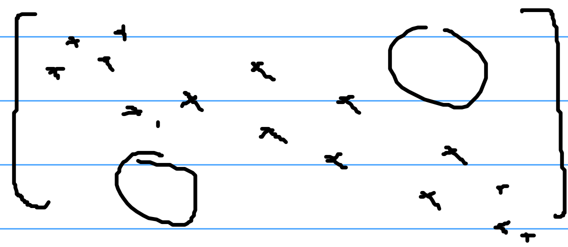
CH # 7 7.1, 7.2, 7.3 7.4

Iterative Methods for linear System 7.3

$A\underline{x} = \underline{b}$, $x = A^{-1} b$

$A_{50 \times 50}$ if A is Sparse matrix

$A =$  50×50

$A\underline{x} = b$

$|2| = 2$

$x^{(0)}$, $A x^{(0)} = b$, $x^{(1)}$

$f(x) = 0$

$\underline{x}^{(0)}$, $\underline{x}^{(1)}$ $\underline{x}^{(2)}$ $\longrightarrow$ $\underline{x}^{*}$

$\| f(x) \| < \varepsilon$

$|x_n - x_{n-1}| < \varepsilon$

$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\|x\|_{l_2} = \sqrt{x_1^2 + x_2^2 + x_3^2}$ (l_2 norm)

$$\|x\|_\infty = \max_{1 \le i \le 3} |x_i|$$

Stopping criteria $$\frac{\| x^{(k)} - x^{(k-1)} \|}{\| x^{(k)} \|} < \varepsilon$$

Jacobi's Method

$$10x_1 - x_2 + 2x_3 = 6 \quad \text{--- (1)}$$
$$-x_1 + 11x_2 - x_3 + 3x_4 = 25 \quad \text{--- (2)}$$
$$2x_1 - x_2 + 10x_3 - x_4 = -11 \quad \text{--- (3)}$$
$$8x_2 - x_3 + 8x_4 = 15 \quad \text{--- (4)}$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$x = \vec{g}(x) \quad f(x) = 0$

$x_n = g(x_{n-1})$

$x_1 = g(x_0)$

$x_2 = g(x_1)$

$\vdots$

(1) $x_1 = \frac{1}{10}(x_2 - 2x_3 + 6)$

$x_2 = \frac{1}{11}(x_1 + x_3 - 3x_4 + 25)$

$x_3 = \frac{1}{10}(-2x_1 + x_2 + x_4 - 11)$

$x_4 = \frac{1}{8}(-3x_2 + x_3 + 15)$

$$\underline{x} = \underline{g}(x)$$

$$x_1^{(k)} = \frac{1}{10}\left(x_2^{(k-1)} - 2x_3^{(k-1)} + 6\right)$$
$$x_2^{(k)} = \frac{1}{11}\left(x_1^{(k-1)} + x_3^{(k-1)} - 3x_4^{(k-1)} + 25\right)$$
$$x_3^{(k)} = \frac{1}{10}\left(-2x_1^{(k-1)} + x_2^{(k-1)} + x_4^{(k-1)} - 11\right)$$
$$x_4^{(k)} = \frac{1}{8}\left(-3x_2^{(k-1)} + x_3^{(k-1)} + 15\right)$$

$k = 1, 2, 3, \ldots$

$$\underline{x}^{(0)} = \left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)}\right) \qquad k = 0$$
$$= (0, 0, 0, 0)$$

$$\underline{x}^{(1)} = \left(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}\right)$$
$$= \left(6/10, 25/11, -11/10, 15/8\right)$$

$\underline{x}^{(2)}$

$\vdots$

$$\frac{\|x^{(k)} - x^{(k-1)}\|}{\|x^{k}\|} < \varepsilon$$

consider a linear system

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$$

$$i = 1, 2, 3, \ldots n$$

$$x_i = \frac{-1}{a_{ii}} \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j + \frac{b_i}{a_{ii}}, \quad i = 1, 2, 3, \ldots n$$

$$x_i^{(k)} = -\frac{1}{a_{ii}} \sum_{\substack{j=1 \\ i \neq j}}^{n} a_{ij}\, x_i^{(k-1)} + \frac{b_i}{a_{ii}} \qquad i = 1, 2, 3, \cdots n$$

Jacobi iteration

$A\underline{x} = b$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$-L = \begin{bmatrix} 0 & & & O \\ -a_{21} & 0 & & \\ \vdots & & 0 & \\ -a_{n1} & \cdots & -a_{n,n-1} & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} a_{11} & & O \\ & a_{12} & \\ O & & a_{nn} \end{bmatrix}$$

$$-U = \begin{bmatrix} 0 & -a_{12} & \cdots & -a_{n1} \\ & 0 & & \vdots \\ & & \ddots & -a_{n-1,n} \\ O & & & 0 \end{bmatrix}$$

$$A = D - L - U$$

$$A X = b$$

$$(D - L - U) X = b$$

$$Dx - (L + U) x = b$$

$$D x = (L + U) x + b$$

$$\underline{X} = D^{-1}(L + U) x + D^{-1} b$$

$$\underline{X}^{(k)} = \underbrace{D^{-1}(L + U)}_{T_j} X^{(k-1)} + \underbrace{D^{-1} b}_{\underline{c}_j}$$

$$\underline{X}^{(k)} = T_j \underline{X}^{(k-1)} + c_j$$

Jacobi Iterative Method

$k = 1, 2, 3, \cdots$

Gauss Seidal Method

$$\rightarrow x_1^{(k)} = \frac{1}{2}\left(x_2^{(k-1)} + 3x_3^{(k-1)} + 10\right),$$

$$x_2^{(k)} = \frac{1}{5}\left(x_1^{(k)} + x_3^{(k-1)} - 2\right)$$

$$x_3^{(k)} = -\frac{1}{10}\left(x_1^{(k)} + 5x_2^{(k)} - 1\right)$$

$$\underbrace{\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right)} \rightarrow \underbrace{\left(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}\right)}$$

$$\underline{x}^{(0)} = (0, 0, 0)$$

$$x_1^{(1)} = \frac{1}{2}(0 + 0 + 10) = \frac{10}{2} = 5$$

$$x_2^{(1)} = \frac{1}{5}(5 + 0 - 2) = 3/5$$

$$x_3^{(1)} = -\frac{1}{10}\left(5 + 5(3/5) - 1\right) =$$

$$Ax = b$$

$$A = D - L - U$$

$$(D - L - U)X = b$$

$$(D - L)X = UX + b$$

$$\underline{X} = (D-L)^{-1}U\underline{X} + (D-L)^{-1}b$$

$$\underline{X}^{(k)} = \underbrace{(D-L)^{-1}U}_{T_g} X^{(k-1)} + \underbrace{(D-L)^{-1}b}_{c_g}$$

$$\underline{X}^{(k)} = \underline{T_g}\,\underline{X}^{(k-1)} + \underline{c_g} \quad \text{(Gauss Seidal Method)}$$

$$k = 1, 2, 3, 4, \ldots$$