HW2

Recall that we use $|0\rangle$ to denote the vector $\binom{1}{0}$, and $|1\rangle$ to denote the vector $\binom{0}{1}$. Therefore, we can express every qubit as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 + |\beta|^2 = 1$. In other words, $|\psi\rangle$ is the vector $\binom{\alpha}{\beta}$. We have the following basic single-qubit unitary matrices:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

(a) Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ be any single-qubit quantum state. Write the following states as a linear combination of $|0\rangle$ and $|1\rangle$: $H|\psi\rangle$, $X|\psi\rangle$, and $HZH|\psi\rangle$. Based on these computations, what identity between single-qubit operators can you prove?

For any two qubits $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$ and $|\varphi\rangle = \beta_0|0\rangle + \beta_1|1\rangle$, we define their *tensor product* $|\psi\rangle \otimes |\varphi\rangle$ as the natural state that is a combination of both qubits:

$$|\psi\rangle \otimes |\varphi\rangle = \alpha_0\beta_0|0\rangle\otimes|0\rangle + \alpha_0\beta_1|0\rangle\otimes|1\rangle + \alpha_1\beta_0|1\rangle\otimes|0\rangle + \alpha_1\beta_1|1\rangle\otimes|1\rangle$$

$$= \alpha_0\beta_0\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \alpha_0\beta_1\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \alpha_1\beta_0\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + \alpha_1\beta_1\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}\alpha_0\beta_0\\ \alpha_0\beta_1\\ \alpha_1\beta_0\\ \alpha_1\beta_1\end{pmatrix}$$

where for each $x, y \in \{0, 1\}$, we've defined $|x\rangle \otimes |y\rangle$ to be the length-4 vector with a 1 in position $(xy)_2$ (i.e., when the number is written in binary). We say that a two-qubit state is *entangled* if it cannot be written as the tensor product of two single-qubit states.

(b) Give an example of a 2-qubit operation U such that $U(|0\rangle \otimes |0\rangle)$ is entangled. (Make sure to check that U is unitary!)

a) i) $H|\psi\rangle = H(\alpha|0\rangle + \beta|1\rangle)$
$= \alpha H|0\rangle + \beta H|1\rangle$
$= \alpha \frac{|0\rangle + |1\rangle}{\sqrt{2}} + \beta \frac{|0\rangle - |1\rangle}{\sqrt{2}}$
$= \frac{\alpha+\beta}{\sqrt{2}}|0\rangle + \frac{\alpha-\beta}{\sqrt{2}}|1\rangle$

ii) $X|\psi\rangle = X(\alpha|0\rangle + \beta|1\rangle)$ $\quad X|0\rangle = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$
$\quad X|1\rangle = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$
$= \alpha X|0\rangle + \beta X|1\rangle$
$= \beta|0\rangle + \alpha|1\rangle$

iii) $HZH|\psi\rangle = HZH(\alpha|0\rangle + \beta|1\rangle)$ $\quad Z|0\rangle = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} = |0\rangle$
$= HZ\left(\frac{\alpha+\beta}{\sqrt{2}}|0\rangle + \frac{\alpha-\beta}{\sqrt{2}}|1\rangle\right)$ $\quad Z|1\rangle = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix} = -|1\rangle$
$= H\left(\frac{\alpha+\beta}{\sqrt{2}}|0\rangle + \frac{\beta-\alpha}{\sqrt{2}}|1\rangle\right)$
$= \frac{\alpha+\beta}{\sqrt{2}} \cdot \frac{|0\rangle+|1\rangle}{\sqrt{2}} + \frac{-(\alpha-\beta)(|0\rangle - |1\rangle)}{\sqrt{2}\,\sqrt{2}}$
$= \frac{2}{2}\beta|0\rangle + \frac{2\alpha}{2}|1\rangle = \beta|0\rangle + \alpha|1\rangle$ $\quad \therefore X = HZH$

(b) $U = CNOT(H\otimes I) = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix} \otimes \begin{pmatrix}1&0\\0&1\end{pmatrix}$

$(H\otimes I)^{\dagger}$ $\quad = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&-1&0&1\end{pmatrix}$ $\quad U^{\dagger} = \frac{1}{\sqrt{2}}\begin{pmatrix}1&0&1&0\\0&1&0&-1\\1&0&-1&0\\0&1&0&1\end{pmatrix}$

$UU^{\dagger} = \frac{1}{\sqrt{2}}\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&-1&0&1\end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix}1&0&1&0\\0&1&0&-1\\1&0&-1&0\\0&1&0&1\end{pmatrix} = \begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix} = I$

$a, b, c, d \in \mathbb{C}$

$U(|0\rangle|0\rangle) = U|00\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&-1&0&1\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}\\0\\0\\1/\sqrt{2}\end{pmatrix} \neq \begin{pmatrix}a\\b\end{pmatrix}\otimes\begin{pmatrix}c\\d\end{pmatrix} = \begin{pmatrix}ac\\ad\\bc\\bd\end{pmatrix}$ $\quad ac = \frac{1}{\sqrt{2}}$, $ad = 0$, $bc = 0$, $bd = 1/\sqrt{2}$

a or $d = 0$ $\quad b$ or $c = 0$

i) $a = 0$
$ac = \frac{1}{\sqrt{2}}$ false

ii) $d = 0$
$bd = \frac{1}{\sqrt{2}}$ false

$\therefore U = CNOT(H\otimes I)$ is a 2-qubit operation such that $U(|0\rangle\otimes|0\rangle)$ is entangled.

c) $U = \begin{pmatrix}u_{11}&u_{12}\\u_{21}&u_{22}\end{pmatrix}$ $\quad V = \begin{pmatrix}v_{11}&v_{12}\\v_{21}&v_{22}\end{pmatrix}$

$U\otimes V = \begin{pmatrix}u_{11}V & u_{12}V\\ u_{21}V & u_{22}V\end{pmatrix} = \begin{pmatrix}u_{11}v_{11} & u_{11}v_{12} & u_{12}v_{11} & u_{12}u_{12}\\ u_{11}v_{21} & u_{11}v_{22} & u_{12}v_{21} & u_{12}v_{22}\\ u_{21}v_{11} & u_{21}v_{12} & u_{22}v_{11} & u_{22}v_{12}\\ u_{21}v_{21} & u_{21}v_{22} & u_{22}v_{21} & u_{22}v_{22}\end{pmatrix}$

d) $U\otimes V(|0\rangle\otimes|0\rangle) = U|0\rangle \otimes V|0\rangle = \begin{pmatrix}u_{11}\\u_{21}\end{pmatrix}\otimes\begin{pmatrix}v_{11}\\v_{21}\end{pmatrix} = \begin{pmatrix}u_{11}v_{11}\\u_{11}v_{21}\\u_{21}v_{11}\\u_{21}v_{21}\end{pmatrix}$

$U\otimes V(|0\rangle\otimes|1\rangle) = U|0\rangle \otimes V|1\rangle = \begin{pmatrix}u_{11}\\u_{21}\end{pmatrix}\otimes\begin{pmatrix}v_{12}\\v_{22}\end{pmatrix} = \begin{pmatrix}u_{11}v_{12}\\u_{11}v_{22}\\u_{21}v_{12}\\u_{21}v_{22}\end{pmatrix}$

$U\otimes V(|0\rangle\otimes|0\rangle) = U|1\rangle \otimes V|0\rangle = \begin{pmatrix}u_{12}\\u_{22}\end{pmatrix}\otimes\begin{pmatrix}v_{11}\\v_{21}\end{pmatrix} = \begin{pmatrix}u_{12}v_{11}\\u_{12}v_{21}\\u_{22}v_{11}\\u_{22}v_{21}\end{pmatrix}$

$U\otimes V(|0\rangle\otimes|0\rangle) = U|1\rangle \otimes V|1\rangle = \begin{pmatrix}u_{12}\\u_{22}\end{pmatrix}\otimes\begin{pmatrix}v_{12}\\v_{22}\end{pmatrix} = \begin{pmatrix}u_{12}v_{12}\\u_{12}v_{22}\\u_{22}v_{12}\\u_{22}v_{22}\end{pmatrix}$

$\therefore U\otimes V\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\} = \begin{pmatrix}u_{11}v_{11} & u_{11}v_{12} & u_{12}v_{11} & u_{12}u_{12}\\ u_{11}v_{21} & u_{11}v_{22} & u_{12}v_{21} & u_{12}v_{22}\\ u_{21}v_{11} & u_{21}v_{12} & u_{22}v_{11} & u_{22}v_{12}\\ u_{21}v_{21} & u_{21}v_{22} & u_{22}v_{21} & u_{22}v_{22}\end{pmatrix} = U\otimes V$ ◻

2-a) $|\psi_1\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$ $|\psi_2\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}$ $|\psi_3\rangle = \frac{|0\rangle + \sqrt{2}|1\rangle}{\sqrt{3}}$ $\langle 1|0\rangle = (0\ 1)\begin{pmatrix}1\\0\end{pmatrix} = 0$

$\langle 0|1\rangle = (1\ 0)\begin{pmatrix}0\\1\end{pmatrix} = 0$

$$\langle\psi_1|\psi_2\rangle = \frac{\langle 0| - i\langle 1|}{\sqrt{2}} \cdot \frac{|0\rangle - i|1\rangle}{\sqrt{2}} = \frac{\langle 0|0\rangle - i\langle 1|0\rangle - i\langle 0|1\rangle - \langle 1|1\rangle}{2}$$
$$= \frac{1-0-0-1}{2} = 0.$$

$$\langle\psi_1|\psi_3\rangle = \frac{\langle 0| - i\langle 1|}{\sqrt{2}} \cdot \frac{|0\rangle + \sqrt{2}|1\rangle}{\sqrt{3}} = \frac{\langle 0|0\rangle - i\langle 1|0\rangle + \sqrt{2}\langle 0|1\rangle - \sqrt{2}i\langle 1|1\rangle}{\sqrt{6}}$$
$$= \frac{1-\sqrt{2}i}{\sqrt{6}}.$$

$$\langle\psi_2|\psi_3\rangle = \frac{\langle 0| + i\langle 1|}{\sqrt{2}} \cdot \frac{|0\rangle + \sqrt{2}|1\rangle}{\sqrt{3}} = \frac{\langle 0|0\rangle + i\langle 1|0\rangle + \sqrt{2}\langle 0|1\rangle + \sqrt{2}i\langle 1|1\rangle}{\sqrt{6}}$$
$$= \frac{1+\sqrt{2}i}{\sqrt{6}}.$$

(b) $|\psi\rangle = \sum_{x\in\{0,1\}^n} a_x|x\rangle$, $\langle\psi| = \sum_{x\in\{0,1\}^n} a_x^*\langle x|$

$\langle\psi|\psi\rangle = \sum_{x\in\{0,1\}^n} a_x^* a_x = \sum_{x\in\{0,1\}^n} |a_x|^2$

Since $|\psi\rangle$ has to be a valid n-qubit state, $\sum_{x\in\{0,1\}^n} |a_x|^2 = 1$.

So, $\langle\psi|\psi\rangle = 1$. Q.E.D.

(c) $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$ $|\varphi\rangle = \beta_0|0\rangle + \beta_1|1\rangle$ $\langle\psi|\varphi\rangle = 0$ $\alpha_0^2+\alpha_1^2=1$ $\beta_0^2+\beta_1^2=1$

$\langle\psi|\varphi\rangle = (\alpha_0^*\langle 0| + \alpha_1^*\langle 1|)(\beta_0|0\rangle + \beta_1|1\rangle) = \alpha_0^*\beta_0 + \alpha_1^*\beta_1 = 0$

$\alpha_0^*\beta_0 = -\alpha_1^*\beta_1$, $-\frac{\alpha_0^*}{\alpha_1^*} = \frac{\beta_1}{\beta_0}$

$*$ $\alpha_0\beta_0^* = -\alpha_1\beta_1^*$ $-\frac{\beta_0^*}{\beta_1^*} = \frac{\alpha_1}{\alpha_0}$

Let $U = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$

$U|\psi\rangle = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix} = \begin{pmatrix}a\alpha_0 + b\alpha_1\\ c\alpha_0 + d\alpha_1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$ $a\alpha_0 + b\alpha_1 = 1$, $c\alpha_0 = -d\alpha_1$

$U|\varphi\rangle = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix} = \begin{pmatrix}a\beta_0 + b\beta_1\\ c\beta_0 + d\beta_1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$ $a\beta_0 = -b\beta_1$, $c\beta_0 + d\beta_1 = 1$

$-\frac{a}{b} = \frac{\beta_1}{\beta_0} = -\frac{\alpha_0^*}{\alpha_1^*}$ $\frac{a}{b}\alpha_0 + \alpha_1 = \frac{1}{b}$

$\frac{\alpha_0^*}{\alpha_1^*}\alpha_0 + \alpha_1 = \frac{1}{b}$

$\alpha_0^*\alpha_0 + \alpha_1\alpha_1^* = \frac{\alpha_1^*}{b}$

$||\alpha_0||^2 + ||\alpha_1||^2 = 1 = \frac{\alpha_1^*}{b}$ $b = \alpha_1^* \Rightarrow a = \alpha_0^*$

$-\frac{c}{d} = \frac{\alpha_1}{\alpha_0} = -\frac{\beta_0^*}{\beta_1^*}$ $\frac{c}{d}\beta_0 + \beta_1 = \frac{1}{d}$

$\frac{\beta_0^*}{\beta_1^*}\beta_0 + \beta_1 = \frac{1}{d}$

$\beta_0\beta_0^* + \beta_1\beta_1^* = \frac{\beta_1^*}{d}$

$||\beta_0||^2 + ||\beta_1||^2 = 1 = \frac{\beta_1^*}{d}$ $d = \beta_1^* \Rightarrow c = \beta_0^*$

$\therefore U = \begin{pmatrix}\alpha_0^* & \alpha_1^*\\ \beta_0^* & \beta_1^*\end{pmatrix}$

Since $\langle\psi|\varphi\rangle = 0$, U maps the orthogonal basis $\{|\psi\rangle, |\varphi\rangle\}$ to $\{|0\rangle, |1\rangle\}$. Therefore, U serves as a unitary computation to determine whether or not some state $|x\rangle$ is $|\psi\rangle$ or $|\varphi\rangle$.

d) Let $|\psi\rangle = \sum_{i=1}^{2n} \alpha_i|\psi_i\rangle$.

Assume that $|\varphi\rangle = \sum_{i=1}^{2n} \beta_i|\psi_i\rangle$ is true.

Let $\langle\psi| = \sum_{i=1}^{2n} \alpha_i^*\langle\psi_i|$. So the inner product of $|\varphi\rangle$ is $\langle\psi|\varphi\rangle = \sum_{i=1}^{2n} \alpha_i^*\beta_i$.

e)

f)

g) $|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$ $\quad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$ $\quad \frac{|0\rangle + i|1\rangle}{\sqrt{2}} \rightarrow \{|+\rangle, |-\rangle\}$

$$\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\} \longrightarrow \left\{\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}, \begin{pmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{pmatrix}\right\}$$

P from I to F $\quad \left[\begin{array}{cc|cc} 1/\sqrt{2} & 1/\sqrt{2} & 1 & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} \sqrt{2} & 0 & 1 & 1 \\ 0 & -\sqrt{2} & -1 & 1 \end{array}\right]$

$I = FP \quad F^{-1} = P$ $\quad = \left[\begin{array}{cc|cc} 1 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1 & 1/\sqrt{2} & -1/\sqrt{2} \end{array}\right]$

$|0\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$ $\quad |1\rangle = \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle$

$P[x]_I = [x]_F$

$$\frac{|0\rangle + i|1\rangle}{\sqrt{2}} = \frac{|+\rangle + |-\rangle + i(|+\rangle - |-\rangle)}{2} = \boxed{\frac{(1+i)|+\rangle + (1-i)|-\rangle}{2}}.$$

$$\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}\begin{pmatrix}1/\sqrt{2}\\ i/\sqrt{2}\end{pmatrix} = \begin{pmatrix}\frac{1+i}{2}\\ \frac{1-i}{2}\end{pmatrix}$$

$\frac{1+i}{2}|+\rangle + \frac{1-i}{2}|-\rangle$