Quantum Phase Estimation

Phase estimation is a fundamental procedure in quantum computing that allows us to estimate an unknown phase angle of a specific quantum state. To understand phase estimation, let's consider a unitary operator called U that has an eigenvector |u⟩ with an eigenvalue $e^{2\Pi i\theta}$, where θ is the unknown phase we want to estimate. Note that in this documentation we use big endian notation.

The goal of the phase estimation algorithm is to determine the value of θ. To do this, we assume that we have access to special black boxes or oracles that can prepare the state |u⟩ and perform a controlled-U operation, controlled by the qubits in the first register. These controlled-U operations are performed for specific powers of two, denoted as j.

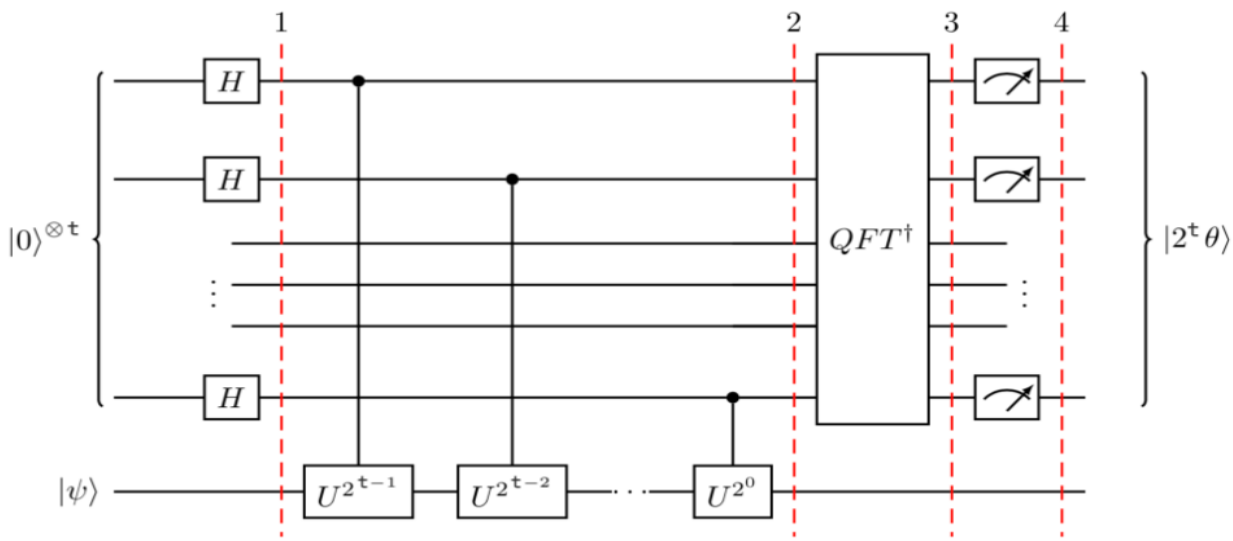
The phase estimation procedure itself is not a complete quantum algorithm but rather a subroutine or module that can be combined with other subroutines to perform interesting computational tasks. In specific applications, we describe how to perform the black box operations and combine them with the phase estimation procedure to achieve useful tasks.

The phase estimation procedure involves two registers: the first register contains t qubits initially set to the |0⟩ state, and the second register holds the |u⟩ state. We choose t depends on two things: the number of digits of accuracy we wish to have in our estimate for θ, and with what probability we wish the phase estimation procedure to be successful.

The phase estimation process consists of two stages. In the first stage, we apply a circuit that includes a Hadamard transform to the first register, followed by controlled-U operations on the second register with U raised to successive powers of two. This circuit prepares an entangled state between the two registers.

The final state of the first register, after applying the circuit, provides information about the unknown phase θ. The dependence of the number of qubits (t) in the first register on the desired accuracy of the estimate and the probability of success emerges naturally from the analysis of the phase estimation procedure.

The general quantum circuit for phase estimation is shown below



Quantum Phase Estimation procedure can be summarized in 5 steps :

1. Initial state
2. Superposition
3. Black Box
4. Inverse QFT
5. Measurement

1.Initial state

In this step a quantum circuit containing two registers is created. First register containing ***t***-number of counting qubits for estimating phase. Second register contains qubit(s) which has the state Ψ encoded in it.

$$|0\rangle^{\otimes t} |\psi\rangle$$

2.Superposition

In this step a ***t***- qubit Hadamard gate is applied on the first register which creates a superposition on the first ***t***- counting qubits.

$$\frac{1}{2^{\frac{t}{2}}}(|0\rangle + |1\rangle)^{\otimes t} |\psi\rangle$$

3. Black Box (Controlled U operations)

This step performs a black box operation on the circuit. This black box performs the main procedure of the phase estimation algorithm by writing the phase on the counting qubits with controlled unitary gates.

The Black Box:

This black box performs controlled unitary operations which are controlled by counting qubits and target as ψ qubit. The quantum phase estimation algorithm uses phase kickback to write the phase of U (in the Fourier basis) to the t qubits in the counting register.

When we use a qubit to control the U-gate, the qubit will turn (due to kickback) proportionally to the phase $e^{2\Pi i\theta}$. We can use successive CU-gates to repeat this rotation an appropriate number of times until we have encoded the phase theta as a number between 0 and 2^t in the Fourier basis.

Since U is a unitary operator with eigenvector $|\psi\rangle$ such that $U\,|\psi\rangle\ = e^{2\pi i\theta}|\psi\rangle$, this means:

$$U^{2^j}|\psi\rangle = U^{2^j-1}U|\psi\rangle = U^{2^j-1}e^{2\pi i\theta}|\psi\rangle = \cdots = e^{2\pi i2^j\theta}|\psi\rangle$$

Applying all the t controlled operations CU^{2^j} with $0 \le j \le t-1$, and using the relation

$$|0\rangle \otimes |\psi\rangle + |1\rangle \otimes e^{2\pi i\theta}|\psi\rangle = \big(|0\rangle + e^{2\pi i\theta}|1\rangle\big) \otimes |\psi\rangle$$

we get

$$\frac{1}{2^{\frac{t}{2}}}(|0\rangle + e^{2\pi i\theta 2^{t-1}}\,|1\rangle) \otimes ... \otimes (|0\rangle + e^{2\pi i\theta 2^{1}}\,|1\rangle) \otimes (|0\rangle + e^{2\pi i\theta 2^{0}}\,|1\rangle) \otimes |\psi\rangle$$

Which is equivalent to

$$\frac{1}{2^{\frac{t}{2}}}\sum_{k=0}^{2^t-1} e^{2\pi i\theta k}\,|k\rangle \otimes |\psi\rangle$$

where k denotes the integer representation of n-bit binary numbers.

4.Inverse QFT

To determine the phase encoded in our counting qubits, which are in the Fourier basis due to the CU gate operation, we need to convert the qubits from the Fourier basis to the computational basis. To achieve this conversion, we apply the inverse quantum Fourier transform (QFT). The inverse QFT allows us to retrieve the phase information in a form that is easily interpretable in the computational basis. By adding the inverse QFT to our circuit, we can extract the desired phase information and further analyze it.

Recall that QFT maps an t-qubit input state |j ⟩ to an output as

$$\frac{1}{2^{\frac{t}{2}}}(|0\rangle + e^{\frac{2\pi ij}{2}}\,|1\rangle) \otimes (|0\rangle + e^{\frac{2\pi ij}{2^2}}\,|1\rangle) \otimes ... \otimes (|0\rangle + e^{\frac{2\pi ij}{2^{t-1}}}\,|1\rangle) \otimes (|0\rangle + e^{\frac{2\pi ij}{2^t}}\,|1\rangle) \otimes |\psi\rangle$$

Replacing x by $2^t\theta$ in the above expression gives exactly the expression derived in step 2 above. Therefore, to recover the state $|2^t\theta\rangle$, apply an inverse Fourier transform on the auxiliary register. Doing so, we get

$$\frac{1}{2^{\frac{t}{2}}}\sum_{k=0}^{2^t-1} e^{2\pi i\theta k}\,|k\rangle \otimes |\psi\rangle \xrightarrow{QFT_t^{-1}} \frac{1}{2^t}\sum_{j=0}^{2^t-1}\sum_{k=0}^{2^t-1} e^{\frac{-2\pi ik}{2^t}(x-2^t\theta)}\,|x\rangle \otimes |\psi\rangle$$

4.Measurement

Measure the counting qubits in the computational basis.

In the above expression whenever the **j**=$2^t\theta$, the amplitude of measuring $2^t\theta$ reaches maximum. In situations where $2^t\theta$ is an integer,, if we measure the system in the computational basis, we will likely obtain the phase in the auxiliary register with a high probability.

$$|2^t\theta\rangle \otimes |\psi\rangle$$

However, when 2^t times θ is not a whole number, the expression still reaches its peak close to $2^t\theta$, but with a probability that is better than 40%. This means that even if the exact value is not obtained, we can still estimate the phase with reasonable accuracy.

Summary :

Phase estimation is a highly valuable technique in quantum computing, primarily because it enables us to accurately estimate eigenvalues associated with specific eigenvectors of unitary operators. However, its true significance lies in the fact that phase estimation serves as a fundamental building block for solving various complex quantum computing problems. By leveraging the power of phase estimation, we can reduce and tackle a diverse range of computational challenges, making it a crucial tool in the quantum computing toolbox.