Complete Solutions to **Miscellaneous Exercise 1**

1. (i) *How do we find* $(\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{B})$ *given* $\mathbf{A}=\begin{pmatrix}1&2\\3&4\end{pmatrix}$ *and* $\mathbf{B}=\begin{pmatrix}5&6\\7&8\end{pmatrix}$ *?*

Evaluate $\mathbf{A}-\mathbf{B}$, $\mathbf{A}+\mathbf{B}$ and then multiply them together.

$$\mathbf{A}-\mathbf{B}=\begin{pmatrix}1&2\\3&4\end{pmatrix}-\begin{pmatrix}5&6\\7&8\end{pmatrix}=\begin{pmatrix}-4&-4\\-4&-4\end{pmatrix}=-4\begin{pmatrix}1&1\\1&1\end{pmatrix}$$

$$\mathbf{A}+\mathbf{B}=\begin{pmatrix}1&2\\3&4\end{pmatrix}+\begin{pmatrix}5&6\\7&8\end{pmatrix}=\begin{pmatrix}6&8\\10&12\end{pmatrix}=2\begin{pmatrix}3&4\\5&6\end{pmatrix}$$

Multiplying these together gives

$$\begin{aligned}(\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{B})&=-4\begin{pmatrix}1&1\\1&1\end{pmatrix}2\begin{pmatrix}3&4\\5&6\end{pmatrix}\\&=-8\begin{pmatrix}8&10\\8&10\end{pmatrix}=-16\begin{pmatrix}4&5\\4&5\end{pmatrix}\end{aligned}$$

(ii) *How do we determine* $\mathbf{A}^2-\mathbf{B}^2$ *?*

$$\begin{aligned}\mathbf{A}^2-\mathbf{B}^2&=\begin{pmatrix}1&2\\3&4\end{pmatrix}^2-\begin{pmatrix}5&6\\7&8\end{pmatrix}^2\\&=\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}1&2\\3&4\end{pmatrix}-\begin{pmatrix}5&6\\7&8\end{pmatrix}\begin{pmatrix}5&6\\7&8\end{pmatrix}\\&=\begin{pmatrix}7&10\\15&22\end{pmatrix}-\begin{pmatrix}67&78\\91&106\end{pmatrix}\\&=\begin{pmatrix}-60&-68\\-76&-84\end{pmatrix}=-4\begin{pmatrix}15&17\\19&21\end{pmatrix}\end{aligned}$$

$\mathbf{A}^2-\mathbf{B}^2\neq(\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{B})$ because

$$(\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{B})=\mathbf{A}^2+\underbrace{\mathbf{AB}-\mathbf{BA}}_{\neq\mathbf{O}\text{ because }\mathbf{AB}\neq\mathbf{BA}}-\mathbf{B}^2$$

Remember matrix multiplication is **not** commutative, that is

$$\mathbf{AB}\neq\mathbf{BA}$$

so we **cannot** have $\mathbf{A}^2-\mathbf{B}^2=(\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{B})$.

2. We have an error in the first line:

$$\mathbf{AB}(\mathbf{AB})^{-1}=\mathbf{ABA}^{-1}\mathbf{B}^{-1}$$

because $(\mathbf{AB})^{-1}=\mathbf{B}^{-1}\mathbf{A}^{-1}\neq\mathbf{A}^{-1}\mathbf{B}^{-1}$ so the above are **not** equal. Again there is an error in the second line of the derivation:

$$\mathbf{ABA}^{-1}\mathbf{B}^{-1}=\mathbf{AA}^{-1}\mathbf{BB}^{-1}$$

because $\mathbf{BA}^{-1}\neq\mathbf{A}^{-1}\mathbf{B}$ [Not Equal]. Matrix multiplication is **not** commutative.

3. We are given the matrix $\mathbf{A}=\dfrac{1}{7}\begin{pmatrix}3&-2&-6\\-2&6&-3\\-6&-3&-2\end{pmatrix}$.

(a) The matrix $\mathbf{A}^2$ is evaluated by

$$\mathbf{A}^2 = \frac{1}{7}\begin{pmatrix} 3 & -2 & -6 \\ -2 & 6 & -3 \\ -6 & -3 & -2 \end{pmatrix}\frac{1}{7}\begin{pmatrix} 3 & -2 & -6 \\ -2 & 6 & -3 \\ -6 & -3 & -2 \end{pmatrix}$$

$$= \frac{1}{7}\times\frac{1}{7}\begin{pmatrix} 3 & -2 & -6 \\ -2 & 6 & -3 \\ -6 & -3 & -2 \end{pmatrix}\begin{pmatrix} 3 & -2 & -6 \\ -2 & 6 & -3 \\ -6 & -3 & -2 \end{pmatrix}$$

$$= \frac{1}{49}\begin{pmatrix} 49 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & 49 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}$$

Hence $\mathbf{A}^2 = \mathbf{I}$. *What is* $\mathbf{A}^3$ *equal to?*

$$\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = \mathbf{IA} = \mathbf{A} = \frac{1}{7}\begin{pmatrix} 3 & -2 & -6 \\ -2 & 6 & -3 \\ -6 & -3 & -2 \end{pmatrix}$$

(b) Since we have $\mathbf{A}^2 = \mathbf{I}$ therefore $\mathbf{A}^{-1} = \mathbf{A}$ and

$$\mathbf{A}^{2004} = \left(\mathbf{A}^2\right)^{1002} = \mathbf{I}^{1002} = \mathbf{I}$$

where **I** is the identity 3 by 3 matrix.

4. We have the following:

$$\mathbf{A}^t = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}^t = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ -1 & 1 & 3 \end{pmatrix}$$

The matrix multiplication **AB** is **not** valid because the number of columns (3) of matrix **A** does **not** equal the number of rows (4) of matrix **B**.
The matrix addition $\mathbf{B}+\mathbf{C}$ is **not** valid because matrices **B** and **C** are different sizes.
The matrix subtraction $\mathbf{A}-\mathbf{B}$ is **not** valid because matrices **A** and **B** are different sizes.
The matrix multiplication **CB** is given by

$$\mathbf{CB} = \begin{pmatrix} 1 & -1 & 1 & 0 \\ -2 & 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 0 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ -1 & 1 \end{pmatrix}$$

The matrix multiplication $\mathbf{BC}^t$ is **not** valid because

$$\mathbf{BC}^t = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 0 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 & 0 \\ -2 & 1 & 2 & 3 \end{pmatrix}^t = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 0 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -1 & 1 \\ 1 & 2 \\ 0 & 3 \end{pmatrix}$$

This matrix multiplication is impossible because the number of columns (2) of the matrix **B** does **not** equal the number of rows (4) of the matrix $\mathbf{C}^t$.
The matrix multiplication $\mathbf{A}^2$ is given by

$$\mathbf{A}^2=\begin{pmatrix}1&0&-1\\1&1&1\\1&2&3\end{pmatrix}\begin{pmatrix}1&0&-1\\1&1&1\\1&2&3\end{pmatrix}=\begin{pmatrix}0&-2&-4\\3&3&3\\6&8&10\end{pmatrix}$$

5. We need to show that $\mathbf{A}^T\mathbf{A}$ is symmetric for all matrices **A**.
Proof.
First we need to establish that $\mathbf{A}^T\mathbf{A}$ is a valid multiplication operation. Let matrix **A** be of size $m\times n$. Then $\mathbf{A}^T$ is of size $n\times m$. The multiplication operation $\mathbf{A}^T\mathbf{A}$ is valid provided the number of columns of the left hand matrix is equal to the number of rows of the right hand matrix. *How many columns does* $\mathbf{A}^T$ *have?*
m. How many rows does ***A*** *have?*
m. Hence the matrix multiplication $\mathbf{A}^T\mathbf{A}$ is valid.
How do we show that $\mathbf{A}^T\mathbf{A}$ *is symmetric?*
We need to prove that $\left(\mathbf{A}^T\mathbf{A}\right)^T=\mathbf{A}^T\mathbf{A}$. We have

$$\begin{aligned}\left(\mathbf{A}^T\mathbf{A}\right)^T&=\mathbf{A}^T\left(\mathbf{A}^T\right)^T && \left[\text{Because } \left(\mathbf{XY}\right)^T=\mathbf{Y}^T\mathbf{X}^T\right]\\&=\mathbf{A}^T\mathbf{A} && \left[\text{Because } \left(\mathbf{X}^T\right)^T=\mathbf{X}\right]\end{aligned}$$

Hence $\mathbf{A}^T\mathbf{A}$ is symmetric.

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6. We need to find conditions on a, b, c and d so that

$$\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}1&2\\3&0\end{bmatrix}=\begin{bmatrix}1&2\\3&0\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}$$

Expanding both of these separately we have

$$\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}1&2\\3&0\end{bmatrix}=\begin{bmatrix}a+3b&2a\\c+3d&2c\end{bmatrix}$$

$$\begin{bmatrix}1&2\\3&0\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}=\begin{bmatrix}a+2c&b+2d\\3a&3b\end{bmatrix}$$

Equating each corresponding entry of the matrices on the right hand side gives:

$$\left.\begin{aligned}a+3b&=a+2c\\2a&=b+2d\\c+3d&=3a\\2c&=3b\end{aligned}\right\}\quad\text{gives } a=c \text{ and } b=d=\frac{2}{3}c$$

7. We can find (many) example(s) such that 2×2 matrix **A** and 2×1 nonzero vectors **u** and **v** such that $\mathbf{Au}=\mathbf{Av}$ yet $\mathbf{u}\neq\mathbf{v}$.
Let $\mathbf{A}=\begin{pmatrix}1&2\\2&4\end{pmatrix}$, $\mathbf{u}=\begin{pmatrix}-2\\1\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}2\\-1\end{pmatrix}$ then

$$\mathbf{Au}=\mathbf{Av}=\begin{pmatrix}0\\0\end{pmatrix}=\mathbf{O}\text{ but }\mathbf{u}=\begin{pmatrix}-2\\1\end{pmatrix}\neq\begin{pmatrix}2\\-1\end{pmatrix}=\mathbf{v}$$

8. (a) We are given the matrices $\mathbf{A}=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$:

$$\mathbf{A}^2=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}=\begin{pmatrix}7 & 10\\ 15 & 22\end{pmatrix}$$

$$\mathbf{B}^2=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$$

$$\mathbf{AB}=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}=\begin{pmatrix}-2 & 1\\ -4 & 3\end{pmatrix}$$

$$\mathbf{BA}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}=\begin{pmatrix}3 & 4\\ -1 & -2\end{pmatrix}$$

(b) We are given $\mathbf{A}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}e & f\\ g & h\end{pmatrix}$:

$$\mathbf{AB}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}e & f\\ g & h\end{pmatrix}=\begin{pmatrix}ae+bg & af+bh\\ ce+dg & cf+dh\end{pmatrix}$$

$$\mathbf{BA}=\begin{pmatrix}e & f\\ g & h\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}ae+cf & be+df\\ ag+ch & bg+dh\end{pmatrix}$$

Subtracting each corresponding entry gives

$$\begin{aligned}\mathbf{AB}-\mathbf{BA}&=\begin{pmatrix}ae+bg & af+bh\\ ce+dg & cf+dh\end{pmatrix}-\begin{pmatrix}ae+cf & be+df\\ ag+ch & bg+dh\end{pmatrix}\\ &=\begin{pmatrix}bg-cf & af+bh-be-df\\ ce+dg-ag-ch & cf-bg\end{pmatrix}\end{aligned}$$

9. We need to evaluate the first four indices of the given matrix $\mathbf{M}=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$:

$$\mathbf{M}^2=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}=\begin{pmatrix}2 & 2\\ 2 & 2\end{pmatrix}=2\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}=2\mathbf{M}$$

$$\begin{aligned}\mathbf{M}^3=\mathbf{M}^2\mathbf{M}&=\begin{pmatrix}2 & 2\\ 2 & 2\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\\ &=\begin{pmatrix}4 & 4\\ 4 & 4\end{pmatrix}=4\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}=4\mathbf{M}\end{aligned}$$

$$\begin{aligned}\mathbf{M}^4=\mathbf{M}^3\mathbf{M}&=\begin{pmatrix}4 & 4\\ 4 & 4\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\\ &=\begin{pmatrix}8 & 8\\ 8 & 8\end{pmatrix}=8\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}=8\mathbf{M}\end{aligned}$$

We need to predict a formula for $\mathbf{M}^n=c(n)\mathbf{M}$. *What do you notice about the above results?*

$$\mathbf{M}^2=2\mathbf{M},\ \mathbf{M}^3=4\mathbf{M}\ \text{and}\ \mathbf{M}^4=8\mathbf{M}$$

The predicted formula is $\mathbf{M}^n=2^{n-1}\mathbf{M}$. Thus $c(n)=2^{n-1}$.

10. We are given $\mathbf{A}=\begin{pmatrix}\frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3}\end{pmatrix}$ which we can rewrite as $\mathbf{A}=\frac{1}{3}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$.

(i) We have

$$\begin{aligned}\mathbf{A}^2 &= \frac{1}{3}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\frac{1}{3}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\\ &= \frac{1}{3^2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}=\frac{1}{3^2}\begin{pmatrix}2 & 2\\ 2 & 2\end{pmatrix}\\ &= \frac{2}{3^2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\end{aligned}$$

(ii) Similarly, we have

$$\begin{aligned}\mathbf{A}^3=\mathbf{A}^2\mathbf{A} &= \frac{1}{3^2}\begin{pmatrix}2 & 2\\ 2 & 2\end{pmatrix}\frac{1}{3}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\\ &= \frac{1}{3^3}\begin{pmatrix}4 & 4\\ 4 & 4\end{pmatrix}=\frac{2^2}{3^3}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}=\frac{1}{2}\frac{2^3}{3^3}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}=\frac{1}{2}\left(\frac{2}{3}\right)^3\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\end{aligned}$$

To prove $\mathbf{A}^n=\frac{1}{2}\left(\frac{2}{3}\right)^n\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$ we use mathematical induction.

Proof.
Clearly the result is true for $n=1$ because we have

$$\mathbf{A}=\frac{1}{2}\left(\frac{2}{3}\right)\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}=\frac{1}{3}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \quad \surd$$

We assume the given result is true for $n=k$, that is

$$\mathbf{A}^k=\frac{1}{2}\left(\frac{2}{3}\right)^k\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \qquad (*)$$

Required to prove this for $n=k+1$:

$$\begin{aligned}\mathbf{A}^{k+1}=\mathbf{A}^k\mathbf{A} &= \underbrace{\frac{1}{2}\left(\frac{2}{3}\right)^k\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}}_{\text{By }(*)}\frac{1}{3}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\\ &= \frac{1}{2}\frac{1}{3}\left(\frac{2}{3}\right)^k\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\\ &= \frac{1}{2}\frac{1}{3}\left(\frac{2}{3}\right)^k\begin{pmatrix}2 & 2\\ 2 & 2\end{pmatrix}=\frac{1}{2}\frac{2}{3}\left(\frac{2}{3}\right)^k\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}=\frac{1}{2}\left(\frac{2}{3}\right)^{k+1}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}=\frac{1}{2}\left(\frac{2}{3}\right)^{k+1}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\end{aligned}$$

Hence by mathematical induction we have our result $\mathbf{A}^n=\frac{1}{2}\left(\frac{2}{3}\right)^n\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$.

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11. In general if **B** is a $m\times p$ matrix and **C** is a $p\times n$ matrix then **BC** is a $m\times n$ matrix. Since we are told that **BC** is a 4×6 matrix therefore the matrix **B** has 4 rows.

12. (a) We are given the matrix $\mathbf{A}=\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$. *How can we check that this is invertible or not?*

By carrying out row operations:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Execute the following row operations $2R_2+R_1$ and $2R_3+R_1$:

$$\begin{array}{c} R_1 \\ R'_2=2R_2+R_1 \\ R'_3=2R_3+R_1 \end{array}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{bmatrix}$$

Adding the bottom two rows $R'_3+R'_2$ gives

$$\begin{array}{c} R_1 \\ R'_2 \\ R''_3=R'_3+R'_2 \end{array}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Since we have a row of zeros therefore we conclude that the matrix **A** is **non-invertible (singular)**.

(b) We are given that $\mathbf{ABC}=\mathbf{I}_n$. Pre-multiplying this by $\mathbf{A}^{-1}$ and post multiplying by $\mathbf{C}^{-1}$ gives

$$\mathbf{A}^{-1}(\mathbf{ABC})\mathbf{C}^{-1}=\mathbf{A}^{-1}\mathbf{I}_n\mathbf{C}^{-1}$$

$$\underbrace{(\mathbf{A}^{-1}\mathbf{A})}_{=\mathbf{I}}\mathbf{B}\underbrace{(\mathbf{CC}^{-1})}_{=\mathbf{I}}=\mathbf{A}^{-1}\mathbf{C}^{-1}=(\mathbf{CA})^{-1}$$

$$\mathbf{B}=(\mathbf{CA})^{-1}$$

Since $\mathbf{B}=(\mathbf{CA})^{-1}$ and we are given that **C** and **A** are invertible, so **CA** is invertible and

$$\mathbf{B}^{-1}=\left((\mathbf{CA})^{-1}\right)^{-1}=\mathbf{CA} \qquad \left[\text{Because } \left[\mathbf{X}^{-1}\right]^{-1}=\mathbf{X}\right]$$

Hence matrix **B** is invertible.

13. (a) An $n\times n$ matrix **A** is invertible if and only if there exists a $n\times n$ matrix **B** such that

$$\mathbf{AB}=\mathbf{BA}=\mathbf{I}$$

The matrix **B** is normally denoted by $\mathbf{A}^{-1}$.

(b) To find the inverse of the given matrix $\mathbf{A}=\begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}$ we use the Gaussian process given by:

<u>Step 1</u>: Formulate the augmented matrix $\left[\mathbf{A} \mid \mathbf{I}\right]$.

<u>Step 2</u>: Use row operations to convert this $\left[\mathbf{A} \mid \mathbf{I}\right]$ to $\left[\mathbf{I} \mid \mathbf{B}\right]$. If this is possible then **B** is the inverse of the matrix **A**.

We have $\left[\mathbf{A} \mid \mathbf{I}\right]$ is given by

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right]$$

Carrying out the row operations $R_2 - R_1$ and $R_3 - R_1$:

$$\begin{array}{c} R_1 \\ R'_2 = R_2 - R_1 \\ R'_3 = R_3 - R_1 \end{array}\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$$

Executing the row operation $R'_3 - R'_2$:

$$\begin{array}{c} R_1 \\ R'_2 \\ R''_3 = R'_3 - R'_2 \end{array}\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

Executing the row operation $R_1 - 6R'_2$:

$$\begin{array}{c} R_1{*} = R_1 - 6R'_2 \\ R'_2 \\ R''_3 \end{array}\left[\begin{array}{ccc|ccc} 2 & 0 & 6 & 7 & -6 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

Executing the row operation $R_1{*} - 6R''_3$:

$$\begin{array}{c} R_1{**} = R{*}_1 - 6R''_3 \\ R'_2 \\ R''_3 \end{array}\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 7 & 0 & -6 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

Dividing the top row by 2 gives us our result of $\left[\mathbf{I} \mid \mathbf{B}\right]$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/2 & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

Hence $\mathbf{A}^{-1} = \begin{bmatrix} 7/2 & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$. We can check this is indeed the inverse of **A** by carrying out the matrix multiplication:

$$\mathbf{AA}^{-1} = \begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}\begin{bmatrix} 7/2 & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

14. We are given the linear equations

$$\begin{aligned} x + y + 2z &= 8 \\ -x - 2y + 3z &= 1 \\ 3x - 7y + 4z &= 10 \end{aligned}$$

We can write these in matrix form as

$$\begin{pmatrix} 1 & 1 & 2 \\ -1 & -2 & 3 \\ 3 & -7 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \\ 10 \end{pmatrix}$$

We write this as an augmented matrix and then carry out row operations.

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left(\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right)$$

Executing $R_2 + R_1$ and $R_3 - 3R_1$:

$$\begin{array}{c} R_1 \\ R_2{*} = R_2 + R_1 \\ R_3{*} = R_3 - 3R_1 \end{array}\left(\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right)$$

Carrying out the row operation $R_3{*} - 10R_2{*}$:

$$\begin{array}{c} R_1 \\ R_2{*} \\ R_3{**} = R_3{*} - 10R_2{*} \end{array}\left(\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & 0 & -52 & -104 \end{array}\right)$$

Dividing the bottom row by -52 and the middle row by -1 gives

$$\begin{array}{c} R_1 \\ R_2^{\dagger} \\ R_3^{\dagger} \end{array}\left(\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

Executing $R_2^{\dagger} + 5R_3^{\dagger}$ gives

$$\begin{array}{c} R_1 \\ R_2^{\dagger\dagger} = R_2^{\dagger} + 5R_3^{\dagger} \\ R_3^{\dagger} \end{array}\left(\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

Carrying out the row operation $R_1 - R_2^{\dagger\dagger}$:

$$\begin{array}{c} R_1{*} = R_1 - R_2^{\dagger\dagger} \\ R_2^{\dagger\dagger} \\ R_3^{\dagger} \end{array}\left(\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

Carrying out the row operation $R_1{*} - 2R_3^{\dagger}$

$$\begin{array}{cc} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} R_1{**} = R_1{*} - 2R_3^{\dagger} \\ R_2^{\dagger\dagger} \\ R_3^{\dagger} \end{array} & \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right) \end{array}$$

Hence we have the solution $x = 3,\ y = 1$ and $z = 2$.

Check by substituting these into the given equations:

$$3 + 1 + 2(2) = 8$$

$$-3 - 2(1) + 3(2) = 1$$

$$3(3) - 7(1) + 4(2) = 10$$

15. (a) The linear system $\mathbf{Ax}=\mathbf{b}$ where $\mathbf{A}=\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix}$, $\mathbf{x}=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{b}=\begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$ has it's augmented matrix given by:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 7 & 3 \\ -2 & 5 & 4 & 3 \\ -5 & 6 & -3 & 1 \end{array}\right]$$

Carrying out the row operations R_2+2R_1 and R_3+5R_1:

$$\begin{array}{r} R_1 \\ R_2^* = R_2+2R_1 \\ R_3^* = R_3+5R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 7 & 3 \\ 0 & 9 & 18 & 9 \\ 0 & 16 & 32 & 16 \end{array}\right]$$

Dividing the middle row by 9 and the bottom row by 16 gives

$$\begin{array}{c} R_1 \\ R_2^{**} \\ R_3^{**} \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 7 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

Subtracting the bottom two rows $R_3^{**}-R_2^{**}$ gives

$$\begin{array}{c} R_1 \\ R_2^{**} \\ R_3^{\dagger} \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 7 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This is **not** in reduced row echelon form. *Why not?*
Because of the 2 in the top row. Remember any non-zero entry in a column containing a leading 1 is the leading 1. *How can we remove the 2?*
By executing the row operation $R_1-2R_2^{**}$:

$$\begin{array}{c} \\ R_1-2R_2^{**} \\ R_2^{**} \\ R_3^{\dagger} \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

This is now in reduced row echelon form.
(b) We have 2 non-zero rows and 3 unknowns so there is $3-2=1$ free variable. *Which one is the free variable?*
z. Let $z=t$ then from the middle row R_2** we have

$$y+2z=1 \quad \Rightarrow \quad y=1-2z=1-2t$$

From the top row we have

$$x+3z=1 \quad \Rightarrow \quad x=1-3z=1-3t$$

Hence our solution set is $x=1-3t,\ y=1-2t$ and $z=t$. In parametric vector form we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1-3t \\ 1-2t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$$

16. We are given the linear system:

$$\begin{aligned} x_1 - x_2 - 2x_3 - 8x_4 &= -3 \\ -2x_1 + x_2 + 2x_3 + 9x_4 &= 5 \\ 3x_1 - 2x_2 - 3x_3 - 15x_4 &= -9 \end{aligned}$$

The augmented matrix is given by

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left(\begin{array}{cccc|c} 1 & -1 & -2 & -8 & -3 \\ -2 & 1 & 2 & 9 & 5 \\ 3 & -2 & -3 & -15 & -9 \end{array}\right)$$

Executing the row operations $R_2 + 2R_1$ and $R_3 - 3R_1$:

$$\begin{array}{c} R_1 \\ R_2' = R_2 + 2R_1 \\ R_3' = R_3 - 3R_1 \end{array} \left(\begin{array}{cccc|c} 1 & -1 & -2 & -8 & -3 \\ 0 & -1 & -2 & -7 & -1 \\ 0 & 1 & 3 & 9 & 0 \end{array}\right)$$

Adding the bottom 2 rows $R_3' + R_2'$:

$$\begin{array}{c} R_1 \\ R_2' \\ R_3'' = R_3' + R_2' \end{array} \left(\begin{array}{cccc|c} 1 & -1 & -2 & -8 & -3 \\ 0 & -1 & -2 & -7 & -1 \\ 0 & 0 & 1 & 2 & -1 \end{array}\right)$$

Carrying out the row operation $R_1 - R_2'$:

$$\begin{array}{c} R'_1 = R_1 - R_2' \\ R_2' \\ R_3'' \end{array} \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -2 \\ 0 & -1 & -2 & -7 & -1 \\ 0 & 0 & 1 & 2 & -1 \end{array}\right)$$

Multiplying the middle row by -1 gives

$$\begin{array}{cc} & \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{array}{c} R'_1 \\ -R_2' \\ R_3'' \end{array} & \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 2 & 7 & 1 \\ 0 & 0 & 1 & 2 & -1 \end{array}\right) \end{array}$$

It is **not** in reduced row echelon form but we can find a solution set from this.
Since we have 4 unknowns and 3 non-zero rows therefore there is $4 - 3 = 1$ free variable.
Which variable is free?
x_4 because it **does not** appear at the beginning of any row. Let $x_4 = t$. From the bottom row R_3'' we have

$$x_3 + 2x_4 = -1 \quad \text{implies} \quad x_3 = -1 - 2x_4 = -1 - 2t$$

From the middle row we have

$$\begin{aligned} x_2 + 2x_3 + 7x_4 = 1 \quad \text{gives} \quad x_2 &= 1 - 2x_3 - 7x_4 \\ &= 1 - 2(-1 - 2t) - 7t = 3 - 3t \end{aligned}$$

From the top row we have

$$x_1 - x_4 = -2 \quad \Rightarrow \quad x_1 = -2 + x_4 = -2 + t$$

Our solution set is $x_1 = -2 + t$, $x_2 = 3 - 3t$, $x_3 = -1 - 2t$ and $x_4 = t$. We can write this in vector form as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2+t \\ 3-3t \\ -1-2t \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -3 \\ -2 \\ 1 \end{bmatrix}$$

17. The question says solve the given linear system by using Gauss-Jordan Elimination.
What does this mean?
Means you have to place the augmented matrix in **reduced** row echelon form. We are given the linear system:

$$\begin{aligned} x_1 + 2x_2 + x_3 - 4x_4 &= 1 \\ x_1 + 3x_2 + 7x_3 + 2x_4 &= 2 \\ x_1 - 11x_3 - 16x_4 &= -1 \end{aligned}$$

The augmented matrix is

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left(\begin{array}{cccc|c} 1 & 2 & 1 & -4 & 1 \\ 1 & 3 & 7 & 2 & 2 \\ 1 & 0 & -11 & -16 & -1 \end{array}\right)$$

Exchange rows R_1 and R_3:

$$\begin{array}{c} R^*_1 = R_3 \\ R_2 \\ R^*_3 = R_1 \end{array} \left(\begin{array}{cccc|c} 1 & 0 & -11 & -16 & -1 \\ 1 & 3 & 7 & 2 & 2 \\ 1 & 2 & 1 & -4 & 1 \end{array}\right)$$

Executing the row operations $R_2 - R^*_1$ and $R^*_3 - R^*_1$:

$$\begin{array}{c} R^*_1 \\ R^*_2 = R_2 - R^*_1 \\ R^{**}_3 = R^*_3 - R^*_1 \end{array} \left(\begin{array}{cccc|c} 1 & 0 & -11 & -16 & -1 \\ 0 & 3 & 18 & 18 & 3 \\ 0 & 2 & 12 & 12 & 2 \end{array}\right)$$

Dividing the middle row by 3 and the bottom row by 2 gives

$$\begin{array}{c} R^*_1 \\ R_2^\dagger \\ R_3^\dagger \end{array} \left(\begin{array}{cccc|c} 1 & 0 & -11 & -16 & -1 \\ 0 & 1 & 6 & 6 & 1 \\ 0 & 1 & 6 & 6 & 1 \end{array}\right)$$

Carrying out the row operation $R_3^\dagger - R_2^\dagger$:

$$\begin{array}{cc} & \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{array}{c} R^*_1 \\ R_2^\dagger \\ R_3^{\dagger\dagger} = R_3^\dagger - R_2^\dagger \end{array} & \left(\begin{array}{cccc|c} 1 & 0 & -11 & -16 & -1 \\ 0 & 1 & 6 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

We have 2 free variables. *Which of these variables are free?*
x_3 and x_4. Let $x_3 = s$ and $x_4 = t$. From the middle row $R_2^\dagger$ we have

$$x_2 + 6x_3 + 6x_4 = 1 \quad \Rightarrow \quad x_2 = 1 - 6x_3 - 6x_4 = 1 - 6s - 6t$$

From the top row we have

$$x_1 - 11x_3 - 16x_4 = -1 \quad \Rightarrow \quad x_1 = -1 + 11x_3 + 16x_4 = -1 + 11s + 16t$$

Our solution set is $x_1 = -1 + 11s + 16t$, $x_2 = 1 - 6s - 6t$, $x_3 = s$ and $x_4 = t$. In vector form we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1+11s+16t \\ 1-6s-6t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 11 \\ -6 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 16 \\ -6 \\ 0 \\ 1 \end{bmatrix}$$

18. We are given

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\begin{bmatrix} 1 & 2 & 5 & 6 \\ 3 & 7 & 3 & 2 \\ 2 & 5 & 11 & 22 \end{bmatrix}$$

(i) Carrying out the row operations $R_2 - 3R_1$ and $R_3 - 2R_1$:

$$\begin{array}{c} R_1 \\ R_2{*} = R_2 - 3R_1 \\ R_3{*} = R_3 - 2R_1 \end{array}\begin{bmatrix} 1 & 2 & 5 & 6 \\ 0 & 1 & -12 & -16 \\ 0 & 1 & 1 & 10 \end{bmatrix}$$

Executing the row operation $R_3{*} - R_2{*}$ gives

$$\begin{array}{c} R_1 \\ R_2{*} \\ R_3{**} = R_3{*} - R_2{*} \end{array}\begin{bmatrix} 1 & 2 & 5 & 6 \\ 0 & 1 & -12 & -16 \\ 0 & 0 & 13 & 26 \end{bmatrix}$$

This is now in echelon form.

(ii) *What do we need to do in order to place the above matrix into row echelon form?*

Divide the bottom row $R_3{**}$ by 13:

$$\begin{array}{c} R_1 \\ R_2{*} \\ R_3{***} = R_3{**}/13 \end{array}\begin{bmatrix} 1 & 2 & 5 & 6 \\ 0 & 1 & -12 & -16 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

(iii) For reduced row echelon form we need to ensure that the only non-zero entry containing a leading zero is that leading zero. *What does this mean in this case?*

Need to convert the $2,\ 5$ and -12 to zero.

Carrying out the row operation $R_2{*} + 12R_3{***}$:

$$\begin{array}{c} R_1 \\ R_2{**} = R_2{*} + 12R_3{***} \\ R_3{***} \end{array}\begin{bmatrix} 1 & 2 & 5 & 6 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Executing the row operations $R_1 - 2R_2{**}$:

$$\begin{array}{c} R_1{*} = R_1 - 2R_2{**} \\ R_2{**} \\ R_3{***} \end{array}\begin{bmatrix} 1 & 0 & 5 & -10 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The last row operation is $R_1{*} - 5R_3{***}$:

$$\begin{array}{c} R_1{**} = R_1{*} - 5R_3{***} \\ R_2{**} \\ R_3{***} \end{array}\begin{bmatrix} 1 & 0 & 0 & -20 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

This final matrix is in reduced row echelon form.

19. The given linear system written in augmented matrix form is

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}\left(\begin{array}{cccc|c} 1 & 2 & -3 & -5 & -13 \\ 3 & 1 & 4 & -4 & 5 \\ 2 & -2 & 3 & -10 & 7 \\ 1 & 1 & 2 & 2 & 0 \end{array}\right)$$

Interchange top and bottom rows because it is easier to deal with the numbers on the bottom row:

$$\begin{array}{c} R_1^* = R_4 \\ R_2 \\ R_3 \\ R_4^* = R_1 \end{array}\left(\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 0 \\ 3 & 1 & 4 & -4 & 5 \\ 2 & -2 & 3 & -10 & 7 \\ 1 & 2 & -3 & -5 & -13 \end{array}\right)$$

Carrying out the row operations $R_2 - 3R_1^*$, $R_3 - 2R_1^*$ and $R_4^* - R_1^*$:

$$\begin{array}{c} R_1^* \\ R_2^* = R_2 - 3R_1^* \\ R_3^* = R_3 - 2R_1^* \\ R_4^{**} = R_4^* - R_1^* \end{array}\left(\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 0 \\ 0 & -2 & -2 & -10 & 5 \\ 0 & -4 & -1 & -14 & 7 \\ 0 & 1 & -5 & -7 & -13 \end{array}\right)$$

Execute the row operation $R_3^* - 2R_2^*$:

$$\begin{array}{c} R_1^* \\ R_2^* \\ R_3^{**} = R_3^* - 2R_2^* \\ R_4^{**} \end{array}\left(\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 0 \\ 0 & -2 & -2 & -10 & 5 \\ 0 & 0 & 3 & 6 & -3 \\ 0 & 1 & -5 & -7 & -13 \end{array}\right)$$

Carrying out the row operation $2R_4^{**} + R_2^*$:

$$\begin{array}{c} R_1^* \\ R_2^* \\ R_3^{**} \\ R_4^{\dagger} = 2R_4^* + R_2^* \end{array}\left(\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 0 \\ 0 & -2 & -2 & -10 & 5 \\ 0 & 0 & 3 & 6 & -3 \\ 0 & 0 & -12 & -24 & -21 \end{array}\right)$$

Executing $R_4^{\dagger} + 4R_3^{**}$:

$$\begin{array}{cc} & \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \end{array} \\ \begin{array}{c} R_1^* \\ R_2^* \\ R_3^{**} \\ R_4^{\dagger\dagger} = R_4^{\dagger} + 4R_3^{**} \end{array} & \left(\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 0 \\ 0 & -2 & -2 & -10 & 5 \\ 0 & 0 & 3 & 6 & -3 \\ 0 & 0 & 0 & 0 & -33 \end{array}\right) \end{array}$$

From the last row we have

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = -33 \text{ which means that } 0 = -33$$

This is impossible so the linear system is inconsistent.

20. Let $\mathbf{A}=\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ then we have

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}=\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix},\quad \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}=\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix},\quad \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Expanding each of these out

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}=\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \text{ gives } \begin{matrix} b+c=-1 \\ e+f=0 \\ h+i=2 \end{matrix}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}=\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \text{ gives } \begin{matrix} a+c=0 \\ d+f=-1 \\ g+i=2 \end{matrix}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \text{ gives } \begin{matrix} a+b=1 \\ d+e=1 \\ g+h=2 \end{matrix}$$

Solving these gives

$$a=1,\ b=0,\ c=-1,\ d=0,\ e=1,\ f=-1,\ g=1,\ h=1 \text{ and } i=1$$

Putting these values into the matrix $\mathbf{A}$ and checking:

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}=\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix},\quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}=\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix},\quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Thus $\mathbf{A}=\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$.

21. We need to find the reduced row echelon form, rref, of

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}\begin{pmatrix} 2 & 2 & 0 & 2 \\ -1 & -1 & 2 & 1 \\ 2 & 2 & -1 & 1 \\ -1 & -1 & 1 & 0 \end{pmatrix}$$

Interchanging rows and multiplying by -1 gives:

$$\begin{array}{c} R_1'=-R_4 \\ R_2'=R_1 \\ R_3 \\ R_4'=-R_2 \end{array}\begin{pmatrix} 1 & 1 & -1 & 0 \\ 2 & 2 & 0 & 2 \\ 2 & 2 & -1 & 1 \\ 1 & 1 & -2 & -1 \end{pmatrix}$$

Executing the row operations $R_2'-2R_1'$, R_3-2R_1' and $R_4'-R_1'$:

$$\begin{array}{r} R_1' \\ R_2''=R_2'-2R_1' \\ R_3'=R_3-2R_1' \\ R_4''=R_4'-R_1' \end{array}\begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}$$

Carrying out the row operations $R_2''/2$ and $-R_4''$ gives

$$\begin{array}{r} R_1' \\ R_2'''=R_2''/2 \\ R_3'=R_3-2R_1' \\ R_4'''=-R_4'' \end{array}\begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Executing the row operations $R_3'-R_2'''$ and $R_4'''-R_2'''$ gives:

$$\begin{array}{r} R_1' \\ R_2'''=R_2''/2 \\ R_3^*=R_3'-R_2''' \\ R_4^*=R_4'''-R_2''' \end{array}\begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Executing the row operation $R_1'+R_2'''$ gives

$$\begin{array}{r} R_1'+R_2''' \\ R_2'''=R_2''/2 \\ R_3^* \\ R_4^* \end{array}\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This is now in reduced row echelon form (rref).

22. Writing the augmented matrix for the given linear system:

$$\begin{array}{r} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}\left(\begin{array}{ccccc|c} 2 & 3 & 1 & 4 & -9 & 17 \\ 1 & 1 & 1 & 1 & -3 & 6 \\ 1 & 1 & 1 & 2 & -5 & 8 \\ 2 & 2 & 2 & 3 & -8 & 14 \end{array}\right)$$

Interchanging rows 1 and 2 gives

$$\begin{array}{r} R_1^*=R_2 \\ R_2^*=R_1 \\ R_3 \\ R_4 \end{array}\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -3 & 6 \\ 2 & 3 & 1 & 4 & -9 & 17 \\ 1 & 1 & 1 & 2 & -5 & 8 \\ 2 & 2 & 2 & 3 & -8 & 14 \end{array}\right)$$

Carrying out the row operations $R_2^*-2R_1^*$, $R_3-R_1^*$ and $R_4-2R_1^*$:

$$\begin{array}{r} R_1^* \\ R_2^{**}=R_2^*-2R_1^* \\ R_3^*=R_3-R_1^* \\ R_4^*=R_4-2R_1^* \end{array}\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -3 & 6 \\ 0 & 1 & -1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 1 & -2 & 2 \end{array}\right)$$

Subtracting the bottom two rows, $R_4^*-R_3^*$:

$$\begin{array}{r} R_1* \\ R_2** \\ R_3* \\ R_4**=R_4*-R_3* \end{array}\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -3 & 6 \\ 0 & 1 & -1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Executing the row operation R_2**-2R_3*:

$$\begin{array}{r} R_1* \\ R_2***=R_2**-2R_3* \\ R_3* \\ R_4** \end{array}\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -3 & 6 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Carrying out the row operation R_1*-R_3* gives

$$\begin{array}{r} R_1**=R_1*-R_3* \\ R_2*** \\ R_3* \\ R_4** \end{array}\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & -1 & 4 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Carrying out the row operation R_1**-R_2***:

$$\begin{array}{rc} & \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{array}{r} R_1**-R_2*** \\ R_2*** \\ R_3* \\ R_4** \end{array} & \left(\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -2 & 3 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

This is now in reduced row echelon form. We have 5 unknowns and only 3 non-zero equations so there are $5-3=2$ free variables. *Which of the above are free variables?* x_3 and x_5. Let $x_3=s$ and $x_5=t$. From the third row R_3* we have

$$x_4-2x_5=2 \quad \Rightarrow \quad x_4=2x_5+2=2t+2$$

From the second row R_2*** we have

$$x_2-x_3+x_5=1 \quad \Rightarrow \quad x_2=1+x_3-x_5=1+s-t$$

From the top row we have

$$x_1+2x_3-2x_5=3 \quad \Rightarrow \quad x_1=3+2x_5-2x_3=3+2t-2s$$

Hence our solution in vector form is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}=\begin{pmatrix} 3+2t-2s \\ 1+s-t \\ s \\ 2t+2 \\ t \end{pmatrix}=\begin{pmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}+s\begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}+t\begin{pmatrix} 2 \\ -1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$$

23. (a) We need to find the inverse of the given matrix **A** by applying elementary row operations on $\left[\mathbf{A} \mid \mathbf{I}\right]$. Need to convert (if possible) this to $\left[\mathbf{I} \mid \mathbf{B}\right]$ then **B** is the inverse of matrix **A**.

$$\left[\mathbf{A} \mid \mathbf{I}\right] = \begin{array}{c} R_1 \\ R_2 \end{array} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ -5 & 2 & 0 & 1 \end{array}\right]$$

Carrying out the row operation $R_2 + 5R_1$ gives

$$\begin{array}{c} R_1 \\ R_2{*} = R_2 + 5R_1 \end{array} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 2 & 5 & 1 \end{array}\right]$$

Divide the bottom row by 2, that is $R_2{*}/2$ gives

$$\begin{array}{c} R_1 \\ R_2{*}/2 \end{array} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 5/2 & 1/2 \end{array}\right]$$

Hence the inverse of the matrix **A** is $\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 \\ 5/2 & 1/2 \end{bmatrix}$. The elementary matrices are given by examining the stated row operations above. The matrix $\mathbf{E}_1$ is given by $R_2 + 5R_1$ which is

$$\mathbf{E}_1 = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$$

The elementary matrix $\mathbf{E}_2$ is defined by the above row operation $R_2/2$:

$$\mathbf{E}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$$

Hence the matrix $\mathbf{A}^{-1} = \mathbf{E}_2\mathbf{E}_1$. You may like to check that this is indeed the case:

$$\mathbf{E}_2\mathbf{E}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 5/2 & 1/2 \end{pmatrix} = \mathbf{A}^{-1}$$

(b) *How can we write the matrix **A** as a product of elementary matrices?*
From part (a) we have $\mathbf{A}^{-1} = \mathbf{E}_2\mathbf{E}_1$. Therefore

$$\mathbf{A} = \left(\mathbf{A}^{-1}\right)^{-1} = \left(\mathbf{E}_2\mathbf{E}_1\right)^{-1} = \mathbf{E}_1^{-1}\mathbf{E}_2^{-1}$$

$\mathbf{E}_1 = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$ therefore $\mathbf{E}_1^{-1} = \begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix}$. Similarly

$$\mathbf{E}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \text{ gives } \mathbf{E}_2^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Hence $\mathbf{E}_1^{-1}\mathbf{E}_2^{-1} = \begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -5 & 2 \end{pmatrix} = \mathbf{A}$.

24. *What do you notice about the given matrices?*

$$\mathbf{A} = \begin{bmatrix} 5 & 1 & 9 \\ 2 & 4 & 0 \\ 1 & 1 & 0 \\ -1 & 3 & -2 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 4 & 0 \\ 5 & 1 & 9 \\ -1 & 3 & -2 \end{bmatrix}$$

Matrix **B** is matrix **A** with the first and third rows swapped over. *How do we write this as an elementary matrix?*
The elementary matrix is the 4 by 4 identity matrix with a single row operation of swapping the first and third rows which is given by:

$$\mathbf{E} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

25. We are given

$$\left(\mathbf{A}^T - 3\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

How do we find the matrix $\mathbf{A}$*?*
First by taking the inverse of both sides:

$$\left(\mathbf{A}^T - 3\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \qquad \left[\text{Because } \left(\mathbf{X}^{-1}\right)^{-1} = \mathbf{X}\right]$$

$$\mathbf{A}^T = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} + 3\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

In order to find $\mathbf{A}$ we need to take the transpose of both sides because $\left(\mathbf{A}^T\right)^T = \mathbf{A}$.
We have

$$\begin{aligned} \mathbf{A}^T &= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} + 3\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} + 3\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ -3 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ -4 & 11 \end{bmatrix} \end{aligned}$$

Therefore $\mathbf{A} = \left(\mathbf{A}^T\right)^T = \begin{bmatrix} 4 & 5 \\ -4 & 11 \end{bmatrix}^T = \begin{bmatrix} 4 & -4 \\ 5 & 11 \end{bmatrix}$. This is option D.

26. *How can we find the matrix* $\mathbf{X}$ *from the given formula*

$$\mathbf{B}\left(\mathbf{X} + \mathbf{C}\right) = \mathbf{D} \text{ ?}$$

By transposing this formula:

$$\left(\mathbf{X} + \mathbf{C}\right) = \mathbf{B}^{-1}\mathbf{D}$$

$$\mathbf{X} = \mathbf{B}^{-1}\mathbf{D} - \mathbf{C}$$

We need to find the inverse matrix $\mathbf{B}^{-1}$. *How?*
By converting the augmented matrix $\left[\mathbf{B} \mid \mathbf{I}\right]$ to $\left[\mathbf{I} \mid \mathbf{A}\right]$. The matrix $\mathbf{A}$ is the inverse of $\mathbf{B}$.
We have $\left[\mathbf{B} \mid \mathbf{I}\right]$ is equal to

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 1 & 3 & 6 & 0 & 0 & 1 \end{array}\right]$$

Carrying out the row operation $R_3 - R_1$ gives:

$$\begin{array}{r} R_1 \\ R_2 \\ R_3' = R_3 - R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

Carrying out the row operation $R_2 - 2R_3'$:

$$\begin{array}{r} R_1 \\ R_2' = R_2 - 2R_3' \\ R_3' \end{array} \left[\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

Executing the row operation $R_1 - 3R_2'$:

$$\begin{array}{r} R_1' = R_1 - 3R_2' \\ R_2' \\ R_3' \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & -5 & -3 & 6 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

The last row operation is $R_1' - 5R_3'$:

$$\begin{array}{r} R_1'' = R_1' - 5R_3' \\ R_2' \\ R_3' \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -3 & 1 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

Thus $\mathbf{B}^{-1} = \begin{bmatrix} 0 & -3 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix}$. Substituting this $\mathbf{B}^{-1} = \begin{bmatrix} 0 & -3 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 5 & 0 & -2 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} -3 & 1 & 2 \\ 1 & 0 & -1 \\ 0 & -2 & 0 \end{bmatrix}$ into $\mathbf{X} = \mathbf{B}^{-1}\mathbf{D} - \mathbf{C}$ gives

$$\mathbf{X} = \begin{bmatrix} 0 & -3 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} -3 & 1 & 2 \\ 1 & 0 & -1 \\ 0 & -2 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 5 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -2 & 3 \\ -5 & 6 & 3 \\ 3 & -3 & -2 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 5 & 0 & -2 \end{bmatrix} = \begin{bmatrix} -6 & -2 & 2 \\ -5 & 4 & 3 \\ -2 & -3 & 0 \end{bmatrix}$$

27. a. The augmented matrix for the given linear system is

$$\begin{array}{r} R_1 \\ R_2 \\ R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ -1 & 3 & 2 & -2 \\ 2 & 1 & 1 & 1 \end{array}\right)$$

Executing the row operations $R_2 + R_1$ and $R_3 - 2R_1$:

$$\begin{array}{r} R_1 \\ R_2' = R_2 + R_1 \\ R_3' = R_3 - 2R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 4 & 3 & 3 \\ 0 & -1 & -1 & -9 \end{array}\right)$$

Carrying out the row operation $4R_3' + R_2'$:

$$\begin{array}{r} \\ R_1 \\ R_2' \\ R_3''=4R_3'+R_2' \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 4 & 3 & 3 \\ 0 & 0 & -1 & -33 \end{array}\right) \end{array}$$

From the bottom row we have $z = 33$. Substituting this into the expansion of the middle row

$$4y + 3z = 3 \quad \Rightarrow \quad 4y + 99 = 3 \quad \Rightarrow \quad y = -24$$

Substituting these two results $y = -24$ and $z = 33$ into the expansion of the first row gives

$$x + y + z = 5 \quad \Rightarrow \quad x - 24 + 33 = 5 \quad \Rightarrow \quad x = -4$$

Thus our solution to the given linear system is $x = -4$, $y = -24$ and $z = 33$.

b. Writing the given linear system in matrix form is

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 3 & 2 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$$

The inverse of the 3 by 3 matrix is (use row operations to find the inverse):

$$\begin{pmatrix} -1 & 0 & 1 \\ -5 & 1 & 3 \\ 7 & -1 & -4 \end{pmatrix}$$

Hence we have

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ -5 & 1 & 3 \\ 7 & -1 & -4 \end{pmatrix} \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ -24 \\ 33 \end{pmatrix} \text{ gives } x = -4,\ y = -24 \text{ and } z = 33$$

c. The given system will have **no** solution if in the above row operations the -1 in the bottom row is zero but the right hand side of the vertical bar is non-zero.

$$\begin{array}{r} \\ R_1 \\ R_2' \\ R_3''=4R_3'+R_2' \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 4 & 3 & 3 \\ 0 & 0 & -1 & -33 \end{array}\right) \end{array}$$

No solution
if this is
zero.

We use the row operations stated above in the opposite direction. In order to achieve a 0 in this position we need to nominate c_1 the coefficient of the *z* variable in the operation before, that is

$$\begin{array}{r} R_1 \\ R_2'=R_2+R_1 \\ R_3'=R_3-2R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 4 & 3 & 3 \\ 0 & -1 & c_1 & -9 \end{array}\right)$$

The row operation was $R_3'' = 4R_3' + R_2'$ so we have $4c_1 + 3 = 0$ which gives $c_1 = -3/4$. In order to get $c_1 = -3/4$ in this position we let c_2 be the *z* coefficient in the previous row operation, that is

$$\begin{array}{r} R_1 \\ R_2 \\ R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ -1 & 3 & 2 & -2 \\ 2 & 1 & c_2 & 1 \end{array}\right)$$

The row operation was $R_3' = R_3 - 2R_2$ so we have $c_2 - 2 = -3/4$ which gives $c_2 = 5/4$. Thus our z coefficient is 5/4 in order for the linear system to be **inconsistent** which means that it has **no** solution.

28. We need to prove that $\mathbf{A}^{-1}$ is symmetric provided **A** is symmetric. *How do we prove this?*

Required to show that $\left(\mathbf{A}^{-1}\right)^T = \mathbf{A}^{-1}$

Proof. We have

$$\begin{aligned}\left(\mathbf{A}^{-1}\right)^T &= \left(\mathbf{A}^T\right)^{-1} && \left[\text{Because } \left(\mathbf{X}^{-1}\right)^T = \left(\mathbf{X}^T\right)^{-1}\right] \\ &= \mathbf{A}^{-1} && \left[\text{Because } \mathbf{A} \text{ is symmetric}\right]\end{aligned}$$

This is our required result.

■

29. We are given $\mathbf{A}^2 - \mathbf{A} + \mathbf{I} = \mathbf{O}$ and need to show that the inverse of the matrix **A** is given by $\mathbf{I} - \mathbf{A}$.

Proof.

We pre-multiply **A** be $\mathbf{I} - \mathbf{A}$:

$$\begin{aligned}\left(\mathbf{I} - \mathbf{A}\right)\mathbf{A} &= \mathbf{IA} - \mathbf{A}^2 \\ &= \mathbf{A} - \mathbf{A}^2 = \mathbf{A} - \left(\mathbf{A} - \mathbf{I}\right) && \left[\text{Because we are given } \mathbf{A}^2 - \mathbf{A} + \mathbf{I} = \mathbf{O}\right] \\ &= \mathbf{I}\end{aligned}$$

We have $\left(\mathbf{I} - \mathbf{A}\right)\mathbf{A} = \mathbf{I}$. Next we post-multiply **A** be $\mathbf{I} - \mathbf{A}$:

$$\mathbf{A}\left(\mathbf{I} - \mathbf{A}\right) = \mathbf{AI} - \mathbf{A}^2 = \mathbf{A} - \left(\mathbf{A} - \mathbf{I}\right) = \mathbf{I}$$

Thus we have $\mathbf{A}\left(\mathbf{I} - \mathbf{A}\right) = \mathbf{I}$. Combining these results shows that the matrix $\mathbf{I} - \mathbf{A}$ is the inverse of the matrix **A**.

■

30. (a) A square matrix **A** is invertible if and only if there exists a square matrix **B** such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I} \quad \text{where } \mathbf{I} \text{ is the identity matrix}$$

Need to prove that this **B** is unique.

Proof.

Suppose there exists a square matrix **C** such that

$$\mathbf{AC} = \mathbf{CA} = \mathbf{I}$$

What do we need to prove?

Required to prove that $\mathbf{B} = \mathbf{C}$.

We have $\mathbf{AB} = \mathbf{AC} = \mathbf{I}$. Pre-multiply both sides by **B**:

$$\begin{aligned}\mathbf{B}\left(\mathbf{AB}\right) &= \mathbf{B}\left(\mathbf{AC}\right) \\ \left(\mathbf{BA}\right)\mathbf{B} &= \left(\mathbf{BA}\right)\mathbf{C} \\ \mathbf{IB} &= \mathbf{IC} && \left[\text{Because from above } \mathbf{BA} = \mathbf{I}\right] \\ \mathbf{B} &= \mathbf{C}\end{aligned}$$

Hence the inverse matrix **B** is unique.

■

(b) We need to prove that the product of any finite number of invertible matrices is invertible.

Proof.

Let $\mathbf{A}_1,\ \mathbf{A}_2,\ \mathbf{A}_3,\ \cdots,\ \mathbf{A}_n$ be a sequence of invertible matrices. Consider the product $\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\cdots\mathbf{A}_n$. We use mathematical induction to show

$$\left(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\cdots\mathbf{A}_n\right)^{-1}=\mathbf{A}_n^{-1}\mathbf{A}_{n-1}^{-1}\cdots\mathbf{A}_2^{-1}\mathbf{A}_1^{-1}$$

Consider the result for $n=2$:

$$\left(\mathbf{A}_1\mathbf{A}_2\right)^{-1}=\mathbf{A}_2^{-1}\mathbf{A}_1^{-1}\quad\text{because }\left(\mathbf{AB}\right)^{-1}=\mathbf{B}^{-1}\mathbf{A}^{-1}$$

Assume the result is true for $n=k$, that is

$$\left(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\cdots\mathbf{A}_k\right)^{-1}=\mathbf{A}_k^{-1}\mathbf{A}_{k-1}^{-1}\cdots\mathbf{A}_2^{-1}\mathbf{A}_1^{-1}\qquad(*)$$

We apply these two to show the result for $n=k+1$. *What do we need to prove?*
Required to prove

$$\left(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\cdots\mathbf{A}_k\mathbf{A}_{k+1}\right)^{-1}=\mathbf{A}_{k+1}^{-1}\mathbf{A}_k^{-1}\cdots\mathbf{A}_2^{-1}\mathbf{A}_1^{-1}$$

Starting with the Left Hand Side we have

$$\begin{aligned}\left(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\cdots\mathbf{A}_k\mathbf{A}_{k+1}\right)^{-1}&=\mathbf{A}_{k+1}^{-1}\left(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\cdots\mathbf{A}_k\right)^{-1} && \left[\text{Because }\left(\mathbf{AB}\right)^{-1}=\mathbf{B}^{-1}\mathbf{A}^{-1}\right]\\&=\mathbf{A}_{k+1}^{-1}\mathbf{A}_k^{-1}\cdots\mathbf{A}_2^{-1}\mathbf{A}_1^{-1} && \left[\text{By }(*)\right]\end{aligned}$$

Hence we have our result. Therefore by mathematical induction we conclude that the product of a finite number of invertible matrices is invertible.

■

(c) The inverse of the given matrix is obtained by converting $\left[\mathbf{A}\ \middle|\ \mathbf{I}\right]$ to $\left[\mathbf{I}\ \middle|\ \mathbf{B}\right]$ and $\mathbf{B}$ would be the inverse matrix of $\mathbf{A}$:

$$\begin{array}{c}R_1\\R_2\\R_3\end{array}\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\-1&1&1&0&1&0\\0&1&0&0&0&1\end{array}\right]$$

Interchanging middle and bottom rows, R_2 and R_3, gives

$$\begin{array}{c}R_1\\R_2'=R_3\\R_3'=R_2\end{array}\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&1&0&0&0&1\\-1&1&1&0&1&0\end{array}\right]$$

Executing the row operation $R_3'+R_1$ gives

$$\begin{array}{c}R_1\\R_2'\\R_3^*=R_3'+R_1\end{array}\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&1&0&0&0&1\\0&1&2&1&1&0\end{array}\right]$$

What are we trying to achieve?
The identity matrix on the Left Hand Side. *How?*
Carry out the row operation $R_3^*-R_2'$:

$$\begin{array}{c}R_1\\R_2'\\R_3^{**}=R_3^*-R_2'\end{array}\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&1&0&0&0&1\\0&0&2&1&1&-1\end{array}\right]$$

Carrying out the row operation $2R_1-R_3^{**}$:

$$\begin{array}{c} R_1' = 2R_1 - R_3^{**} \\ R_2' \\ R_3^{**} \end{array} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 & -1 \end{array}\right]$$

Divide the top and bottom row by 2:

$$\begin{array}{c} R_1'/2 \\ R_2' \\ R_3^{**}/2 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right]$$

The inverse of the matrix **A** is the matrix on the Right Hand Side:

$$\mathbf{A}^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & -1/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

You may check that $\mathbf{AA}^{-1} = \mathbf{I}$.

31. We are given $\mathbf{AXA}^{-1} = \mathbf{B}$ and **A**, **B** are invertible (non-singular) matrices.
We have

$$\mathbf{AXA}^{-1} = \mathbf{B}$$

$$\mathbf{A}^{-1}\left(\mathbf{AXA}^{-1}\right) = \mathbf{A}^{-1}\mathbf{B} \qquad \left[\text{Pre multiplying both sides by } \mathbf{A}^{-1}\right]$$

$$\underbrace{\left(\mathbf{A}^{-1}\mathbf{A}\right)}_{=\mathbf{I}}\mathbf{XA}^{-1} = \mathbf{A}^{-1}\mathbf{B}$$

$$\mathbf{XA}^{-1} = \mathbf{A}^{-1}\mathbf{B}$$

$$\left(\mathbf{XA}^{-1}\right)\mathbf{A} = \mathbf{A}^{-1}\mathbf{BA} \qquad \left[\text{Post multiplying both sides by } \mathbf{A}\right]$$

$$\mathbf{X}\underbrace{\left(\mathbf{A}^{-1}\mathbf{A}\right)}_{=\mathbf{I}} = \mathbf{A}^{-1}\mathbf{BA}$$

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{BA}$$

Since $\mathbf{X} = \mathbf{A}^{-1}\mathbf{BA}$ and matrices **A**, **B** and $\mathbf{A}^{-1}$ are invertible therefore $\mathbf{A}^{-1}\mathbf{BA}$ and **X** are invertible.

What is $\mathbf{X}^{-1}$ *equal to?*

$$\begin{aligned} \mathbf{X}^{-1} &= \left(\mathbf{A}^{-1}\mathbf{BA}\right)^{-1} \\ &= \left(\mathbf{BA}\right)^{-1}\left(\mathbf{A}^{-1}\right)^{-1} \\ &= \mathbf{A}^{-1}\mathbf{B}^{-1}\mathbf{A} \qquad \left[\text{Because } \left(\mathbf{BA}\right)^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1} \text{ and } \left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}\right] \end{aligned}$$

Hence $\mathbf{X}^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}\mathbf{A}$.

32. We are given the linear system

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 & 0 & 5 \\ 0 & 1 & 1 & 0 & 2 & 1 & 10 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 16 \end{bmatrix}$$

The augmented matrix is given by

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left[\begin{array}{ccccccc|c} 0 & 1 & 1 & 0 & 2 & 0 & 4 & 7 \\ 0 & 0 & 0 & 1 & 3 & 0 & 5 & 8 \\ 0 & 1 & 1 & 0 & 2 & 1 & 10 & 16 \end{array}\right]$$

Carrying out the row operation $R_3 - R_1$ gives

$$\begin{array}{cc} & \begin{array}{ccccccccc} a & b & c & d & e & f & g & \end{array} \\ \begin{array}{c} R_1 \\ R_2 \\ R'_3 = R_3 - R_1 \end{array} & \left[\begin{array}{ccccccc|c} 0 & 1 & 1 & 0 & 2 & 0 & 4 & 7 \\ 0 & 0 & 0 & 1 & 3 & 0 & 5 & 8 \\ 0 & 0 & 0 & 0 & 0 & 1 & 6 & 9 \end{array}\right] \end{array}$$

How many free variables do we have in this case?

$7-3=4$. *Which of these variables are free?*

a, c, e and g because corresponding columns do **not** contain a leading 1. Let $a = p,\ c = q,\ e = r$ and $g = t$. From the bottom row $R_3{}'$ we have

$$f + 6g = 9 \quad \text{which gives } f = 9 - 6g = 9 - 6t$$

From the middle row R_2 we have

$$d + 3e + 5g = 8 \text{ which gives } d = 8 - 3e - 5g = 8 - 3r - 5t$$

From the top row R_1 we have

$$b + c + 2e + 4g = 7 \text{ implies that } b = 7 - c - 2e - 4g = 7 - q - 2r - 4t$$

Thus our solution set is given by

$$\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \end{bmatrix} = \begin{bmatrix} p \\ 7-q-2r-4t \\ q \\ 8-3r-5t \\ r \\ 9-6t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 0 \\ 8 \\ 0 \\ 9 \\ 0 \end{bmatrix} + p\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + q\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + r\begin{bmatrix} 0 \\ -2 \\ 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -4 \\ 0 \\ -5 \\ 0 \\ -6 \\ 1 \end{bmatrix}$$

33. We need to prove that $\mathbf{AB} = \mathbf{I} \quad \Leftrightarrow \quad \mathbf{BA} = \mathbf{I}$.

Proof. $(\Rightarrow)$. Assume that $\mathbf{AB} = \mathbf{I}$. This means that matrix **B** is the right inverse of the matrix **A**. Using this

$$\begin{aligned} \mathbf{B} &= \mathbf{BI} \\ &= \mathbf{B}(\mathbf{AB}) \\ &= (\mathbf{BA})\mathbf{B} \end{aligned}$$

Post-multiply this $\mathbf{B} = (\mathbf{BA})\mathbf{B}$ by $\mathbf{B}^{-1}$:

$$\mathbf{BB}^{-1} = (\mathbf{BA})\mathbf{BB}^{-1}$$
$$\mathbf{I} = \mathbf{BAI} = \mathbf{BA}$$

We have $\mathbf{BA} = \mathbf{I}$.

$(\Leftarrow)$. Similarly we go the other way, that is we assume $\mathbf{BA} = \mathbf{I}$ and deduce $\mathbf{AB} = \mathbf{I}$.

$$\begin{aligned}\mathbf{A} &= \mathbf{AI}\\ &= \mathbf{A}(\mathbf{BA})\\ &= (\mathbf{AB})\mathbf{A}\end{aligned}$$

Post-multiply this $\mathbf{A} = (\mathbf{AB})\mathbf{A}$ by $\mathbf{A}^{-1}$:

$$\mathbf{AA}^{-1} = (\mathbf{AB})\mathbf{AA}^{-1}$$
$$\mathbf{I} = \mathbf{ABI} = \mathbf{AB}$$

We have $\mathbf{AB} = \mathbf{I}$.

■

34. We need to prove if **AB** are square matrices and **AB** is invertible (non-singular) then both **A** and **B** are invertible with $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

What do we need to show?

It is enough to prove that $(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{I}$ because by result of question 33 we have $(\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) = \mathbf{I}$.

Proof.

Proving the first result:

$$\begin{aligned}(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) &= \mathbf{A}\underbrace{(\mathbf{BB}^{-1})}_{=\mathbf{I}}\mathbf{A}^{-1}\\ &= \mathbf{AIA}^{-1} = \mathbf{AA}^{-1} = \mathbf{I}\end{aligned}$$

By the result of question 33 $(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{I} \Leftrightarrow (\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) = \mathbf{I}$

Thus $\mathbf{B}^{-1}\mathbf{A}^{-1}$ is the inverse of the matrix multiplication **AB**. Hence

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

We have proven our required result.

■

35. (i) We are given the diagonal matrix $\mathbf{D} = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \vdots & 0 & a_{nn} \end{pmatrix}$ and need to prove that

$$\mathbf{D}^p = \begin{pmatrix} a_{11}{}^p & 0 & \cdots & 0 \\ 0 & a_{22}{}^p & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \vdots & 0 & a_{nn}{}^p \end{pmatrix}$$

How?

Using mathematical induction. Clearly the result is true for $p=1$. Assume the result is true for $p=k$:

$$\mathbf{D}^k=\begin{pmatrix} a_{11}{}^k & 0 & \cdots & 0 \\ 0 & a_{22}{}^k & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \vdots & 0 & a_{nn}{}^k \end{pmatrix}$$

Required to prove the result for $p=k+1$. We have

$$\mathbf{D}^{k+1}=\mathbf{D}^k\mathbf{D}=\begin{pmatrix} a_{11}{}^k & 0 & \cdots & 0 \\ 0 & a_{22}{}^k & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \vdots & 0 & a_{nn}{}^k \end{pmatrix}\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \vdots & 0 & a_{nn} \end{pmatrix}$$

$$=\begin{pmatrix} a_{11}{}^k a_{11} & 0 & \cdots & 0 \\ 0 & a_{22}{}^k a_{22} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \vdots & 0 & a_{nn}{}^k a_{nn} \end{pmatrix}=\begin{pmatrix} a_{11}{}^{k+1} & 0 & \cdots & 0 \\ 0 & a_{22}{}^{k+1} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \vdots & 0 & a_{nn}{}^{k+1} \end{pmatrix}$$

Hence by mathematical induction we have our result.

■

(ii) Multiplying out the matrices **D** and $\mathbf{D}^{-1}$:

$$\mathbf{D}\mathbf{D}^{-1}=\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \vdots & 0 & a_{nn} \end{pmatrix}\begin{pmatrix} a_{11}{}^{-1} & 0 & \cdots & 0 \\ 0 & a_{22}{}^{-1} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \vdots & 0 & a_{nn}{}^{-1} \end{pmatrix}=\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \vdots & 0 & 1 \end{pmatrix}=\mathbf{I}$$

We have $\mathbf{D}\mathbf{D}^{-1}=\mathbf{I}$. By the result of question 33 we have $\mathbf{D}\mathbf{D}^{-1}=\mathbf{I} \Leftrightarrow \mathbf{D}^{-1}\mathbf{D}=\mathbf{I}$. Hence the inverse matrix is given by $\mathbf{D}^{-1}=\begin{pmatrix} a_{11}{}^{-1} & 0 & \cdots & 0 \\ 0 & a_{22}{}^{-1} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \vdots & 0 & a_{nn}{}^{-1} \end{pmatrix}$.

■