Complete Solutions to Exercises 2.3

1. (a) Using scalars k and c and equating the linear combination to zero $k\mathbf{e}_1 + c\mathbf{e}_2 = \mathbf{O}$ we have

$$k\mathbf{e}_1 + c\mathbf{e}_2 = k\begin{pmatrix}1\\0\end{pmatrix} + c\begin{pmatrix}0\\1\end{pmatrix}$$
$$= \begin{pmatrix}k\\0\end{pmatrix} + \begin{pmatrix}0\\c\end{pmatrix} = \begin{pmatrix}k\\c\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

This gives $k = 0$ and $c = 0$ which means **all** the scalars are zero therefore $\mathbf{e}_1$ and $\mathbf{e}_2$ are linearly independent.

(b) We have the linear combination $k\mathbf{u} + c\mathbf{v} = \mathbf{O}$:

$$k\mathbf{u} + c\mathbf{v} = k\begin{pmatrix}3\\4\end{pmatrix} + c\begin{pmatrix}-6\\-8\end{pmatrix}$$
$$= \begin{pmatrix}3k\\4k\end{pmatrix} + \begin{pmatrix}-6c\\-8c\end{pmatrix} = \begin{pmatrix}3k-6c\\4k-8c\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

We have the simultaneous equations

$$3k - 6c = 0 \qquad (*)$$
$$4k - 8c = 0 \qquad (**)$$

From the first equation (*) we have

$$3k = 6c \quad \text{which gives} \quad k = 2c$$

Let $c = 1$ and then substituting this, $c = 1$, into $k = 2c = 2(1) = 2$. Checking that this satisfies the second equation (**):

$$4(2) - 8(1) = 0 \qquad \surd$$

Since the scalars, $c = 1$ and $k = 2$, are nonzero and which satisfy the linear combination $k\mathbf{u} + c\mathbf{v} = \mathbf{O}$ therefore the given vectors **u** and **v** are linearly dependent.

(c) Given $\mathbf{u} = \begin{pmatrix}6\\10\end{pmatrix}$, $\mathbf{v} = \begin{pmatrix}-3\\-5\end{pmatrix}$ we have vector **u** is a multiple of vector **v**, actually $\mathbf{u} = -2\mathbf{v}$ which implies that $\mathbf{u} + 2\mathbf{v} = \mathbf{O}$. There are non-zero scalars 1 and 2 such that

$$(1)\mathbf{u} + 2\mathbf{v} = \mathbf{O}$$

Hence the given vectors **u** and **v** are linearly dependent.

(d) Similarly $\mathbf{u} = \begin{pmatrix}\pi\\-2\pi\end{pmatrix}$, $\mathbf{v} = \begin{pmatrix}-1\\2\end{pmatrix}$ are multiples of each other because $\mathbf{u} = -\pi\mathbf{v}$.

From this we have $\mathbf{u} + \pi\mathbf{v} = \mathbf{O}$ which means there are non-zero scalars 1 and π such that

$$\mathbf{u} + \pi\mathbf{v} = \mathbf{O}$$

The given vectors **u** and **v** are linearly dependent.

(e) Since one of the vectors, $\mathbf{u} = \begin{pmatrix}0\\0\end{pmatrix}$, is the zero vector therefore by Proposition (2.21) we have if one (or more) of vectors is the zero vector then the vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots$ and $\mathbf{v}_n$ are linearly dependent. Hence vectors **u** and **v** are linearly dependent.

2. (a) Consider the linear combination

$$k_1\mathbf{e}_1 + k_2\mathbf{e}_2 + k_3\mathbf{e}_3 = \mathbf{O}$$

We have

$$k_1\mathbf{e}_1 + k_2\mathbf{e}_2 + k_3\mathbf{e}_3 = k_1\begin{pmatrix}1\\0\\0\end{pmatrix} + k_2\begin{pmatrix}0\\1\\0\end{pmatrix} + k_3\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= \begin{pmatrix}k_1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\k_2\\0\end{pmatrix} + \begin{pmatrix}0\\0\\k_3\end{pmatrix} = \begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

This gives $k_1 = k_2 = k_3 = 0$ which means **all** the scalars are zero therefore $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are linearly independent.

(b) We have the linear combination $k_1\mathbf{u} + k_2\mathbf{v} + k_3\mathbf{w} = \mathbf{O}$:

$$k_1\mathbf{u} + k_2\mathbf{v} + k_3\mathbf{w} = k_1\begin{pmatrix}2\\2\\2\end{pmatrix} + k_2\begin{pmatrix}1\\2\\-1\end{pmatrix} + k_3\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

The augmented matrix $\left(\mathbf{A} \mid \mathbf{O}\right)$ is given by

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3 \end{array}\left(\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & -1 & 1 & 0 \end{array}\right)$$

Carrying out the row operations:

$$\begin{array}{c} \\ \mathrm{R}_1 \\ \mathrm{R}_2 - \mathrm{R}_1 \\ \mathrm{R}_3 \end{array}\begin{array}{c} \begin{array}{ccc} k_1 & k_2 & k_3 \end{array} \\ \left(\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \end{array}\right) \end{array}$$

From the middle row we have

$$k_2 = 0$$

Substituting this $k_2 = 0$ into the other rows yields $k_1 = 0$ and $k_3 = 0$.

All the scalars are equal to zero, $k_1 = k_2 = k_3 = 0$ therefore the given vectors, $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, are linearly independent.

(c) By examining the given vectors $\mathbf{u} = \begin{pmatrix}1\\1\\1\end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix}-2\\-2\\-2\end{pmatrix}$ we note that $\mathbf{v} = -2\mathbf{u}$

because $\mathbf{v} = \begin{pmatrix}-2\\-2\\-2\end{pmatrix} = -2\begin{pmatrix}1\\1\\1\end{pmatrix} = -2\mathbf{u}$. Hence we have $\mathbf{v} + 2\mathbf{u} = \mathbf{O}$ which means the scalars are **not** zero therefore the given vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent.

(d) We have the linear combination $k_1\mathbf{u} + k_2\mathbf{v} + k_3\mathbf{w} = \mathbf{O}$:

$$k_1\mathbf{u}+k_2\mathbf{v}+k_3\mathbf{w}=k_1\begin{pmatrix}-1\\2\\-3\end{pmatrix}+k_2\begin{pmatrix}0\\-4\\6\end{pmatrix}+k_3\begin{pmatrix}2\\0\\6\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

The augmented matrix $(\mathbf{A} \mid \mathbf{O})$ is given by

$$\begin{array}{c}\mathrm{R}_1\\\mathrm{R}_2\\\mathrm{R}_3\end{array}\left(\begin{array}{ccc|c}-1&0&2&0\\2&-4&0&0\\3&6&6&0\end{array}\right)$$

Executing the following row operations:

$$\begin{array}{c}\mathrm{R}_1\\\mathrm{R}_2\text{*=}\mathrm{R}_2+2\mathrm{R}_1\\\mathrm{R}_3\text{*=}\mathrm{R}_3+3\mathrm{R}_1\end{array}\left(\begin{array}{ccc|c}-1&0&2&0\\0&-4&4&0\\0&6&6&0\end{array}\right)$$

Carry out the row operation $\mathrm{R}_3*+\frac{3}{2}\mathrm{R}_2*$:

$$\begin{array}{cc} & \begin{array}{ccc}k_1&k_2&k_3\end{array}\\ \begin{array}{c}\mathrm{R}_1\\\mathrm{R}_2*\\\mathrm{R}_3*+3\mathrm{R}_2*/2\end{array} & \left(\begin{array}{ccc|c}-1&0&2&0\\0&-4&4&0\\0&0&12&0\end{array}\right)\end{array}$$

From the bottom row we have $k_3=0$. Using back substitution gives $k_2=k_3=0$. Hence $k_1=k_2=k_3=0$ which means that the given vectors are linearly independent.

3. (a) We examine the linear combination $k_1\mathbf{u}+k_2\mathbf{v}+k_3\mathbf{w}+k_4\mathbf{x}=\mathbf{O}$

$$k_1\mathbf{u}+k_2\mathbf{v}+k_3\mathbf{w}+k_4\mathbf{x}=k_1\begin{pmatrix}0\\-1\\0\\3\end{pmatrix}+k_2\begin{pmatrix}1\\0\\5\\0\end{pmatrix}+k_3\begin{pmatrix}2\\1\\0\\0\end{pmatrix}+k_4\begin{pmatrix}0\\1\\0\\-4\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

The augmented matrix is given by:

$$\begin{array}{cc} & \begin{array}{cccc}k_1&k_2&k_3&k_4\end{array}\\ \begin{array}{c}\mathrm{R}_1\\\mathrm{R}_2\\\mathrm{R}_3\\\mathrm{R}_4\end{array} & \left(\begin{array}{cccc|c}0&1&2&0&0\\-1&0&1&1&0\\0&5&0&0&0\\3&0&0&-4&0\end{array}\right)\end{array}$$

From the third row we have

$$5k_2=0 \text{ which gives } k_2=0$$

Substituting this $k_2=0$ into the top row we have

$$0+2k_3=0 \text{ which gives } k_3=0$$

Substituting $k_3=0$ into the second row:

$$-k_1+k_4=0 \text{ gives } k_4=k_1$$

Substituting $k_4=k_1$ into the bottom row:

$$3k_4 - 4k_4 = -k_4 = 0 \text{ gives } k_4 = 0$$

Since $k_4 = k_1$ therefore $k_1 = 0$. All the scalars, $k_1 = k_2 = k_3 = k_4 = 0$ which means that the given vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{x}$ are linearly independent.

(b) *What do you notice about the first vector* **u** *and the last vector* **x** *of the given vectors?*

$$\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 3 \\ 3 \end{pmatrix}, \ \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 5 \\ 0 \end{pmatrix}, \ \mathbf{w} = \begin{pmatrix} -3 \\ -6 \\ -9 \\ -4 \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} -5 \\ 5 \\ -15 \\ -15 \end{pmatrix}$$

$\mathbf{x} = \begin{pmatrix} -5 \\ 5 \\ -15 \\ -15 \end{pmatrix} = -5\underbrace{\begin{pmatrix} 1 \\ -1 \\ 3 \\ 3 \end{pmatrix}}_{=\mathbf{u}} = -5\mathbf{u}$. Since $\mathbf{x} = -5\mathbf{u}$ or $\mathbf{x} + 5\mathbf{u} = \mathbf{O}$ therefore we have the linear combination

$$5\mathbf{u} + 0\mathbf{v} + 0\mathbf{u} + \mathbf{x} = 5\mathbf{u} + \mathbf{x} = \mathbf{O}$$

We have nonzero scalars which give the zero vector therefore the given vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{x}$ are linearly dependent.

(c) We examine the linear combination $k_1\mathbf{u} + k_2\mathbf{v} + k_3\mathbf{w} + k_4\mathbf{x} = \mathbf{O}$

$$k_1\mathbf{u} + k_2\mathbf{v} + k_3\mathbf{w} + k_4\mathbf{x} = k_1\begin{pmatrix} -2 \\ 2 \\ 3 \\ 4 \end{pmatrix} + k_2\begin{pmatrix} 0 \\ 3 \\ -2 \\ -3 \end{pmatrix} + k_3\begin{pmatrix} 2 \\ -2 \\ -1 \\ 0 \end{pmatrix} + k_4\begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2k_1 \\ 2k_1 \\ 3k_1 \\ 4k_1 \end{pmatrix} + \begin{pmatrix} 0 \\ 3k_2 \\ -2k_2 \\ -3k_2 \end{pmatrix} + \begin{pmatrix} 2k_3 \\ -2k_3 \\ -k_3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 3k_4 \\ 0 \\ k_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Writing out the simultaneous equations we have

$$-2k_1 + 2k_3 = 0 \quad (1)$$

$$2k_1 + 3k_2 - 2k_3 + 3k_4 = 0 \quad (2)$$

$$3k_1 - 2k_2 - k_3 = 0 \quad (3)$$

$$4k_1 - 3k_2 + k_4 = 0 \quad (4)$$

From the first equation (1) we have $k_3 = k_1$. Let $k_1 = 1$ then $k_3 = 1$. Substituting this $k_1 = 1$ and $k_3 = 1$ into the third equation (3) gives

$$3 - 2k_2 - 1 = 0 \Rightarrow 2k_2 = 2 \text{ which gives } k_2 = 1$$

Substituting $k_1 = 1$ and $k_2 = 1$ into the bottom equation

$$4 - 3 + k_4 = 0 \text{ gives } k_4 = -1$$

Just need to check that these scalar values, $k_1 = 1$, $k_2 = 1$, $k_3 = 1$ and $k_4 = -1$ satisfy the second equation (2):

$$2 + 3 - 2 - 3 = 0$$

Since these nonzero scalars, $k_1 = 1,\ k_2 = 1,\ k_3 = 1$ and $k_4 = -1$, satisfy $k_1\mathbf{u} + k_2\mathbf{v} + k_3\mathbf{w} + k_4\mathbf{x} = \mathbf{O}$ therefore the given vectors $\mathbf{u},\ \mathbf{v},\ \mathbf{w}$ and $\mathbf{x}$ are linearly dependent.

4. We need to prove that if $\mathbf{u} = k\mathbf{v}$ then the vectors **u** and **v** are linearly dependent.
Proof. Since $\mathbf{u} = k\mathbf{v}$ therefore $(1)\mathbf{u} - k\mathbf{v} = \mathbf{O}$. Hence we have nonzero scalars which give the zero vector therefore vectors **u** and **v** are linearly dependent.

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5. We need to prove the vectors $\mathbf{u}+\mathbf{v}$, $\mathbf{v}+\mathbf{w}$ and $\mathbf{u}-\mathbf{w}$ are linearly dependent.
Proof. Since

$$(\mathbf{u}+\mathbf{v}) - (\mathbf{v}+\mathbf{w}) - (\mathbf{u}-\mathbf{w}) = \mathbf{O}$$

therefore $\mathbf{u}+\mathbf{v}$, $\mathbf{v}+\mathbf{w}$ and $\mathbf{u}-\mathbf{w}$ are linearly dependent because

$$k_1(\mathbf{u}+\mathbf{v}) + k_2(\mathbf{v}+\mathbf{w}) + k_3(\mathbf{u}-\mathbf{w}) = \mathbf{O} \text{ where } k_1 = 1,\ k_2 = -1 \text{ and } k_3 = -1$$

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6. We need to show that $\mathbf{e}_1$ and $\mathbf{e}_1 + \mathbf{e}_2$ are linearly independent.
Proof.
We know $\mathbf{e}_1$ and $\mathbf{e}_2$ are linearly independent. Consider the linear combination

$$k\mathbf{e}_1 + c(\mathbf{e}_1 + \mathbf{e}_2) = \mathbf{O}$$

Expanding this out yields

$$k\mathbf{e}_1 + c\mathbf{e}_1 + c\mathbf{e}_2 = \mathbf{O}$$
$$(k+c)\mathbf{e}_1 + c\mathbf{e}_2 = \mathbf{O}$$

Since $\mathbf{e}_1$ and $\mathbf{e}_2$ are linearly independent so all the scalars in the bottom equation are zero, hence $k + c = 0$ and $c = 0$. This implies $k = c = 0$.
Hence $k\mathbf{e}_1 + c(\mathbf{e}_1 + \mathbf{e}_2) = \mathbf{O}$ gives $k = c = 0$ therefore $\mathbf{e}_1$ and $\mathbf{e}_1 + \mathbf{e}_2$ are linearly independent because all scalars are zero.

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7. We need to prove that $\mathbf{e}_1$, $\mathbf{e}_1 + \mathbf{e}_2$ and $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ are linearly independent in $\mathbb{R}^3$.
Proof.
We know that $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ the standard unit vectors in $\mathbb{R}^3$ are linearly independent. Consider the linear combination

$$k_1\mathbf{e}_1 + k_2(\mathbf{e}_1 + \mathbf{e}_2) + k_3(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = \mathbf{O}$$

Expanding these out

$$k_1\mathbf{e}_1 + k_2\mathbf{e}_1 + k_2\mathbf{e}_2 + k_3\mathbf{e}_1 + k_3\mathbf{e}_2 + k_3\mathbf{e}_3 = \mathbf{O}$$
$$(k_1 + k_2 + k_3)\mathbf{e}_1 + (k_2 + k_3)\mathbf{e}_2 + k_3\mathbf{e}_3 = \mathbf{O}$$

Vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are linearly independent therefore

$$k_1 + k_2 + k_3 = 0,\ k_2 + k_3 = 0 \text{ and } k_3 = 0$$
$$k_1 = -k_2 - k_3,\ k_2 = -k_3 \text{ and } k_3 = 0$$

We have $k_3 = 0$, $k_2 = -k_3 = 0$ and $k_1 = -k_2 - k_3 = 0 - 0 = 0$. We have

$$k_1\mathbf{e}_1 + k_2(\mathbf{e}_1 + \mathbf{e}_2) + k_3(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = \mathbf{O} \text{ gives } k_1 = k_2 = k_3 = 0$$

Hence $\mathbf{e}_1$, $\mathbf{e}_1 + \mathbf{e}_2$ and $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ are linearly independent.

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8. Required to prove that $\mathbf{u}+\mathbf{v}$, $\mathbf{v}+\mathbf{w}$, $\mathbf{w}+\mathbf{x}$ and $\mathbf{u}+\mathbf{x}$ are linearly dependent given that **u**, **v**, **w** and **x** be linearly independent.
Proof.
Consider the linear combination

$$k_1(\mathbf{u}+\mathbf{v})+k_2(\mathbf{v}+\mathbf{w})+k_3(\mathbf{w}+\mathbf{x})+k_4(\mathbf{u}+\mathbf{x})=\mathbf{O}$$

Expanding this out gives

$$k_1\mathbf{u}+k_1\mathbf{v}+k_2\mathbf{v}+k_2\mathbf{w}+k_3\mathbf{w}+k_3\mathbf{x}+k_4\mathbf{u}+k_4\mathbf{x}=\mathbf{O}$$

$$(k_1+k_4)\mathbf{u}+(k_1+k_2)\mathbf{v}+(k_2+k_3)\mathbf{w}+(k_3+k_4)\mathbf{x}=\mathbf{O} \qquad [\text{Factorizing}]$$

We are given that the vectors **u**, **v**, **w** and **x** are linearly independent therefore all the scalars in brackets are zero, that is

$$k_1+k_4=0,\ k_1+k_2=0,\ k_2+k_3=0 \text{ and } k_3+k_4=0$$

$$k_1=-k_4,\ k_1=-k_2,\ k_2=-k_3 \text{ and } k_3=-k_4$$

Let $k_4=1$ then substituting this and the resulting k's into the above we have

$$k_1=-1,\quad k_2=-(-1)=1 \quad\text{and}\quad k_3=-1$$

Since the linear combination

$$k_1(\mathbf{u}+\mathbf{v})+k_2(\mathbf{v}+\mathbf{w})+k_3(\mathbf{w}+\mathbf{x})+k_4(\mathbf{u}+\mathbf{x})=\mathbf{O} \text{ gives}$$

$$k_1=-1,\quad k_2=1,\ k_3=-1 \text{ and } k_4=1$$

which means all the scalars are **not** zero. Hence the vectors

$$\mathbf{u}+\mathbf{v},\ \mathbf{v}+\mathbf{w},\ \mathbf{w}+\mathbf{x} \text{ and } \mathbf{u}+\mathbf{x}$$

are linearly dependent.

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9. We need to show that $\mathbf{x}=k_1\mathbf{u}+k_2\mathbf{v}+k_3\mathbf{w}$ where **u**, **v** and **w** are linearly independent is unique.
Proof.
Suppose we also have $\mathbf{x}=c_1\mathbf{u}+c_2\mathbf{v}+c_3\mathbf{w}$. Required to prove

$$c_1=k_1,\ c_2=k_2 \text{ and } c_3=k_3$$

Equating the two linear combinations we have

$$k_1\mathbf{u}+k_2\mathbf{v}+k_3\mathbf{w}=c_1\mathbf{u}+c_2\mathbf{v}+c_3\mathbf{w}$$

$$k_1\mathbf{u}+k_2\mathbf{v}+k_3\mathbf{w}-c_1\mathbf{u}-c_2\mathbf{v}-c_3\mathbf{w}=\mathbf{O} \qquad [\text{Collecting vectors}]$$

$$(k_1-c_1)\mathbf{u}+(k_2-c_2)\mathbf{v}+(k_3-c_3)\mathbf{w}=\mathbf{O} \qquad [\text{Factorizing}]$$

Since **u**, **v** and **w** are linearly independent therefore all the scalars are zero:

$$k_1-c_1=0,\ k_2-c_2=0 \text{ and } k_3-c_3=0$$

$$k_1=c_1,\ k_2=c_2 \text{ and } k_3=c_3$$

Hence the representation $\mathbf{x}=k_1\mathbf{u}+k_2\mathbf{v}+k_3\mathbf{w}$ is unique.

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10. *Proof.* Consider the linear combination

$$k_1(c_1\mathbf{v}_1)+k_2(c_2\mathbf{v}_2)+k_3(c_3\mathbf{v}_3)+\cdots+k_n(c_n\mathbf{v}_n)=\mathbf{O}$$

Note that $c_1\mathbf{v}_1,\ c_2\mathbf{v}_2,\ c_3\mathbf{v}_3,$ and $c_n\mathbf{v}_n$ are all vectors.

Required to prove that $k_1=k_2=k_3=\cdots=k_n=0$. Expanding out the above and rearranging yields

$$(k_1c_1)\mathbf{v}_1+(k_2c_2)\mathbf{v}_2+(k_3c_3)\mathbf{v}_3+\cdots+(k_nc_n)\mathbf{v}_n=\mathbf{O}$$

Since the vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,$ and $\mathbf{v}_n$ are linearly independent therefore

$$k_1c_1 = k_2c_2 = k_3c_3 = \cdots = k_nc_n = 0$$

The scalar $c_j \neq 0$ [Not Zero] for any j between 1 to *n* because we are given that *c*'s are real non-zero scalars. Therefore $k_1 = k_2 = k_3 = \cdots = k_n = 0$.

Since the linear combination

$$k_1(c_1\mathbf{v}_1) + k_2(c_2\mathbf{v}_2) + k_3(c_3\mathbf{v}_3) + \cdots + k_n(c_n\mathbf{v}_n) = \mathbf{O} \text{ gives } k_1 = k_2 = k_3 = \cdots = k_n = 0$$

therefore we conclude that the vectors $c_1\mathbf{v}_1,\ c_2\mathbf{v}_2,\ c_3\mathbf{v}_3,$ and $c_n\mathbf{v}_n$ are linearly independent.

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11. We need to prove if $S = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n\}$ is linearly independent then any subset $T = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_m\}$ where $m < n$ is also linearly independent.

Proof.

We are given that $S = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n\}$ is linearly independent therefore we have

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + \cdots + k_n\mathbf{v}_n = \mathbf{O} \quad \Rightarrow \quad k_1 = k_2 = k_3 = \cdots = k_n = 0$$

Consider the linear combination

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_m\mathbf{v}_m = \mathbf{O}$$

Required to prove that $c_1 = c_2 = c_3 = \cdots = c_n = 0$. Equating the two linear combinations and remembering that $m < n$ we have

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + \cdots + k_n\mathbf{v}_n = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_m\mathbf{v}_m = \mathbf{O}$$

$$(k_1 - c_1)\mathbf{v}_1 + (k_2 - c_2)\mathbf{v}_2 + \cdots + (k_m - c_m)\mathbf{v}_m + k_{m+1}\mathbf{v}_{m+1} + \cdots + k_n\mathbf{v}_n = \mathbf{O}$$

Since $S = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n\}$ is linearly independent therefore **all** the scalars in the last line are zero, that is

$$k_1 - c_1 = k_2 - c_2 = \cdots = k_m - c_m = k_{m+1} = \cdots = k_n = 0$$

In particular we have the first *m* scalars

$$k_1 - c_1 = k_2 - c_2 = \cdots = k_m - c_m = 0$$

$$k_1 = c_1,\ k_2 = c_2,\ \cdots \text{ and } k_m = c_m$$

Because all the *k*'s are zero therefore $c_1 = c_2 = c_3 = \cdots = c_m = 0$. Hence $S_1 = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_m\}$ is linearly independent.

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12. *Proof.* Consider the linear combination

$$k_1\mathbf{u} + k_2\mathbf{v} + k_3\mathbf{w} = \mathbf{O}$$

$$k_1\begin{pmatrix} t \\ 1 \\ 1 \end{pmatrix} + k_2\begin{pmatrix} -1 \\ t \\ 1 \end{pmatrix} + k_3\begin{pmatrix} 1 \\ 1 \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \left[\text{Substituting the given values of the vectors } \mathbf{u},\ \mathbf{v} \text{ and } \mathbf{w}\right]$$

The augmented matrix is

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3 \end{array} \left(\begin{array}{ccc|c} t & -1 & 1 & 0 \\ 1 & t & 1 & 0 \\ 1 & 1 & t & 0 \end{array}\right)$$

Executing the following row operations:

$$\begin{array}{r} \mathrm{R}_1 \\ \mathrm{R}_2^*=\mathrm{R}_2-\mathrm{R}_1 \\ \mathrm{R}_3^*=\mathrm{R}_3+\mathrm{R}_1 \end{array}\left(\begin{array}{ccc|c} t & -1 & 1 & 0 \\ 1-t & t+1 & 0 & 0 \\ 1+t & 0 & t+1 & 0 \end{array}\right)$$

Multiply the bottom row R_3^* by $1/(1+t)$ provided $t \neq -1$:

$$\begin{array}{r} \mathrm{R}_1 \\ \mathrm{R}_2^* \\ \mathrm{R}_3^{**}=\mathrm{R}_3^*/(1+t) \end{array}\left(\begin{array}{ccc|c} t & -1 & 1 & 0 \\ 1-t & t+1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right)$$

Carrying out the row operation $\mathrm{R}_1-\mathrm{R}_3^{**}$:

$$\begin{array}{r} \mathrm{R}_1^*=\mathrm{R}_1-\mathrm{R}_3^{**} \\ \mathrm{R}_2^* \\ \mathrm{R}_3^{**} \end{array}\left(\begin{array}{ccc|c} t-1 & -1 & 0 & 0 \\ 1-t & t+1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right)$$

Carrying out the row operation $\mathrm{R}_2^*+\mathrm{R}_1^*$:

$$\begin{array}{rc} & \begin{array}{ccc} k_1 & k_2 & k_3 \end{array} \\ \begin{array}{r} \mathrm{R}_1^* \\ \mathrm{R}_2^*=\mathrm{R}_2^*+\mathrm{R}_1^* \\ \mathrm{R}_3^{**} \end{array} & \left(\begin{array}{ccc|c} t-1 & -1 & 0 & 0 \\ 0 & t & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \end{array}$$

From the middle row we have $k_2\,t=0$. Remember for linear independence we need all the scalars to be zero. So $k_2=0$ which means that $t \neq 0$ because if $t=0$ then we could take $k_2 \neq 0$.

From the top row we have

$$(t-1)k_1-k_2=0$$

We already have $k_2=0$ and so substituting this into this $(t-1)k_1-k_2=0$ gives

$$(t-1)k_1=0$$

Again we have $k_1=0$ so $t-1\neq 0$ or $t\neq 1$.

Hence the vectors **u**, **v** and **w** are linearly independent whenever $t\neq 0,\ t\neq 1$ or $t\neq -1$.

13. We need to prove the following result:

Let $S=\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n\}$ be n vectors in the n - space $\mathbb{R}^n$. Let **A** be the n by n matrix whose columns are given by the vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots$ and $\mathbf{v}_n$:

$$\mathbf{A}=\begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n\end{pmatrix}$$

Then vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n$ are linearly independent $\Leftrightarrow$ matrix **A** is invertible.

Proof.

Consider the linear combination:

$$k_1\mathbf{v}_1+k_2\mathbf{v}_2+\cdots+k_n\mathbf{v}_n=\mathbf{O}$$

where the k's are real scalars. Let us write this linear combination in matrix form $\mathbf{Ax}=\mathbf{O}$ where $\mathbf{A}=\begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n\end{pmatrix}$:

$$\begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n\end{pmatrix}\begin{pmatrix}k_1\\k_2\\\vdots\\k_n\end{pmatrix}=\mathbf{O} \qquad \left[\mathbf{x}=\begin{pmatrix}k_1\\k_2\\\vdots\\k_n\end{pmatrix}\right]$$

$(\Leftarrow)$. Let us assume that matrix **A** is invertible. Then by the following Theorem of chapter 1:

> Theorem (1.35). Let **A** be an *n* by *n* matrix, then the following statements are equivalent:
> (a) The matrix **A** is invertible (non-singular).
> (b) The linear system $\mathbf{Ax}=\mathbf{O}$ only has the trivial solution $\mathbf{x}=\mathbf{O}$.

We have that $\mathbf{x}=\begin{pmatrix}k_1\\k_2\\\vdots\\k_n\end{pmatrix}=\mathbf{O}$ which means that

$$k_1=k_2=\cdots=k_n=0$$

Therefore the vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots$ and $\mathbf{v}_n$ are linearly independent.

$(\Rightarrow)$. Now we assume that the vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n$ are linearly independent.

Consider the matrix $\mathbf{A}=\begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n\end{pmatrix}$. Required to prove that matrix **A** is invertible.

Suppose matrix **A** is non-invertible. Then by the following proposition of chapter 1:

> Proposition (1.39). Let **A** be a square matrix and **R** be the reduced row echelon form of **A**. Then **R** has at least one row of zeros $\Leftrightarrow$ **A** is non-invertible (singular).

The reduced row echelon form of matrix **A** has at least one row of zeros. This means that the linear system $\mathbf{Ax}=\mathbf{O}$ which is equivalent to $\mathbf{Rx}=\mathbf{O}$ where **R** is the reduced row echelon form of matrix **A** has less equations than unknowns so we have an infinite number of solutions. This implies all the scalars (*k*'s) are not zero which suggests that the vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n$ are linearly dependent. This is a contradiction because we are assuming the vectors are linearly independent. Hence our supposition matrix **A** is non-invertible must be wrong so matrix **A** is invertible. This completes our proof.

■