Linear Algebra
Answer Key of Assignment#01

1. Use back substitution to solve the following system of equations and write out the coefficient matrix:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 5 \\ 2x_2 + x_3 - 2x_4 + x_5 &= 1 \\ 4x_3 + x_4 - 2x_5 &= 1 \\ x_4 - 3x_5 &= 0 \\ 2x_5 &= 2 \end{aligned}$$

Solution:
(-2,3,0,3,1).

2. Write out the system of equations that corresponds to the following augmented matrix:

$$\left(\begin{array}{cccc|c} 4 & -3 & 1 & 2 & 4 \\ 3 & 1 & -5 & 6 & 5 \\ 1 & 1 & 2 & 4 & 8 \\ 5 & 1 & 3 & -2 & 7 \end{array}\right)$$

Solution:
$4x_1 - 3x_2 + x_3 + 2x_4 = 4$
$3x_1 + x_2 - 5x_3 + 6x_4 = 5$
$x_1 + x_2 + 2x_3 + 4x_4 = 8$
$5x_1 + x_2 + 3x_3 - 2x_4 = 7$

3. Given a system of the form:

$$\begin{aligned} -m_1x_1 + x_2 &= b_1 \\ -m_2x_1 + x_2 &= b_2 \end{aligned}$$

where m_1, m_2, b_1, and b_2 are constants:

(a) Show that the system will have a unique solution if $m_1 \neq m_2$.

(b) Show that if $m_1 = m_2$, then the system will be consistent only if $b_1 = b_2$.

(c) Give a geometric interpretation of parts (a) and (b).

Solution:

Given the system

$$\begin{aligned} -m_1x_1 + x_2 &= b_1 \\ -m_2x_1 + x_2 &= b_2 \end{aligned}$$

one can eliminate the variable x_2 by subtracting the first row from the second. One then obtains the equivalent system

$$\begin{aligned} -m_1x_1 + x_2 &= b_1 \\ (m_1 - m_2)x_1 &= b_2 - b_1 \end{aligned}$$

(a) If $m_1 \neq m_2$, then one can solve the second equation for x_1

$$x_1 = \frac{b_2 - b_1}{m_1 - m_2}$$

One can then plug this value of x_1 into the first equation and solve for x_2. Thus, if $m_1 \neq m_2$, there will be a unique ordered pair (x_1, x_2) that satisfies the two equations.

(b) If $m_1 = m_2$, then the x_1 term drops out in the second equation

$$0 = b_2 - b_1$$

This is possible if and only if $b_1 = b_2$.

(c) If $m_1 \neq m_2$, then the two equations represent lines in the plane with different slopes. Two nonparallel lines intersect in a point. That point will be the unique solution to the system. If $m_1 = m_2$ and $b_1 = b_2$, then both equations represent the same line and consequently every point on that line will satisfy both equations. If $m_1 = m_2$ and $b_1 = b_2$, then the equations represent parallel lines. Since parallel lines do not intersect, there is no point on both lines and hence no solution to the system.

4. Consider a system of the form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= 0 \\ a_{21}x_1 + a_{22}x_2 &= 0 \end{aligned}$$

where a_{11}, a_{12}, a_{21}, and a_{22} are constants. Explain why a system of this form must be consistent.

Solution:
The system must be consistent since (0, 0) is always a solution.

5. Give a geometrical interpretation of a linear equation in three unknowns. Give a geometrical description of the possible solution sets for a 3×3 linear system.

Solution:
A linear equation in 3 unknowns represents a plane in three space. The solution set to a this linear system would be the set of all points that lie on all three planes. If the planes are parallel or one plane is parallel to the line of intersection of the other two, then the solution set will be empty. The three equations could represent the same plane or the three planes could all intersect in a line. In either case the solution set will contain infinitely many points. If the three planes intersect in a point, then the solution set will

contain only that point.

6. Make a list of the lead variables and a second list of the free variables:

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 5 \\ 0 & 0 & 1 & 3 & 4 \end{array}\right]$$

Solution:
The variables x_1 and x_3 are lead variables and x_2 and x_4 are free variables.

7. For each of the systems of equations that follow, use Gaussian elimination to obtain solution.
(a)

$$\begin{aligned} x_1 - 2x_2 &= 3 \\ 2x_1 + x_2 &= 1 \\ -5x_1 + 8x_2 &= 4 \end{aligned}$$

(b)

$$\begin{aligned} -x_1 + 2x_2 - x_3 &= 2 \\ -2x_1 + 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 + 2x_3 &= 5 \\ -3x_1 + 8x_2 + 5x_3 &= 17 \end{aligned}$$

(c)

$$\begin{aligned} x_1 + 2x_2 - 3x_3 + x_4 &= 1 \\ -x_1 - x_2 + 4x_3 - x_4 &= 6 \\ -2x_1 - 4x_2 + 7x_3 - x_4 &= 1 \end{aligned}$$

Solution:
(a) Inconsistent.
(b) (0,1.5,1)
(c) $\{(2 - 6\alpha, 4 + \alpha, 3 - \alpha, \alpha) | \alpha$ is a real number$\}$.

8. Consider a linear system whose augmented matrix is of the form:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ -1 & 4 & 3 & 2 \\ 2 & -2 & a & 3 \end{array}\right]$$

For what values of a will the system have a unique solution?

Solution:
After applying Gaussian elimination method to given system, we get a unique system for $a \neq 2$.

9. Consider a linear system whose augmented matrix is of the form:

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & a & b \end{array}\right]$$

(a) For what values of a and b will the system have infinitely many solutions?

(b) For what values of a and b will the system be inconsistent?

Solution:
After applying Gaussian elimination method to given system, we get
(a) infinitely many solutions for a=5, b=4.
(b) an inconsistent system for a=5, $b \neq 4$.

10. (a)
Is the vector $\mathbf{b} = (7, 4, -3)$ a linear combination of the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}?$$

(b)
Is the vector $\mathbf{b} = (1, 0, 1)$ a linear combination of the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}?$$

(c)
Is the vector $\mathbf{b} = (8, 8, 12)$ a linear combination of the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 6 \\ 4 \\ 9 \end{bmatrix}?$$

Solution:
(a)

$$\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{b}\end{bmatrix} = \begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix}$$

The RREF of the augmented matrix is

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

and therefore the solution is $x_1 = 3$ and $x_2 = 2$. Therefore, yes, $\mathbf{b}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2$:

$$3\mathbf{v}_1 + 2\mathbf{v}_2 = 3\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} = \mathbf{b}$$

Notice that the solution set does not contain any free parameters because $n = 2$ (unknowns) and $r = 2$ (rank) and so $d = 0$. Therefore, the above linear combination is the only way to write $\mathbf{b}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. □

(b) The augmented matrix of the corresponding linear system is:

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 2 & 0 & 4 & 1 \end{bmatrix}.$$

After row reducing we obtain that

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

The last row is inconsistent, and therefore the linear system does not have a solution. Therefore, no, $\mathbf{b}$ is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. □

(c) The augmented matrix of the corresponding linear system is:

$$\begin{bmatrix} 2 & 4 & 6 & 8 \\ 1 & 2 & 4 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix} \xrightarrow{\text{REF}} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The system is consistent and therefore $\mathbf{b}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. In this case, the solution set contains $d = 1$ free parameters and therefore, it is possible to write $\mathbf{b}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in infinitely many ways. In terms of the parameter t, the solution set is

$$\begin{aligned} x_1 &= -8 - 2t \\ x_2 &= t \\ x_3 &= 4 \end{aligned}$$

Choosing any t gives scalars that can be used to write $\mathbf{b}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. For example, choosing $t = 1$ we obtain $x_1 = -10$, $x_2 = 1$, and $x_3 = 4$, and you can verify that

$$-10\mathbf{v}_1 + \mathbf{v}_2 + 4\mathbf{v}_3 = -10 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix} + 4 \begin{bmatrix} 6 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \\ 12 \end{bmatrix} = \mathbf{b}$$

Or, choosing $t = -2$ we obtain $x_1 = -4$, $x_2 = -2$, and $x_3 = 4$, and you can verify that

$$-4\mathbf{v}_1 - 2\mathbf{v}_2 + 4\mathbf{v}_3 = -4 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} - 2 \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix} + 4 \begin{bmatrix} 6 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \\ 12 \end{bmatrix} = \mathbf{b}$$