Bisection Method

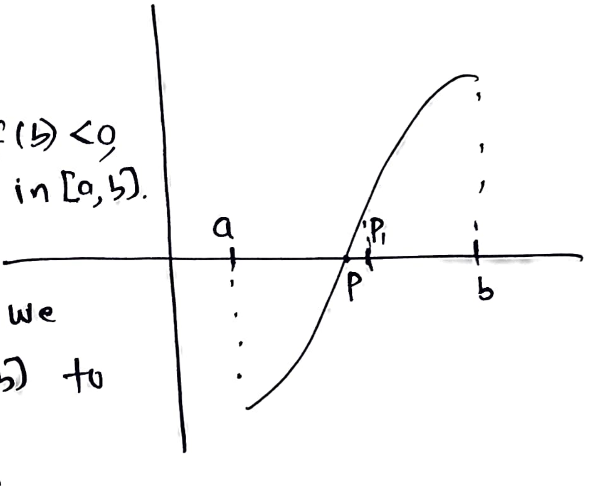
In this course one of the main interests is to be able to find root of a nonlinear function $y = f(x)$ for which existing analytical methods don't work. For example for

$$f(x) = x^3 + \sin x^2 + \frac{\ln x}{1+x^2} + 3,$$ we don't know

of any method which can provid analytic solution to $f(x) = 0$. In such situations, we resort to numerical methods, which generate sequence of successive approximations $P_1, P_2 \dots P_n \to P$, converging to p such that $f(P) = 0$.

The main idea in Bisection Method is to begin with an interval $[a, b]$, which contains the root of ~~$y=f(x)$~~ given function $y = f(x)$ and then interval is halved to determine the first approximation.

Consider $y = f(x)$ as given in the graph, which has root at $x = p$.



Since f is continuous and $f(a)f(b) < 0$ IVT gaurantees that f has root in $[a, b]$.

According to bisection method, we consider the mid point of $[a, b]$ to be our first approximation.

If we set $a_1 = a$, $b_1 = b$,

then $P_1 = \frac{a_1 + b_1}{2}$. Now we have two subintervals $[a, P_1]$ and $[P_1, b]$ we will choose the one which contains the root P. Since $f(a)f(P_1) < 0$, so our interval of interest is $[a, P_1]$

Rename $a_2 = a_1$, $b_2 = P_1$.

The next approximation is $P_2 = \dfrac{a_2 + b_2}{2}$.

We continue the process untill we reach a certain level of accuracy.

The term 'stopping criteria' is used to terminate an iterative process.

Following are three main stopping criteria we will be using in this course.

1. $|P_n - P| < \varepsilon$ (Absolute error) as $n \to \infty$

2. $\dfrac{|P_n - P|}{|P|} < \varepsilon$ (Relative error) as $n \to \infty$

3. $|f(P_n)| < \varepsilon$ (Bound on the function values) as $n \to \infty$

Here 'ε' is the pre-specified desired accuracy.

Some knowledge of function's behaviour is required around the root 'P', to be able to choose the stopping criteria which gives the best result for the given function. However, if no information is available, the best is to use 2.

Example

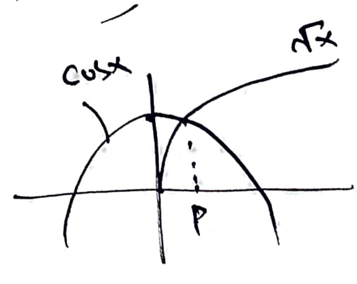
Find the root of $f(x) = \sqrt{x} - \cos x$ in $[0, 1]$ using bisection method, accurate 10^{-3}.

$f(0) = -1$

$f(1) = 0.4596$

Since $f(0) f(1) < 0$, f has root in $[0, 1]$.

$f(x) = 0$, $\sqrt{x} = \cos x$



After applying the bisection method and using (3) as the stopping criteria we have the following values.

n	a_n	b_n	P_n	$\lvert f(P_n)\rvert$
1	0.0000	1.0000	0.5000	0.1705
2	0.5000	1.0000	0.7500	0.1343
3	0.5000	0.7500	0.62500	0.0203
4	⋮	⋮	⋮	⋮
5	⋮	⋮	⋮	⋮
6	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮
10	0.6406	0.6426	0.6416	0.000138

After 10 iterations we have $\lvert f(P_{10})\rvert \leq 0.000138 < 0.001$

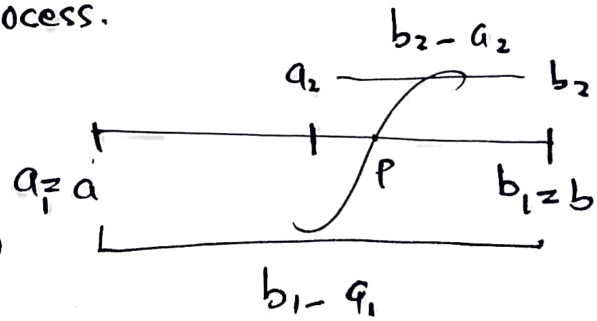
and the approximate root is $P = 0.6416$

Theorem (2.1)

Suppose $f \in C[a,b]$ and $f(a)\,f(b) < 0$, then bisection method generates a sequence $\{P_n\}_{n\geq 0}^{\infty}$ that converges to the root p of f and

$$\lvert P_n - P\rvert \leq \frac{1}{2^n}(b-a), \qquad n \geq 1.$$

This ~~theo~~ theorem provides upper bound on the error $\lvert P_n - P\rvert$ at every step of the iterative process.



Proof

$$b_1 - a_1 = b - a$$

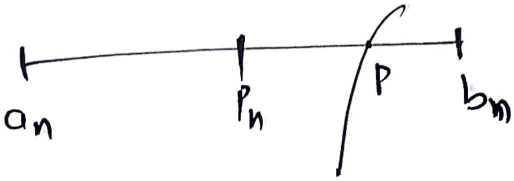
$$b_2 - a_2 = \frac{1}{2}(b_1 - a_1) = \frac{1}{2}(b-a)$$

$$\vdots$$

$$b_n - a_n = \frac{1}{2^{n-1}}(b-a) \quad \text{(1)} \qquad \text{and} \qquad P_n = \frac{a_n + b_n}{2}$$

Consider the nth step of the iteration,

from the diagram one can see that



$$|p_n - p| \leq \frac{b_n - a_n}{2}$$ and from the above eq ① we have $b_n - a_n = \frac{1}{2^{n-1}}(b-a)$. Substituting this in the above inequality we have

$$|p_n - p| \leq \frac{\frac{(b-a)}{2^{n-1}}}{2} = \frac{(b-a)}{2^n}$$

$$\boxed{|p_n - p| \leq \frac{(b-a)}{2^n} \quad n \geq 1} \quad ②$$

Hence proved.

If we recall the definition of rate of convergence, ② suggests that sequence generated by the bisection method converges to the root 'p' with rate of convergence $O(\frac{1}{2^n})$, i.e.,

$$p_n = p + O\left(\frac{1}{2^n}\right).$$

Rate of convergence

$|\alpha_n - \alpha| \leq c\,\beta_n$

$\beta_n \to 0$ as $n \to \infty$

$\alpha_n = \alpha + O(\beta_n)$

We have yet to find the order of convergence of the bisection method.