Solutions of Equations of One Variable

Exercise Set 2.1, page 54

*1. $p_3 = 0.625$

2. *(a) $p_3 = -0.6875$

   (b) $p_3 = 1.09375$

3. The Bisection method gives:

   (a) $p_7 = 0.5859$

   (b) $p_8 = 3.002$

   (c) $p_7 = 3.419$

4. The Bisection method gives:

   (a) $p_7 = -1.414$

   (b) $p_8 = 1.414$

   (c) $p_7 = 2.727$

   (d) $p_7 = -0.7265$

5. The Bisection method gives:

   (a) $p_{17} = 0.641182$

   (b) $p_{17} = 0.257530$

   (c) For the interval $[-3, -2]$, we have $p_{17} = -2.191307$, and for the interval $[-1, 0]$, we have $p_{17} = -0.798164$.

   (d) For the interval $[0.2, 0.3]$, we have $p_{14} = 0.297528$, and for the interval $[1.2, 1.3]$, we have $p_{14} = 1.256622$.

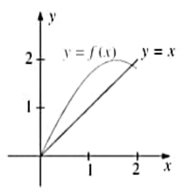
6. (a) $p_{17} = 1.51213837$

   (b) $p_{17} = 0.97676849$

   (c) For the interval $[1, 2]$, we have $p_{17} = 1.41239166$, and for the interval $[2, 4]$, we have $p_{18} = 3.05710602$.

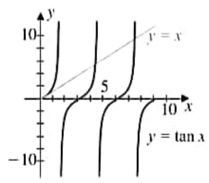
   (d) For the interval $[0, 0.5]$, we have $p_{16} = 0.20603180$, and for the interval $[0.5, 1]$, we have $p_{16} = 0.68196869$.

7. (a)



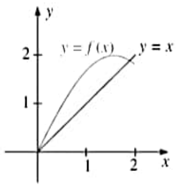
   (b) Using $[1.5, 2]$ from part (a) gives $p_{16} = 1.89550018$.

*8. (a)



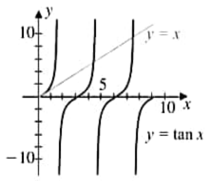
(d) For the interval $[0, 0.5]$, we have $p_{16} = 0.20603180$, and for the interval $[0.5, 1]$, we have $p_{16} = 0.68196869$.

7. (a)



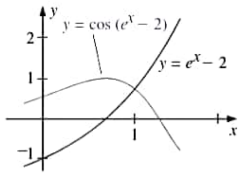
(b) Using $[1.5, 2]$ from part (a) gives $p_{16} = 1.89550018$.

*8. (a)



(b) Using $[4.2, 4.6]$ from part (a) gives $p_{16} = 4.4934143$.

9. (a)



(b) $p_{17} = 1.00762177$

10. (a) 0
(b) 0
(c) 2
(d) -2

11. *(a) 2
(b) -2
*(c) -1
(d) 1

*12. We have $\sqrt{3} \approx p_{14} = 1.7320$, using $[1, 2]$.

13. The third root of 25 is approximately $p_{14} = 2.92401$, using $[2, 3]$.

*14. A bound for the number of iterations is $n \geq 12$ and $p_{12} = 1.3787$.

15. A bound is $n \geq 14$, and $p_{14} = 1.32477$.

11. *(a) 2

    (b) -2

    *(c) -1

    (d) 1

*12. We have $\sqrt{3} \approx p_{14} = 1.7320$, using $[1, 2]$.

13. The third root of 25 is approximately $p_{14} = 2.92401$, using $[2, 3]$.

*14. A bound for the number of iterations is $n \geq 12$ and $p_{12} = 1.3787$.

15. A bound is $n \geq 14$, and $p_{14} = 1.32477$.

16. For $n > 1$,

$$|f(p_n)| = \left(\frac{1}{n}\right)^{10} \leq \left(\frac{1}{2}\right)^{10} = \frac{1}{1024} < 10^{-3},$$

so

$$|p - p_n| = \frac{1}{n} < 10^{-3} \Leftrightarrow 1000 < n.$$

*17. Since $\lim_{n\to\infty}(p_n - p_{n-1}) = \lim_{n\to\infty} 1/n = 0$, the difference in the terms goes to zero. However, p_n is the nth term of the divergent harmonic series, so $\lim_{n\to\infty} p_n = \infty$.

18. Since $-1 < a < 0$ and $2 < b < 3$, we have $1 < a + b < 3$ or $1/2 < 1/2(a + b) < 3/2$ in all cases. Further,

$$\begin{aligned} f(x) < 0, &\quad \text{for } -1 < x < 0 \quad \text{and} \quad 1 < x < 2; \\ f(x) > 0, &\quad \text{for } 0 < x < 1 \quad \text{and} \quad 2 < x < 3. \end{aligned}$$

Thus, $a_1 = a$, $f(a_1) < 0$, $b_1 = b$, and $f(b_1) > 0$.

(a) Since $a + b < 2$, we have $p_1 = \frac{a+b}{2}$ and $1/2 < p_1 < 1$. Thus, $f(p_1) > 0$. Hence, $a_2 = a_1 = a$ and $b_2 = p_1$. The only zero of f in $[a_2, b_2]$ is $p = 0$, so the convergence will be to 0.

(b) Since $a + b > 2$, we have $p_1 = \frac{a+b}{2}$ and $1 < p_1 < 3/2$. Thus, $f(p_1) < 0$. Hence, $a_2 = p_1$ and $b_2 = b_1 = b$. The only zero of f in $[a_2, b_2]$ is $p = 2$, so the convergence will be to 2.

(c) Since $a + b = 2$, we have $p_1 = \frac{a+b}{2} = 1$ and $f(p_1) = 0$. Thus, a zero of f has been found on the first iteration. The convergence is to $p = 1$.

*19. The depth of the water is 0.838 ft.

20. The angle θ changes at the approximate rate $w = -0.317059$.

Exercise Set 2.2, page 64

1. For the value of x under consideration we have

   (a) $x = (3 + x - 2x^2)^{1/4} \Leftrightarrow x^4 = 3 + x - 2x^2 \Leftrightarrow f(x) = 0$

   (b) $x = \left(\frac{x + 3 - x^4}{2}\right)^{1/2} \Leftrightarrow 2x^2 = x + 3 - x^4 \Leftrightarrow f(x) = 0$

   (c) $x = \left(\frac{x + 3}{x^2 + 2}\right)^{1/2} \Leftrightarrow x^2(x^2 + 2) = x + 3 \Leftrightarrow f(x) = 0$

   (d) $x = \frac{3x^4 + 2x^2 + 3}{4x^3 + 4x - 1} \Leftrightarrow 4x^4 + 4x^2 - x = 3x^4 + 2x^2 + 3 \Leftrightarrow f(x) = 0$

2. (a) $p_4 = 1.10782$; (b) $p_4 = 0.987506$; (c) $p_4 = 1.12364$; (d) $p_4 = 1.12412$;
   (b) Part (d) gives the best answer since $|p_4 - p_3|$ is the smallest for (d).

*3. The order in descending speed of convergence is (b), (d), and (a). The sequence in (c) does not converge.

Exercise Set 2.2, page 64

1. For the value of x under consideration we have

   (a) $x = (3 + x - 2x^2)^{1/4} \Leftrightarrow x^4 = 3 + x - 2x^2 \Leftrightarrow f(x) = 0$

   (b) $x = \left(\dfrac{x+3-x^4}{2}\right)^{1/2} \Leftrightarrow 2x^2 = x + 3 - x^4 \Leftrightarrow f(x) = 0$

   (c) $x = \left(\dfrac{x+3}{x^2+2}\right)^{1/2} \Leftrightarrow x^2(x^2+2) = x + 3 \Leftrightarrow f(x) = 0$

   (d) $x = \dfrac{3x^4+2x^2+3}{4x^3+4x-1} \Leftrightarrow 4x^4 + 4x^2 - x = 3x^4 + 2x^2 + 3 \Leftrightarrow f(x) = 0$

2. (a) $p_4 = 1.10782$; (b) $p_4 = 0.987506$; (c) $p_4 = 1.12364$; (d) $p_4 = 1.12412$;
   (b) Part (d) gives the best answer since $|p_4 - p_3|$ is the smallest for (d).

*3. The order in descending speed of convergence is (b), (d), and (a). The sequence in (c) does not converge.

4. The sequence in (c) converges faster than in (d). The sequences in (a) and (b) diverge.

5. With $g(x) = (3x^2 + 3)^{1/4}$ and $p_0 = 1$, $p_6 = 1.94332$ is accurate to within 0.01.

6. With $g(x) = \sqrt{1 + \frac{1}{x}}$ and $p_0 = 1$, we have $p_4 = 1.324$.

7. Since $g'(x) = \frac{1}{4}\cos\frac{x}{2}$, g is continuous and g' exists on $[0, 2\pi]$. Further, $g'(x) = 0$ only when $x = \pi$, so that $g(0) = g(2\pi) = \pi \le g(x) =\le g(\pi) = \pi + \frac{1}{2}$ and $|g'(x)| \le \frac{1}{4}$, for $0 \le x \le 2\pi$. Theorem 2.3 implies that a unique fixed point p exists in $[0, 2\pi]$. With $k = \frac{1}{4}$ and $p_0 = \pi$, we have $p_1 = \pi + \frac{1}{2}$. Corollary 2.5 implies that

   $$|p_n - p| \le \frac{k^n}{1-k}|p_1 - p_0| = \frac{2}{3}\left(\frac{1}{4}\right)^n.$$

   For the bound to be less than 0.1, we need $n \ge 4$. However, $p_3 = 3.626996$ is accurate to within 0.01.

8. Using $p_0 = 1$ gives $p_{12} = 0.6412053$. Since $|g'(x)| = 2^{-x}\ln 2 \le 0.551$ on $\left[\frac{1}{3}, 1\right]$ with $k = 0.551$, Corollary 2.5 gives a bound of 16 iterations.

*9. For $p_0 = 1.0$ and $g(x) = 0.5(x + \frac{3}{x})$, we have $\sqrt{3} \approx p_4 = 1.73205$.

10. For $g(x) = 5/\sqrt{x}$ and $p_0 = 2.5$, we have $p_{14} = 2.92399$.

11. (a) With $[0, 1]$ and $p_0 = 0$, we have $p_9 = 0.257531$.
    (b) With $[2.5, 3.0]$ and $p_0 = 2.5$, we have $p_{17} = 2.690650$.
    (c) With $[0.25, 1]$ and $p_0 = 0.25$, we have $p_{14} = 0.909999$.
    (d) With $[0.3, 0.7]$ and $p_0 = 0.3$, we have $p_{39} = 0.469625$.
    (e) With $[0.3, 0.6]$ and $p_0 = 0.3$, we have $p_{48} = 0.448059$.

    (f) With $[0, 1]$ and $p_0 = 0$, we have $p_6 = 0.704812$.

12. The inequalities in Corollary 2.4 give $|p_n - p| < k^n \max(p_0 - a, b - p_0)$. We want

    $$k^n \max(p_0 - a, b - p_0) < 10^{-5} \quad \text{so we need} \quad n > \frac{\ln(10^{-5}) - \ln(\max(p_0 - a, b - p_0))}{\ln k}.$$

    (a) Using $g(x) = 2 + \sin x$ we have $k = 0.9899924966$ so that with $p_0 = 2$ we have $n > \ln(0.00001)/\ln k = 1144.663221$. However, our tolerance is met with $p_{63} = 2.5541998$.

    (b) Using $g(x) = \sqrt[3]{2x+5}$ we have $k = 0.1540802832$ so that with $p_0 = 2$ we have $n > \ln(0.00001)/\ln k = 6.155718005$. However, our tolerance is met with $p_6 = 2.0945503$.

(f) With $[0, 1]$ and $p_0 = 0$, we have $p_6 = 0.704812$.

12. The inequalities in Corollary 2.4 give $|p_n - p| < k^n \max(p_0 - a, b - p_0)$. We want

$$k^n \max(p_0 - a, b - p_0) < 10^{-5} \quad \text{so we need} \quad n > \frac{\ln(10^{-5}) - \ln(\max(p_0 - a, b - p_0))}{\ln k}.$$

(a) Using $g(x) = 2 + \sin x$ we have $k = 0.9899924966$ so that with $p_0 = 2$ we have $n > \ln(0.00001)/\ln k = 1144.663221$. However, our tolerance is met with $p_{63} = 2.5541998$.

(b) Using $g(x) = \sqrt[3]{2x + 5}$ we have $k = 0.1540802832$ so that with $p_0 = 2$ we have $n > \ln(0.00001)/\ln k = 6.155718005$. However, our tolerance is met with $p_6 = 2.0945503$.

*(c) Using $g(x) = \sqrt{e^x/3}$ and the interval $[0, 1]$ we have $k = 0.4759448347$ so that with $p_0 = 1$ we have $n > \ln(0.00001)/\ln k = 15.50659829$. However, our tolerance is met with $p_{12} = 0.91001496$.

(d) Using $g(x) = \cos x$ and the interval $[0, 1]$ we have $k = 0.8414709848$ so that with $p_0 = 0$ we have $n > \ln(0.00001)/\ln k > 66.70148074$. However, our tolerance is met with $p_{30} = 0.73908230$.

13. For $g(x) = (2x^2 - 10\cos x)/(3x)$, we have the following:

$$p_0 = 3 \Rightarrow p_8 = 3.16193; \quad p_0 = -3 \Rightarrow p_8 = -3.16193.$$

For $g(x) = \arccos(-0.1x^2)$, we have the following:

$$p_0 = 1 \Rightarrow p_{11} = 1.96882; \quad p_0 = -1 \Rightarrow p_{11} = -1.96882.$$

*14. For $g(x) = \dfrac{1}{\tan x} - \dfrac{1}{x} + x$ and $p_0 = 4$, we have $p_4 = 4.493409$.

15. With $g(x) = \dfrac{1}{\pi}\arcsin\left(-\dfrac{x}{2}\right) + 2$, we have $p_5 = 1.683855$.

16. (a) If fixed-point iteration converges to the limit p, then

$$p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} 2p_{n-1} - Ap_{n-1}^2 = 2p - Ap^2.$$

Solving for p gives $p = \dfrac{1}{A}$.

(b) Any subinterval $[c, d]$ of $\left(\dfrac{1}{2A}, \dfrac{3}{2A}\right)$ containing $\dfrac{1}{A}$ suffices.

Since

$$g(x) = 2x - Ax^2, \quad g'(x) = 2 - 2Ax,$$

so $g(x)$ is continuous, and $g'(x)$ exists. Further, $g'(x) = 0$ only if $x = \dfrac{1}{A}$.

Since

$$g\left(\frac{1}{A}\right) = \frac{1}{A}, \quad g\left(\frac{1}{2A}\right) = g\left(\frac{3}{2A}\right) = \frac{3}{4A}, \quad \text{and we have} \quad \frac{3}{4A} \le g(x) \le \frac{1}{A}.$$

For x in $\left(\frac{1}{2A}, \frac{3}{2A}\right)$, we have

$$\left|x - \frac{1}{A}\right| < \frac{1}{2A} \quad \text{so} \quad |g'(x)| = 2A\left|x - \frac{1}{A}\right| < 2A\left(\frac{1}{2A}\right) = 1.$$

17. One of many examples is $g(x) = \sqrt{2x-1}$ on $\left[\frac{1}{2}, 1\right]$.

*18. (a) The proof of existence is unchanged. For uniqueness, suppose p and q are fixed points in $[a,b]$ with $p \neq q$. By the Mean Value Theorem, a number ξ in (a,b) exists with

$$p - q = g(p) - g(q) = g'(\xi)(p-q) \leq k(p-q) < p - q,$$

giving the same contradiction as in Theorem 2.3.

(b) Consider $g(x) = 1 - x^2$ on $[0,1]$. The function g has the unique fixed point

$$p = \frac{1}{2}\left(-1 + \sqrt{5}\right).$$

With $p_0 = 0.7$, the sequence eventually alternates between 0 and 1.

*19. (a) Suppose that $x_0 > \sqrt{2}$. Then

$$x_1 - \sqrt{2} = g(x_0) - g\left(\sqrt{2}\right) = g'(\xi)\left(x_0 - \sqrt{2}\right),$$

where $\sqrt{2} < \xi < x_0$. Thus, $x_1 - \sqrt{2} > 0$ and $x_1 > \sqrt{2}$. Further,

$$x_1 = \frac{x_0}{2} + \frac{1}{x_0} < \frac{x_0}{2} + \frac{1}{\sqrt{2}} = \frac{x_0 + \sqrt{2}}{2}$$

and $\sqrt{2} < x_1 < x_0$. By an inductive argument,

$$\sqrt{2} < x_{m+1} < x_m < \ldots < x_0.$$

Thus, $\{x_m\}$ is a decreasing sequence which has a lower bound and must converge. Suppose $p = \lim_{m\to\infty} x_m$. Then

$$p = \lim_{m\to\infty}\left(\frac{x_{m-1}}{2} + \frac{1}{x_{m-1}}\right) = \frac{p}{2} + \frac{1}{p}. \quad \text{Thus} \quad p = \frac{p}{2} + \frac{1}{p},$$

which implies that $p = \pm\sqrt{2}$. Since $x_m > \sqrt{2}$ for all m, we have $\lim_{m\to\infty} x_m = \sqrt{2}$.

(b) We have

$$0 < \left(x_0 - \sqrt{2}\right)^2 = x_0^2 - 2x_0\sqrt{2} + 2,$$

so $2x_0\sqrt{2} < x_0^2 + 2$ and $\sqrt{2} < \frac{x_0}{2} + \frac{1}{x_0} = x_1$.

(c) Case 1: $0 < x_0 < \sqrt{2}$, which implies that $\sqrt{2} < x_1$ by part (b). Thus,

$$0 < x_0 < \sqrt{2} < x_{m+1} < x_m < \ldots < x_1 \quad \text{and} \quad \lim_{m\to\infty} x_m = \sqrt{2}.$$

Case 2: $x_0 = \sqrt{2}$, which implies that $x_m = \sqrt{2}$ for all m and $\lim_{m\to\infty} x_m = \sqrt{2}$.
Case 3: $x_0 > \sqrt{2}$, which by part (a) implies that $\lim_{m\to\infty} x_m = \sqrt{2}$.

20. (a) Let

$$g(x) = \frac{x}{2} + \frac{A}{2x}.$$

Note that $g\left(\sqrt{A}\right) = \sqrt{A}$. Also,

$$g'(x) = 1/2 - A/\left(2x^2\right) \text{ if } x \neq 0 \quad \text{and} \quad g'(x) > 0 \text{ if } x > \sqrt{A}.$$

If $x_0 = \sqrt{A}$, then $x_m = \sqrt{A}$ for all m and $\lim_{m\to\infty} x_m = \sqrt{A}$.

20. (a) Let

$$g(x) = \frac{x}{2} + \frac{A}{2x}.$$

Note that $g\left(\sqrt{A}\right) = \sqrt{A}$. Also,

$$g'(x) = 1/2 - A/\left(2x^2\right) \text{ if } x \neq 0 \quad \text{and} \quad g'(x) > 0 \text{ if } x > \sqrt{A}.$$

If $x_0 = \sqrt{A}$, then $x_m = \sqrt{A}$ for all m and $\lim_{m\to\infty} x_m = \sqrt{A}$.
If $x_0 > A$, then

$$x_1 - \sqrt{A} = g(x_0) - g\left(\sqrt{A}\right) = g'(\xi)\left(x_0 - \sqrt{A}\right) > 0.$$

Further,

$$x_1 = \frac{x_0}{2} + \frac{A}{2x_0} < \frac{x_0}{2} + \frac{A}{2\sqrt{A}} = \frac{1}{2}\left(x_0 + \sqrt{A}\right).$$

Thus, $\sqrt{A} < x_1 < x_0$. Inductively,

$$\sqrt{A} < x_{m+1} < x_m < \ldots < x_0$$

and $\lim_{m\to\infty} x_m = \sqrt{A}$ by an argument similar to that in Exercise 19(a).
If $0 < x_0 < \sqrt{A}$, then

$$0 < \left(x_0 - \sqrt{A}\right)^2 = x_0^2 - 2x_0\sqrt{A} + A \quad \text{and} \quad 2x_0\sqrt{A} < x_0^2 + A,$$

which leads to

$$\sqrt{A} < \frac{x_0}{2} + \frac{A}{2x_0} = x_1.$$

Thus

$$0 < x_0 < \sqrt{A} < x_{m+1} < x_m < \ldots < x_1,$$

and by the preceding argument, $\lim_{m\to\infty} x_m = \sqrt{A}$.

(b) If $x_0 < 0$, then $\lim_{m\to\infty} x_m = -\sqrt{A}$.

21. Replace the second sentence in the proof with: "Since g satisfies a Lipschitz condition on $[a, b]$ with a Lipschitz constant $L < 1$, we have, for each n,

$$|p_n - p| = |g(p_{n-1}) - g(p)| \leq L|p_{n-1} - p|."$$

The rest of the proof is the same, with k replaced by L.

22. Let $\varepsilon = (1 - |g'(p)|)/2$. Since g' is continuous at p, there exists a number $\delta > 0$ such that for $x \in [p-\delta, p+\delta]$, we have $|g'(x) - g'(p)| < \varepsilon$. Thus, $|g'(x)| < |g'(p)| + \varepsilon < 1$ for $x \in [p-\delta, p+\delta]$. By the Mean Value Theorem

$$|g(x) - g(p)| = |g'(c)||x - p| < |x - p|,$$

for $x \in [p - \delta, p + \delta]$. Applying the Fixed-Point Theorem completes the problem.

23. With $g(t) = 501.0625 - 201.0625e^{-0.4t}$ and $p_0 = 5.0$, $p_3 = 6.0028$ is within 0.01 s of the actual time.

*24. Since g' is continuous at p and $|g'(p)| > 1$, by letting $\epsilon = |g'(p)| - 1$ there exists a number $\delta > 0$ such that $|g'(x) - g'(p)| < |g'(p)| - 1$ whenever $0 < |x - p| < \delta$. Hence, for any x satisfying $0 < |x - p| < \delta$, we have

$$|g'(x)| \geq |g'(p)| - |g'(x) - g'(p)| > |g'(p)| - (|g'(p)| - 1) = 1.$$

If p_0 is chosen so that $0 < |p - p_0| < \delta$, we have by the Mean Value Theorem that

$$|p_1 - p| = |g(p_0) - g(p)| = |g'(\xi)||p_0 - p|,$$

for some ξ between p_0 and p. Thus, $0 < |p - \xi| < \delta$ so $|p_1 - p| = |g'(\xi)||p_0 - p| > |p_0 - p|$.

Exercise Set 2.3, page 75

*1. $p_2 = 2.60714$

2. $p_2 = -0.865684$; If $p_0 = 0$, $f'(p_0) = 0$ and p_1 cannot be computed.

*3. (a) 2.45454
(b) 2.44444
(c) Part (a) is better.

4. (a) -1.25208
(b) -0.841355

5. (a) For $p_0 = 2$, we have $p_5 = 2.69065$.
(b) For $p_0 = -3$, we have $p_3 = -2.87939$.
*(c) For $p_0 = 0$, we have $p_4 = 0.73909$.
(d) For $p_0 = 0$, we have $p_3 = 0.96434$.

6. (a) For $p_0 = 1$, we have $p_8 = 1.829384$.
(b) For $p_0 = 1.5$, we have $p_4 = 1.397748$.
(c) For $p_0 = 2$, we have $p_4 = 2.370687$; and for $p_0 = 4$, we have $p_4 = 3.722113$.
(d) For $p_0 = 1$, we have $p_4 = 1.412391$; and for $p_0 = 4$, we have $p_5 = 3.057104$.
(e) For $p_0 = 1$, we have $p_4 = 0.910008$; and for $p_0 = 3$, we have $p_9 = 3.733079$.
(f) For $p_0 = 0$, we have $p_4 = 0.588533$; for $p_0 = 3$, we have $p_3 = 3.096364$; and for $p_0 = 6$, we have $p_3 = 6.285049$.

7. Using the endpoints of the intervals as p_0 and p_1, we have:
(a) $p_{11} = 2.69065$
(b) $p_7 = -2.87939$
*(c) $p_6 = 0.73909$

(d) $p_5 = 0.96433$

8. Using the endpoints of the intervals as p_0 and p_1, we have:
(a) $p_7 = 1.829384$
(b) $p_9 = 1.397749$

(d) $p_5 = 0.96433$

8. Using the endpoints of the intervals as p_0 and p_1, we have:

   (a) $p_7 = 1.829384$
   (b) $p_9 = 1.397749$
   (c) $p_6 = 2.370687; p_7 = 3.722113$
   (d) $p_8 = 1.412391; p_7 = 3.057104$
   (e) $p_6 = 0.910008; p_{10} = 3.733079$
   (f) $p_6 = 0.588533; p_5 = 3.096364; p_5 = 6.285049$

9. Using the endpoints of the intervals as p_0 and p_1, we have:

   (a) $p_{16} = 2.69060$
   (b) $p_6 = -2.87938$
   *(c) $p_7 = 0.73908$
   (d) $p_6 = 0.96433$

10. Using the endpoints of the intervals as p_0 and p_1, we have:

    (a) $p_8 = 1.829383$
    (b) $p_9 = 1.397749$
    (c) $p_6 = 2.370687; p_8 = 3.722112$
    (d) $p_{10} = 1.412392; p_{12} = 3.057099$
    (e) $p_7 = 0.910008; p_{29} = 3.733065$
    (f) $p_9 = 0.588533; p_5 = 3.096364; p_5 = 6.285049$

11. (a) Newton's method with $p_0 = 1.5$ gives $p_3 = 1.51213455$.
    The Secant method with $p_0 = 1$ and $p_1 = 2$ gives $p_{10} = 1.51213455$.
    The Method of False Position with $p_0 = 1$ and $p_1 = 2$ gives $p_{17} = 1.51212954$.

    (b) Newton's method with $p_0 = 0.5$ gives $p_5 = 0.976773017$.
    The Secant method with $p_0 = 0$ and $p_1 = 1$ gives $p_5 = 10.976773017$.
    The Method of False Position with $p_0 = 0$ and $p_1 = 1$ gives $p_5 = 0.976772976$.

12. (a) We have

	Initial Approximation	Result	Initial Approximation	Result
Newton's	$p_0 = 1.5$	$p_4 = 1.41239117$	$p_0 = 3.0$	$p_4 = 3.05710355$
Secant	$p_0 = 1, p_1 = 2$	$p_8 = 1.41239117$	$p_0 = 2, p_1 = 4$	$p_{10} = 3.05710355$
False Position	$p_0 = 1, p_1 = 2$	$p_{13} = 1.41239119$	$p_0 = 2, p_1 = 4$	$p_{19} = 3.05710353$

Secant	$p_0 = 1, p_1 = 2$	$p_8 = 1.41239117$	$p_0 = 2, p_1 = 4$	$p_{10} = 3.05710355$
False Position	$p_0 = 1, p_1 = 2$	$p_{13} = 1.41239119$	$p_0 = 2, p_1 = 4$	$p_{19} = 3.05710353$

(b) We have

	Initial Approximation	Result	Initial Approximation	Result
Newton's	$p_0 = 0.25$	$p_4 = 0.206035120$	$p_0 = 0.75$	$p_4 = 0.681974809$
Secant	$p_0 = 0, p_1 = 0.5$	$p_9 = 0.206035120$	$p_0 = 0.5, p_1 = 1$	$p_8 = 0.681974809$
False Position	$p_0 = 0, p_1 = 0.5$	$p_{12} = 0.206035125$	$p_0 = 0.5, p_1 = 1$	$p_{15} = 0.681974791$

*13. For $p_0 = 1$, we have $p_5 = 0.589755$. The point has the coordinates $(0.589755, 0.347811)$.

14. For $p_0 = 2$, we have $p_2 = 1.866760$. The point is $(1.866760, 0.535687)$.

15. The equation of the tangent line is

$$y - f(p_{n-1}) = f'(p_{n-1})(x - p_{n-1}).$$

To complete this problem, set $y = 0$ and solve for $x = p_n$.

*16. Newton's method gives $p_{15} = 1.895488$, for $p_0 = \frac{\pi}{2}$; and $p_{19} = 1.895489$, for $p_0 = 5\pi$. The sequence does not converge in 200 iterations for $p_0 = 10\pi$. The results do not indicate the fast convergence usually associated with Newton's method.

17. (a) For $p_0 = -1$ and $p_1 = 0$, we have $p_{17} = -0.04065850$, and for $p_0 = 0$ and $p_1 = 1$, we have $p_9 = 0.9623984$.

(b) For $p_0 = -1$ and $p_1 = 0$, we have $p_5 = -0.04065929$, and for $p_0 = 0$ and $p_1 = 1$, we have $p_{12} = -0.04065929$.

(c) For $p_0 = -0.5$, we have $p_5 = -0.04065929$, and for $p_0 = 0.5$, we have $p_{21} = 0.9623989$.

18. (a) The Bisection method yields $p_{10} = 0.4476563$.

(b) The method of False Position yields $p_{10} = 0.442067$.

(c) The Secant method yields $p_{10} = -195.8950$.

*19. This formula involves the subtraction of nearly equal numbers in both the numerator and denominator if p_{n-1} and p_{n-2} are nearly equal.

20. Newton's method for the various values of p_0 gives the following results.

(a) $p_8 = -1.379365$

(b) $p_7 = -1.379365$

(c) $p_7 = 1.379365$

(d) $p_7 = -1.379365$

(e) $p_7 = 1.379365$

(f) $p_8 = 1.379365$

21. Newton's method for the various values of p_0 gives the following results.

(a) $p_0 = -10, p_{11} = -4.30624527$
(b) $p_0 = -5, p_5 = -4.30624527$
(c) $p_0 = -3, p_5 = 0.824498585$
(d) $p_0 = -1, p_4 = -0.824498585$
(e) $p_0 = 0$, p_1 cannot be computed because $f'(0) = 0$
(f) $p_0 = 1, p_4 = 0.824498585$
(g) $p_0 = 3, p_5 = -0.824498585$
(h) $p_0 = 5, p_5 = 4.30624527$
(i) $p_0 = 10, p_{11} = 4.30624527$

*22. The required accuracy is met in 7 iterations of Newton's method.

*23. For $f(x) = \ln(x^2 + 1) - e^{0.4x} \cos \pi x$, we have the following roots.

(a) For $p_0 = -0.5$, we have $p_3 = -0.4341431$.
(b) For $p_0 = 0.5$, we have $p_3 = 0.4506567$.
For $p_0 = 1.5$, we have $p_3 = 1.7447381$.
For $p_0 = 2.5$, we have $p_5 = 2.2383198$.
For $p_0 = 3.5$, we have $p_4 = 3.7090412$.
(c) The initial approximation $n - 0.5$ is quite reasonable.
(d) For $p_0 = 24.5$, we have $p_2 = 24.4998870$.

24. We have $\lambda \approx 0.100998$ and $N(2) \approx 2{,}187{,}950$.

25. The two numbers are approximately 6.512849 and 13.487151.

*26. The minimal annual interest rate is 6.67%.

27. The borrower can afford to pay at most 8.10%.

*28. (a) $\frac{1}{3}e, t = 3$ hours
(b) 11 hours and 5 minutes
(c) 21 hours and 14 minutes

*29. (a) First define the function by
$f := x- > 3^{3x+1} - 7 \cdot 5^{2x}$

$$f := x \to 3^{(3x+1)} - 7\ 5^{2x}$$

solve$(f(x) = 0, x)$

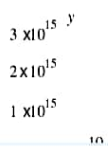
$$-\frac{\ln(3/7)}{\ln(27/25)}$$

fsolve$(f(x) = 0, x)$

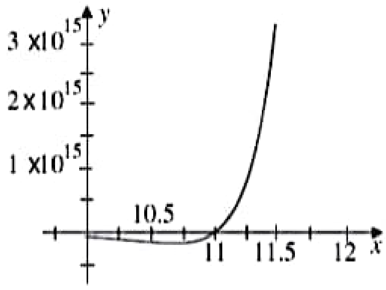
$$\mathrm{fsolve}(3^{(3x+1)} - 7\ 5^{(2x)} = 0, x)$$

The procedure *solve* gives the exact solution, and *fsolve* fails because the negative x-axis is an asymptote for the graph of $f(x)$.

(b) Using the Maple command *plot*$(\{f(x)\}, x = 10.5..11.5)$ produces the following graph.



(b) Using the Maple command *plot*($\{f(x)\}, x = 10.5..11.5$) produces the following graph.



(c) Define $f'(x)$ using
$fp := x- > (D)(f)(x)$

$$fp := x \to 3\ 3^{(3x+1)} \ln(3) - 14\ 5^{(2x)} \ln(5)$$

$Digits := 18; p0 := 11$

$$\text{Digits} := 18$$
$$p0 := 11$$

for i from 1 *to* 5 *do*
$p1 := evalf(p0 - f(p0)/fp(p0))$
$err := abs(p1 - p0)$
$p0 := p1$
od
The results are given in the following table.

i	p_i	$\lvert p_i - p_{i-1}\rvert$
1	11.0097380401552503	.0097380401552503
2	11.0094389359662827	.0002991041889676
3	11.0094386442684488	$.2916978339\ 10^{-6}$
4	11.0094386442681716	$.2772\ 10^{-2}$
5	11.0094386442681716	0

(d) We have $3^{3x+1} = 7 \cdot 5^{2x}$. Taking the natural logarithm of both sides gives

$$(3x + 1)\ln 3 = \ln 7 + 2x \ln 5.$$

Thus

$$3x \ln 3 - 2x \ln 5 = \ln 7 - \ln 3, \quad x(3 \ln 3 - 2 \ln 5) = \ln \frac{7}{3},$$

and

$$x = \frac{\ln 7/3}{\ln 27 - \ln 25} = \frac{\ln 7/3}{\ln 27/25} = -\frac{\ln 3/7}{\ln 27/25}.$$

This agrees with part (a).

30. (a) $solve(2^{x^2} - 3 \cdot 7^{(x+1)}, x)$ fails and $fsolve(2^{x^2} - 3 \cdot 7^{(x+1)}, x)$ returns -1.118747530.
    (b) $plot(2^{x^2} - 3 \cdot 7^{(x+1)}, x = -2..4)$ shows there is also a root near $x = 4$.
    (c) With $p_0 = 1$, $p_1 = -1.1187475303988963$ is accurate to 10^{-16}; with $p_0 = 4$, $p_6 = 3.9261024524565005$ is accurate to 10^{-16}
    (d) The roots are
$$\frac{\ln(7) \pm \sqrt{[\ln(7)]^2 + 4\ln(2)\ln(4)}}{2\ln(2)}.$$
31. We have $P_L = 265816$, $c = -0.75658125$, and $k = 0.045017502$. The 1980 population is $P(30) = 222{,}248{,}320$, and the 2010 population is $P(60) = 252{,}967{,}030$.
32. $P_L = 290228$, $c = 0.6512299$, and $k = 0.03020028$;
The 1980 population is $P(30) = 223,069,210$, and the 2010 population is $P(60) = 260,943,806$.
33. Using $p_0 = 0.5$ and $p_1 = 0.9$, the Secant method gives $p_5 = 0.842$.
34. (a) We have, approximately,
$$A = 17.74, \quad B = 87.21, \quad C = 9.66, \quad \text{and} \quad E = 47.47$$
With these values we have
$$A \sin\alpha \cos\alpha + B \sin^2\alpha - C\cos\alpha - E\sin\alpha = 0.02.$$
    (b) Newton's method gives $\alpha \approx 33.2°$.

Exercise Set 2.4, page 85

1. *(a) For $p_0 = 0.5$, we have $p_{13} = 0.567135$.
    (b) For $p_0 = -1.5$, we have $p_{23} = -1.414325$.
    (c) For $p_0 = 0.5$, we have $p_{22} = 0.641166$.
    (d) For $p_0 = -0.5$, we have $p_{23} = -0.183274$.
2. (a) For $p_0 = 0.5$, we have $p_{15} = 0.739076589$.
    (b) For $p_0 = -2.5$, we have $p_9 = -1.33434594$.
    (c) For $p_0 = 3.5$, we have $p_5 = 3.14156793$.
    (d) For $p_0 = 4.0$, we have $p_{44} = 3.37354190$.
3. Modified Newton's method in Eq. (2.11) gives the following:
    *(a) For $p_0 = 0.5$, we have $p_3 = 0.567143$.
    (b) For $p_0 = -1.5$, we have $p_2 = -1.414158$.
    (c) For $p_0 = 0.5$, we have $p_3 = 0.641274$.
    (d) For $p_0 = -0.5$, we have $p_5 = -0.183319$.

4. (a) For $p_0 = 0.5$, we have $p_4 = 0.739087439$.
    (b) For $p_0 = -2.5$, we have $p_{53} = -1.33434594$.
    (c) For $p_0 = 3.5$, we have $p_5 = 3.14156793$.
    (d) For $p_0 = 4.0$, we have $p_3 = -3.72957639$.
5. Newton's method with $p_0 = -0.5$ gives $p_{13} = -0.169607$. Modified Newton's method in Eq. (2.11) with $p_0 = -0.5$ gives $p_{11} = -0.169607$.
6. *(a) Since
$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1,$$

(c) For $p_0 = 0.5$, we have $p_3 = 0.641274$.
(d) For $p_0 = -0.5$, we have $p_5 = -0.183319$.

4. (a) For $p_0 = 0.5$, we have $p_4 = 0.739087439$.
   (b) For $p_0 = -2.5$, we have $p_{53} = -1.33434594$.
   (c) For $p_0 = 3.5$, we have $p_5 = 3.14156793$.
   (d) For $p_0 = 4.0$, we have $p_3 = -3.72957639$.
5. Newton's method with $p_0 = -0.5$ gives $p_{13} = -0.169607$. Modified Newton's method in Eq. (2.11) with $p_0 = -0.5$ gives $p_{11} = -0.169607$.
6. *(a) Since

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1,$$

   we have linear convergence. To have $|p_n - p| < 5 \times 10^{-2}$, we need $n \geq 20$.

   (b) Since

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = 1,$$

   we have linear convergence. To have $|p_n - p| < 5 \times 10^{-2}$, we need $n \geq 5$.
7. (a) For $k > 0$,

$$\lim_{n\to\infty} \frac{|p_{n+1} - 0|}{|p_n - 0|} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^k}}{\frac{1}{n^k}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^k = 1,$$

   so the convergence is linear.

   (b) We need to have $N > 10^{m/k}$.

*8. (a) Since

$$\lim_{n\to\infty} \frac{|p_{n+1} - 0|}{|p_n - 0|^2} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{(10^{-2^n})^2} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{10^{-2^{n+1}}} = 1,$$

   the sequence is quadratically convergent.

   (b) We have

$$\lim_{n\to\infty} \frac{|p_{n+1} - 0|}{|p_n - 0|^2} = \lim_{n\to\infty} \frac{10^{-(n+1)^k}}{\left(10^{-n^k}\right)^2} = \lim_{n\to\infty} \frac{10^{-(n+1)^k}}{10^{-2n^k}}$$
$$= \lim_{n\to\infty} 10^{2n^k - (n+1)^k} = \lim_{n\to\infty} 10^{n^k\left(2 - \left(\frac{n+1}{n}\right)^k\right)} = \infty,$$

   so the sequence $p_n = 10^{-n^k}$ does not converge quadratically.

9. Typical examples are
   (a) $p_n = 10^{-3^n}$
   (b) $p_n = 10^{-\alpha^n}$

*10. Suppose $f(x) = (x - p)^m q(x)$. Since

$$g(x) = x - \frac{m(x-p)q(x)}{mq(x) + (x-p)q'(x)},$$

   we have $g'(p) = 0$.