2. For the equation $\tan x + \lambda x = 0$, where $0 < \lambda < 1$, show graphically that there is exactly one root p in $[\frac{\pi}{2}, \pi]$. Does the Newton's method converge to the root p in $[\frac{\pi}{2}, \pi]$ if we start with initial approximation $p_0 = \pi$? Justify your answer

$y = \tan x + \lambda x$

$y = \sec^2 x + \lambda$

$\tan x + \lambda x = 0$

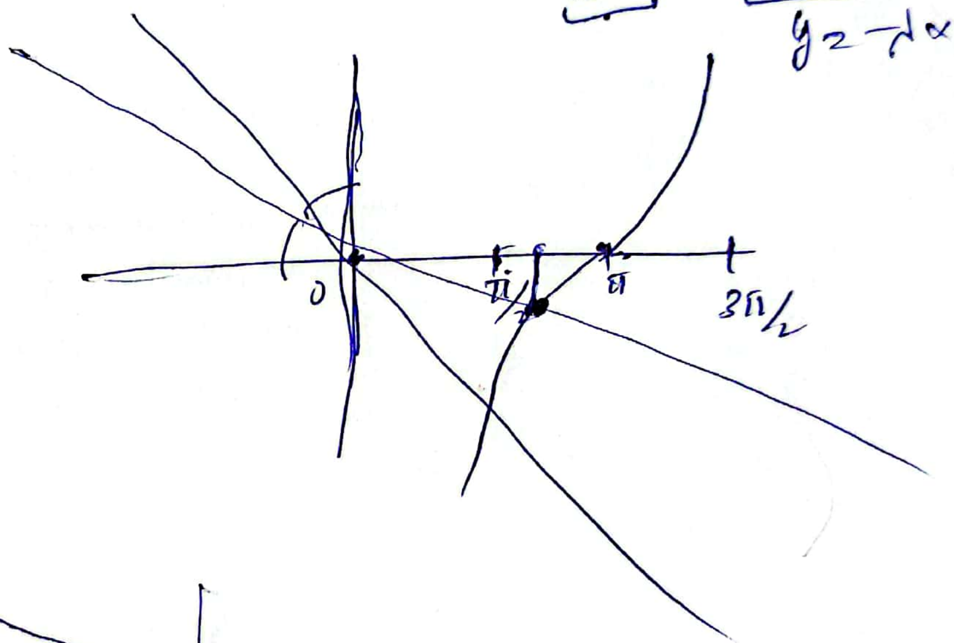
$\tan x = -\lambda x$

$y = -\lambda x$

$$P_n = P_{n-1} - \frac{f(P_{n-1})}{f'(P_{n-1})}$$

$$P_1 = P_0 - \frac{f(P_0)}{f'(P_0)}$$

$$P_1 = \pi - \frac{\tan \pi + \lambda \pi}{\sec^2 \pi + \lambda}$$

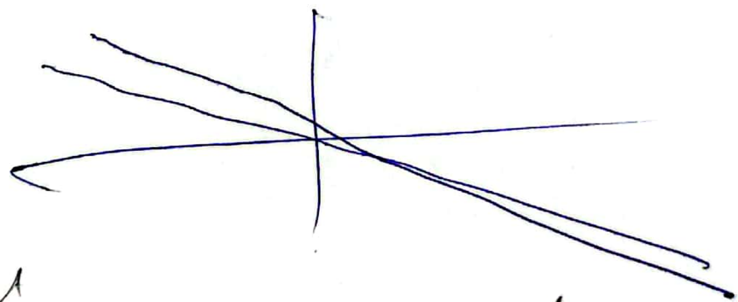


✓ $f'(x)$ $\sec^2 x + 0.5 \neq 0$

✓ $f''(x) = 2\sec^2 x \tan \neq 0$

✓ P is on (a,b)

✓ $f(p) = 0$



$y = \lambda x$

If all the above conditions are satisfied then there exist $\delta > 0$ such that for any $P_0 \in [p-\delta, p+\delta]$ sequence P_n generated by Newton method converge to root p.

Solve it again

2. Let x_n be the iterative sequence generated by the Newton-Raphson method in finding the root of the equation $e^{-ax} = x$, where a in the range $0 < a \leq 1$. If x^* denoted the exact root of this equation and $x_0 > 0$, then show that

$$|x^* - x_{n+1}| \leq \frac{1}{2}(x^* - x_n)^2$$

$f(x) = e^{-ax} - x$ assume $a=1$

$f(x) = e^{-x} - x$

$$\frac{|x^* - x_{n+1}|}{|x^* - x_n|^2} \leq \frac{1}{2}$$ $x_0 = 0.5$ $f'(x) = -e^{-x} - 1$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ ← equation.

$x_1 = x_0 + \frac{e^{-x} - x}{+e^{-x} + 1}$ $g(\ ) < 1/2$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$\cancel{x = x}_n$.

$$\frac{|x_{n+1} - x^*|}{|x'_n - x^*|} = \frac{g''(p)}{2} < \frac{1}{2}$$

$f(x_n) = f(x_n) + \cancel{(x - x_n)} f'(x_n) + \frac{(x-x_n)^2 f''(x_n)}{2!} +$

$n \to \infty$

$f(x_n) = x^*$ $x^{*2} - 2x^* x_n + x_n^2$

$g(x) = x - \frac{f(x)}{f'(x)}$ $e^{-ax} - x = 0$ $e^{-ax} = x$

$f(x) = e^{-ax} - x$

$f'(x) = -a e^{-ax} - 1$ → $f''(x)$ $a^2 e^{-ax}$

$1e^{-x}$

at $x = 1$

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

$$x_n = x_{n-1} - \frac{e^{-ax_{n-1}} - x_{n-1}}{-a e^{-ax_{n-1}} - 1}$$

Scanned with CamScanner

4. Find the largest error that occurs if linear Lagrange interpolation is used to approximate $f(x) = x^2$ for $N \le x \le N+1$, where N is an integer.

$f(x) = x^2$

Formula

$E(x) = f(x) - P_n(x)$

$$\boxed{E_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)(x-x_2)\dots(x-x_n)}$$

$E_1(x) = f$

$x_0 = N$

$x_1 = N+1$

$$E(x) = \frac{f^{(2)}(\xi(x))}{2!}(x-x_0)(x-x_1)$$

0 1 2 3

$$E(x) = \frac{2}{2}(x-N)(x-N+1)$$

$$= \boxed{(x-N)(x-N+1)}$$

$\frac{1}{2}$

Max $(E(x))$

N, N+1

$\frac{N+1}{2}$

$g(x) = x^2 - x(N+1) - Nx + N(N+1)$

$\frac{3}{4}$

$g\left(\frac{N+1}{2}\right)$

$g(x) = x^2 - 2Nx - x + N(N+1)$

$= \left(\frac{N+1}{2} - N\right)\left(\frac{N+1}{2} - N+1\right)$

$g(x) = x^2 + (-2N-1)x + N(N+1)$

$= \left(\frac{N+1-2N}{2}\right)\left(\frac{-N+1}{2}\right)$

$g'(x) = 2x - 2N - 1 = 0$, $\boxed{x = \frac{N+1}{2}}$

$= \left(-\frac{N+1}{2}\right)\left(-\frac{N+1}{2}\right)$