Note Title — 9/25/2023

Linear Algebra — Lec #8 Mon-Wed

Quiz #01 — Oct 02, Assignment #1

Quiz #02 — Oct 09 — Assignment #2

Recall:

$$\boxed{B \text{ is a basis of } \mathbb{R}^n}$$

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Note Title 9/23/2023

B is a L.I set $\qquad$ Span (B) = $\mathbb{R}^n$

Theorem: If B is a basis of $\mathbb{R}^n$, then $\boxed{\#(B) = n}$

Example $S = \left\{ v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

$S \subset \mathbb{R}^3$

Is S a basis of $\mathbb{R}^3$?

$\#(S) = 4 > 3 \Rightarrow S$ is not a basis

$\boxed{\text{Example}}$

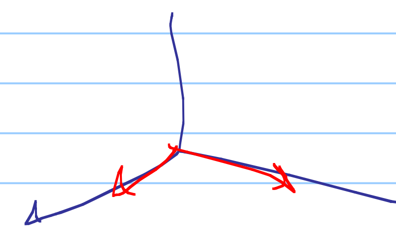
$$A = \left\{ v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$\#(A) = 2 < 3 \Rightarrow A$ is not a basis of $\mathbb{R}^3$

Example $D = \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \subset \mathbb{R}^3$

D is not a basis of $\mathbb{R}^3$

$\mathrm{Span}(e_1, e_2) =$ xy-plane



Example

$$F = \left\{ v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\neq} = \vec{0}, v_3 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \right\}$$

Is F a basis of $\mathbb{R}^3$?

$\boxed{F \text{ is not a basis of } \mathbb{R}^3}$

Since $\vec{0} \in F$

Remark

$\vec{0} \in S \Rightarrow \boxed{S \text{ is a L.D set}}$

Def:

Reduced Row Echelon form (R.R.E.F)

Let A be an mxn matrix

A R.R.E.F of A is a R.E.F

with the following additional property:

"All entries above a leading one are zero".

$A = \begin{bmatrix} 1 & 2 & \underline{\underline{3}} \\ 0 & 1 & 4 \\ 0 & 0 & \underline{\underline{1}} \end{bmatrix}$ Not a R.R.E.F

3×3

$-4R_3 + R_2 \; , \; -3R_3 + R_1$

$\begin{bmatrix} 1 & 2 & 0 \\ 0 & \underline{\underline{1}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$-2R_2+R_1 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

$$B = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

R.R.E.F

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ \cdot & \cdot & 0 & 1 \end{bmatrix}$$

Remark: If A is an $n \times n$ matrix such that a R.E.F of A contains "n" leading one's, then the R.R.E.F of A is I_n.

Example $B = \left\{ v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

$\subset \mathbb{R}^3$

$-$ B a basis of $\mathbb{R}^3$

$$\boxed{\text{L.I:} \quad x_1 v_1 + x_2 v_2 + x_3 v_3 = \vec{0}}$$

$$\left[\underline{v_1 \quad v_2 \quad v_3} \,\middle|\, \vec{0}\right] \sim \left[\underline{I_n} \,\middle|\, \vec{0}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right]$$

$-R_1 + R_2$

$-R_1 + R_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right]$$

$$R_2 \leftrightarrow R_3 \qquad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$-R_2,\ -R_3 \qquad \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \end{array}\right]$$

$$\begin{array}{l} -R_3+R_2 \\ -R_3+R_1 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$-R_2 + R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The system has a unique (trivial) solution

$\Rightarrow B = \{v_1, v_2, v_3\}$ ~~are~~ is a L.I set

$$\boxed{A \sim I_n}$$

(ii) $\boxed{\mathrm{Span}(B) = \mathbb{R}^3}$:

$B = \{v_1, v_2, v_3\}$

$\boxed{\text{For } v = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3}$

Claim : $\boxed{v \text{ is a L.C of } v_1\ v_2\ \&\ v_3}$

i.e $\boxed{v \in \mathrm{Span}(v_1, v_2, v_3)}$

Solve : $x_1 v_1 + x_2 v_2 + x_3 v_3 = v \Rightarrow \left[\, v_1 \ \ v_2 \ \ v_3 \mid \vec{v} \,\right]$

$\sim \left[\, I_n \mid u \,\right]$

→ all variable are leading, the system has unique solution

⇒ V is a L.C of V_1, V_2, & V_3.

Theorem: If $B = \{ v_1, v_2, v_n \} \subset \mathbb{R}^n$ is a Linearly independent, then $\text{Span}(B) = \mathbb{R}^n$

i.e. B is basis of $\mathbb{R}^n$.

Proof: Given $B = \{ v_1, v_2, .., v_n \}$ is a $\boxed{\text{L.I set}}$

$$[\underline{v_1 \quad v_2 \quad \dots \quad v_n} \mid \vec{0}] \sim [I_n \mid \vec{0}]$$

(has a unique trivial solution)

i.e $[v_1 \quad v_2 \quad v_n] \sim I_n$ } $\boxed{\text{Assumption}}$

Claim: $\operatorname{Span}(B) = \mathbb{R}^n$

For any $v \in \mathbb{R}^n$, $v \in \operatorname{Span}(B)$ ie v is a L.C of $v_1, v_2, \dots, v_n$

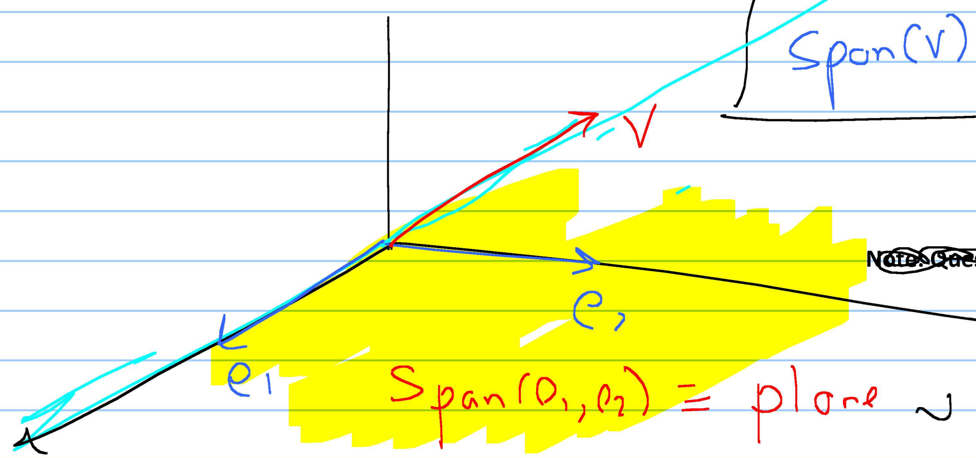
Solve: $x_1 v_1 + x_2 v_2 + \dots x_n v_n = v$

$$\left[\begin{array}{cccc|c} v_1 & v_2 & & v_n & v \end{array}\right] \xrightarrow[\text{operation}]{\text{Row}} \underbrace{\left[\begin{array}{c|c} I_n & u \end{array}\right]}_{\text{(all variables are leading)}}$$

$\Rightarrow$ $[v_1 \quad v_n \,|\, v]$ is consistent & has a unique Solution

$\Rightarrow$ v is a L.C of $v_1, v_2, \dots v_n$.

"Subspace of $\mathbb{R}^n$"



$\boxed{\mathrm{Span}(v) = \text{Line} \sim \mathbb{R}}$ 1-dim

~~Note: Questions marked with * are Theoretical Problems~~

$\mathrm{Span}(e_1, e_2) = \text{plane} \sim \mathbb{R}^n$ (2-dim)

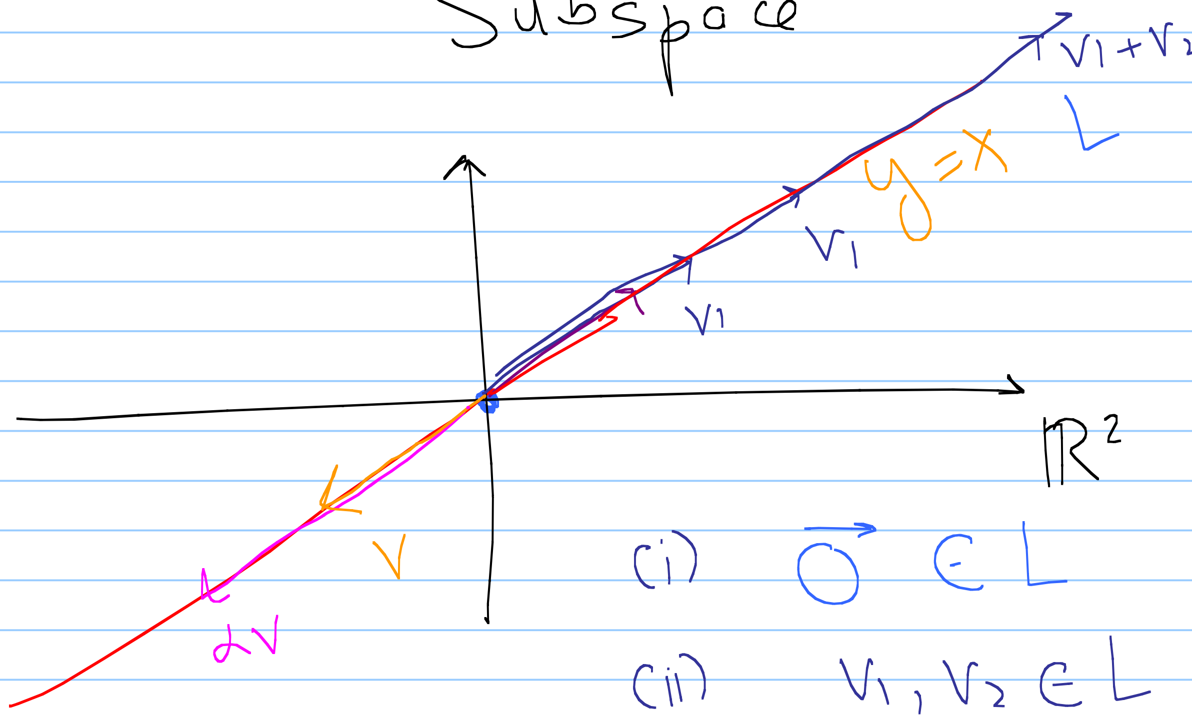
Def: Dimension of $\mathbb{R}^n$

Let B be a basis of $\mathbb{R}^n$,

dimension of $\mathbb{R}^n$ = $\dim(\mathbb{R}^n) = \#(B) = n$

$\mathbb{R}^n$ is an n-dimensional Space

Subspace



$\text{Span}(v) = \{\alpha v \mid \alpha \in \mathbb{R}\}$

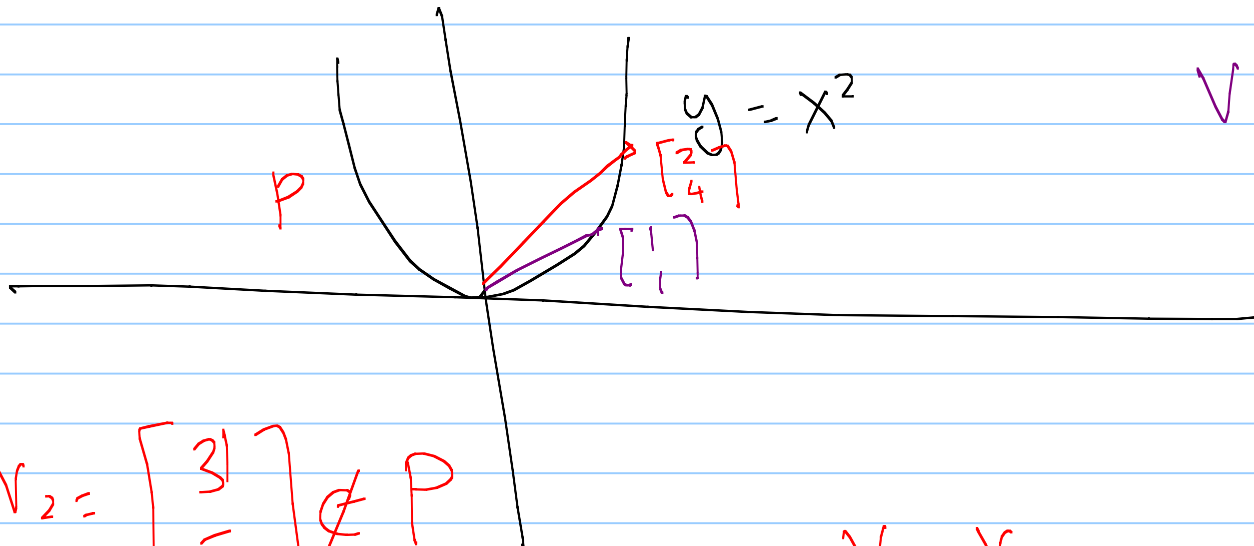
$\alpha = 0 \quad 0 \cdot v = \vec{0}$

$\in \text{Span}(v)$

(i) $\vec{0} \in L$

(ii) $v_1, v_2 \in L \qquad v_1 + v_2 \in L$

(ii) $\alpha \in \mathbb{R}$, $v \in L$, $\alpha v \in L$



$$V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$V_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$V_1 + V_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \notin P$$

$$V_1 + V_2 =$$

Subspace of $\mathbb{R}^n$

Def: A subset W of $\mathbb{R}^n$ is a

Subspace of $\mathbb{R}^n$:

① $\boxed{\vec{0} \in W}$ ② For all $\underline{u, v} \in W$, $\boxed{u+v \in W}$

③ For all $\alpha \in \mathbb{R}$, $u \in W$, $\boxed{\alpha u \in W}$

Example