Differentiation (Part – 1)

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Function

If x and y are two variables related to one another in such a way that each value of x determines exactly one value of y, then we say that y is a function of x and it is simply written as $y = f(x)$, where x is an independent variable and y is a dependent variable. The value of y or $f(x)$ is called functional value.

$y = x^2$ is a function

For $x = \pm1, \pm2, \pm3, \dots \dots \dots$

We get, $y = 1, 4, 9, \dots \dots \dots$

x	y
1	1
-1	1
2	4
-2	4
3	9
-3	9

$y^2 = x\ or, (y = \pm\sqrt{x})$ is not a function

For $x = 1, 4, 9, \dots \dots \dots$

We get, $y = \pm1, \pm2, \pm3, \dots \dots \dots$

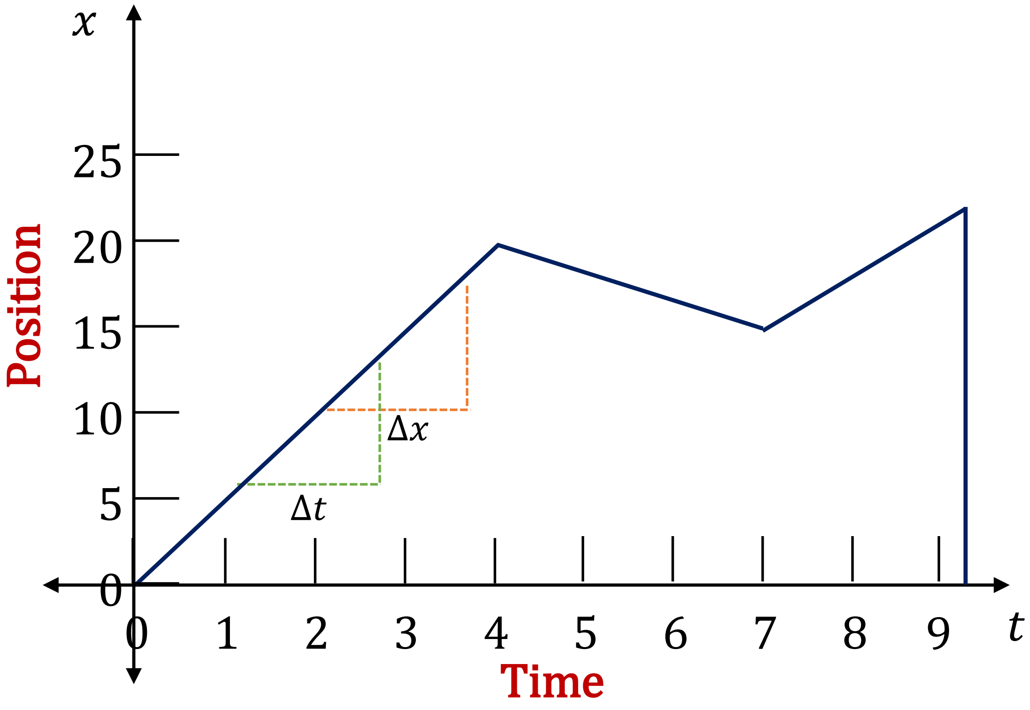
x	y
1	1
1	-1
4	2
4	-2
9	3
9	-3

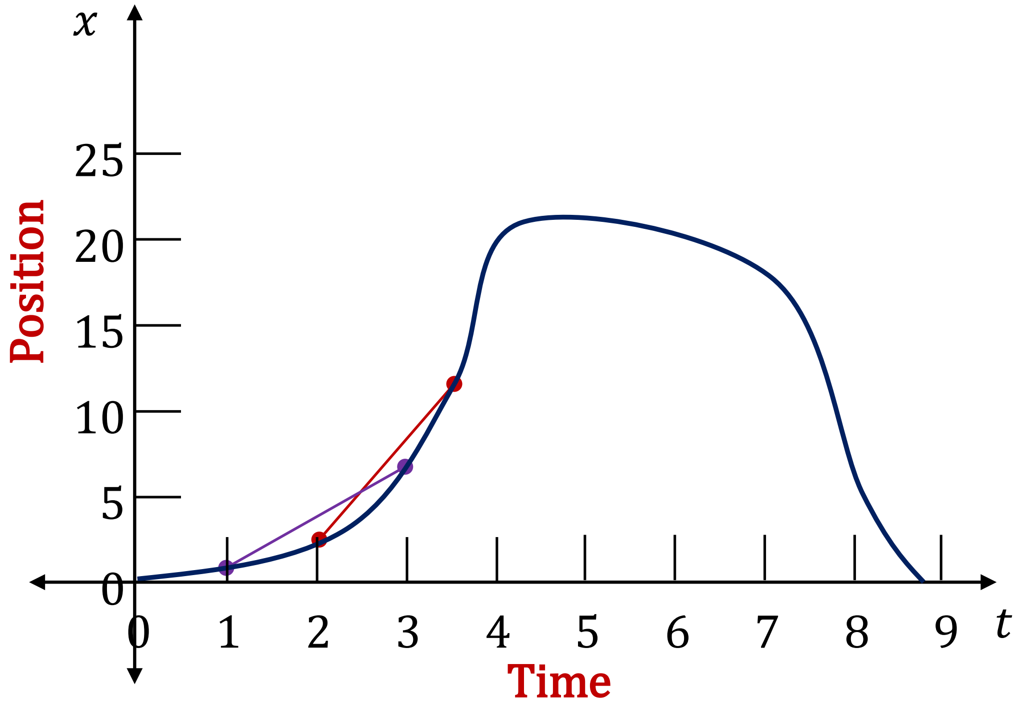
Concept of Derivative

Differentiation → Instantaneous rate of change of a function with respect to one of its variables

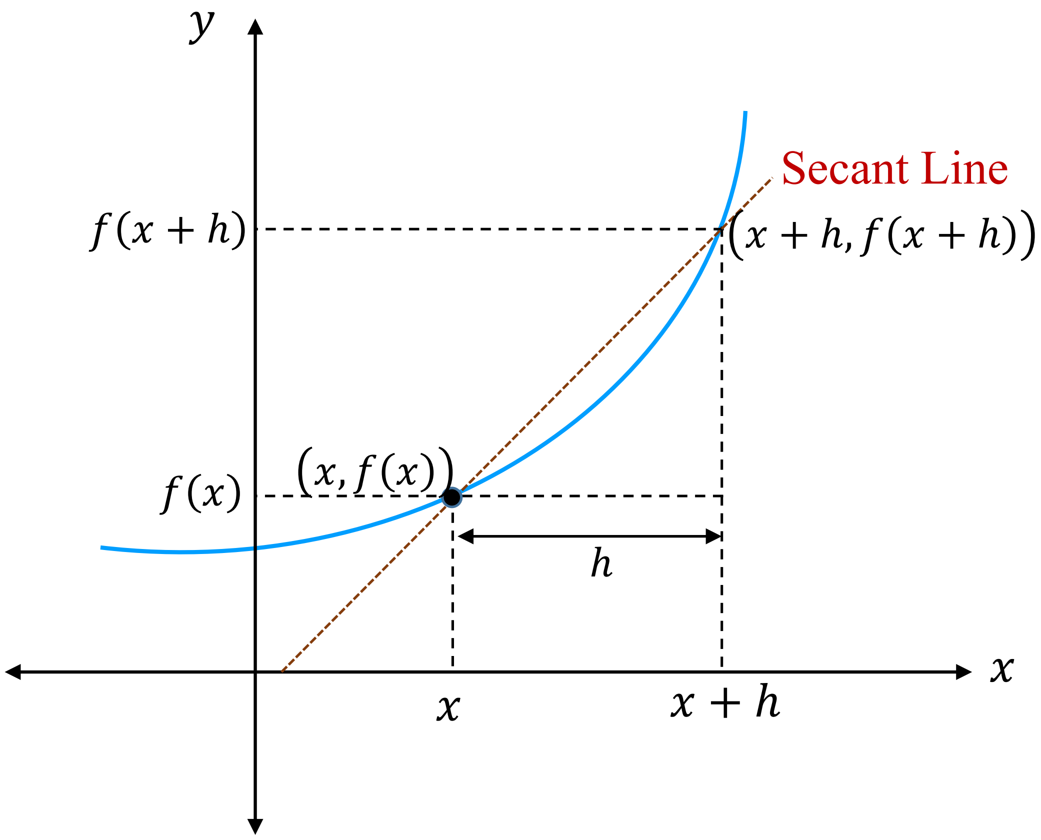
≈ to find the slope of the tangent line to the function at a point.

$$Speed = \frac{change\ of\ position}{change\ of\ time} = \frac{\Delta x}{\Delta t}$$





Geometrical Representation of Derivative

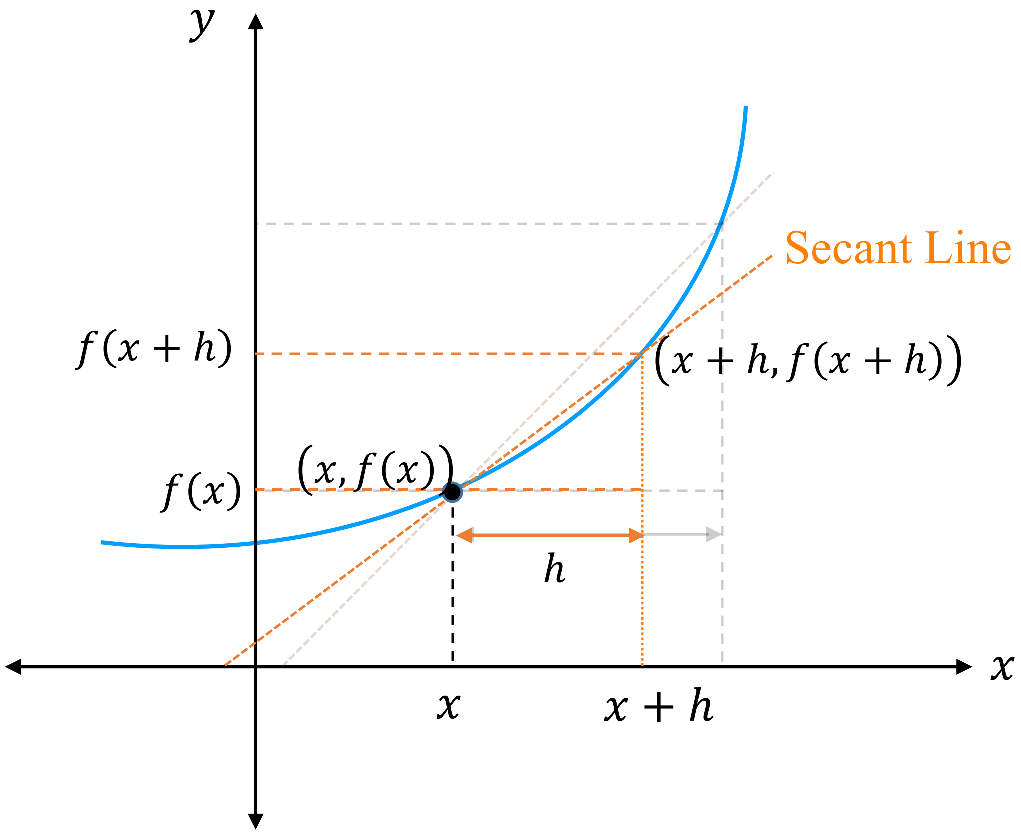


$$\text{Gradient/Slope} = \frac{\text{change in } y}{\text{change in } x}$$

$$= \frac{f(x+h) - f(x)}{x+h-x}$$

$$= \frac{f(x+h) - f(x)}{\text{h}}$$

Geometrical Representation of Derivative



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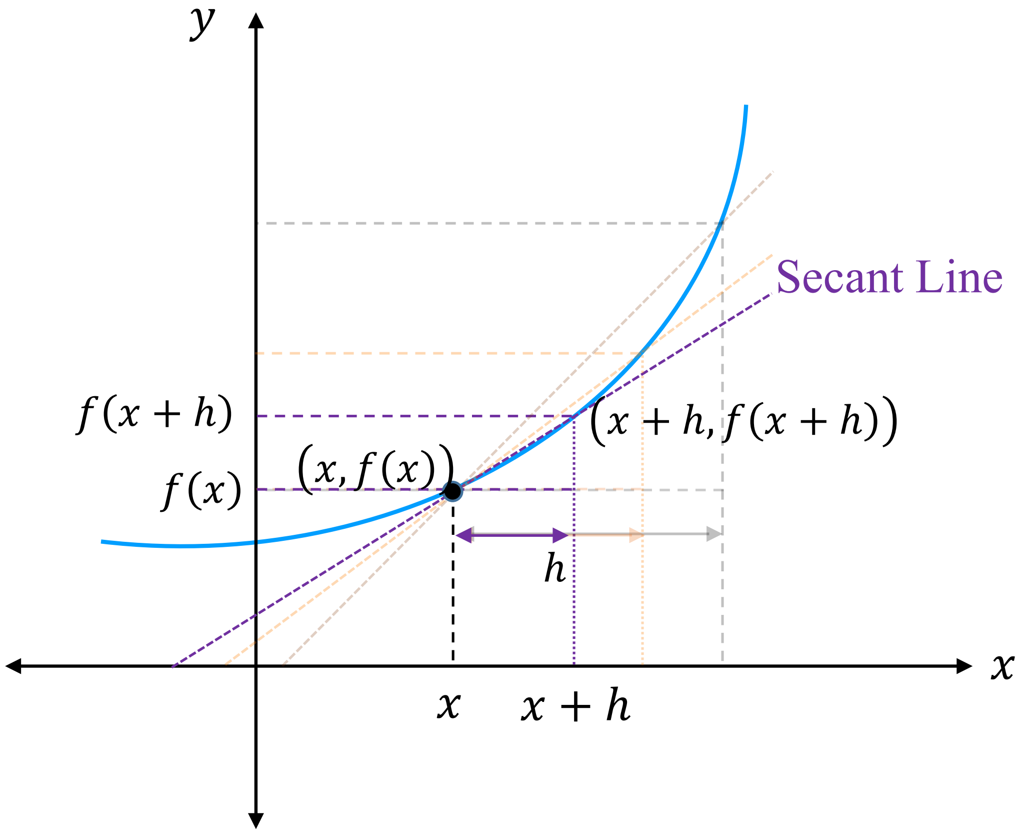

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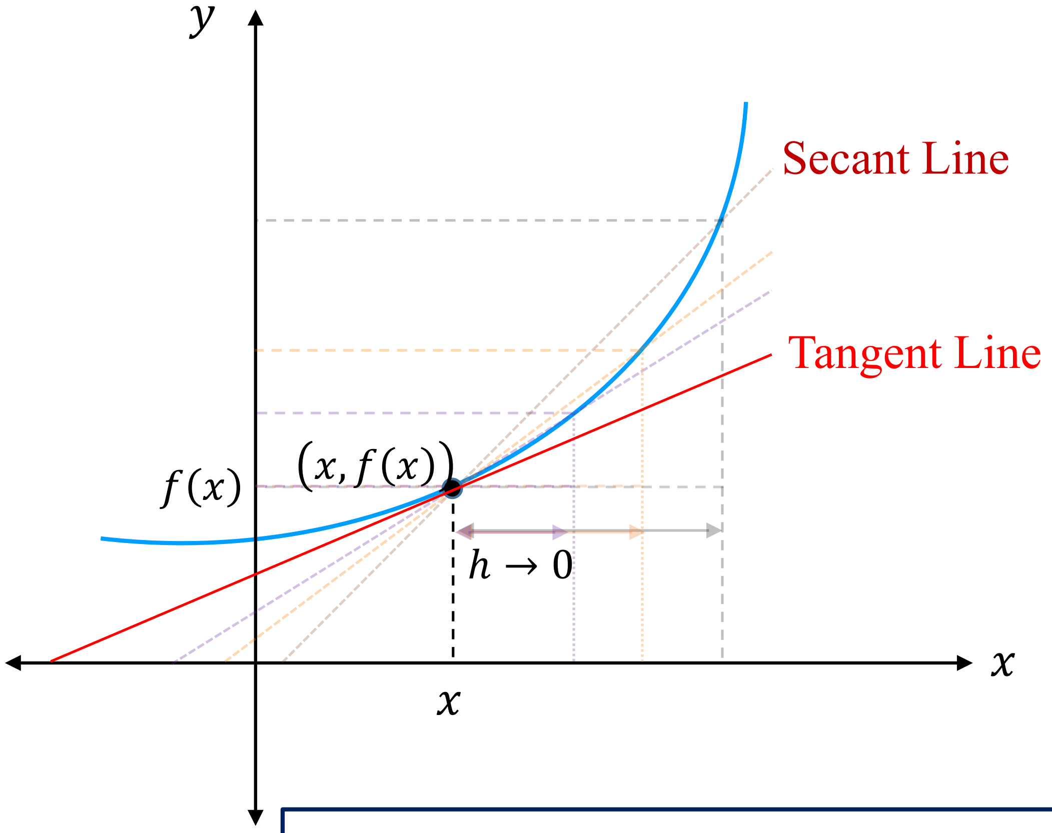

$$\text{Gradient/Slope} = \frac{\text{change in } y}{\text{change in } x}$$

$$= \frac{f(x+h) - f(x)}{x + h - x}$$

$$= \frac{f(x+h) - f(x)}{h}$$

$$\therefore \frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

First Principle: A function $f(x)$ is differentiable at a point x if

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists (is finite).

Example 1: Find the derivative of $f(x) = \mathbf{c}$ using first principle

Let $y = f(x) = c$

From **First Principle**,

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{c-c}{h}$$

$$= \lim_{h\to 0} \frac{0}{h}$$

$$= \lim_{h\to 0} 0$$

$$= 0$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(c) = 0$$

$$\boxed{\begin{aligned} f(x) &= c \\ f(x+h) &= c \end{aligned}}$$

Example 2: Find the derivative of $f(x) = x$ using first principle

Let $y = f(x) = x$

From **First Principle**,

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{x + h - x}{h}$$

$$= \lim_{h \to 0} \frac{h}{h}$$

$$= \lim_{h \to 0} 1$$

$$= 1$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(x) = 1$$

$f(x) = x$
$f(x + h) = x + h$

Example 3: Find the derivative of $f(x) = x^n$ using first principle

Let $y = f(x) = x^n$

$$f(x) = x^n$$
$$f(x+h) = (x+h)^n$$

From **First Principle**,

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0}\frac{\left\{x\left(1+\frac{h}{x}\right)\right\}^n - x^n}{h}$$

$$= \lim_{h\to 0}\frac{x^n\left(1+\frac{h}{x}\right)^n - x^n}{h}$$

$$= x^n\lim_{h\to 0}\frac{\left(1+\frac{h}{x}\right)^n - 1}{h}$$

$$\frac{dy}{dx} = x^n\lim_{h\to 0}\frac{1+n\left(\frac{h}{x}\right)+\frac{n(n-1)}{2!}\left(\frac{h}{x}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{h}{x}\right)^3+\cdots-1}{h}$$

$$= x^n\lim_{h\to 0}\frac{n\left(\frac{h}{x}\right)+\frac{n(n-1)}{2!}\left(\frac{h}{x}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{h}{x}\right)^3+\cdots}{h}$$

$$= x^n\lim_{h\to 0}\frac{h\left\{n\left(\frac{1}{x}\right)+\frac{n(n-1)}{2!}\left(\frac{h}{x^2}\right)+\frac{n(n-1)(n-2)}{3!}\left(\frac{h^2}{x^3}\right)+\cdots\right\}}{h}$$

$$= x^n\lim_{h\to 0}\left\{n\left(\frac{1}{x}\right)+\frac{n(n-1)}{2!}\left(\frac{h}{x^2}\right)+\frac{n(n-1)(n-2)}{3!}\left(\frac{h^2}{x^3}\right)+\cdots\right\}$$

$$= x^n n\left(\frac{1}{x}\right) = x^n\, nx^{-1} = nx^{n-1}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(x^n) = nx^{n-1}$$

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Example 4: Find the derivative of $f(x) = e^x$ using first principle

Let $y = f(x) = e^x$

$f(x) = e^x$

$f(x+h) = e^{(x+h)}$

From **First Principle**,

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{e^{(x+h)} - e^x}{h}$$

$$= \lim_{h\to 0}\frac{e^x . e^h - e^x}{h}$$

$$= \lim_{h\to 0}\frac{e^x\left(e^h - 1\right)}{h}$$

$$= e^x \lim_{h\to 0}\frac{1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \frac{h^4}{4!} + \cdots - 1}{h}$$

$$\left[\because e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right]$$

$$\frac{dy}{dx} = e^x \lim_{h\to 0}\frac{h + \frac{h^2}{2!} + \frac{h^3}{3!} + \frac{h^4}{4!} + \cdots}{h}$$

$$= e^x \lim_{h\to 0}\frac{h\left(1 + \frac{h}{2!} + \frac{h^2}{3!} + \frac{h^3}{4!} + \cdots\right)}{h}$$

$$= e^x \lim_{h\to 0}\left(1 + \frac{h}{2!} + \frac{h^2}{3!} + \frac{h^3}{4!} + \cdots\right)$$

$$= e^x . 1$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(e^x\right) = e^x$$

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Example 5: Find the derivative of $f(x) = \ln(x)$ using first principle

Let $y = f(x) = \ln(x)$

$$f(x) = \ln(x)$$
$$f(x+h) = \ln(x+h)$$

From **First Principle**,

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\ln\left(\frac{x+h}{x}\right)}{h}$$

$$= \lim_{h\to 0}\frac{\ln\left(1+\frac{h}{x}\right)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{h}{x} - \frac{1}{2}\frac{h^2}{x^2} + \frac{1}{3}\frac{h^3}{x^3} - \frac{1}{4}\frac{h^4}{x^4} + \cdots}{h}$$

$$\left[\because \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right]$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{h\left(\frac{1}{x} - \frac{1}{2}\frac{h}{x^2} + \frac{1}{3}\frac{h^2}{x^3} - \frac{1}{4}\frac{h^3}{x^4} + \cdots\right)}{h}$$

$$= \lim_{h\to 0}\left(\frac{1}{x} - \frac{1}{2}\frac{h}{x^2} + \frac{1}{3}\frac{h^2}{x^3} - \frac{1}{4}\frac{h^3}{x^4} + \cdots\right)$$

$$= \frac{1}{x}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

Example 6: Find the derivative of $f(x) = \cos x$ using first principle

$$\lim_{h\to 0}\frac{\sin h}{h} = \lim_{h\to 0}\frac{h}{\sin h} = 1$$

Let $y = f(x) = \cos x$

From **First Principle**,

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f(x) = \cos x$$
$$f(x+h) = \cos(x+h)$$

$$= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$

$$= \lim_{h\to 0}\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

$$\cos a - \cos b = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$

$$= -\lim_{h\to 0}\sin\left(\frac{2x+h}{2}\right)\times\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}$$

$\because h\to 0 \Longrightarrow \frac{h}{2}\to 0$????

$$h\to 0$$
$$\Longrightarrow \frac{1}{2}\times h \to \frac{1}{2}\times 0$$
$$\Longrightarrow \frac{h}{2}\to 0$$

$$= -\sin\left(\frac{2x+0}{2}\right)\times 1$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\cos x) = -\sin x$$

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Example 7: Find the derivative of $f(x) = \sin ax$ using first principle

Let $y = f(x) = \sin ax$

From **First Principle**,

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f(x) = \sin ax$$
$$f(x+h) = \sin a(x+h) = \sin(ax+ah)$$

$$\sin a - \sin b = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$

$$\lim_{h\to 0}\frac{\sin h}{h} = \lim_{h\to 0}\frac{h}{\sin h} = 1$$

$$= \lim_{h\to 0}\frac{\sin(ax+ah)-\sin ax}{h}$$

$$= \lim_{h\to 0}\frac{2\cos\left(\frac{2ax+ah}{2}\right)\sin\left(\frac{ah}{2}\right)}{h}$$

$$= \lim_{h\to 0}\cos\left(\frac{2ax+ah}{2}\right)\times\lim_{h\to 0}\frac{\sin\left(\frac{ah}{2}\right)}{\frac{h}{2}}$$

$$= \cos\left(\frac{2ax+a.0}{2}\right)\times\lim_{h\to 0}\frac{a\sin\left(\frac{ah}{2}\right)}{\left(\frac{ah}{2}\right)}$$

$$\frac{dy}{dx} = (\cos ax)\times a\lim_{\frac{ah}{2}\to 0}\frac{\sin\left(\frac{ah}{2}\right)}{\left(\frac{ah}{2}\right)}$$

$$\left[\because h\to 0 \Rightarrow \frac{ah}{2}\to 0\right]$$

$$= (\cos ax)\times a(1)$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\sin ax) = a\cos ax$$

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Example 8: Find the derivative of $f(x) = \tan x$ using first principle

Let $y = f(x) = \tan x$

$\sin a \cos b - \cos a \sin b = \sin(a-b)$

From **First Principle**,

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$f(x) = \tan x$

$f(x+h) = \tan(x+h)$

$$= \lim_{h\to 0}\frac{\tan(x+h)-\tan x}{h}$$

$$= \lim_{h\to 0}\frac{\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}}{h}$$

$$= \lim_{h\to 0}\frac{\frac{\sin(x+h)\cos x - \sin x\cos(x+h)}{\cos(x+h)\cos x}}{h}$$

$$= \lim_{h\to 0}\frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{h\cos(x+h)\cos x}$$

$$= \lim_{h\to 0}\frac{\sin(x+h-x)}{h\cos(x+h)\cos x}$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{\sin h}{h\cos(x+h)\cos x}$$

$$= \lim_{h\to 0}\frac{\sin h}{h} \times \lim_{h\to 0}\frac{1}{\cos(x+h)\cos x}$$

$$= 1 \times \frac{1}{\cos(x+0)\cos x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\tan x) = \sec^2 x$$

Example 9: Find the derivative of $f(x) = \tan^{-1} x$ using first principle

Let $y = f(x) = \tan^{-1} x$

From **First Principle**,

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$f(x) = \tan^{-1} x$

$f(x+h) = \tan^{-1}(x+h)$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{\tan^{-1}(x+h) - \tan^{-1} x}{h} \cdots (i)$$

Suppose $\tan^{-1} x = y \Longrightarrow x = \tan y$

and $\tan^{-1}(x+h) = y + k \Longrightarrow x + h = \tan(y+k)$

$\Longrightarrow h = \tan(y+k) - x$

$\Longrightarrow h = \tan(y+k) - \tan y$

$x + h = \tan(y+k)$

When $h \to 0$, $x = \tan(y+k)$

$\Longrightarrow \tan^{-1} x = y + k$

$\Longrightarrow \tan^{-1} x = \tan^{-1} x + k$

$\Longrightarrow k \to 0$

Thus, $h \to 0 \Longrightarrow k \to 0$

In concept of Limit

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{y+k-y}{h}$$

$$\frac{dy}{dx} = \lim_{k\to 0} \frac{k}{\tan(y+k) - \tan y} \quad [\because\ h \to 0 \Longrightarrow k \to 0\ ???]$$

Example 9: Find the derivative of $f(x) = \tan^{-1} x$ using first principle (cont.…)

$$\frac{dy}{dx} = \lim_{k\to 0} \frac{k}{\tan(y+k) - \tan y} \quad [\because\ h \to 0 \Rightarrow k \to 0]$$

$$= \lim_{k\to 0} \frac{k}{\frac{\sin(y+k)}{\cos(y+k)} - \frac{\sin y}{\cos y}}$$

$$= \lim_{k\to 0} \frac{k}{\frac{\sin(y+k)\cos y - \sin y \cos(y+k)}{\cos(y+k)\cos y}}$$

$$= \lim_{k\to 0} \frac{k \cos(y+k)\cos y}{\sin(y+k)\cos y - \cos(y+k)\sin y}$$

$$= \lim_{k\to 0} \frac{k \cos(y+k)\cos y}{\sin(y+k-y)}$$

$$= \lim_{k\to 0} \frac{k \cos(y+k)\cos y}{\sin k}$$

$$\boxed{\sin a \cos b - \cos a \sin b = \sin(a-b)}$$

$$\boxed{\lim_{k\to 0} \frac{\sin k}{k} = \lim_{k\to 0} \frac{k}{\sin k} = 1}$$

$$\frac{dy}{dx} = \lim_{k\to 0} \frac{k}{\sin k} \times \lim_{k\to 0} \cos(y+k)\cos y$$

$$= 1 \times \cos(y+0)\cos y$$

$$= \cos^2 y$$

$$= \frac{1}{\sec^2 y}$$

$$= \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

$$[\because\ \tan y = x]$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

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Example 10: Find $\frac{dy}{dx}$ for $y = \sqrt{x}$ using first principle

Let $y = f(x) = \sqrt{x}$

From **First Principle**,

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f(x) = \sqrt{x}$$
$$f(x+h) = \sqrt{x+h}$$

$$= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$= \lim_{h\to 0}\frac{\left(\sqrt{x+h}-\sqrt{x}\right)\left(\sqrt{x+h}+\sqrt{x}\right)}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= \lim_{h\to 0}\frac{\left(\sqrt{x+h}\right)^2-(\sqrt{x})^2}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= \lim_{h\to 0}\frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= \lim_{h\to 0}\frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$= \frac{1}{\sqrt{x+0}+\sqrt{x}}$$

$$= \frac{1}{\sqrt{x}+\sqrt{x}}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$$

Exercise

Find the derivative of the following functions using first principle:

1. $f(x) = a^x$
2. $f(x) = \cos ax$
3. $f(x) = \sin^{-1} x$
4. $f(x) = \cos^{-1} x$
5. $f(x) = \sin(x^2)$