Chapter: General Vector Spaces

1. Let $P(t)$ be the set of polynomials of degree equal to n where n is a positive integer. Show that this set $P(t)$ is not a vector space.

2. Consider the set of vectors $B = \{v_1, v_2, v_3, ..., v_n\}$ in a vector space V. Prove that if the set B is a basis for V and $S = \{v_1, v_2, v_3, ..., v_m\}$ is a set of linearly independent vectors in V then $m \leq n$.

3. Show that the set of matrices of the form $\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$ does not form a vector space.

4. Let S be the subset of vectors of the form $\begin{pmatrix} x \\ y \end{pmatrix}$ where $x \geq 0$ in the vector space R^2. Show that S is not a subspace of R^2.

5. Let P_2 be the set of all polynomials of degree less than or equal to 2. Let $v_1 = t^2 - 1$, $v_2 = t^2 + 3t - 5$ and $v_3 = t$ be vectors in P_2. Show that the quadratic polynomial $x = 7t^2 - 15$ is a linear combination of $\{v_1, v_2, v_3\}$.

6. Let M_{22} be the vector space containing matrices of size 2 by 2. Show that the matrices $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are not a basis for M_{22}.

7. Let M_{22} be the vector space of 2 by 2 matrices. Consider the matrices
$A = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$ $\quad B = \begin{pmatrix} 0 & 2 \\ 0 & -1 \end{pmatrix}$ $\quad C = \begin{pmatrix} 0 & 1 \\ 5 & 2 \end{pmatrix}$
Determine whether the matrix $D = \begin{pmatrix} 1 & 2 \\ 4 & -2 \end{pmatrix}$ is within the span $\{A, B, C\}$.

8. Define conditions with one example which shows that vectors v_1, v_2 and v_3 are linearly dependent vectors in R^3.

9. Define conditions for n vectors in vector space V to form basis of V.

10. Prove that the zero vector, **O**, on its own is a linearly dependent vector in a vector space V.

11. Let A be an n × n matrix such that Ax = b has exactly one solution for each b in R^n. What three other things (or facts) can you say about A.

Chapter: Determinants

12. Find the determinant of $D = \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix}$

13. Find the determinant of $A = \begin{pmatrix} -1 & 5 & -2 \\ -6 & 6 & 0 \\ 3 & -7 & 1 \end{pmatrix}$ by using elementary operations.

14. Define upper and lower triangular matrix in R^{nxn} with one example of each.

15. Let A, B, C be n x n matrices. Suppose that $detA = 3$, $detB = 0$, and $detC = 7$. (i) Is AC invertible? (ii) Is AB invertible?

16. Prove that if a square matrix A contains a zero row or zero column then $det(A) = 0$.

17. Define Elementary matrices with one example of 2 by 2 and 3 by 3 matrices.

18. Define invertible matrix along with its condition for invertibility.

19. Find the determinant of the following 4 by 4 matrix by using row operations:
$$M = \begin{pmatrix} 1 & 2 & 2 & 4 \\ 7 & 8 & 3 & 0 \\ 3 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

20. Let A and B be 3 by 3 matrices with $det(A) = 3$, $det(B) = -4$. Determine
$(a) det(-2AB)$ $(b) det(A^5 B^6)$.

21. Find the determinants of the following matrices by using row operations:
i) $P = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 6 \\ 1 & 4 & 3 & 7 \\ 1 & 6 & 1 & 9 \end{pmatrix}$
ii) $R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

Chapter: Linear Transformation

22. Consider the transformation $T : R^3 \to R^2$ defined by
$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - y \\ x - z \end{pmatrix}$. Determine $T\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $T\begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix}$.

23. Show that $T : R^2 \to R^2$ defined by $T(x) = Ax$ where $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is a linear transformation.

24. Let $T : V \to W$ be a linear transformation of an n-dimensional vector space into a vector space W. Let $\{v_1, v_2, v_3, ..., v_n\}$ be a basis for V. Prove that if u is any vector in V then we can write T (u) as a linear combination of $\{T(v1), T(v2), T(v3), ..., T(v_n)\}$.

25. Show that the transformation $T : R^2 \to R^3$ defined by
$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ x - y \\ y \end{pmatrix}$ is linear.

26. Explain why $T : U \to V$ given by $T(\mathbf{u}) = \pm\sqrt{\mathbf{u}}$ is not a transformation.

27. Let P_2 be the vector space of polynomials of degree 2 or less. Decide whether the following transformations are linear:
(a) $T : P_2 \to P_2$ given by $T(c_2x^2 + c_1x + c_0) = c_0x^2 + c_1x + c_2$.
(b) $T : P_2 \to P_2$ given by $T(c_2x^2 + c_1x + c_0) = c_0^2x^2 + c_1^2x + c_2^2$.

28. Let $T : V \to W$ be a transformation such that $T(\mathbf{O}) = \mathbf{O}$. Prove that T is not a linear transformation.
Where **0** is a zero vector in V and W respectively.

29. Show that the transpose of any square matrix is a linear transformation (mapping).

30. Let $T : R^3 \to R^2$ be given by $T(x) = Ax$ where x is in R^3 and
$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

Find $i) ker(T)$ $ii)$ Null space of T.

31. Consider the zero linear transformation $T : V \to W$ defined by $T(v) = \mathbf{O}$ for all v in the domain V. Prove that $ker(T) = V$.

32. Consider the zero linear transformation $T : V \to W$ such that $T(\mathbf{v}) = \mathbf{O}$ for all vectors v in V. Find range(T) or in other words the image of T.

33. Let $T : V \to W$ be a linear transformation. Show that if $u \in ker(T)$ and $v \in ker(T)$ then for any scalars k and c the vector $(ku + cv) \in ker(T)$.

Chapter: Coordinate bases

34. Verify that $B = \{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\}$ is a basis for R^2. Further, find the coordinates of $v = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ relative to B.

35. Let $V_1 = \begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix}$,$V_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $X = \begin{pmatrix} 3 \\ 12 \\ 7 \end{pmatrix}$. And let $B = \{V_1, V_2\}$. Show that B is linearly independent and therefore a basis for $W = span\{v_1, V_2\}$. Determine if X is in W and if so find the coordinate vector of X relative to B.

36. What are the coordinates of $v = \begin{pmatrix} 3 \\ 11 \\ -7 \end{pmatrix}$ in the standard basis $E = \{e_1, e_2, e_3\}$?

37. Let $P_3[t]$ be the vector space of polynomials of degree at most 3. Find the coordinates of $v(t) = 3 - t^2 - 7t^3$ relative to $B = \{1, t, t^2, t^3\}$.

38. Let $B = \{v_1, ..., v_n\}$ be a basis for V and let $x \in V$. Define coordinates of x relative to the basis B.

39. The columns of the matrix P form a basis B for R^3
$P = \begin{pmatrix} 1 & 3 & 3 \\ -1 & -4 & -2 \\ 0 & 0 & -1 \end{pmatrix}$
i) What vector x $\in R^3$ has B-coordinates $[x]_B = (1, 0, -1)$.
ii) Find the B-coordinates of $v = (2, -1, 0)$.

40. Define the change-of-coordinates matrix P from the basis B to the standard basis in R^n.