CSE-303: COMPUTER GRAPHICS

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2D TRANSFORMATIONS

2D Transformations

- What are transformations?
  - Geometrical changes of an object from a current state to modified state.
- Why are transformations is needed?
  - To manipulate the initially created object and to display the modified object without having to redraw it.

2D Transformations

- 2 ways
  - Object Transformation
    - Alter the coordinates descriptions an object
    - Translation, rotation, scaling etc.
    - Coordinate system unchanged
  - Coordinate transformation
    - Produce a different coordinate system

Matrix Math

- Why do we use matrix?
  - More convenient organization of data.
  - More efficient processing
  - Enable the combination of various concatenations
- Matrix addition and subtraction

$$\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} \pm \begin{pmatrix} \mathbf{c} \\ \mathbf{d} \end{pmatrix} = \begin{pmatrix} \mathbf{a} \pm \mathbf{c} \\ \mathbf{b} \pm \mathbf{d} \end{pmatrix}$$

Matrix Math

- How about it?
- $\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} \pm \begin{pmatrix} \mathbf{c} & \mathbf{d} \\ \mathbf{e} & \mathbf{f} \end{pmatrix}$

- Matrix Multiplication
  - Element-wise product

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{e} & \mathbf{f} \\ \mathbf{g} & \mathbf{h} \end{pmatrix} = \begin{pmatrix} \mathbf{a.e + b.g} & \mathbf{a.f + b.h} \\ \mathbf{c.e + d.g} & \mathbf{c.f + d.h} \end{pmatrix}$$

Matrix Math

- What about this?

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = ?$$

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = ?$$

- Type of matrix

$$\begin{pmatrix} a & b \end{pmatrix} \qquad \begin{pmatrix} a \\ b \end{pmatrix}$$

Row-vector **Column-vector**

Matrix Math

- Is there a difference between possible representations?

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \bullet \begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} ae + bf \\ ce + df \end{bmatrix}$$

$$\begin{bmatrix} e & f \end{bmatrix} \bullet \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ae + cf & be + df \end{bmatrix}$$

$$\begin{bmatrix} e & f \end{bmatrix} \bullet \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} ae + bf & ce + df \end{bmatrix}$$

Matrix Math

- We'll use the column-vector representation for a point.
- Which implies that we use pre-multiplication of the transformation – it appears before the point to be transformed in the equation.

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Ax + By \\ Cx + Dy \end{bmatrix}$$

- What if we needed to switch to the other convention?

Translation

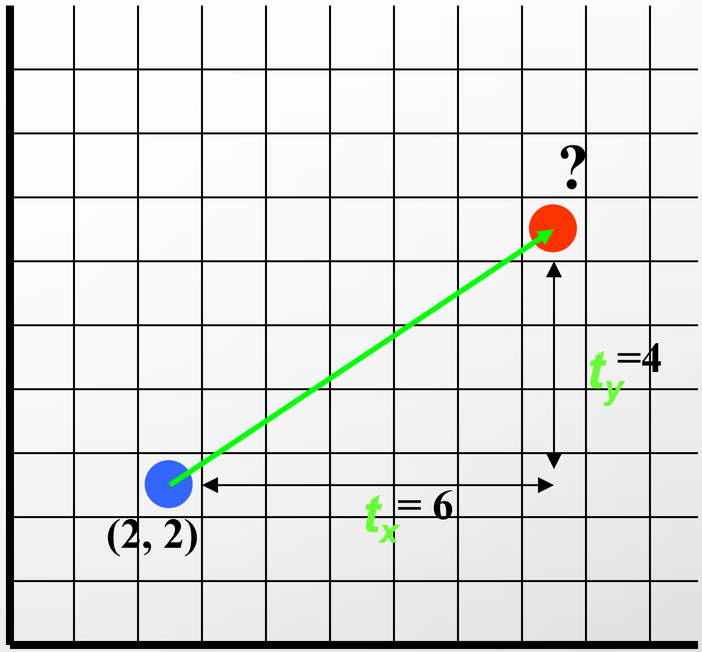
- A translation moves all points in an object along the same straight-line path to new positions.
- The path is represented by a vector, called the translation or shift vector.
- We can write the components:

$$p'_x = p_x + t_x$$
$$p'_y = p_y + t_y$$

- or in matrix form:

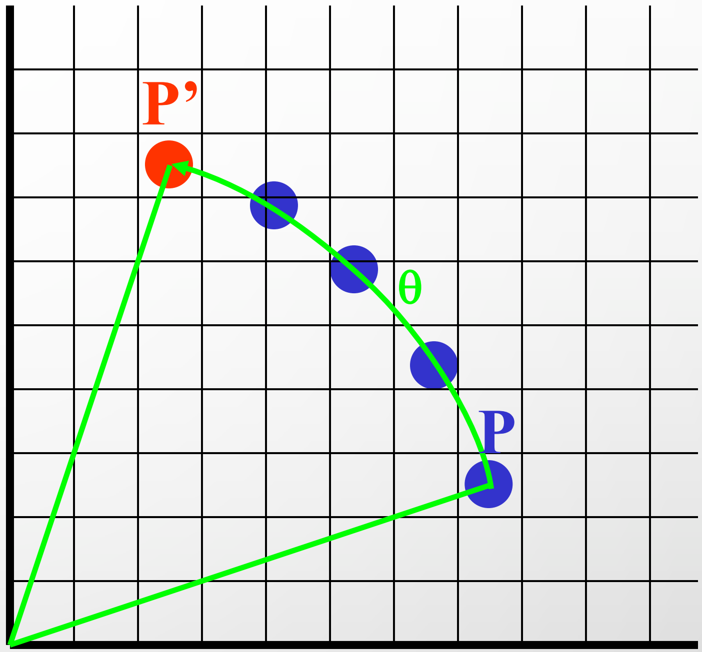
$$P' = P + T$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$



Rotation

- A rotation repositions all points in an object along a circular path in the plane centered at the pivot point.
- First, we'll assume the pivot is at the origin.



Rotation

- Review Trigonometry

=> $\cos\phi = x/r$, $\sin\phi = y/r$

- $x = r.\cos\phi$, $y = r.\sin\phi$

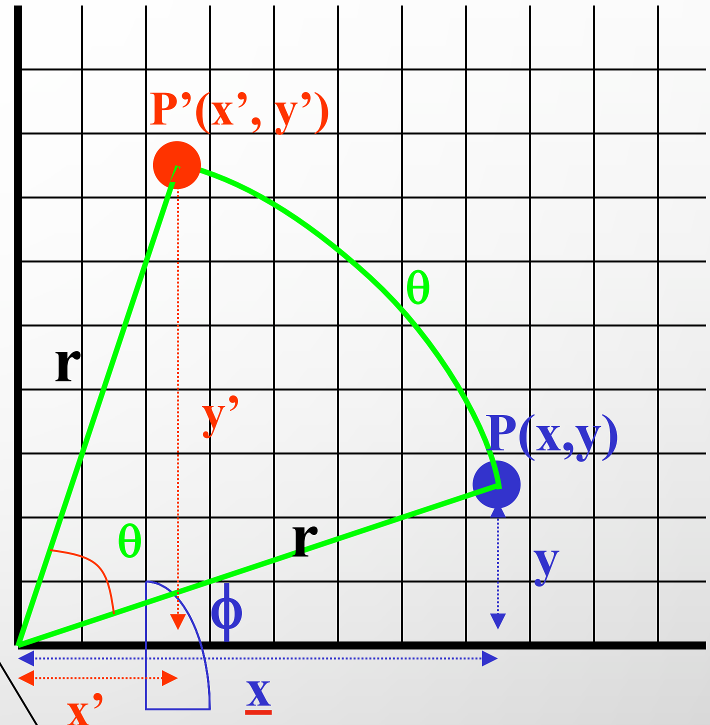
=> $\cos(\phi + \theta) = x'/r$

- $x' = r.\cos(\phi + \theta)$
- $x' = r.\cos\phi\cos\theta - r.\sin\phi\sin\theta$
- $x' = x.\cos\theta - y.\sin\theta$

=> $\sin(\phi + \theta) = y'/r$

$y' = r.\sin(\phi + \theta)$

- $y' = r.\cos\phi\sin\theta + r.\sin\phi\cos\theta$
- $y' = x.\sin\theta + y.\cos\theta$



Identity of Trigonometry

Rotation

- **We can write the components:**

$$p'_x = p_x \cos\theta - p_y \sin\theta$$
$$p'_y = p_x \sin\theta + p_y \cos\theta$$

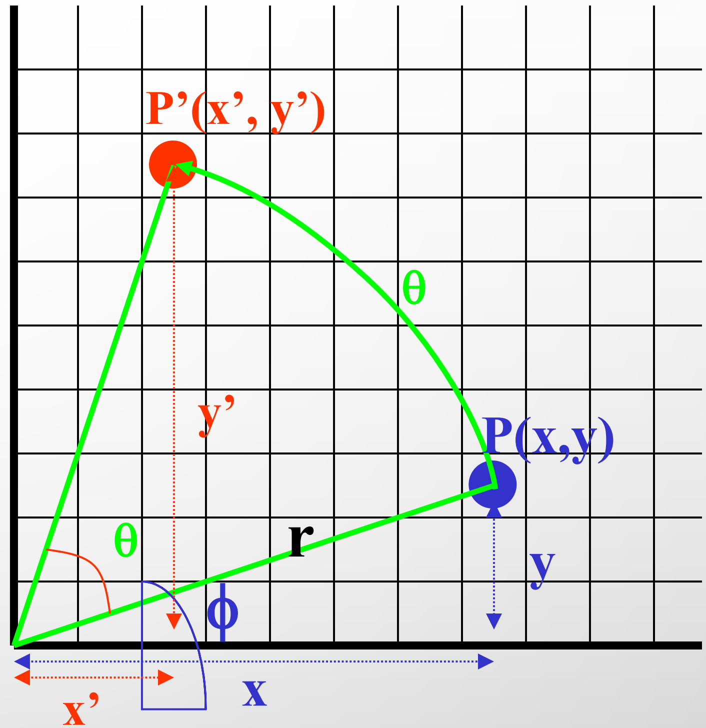
- **or in matrix form:**

$$P' = R \bullet P$$

∀ **θ can be clockwise (-ve) or counterclockwise (+ve as our example).**

- **Rotation matrix**

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



Rotation

- Example
  - Find the transformed point, P', caused by rotating P= (5, 1) about the origin through an angle of 90°.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} x \cdot \cos\theta - y \cdot \sin\theta \\ x \cdot \sin\theta + y \cdot \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 5 \cdot \cos 90 - 1 \cdot \sin 90 \\ 5 \cdot \sin 90 + 1 \cdot \cos 90 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \cdot 0 - 1 \cdot 1 \\ 5 \cdot 1 + 1 \cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

Scaling

- Scaling changes the size of an object and involves two scale factors, S_x and S_y for the x- and y-coordinates respectively.
- Scales are about the origin.
- We can write the components:

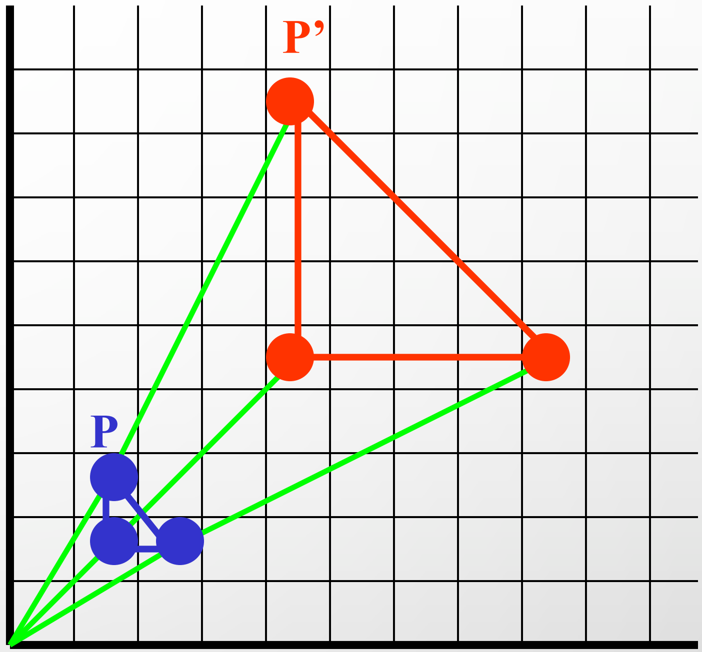
$$p'_x = s_x \bullet p_x$$
$$p'_y = s_y \bullet p_y$$

or in matrix form:

$$P' = S \bullet P$$

Scale matrix as:

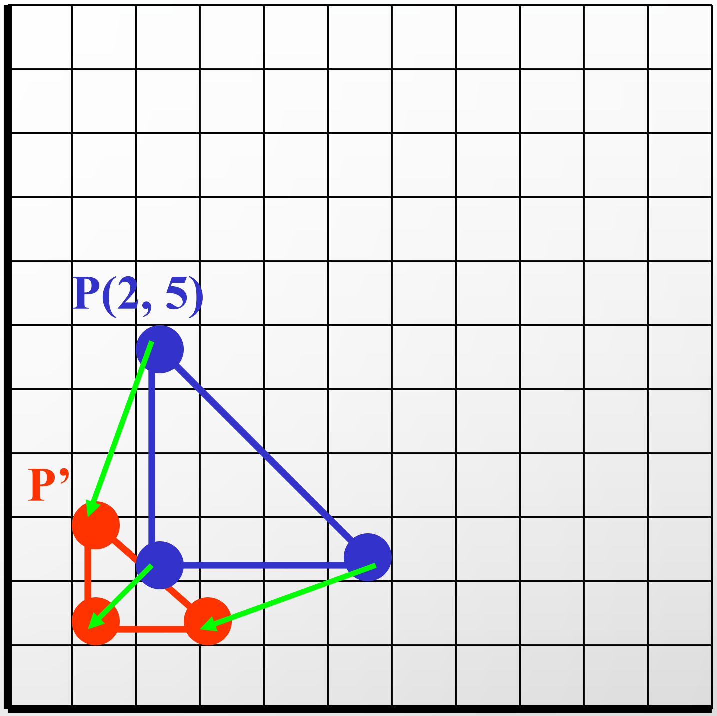
$$S = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$



Scaling

- If the scale factors are in between 0 and 1 ➔ the points will be moved closer to the origin ➔ the object will be smaller.

- Example :
    - P(2, 5), Sx = 0.5, Sy = 0.5
    - Find P' ?



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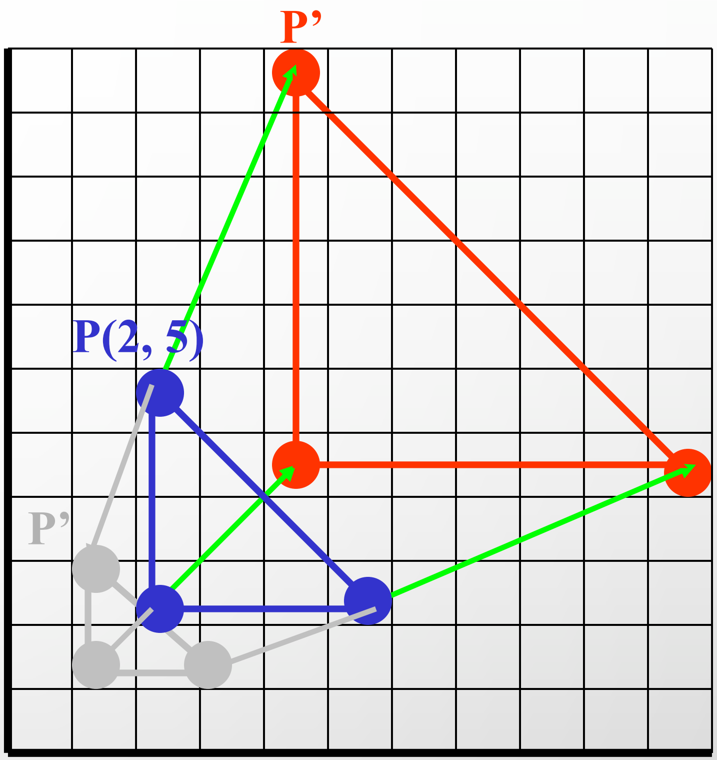


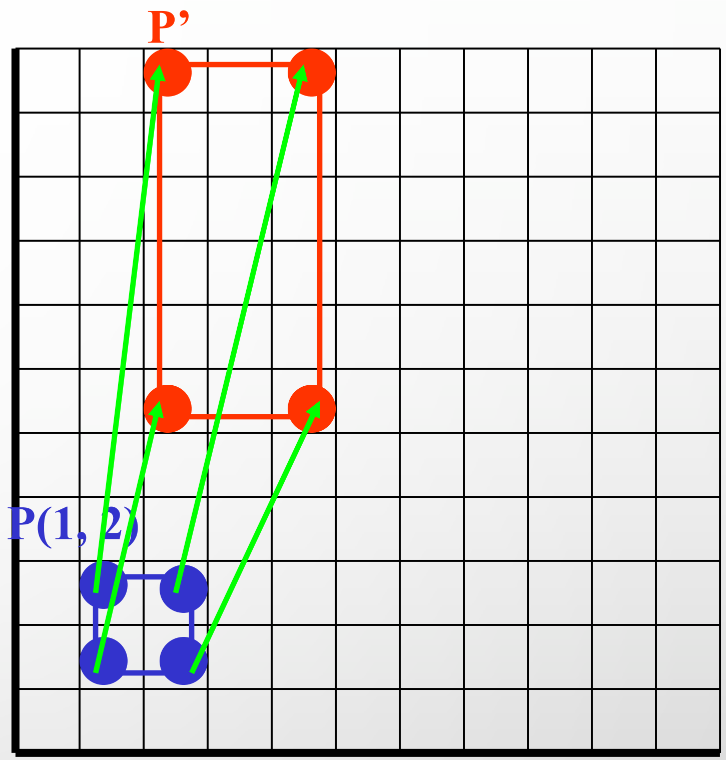
- If the scale factors are larger than 1 ➔ the points will be moved away from the origin ➔ the object will be larger.

- Example :
  - P(2, 5), Sx = 2, Sy = 2
  - Find P' ?

Scaling

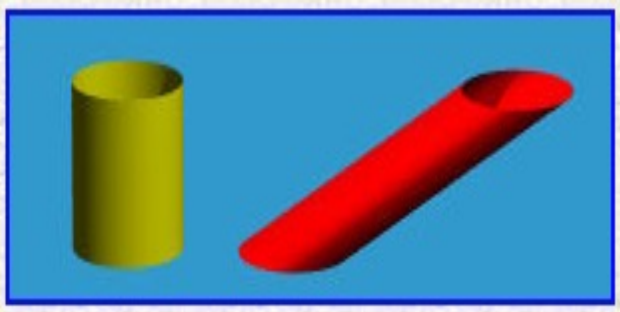
- If the scale factors are the same, $S_x = S_y$ ➔ uniform scaling
- Only change in size (as previous example)

- If $S_x \neq S_y$ ➔ differential scaling.
- Change in size and shape
- Example : square → rectangle
  - P(1, 3), $S_x = 2$, $S_y = 5$, P' ?



Shearing Transformation

The searing transformation when applied to the object it results distortion of shape. In sharing the parallel layers of any objects are simply slided with respect to each other.



Types of Shearing Transformation

X- shear: In X-shear y coordinate remain unchanged, but x is changed
Y- shear: In Y-shear x coordinate remain unchanged, but y is changed

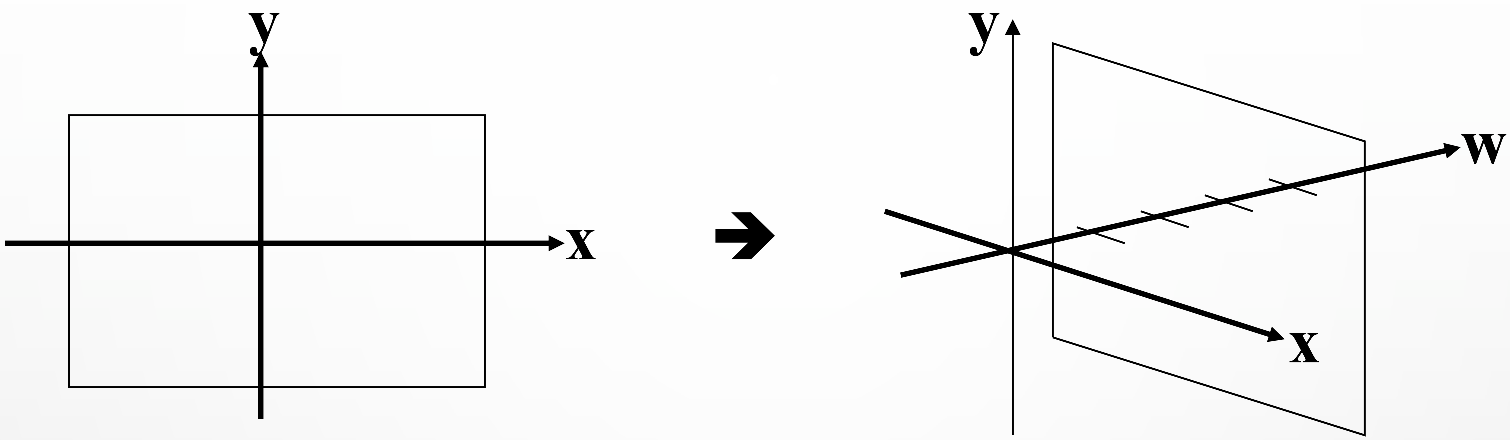
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ shx & 1 \end{bmatrix}$$

X- Shear

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} 1 & Shy \\ 0 & 1 \end{bmatrix}$$

Y- Shear

Homogenous Coordinates



- Introduced in mathematics:
  - for projections and drawings
  - used in artillery, architecture
  - used to be classified material
- Add a third coordinate, w
- A 2D point is a 3 coordinates vector: $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$

Homogenous Coordinates

- Two points are equal if and only if: $x'/w' = x/w$ and $y'/w' = y/w$
  - $w = 0$: points at infinity
  - useful for projections and curve drawing
- Homogenize = divide by w
- Homogenized points: $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$

Matrix Representation

- Point in column-vector:

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- Our point now has three coordinates. So our matrix is needs to be 3x3.
- Translation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Matrix Representation

- **Rotation**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- **Scaling**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Composite Transformation

- We can represent any sequence of transformations as a single matrix.
  - No special cases when transforming a point – matrix • vector.
  - Composite transformations – matrix • matrix.
- Composite transformations:
  - Rotate about an arbitrary point – translate, rotate, translate
  - Scale about an arbitrary point – translate, scale, translate
  - Change coordinate systems – translate, rotate, scale
- Does the order of operations matter?

Composition Properties

- Is matrix multiplication associative?
  - $(A.B).C \overset{?}{=} A.(B.C)$

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \bullet \begin{bmatrix} e & f \\ g & h \end{bmatrix}\right) \bullet \begin{bmatrix} i & j \\ k & l \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix} \bullet \begin{bmatrix} i & j \\ k & l \end{bmatrix}$$

$$= \begin{bmatrix} aei+bgi+afk+bhk & aej+bgj+afl+bhl \\ cei+dgi+cfk+dhk & cej+dgj+cfl+dhl \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \bullet \left(\begin{bmatrix} e & f \\ g & h \end{bmatrix} \bullet \begin{bmatrix} i & j \\ k & l \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \bullet \begin{bmatrix} ei+fk & ej+fl \\ gi+hk & gj+hl \end{bmatrix}$$

$$= \begin{bmatrix} aei+afk+bgi+bhk & aej+afl+bgj+bhl \\ cei+cfk+dgi+dhk & cej+cfl+dgj+dhl \end{bmatrix}$$

Composition Properties

- Is matrix multiplication commutative?
  - $A \,.\, B \stackrel{?}{=} B \,.\, A$

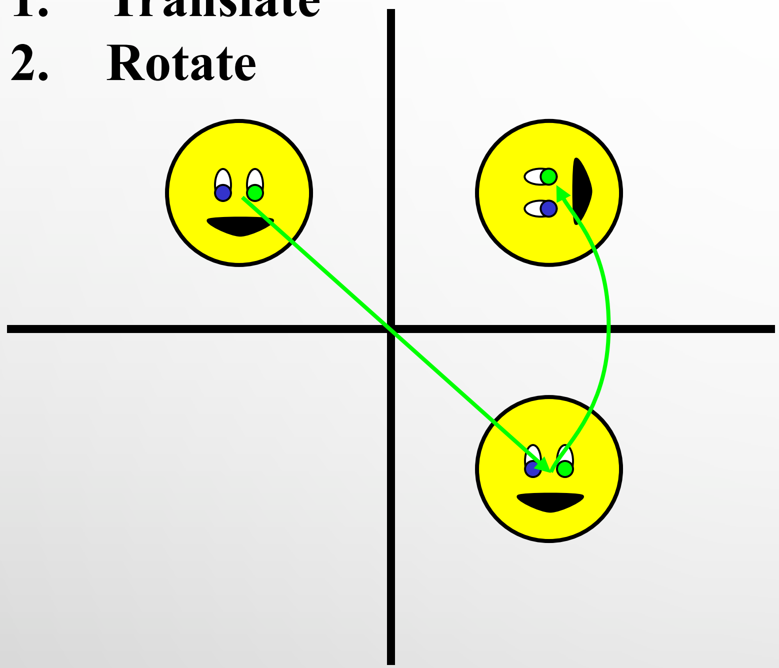
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \text{ae} + \text{bg} & \text{af} + \text{bh} \\ \text{ce} + \text{dg} & \text{cf} + \text{dh} \end{bmatrix}$$

$$\begin{bmatrix} e & f \\ g & h \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \text{ea} + \text{fc} & \text{eb} + \text{fd} \\ \text{ga} + \text{hc} & \text{gb} + \text{hd} \end{bmatrix}$$

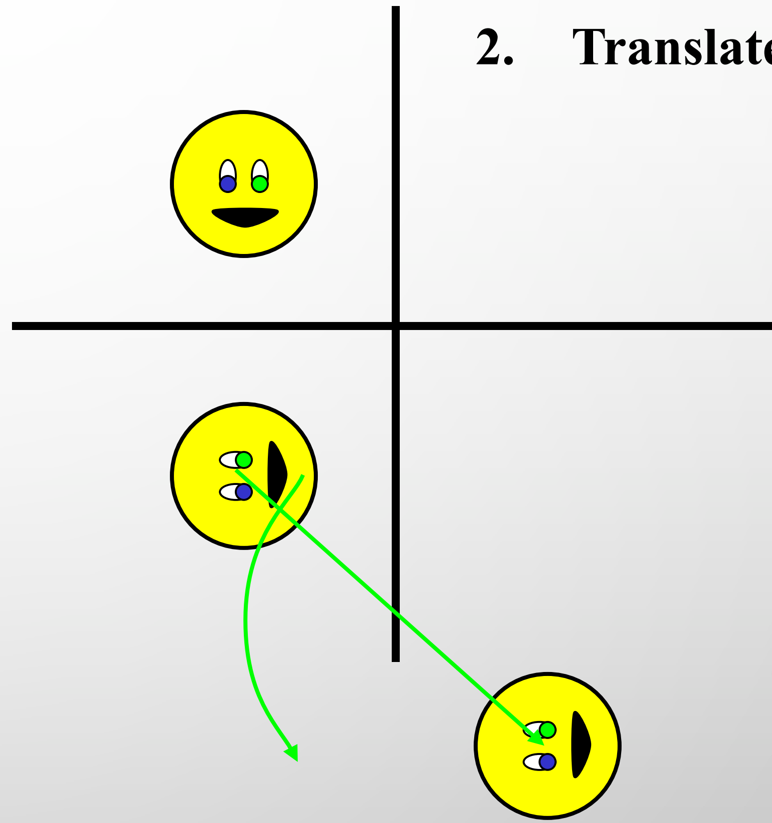
Order of operations

- So, it does matter. Let’s look at an example:

1. **Translate**
2. **Rotate**



1. **Rotate**
2. **Translate**



Composite Transformation Matrix

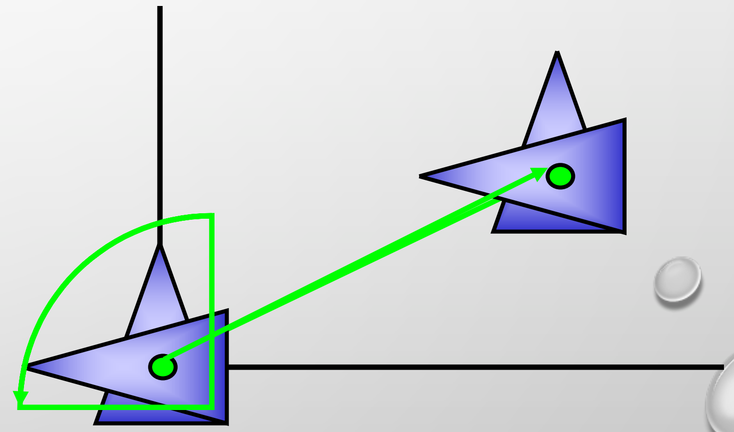
- **Arrange the transformation matrices in order from right to left.**
- **General Pivot- Point Rotation**
  - **Operation :-**
    1. **Translate (pivot point is moved to origin)**
    2. **Rotate about origin**
    3. **Translate (pivot point is returned to original position)**

T(pivot) • R(θ) • T(–pivot)

$$\begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \bullet \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \bullet \begin{pmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \bullet \begin{pmatrix} \cos\theta & -\sin\theta & -t_x \cos\theta + t_y \sin\theta \\ \sin\theta & \cos\theta & -t_x \sin\theta - t_y \cos\theta \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta & -\sin\theta & -t_x \cos\theta + t_y \sin\theta + t_x \\ \sin\theta & \cos\theta & -t_x \sin\theta - t_y \cos\theta + t_y \\ 0 & 0 & 1 \end{pmatrix}$$



Composite Transformation Matrix

- Example
  - Perform 60° rotation of a point P(2, 5) about a pivot point (1,2). Find P'?

$$\begin{pmatrix} \cos\theta & -\sin\theta & -t_x\cos\theta + t_y\sin\theta + t_x \\ \sin\theta & \cos\theta & -t_x\sin\theta - t_y\cos\theta + t_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{1} \end{pmatrix}$$

Sin 60 = 0.8660
Cos 60 = 1/2

$$\begin{pmatrix} 0.5 & -0.866 & -1.0.5 + 2.0.866 + 1 \\ 0.866 & 0.5 & -1.0.866 - 2.0.5 + 2 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{2} \\ \mathbf{5} \\ \mathbf{1} \end{pmatrix}$$

$$\begin{pmatrix} 0.5 & -0.866 & 2.232 \\ 0.866 & 0.5 & 0.134 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{2} \\ \mathbf{5} \\ \mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{-1.098} \\ \mathbf{4.366} \\ \mathbf{1} \end{pmatrix}$$

P' = (-1, 4)

Composite Transformation Matrix

General Fixed-Point Scaling

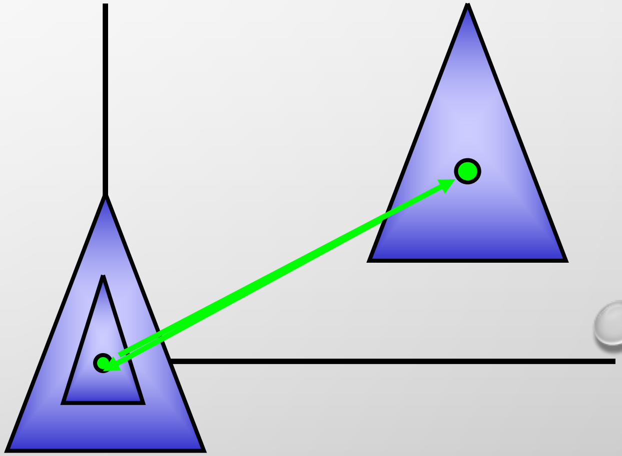
Operation :-

1. Translate (fixed point is moved to origin)
2. Scale with respect to origin
3. Translate (fixed point is returned to original position)

T(fixed) • S(scale) • T(–fixed)

Find the matrix that represents scaling of an object with respect to any fixed point?

Given P(6, 8) , Sx = 2, Sy = 3 and fixed point (2, 2). Use that matrix to find P'?



Answer

$$\begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} Sx & 0 & 0 \\ 0 & Sy & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} Sx & 0 & -t_x\,Sx \\ 0 & Sy & -t_y\,Sy \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} Sx & 0 & -t_x\,Sx + t_x \\ 0 & Sy & -t_y\,Sy + t_y \\ 0 & 0 & 1 \end{pmatrix}$$

$x = 6$, $y = 8$, $Sx = 2$, $Sy = 3$, $t_x = 2$, $t_y = 2$

$$\begin{pmatrix} 2 & 0 & -2(2) + 2 \\ 0 & 3 & -2(3) + 2 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{6} \\ \mathbf{8} \\ \mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{10} \\ \mathbf{20} \\ \mathbf{1} \end{pmatrix}$$

Composite Transformation Matrix

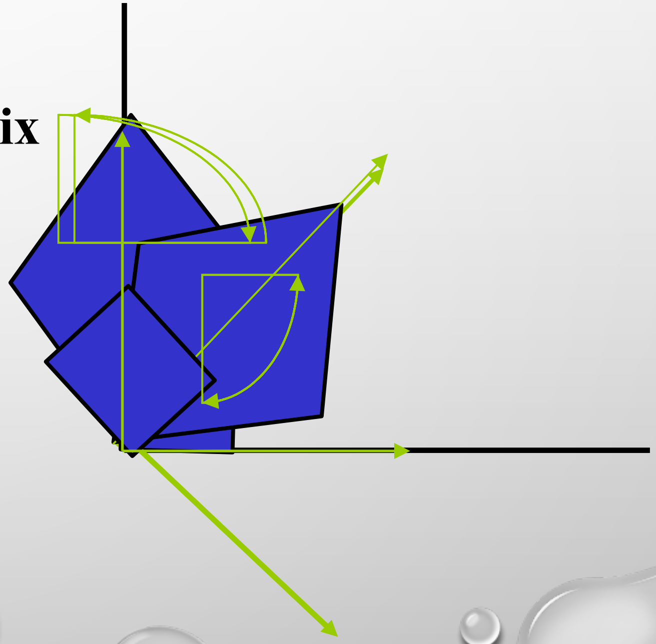
General Scaling Direction

Operation :-

1. Rotate (scaling direction align with the coordinate axes)
2. Scale with respect to origin
3. Rotate (scaling direction is returned to original position)

R(–θ) • S(scale) • R(θ)

Find the composite transformation matrix by yourself !!



S . T . R

$$\begin{pmatrix} Sx & 0 & 0 \\ 0 & Sy & 0 \\ 0 & 0 & 1 \end{pmatrix} \bullet \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \bullet \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} Sx & 0 & 0 \\ 0 & Sy & 0 \\ 0 & 0 & 1 \end{pmatrix} \bullet \begin{pmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} Sx\cos\theta & Sx(-\sin\theta) & Sx\, t_x \\ Sy\, \sin\theta & Sy\, \cos\theta & Sy\, t_y \\ 0 & 0 & 1 \end{pmatrix}$$