Example: $A = \begin{bmatrix} -4 & -6 & -7 \\ 3 & 5 & 3 \\ 0 & 0 & 3 \end{bmatrix}$

(i) Find Eigenvalues of A

(ii) Find Eigenspaces and basis Eigenvectors for each Eigenvalue of A

$$A - \lambda I = \begin{bmatrix} -4-\lambda & -6 & -7 \\ 3 & 5-\lambda & 3 \\ \boxed{0 \quad 0 \quad 3-\lambda} \end{bmatrix}.$$

Evalute the determinant using the third row of A

$$\det(A-\lambda I) = 0 \begin{vmatrix} A & A \\ A & A \end{vmatrix} - 0 \begin{vmatrix} A & A \\ A & A \end{vmatrix} + 3-\lambda \begin{vmatrix} -4-\lambda & -6 \\ 3 & 5-\lambda \end{vmatrix}$$

$$= 3-\lambda \begin{vmatrix} -4-\lambda & -6 \\ 3 & 5-\lambda \end{vmatrix}$$

$$= (3-\lambda)\left[(-4-\lambda)(5-\lambda) + 18\right]$$

$$= (3-\lambda)\left[-20+4\lambda \pm 5\lambda + \lambda^2 + 18\right]$$

$$\boxed{\begin{gathered}\lambda^2 - 2\lambda + \lambda - 2 \\ = \lambda(\lambda-2) + (\lambda-2)\end{gathered}}$$

$$(3-\lambda)\,(\lambda^2 - \lambda - 2)$$

$$(3-\lambda)(\lambda-2)(\lambda+1)$$

C.E.: $(3-\lambda)\,(\lambda-2)\,(\lambda+1) = 0$

$\lambda = 3,\ 2,\ \lambda = -1$ } Eigen values

Eigenvalues: $\lambda = 3,\ 2,\ -1$

Verify:

$$E_{\lambda=3} = \mathrm{Null}(A - 3I) = \mathrm{Span}\left(\underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{V_{\lambda=3}}\right)$$

$$E_{\lambda=2} = \mathrm{Null}(A - 2I) = \mathrm{Span}\left(\underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{= V_{\lambda=2}}\right)$$

$$E_{\lambda=-1} = \mathrm{Null}(A + I) = \mathrm{Span}\left(\underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}}_{= V_{\lambda=-1}}\right)$$

Basis Eigenvectors : $V_{\lambda=3} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $V_{\lambda=2} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $V_{\lambda=-1} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

Computation

For $\lambda = 3$ $\quad E_{\lambda=3} = \text{Null}(A - 3I)$

Solve: $(A - 3I)v = 0$

$$A - 3I = \begin{bmatrix} -7 & -6 & -7 \\ 3 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{2R_2 + R_1} \begin{bmatrix} -1 & 0 & -1 \\ 3 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{3R_1 + R_2} \begin{bmatrix} -1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[\frac{1}{2}R_2]{-1R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x = -t$
$y = 0$
$z = t$

$$E_{\lambda=3} = \left\{ \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} \middle| \ t \in \mathbb{R} \right\} = \text{Span}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$\boxed{V_{\lambda=3} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \text{Basis Eigenvector}}$$

For $\lambda = 2$, $E_{\lambda=2} = \text{Null}(A - 2I)$

Solve: $(A - 2I)v = \vec{0}$

$$A - 2I = \begin{bmatrix} -6 & -6 & 7 \\ 3 & 3 & 3 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{2R_2 + R_1} \begin{bmatrix} 0 & 0 & 13 \\ 3 & 3 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2 \quad \begin{bmatrix} 3 & 3 & 3 \\ 0 & 0 & 13 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\frac{1}{3} R_1]{\frac{1}{13} R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$-R_2 + R_3 \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x = -t$
$y = t$
$z = 0$

$$E_{\lambda=2} = \left\{ \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$$

$$= \text{Span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right)$$

$$\boxed{V_{\lambda=2} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}$$

basis Eigenvector for $\lambda = 2$

For $\lambda = -1$ $\quad E_{\lambda=-1} = \text{Null}(A + I)$

Solve: $(A + I)v = \vec{0}$

$$A + I = \begin{bmatrix} -3 & -6 & -7 \\ 3 & 6 & 3 \\ 0 & 0 & 4 \end{bmatrix} \xrightarrow{1R_1 + R_2} \begin{bmatrix} -3 & -6 & -7 \\ 0 & 0 & -4 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\xrightarrow{1R_2+R_3} \begin{bmatrix} -3 & -6 & -7 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{4}R_2} \begin{bmatrix} -3 & -6 & -7 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x = -2t$

$y = t$

$z = 0$

$$E_{\lambda=-1} = \left\{ \begin{bmatrix} -2t \\ t \\ 0 \end{bmatrix} \middle| \, t \in \mathbb{R} \right\} = \operatorname{Span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right)$$

$V_{\lambda=-1} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ is a basis eigenvector for $\lambda = -1$