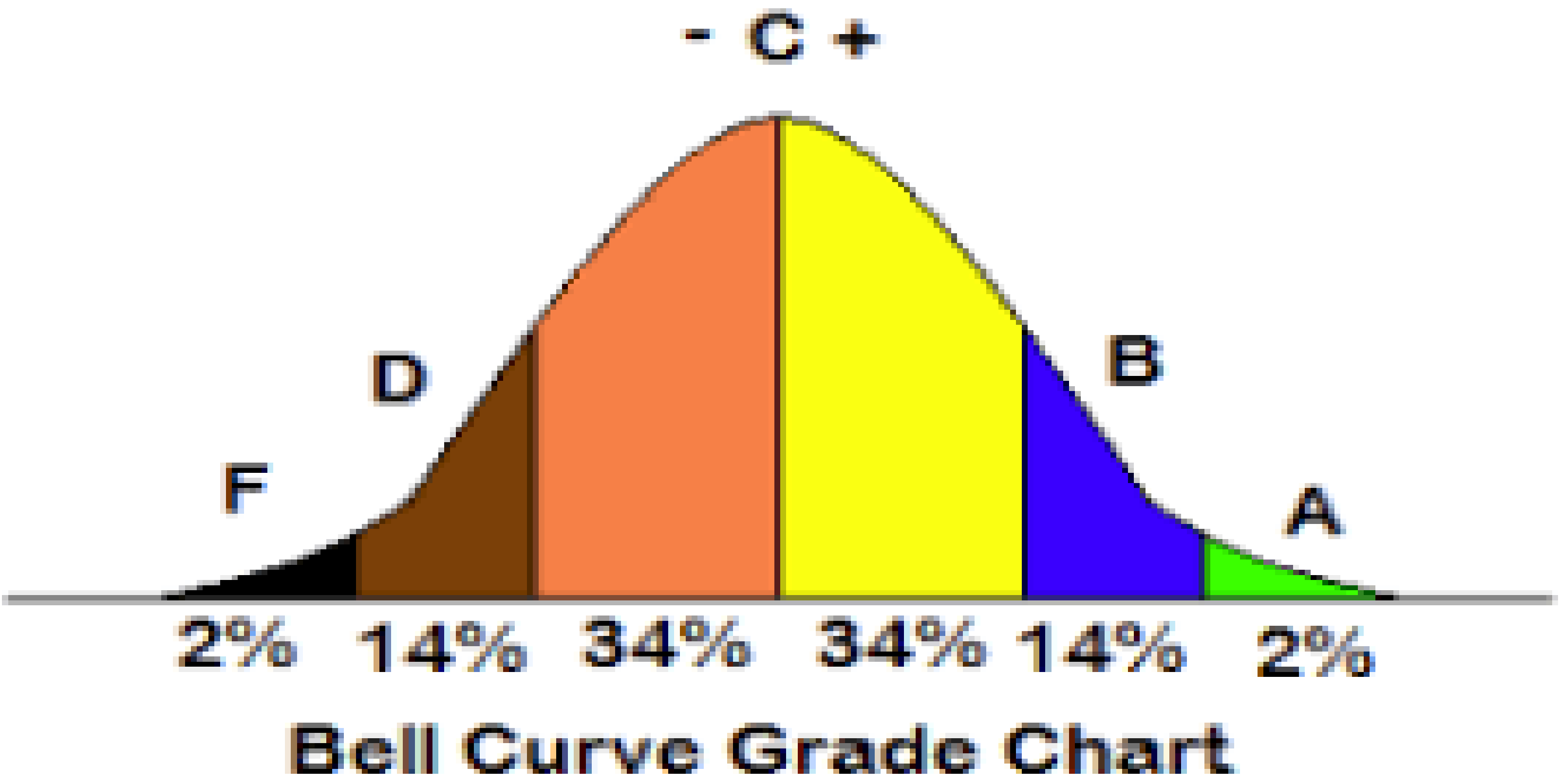
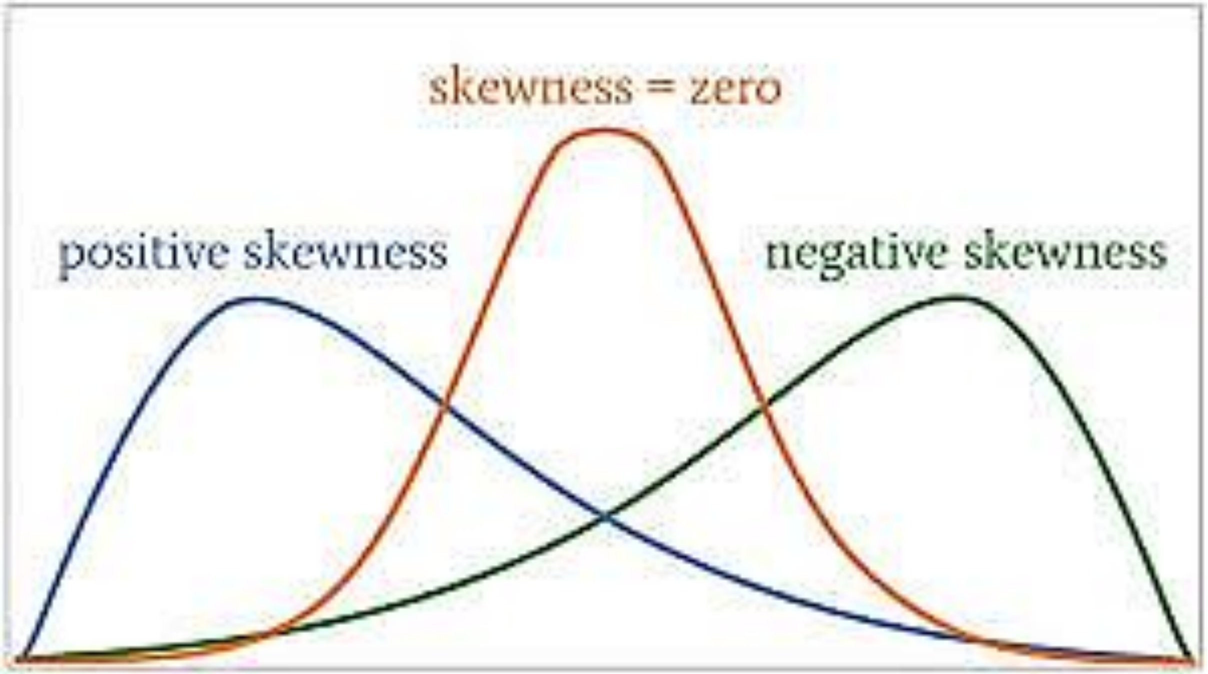
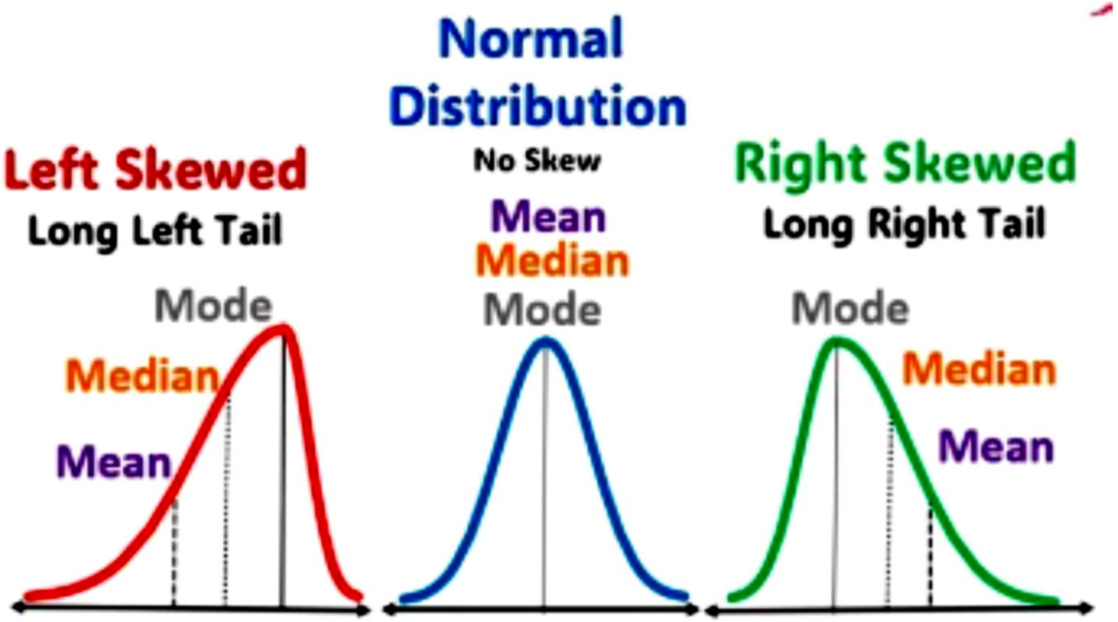
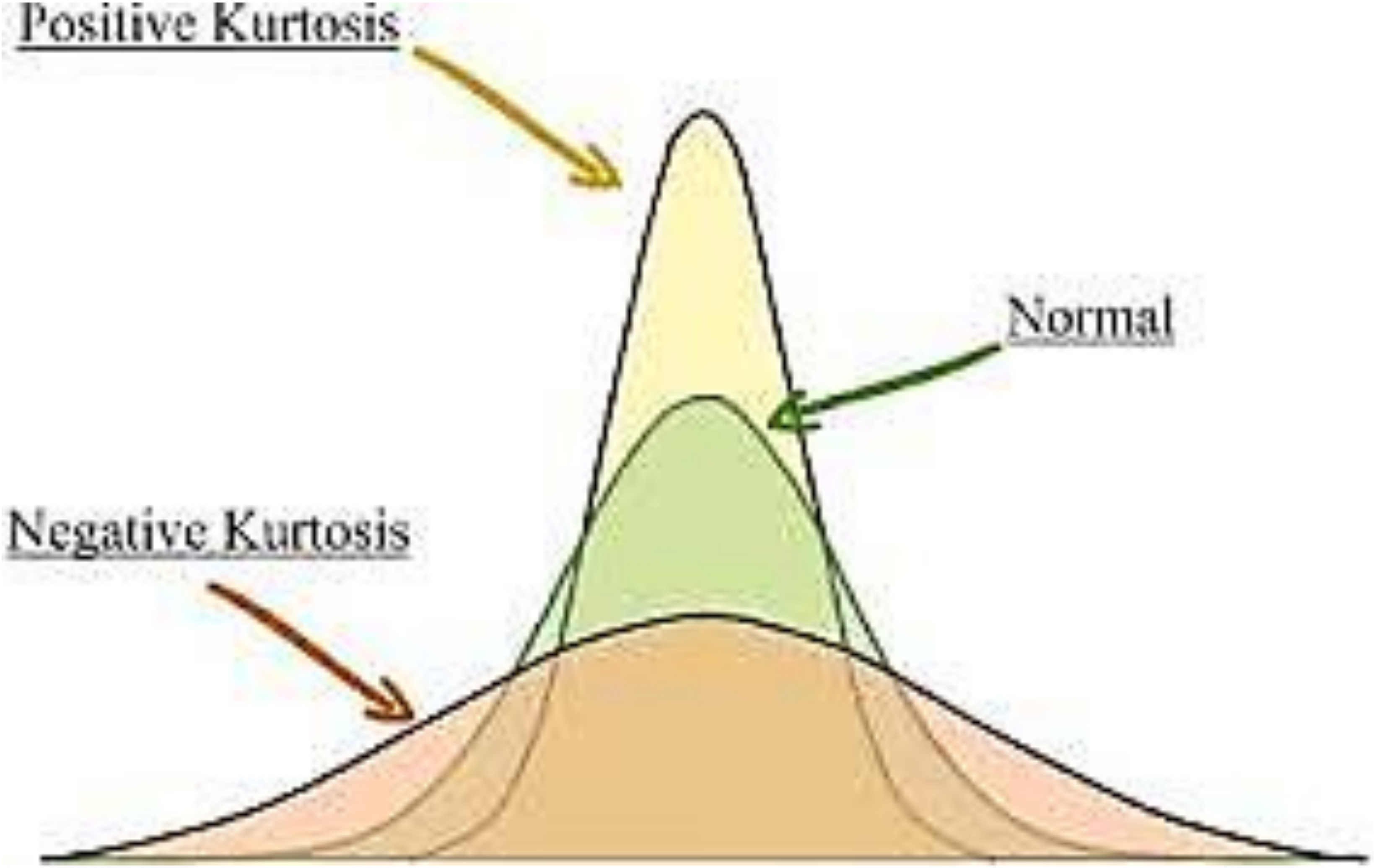
The Central Limit Theorem



The Central Limit Theorem






Positive Kurtosis
Normal
Negative Kurtosis

The Central Limit Theorem

In addition to knowing how individual data values vary about the mean for a population,
Statisticians are interested in knowing how the means of samples of the same size taken from the same population vary about the population mean.

Distribution of Sample Means

A **Sampling distribution of sample means** is a distribution obtained by using the means computed from random samples of a specific size taken from a population.

Sampling error is the difference between the sample measure and the corresponding population measure due to the fact that the sample is not a perfect representation of the population.

Properties of the Distribution of Sample Means

- ✓ **The mean of the sample means will be the same as the population mean.**

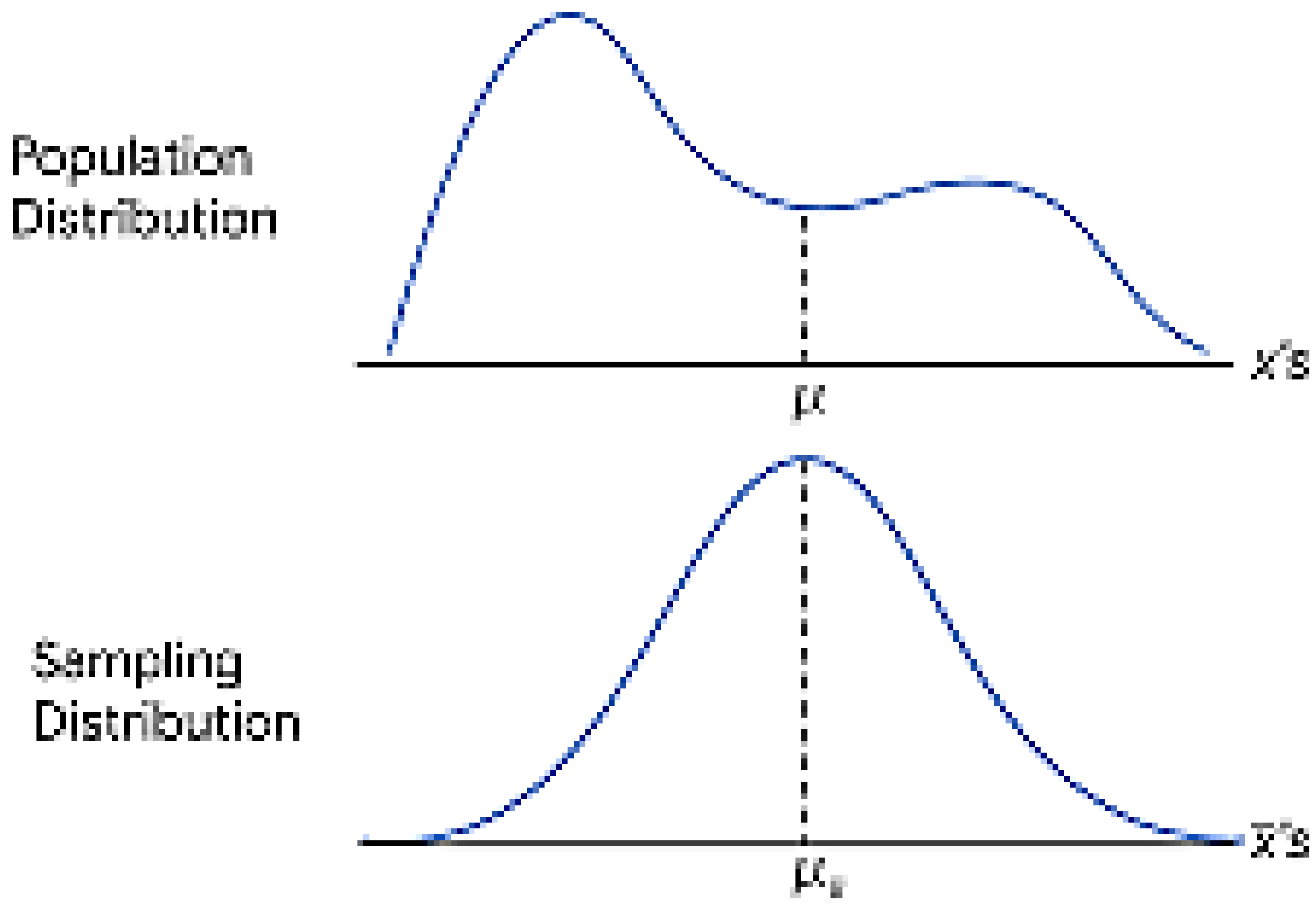
$$\text{Sample Mean} = \text{Population Mean} = \mu$$

- ✓ **The standard deviation of the sample means will be smaller than the standard deviation of the population, and will be equal to the population standard deviation divided by the square root of the sample size.**

$$\text{Sample Standard Deviation} = \frac{\text{Standard Deviation}}{n}$$

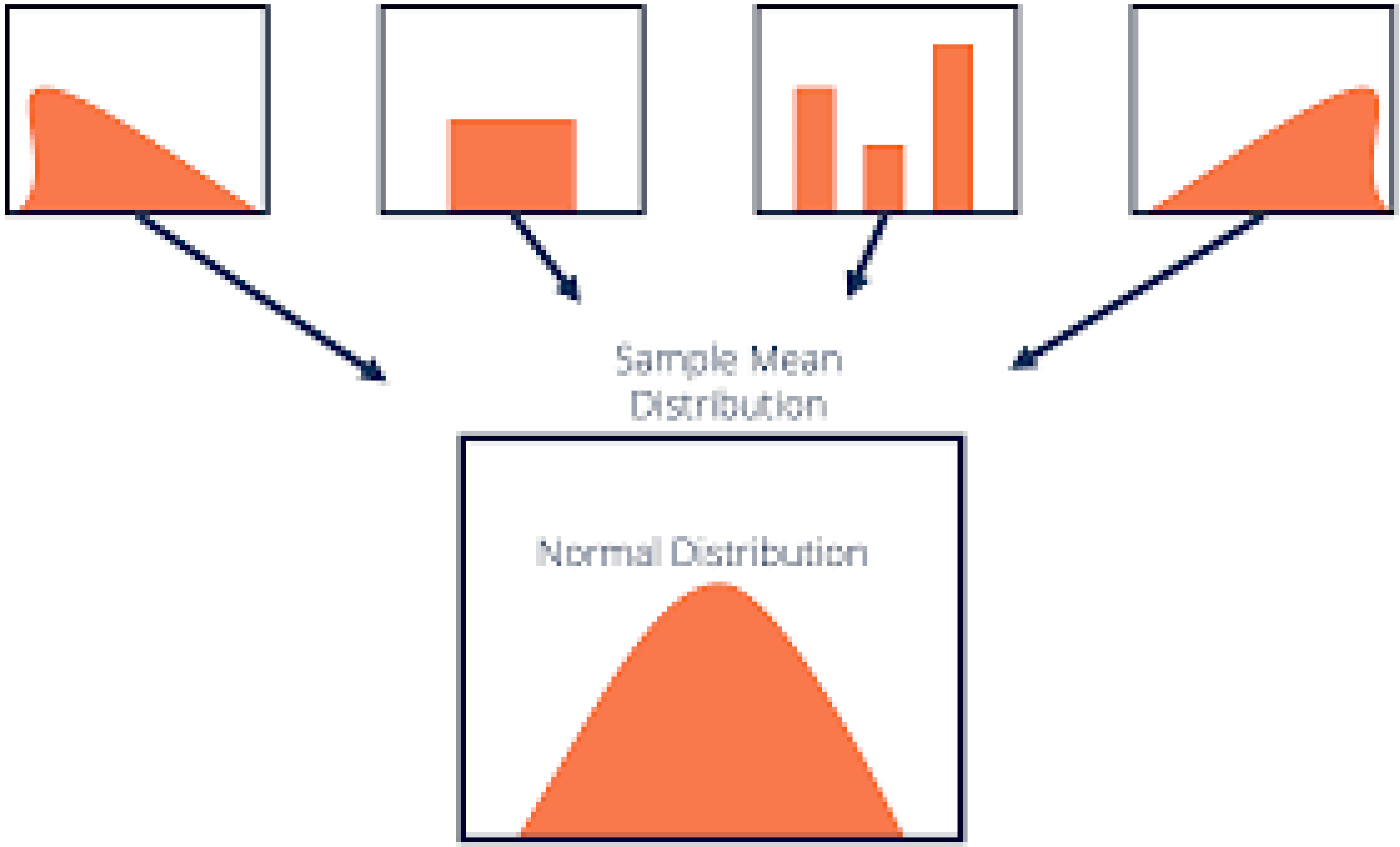
OR

$$\text{Sample Standard Deviation} = \frac{\sigma}{\sqrt{n}}$$

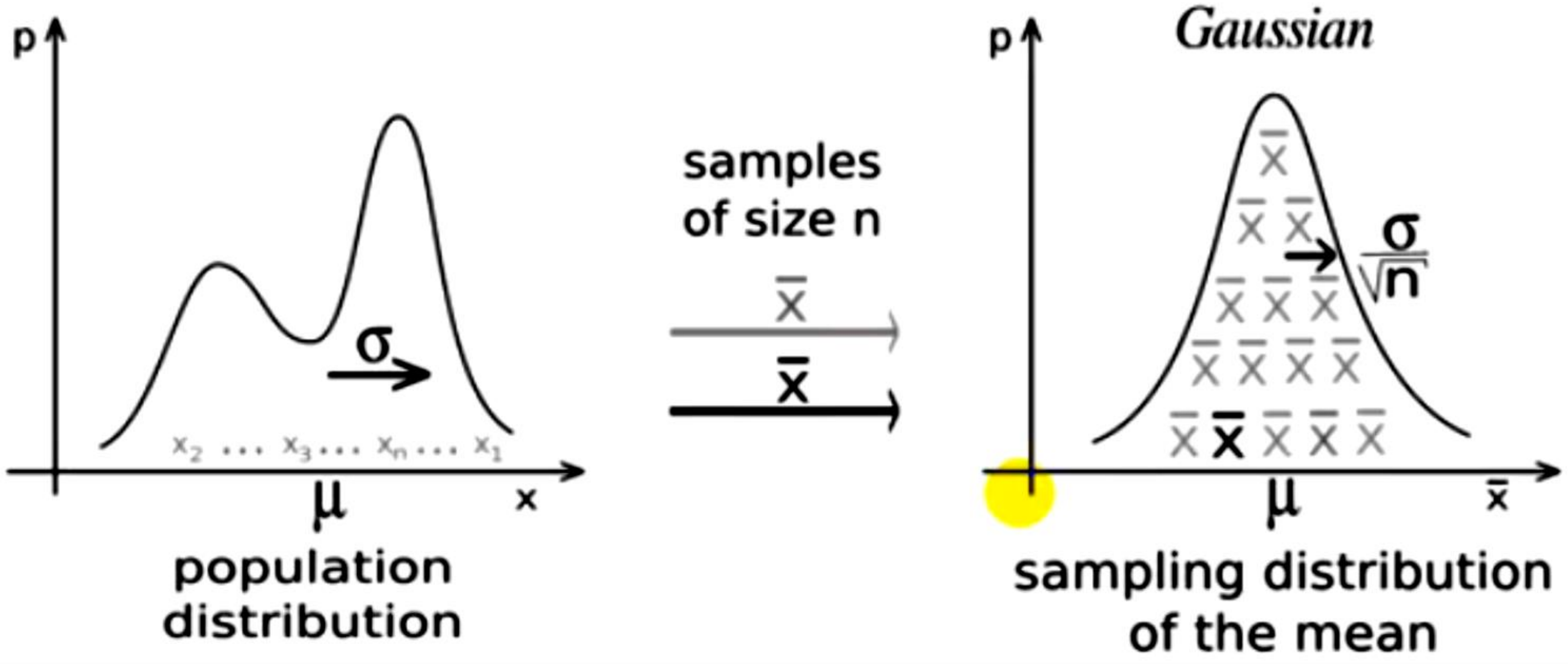


Population
Distribution
x's
μ
Sampling
Distribution
$\bar{x}$'s
$\mu_{\bar{x}}$

The Central Limit Theorem



Central Limit Theorem



Positively Skewed Distribution (μ, σ)

S1= x1,x2,x3.......x30 Mean= $\bar{x}_1$

S2= x1,x2,x3.......x30 Mean= $\bar{x}_2$

S3= x1,x2,x3.......x30 Mean= $\bar{x}_3$

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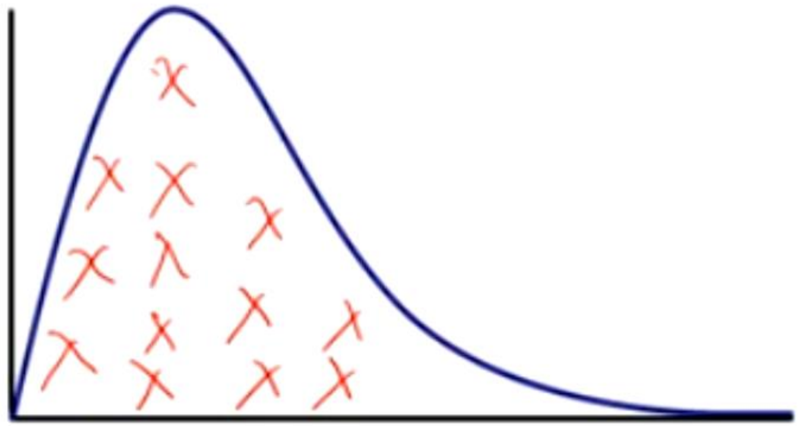
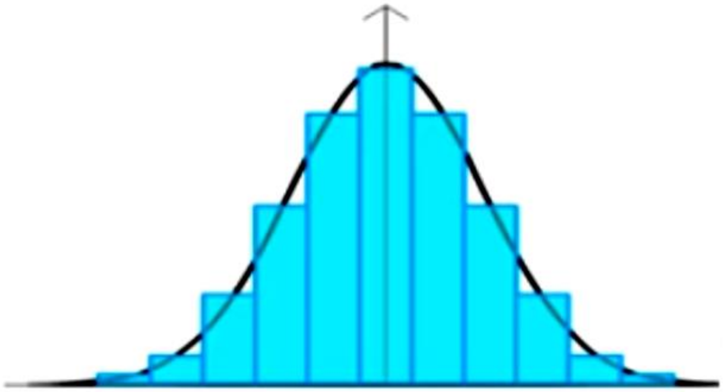
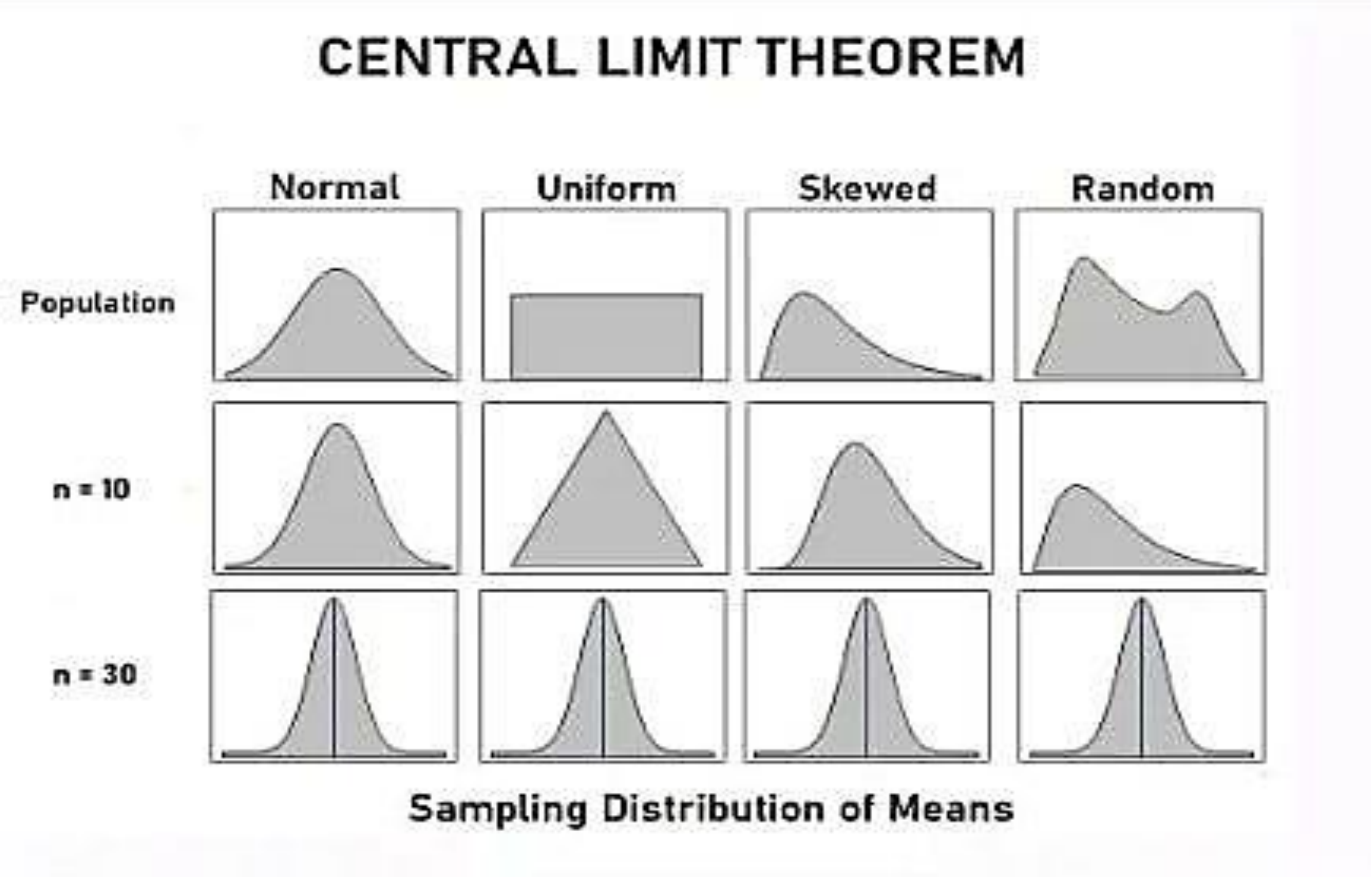
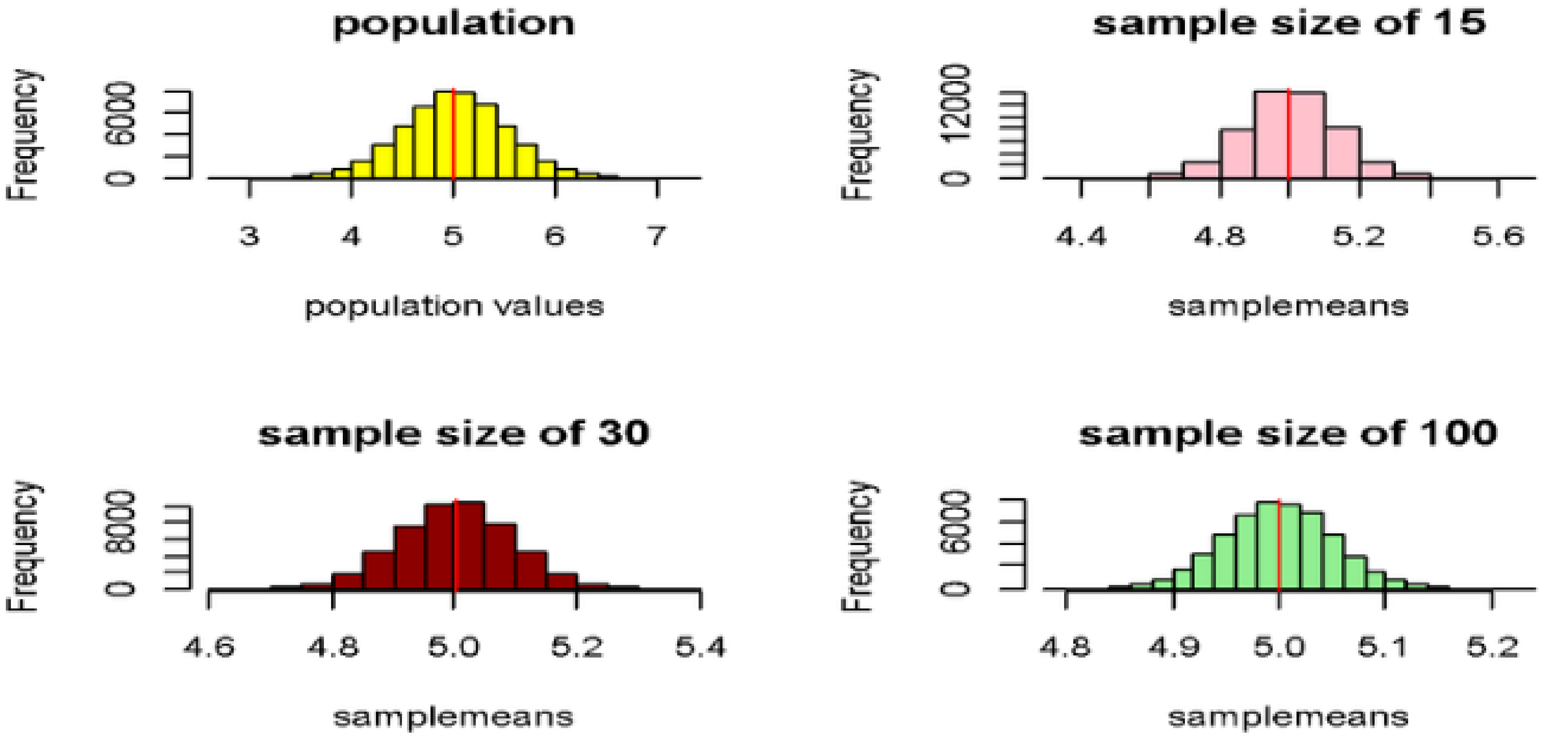
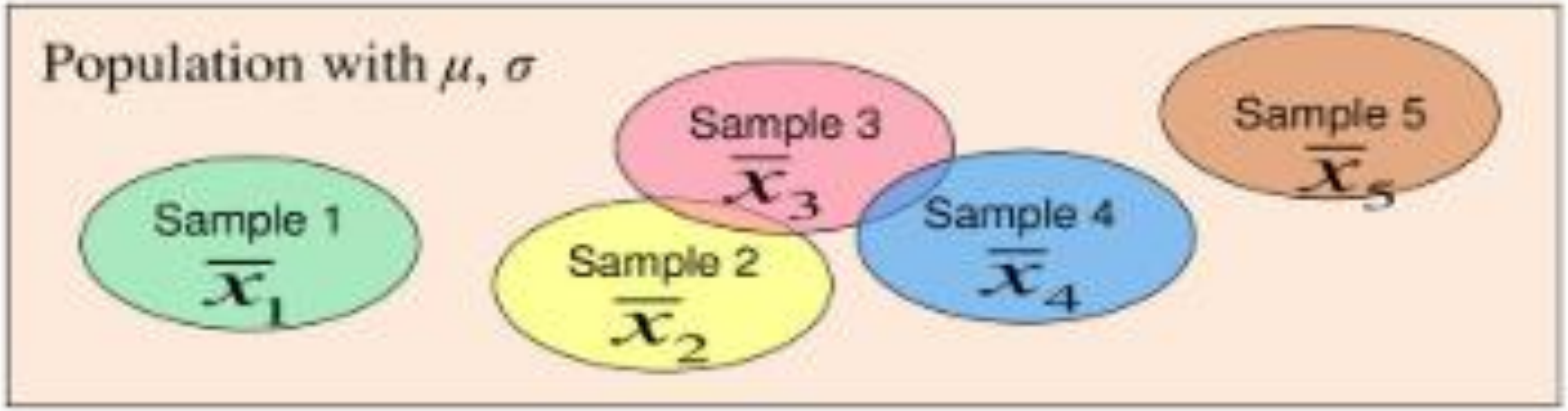
S100= x1,x2,x3.......x30 Mean= $\overline{x_{100}}$



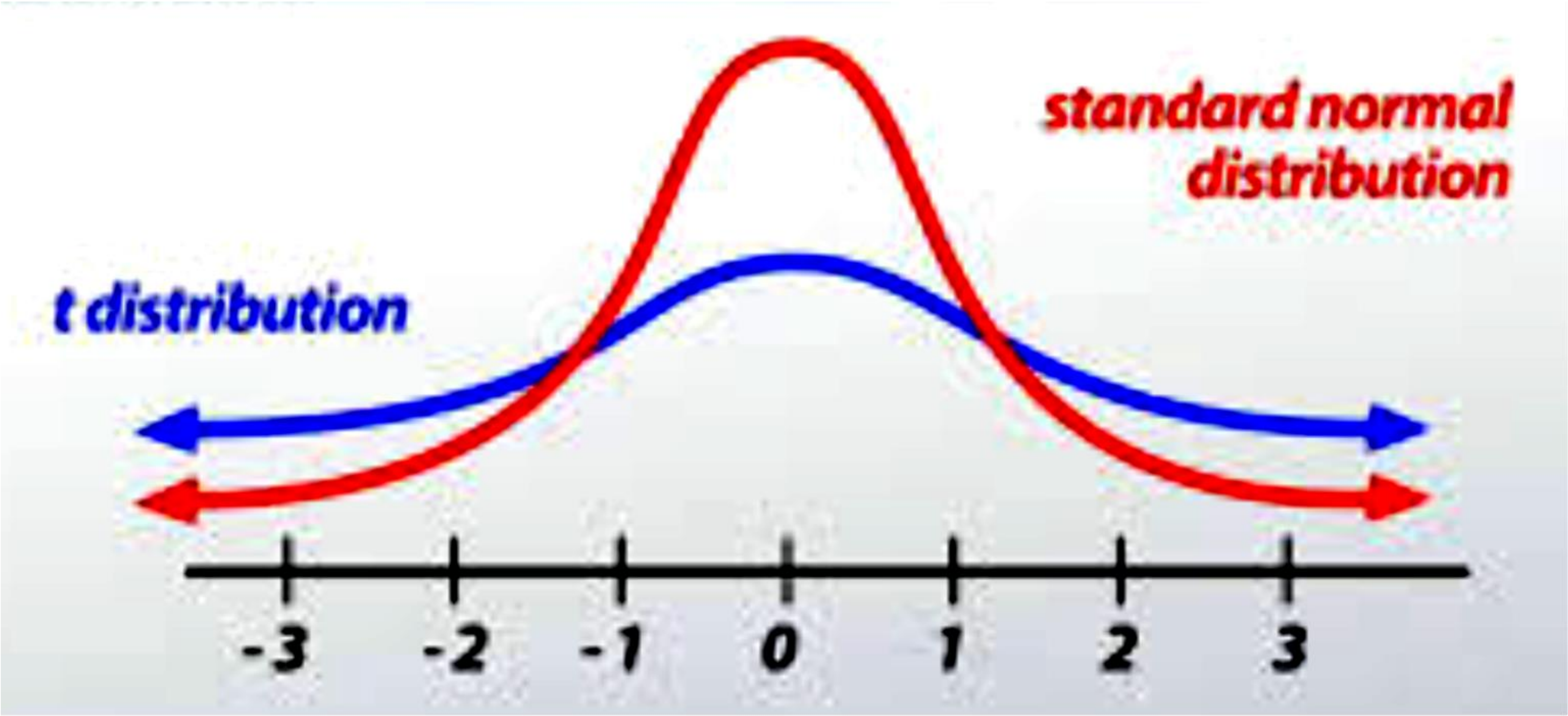



CENTRAL LIMIT THEOREM
Normal
Uniform
Skewed
Random
Population
n = 10
n = 30
Sampling Distribution of Means


population
Frequency
0 6000
3 4 5 6 7
population values
sample size of 15
Frequency
0 12000
4.4 4.8 5.2 5.6
samplemeans
sample size of 30
Frequency
0 8000
4.6 4.8 5.0 5.2 5.4
samplemeans
sample size of 100
Frequency
0 6000
4.8 4.9 5.0 5.1 5.2
samplemeans



The sampling distribution consists of the values of the sample means, $\overline{x}_1, \overline{x}_2, \overline{x}_3, \overline{x}_4, \overline{x}_5, \ldots$


standard normal distribution
t distribution
-3
-2
-1
0
1
2
3

Central Limit Theorem: If $\bar{X}$ is the mean of a random sample of size n taken from a population with mean μ and finite variance σ^2, then the limiting form of the distribution of

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}},$$

as $n \to \infty$, is the standard normal distribution $n(z; 0, 1)$.

$$z = \frac{\bar{x} - \mu_{\bar{x}}}{\sigma_{\bar{x}}} = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

Example 1 Normal Distribution

$$z = \frac{x-\mu}{\sigma}, z = \frac{\bar{x}-\mu_{\bar{x}}}{\sigma_{\bar{x}}} = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}}$$

An elevator has a sign stating that the maximum capacity is 4000 lb - 27 passengers. **This converts to a mean passenger weight of 148 lb when the elevator is full.** Assume a worst-case scenario of the elevator being filled with 27 adult males. (Adult males have weights that are **normally distributed** with a mean of 189 lb and a standard deviation of 39 lb.)

a. Find the probability that 1 randomly selected adult male has a weight greater than 148 lb.

b. Find the probability that a sample of 27 randomly selected adult males has a mean weight greater than 148 lb.

Use ND: If the original population is normally distributed or $n > 30$.

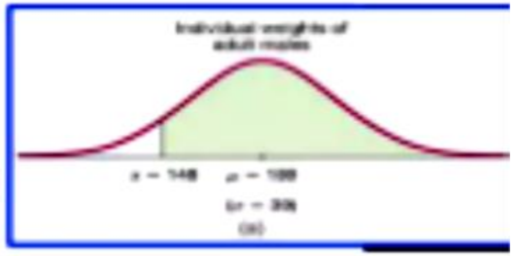
$n < 30$: but the original population of weights of males has a normal distribution, so samples of any size will yield means that are normally distributed.

Given: ND, $\mu = 189\text{lb}$ & $\sigma = 39\text{lb}, \text{n} = 1$

$$z = \frac{x-\mu}{\sigma} = \frac{148-189}{39} = -1.05$$

$$P(x > 148) = P(z > -1.05)$$

$$= 1 - 0.1469 = 0.8531$$



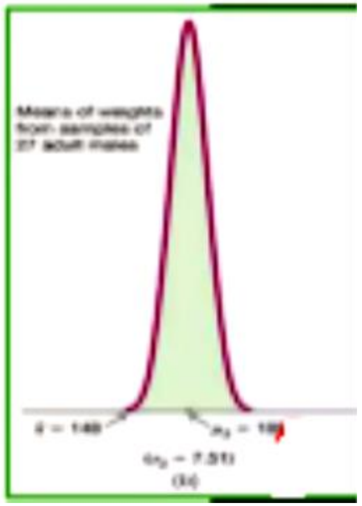
b. ND: $\mu = 189\text{lb}$ & $\sigma = 39\text{lb}, \text{n} = 27$

$P(\bar{x} > 148) =$ $\mu_{\bar{x}} = \mu = 189, \sigma_{\bar{x}} = \sigma/\sqrt{n} = 39/\sqrt{27} = 7.51$

$P(z > -5.46)$

$= 1 - 0.0001 = 0.9999$

$$z = \frac{148-189}{39/\sqrt{27}} = -5.46$$



Interpretation: There is a 0.8534 probability that **an individual** male will weigh more than 148 lb.

There is a 0.99999998 probability that **27 randomly selected males** will have a mean weight of more than 148 lb. Given that the safe capacity of the elevator is 4000 lb, it is almost certain that it will be overweight if it is filled with 27 randomly selected adult males.

Example 2

$$z = \frac{x-\mu}{\sigma}, z = \frac{\overline{x}-\mu_{\overline{x}}}{\sigma_{\overline{x}}} = \frac{\overline{x}-\mu}{\sigma/\sqrt{n}}$$

Assume that the population of human body temperatures has a mean of 98.6°F, as is commonly believed with a standard deviation of 0.62°F. If a sample of size n = 106 is randomly selected, find the probability of getting a **mean** of **98.2°F or lower.**

Given: $\mu = 98.6$°F, & $\sigma = 0.62$°F, n =106 > 30 → *Central Limit Theorem* (*CLT*)

$\mu_{\bar{X}} = \mu = 98.6$

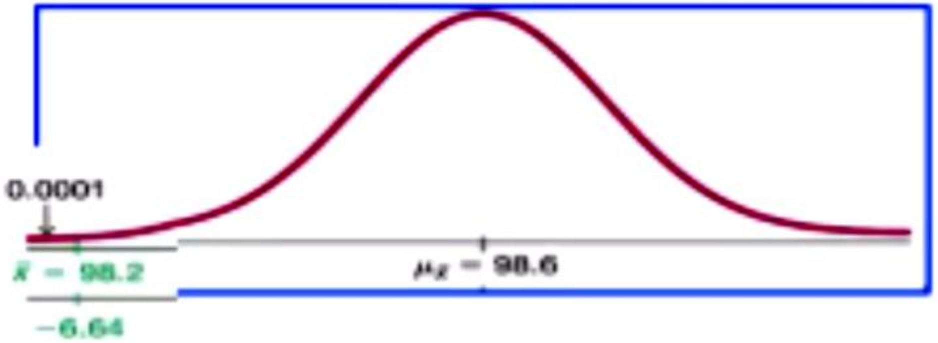
$\sigma/\sqrt{n} = 0.62/\sqrt{106} = 0.06022$

$$z = \frac{\bar{X}-\mu_{\bar{X}}}{\sigma/\sqrt{n}}$$

$$= \frac{98.2-98.6}{0.62/\sqrt{106}} = -6.64$$

$P(\overline{x} \leq 98.2) =$

$P(z \leq -6.64) \;\; = 0.0001$



Example 3

$$z = \frac{x-\mu}{\sigma}, z = \frac{\bar{x}-\mu_{\bar{x}}}{\sigma_{\bar{x}}} = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}}$$

Assume that children under the age of 6 watch an average of 25 hours of television per week. Also, assume the variable is **normally distributed** and the standard deviation is 3 hours. If 20 children between the ages of 2 and 5 are randomly selected, find the probability that the **mean** of the number of hours they watch television will be greater than 26.3 hours.

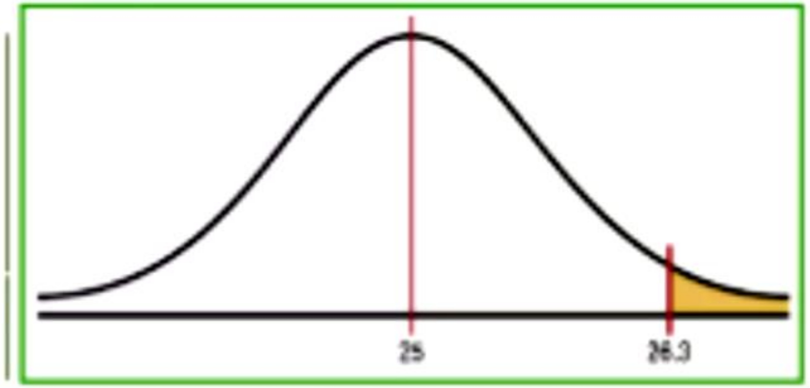
Given: ND: $\mu = 25$hrs, & $\sigma = 3$, n = 20

$$z = \frac{\bar{x}-\mu_{\bar{x}}}{\sigma_{\bar{x}}}$$

$$= \frac{26.3-25}{\frac{3}{\sqrt{20}}} = 1.94$$

$P(\bar{x} > 26.3) =$

$P(z > 1.94)$ $=0.0262$



Example 4

$$z = \frac{x - \mu}{\sigma}, z = \frac{\bar{x} - \mu_{\bar{x}}}{\sigma_{\bar{x}}} = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

The average age of a vehicle registered in the United States is 8 years, or 96 months. Assume the standard deviation is 16 months. If a random sample of 36 vehicles is selected, find the probability that the **mean** of their age is between 90 and 100 months.

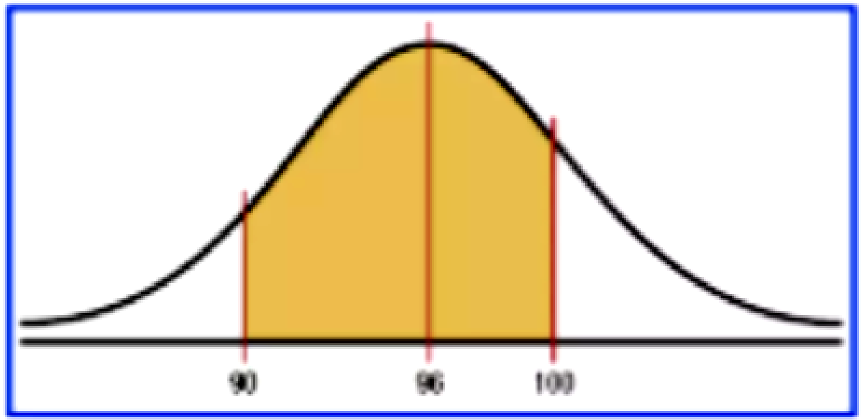
Given: $\mu = 96$ months, & $\sigma = 16$, $n = 36 > 30 \rightarrow CLT$

$$P(90 < \overline{x} < 100) = P(-2.25 < z < 1.5)$$

$$= 0.9332 - 0.0122 = 0.9210$$

$$z = \frac{90 - 96}{\frac{16}{\sqrt{36}}} = -2.25$$

$$z = \frac{100 - 96}{\frac{16}{\sqrt{36}}} = 1.50$$



Example 5: A person consumes 218.4 lb of meat per year on average with a standard deviation of 25 pounds and the distribution is **approximately normal**.

a. Find the probability that a person selected at random consumes less than 224 pounds per year.

b. If a sample of 40 individuals is selected, find the probability the sample mean will be less than 224 pounds per year.

$$z = \frac{x-\mu}{\sigma}, z = \frac{\overline{x}-\mu_{\overline{x}}}{\sigma_{\overline{x}}} = \frac{\overline{x}-\mu}{\sigma/\sqrt{n}}$$

Given: **ND,** $\mu = 218.4$ **lb, &** $\sigma = 25$**lb**

a.

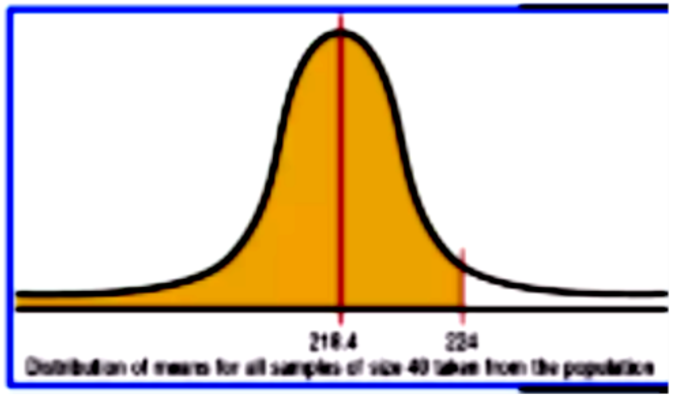
$$z = \frac{x-\mu}{\sigma} = \frac{224-218.4}{25} = 0.22$$

$$P(x < 224)$$
$$= P(z < 0.22)$$
$$= 0.5871$$

b.

$$z = \frac{\overline{x}-\mu_{\overline{x}}}{\sigma_{\overline{x}}} = \frac{224-218.4}{\frac{25}{\sqrt{40}}} = 1.42$$

$$P(\overline{x} < 224)$$
$$= P(z < 1.42)$$
$$= 0.9222$$



Example 8.4: An electrical firm manufactures light bulbs that have a length of life that is approximately normally distributed, with mean equal to 800 hours and a standard deviation of 40 hours. Find the probability that a random sample of 16 bulbs will have an average life of less than 775 hours.

***Solution*:** The sampling distribution of $\bar{X}$ will be approximately normal, with $\mu_{\bar{X}} = 800$ and $\sigma_{\bar{X}} = 40/\sqrt{16} = 10$. The desired probability is given by the area of the shaded

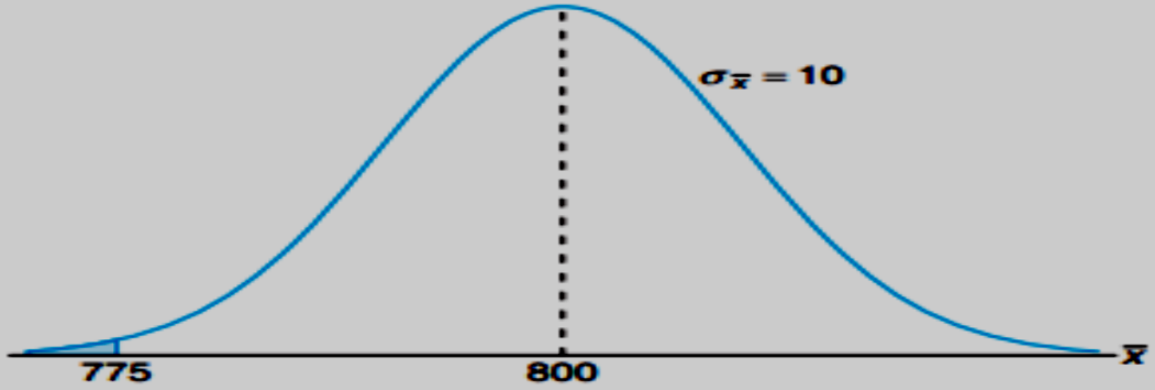


Figure 8.2: Area for Example 8.4.

Corresponding to $\bar{x} = 775$, we find that

$$z = \frac{775 - 800}{10} = -2.5,$$

and therefore

$$P(\bar{X} < 775) = P(Z < -2.5) = 0.0062.$$