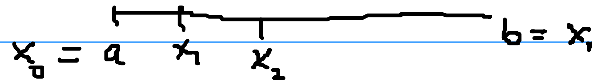
Numerical integration Continued

Composite Simpson's Rule



$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(a) + 2\sum_{j=1}^{n/2-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(b)\right] - \frac{b-a}{180} h^4 f^{(4)}(\xi)$$

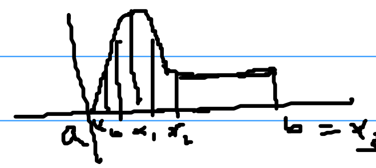
$$h = \frac{b-a}{n}$$

In Composite Simpson's Rule n must be even.

$f \in C^4[a, b]$ $\qquad a < \xi < b$

Because 2 intervals are required to apply Simpson's Rule once

$$\left[\int_a^b f(x)\,dx \simeq \sum \underline{a_i}\, f(\underline{x_i})\right] \text{ Quadrature}$$



Composite Trapezoidal Rule

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] - \frac{b-a}{12} h^2 f''(\xi)$$

$a < \xi < b$

Composite Mid point formula (open Cote)

$$\int_0^1 \ln x \; dx$$

$$\int_a^b f(x)\,dx = 2h \sum_{j=0}^{n/2} f(x_{2j}) + \frac{b-a}{6} h^2 f''(\xi)$$

$a < \xi < b$

$$\int_a^b f(x)\,dx \overset{?}{=} F(b) - F(a)$$

$$h = \frac{b-a}{n+2}$$

Example

$$\int_0^{\pi} \sin x \; dx$$

Find the value of n that will ensure an approximation error less than 0.00002 in Trapezoidal Rule & Simpson's Rule

For Trapezoidal Rule

$$E = \frac{b-a}{12} \cdot h^2 f''(\xi) \quad , \qquad f(x) = \sin x$$

$$f'(x) = \cos x$$

$$a = 0, \quad b = \pi \qquad\qquad f''(x) = -\sin x$$

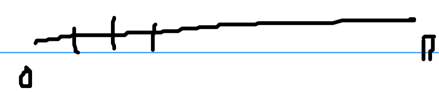
$$h = \frac{b-a}{n} = \pi/n \qquad\qquad 0 < \xi < \pi$$

$$E = \frac{\pi}{12}\left(\pi/n\right)^2 \left(-\sin(\xi)\right)$$

$$|E| = \frac{\pi^3}{12 n^2} \left|\sin(\xi)\right| \leq \frac{\pi^3}{12 n^2} \qquad\qquad |\sin x| \leq 1$$

$$\frac{\pi^3}{12 n^2} < 0.00002 \quad , \qquad n > 359.44$$

$$n \geq 360$$



For Simpson's Rule

$$E = \frac{b-a}{180} h^4 f^{(4)}(\xi) \qquad\qquad f^{(2)}(x) = -\sin x$$

$$f^{(3)}(x) = -\cos x$$

$$h = \frac{b-a}{n} \qquad\qquad f^{(4)}(x) = \sin x$$

$$n > 17.07$$

$$n \geq 18$$

So, in Simpson's Rule we need 18 subintervals while in Trapezoidal Rule we need 360 to get same level of accuracy.

Round-off Error stability

Consider Simpson's rule

$$\int_a^b f(x) \simeq \frac{h}{3}\left[f(x_0) + 2\sum_{j=1}^{n/2-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(x_n) \right]$$

round-off error

$$f(x_i) = \tilde{f}(x_i) + e_i$$

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[\tilde{f}(x_0) + e_0 + 2\sum_{j=1}^{n/2-1}\left(\tilde{f}(x_{2j}) + e_{2j}\right) + 4\sum_{j=1}^{n/2}\left(\tilde{f}(x_{2j-1}) + e_{2j-1}\right) + \tilde{f}(x_n) + e_n \right]$$

$$\int_a^b f(x)\,dx - \frac{h}{3}\left[\tilde{f}(x_0) + 2\sum_{j=1}^{n/2-1} \tilde{f}(x_{2j}) + 4\sum_{j=1}^{n/2} \tilde{f}(x_{2j-1}) + \tilde{f}(x_n) \right]$$

$$= \frac{h}{3}\left[e_0 + 2\sum_{j=1}^{n/2-1} e_{2j} + 4\sum_{j=1}^{n/2} e_{2j-1} + e_n \right]$$

$$|e(h)| = \left| h/3 \left[e_0 + 2\sum_{j=1}^{n/2-1} e_{2j} + 4\sum_{j=1}^{n/2} e_{2j-1} + e_n \right] \right|$$

$$\leq h/3\left(|e_0| + 2\sum_{j=1}^{n/2-1} |e_{2j}| + 4\sum_{j=1}^{n/2} |e_{2j-1}| + |e_n| \right) \qquad |e_j| < \varepsilon$$

$$\leq h/3\left(\varepsilon + 2\sum_{j=1}^{n/2-1} \varepsilon + 4\sum_{j=1}^{n/2} \varepsilon + \varepsilon \right) \qquad \sum_{j=1}^{n} c = nc$$

$$|a+b| \leq |a| + |b|$$

$$= h/3\left(\varepsilon + 2\left(\frac{n}{2} - 1\right)\varepsilon + 4\,\frac{n}{2}\,\varepsilon + \varepsilon \right)$$

$$= \frac{h}{3}\left(\varepsilon + n\varepsilon - 2\varepsilon + 2n\varepsilon + \varepsilon \right)$$

$$= h/3\,(3n\varepsilon) = hn\varepsilon = (b-a)\,\varepsilon \qquad h = \frac{b-a}{n}$$

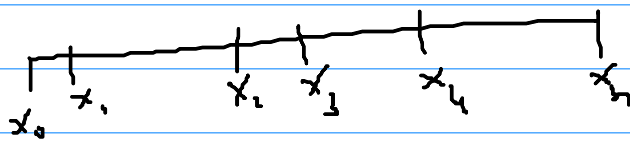
Round-off error does not depend on h $\quad nh = b-a$

Gaussian Quadrature

$$\int_a^b f(x)\,dx \simeq \sum_{i=1}^{n} c_i \, f(x_i)$$

c_i $\qquad$ x_i

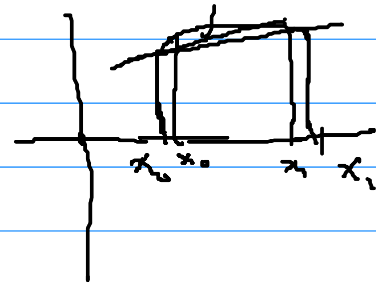
2n



a ———— b

$$h = \frac{b-a}{n}$$

$$x_i = a + ih$$



See the next lecture.