Lecture 28

Euler's Method

$$y' = f(t, y) \qquad y(a) = \alpha$$

$$y = f(x) \qquad a \le t \le b$$

$t_0 = a \quad t_1 \quad t_2 \quad \ldots \quad b = t_n$

t_0	$w(t_0)$
t_1	$w(t_1)$
t_2	

$y(t_0) \simeq w(t_0)$

Analytic Solution

↳ approximate Solution

$$y(t_i) \simeq w(t_i)$$

$$w(t_0) = w_0 = y(a) = \alpha$$

$$w_{i+1} = w_i + h f(t_i, w_i)$$

$$i = 0, 1, 2 \ldots N \qquad \frac{b-a}{N} = h$$

Example

$$y' = y - t^2 + 1 \qquad 0 \le t \le 2$$

$$f(t, y) = y - t^2 + 1 \qquad y(0) = 0.5$$

$$w_0(t_0) = 0.5$$

t_0	t_1	t_2	t_3	t_4
0	0.5	1	1.5	2

$$w_{i+1} = w_i + 0.5\, f(t_i, w_i) \qquad h = \frac{b-a}{4} = \frac{2-0}{4} = 1/2$$

$$w_{i+1} = w_i + 0.5\,(w_i - t_i^2 + 1)$$

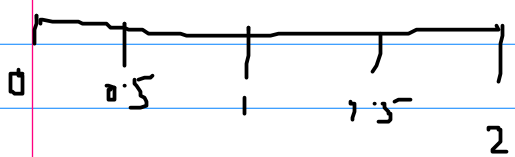
$$w_1 = w_0 + 0.5\,(w_0 - t_0^2 + 1)$$

$$w_1 = (0.5) + (0.5)\,(0.5 - (0)^2 + 1)$$

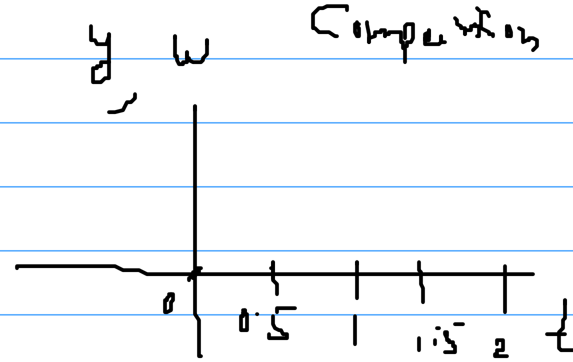
$w_2 =$

$w_3 =$

w_4



Comparison



$w(0) \approx y(0)$ $\qquad$ $|y(0) - w(0)|$

$w(0.5) \approx y(0.5)$ $\qquad$ $|y(0.5) - w(0.5)|$

$$|y(t_i) - w(t_i)|$$

Error bound in Euler formula

$$|y(t_i) - w_i| \le \frac{hM}{2L} \left| e^{L(t_i - a)} - 1 \right|$$

$y' = f(t, y)$, $\quad y(a) = \alpha \qquad a \le t \le b$

$h = \dfrac{b-a}{N} \qquad |y''(t)| \le M,$

f is Lipschitz function and L is Lipschitz constant.

Example

$$y' = \overbrace{y - t^2 + 1}^{f} \qquad y(0) = 0.5$$

Analytic solution is

$$y(t) = (t+1)^2 - 0.5e^t$$

$$\left|\frac{\partial f(t,y)}{\partial y}\right| \le L, \qquad f(t,y) = y - t^2 + 1$$

$$\frac{\partial f(t,y)}{\partial y} = 1$$

$$\left|\frac{\partial f(t,y)}{\partial y}\right| = 1, \qquad L = 1$$

$$y' = 2(t+1) - 0.5e^t$$

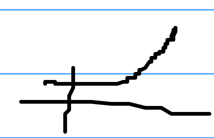
$$y'' = 2 - 0.5e^t, \qquad |y''| = |2 - 0.5e^t|$$

$|a+b| \le |a| + |b|$

$$|y''(t)| \le |2| + 0.5e^t$$

$$\le 2 + 0.5e^t \qquad 0 \le t \le 2$$

$$\le 2 + 0.5e^2$$



$$|y''(t)| \le 2 + 0.5e^2$$

$$\boxed{M = 2 + 0.5e^2}$$

$N = 4$

$$\frac{b-a}{N} = h = \frac{2-0}{4} = h = 0.5$$

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w_i , $y(t_i) = (t_i+1)^2 - 0.5e^{t_i}$

$|y_i - w_i| =$ exact error

$$|y_i - w_i| \le \frac{hM}{2L} \left| e^{L(t_i - a)} - 1 \right|$$

Higher order formula (5.4) 5.3

5.1

5.2

Mid point Method

$f(t, y)$

$w_0 = \alpha$

$$w_{i+1} = w_i + h f\left(t_i + \frac{h}{2},\ w_i + \frac{h}{2} f(t_i, w_i)\right)$$

$i = 0, \ldots N-1$

Modified Euler's Method

$$w_0 = \alpha$$

$$w_{i+1} = w_i + \frac{h}{2}\left[f(t_i, w_i) + f(t_{i+1}, w_i + h f(t_i, w_i)) \right]$$

There some other formulas by the name of Runge-Kutta Methods

Fourth Order Runge-Kutta Method

$$w_0 = \alpha$$

$$y' = \boxed{f(t, y)}$$

$$w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$f(a+h, b+k) = f(a, b) + \cdots$$

$$k_1 = hf(t_i, w_i)$$

$$k_2 = hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_1\right)$$

$$k_3 = hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_2\right)$$

$$k_4 = hf(t_{i+1}, w_i + k_3)$$

$$y' = y - t^2 + 1$$

$$t = 0, 0.5, 1, 1.5, 2$$

$$y(0) = 0.5 \qquad 0 \le t \le 2 \qquad N = 4$$

$w_0 = 0.5$ $\quad i = 0, \ldots N-1$

$w_1 = w_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = h f(t_0, w_0)$ $\quad , f(t, w) = w - t^2 + 1$

$k_1 = 0.5(w_0 - t_0^2 + 1) = 0.5(0.5 - (0)^2 + 1)$

$k_2 = h f(t_0 + h/2, w_0 + \frac{1}{2}k_1)$

$= 0.5 f(0.025, 0.5 + \frac{1}{2}k_1)$

k_3

k_4

8 — 5 $\quad$ 40%