{\theta}{2}\right\}$$

$$= \tan^{-1}\left\{\tan\left(\pi-\frac{\theta}{2}\right)\right\}$$

$$= \pi - \frac{\theta}{2}$$

$$= \pi - \frac{1}{2}\sin^{-1}x \cdots\cdots (1)$$

$\quad and \quad z = \tan^{-1}x \cdots\cdots (2)$

Differentiating (1) *and* (2) *with respect to x we get,*

$$\frac{dy}{dx} = -\frac{1}{2\sqrt{1-x^2}} \qquad and \ \frac{dz}{dx} = \frac{1}{1+x^2}$$

$$Now,\ \frac{dy}{dz} = \frac{dy/dx}{dz/dx}$$

$$= \frac{-\frac{1}{2\sqrt{1-x^2}}}{\frac{1}{1+x^2}}$$

$$= -\frac{1+x^2}{2\sqrt{1-x^2}} \quad (Ans.)$$

8. *Differentiate* $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$ *with respect to* $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$.

$sol: Let,\ y=\sec^{-1}\left(\frac{1}{2x^2-1}\right)$

$$=\sec^{-1}\left(\frac{1}{2\cos^2\theta-1}\right);\left[\begin{array}{l}putting\ \ x=\cos\theta\\ \qquad\therefore\ \theta=\cos^{-1}x\end{array}\right]$$

$$=\sec^{-1}\left(\frac{1}{\cos 2\theta}\right)$$

$$=\sec^{-1}(\sec 2\theta)$$

$$=2\theta$$

$$=2\cos^{-1}x\cdots\cdots(1)$$

$and\ \ z=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

$$=\tan^{-1}\left(\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right);\left[\begin{array}{l}putting\ \ x=\sin\theta\\ \qquad\therefore\ \theta=\sin^{-1}x\end{array}\right]$$

$$=\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos^2\theta}}\right)$$

$$=\tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$$

$$=\tan^{-1}.\tan\theta$$

$$=\theta$$

$$=\sin^{-1}x\cdots\cdots(2)$$

Differentiating (1) *and* (2) *with respect to x we get*,

$$\frac{dy}{dx}=-\frac{2}{\sqrt{1-x^2}}\qquad and\ \frac{dz}{dx}=\frac{1}{\sqrt{1-x^2}}$$

$Now,\ \frac{dy}{dz}=\frac{dy/dx}{dz/dx}$

$$=\frac{-\frac{2}{\sqrt{1-x^2}}}{\frac{1}{\sqrt{1-x^2}}}$$

$$=-2\quad(Ans.)$$

Homework:-

1. $Differentiate\ \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) with\ respect\ to\ \tan^{-1}x.$ Ans: $\frac{1}{2}$

2. $Differentiate\ e^{\sin^{-1}x}\ with\ respect\ to \cos 3x.$ Ans: $-\frac{e^{\sin^{-1}x}}{3\sqrt{1-x^2}.\sin 3x}$

3. $Differentiate\ \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) with\ respect\ to \tan^{-1}\left(\frac{2x}{1-x^2}\right).$ Ans: 1

Successive derivative

Successive derivative: If $y=f(x)$ be a function of x then the first order derivative of y with respect to x is denoted by $\frac{dy}{dx},\ f^{'}(x),\ y_1\ ,\ y^{(1)},\ f^{(1)}(x),\ f_x^{'}(x)\ etc.$

Again the derivative of first ordered derivative of y with respect to x is called second order derivative and is denoted by $\frac{d^2y}{dx^2},\ f^{''}(x),\ y_2\ ,\ y^{(2)},\ f^{(2)}(x),\ f_x^{''}(x)\ etc.$

Similarly, the nth derivative of y with respect to x is denoted by $\frac{d^ny}{dx^n},\ f^n(x),\ y_n\ ,\ y^{(n)},\ f^{(n)}(x),\ f_x^n(x)\ etc.$

❖ Find the nth derivative of the following functions:

1. $y=x^n$

$sol: Given\ that, y=x^n$

$Differentiating\ with\ respect\ to\ x\ we\ get,$

$$y_1=nx^{n-1}$$
$$\therefore\ y_2=n(n-1)x^{n-2}$$
$$\therefore\ y_3=n(n-1)(n-2)x^{n-3}$$

$Similarly,$

$$y_r=n(n-1)(n-2)\cdots\cdots\{n-(r-1)\}x^{n-r}\quad ;where,\ r<n$$
$$\therefore\ y_n=n(n-1)(n-2)\cdots\cdots\{n-(n-1)\}x^{n-n}$$
$$=n(n-1)(n-2)\cdots\cdots 3.2.1$$
$$=n!\quad Ans.$$

2. $y=e^{ax}$

$sol: Given\ that, y=e^{ax}$

$Differentiating\ with\ respect\ to\ x\ we\ get,$

$$y_1=ae^{ax}$$
$$\therefore\ y_2=a^2e^{ax}$$
$$\therefore\ y_3=a^3e^{ax}$$

$Similarly,$

$$y_r=a^re^{ax}\quad ;where,\ r<n$$
$$\therefore\ y_n=a^ne^{ax}\quad Ans.$$

3. $y=(ax+b)^m$

sol : *Given that*, $y=(ax+b)^m$

Differentiating with respect to x we get,

$$y_1=am(ax+b)^{m-1}$$

$$\therefore\ y_2=a^2m(m-1)(ax+b)^{m-2}$$

$$\therefore\ y_3=a^3m(m-1)(m-2)(ax+b)^{m-3}$$

Similarly,

$$y_r=a^r m(m-1)(m-2)\cdots\cdots\{m-(r-1)\}(ax+b)^{m-r}\quad ;\ where,\ r<n$$

$$\therefore\ y_n=a^n m(m-1)(m-2)\cdots\cdots\{m-(n-1)\}(ax+b)^{m-n}$$

$$=\frac{m!}{(m-n)!}\,a^n\,(ax+b)^{m-n}\quad Ans.$$

4. $y=\sin(ax+b)$

sol : *Given that*, $y=\sin(ax+b)$

Differentiating with respect to x we get,

$$y_1=a\cos(ax+b)$$

$$=a\sin\left\{\frac{\pi}{2}+(ax+b)\right\}$$

$$\therefore\ y_2=a^2\cos\left\{\frac{\pi}{2}+(ax+b)\right\}$$

$$=a^2\sin\left\{\frac{\pi}{2}+\frac{\pi}{2}+(ax+b)\right\}$$

$$=a^2\sin\left\{\frac{2\pi}{2}+(ax+b)\right\}$$

$$\therefore\ y_3=a^3\cos\left\{\frac{2\pi}{2}+(ax+b)\right\}$$

$$=a^3\sin\left\{\frac{\pi}{2}+\frac{2\pi}{2}+(ax+b)\right\}$$

$$=a^3\sin\left\{\frac{3\pi}{2}+(ax+b)\right\}$$

Similarly,

$$y_r=a^r\sin\left\{\frac{r\pi}{2}+(ax+b)\right\}\quad ;\ where,\ r<n$$

$$\therefore\ y_n=a^n\sin\left\{\frac{n\pi}{2}+(ax+b)\right\}\quad Ans.$$

5. $y=\cos(ax+b)$

sol : *Given that*, $y=\cos(ax+b)$

Differentiating with respect to x we get,

$$y_1=-a\sin(ax+b)$$

$$=a\cos\left\{\frac{\pi}{2}+(ax+b)\right\}$$

$$\therefore\ y_2=-a^2\sin\left\{\frac{\pi}{2}+(ax+b)\right\}$$

$$=a^2\cos\left\{\frac{\pi}{2}+\frac{\pi}{2}+(ax+b)\right\}$$

$$=a^2\cos\left\{\frac{2\pi}{2}+(ax+b)\right\}$$

$$\therefore\ y_3=-a^3\sin\left\{\frac{2\pi}{2}+(ax+b)\right\}$$

$$=a^3\cos\left\{\frac{\pi}{2}+\frac{2\pi}{2}+(ax+b)\right\}$$

$$=a^3\cos\left\{\frac{3\pi}{2}+(ax+b)\right\}$$

Similarly,

$$y_r=a^r\cos\left\{\frac{r\pi}{2}+(ax+b)\right\}\quad ;\ where,\ r<n$$

$$\therefore\ y_n=a^n\cos\left\{\frac{n\pi}{2}+(ax+b)\right\}\quad Ans.$$

6. $y = e^{ax}\sin(bx+c)$

$sol:$ *Given that,* $y = e^{ax}\sin(bx+c)$

Differentiating with respect to x we get,

$$y_1 = ae^{ax}\sin(bx+c) + be^{ax}\cos(bx+c)$$
$$= e^{ax}\{a\sin(bx+c) + b\cos(bx+c)\}$$

put $a = r\cos\varphi$ *and* $b = r\sin\phi$

$$\therefore r = \sqrt{a^2+b^2} \text{ and } \varphi = \tan^{-1}\left(\frac{b}{a}\right)$$

Now, $y_1 = e^{ax}\{r\cos\varphi\sin(bx+c) + r\sin\varphi\cos(bx+c)\}$

$$= re^{ax}\sin(bx+c+\varphi)$$
$$\therefore y_2 = re^{ax}\{a\sin(bx+c+\varphi) + b\cos(bx+c+\varphi)\}$$
$$= re^{ax}\{r\cos\varphi\sin(bx+c+\varphi) + r\sin\varphi\cos(bx+c+\varphi)\}$$
$$= r^2e^{ax}\sin(bx+c+2\varphi)$$
$$\therefore y_3 = r^3e^{ax}\sin(bx+c+3\varphi)$$

Similarly,

$$y_n = r^ne^{ax}\sin(bx+c+n\varphi)$$
$$= \left(\sqrt{a^2+b^2}\right)^n e^{ax}\sin\left(bx+c+n\tan^{-1}\left(\frac{b}{a}\right)\right) \quad Ans.$$

7. $y = \ln(ax+b)$

$sol:$ *Given that,* $y = \ln(ax+b)$

Differentiating with respect to x we get,

$$y_1 = \frac{a}{(ax+b)}$$
$$\therefore y_2 = -\frac{1.a^2}{(ax+b)^2}$$
$$\therefore y_3 = \frac{1.2\,a^3}{(ax+b)^3}$$
$$\therefore y_4 = -\frac{1.2.3\,a^4}{(ax+b)^4}$$

Similarly,

$$\therefore y_n = \frac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n} \quad Ans.$$

8. *If* $y = \sin nx + \cos nx$ *then show that* $y_r = n^r\left[1+(-1)^r\sin 2nx\right]^{1/2}$.

$sol:$ *Given that,* $y = \sin nx + \cos nx$

Differentiating with respect to x we get,

$$y_1 = n\cos nx - n\sin nx$$
$$= n\sin\left(\frac{\pi}{2}+nx\right) + n\cos\left(\frac{\pi}{2}+nx\right)$$
$$\therefore y_2 = n^2\cos\left(\frac{\pi}{2}+nx\right) - n^2\sin\left(\frac{\pi}{2}+nx\right)$$
$$= n^2\sin\left(\frac{2\pi}{2}+nx\right) + n^2\cos\left(\frac{2\pi}{2}+nx\right)$$
$$\therefore y_3 = n^3\cos\left(\frac{2\pi}{2}+nx\right) - n^3\sin\left(\frac{2\pi}{2}+nx\right)$$
$$= n^3\sin\left(\frac{3\pi}{2}+nx\right) + n^3\cos\left(\frac{3\pi}{2}+nx\right)$$

Similarly,

$$y_r = n^r\sin\left(\frac{r\pi}{2}+nx\right) + n^3\cos\left(\frac{r\pi}{2}+nx\right)$$
$$= n^r\left[\left\{\sin\left(\frac{r\pi}{2}+nx\right) + \cos\left(\frac{r\pi}{2}+nx\right)\right\}^2\right]^{1/2}$$
$$= n^r\left[\sin^2\left(\frac{r\pi}{2}+nx\right) + \cos^2\left(\frac{r\pi}{2}+nx\right) + 2\sin\left(\frac{r\pi}{2}+nx\right)\cos\left(\frac{r\pi}{2}+nx\right)\right]^{1/2}$$
$$= n^r\left[1+\sin 2\left(\frac{r\pi}{2}+nx\right)\right]^{1/2}$$
$$= n^r\left[1+\sin(r\pi+2nx)\right]^{1/2}$$
$$= n^r\left[1+(-1)^r\sin 2nx\right]^{1/2} \quad showed.$$

9. $y = \dfrac{1}{x^2 - 3x + 2}$

$$sol: Given\ that,\ y = \frac{1}{x^2 - 3x + 2}$$

$$= \frac{1}{(x-2)} - \frac{1}{(x-1)}$$

$$= (x-2)^{-1} - (x-1)^{-1}$$

Differentiating with respect to x we get,

$$y_1 = (-1)(x-2)^{-2} - (-1)(x-1)^{-2}$$

$$\therefore\ y_2 = (-1)(-2)(x-2)^{-3} - (-1)(-2)(x-1)^{-3}$$

$$= (-1)^2 2!(x-2)^{-3} - (-1)^2 2!(x-1)^{-3}$$

Similarly,

$$\therefore\ y_n = (-1)^n n!(x-2)^{-(n+1)} - (-1)^n n!(x-1)^{-(n+1)}$$

$$= (-1)^n n!\left\{\frac{1}{(x-2)^{n+1}} - \frac{1}{(x-1)^{n+1}}\right\} \qquad Ans$$

Homework:-

1. Find the nth derivative of the following functions:

   a. $y = \dfrac{1}{x^2 + 5x + 6}$ Ans: $y_n = (-1)^n n!\left[\dfrac{1}{(x+2)^{n+1}} - \dfrac{1}{(x+3)^{n+1}}\right]$

   b. $y = \dfrac{2x+3}{x^2 + 3x + 2}$. Ans: $y_n = (-1)^n n!\left[\dfrac{1}{(x+1)^{n+1}} + \dfrac{1}{(x+2)^{n+1}}\right]$