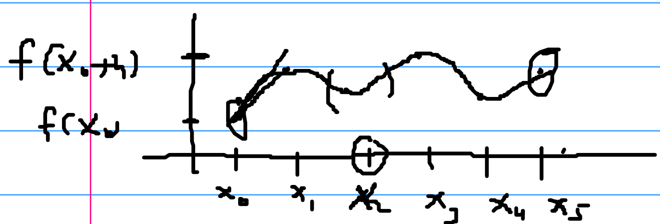
Numerical Differentiation 4.1

$x_0, \; x_1, \; x_2, \; - \text{---} \; x_n$

$h = x_{i+1} - x_i \;, \quad i = 0, \; - - n-1$

$f'(x_0) \qquad , \qquad P(x) \cong f(x)$

$P'(x_0)$



$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = f'(x_0)$$

$x_1 = x_0 + h$

$$f'(x_0) \simeq \frac{f(x_0+h) - f(x_0)}{h}$$

$$\frac{f(x_1) - f(x_0)}{h}$$

If $a = x_0, \; x_1 \; - - \; x_n = b$, $f^{(n+1)} \in C[a, b]$, the Lagrange polynomial is

$$P(x) = \sum_{k=0}^{n} f(x_k) L_k(x) + \frac{(x - x_0)(x - x_1) - - (x - x_n)}{(n+1)!} f^{(n+1)}(\xi(x))$$

After differentiating the polynomial we have $\qquad \xi(x)$ is in between x_0 & x_n

$$P'(x) = \sum_{k=0}^{n} f(x_k) L'_k(x) + D_x \left[\frac{(x - x_0)(x - x_1) - \cdot (x - x_n)}{(n+1)!} \right] f^{(n+1)}(\xi(x))$$

$$+ \frac{(x-x_0)(x-x_1)\cdots(x-x_n)}{(n+1)!} D_x f^{(n+1)}(\xi(x)) \quad \text{—} (1)$$

$x_0, x_1, x_2, \cdots x_n$

$$D_x\left[(x-x_0)(x-x_1)\cdots(x-x_n)\right]$$

$$= (x-x_1)\cdots(x-x_n) + (x-x_0)(x-x_2)\cdots(x-x_n) + \cdots$$

$$+ \cdots (x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

$$D_x\left[(x-x_0)(x-x_1)\cdots(x-x_n)\right]\Big|_{x=x_j} = \prod_{\substack{k=0 \\ k\neq j}}^{n} (x_j - x_k) \quad \text{—} (2)$$

Evaluate (1) for $x = x_j$ and use (2) in (1)

$$P'(x_j) = \sum_{k=0}^{n} f(x_k) L_k'(x_j) + \prod_{\substack{k=0 \\ j\neq k}}^{n} (x_j - x_k) \frac{f^{(n+1)}(\xi(x_j))}{(n+1)!} \quad \text{—} (3)$$

This is called $(n+1)$ points formula

Consider (3) for only x_0, x_1, x_2

$$P'(x_j) = \sum_{k=0}^{2} f(x_k) L_k'(x_j), \quad j = 0, 1, 2$$

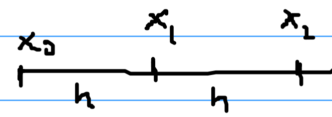
where the Lagrange Coefficients are

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{x^2 - x_2 x - x_1 x + x_1 x_2}{(x_0-x_1)(x_0-x_2)}$$

$$L_0'(x) = \frac{2x - x_2 - x_1}{(x_0-x_1)(x_0-x_2)}$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}, \quad L_1'(x) = \frac{2x - x_0 - x_2}{(x_1-x_0)(x_1-x_2)}$$

$$L_2'(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{2x - x_0 - x_1}{(x_2-x_0)(x_2-x_1)}$$



For $j=0$, $\quad x_0 \quad x_1 = x_0 + h, \quad x_2 = x_0 + 2h$

$$L_0'(x) = \frac{2x - x_1 - x_2}{(x_0-x_1)(x_0-x_2)}, \quad L_0'(x_0) = \frac{2x_0 - x_1 - x_2}{(x_0-x_1)(x_0-x_2)}$$

$$L_0'(x_0) = \frac{x_0 - x_1 + x_0 - x_2}{(x_0-x_1)(x_0-x_2)} = \frac{(-h)+(-2h)}{-h(-2h)} = \frac{-3h}{2h^2} = \frac{-3}{2h}$$

$$L_1'(x) = \frac{2x - x_0 - x_2}{(x_1-x_0)(x_1-x_2)}, \quad L_1'(x_0) = \frac{2x_0 - x_0 - x_2}{(x_1-x_0)(x_1-x_2)} = \frac{x_0 - x_2}{(x_1-x_0)(x_1-x_2)}$$

$$L_1'(x_0) = \frac{-2h}{h(-h)} = \frac{2}{h}$$

$$L_2'(x) = \frac{2x - x_0 - x_1}{(x_2-x_0)(x_2-x_1)}, \quad L_2'(x_0) = \frac{2x_0 - x_0 - x_1}{(x_2-x_0)(x_2-x_1)} = \frac{x_0 - x_1}{2h(h)}$$

$$= \frac{-h}{2h^2} = \frac{-1}{2h}$$

$$f(x_0) = f(x_0)L_0'(x_0) + f(x_1)L_1'(x_0) + f(x_2)L_2'(x_0)$$

$$= f(x_0)\left(\frac{-3}{2h}\right) + f(x_1)\left(\frac{2}{h}\right) + f(x_2)\left(\frac{-1}{2h}\right)$$

$$f'(x_0) = \frac{1}{h}\left[\frac{-3}{2}f(x_0) + 2f(x_1) - \frac{1}{2}f(x_2)\right] + \frac{h^2}{3}f^{3}(\xi_0) \quad a$$

After repeating the same steps for $x_j = x_1$ we have

$$f'(x_1) = \frac{1}{h}\left[-\frac{1}{2} f(x_0) + \frac{1}{2} f(x_2)\right] - \frac{h^2}{6} f^{(3)}(\xi_1) \quad \text{— (B)}$$

Similarly for $x_j = x_2$ we get

$$f'(x_2) = \frac{1}{h}\left[\frac{1}{2} f(x_0) - 2 f(x_1) + \tfrac{3}{2} f(x_2)\right] + \frac{h^2}{3} f^{(3)}(\xi_2) \quad \text{— (C)}$$

Replace $x_1 = x_0 + h$, $x_2 = x_0 + 2h$ in (A), (B) and (C)

$$f'(x_0) = \frac{1}{h}\left[2 f(x_0+h) - \tfrac{3}{2} f(x_0) - \tfrac{1}{2} f(x_0+2h)\right] \quad \text{— (D)}$$

$$f'(x_0+h) = \frac{1}{h}\left[-\frac{1}{2} f(x_0) + \tfrac{1}{2} f(x_0+2h)\right] \quad \text{— (E)}$$

$$\underline{f'(x_0+2h)} = \tfrac{1}{h}\left[\tfrac{1}{2} f(x_0) - 2 f(x_0+h) + \tfrac{3}{2} f(x_0+2h)\right] \quad \text{— (F)}$$

$$\text{(E)} \Rightarrow f'(x_0) = \tfrac{1}{h}\left[-\tfrac{1}{2} f(x_0-h) + \tfrac{1}{2} f(x_0+h)\right] - \frac{h^2}{6} f^{(3)}(\xi_0)$$

This is called Three-Point Midpoint formula

$$\text{(F)} \Rightarrow f'(x_0) = \tfrac{1}{h}\left[\frac{1}{2} f(x_0-2h) - 2 f(x_0-h) + \tfrac{3}{2} f(x_0)\right] \quad \text{— (G)}$$

x_0, x_1, x_2 - — x_n

Formula (G) & (D) are called Three-point Endpoint formulas