Lecture 6

We can use the following inequality to find the upper bound on the number of iterations required to reach a certain level of accuracy in the bisection method.

$$|P_n - P| \leq \frac{(b-a)}{2^n} \qquad n \geq 1 \qquad \text{--- (1)}$$

Example

Find a bound for the number of iterations required to achieve an approximation with accuracy 10^{-3} to the solution of $x^3 + x - 4 = 0$ lying in the interval $[1, 4]$.

Sol

Here $\varepsilon = 10^{-3} = 0.001$. Ideally, we would set $|P_n - P| \leq \varepsilon$ and solve this inequality to find the value of n. However, since we don't know the value of $|P_n - P|$, we will not be able to find n. Instead, we set $|P_n - P| \leq \frac{b-a}{2^n} \leq \varepsilon$ and solve

$$\frac{b-a}{2^n} \leq \varepsilon \quad \text{for the value of } n.$$

This will give you an upper bound on 'n', because we are using an upper bound $\frac{b-a}{2^n}$ on $|P_n - P|$ to calculate 'n'.

In this example $a = 1$, $b = 4$ therefore

$$\frac{4-1}{2^n} \leq 10^{-3}$$

$\Rightarrow \quad \frac{3}{2^n} \leq 10^{-3}, \quad \cancel{\log\left(\frac{3}{2}\right) \leq \log 10^{-3}}$

$\log\left(\frac{3}{2^n}\right) \leq \log\left(10^{-3}\right)$ $\quad \because$ log is an increasing function.

$\log(3) - \log(2^n) \leq -3 \log(10)$

$-\log(2^n) \leq \log(3) - 3$

$-n \log(2) \leq \log(3) - 3$

$n \log(2) \geq -\log(3) + 3$

$n \geq \frac{-\log(3) + 3}{\log(2)}$ $\quad \log(2) > 0$

$n \geq 8.380$. Since 'n' is an integer we take, $n = 9$. This is an upper bound on the number of iterations and actual number of iterations may be less than this value.

Some Remarks

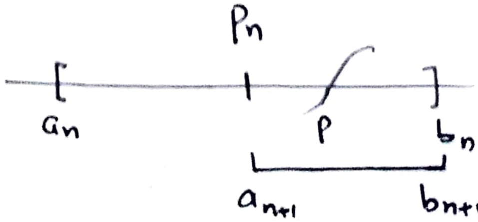
1. Bisection Method is slow converging method. It converges linearly to the root. (you have to prove it).

2. Bisection method always converges.

3. At every step of Bisection Method we have

$$\frac{|p_n - p|}{|p|} \leq \frac{|b_{n+1} - a_{n+1}|}{|a_{n+1}|},$$

meaning that we can calculate relative error at each iteration.



4. Considering the round off error issue, it is less erraneous to use

$$P_n = a_n + \frac{b_n - a_n}{2} \quad \text{instead of} \quad P_n = \frac{a_n + b_n}{2}$$

to calculate the root at every iteration.

Some questions from article 2.1

16. Let $f(x) = (x-1)^{10}$, $P = 1$ and $P_n = 1 + \frac{1}{n}$.
Show that $|f(P_n)| < 10^{-3}$ whenever $n > 1$ but $|P - P_n| < 10^{-3}$ requires $n > 100$.

Sol
=

The crux of question is to show that sometimes $|f(P_n)| < \varepsilon$ stopping criterian con be misleading.

The root of $f(x) = (x-1)^{10}$ is $P = 1$, so if we try to use $P_n = 1 + \frac{1}{n}$ to find root of f accurate to 10^{-3}, $|f(P_n)| < \varepsilon$ stopping criterion gives much less number of iterations than the actual iterations.

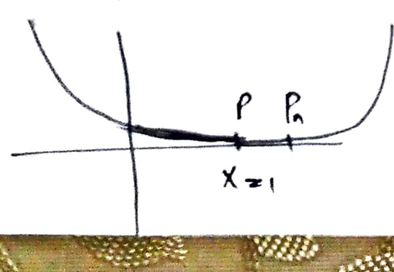
$$|f(P_n)| = \left|f\left(1+\frac{1}{n}\right)\right| = \left|\left(1+\frac{1}{n}-1\right)^{10}\right| = \frac{1}{n^{10}} < 10^{-3}$$

for $\boxed{n > 1.}$

However, if we calculate

$$|P_n - P| = \left|1 + \frac{1}{n} - 1\right| = \frac{1}{n} < 10^{-3} \quad \text{when} \quad \boxed{n > 1000}$$

This is because f is flat around the root.



Compare the two

17 Let $\{P_n\}$ be the sequence defined by $P_n = \sum_{K=1}^{n} \frac{1}{K}$.
Show that $\{P_n\}$ diverges even though $\lim_{n\to\infty} (P_n - P_{n-1}) = 0$

This question explains that if we use $|P_n - P_{n-1}| < \varepsilon$ as our stopping criterion, it may also be misleading. Because it is quite possible that $|P_n - P_{n-1}| \to 0$ as $n \to \infty$ but the sequence $\{P_n\}$ itself is divergent.

Consider $P_n = \sum_{K=1}^{n} \frac{1}{K}$ It is harmonic series

and you know from your calculus coures that it is divergent.

$$P_1 = \frac{1}{1}, \quad P_2 = 1 + \frac{1}{2}, \quad P_3 = 1 + \frac{1}{2} + \frac{1}{3}$$

$$P_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{n}$$

$$P_{n-1} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}$$

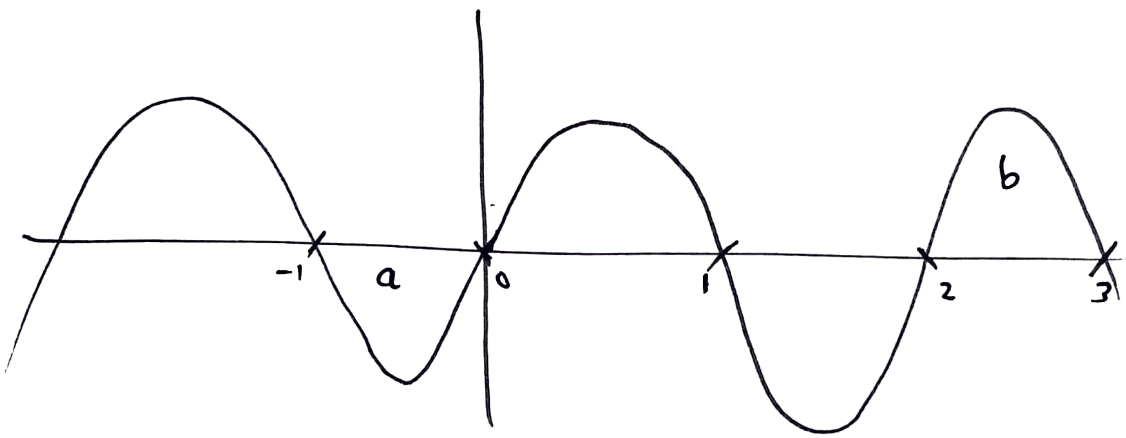
$$|P_n - P_{n-1}| = \frac{1}{n} \to 0 \quad \text{as } n \to \infty$$

(18) The function $f(x) = \sin \pi x$ has zero at every integer. Show that when $-1 < a < 0$ and $2 < b < 3$ the Bisection Method converges to

(a) 0, if $a+b < 2$ (b) 2, if $a+b > 2$ (c) 1, if $a+b = 2$

Solution

$f(x) = \sin \pi x$ is defined on $[a, b]$ where $-1 < a < 0$ and $2 < b < 3$.



(a) Since $P_1 = \frac{a+b}{2}$,

but $a + b < 2$

$\Rightarrow$ $P_1 < 1$. So the next interval is $[a, P_1]$ which contains only '0' root. Hence proved

(b) $P_1 = \frac{a+b}{2}$

but $a + b > 2$

$\Rightarrow$ $P_1 > 1$. So ~~the root~~ the root is in $[P_1, b]$ and it contains only '2' root. Hence proved

(c) $P_1 = \frac{a+b}{2}$

but $a+b = 2$

$\Rightarrow$ $\boxed{P_1 = 1}$

which is one of the root of f. Hence Bisection Method has converged to the root in just single iteration.