Convergence of a sequence

All iterative methods we discuss in this course generate a sequence $\{p_n\}_{n=1}^{\infty}$ of successive approximations that converges to root p of a function f. Therefore, it is very important that we understand convergence behaviour of a sequence in general. We will discuss two related concepts by the name of 'Rate of convergence' and 'Order of Convergence'

Rate of convergence

Suppose a sequence α_n converges to α and an another sequence β_n converges to '0', then, if α_n and β_n are related by the inequality

$$|\alpha_n - \alpha| \leq C\beta_n \quad , \quad C > 0 \quad \text{as } n \to \infty,$$

we say that α_n converges to α with rate of convergence $O(\beta_n)$. This can also be written as $\alpha_n = \alpha + O(\beta_n)$.

Intuitively, this means that α_n has similar convergence behaviour/speed as β_n. Usually, β_n is some known sequence such as $\beta_n = \frac{1}{n^p}$, where $p > 0$.

Example

$\alpha_n = \dfrac{n+1}{n^2}$ converges to $\alpha = 0$, find its rate of convergence.

$$|\alpha_n - \alpha| = \left|\frac{n+1}{n^2} - 0\right| = \left|\frac{n+1}{n^2}\right| \leq \left|\frac{n+n}{n^2}\right| = \frac{2n}{n^2} = \frac{2}{n}$$

$$|\alpha_n - \alpha| \leq 2\left(\frac{1}{n}\right) \quad \text{Here } \beta_n = \frac{1}{n}$$

$\alpha_n = \alpha + O\left(\frac{1}{n}\right)$ $\boxed{\text{This means that convergence behaviour of } \alpha_n \text{ is similar to } \left(\tfrac{1}{n}\right).}$

Order of Convergence.

Order of convergence is slightly different concept from the rate of convergence. In the rate of convergence, we relate behaviour of a sequence with some other sequence, while in order of convergence we quantify the speed of convergence. In essence, both concepts provide information about the convergence speed of a given sequence.

A sequence P_n that converges to P is said to have order of convergence α if

$$\lim_{n\to\infty} \frac{|P_{n+1}-P|}{|P_n-P|^{\alpha}} = \lambda,$$ with asymptotic constant λ.

~~Here~~

Mostly, we will encounter the following two cases.

1. If $\alpha=1$ and $\lambda<1$, the sequence is called linearly convergent.

2. If $\alpha=2$, the sequence is called quadrically convergent.

Example

Consider $P_n = \frac{1}{2n}$ which converges to '0'. Find the order of convergence of P_n. We will calculate the following expression

$$\lim_{n\to\infty} \frac{|P_{n+1}-P|}{|P_n-P|^{\alpha}}$$ and see for which value of α we get a constant number.

$$\lim_{n\to\infty} \frac{\left|\frac{1}{2^{n+1}} - 0\right|}{\left|\frac{1}{2^n} - 0\right|^{\alpha}} = \lim_{n\to\infty} \frac{\frac{1}{2^{n+1}}}{\left(\frac{1}{2^n}\right)^{\alpha}} = \frac{1}{2}, \quad \text{for } \alpha = 1$$

This shows that $P_n = \frac{1}{2^n}$ converges to $P=0$ linearly.

In order to realize and have a feel about the linear convergence we calculate few consecutive value of the error term and see how it reduces.

n	1	2	3	4	5	6	7
$\lvert P_n - P\rvert$	0.5	0.25	0.125	0.0625	0.03125	0.015625	0.0078125

From the above table one can see that error is reducing by the factor of $\frac{1}{2}$, i.e, error is reducing linearly and this kind of convergence is considered very slow convergence.

Example

Show that the sequence $P_n = 10^{-2^n}$ converges to '0' quadratically.

Consider

$$\lim_{n\to\infty} \frac{|P_{n+1} - P|}{|P_n - P|^{\alpha}} = \lim_{n\to\infty} \frac{\left|10^{-2^{n+1}}\right|}{\left|10^{-2^n}\right|^{\alpha}} = 1 \quad \text{for } \alpha = 2.$$

The above expression ~~gi~~ yields a constant value for $\alpha = 2$, so $P_n = 10^{-2^n}$ converges to $P=0$ quadratically.

Like the above example we calculate few error terms and see how the error is reducing.

n	1	2	3	4	5	7
$\lvert P_n - P\rvert$	0.01	0.0001	10^{-8}	10^{-16}	10^{-16}	10^{-32}

From the above table we see that error is reducing with the power of 2, i.e, error is reducing quadritically. This is considered very fast convergence.