Numerical Solutions of Differential Equations chapter 5

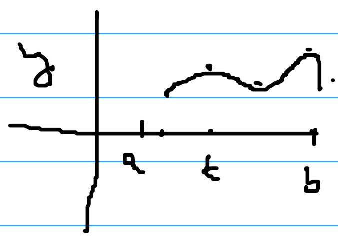
$$\frac{dy}{dt} = e^t \quad , \quad y' = e^t \quad , \quad F' = \sin t$$

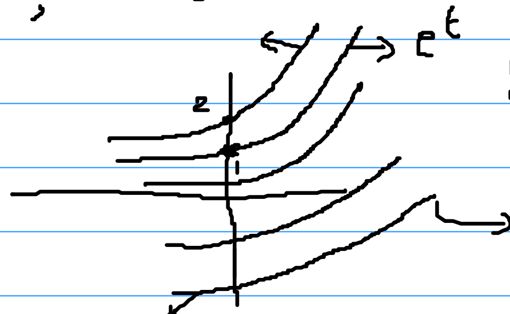
$$y = e^t \qquad\qquad F = -\cos t$$

$$dy = e^t\,dt \qquad\qquad \int_a^b f(x)\,dx$$

$$\int dy = \int e^t\,dt$$

$$y = \int e^t\,dt = e^t + c, \qquad y = e^t + c$$





integral curves
or
antiderivative

$y(0) = 1$ (initial value)

$y(0) = e^0 + c$

$1 = 1 + c$

$c = 0$

$y = e^t$

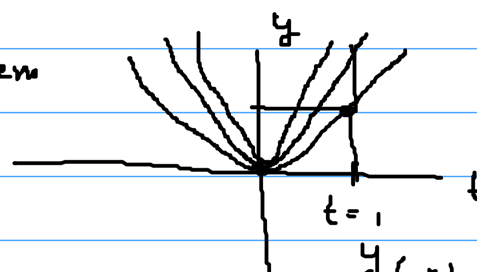
$y(0) = 2$ (initial value)

$y(0) = e^0 + c$

$2 = 1 + c$

$\boxed{c = 1}$ $\quad y = e^t + 1$

$$\left.\begin{aligned} y' &= f(t, y) \qquad a \le t \le b \\ y(a) &= \alpha \end{aligned}\right\}\ \text{initial value problem}$$



$y(0) = 0$

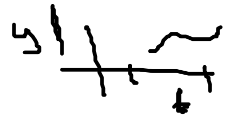
$y(1) = \alpha$

1. Existence
2. Uniqueness
3. Continuous dependence on data

If an initial value problem satisfies conditions 1 – 3, it is called well-posed problem.

$v(0) = \alpha + \varepsilon \qquad \varepsilon \to 0$

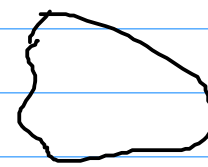
$$y' = f(t,y)$$
$$y(a) = \alpha$$ — ①



$\frac{dy}{dt} = y e^t$

$y(t)$

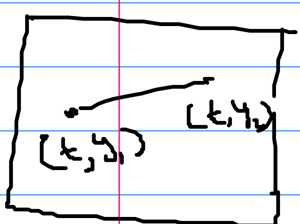
① has a unique solution if f is continuous on $D = \{ a \le t \le b, \ -\infty < y < \infty \}$ and f is a Lipschitz function in y



Lipschitz function

$$|f(t, y_1) - f(t, y_2)| \le L \, |y_2 - y_1|$$

$L > 0 \qquad (t, y_1), (t, y_2) \in D$

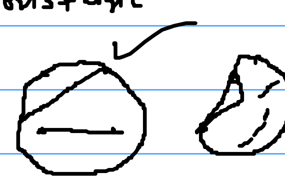


Test for being a Lipschitz function

let $f(t,y)$ is defined on Convex set and there exists a Constant $L>0$ such that

$$\left| \frac{\partial f(t,y)}{\partial y} \right| \le L \quad \text{for all } (t,y) \in D,$$

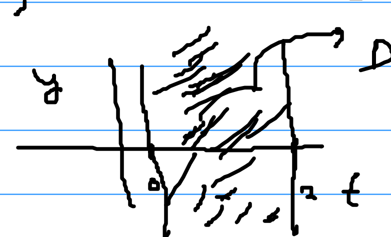
then f is a Lipschitz function.
Where L is called Lipschitz constant



Example
Show that
$$\frac{dy}{dt} = y - t^2 + 1 \qquad 0 \le t \le 2 \qquad y(0) = 0.5$$
→ Ⓐ

has a Unique Solution on $D = \{ 0 \le t \le 2, \ -\infty < y < \infty \}$

Sol
$$f(t,y) = y - t^2 + 1$$
f is Continuous



$$\frac{\partial f}{\partial y} = 1 \quad , \quad \left| \frac{\partial f}{\partial y} \right| = 1 \quad , \quad \left| \frac{\partial f}{\partial y} \right| \le 1$$

$L = 1$, so f is a Lipschitz function.

Therefore, Ⓐ has a Unique Solution.

Numerical Sol of differential equation

$$y' = e^{t^2} \qquad 0 \le t \le 1 \quad , \quad y(0) = 1$$

$$\frac{dy}{dt} = e^{t^2} \quad , \quad dy = e^{t^2}\, dt$$

$$\int dy = \int e^{t^2}\, dt \quad , \quad y = \int e^{t^2}\, dt$$

need some numerical method

$$y' = f(t, y) \qquad a \le t \le b \quad , \quad y(a) = \alpha$$

$t_0 = a$, t_1, t_2, ..., b

$$h = \frac{b-a}{N}$$

$$t_i = a + ih \qquad i = 0, 1, \ldots N$$

t_i, ξ_i, t_{i+1}

expand y using Taylor's Series

$$y(t_{i+1}) = y(t_i + h) = y(t_i) + \frac{h\, y'(t_i)}{1!} + \frac{h^2 y^{(2)}(\xi_i)}{2!}$$

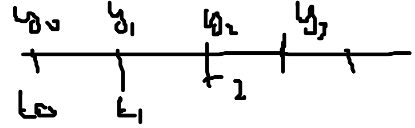
$$y(t_{i+1}) - y(t_i) = h\, y'(t_i) + \frac{h^2}{2!} y^{(2)}(\xi_i)$$

ignore

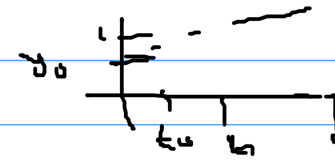
$$y(t_{i+1}) - y(t_i) \simeq h\, y'(t_i)$$

$$y'(t_i) \simeq \frac{y(t_{i+1}) - y(t_i)}{h}$$

$$y' = f(t, y)$$



$$y'(t_i) = f(t_i, y(t_i))$$



$$\frac{y(t_{i+1}) - y(t_i)}{h} \simeq f(t_i, y(t_i))$$

$$y(t_{i+1}) - y(t_i) \simeq h f(t_i, y(t_i))$$

$$y(t_{i+1}) \simeq y(t_i) + h f(t_i, y(t_i))$$

$$w_i \simeq y(t_i) \qquad i = 0, 1, 2, \ldots N$$

$$w_{i+1} = w_i + h f(t_i, w_i)$$

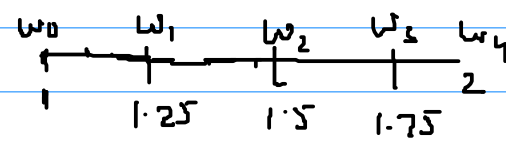
This called Euler's Method

Example

Solve $y' = 1 + y/t \qquad 1 \le t \le 2, \quad y(1) = 2$

$h = 0.25$

$$w_1 = w_0 + h f(t_0, w_0)$$

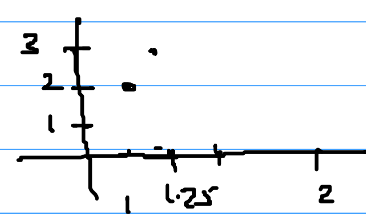


$t_0 = 1 \qquad w_0 = 2$

$$f(t, w) = 1 + w/t$$

$$w_1 = 2 + (0.25)(1 + 2/1)$$

$$w_1 = 2 + 0.25(3) = 2.75$$



$$w_2 = w_1 + h f(t_1, w_1)$$

$$w_2 = 2.75 + 0.25\left(1 + 2.75/1.25\right)$$

⋮

find other values w_i by yourself