CSE-303: COMPUTER GRAPHICS

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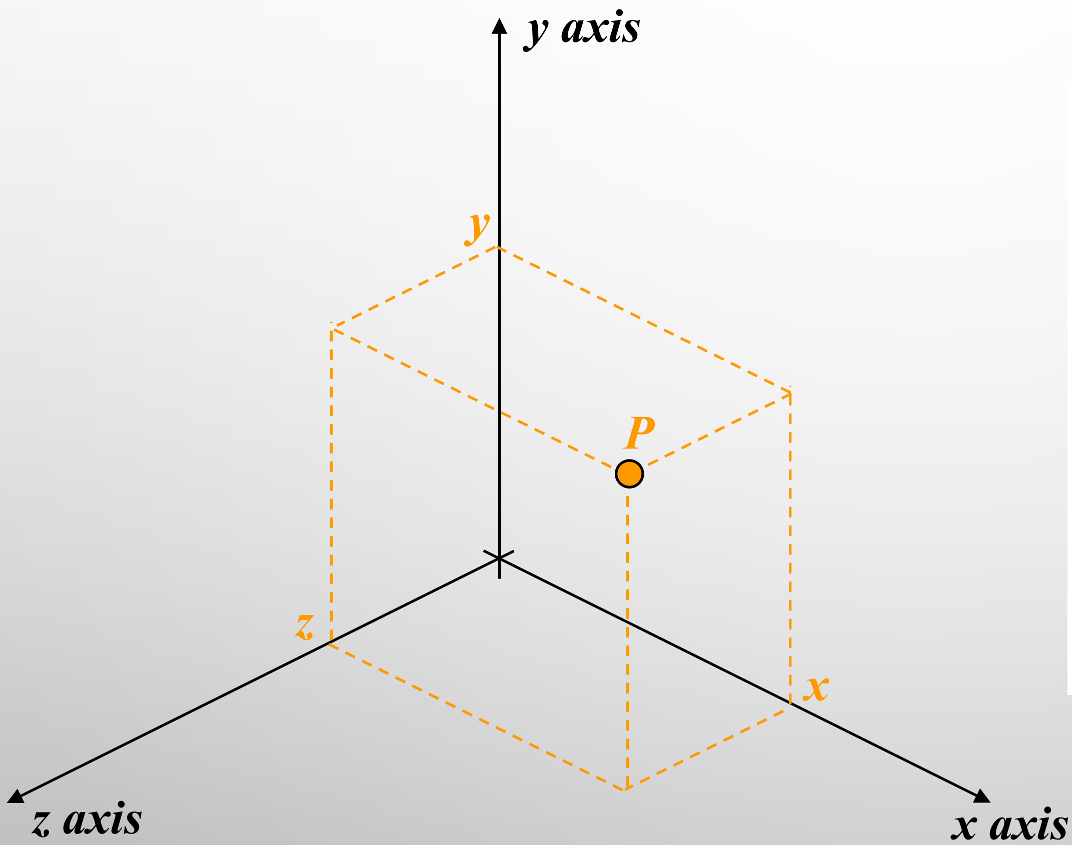
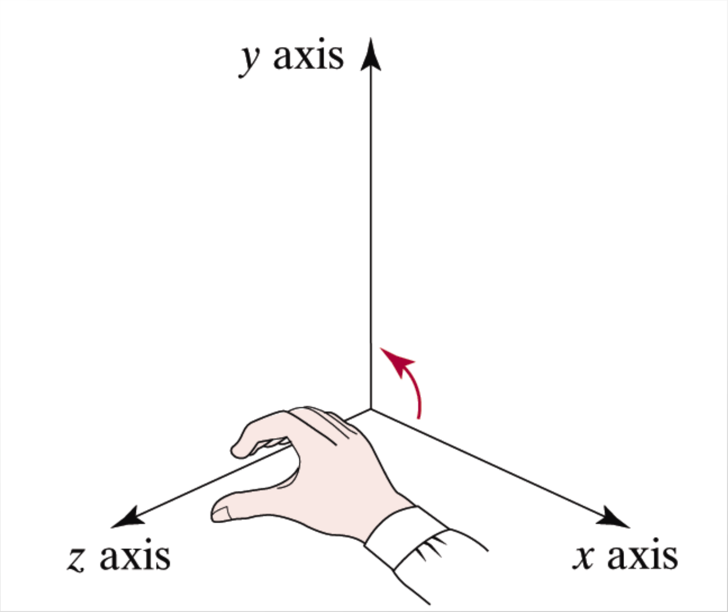
3D TRANSFORMATION

3D COORDINATE SYSTEM

- 3D coordinate system consists of a reference point called origin and three mutually perpendicular passing through origin.

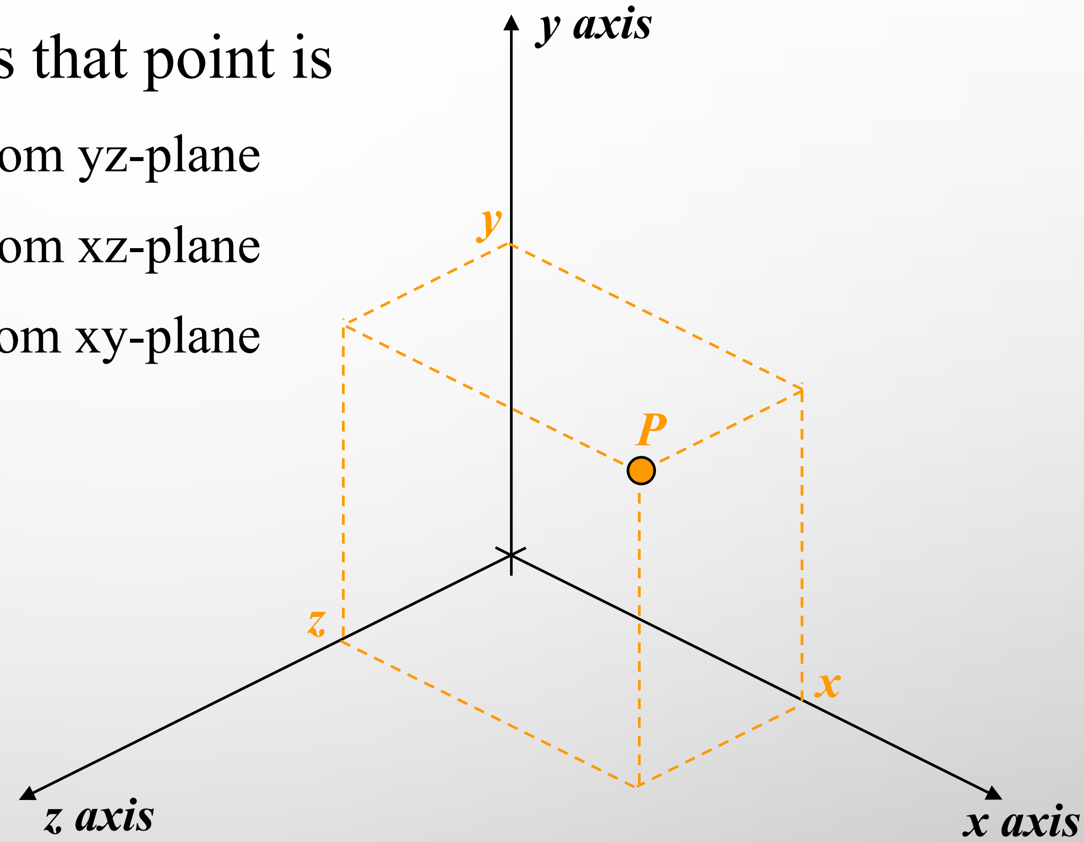




Right-Hand Reference System

3D COORDINATE SYSTEM

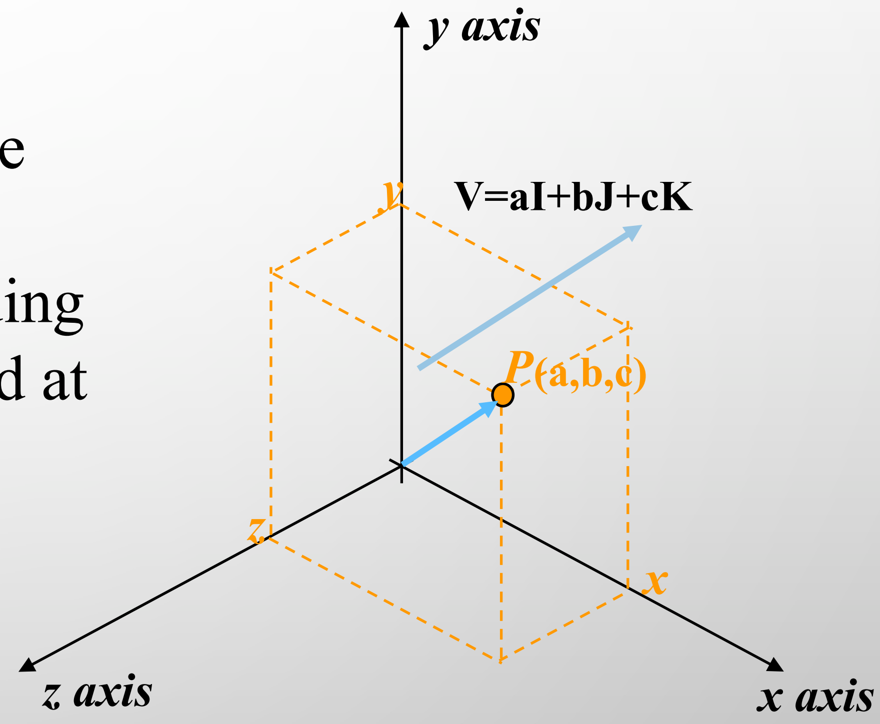
- A point p(x,y,z) specifies that point is
  - At a distance of x units from yz-plane
  - At a distance of y units from xz-plane
  - At a distance of z units from xy-plane



VECTOR FUNDAMENTALS

- In 3D there are 3 natural coordinate vectors I, J and K having unit along X, Y and Z axis respectively.

- Any vector V = aI+bJ+cK can be resolved into three components (a,b,c) that represent corresponding point when the tail of V is placed at the origin.



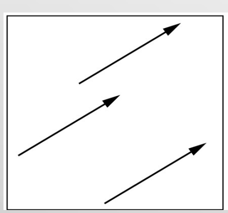
VECTOR FUNDAMENTALS

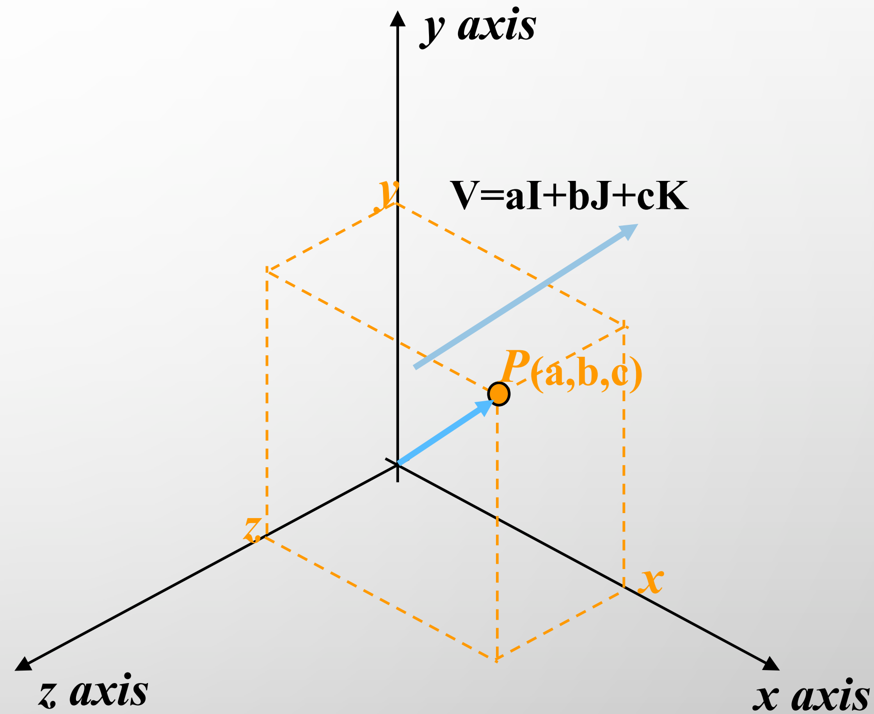
- If $p(x_0,y_0,z_0)$ and $q(x_1,y_1,z_1)$ are two points in space then

  $PQ = (x_1 - x_o)i + (y_1 - y_o)j + (z_1 - z_o)k$

- Vector has two attributes:
  - ***Length***- a scalar denoted by $|v|$
  - ***Direction*** in 3D space

- Equality: two vectors are equal if they have same length and direction

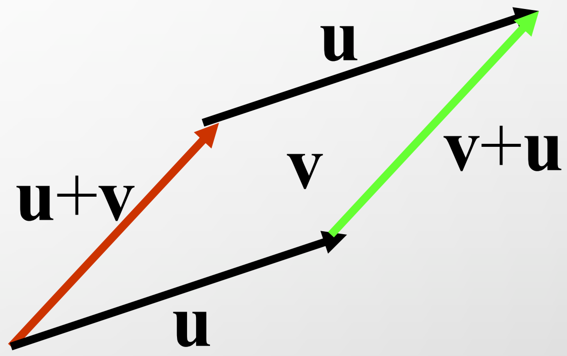




VECTOR FUNDAMENTALS

1. **Vector addition:** perform addition by the head-to-tail rule.

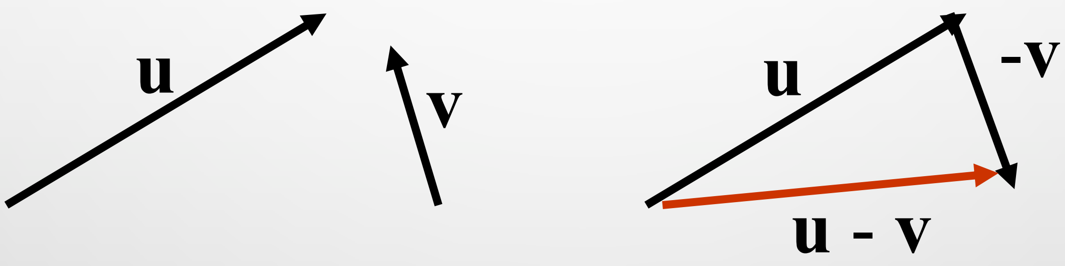
   Note **u+v = v+u**



VECTOR FUNDAMENTALS

2. **Vector subtraction:** to perform subtraction, given **u** and **v**, we define **u** - **v** to be **u** + (-**v**):

note $\mathbf{u} - \mathbf{v} \neq \mathbf{v} - \mathbf{u}$



VECTOR FUNDAMENTALS

3. **Zero vector:** i.E. Vector with zero length and no direction.

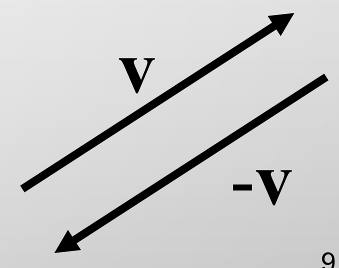
$$\mathbf{V} + (-\mathbf{v}) = \underset{\rightarrow}{\mathbf{0}}$$

4. **Magnitude of vector** of a vector **u** = ***ai+bj+ck*** is a scalar quantity defined as:

$$|u| = \sqrt{a^2 + b^2 + c^2}$$

5. **Unit vector** for any vector **u** is a vector having direction of **u** and unit magnitude. It can be found as

$$u_v = \frac{u}{|u|}$$

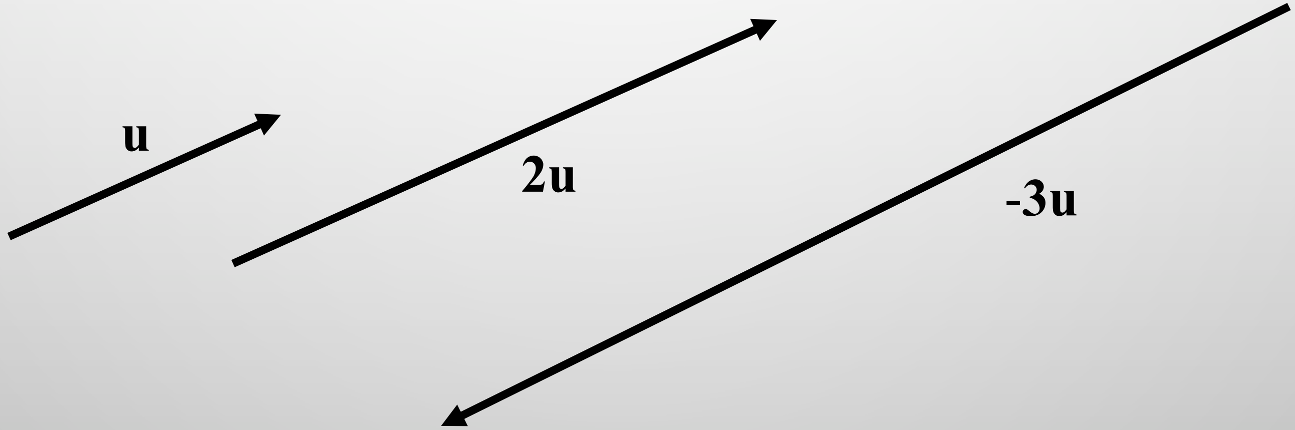


VECTOR FUNDAMENTALS

6. **Scalar multiplication:** (scalar times a vector) let α be a scalar and u a vector we define $\boldsymbol{\alpha}\mathbf{u}$ to be the vector with **length** $|\boldsymbol{\alpha}||\mathbf{u}|$ with

- The **direction** of **u** if $\boldsymbol{\alpha} > 0$
- The opposite of **u** if $\boldsymbol{\alpha} < 0$

If $\boldsymbol{\alpha} = 0$, the vector is the zero vector or a point.



VECTOR FUNDAMENTALS

7. **Dot product:** given two vectors **u** = $\boldsymbol{a_1 i + b_1 j + c_1 k}$ and **v** = $\boldsymbol{a_2 i + b_2 j + c_2 k}$ the dot product of the vectors is defined to be the *scalar* as follows

$$\mathbf{u.v} = \boldsymbol{a_1 . A_2 + b_1 . B_2 + c_1 . C_2}$$

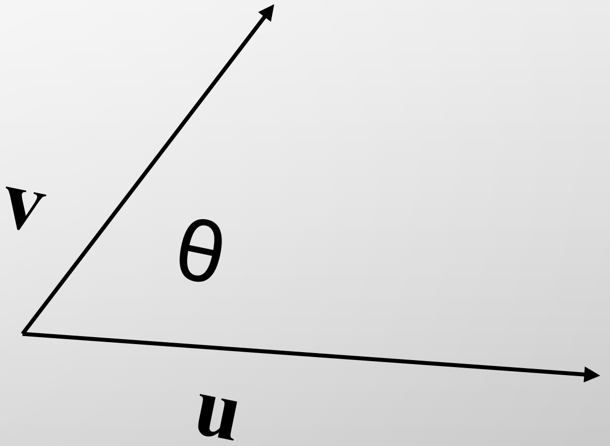
$$\text{Or } \mathbf{u.v} = |\mathbf{u}|\ |\mathbf{v}| \cos \boldsymbol{\theta}\ ,\ 0 \leq \theta \leq \pi$$

where **θ** is the angle between **u** and **v**

Also

$$\cos \boldsymbol{\theta} = \mathbf{u.v} / |\mathbf{u}|\ |\mathbf{v}|$$

$$\text{Or } \boldsymbol{\theta} = \cos^{-1}(\mathbf{u.v} / |\mathbf{u}|\ |\mathbf{v}|)$$



VECTOR FUNDAMENTALS

The **dot product** is a scalar value that tells us something about the relationship between two vectors

- If $\mathbf{u} \cdot \mathbf{v} > 0$ then $0 < \theta < 90^\circ$
  - Vectors point in the same general direction
- If $\mathbf{u} \cdot \mathbf{v} < 0$ then $\theta > 90^\circ$
  - Vectors point in opposite direction
- If $\mathbf{u} \cdot \mathbf{v} = 0$ then $\theta = 90^\circ$
  - Vectors are perpendicular
  - (Or one or both of the vectors is degenerate (0,0,0))

VECTOR FUNDAMENTALS

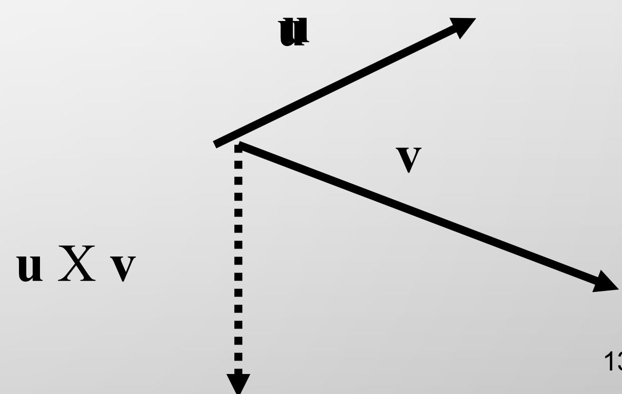
8. **Cross product:** given two vectors **u** = $a_1 i + b_1 j + c_1 k$ and **v** = $a_2 i + b_2 j + c_2 k$ the cross product of the vectors is defined to be the new *vector* whose length is |u||v| sin θ, $0 \leq \theta \leq \pi$ and direction is given by the right hand rule.

$$\mathbf{u} \times \mathbf{v} = (b_1.C_2 - c_1.\,B_2)i + (c_1.A_2 - a_1.\,C_2)j + (a_1.B_2 - b_1.\,A_2)k$$

Or $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin \theta$, $0 \leq \theta \leq \pi$ where θ is the angle b/w **u** & **v**

$$\sin \theta = |\mathbf{u} \times \mathbf{v}| \,/\, |\mathbf{u}|\,|\mathbf{v}|$$

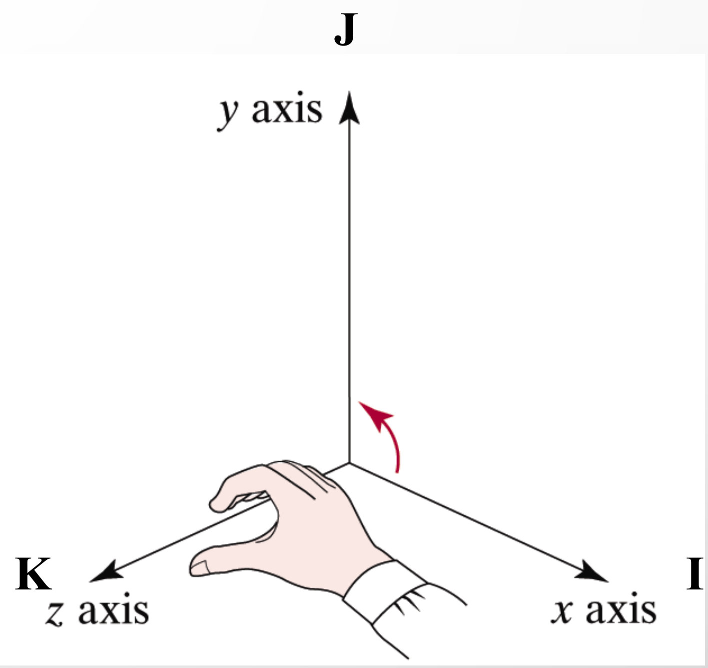
$$\text{Or } \theta = \sin^{-1}(|\mathbf{u} \times \mathbf{v}| \,/\, |\mathbf{u}|\,|\mathbf{v}|)$$



VECTOR FUNDAMENTALS

- In a right handed coordinate system
  - I x J = K
  - J x K = I
  - I x K = -J

    Or K x I = J



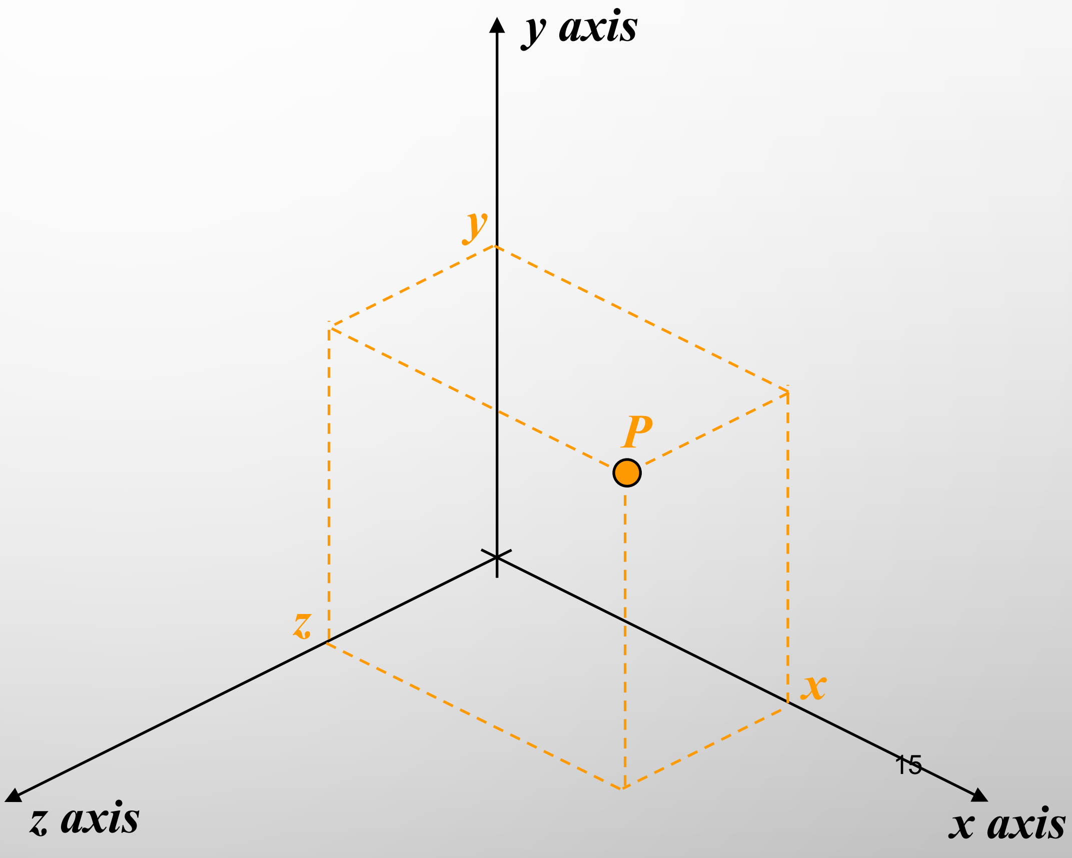
Right-Hand
Reference System

3D HOMOGENEOUS COORDINATES

- Similar to the 2-D situation we can use homogeneous coordinates for 3-D transformations - 4 coordinate column vector

- All transformations can then be represented as matrices

$$P(x, y, z) = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



3D HOMOGENEOUS COORDINATES

- Transform a point:

$$\begin{bmatrix} p'_x \\ p'_y \\ p'_z \\ 1 \end{bmatrix} = \begin{bmatrix} m_{xx} & m_{xy} & m_{xz} & d_x \\ m_{yx} & m_{yy} & m_{yz} & d_y \\ m_{zx} & m_{zy} & m_{zz} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} m_{xx}p_x + m_{xy}p_y + m_{xz}p_z + d_x \\ m_{yx}p_x + m_{yy}p_y + m_{yz}p_z + d_y \\ m_{zx}p_x + m_{zy}p_y + m_{zz}p_z + d_z \\ 0+0+0+1 \end{bmatrix}$$

$$\mathbf{M}\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} + \vec{\mathbf{d}}$$

- Top three rows are the affine transform!
- Bottom row stays 1

3D HOMOGENEOUS COORDINATES

- Transform a vector:

$$\begin{bmatrix} v'_x \\ v'_y \\ v'_z \\ 0 \end{bmatrix} = \begin{bmatrix} m_{xx} & m_{xy} & m_{xz} & d_x \\ m_{yx} & m_{yy} & m_{yz} & d_y \\ m_{zx} & m_{zy} & m_{zz} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \\ 0 \end{bmatrix} = \begin{bmatrix} m_{xx}v_x + m_{xy}v_y + m_{xz}v_z + 0 \\ m_{yx}v_x + m_{yy}v_y + m_{yz}v_z + 0 \\ m_{zx}v_x + m_{zy}v_y + m_{zz}v_z + 0 \\ 0+0+0+0 \end{bmatrix}$$

$$\mathbf{M} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

- Top three rows are the linear transform
  - Displacement **d** is properly ignored
- Bottom row stays 0

3D HOMOGENEOUS COORDINATES

- In homogeneous arithmetic legal operations always end in 0 or 1!

$$\text{vector+vector:} \quad \begin{bmatrix} \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} \vdots \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ 0 \end{bmatrix}$$

$$\text{vector-vector:} \quad \begin{bmatrix} \vdots \\ 0 \end{bmatrix} - \begin{bmatrix} \vdots \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ 0 \end{bmatrix}$$

$$\text{scalar*vector:} \quad s \begin{bmatrix} \vdots \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ 0 \end{bmatrix}$$

$$\text{point+vector:} \quad \begin{bmatrix} \vdots \\ 1 \end{bmatrix} + \begin{bmatrix} \vdots \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ 1 \end{bmatrix}$$

$$\text{point-point:} \quad \begin{bmatrix} \vdots \\ 1 \end{bmatrix} - \begin{bmatrix} \vdots \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ 0 \end{bmatrix}$$

$$\text{point+point:} \quad \begin{bmatrix} \vdots \\ 1 \end{bmatrix} + \begin{bmatrix} \vdots \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ 2 \end{bmatrix}$$

$$\text{scalar*point:} \quad s \begin{bmatrix} \vdots \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ s \end{bmatrix}$$

$$\begin{Bmatrix} \text{weighted average} \\ \text{affine combination} \end{Bmatrix} \text{of points:} \quad \frac{1}{3}\begin{bmatrix} \vdots \\ 1 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} \vdots \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ 1 \end{bmatrix}$$

3D HOMOGENEOUS COORDINATES

- Rotation, scale, and translation of points and vectors unified in a single matrix transformation:

$$\mathbf{p}' = \mathbf{M}\ \mathbf{p}$$

Where m = $\begin{bmatrix} m_{xx} & m_{xy} & m_{xz} & d_x \\ m_{yx} & m_{yy} & m_{yz} & d_y \\ m_{zx} & m_{zy} & m_{zz} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$

- Matrix has the form:
  - Last row always 0,0,0,1

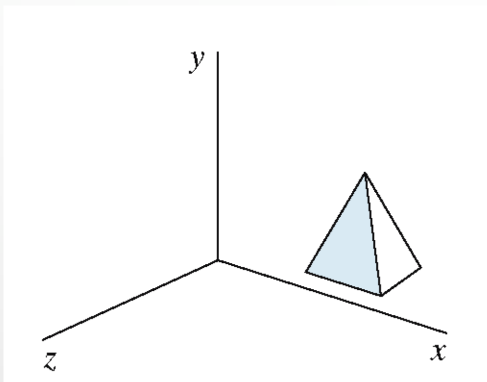
- Transforms compose by matrix multiplication.

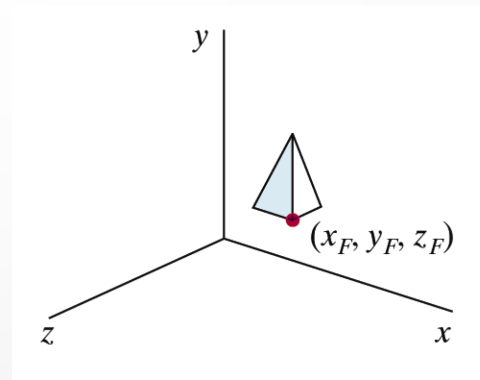
3D GEOMETRIC TRANSFORMATIONS

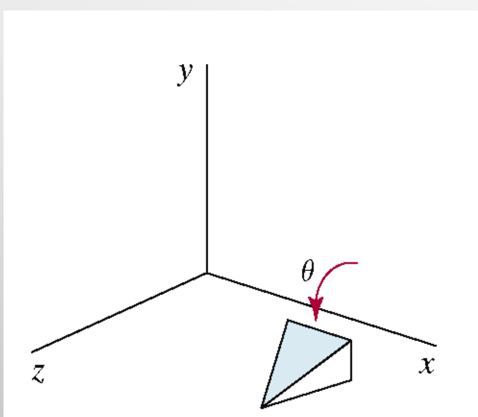
- **Geometric transformations:** in geometric transformation an object itself is moved relative to a stationary coordinate system or background. The mathematical statement of this view point is described by geometric transformation applied to each point of the object. Various geometric transformations are:
    - Translation
    - Scaling
    - Rotation
    - Reflection
    - Shearing

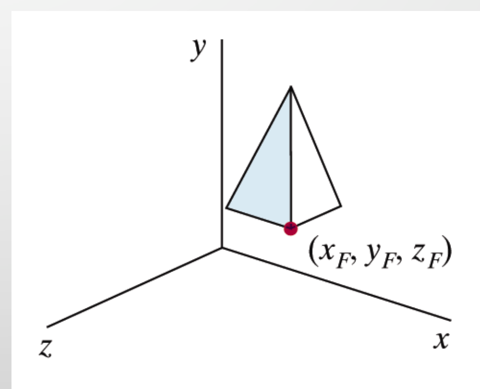
3D GEOMETRIC TRANSFORMATIONS

- Once we have an object described, transformations are used to move that object, scale it and rotate it







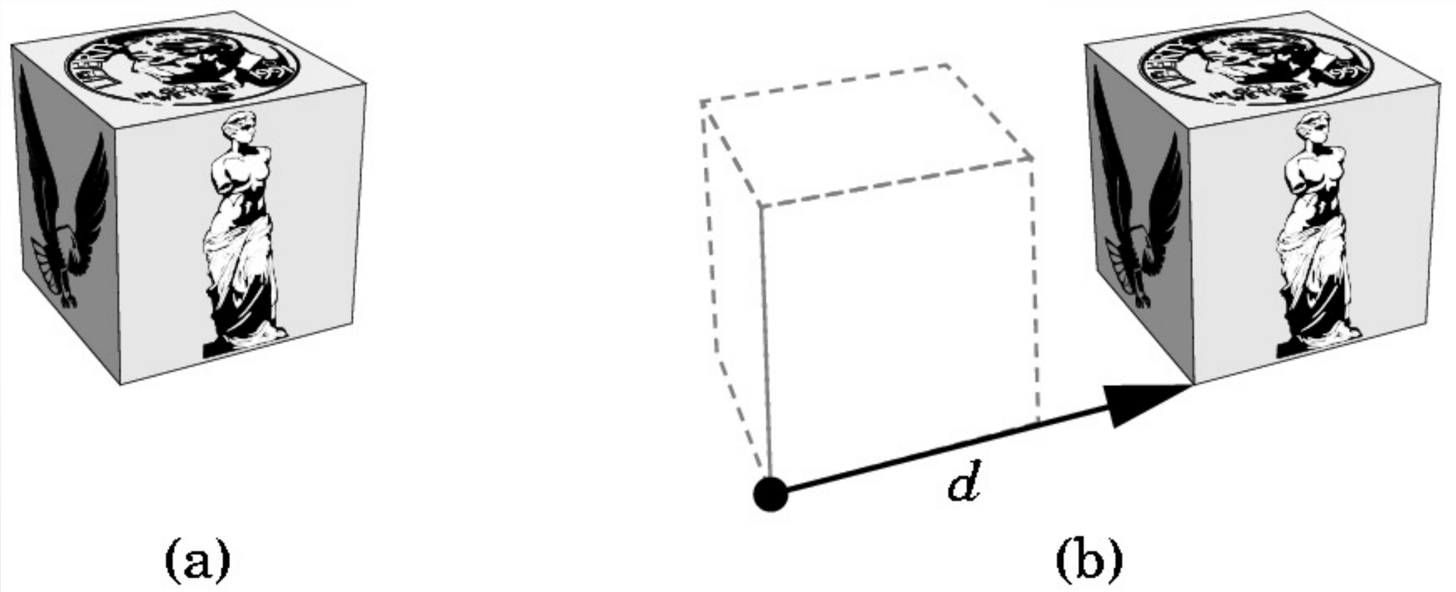


GEOMETRIC TRANSFORMATIONS

- **TRANSLATION**
- SCALING
- ROTATION
- REFLECTION
- SHEARING

3D GEOMETRIC TRANSLATION

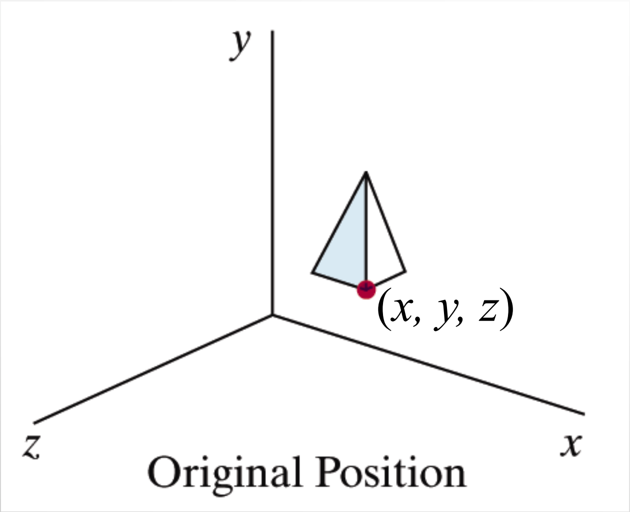
- **Translation** is defined as the displacement of any object by a given distance and direction from its original position

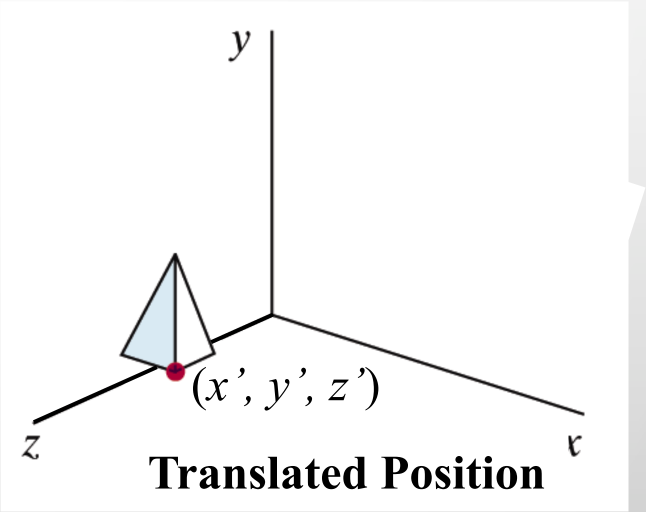


3D GEOMETRIC TRANSLATION

- To translate a point in three dimensions by *tx*, *ty* and *tz* simply calculate the new points as follows:

$$x' = x + tx \qquad y' = y + ty \qquad z' = z + tz$$





3D GEOMETRIC TRANSLATION

- The translation of a point p(x,y,z) by (tx, ty, tz) can be written in matrix form as:

$$P' = T_v(P) \quad where\ v = tx\vec{I} + ty\vec{J} + tz\vec{K}$$

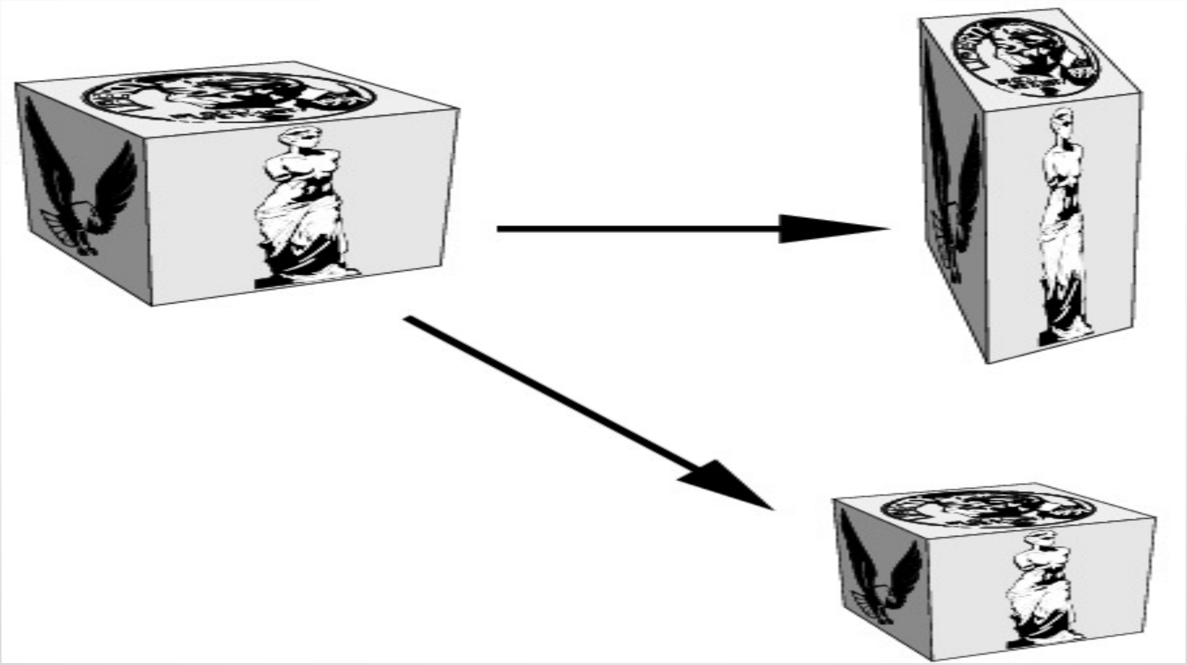
$$T_v = \begin{bmatrix} 1 & 0 & 0 & tx \\ 0 & 1 & 0 & ty \\ 0 & 0 & 1 & tz \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad P' = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} \qquad P = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

GEOMETRIC TRANSFORMATIONS

- TRANSLATION
- **SCALING**
- ROTATION
- REFLECTION
- SHEARING

3D GEOMETRIC SCALING

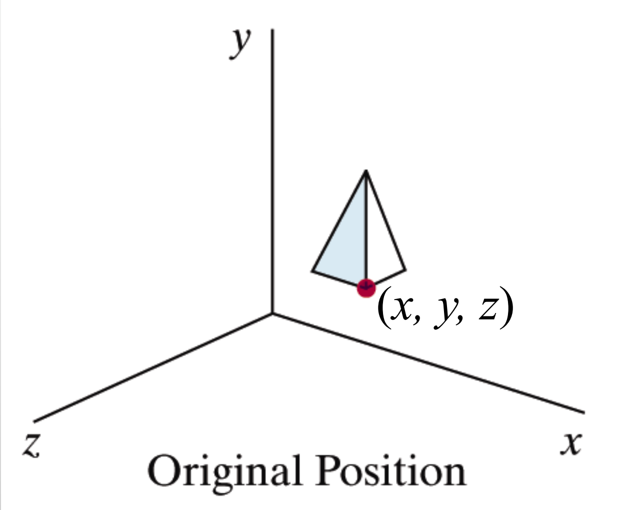
Scaling is the process of expanding or compressing the dimensions of an object determined by the scaling factor.

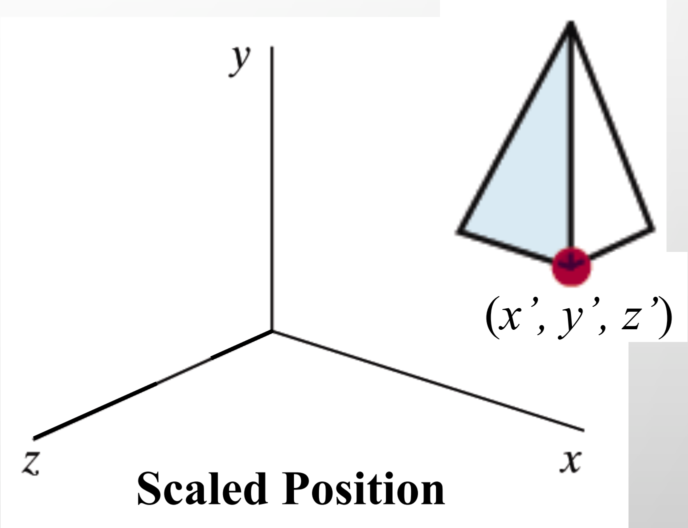


3D GEOMETRIC SCALING

- To scale a point in three dimensions by s_x, s_y and s_z simply calculate the new points as follows:

$$x' = s_x * x \qquad y' = s_y * y \qquad z' = s_z * z$$





3D GEOMETRIC SCALING

- The scaling of a point p*(x,y,z)* by scaling factors s_x, s_y and s_z about origin can be written in matrix form as:

$$P' = S_{sx,\,sy,\,sz}(P) \quad where$$

$$S_{sx,\,sy,\,sz} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad P' = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} \quad P = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

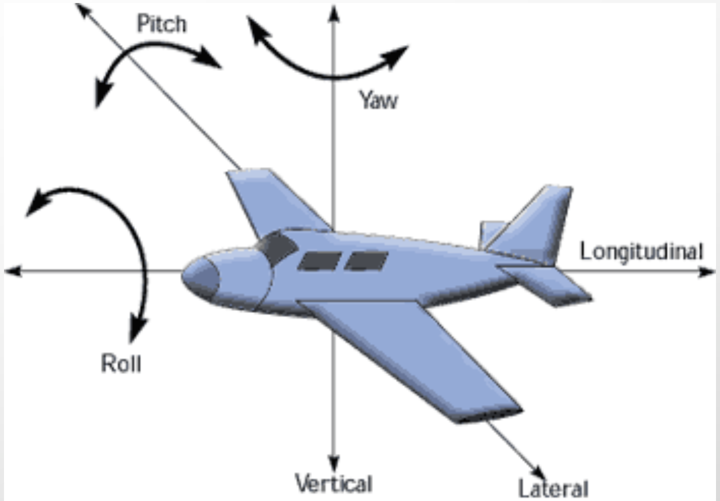
$$such\ that \quad \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} s_x \times x \\ s_y \times y \\ s_z \times z \\ 1 \end{bmatrix}$$

GEOMETRIC TRANSFORMATIONS

- TRANSLATION
- SCALING
- **ROTATION**
- REFLECTION
- SHEARING

3D GEOMETRIC ROTATIONS

- When we performed rotations in two dimensions we only had the choice of rotating about the *z* axis
- In the case of three dimensions we have more options
  - Rotate about *x* – pitch
  - Rotate about *y* – yaw
  - Rotate about *z* - roll



3D GEOMETRIC ROTATIONS

- **Roll** is known as the rising or dipping of the airplane's wing. The movement is done about **longitudinal axis**. The ailerons controlling the roll are located on the trailing edge of both wings.
- **Pitch** refers to the movement of the airplane's nose either up or down. It movement is done about **lateral axis**. The elevator controls the pitch is also located on the rear of the aircraft on the tail, along with the rudder.
- **Yaw** allows the airplane to move towards the left or right while in flight. The movement is done about a **ventricle axis**. The yaw is controlled by the rudder located in the rear of the aircraft on the tail.

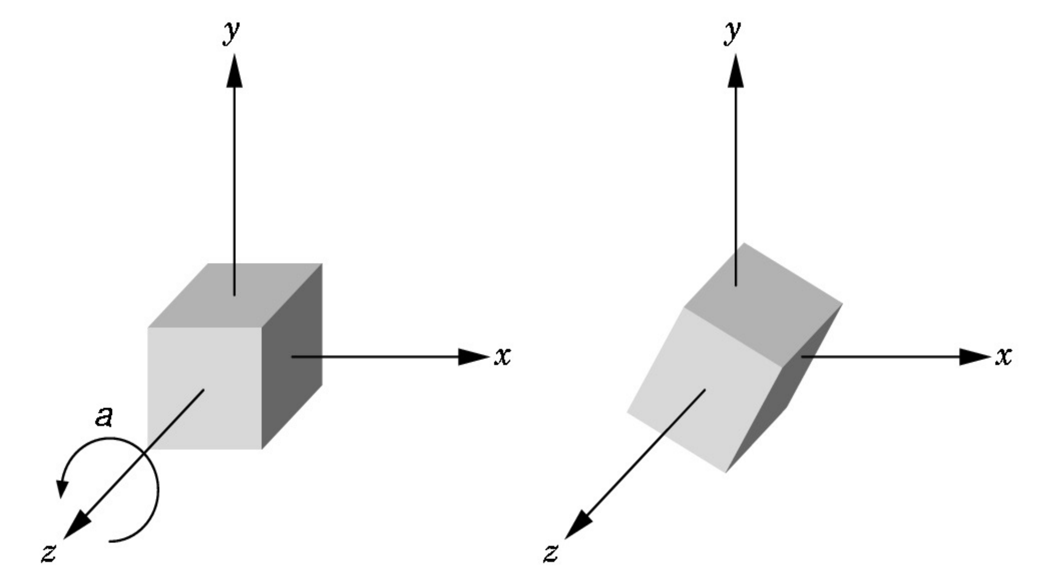
Roll

Pitch

Yaw

3D GEOMETRIC ROTATIONS

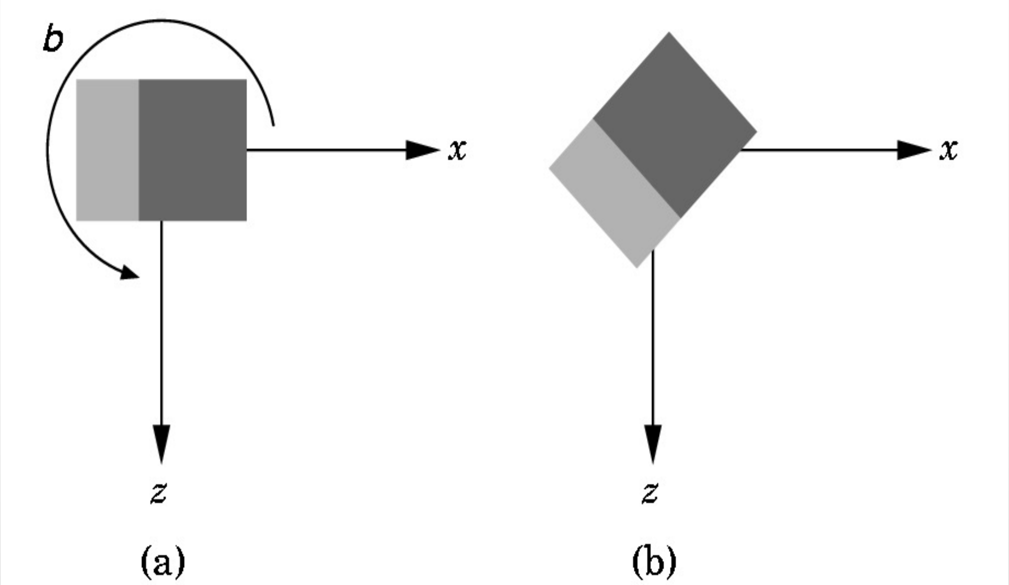
- Rotate about z – axis: the picture shows a z-axis rotation around the origin in a positive angle, (anti-clockwise) as you look down the z-axis towards the origin. The angle is measured in the xy-plane from the x-axis, just as in 2D.



(a) (b)

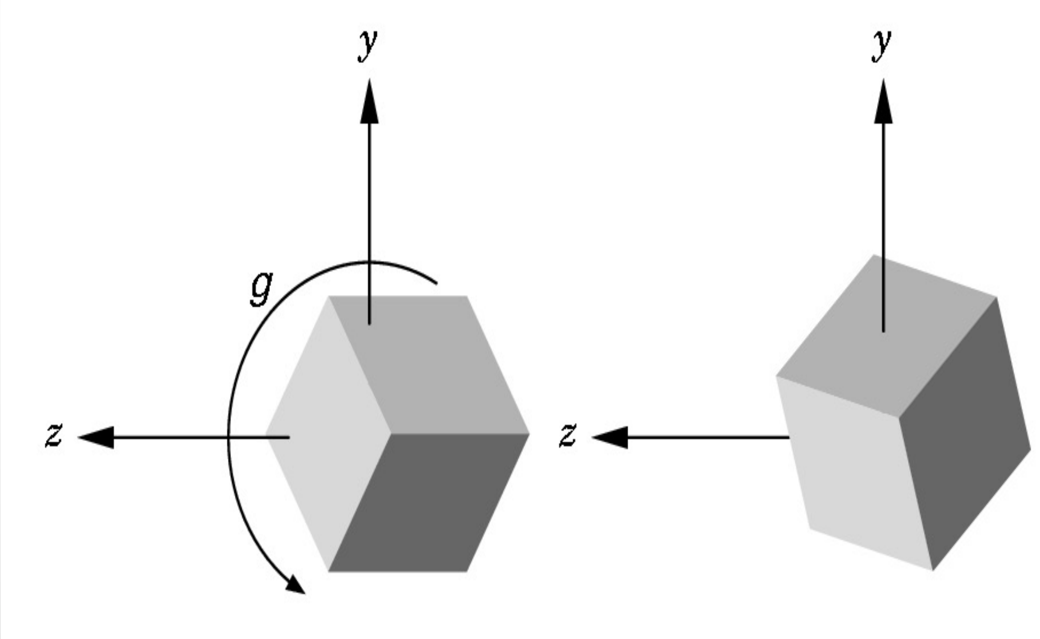
3D GEOMETRIC ROTATIONS

- Rotate about *y* – axis: you are looking down the y-axis which is not shown. A positive (counter-clockwise) angle is shown.



3D GEOMETRIC ROTATIONS

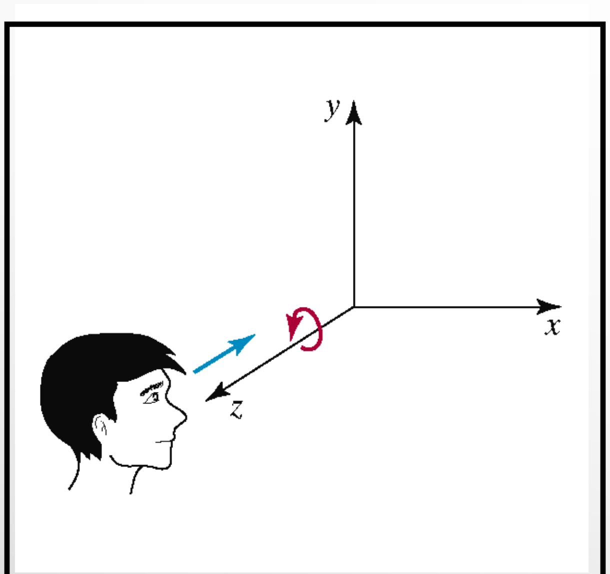
- Rotate about x – axis: you are looking down the x-axis which is not shown. A positive (counter-clockwise) angle is shown.



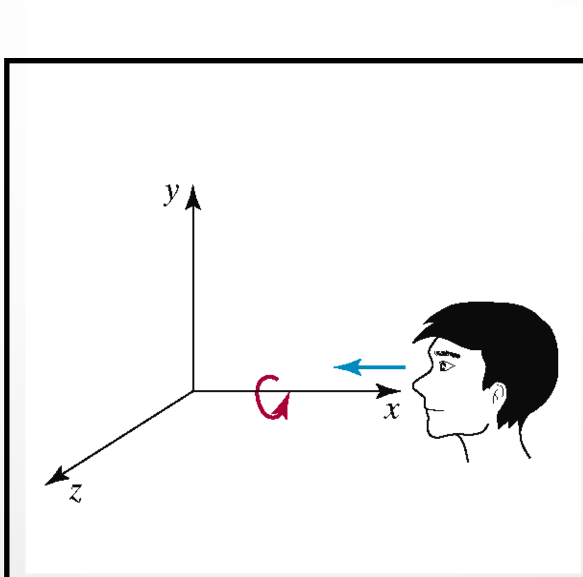
(a) (b)

3D GEOMETRIC ROTATIONS

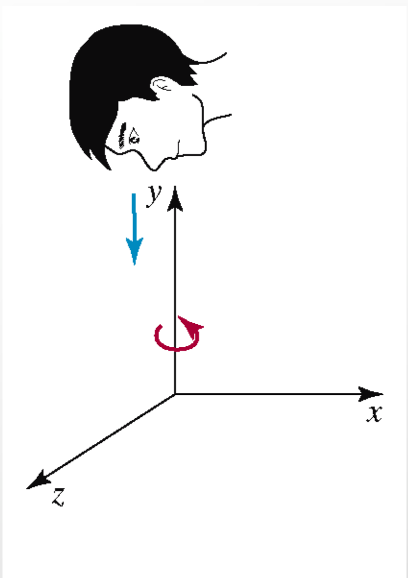
- The equations for the three kinds of rotations in 3-D are



$$x' = x \cdot \cos\theta - y \cdot \sin\theta$$
$$y' = x \cdot \sin\theta + y \cdot \cos\theta$$
$$z' = z$$



$$x' = x$$
$$y' = y \cdot \cos\theta - z \cdot \sin\theta$$
$$z' = y \cdot \sin\theta + z \cdot \cos\theta$$



$$x' = z \cdot \sin\theta + x \cdot \cos\theta$$
$$y' = y$$
$$z' = z \cdot \cos\theta - x \cdot \sin\theta$$

3D GEOMETRIC ROTATIONS

- The scaling of a point p*(x,y,z)* by an angle of θ about different axis about origin can be written in matrix form as:

$$P' = R_{\theta,\, axis}(P)$$

$$where \quad P' = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} \quad P = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \quad R_{\theta,\, K} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_{\theta,\, J} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_{\theta,\, I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

GEOMETRIC TRANSFORMATIONS

- TRANSLATION
- SCALING
- ROTATION
- **REFLECTION**
- SHEARING

3D GEOMETRIC REFLECTIONS

- We can perform reflections relative to a selected *reflection axis* or with respect to a *reflection plane*.
- Reflections relative to a given axis are equivalent to 180° rotations about that axis.
- Reflections with respect to a plane are equivalent to 180° rotations in 4d space.
- 3 standard reflections are
  - About z axis or with respect to xy plane
  - About y axis or with respect to zx plane
  - About x axis or with respect to yz plane

3D GEOMETRIC REFLECTIONS

- Reflection about XY – plane: reflection relative to the xy plane can be obtained from following set of equations:

$$x' = x \qquad y' = y \qquad z' = -z$$

- In homogenous matrix form

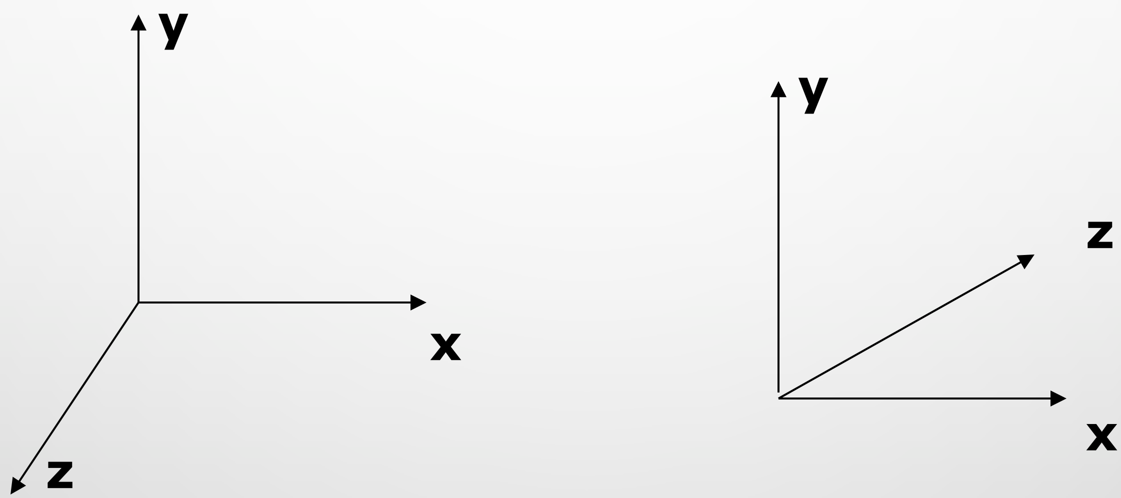
$p' = m_{xy}(p)$ where

$$M_{xy} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Reflections about other planes can also be obtained by symmetry.

3D GEOMETRIC REFLECTIONS

- Reflection about xy – plane is a useful reflection as it converts a right-handed coordinate system into a left-handed coordinate system.



INVERSE TRANSFORMATIONS

- Inverse translation: $T_v^{-1} = T_{-v}$
- Inverse scaling: $S_{sx,sy,sz}^{-1} = S_{1/sx,1/sy,1/sz}$
- Inverse rotations: $R_{\theta,p}^{-1} = R_{-\theta,p}$
- Inverse reflections: $M_{plane}^{-1} = M_{plane}$

3D COORDINATE TRANSFORMATIONS

- **Coordinate transformation:** the object is held stationary while coordinate system is moved relative to the object. These can easily be described in terms of the opposite operation performed by geometric transformation.

3D COORDINATE TRANSFORMATIONS

- Coordinate translation: $\overline{T}_v = T_v^{-1} = T_{-v}$

- Coordinate scaling: $\overline{S}_{sx,sy,sz} = S_{sx,sy,sz}^{-1} = S_{1/sx,1/sy,1/sz}$

- Coordinate rotations: $\overline{R}_{\theta,p} = R_{\theta,p}^{-1} = R_{-\theta,p}$

- Coordinate reflections: $\overline{M}_{plane} = M_{plane}^{-1} = M_{plane}$

COMPOSITE TRANSFORMATION

Various composite transformations are

1. Scaling about arbitrary point P
2. Tilting as rotation about x-axis followed by y-axis
3. Aligning vector V with K (Z – axis)
4. Rotation about arbitrary axis L
5. Mirror reflection about arbitrary plane (N, R_0)