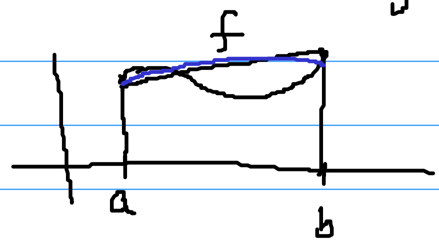
Numerical Integration Continued

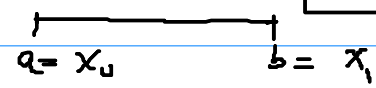


Trapezoidal Rule

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(x_0) + f(x_1)\right] - \frac{h^3}{12} f''(\xi)$$

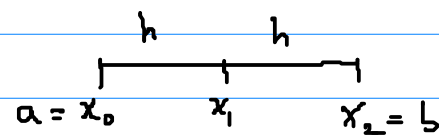
$\boxed{n=1}$

$h = b - a$



Simpson's Rule

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right] - \frac{h^5}{90} f^{(4)}(\xi)$$



$\frac{b-a}{2} = h$

$x_1 = x_0 + h$

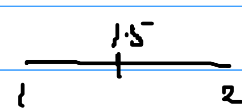
$\boxed{n=2}$

$n=3$, $h = \frac{b-a}{n}$ $a = x_0$, x_1, x_2, $x_3 = b$

$$\int_a^b f(x)dx = \int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right] - \frac{3h^5}{80} f^{(4)}(\xi)$$

Example

$$\int_1^2 e^x\,dx = \frac{1}{2}\left[f(1) + f(2)\right]$$



$h = 2 - 1$

$f(x) = e^x$

$$= \tfrac{1}{2}\left[e^1 + e^2\right]$$

(Trapezoidal Rule)

$\int_1^2 e^x\,dx \qquad f(x) = e^x$ is contin on $[1, 2]$

$\int_0^1 \frac{1}{x(1-x)}\,dx$, $\quad g(x) = \frac{1}{x(1-x)}$ is not defined at $x = 0$ & $x = 1$

Newton's Cotes Famulas

Open Cotes

Closed Cotes
(Trapezoidal Rule
Simpson's Rule

$x_0 \;\; x_1 \;\; x_2 \quad\quad x_n$

$x_{-1} = a \qquad\qquad b = x_{n+1}$

$h = \frac{b-a}{n+2}$

$x_0 \;\; x_1 \;\; x_2$

$a \qquad\qquad b = x_n$

$h = \frac{b-a}{n}$

$a = x_0, \quad x_i = x_0 + ih$

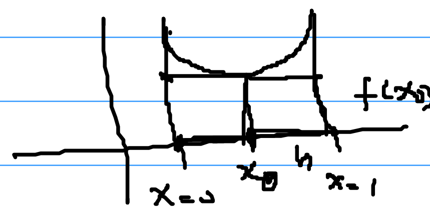
$\boxed{n=0}$

x_0

$a = x_{-1} \qquad x_1 = b$

$h = \frac{b-a}{2}$

$$\int_a^b f(x)\,dx = \int_{x_{-1}}^{x_1} f(x)\,dx = 2h f(x_0) - \frac{h^3}{3} f''(\xi)$$



Midpoint Rule

$x_{-1} < \xi < x_1$

$2h f(x_0)$

$\boxed{n=1}$

$x_0 \quad x_1$

$x_{-1} = a \qquad x_2 = b$

$$\int_{x_{-1}}^{x_2} = \int_a^b f(x)\,dx = \frac{3h}{2}\left[f(x_0) + f(x_1) \right] + \frac{3h^3}{4} f''(\xi)$$

$x_{-1} < \xi < x_2$

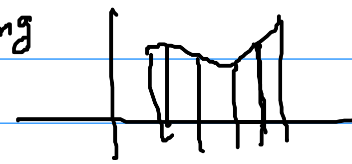
Precision of a quadrature

Degree of precision of a quadrature is the highest power of x^k, $k = 0, 1, 2, \ldots n$ for which quadrature formula gives exact answer.

For example degree of the precision of trapezoidal rule is 1.
Because for $f(x) = x^0, x^1$ the error term, $\frac{h^2}{12} f''(\xi) = 0$, so the Trapezoidal Rule is exact.
Similarly, the Simpson's Rule has degree of precision 3 because for $f(x) = x^0, x^1, x^2, x^3$, the error term $\frac{h^5}{90} f^{(4)}(\xi) = 0$

<u>Example</u> Question 15 of 4.3

Find the degree of precision of the following quadrature formula



$$\int_{-1}^{1} f(x)\,dx = f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right) \quad \text{—— (A)}$$

<u>Sol</u> This formula is of the form of $\int_a^b f(x)dx = \sum c_i f(x_i)$

In order to find the degree of precision (This is general form of any quadrature) of (A) we have to evaluate both sides of (A) for $x^0, x^1, x^2, x^3 \ldots x^n$ and see upto what power x both sides are equal. The highest power for which both sides are equal, would be the degree of precision of (A)

for $f(x) = x^0 = 1$

$$\int_{-1}^{1} 1\,dx = 2, \qquad f(-\sqrt{3}/3) + f(\sqrt{3}/3) = 1 + 1 = 2 \text{ (both sides are equal)}$$

for $f(x) = x$, $\int_{-1}^{1} x\,dx = \left[\frac{x^2}{2}\right]_{-1}^{1} = \frac{0}{2} = 0$

$$f(-\sqrt{3}/3) + f(\sqrt{3}/3) = -\sqrt{3}/3 + \sqrt{3}/3 = 0 \text{ (both sides are equal)}$$

For $f(x) = x^2$

$$\int_{-1}^{1} x^2\,dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = 2/3$$

$$f\left(\frac{-\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right) = \left(-\frac{\sqrt{3}}{3}\right)^2 + \left(\frac{\sqrt{3}}{3}\right)^2 = \frac{3}{9} + \frac{3}{9} = \frac{6}{9}$$
$$= \frac{2}{3}$$

(both sides are equal)

For $f(x) = x^3$

$$\int_{-1}^{1} x^3\,dx = \left|\frac{x^4}{4}\right|_{-1}^{1} = 0$$

$$f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right) = \left(-\frac{\sqrt{3}}{3}\right)^3 + \left(\frac{\sqrt{3}}{3}\right)^3 = 0$$

both sides are equal

For $f(x) = x^4$

$$\int_{-1}^{1} x^4\,dx = \left|\frac{x^5}{5}\right|_{-1}^{1} = \frac{2}{5}$$

$$f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right) = \left(-\frac{\sqrt{3}}{3}\right)^4 + \left(\frac{\sqrt{3}}{3}\right)^4$$
$$= 2\left(\frac{\sqrt{3}}{3}\right)^4 =$$

Both sides are NOT equal. Therefore, the degree of precision is $n = 3$.