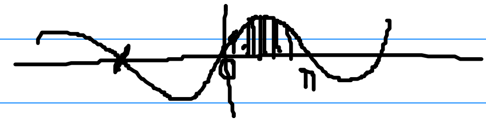
Numerical Integration 4.3

$$\int_a^b f(x)\,dx = F(b) - F(a) \qquad F'(x) = f(x)$$

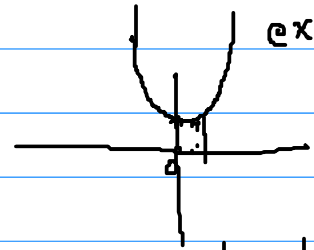
$$\int_0^{\pi} \sin x\,dx$$



$F' = \sin x$

$(-\cos x)' = \sin x$

$$= -\left|\cos x\right|_0^{\pi}$$

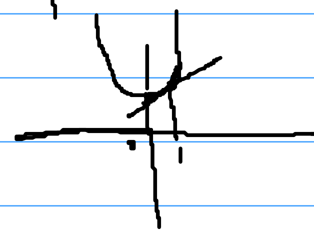


e^{x^2}

$$\int_0^1 e^{x^2}\,dx =$$

$$F' = e^{x^2}$$

$e^{x^2} \simeq P(x)$



Lagrange polynomial

$$f(x) = \sum_{k=0}^{n} L_k(x)\, f(x_k) + \prod_{k=0}^{n} (x - x_k)\, \frac{f^{(n+1)}(\xi(x))}{(n+1)!}$$

$a = x_0,\ x_1,\ x_2,\ - - x_n = b \qquad [a, b]$

Consider only two points $x_0 = a$, $x_1 = b$

$$f(x) = f(x_0)\left(\frac{x - x_1}{x_0 - x_1}\right) + f(x_1)\,\frac{x - x_0}{x_1 - x_0} + \frac{(x - x_0)(x - x_1)}{2!}\, f^{(2)}(\xi(x))$$

$x_0 < \xi(x) < x_1$

$$\int_a^b f(x) = f(x_0)\int_a^b \frac{x - x_1}{x_0 - x_1}\,dx + f(x_1)\int_a^b \frac{x - x_0}{x_1 - x_0} + \frac{1}{2}\int_a^b \underbrace{(x - x_0)(x - x_1)}_{g}\, f^{(2)}(\xi(x))\,dx$$

$$= \frac{f(x_0)}{x_0 - x_1}\left|\frac{(x - x_1)^2}{2}\right|_{x_0}^{x_1} + \frac{f(x_1)}{x_1 - x_0}\left[\frac{(x - x_0)^2}{2}\right|_{x_0}^{x_1} + \frac{1}{2} f^{(2)}(\xi)\int_{x_0}^{x_1} (x - x_0)(x - x_1)\,dx$$

Weighted Mean Value Theorem

$$\int_a^b f(x)\, g(x)$$

if $g(x)$ does not change sign on $[a, b]$, then there exists $c \in [a, b]$ such that

$$\int_a^b f(x)\, g(x) = f(c) \cdot \int_a^b g(x)\, dx$$

$$\int_a^b f(x)dx = -\frac{f(x_0)}{2(x_0 - x_1)} (x_1 - x_0)^2 + \frac{f(x_1)}{2(x_1 - x_0)} (x_1 - x_0)^2 + \frac{1}{2} f^{(2)}(\xi) \int_{x_0}^{x_1} x^2 - x x_1 - x_0 x + x_0 x_1$$

$$= \frac{f(x_0)(x_1 - x_0)}{2} + \frac{f(x_1)(x_1 - x_0)}{2} + \frac{1}{2} f^{(2)}(\xi) \left[\frac{x^3}{3} - \frac{x^2}{2}(x_0 + x_1) + x_0 x_1 x \right]_{x_0}^{x_1}$$

h

x_0 x_1 $\quad h = x_1 - x_0$

$$\int_a^b f(x)dx = \frac{h}{2}\left[f(x_0) + f(x_1) \right] - \frac{h^3}{12} f^{(2)}(\xi), \qquad x_0 < \xi < x_1$$

Trapezoidal Rule —— (A)

x_0 x_1

$f^{(2)}(x) = 0$

$f(x) = ax + b$

$f^{(2)}(x) = 0$

If we divide the interval of integration $[a, b]$ into three points.

h $\quad x_1 \quad$ h

$a = x_0$ $\qquad b = x_2 \qquad h = \dfrac{b-a}{2}$

$x_1 = x_0 + h, \qquad x_2 = x_1 + h$

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right] - \frac{h^5}{90} f^{(4)}(\xi) \quad \text{(B)}$$

$a < \xi < b$

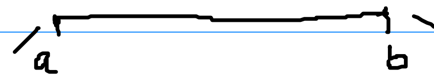
This is called Simpson's Rule

The general term used for formulas like (A) & (B) is

called Newton's Cotes formulas.

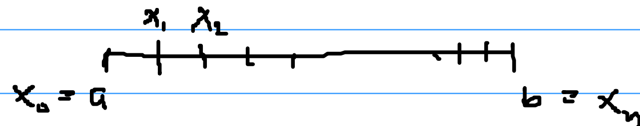
$\int_0^1 e^{x^2}\,dx$, $f(x) = e^{x^2}$ is defined on $[0, 1]$

$\int_0^1 \frac{1}{x(x-1)}$, $g(x) = \frac{1}{x(x-1)}$ $g(0)$ & $g(1)$ are problematic



There are two types of Newton's Cotes formulas

Closed cotes formulas



$\int_{x_0}^{x_n} f(x)\,dx$

$h = \frac{b-a}{n}$, $(n+1)$ points

for $n=1$, it is simply trapezoidal rule x_0, x_1

$n=2$, it is Simpson's Rule

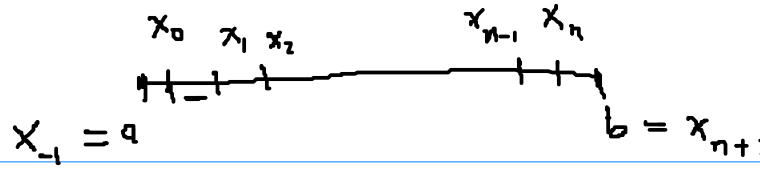
$n=3$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right] - \frac{3h^5}{80} f^{(4)}(\xi)$$

$x_0 < \xi < x_3$

$n=4$ (See the book)

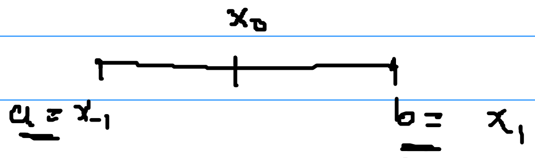
$$\int_0^1 \frac{1}{x(x-1)}\,dx$$



There $n+3$ points

$$h = \frac{b-a}{n+2}$$

$n=0$



$$h = \frac{b-a}{2}$$

$$\int_a^b f(x)\,dx = \int_{x_{-1}}^{x_1} f(x)\,dx = 2h\,f(x_0) + \frac{h^3}{3} f^{(2)}(\xi)$$

$$x_{-1} < \xi < x_1$$

$$\int_0^1 \frac{1}{x(x-1)}\,dx$$

$$\int_a^b f(x) = F(b) - F(a)$$