Linear Algebra
Worksheet 1

1. Does the linear homogeneous system have any nontrivial solutions?

$$\begin{aligned} 3x_1 + x_2 - 9x_3 &= 0 \\ x_1 + x_2 - 5x_3 &= 0 \\ 2x_1 + x_2 - 7x_3 &= 0 \end{aligned}$$

2. Find the general solution of the homogenous system $A\mathbf{x} = \mathbf{0}$ where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 & 1 & 4 \\ 3 & 7 & 7 & 3 & 13 \\ 2 & 5 & 5 & 2 & 9 \end{bmatrix}.$$

3. Given

$$\mathbf{x}_1 = \begin{Bmatrix} -1 \\ 2 \\ 3 \end{Bmatrix}, \quad \mathbf{x}_2 = \begin{Bmatrix} 3 \\ 4 \\ 2 \end{Bmatrix},$$

$$\mathbf{x} = \begin{Bmatrix} 2 \\ 6 \\ 6 \end{Bmatrix}, \quad \mathbf{y} = \begin{Bmatrix} -9 \\ -2 \\ 5 \end{Bmatrix}$$

(a) Is $\mathbf{x} \in \text{Span}(\mathbf{x}_1, \mathbf{x}_2)$?

(b) Is $\mathbf{y} \in \text{Span}(\mathbf{x}_1, \mathbf{x}_2)$?

Prove your answers.

4. For each of the systems of equations that follow, use Gaussian elimination to obtain solution. Also, list down the lead and free vairables, if any.

(a)

$$\begin{aligned} x_1 - 2x_2 &= 3 \\ 2x_1 + x_2 &= 1 \\ -5x_1 + 8x_2 &= 4 \end{aligned}$$

(b)

$$\begin{aligned} -x_1 + 2x_2 - x_3 &= 2 \\ -2x_1 + 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 + 2x_3 &= 5 \\ -3x_1 + 8x_2 + 5x_3 &= 17 \end{aligned}$$

(c)

$$\begin{aligned} x_1 + 2x_2 - 3x_3 + x_4 &= 1 \\ -x_1 - x_2 + 4x_3 - x_4 &= 6 \\ -2x_1 - 4x_2 + 7x_3 - x_4 &= 1 \end{aligned}$$

5. .

Given

$$R = \begin{Bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{Bmatrix}$$

show that R is nonsingular and $R^{-1} = R^T$.

(A matrix is called non-singular if its inverse exist)

6. Let $\mathbf{A}$ be an idempotent matrix, i.e., $\mathbf{A}^2 = \mathbf{A}$.

   (a) Show that $I - A$ is also idempotent.

   (b) Show that $I + A$ is nonsingular and $(I + A)^{-1} = I - \frac{1}{2}A$

7. .

   Find the inverse of $\mathbf{A} = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 1 & 0 \\ -2 & 0 & -7 \end{bmatrix}$ if it exists.

8. Given $\mathbf{A}$ and $\mathbf{C}$ below, show that $\mathbf{C}$ is the inverse of $\mathbf{A}$.

   (a)

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 0 \\ -1 & 2 & -2 \\ -2 & 6 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} -14 & -3 & -6 \\ -5 & -1 & -2 \\ 2 & 0 & 1 \end{bmatrix}$$

   (b) Use the result of part(a) to solve the linear system $\mathbf{Ax} = \mathbf{b}$ if

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 0 \\ -1 & 2 & -2 \\ -2 & 6 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}.$$

9. Prove that if A is non-singular then A^T is non-singular and $(A^T)^{-1} = (A^{-1})^T$.

10. .

Let U be an $n \times n$ upper triangular matrix with nonzero diagonal entries.

(a) Explain why U must be nonsingular.

(b) Explain why U^{-1} must be upper triangular.

11. Determine whether the following sets form subspaces of $\Re^3$:
   (a) $\{(x_1, x_2, x_3)^T \mid x_1 + x_3 = 1\}$
   (b) $\{(x_1, x_2, x_3)^T \mid x_1 = x_2 = x_3\}$

12. Let

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$$

   (a) Show that $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ are linearly dependent.
   (b) Show that $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent.
   (c) What is the dimension of Span($\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$)?
   (d) Give a geometric description of Span($\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$).

13. Let $\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k\}$ be a spanning set for a euclidean space V.
   (a) If we add another vector, $\mathbf{x}_{k+1}$, to the set, will we still have a spanning set? Explain.
   (b) If we delete one of the vectors, say $\mathbf{x}_k$, from the set, will we still have a spanning set? Explain.

14. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be two vectors in a euclidean space V. Show that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent if and only if one of the vectors is a scalar multiple of the other.

15. Let A be an $m \times n$ matrix. Show that if A has linearly independent column vectors, then the null space $N(A) = 0$.

16. .

   Let

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix}$$

   Find a basis for the row space of A and a basis for $N(A)$. Verify that $\dim N(A) = n - r$.

   (Use Rank theorem)

17. Find the coloumn space of the following matrix:

$$\begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{bmatrix}$$

18. Let $\mathbf{A}$ and $\mathbf{B}$ be 6×5 matrices. If dim N$(\mathbf{A}) = 2$, what is the rank of $\mathbf{A}$? If the rank of $\mathbf{B}$ is 4, what is the dimension of N$(\mathbf{B})$?