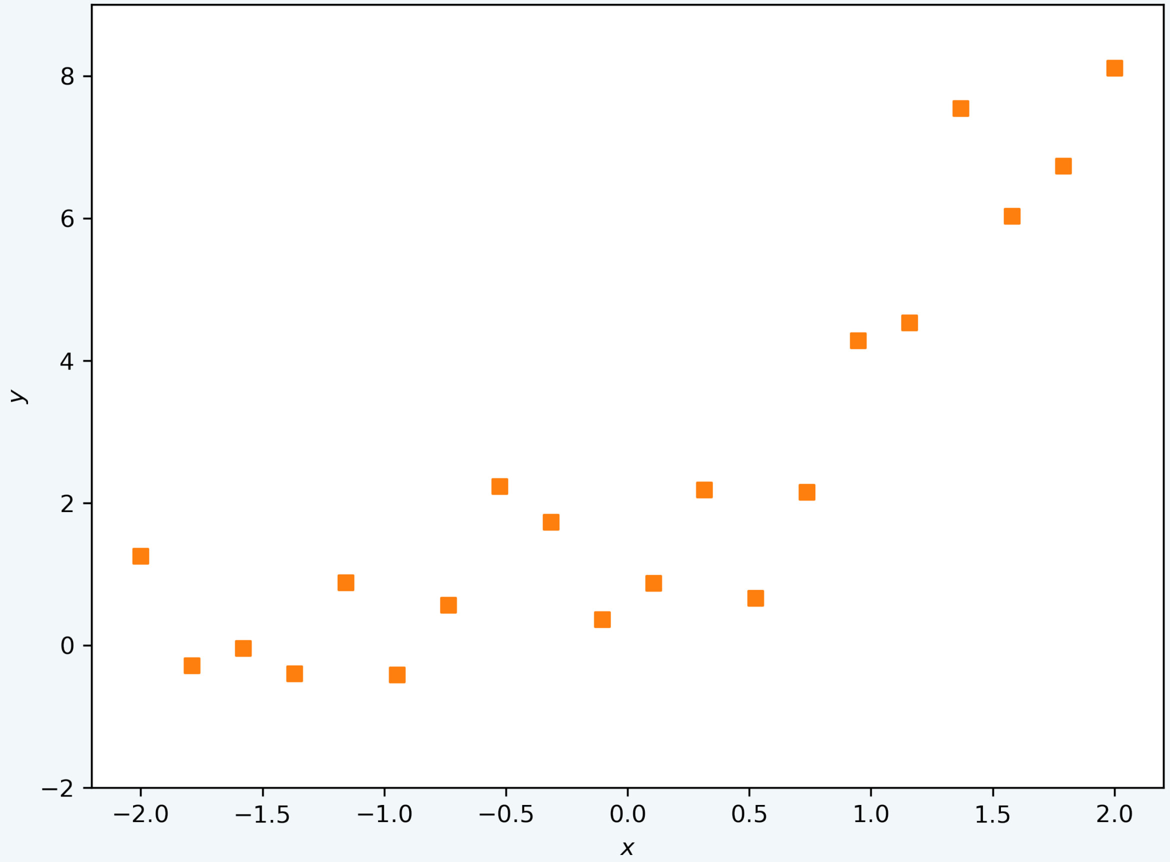
GENERALIZED LINEAR MODELS

Motivation: Normal Linear Models



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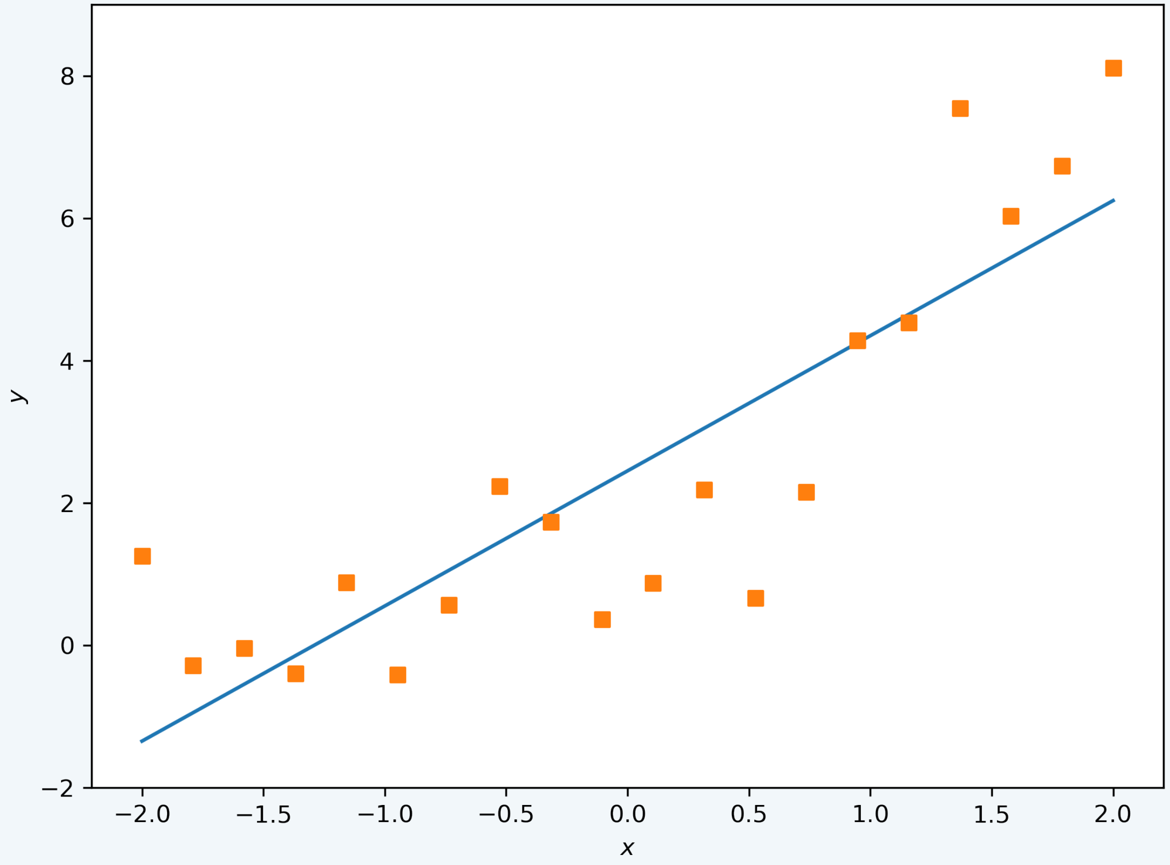

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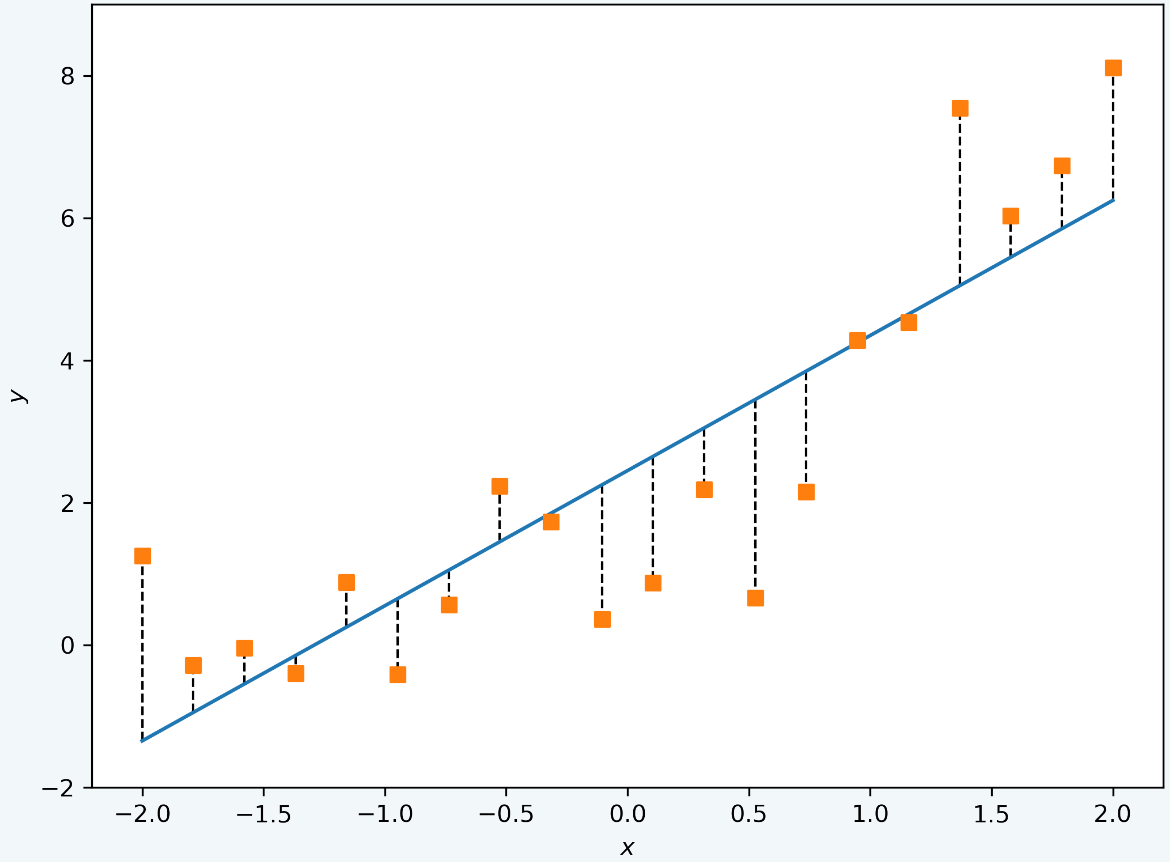

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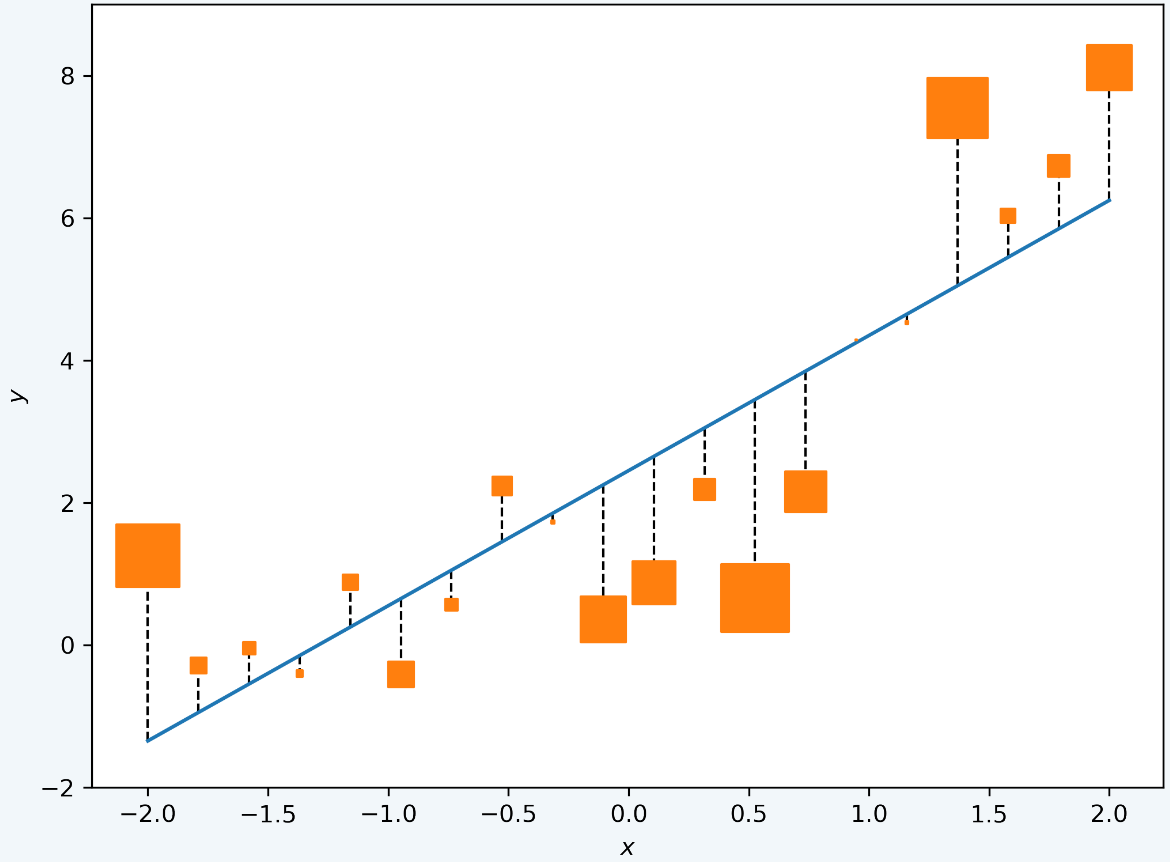

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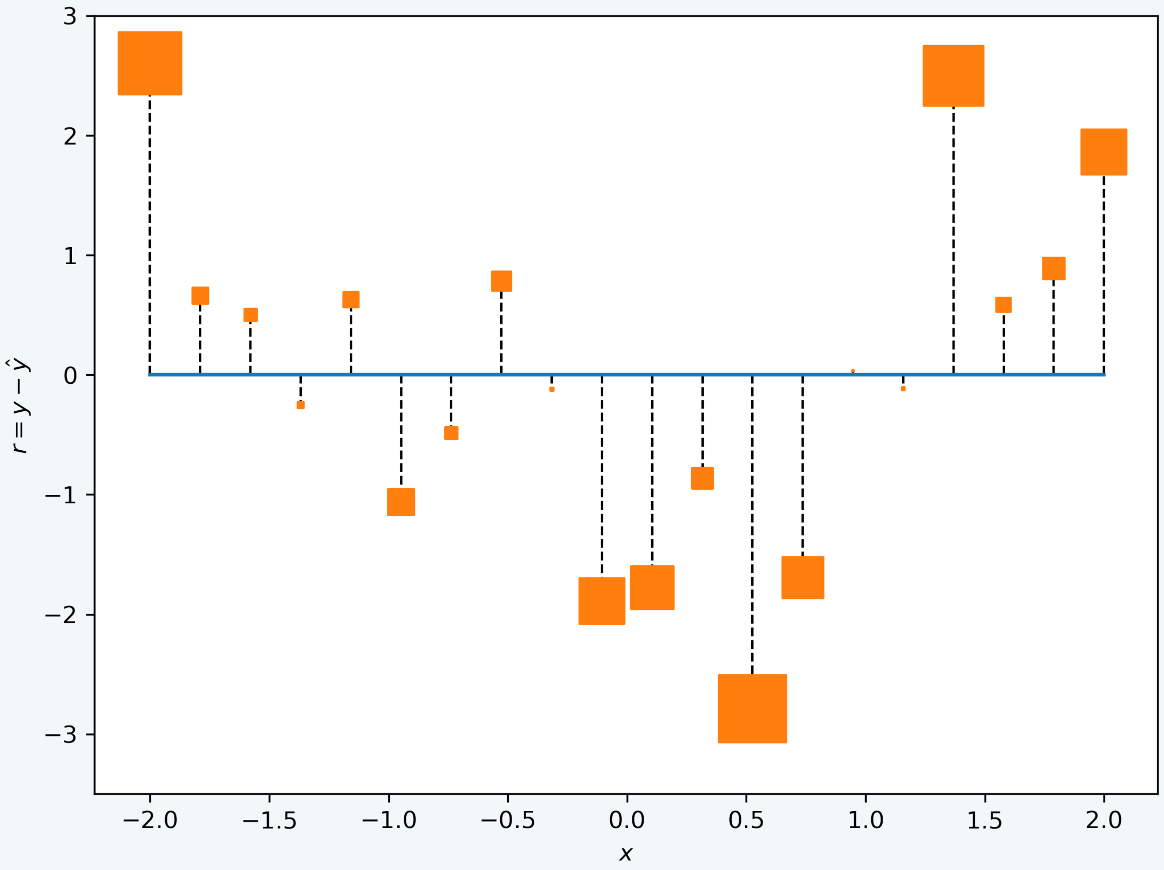

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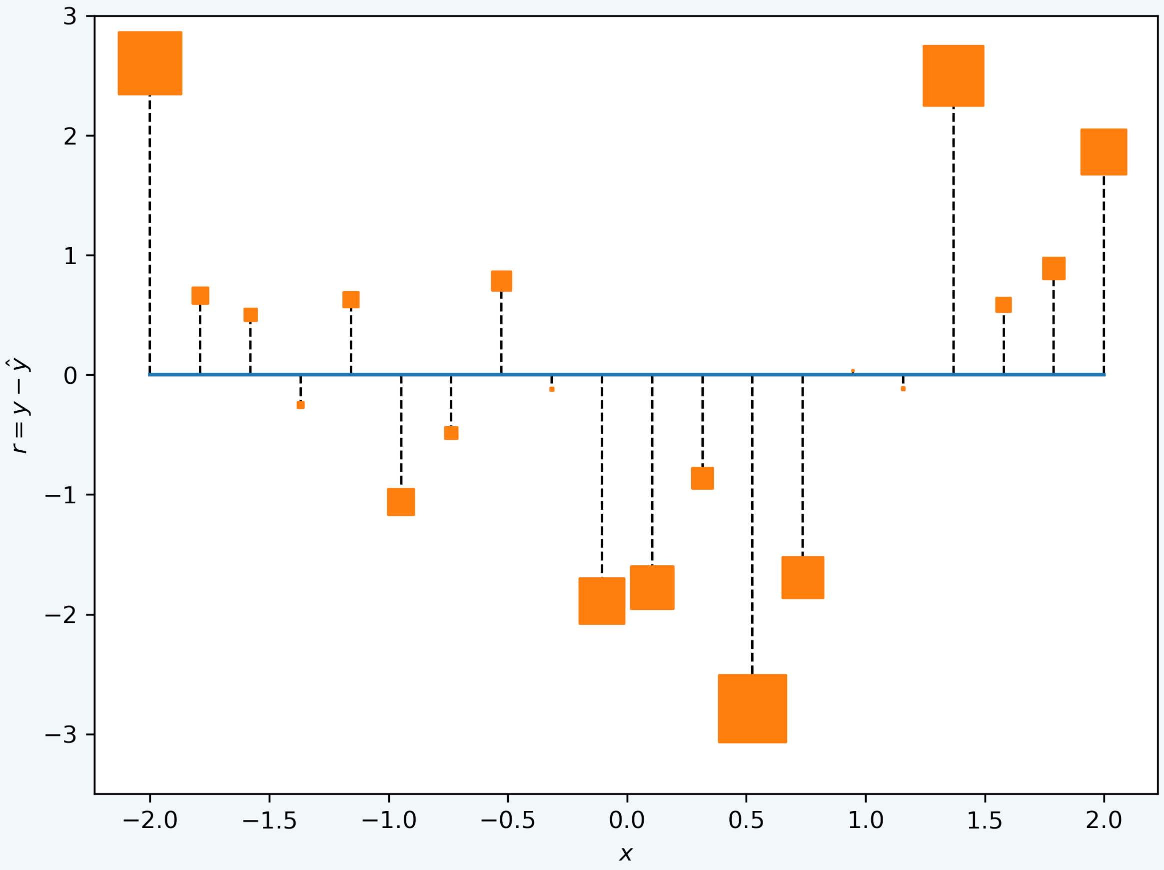

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- A positive (or negative) residual means the data is greater than (or less than) we predicted

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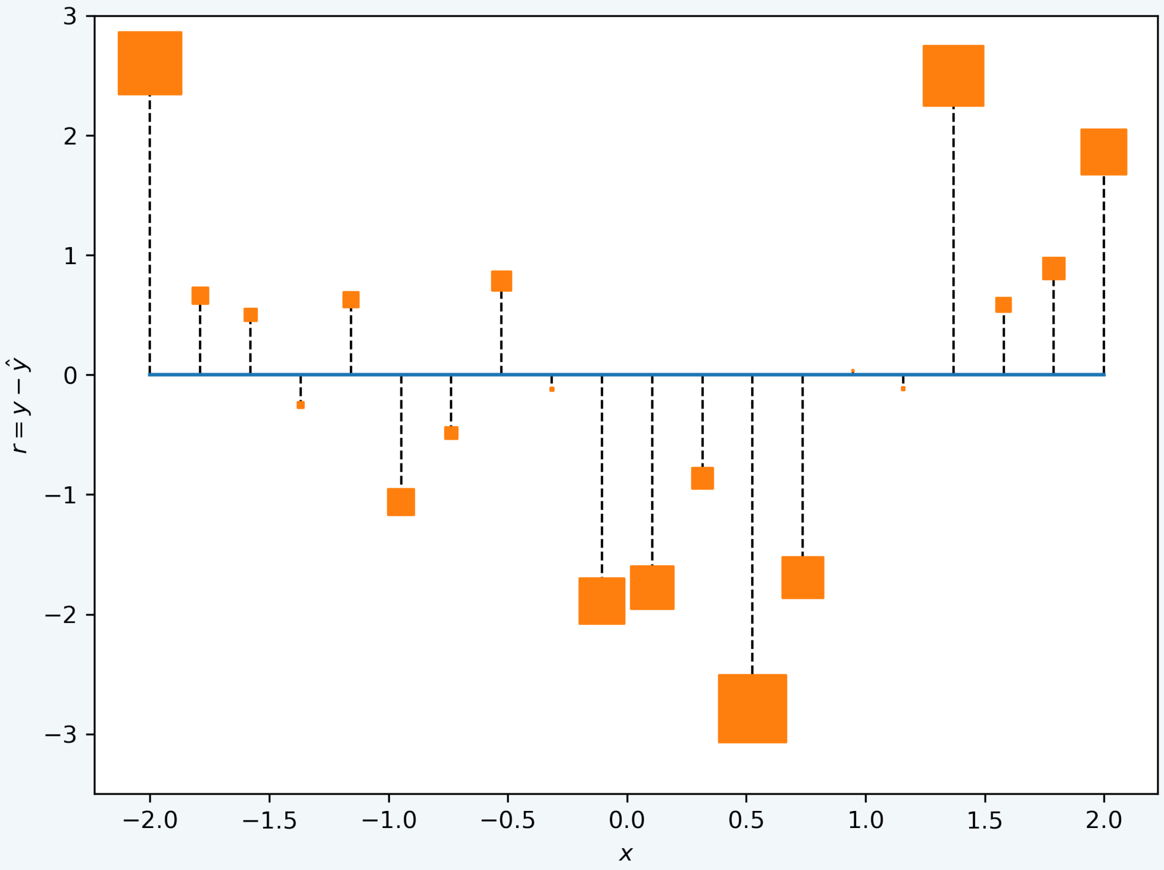

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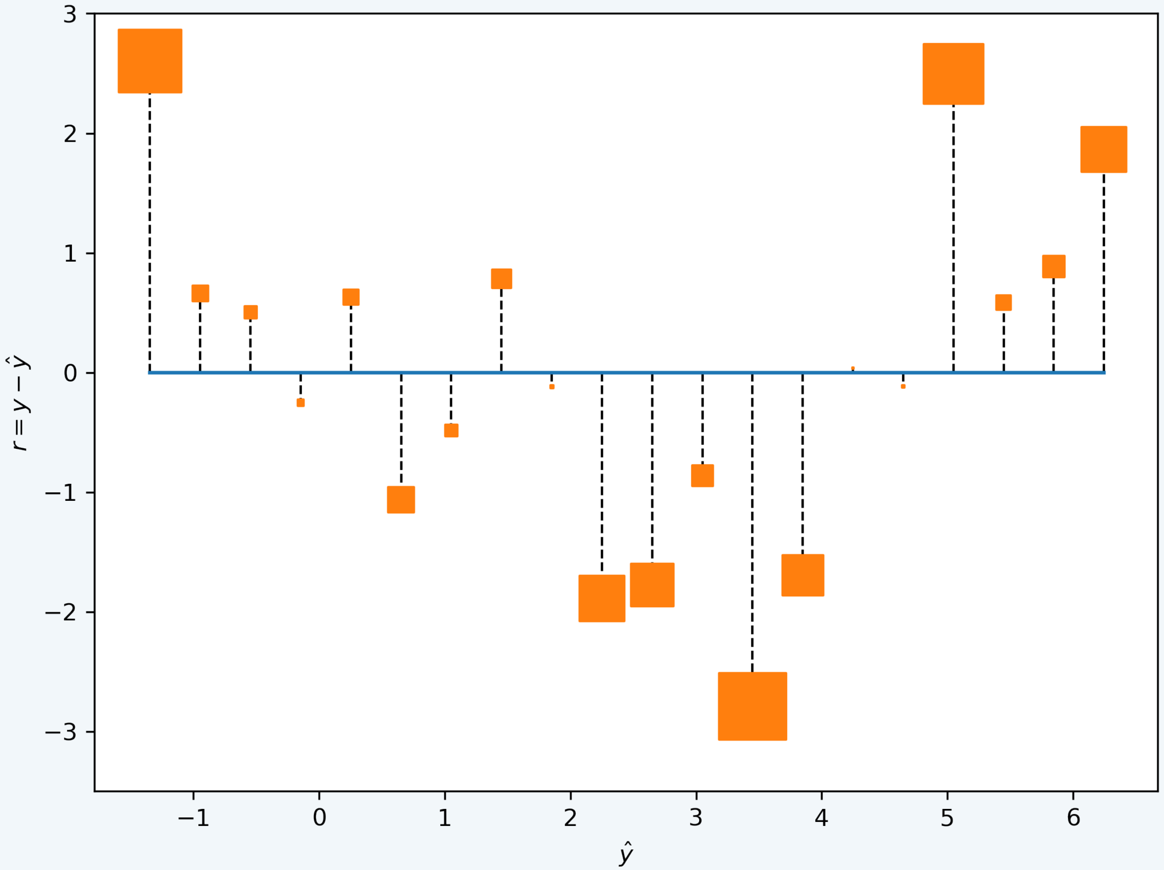

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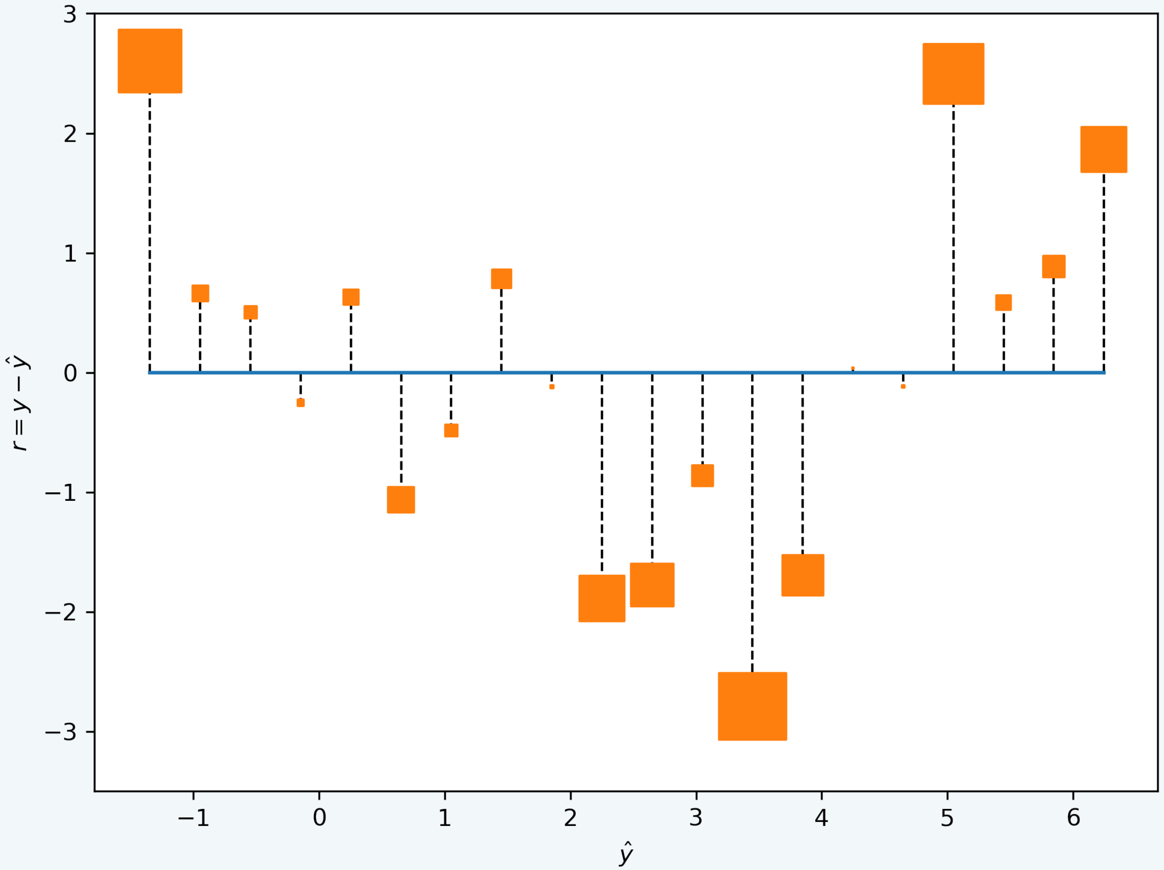

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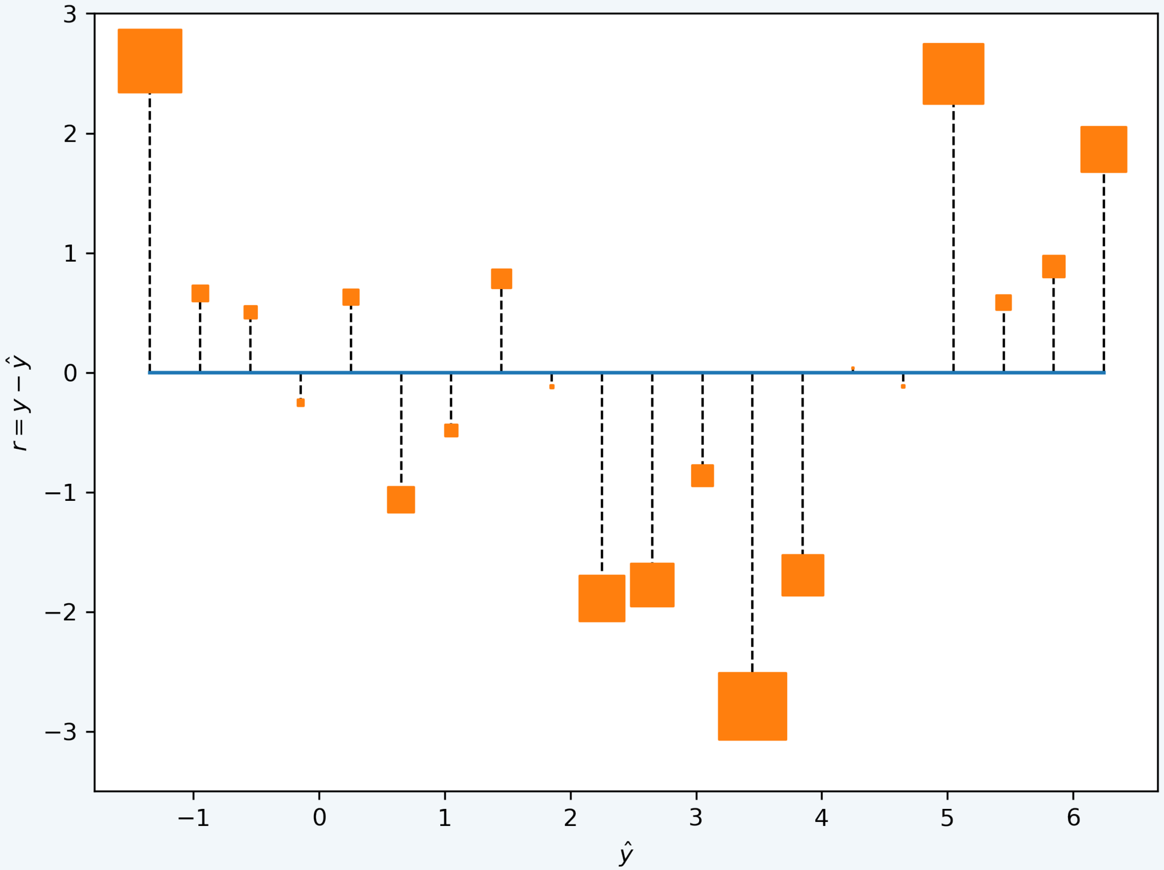

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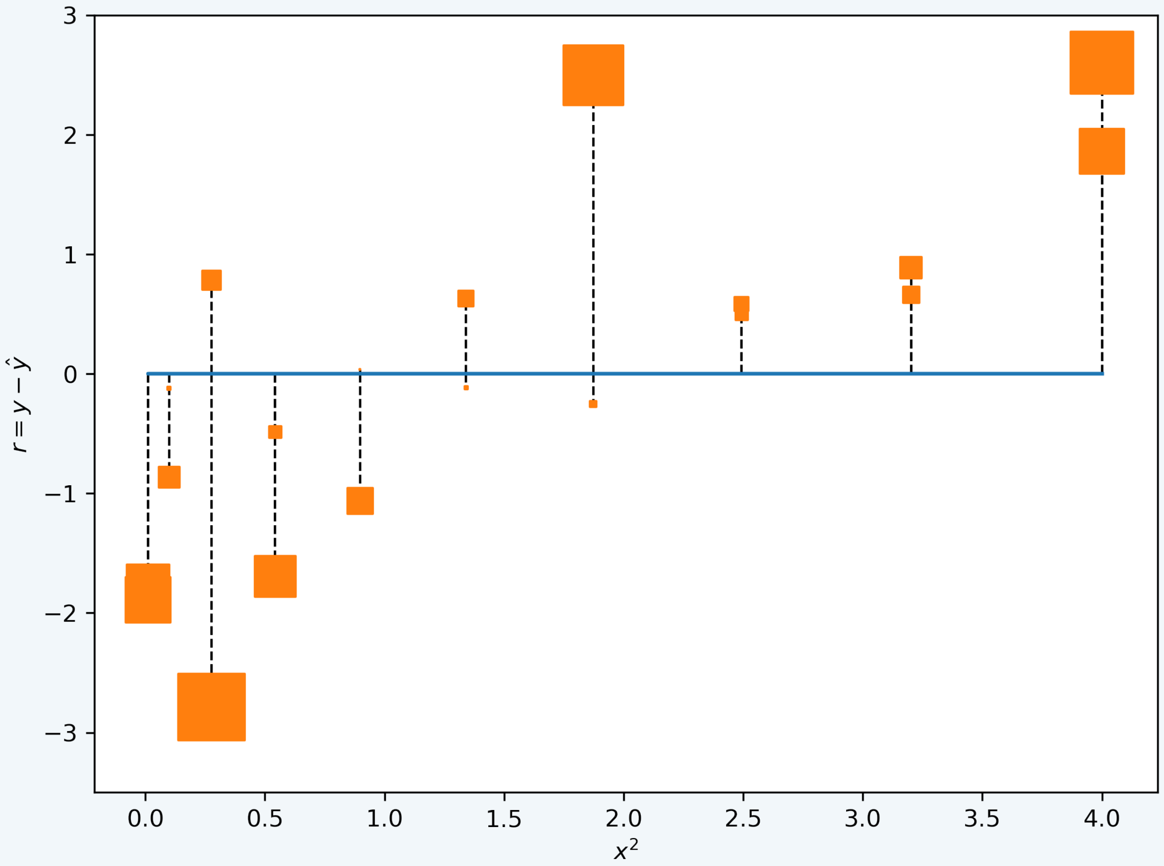

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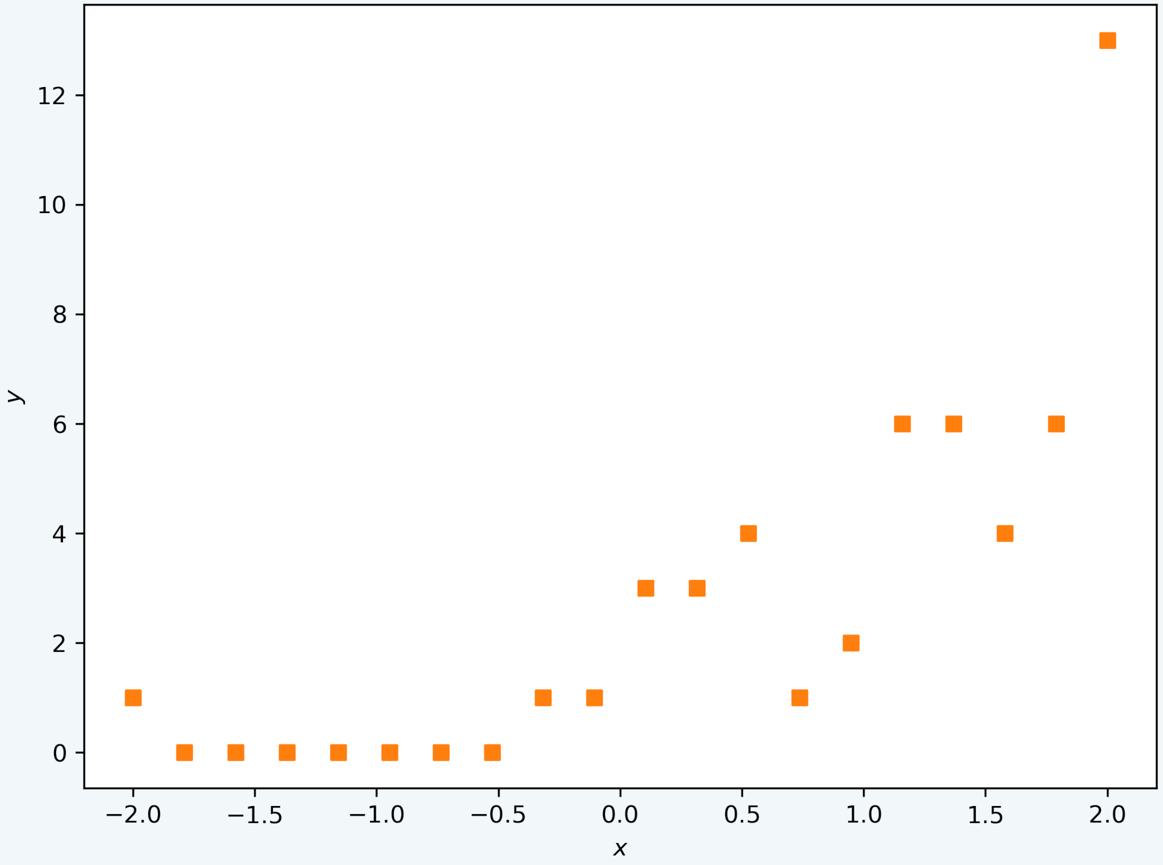
Would an x^2 feature improve the fit?

Looks like a yes!

Can we Generalize?



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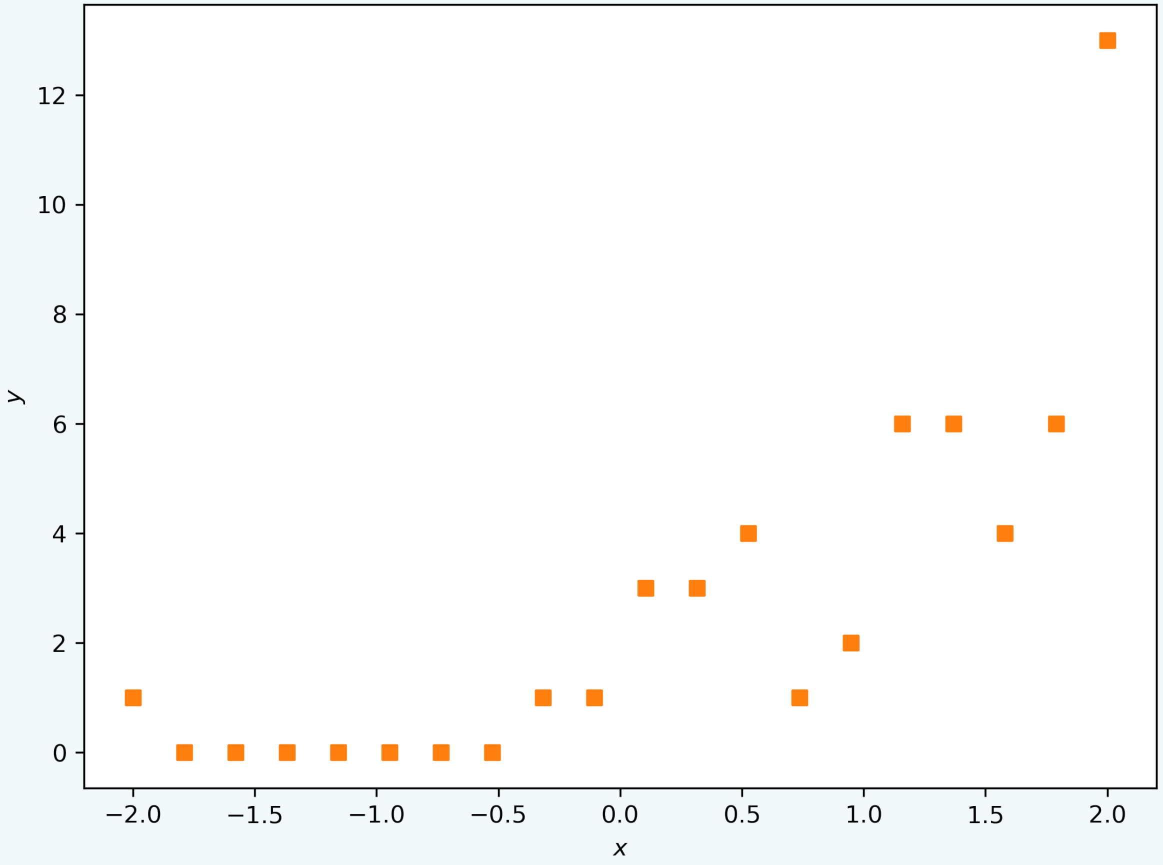

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- Normal (linear regression)
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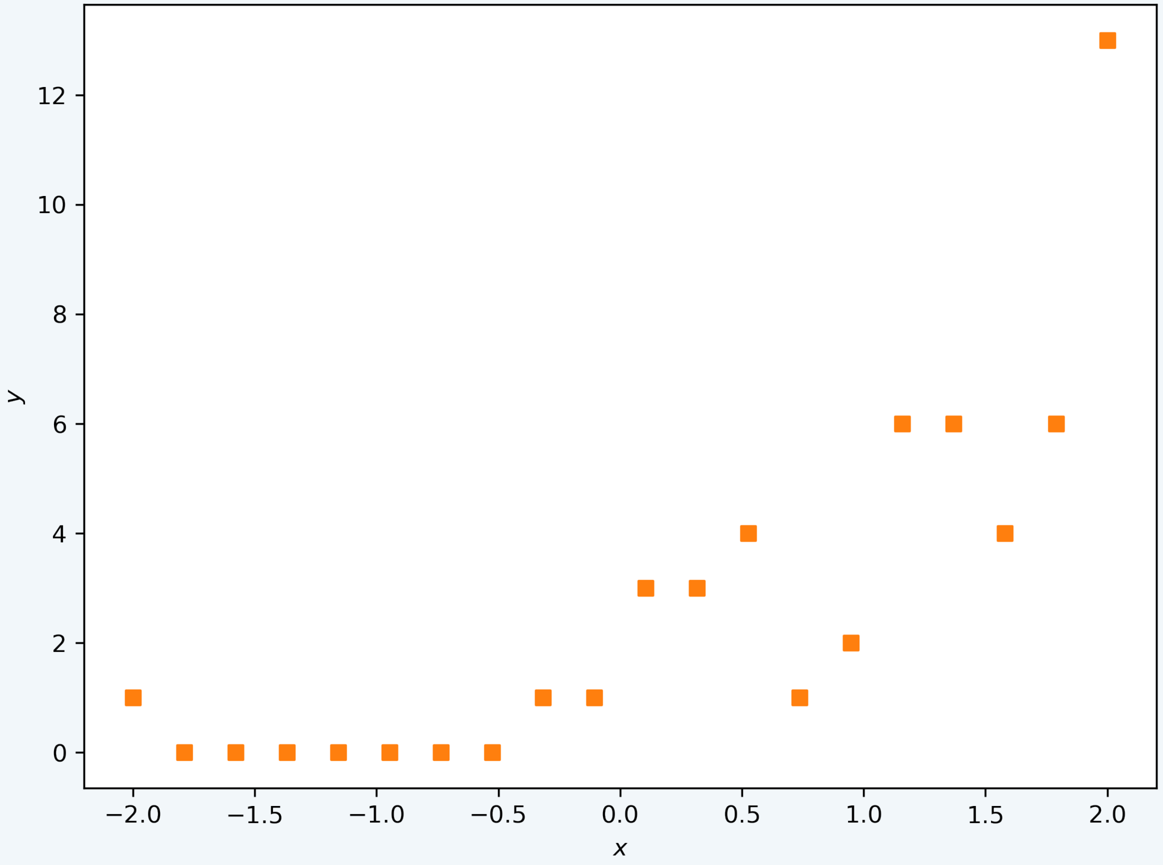

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It would be great to…

- Find something similar for count data
- Find the analog of residuals and squared residuals

Poisson Regression: Just Another GLM

Here are three generalized linear models:

Linear regression	identity $\mathbb{E}(y \mid X) = X\beta$	$y \sim$ Normal$(X\beta, \sigma)$
Logistic regression	logit $\mathbb{E}(y \mid X) = X\beta$	$y \sim$ Bernoulli $\left(\text{logit}^{-1}(X\beta)\right)$
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$X\beta$ is the *systematic component*

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The distribution of *y* is the *stochastic component*

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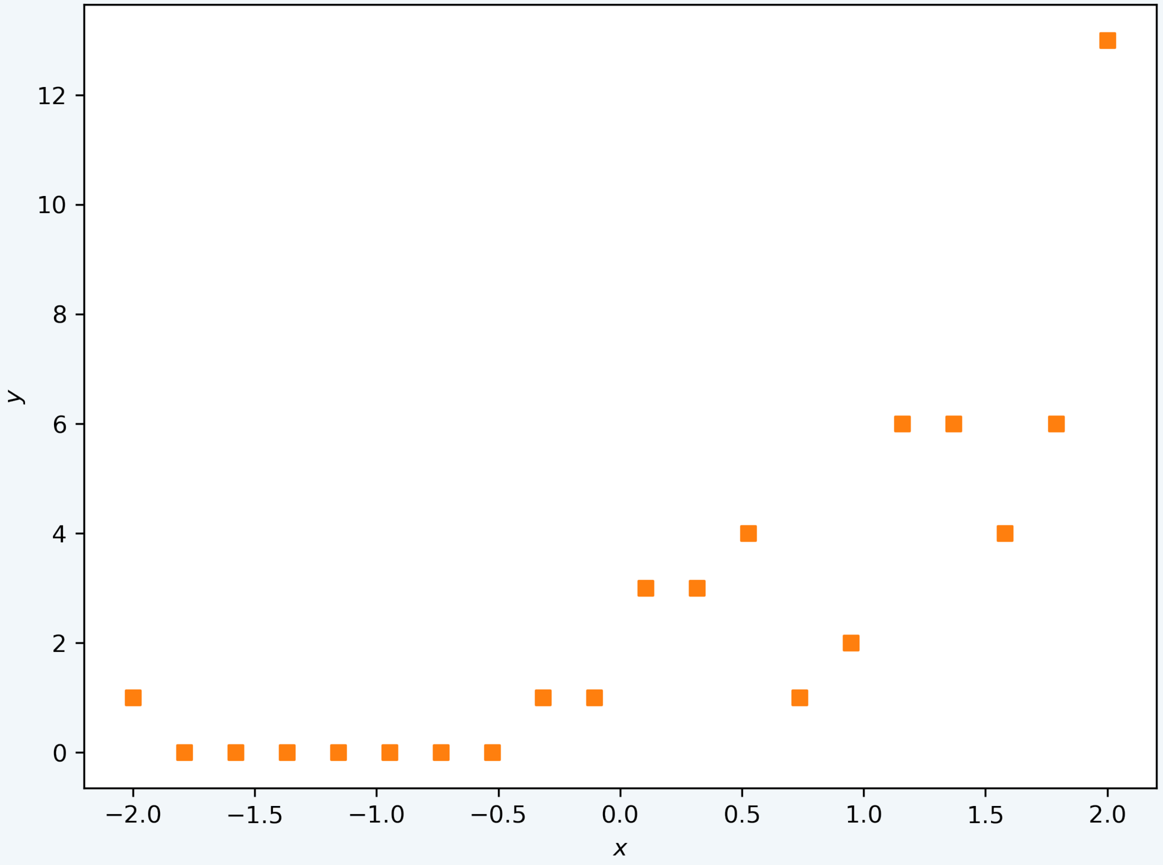
The *link function* connects the systematic and stochastic components

Like Counting Fish in a Barrel



Back to our count data

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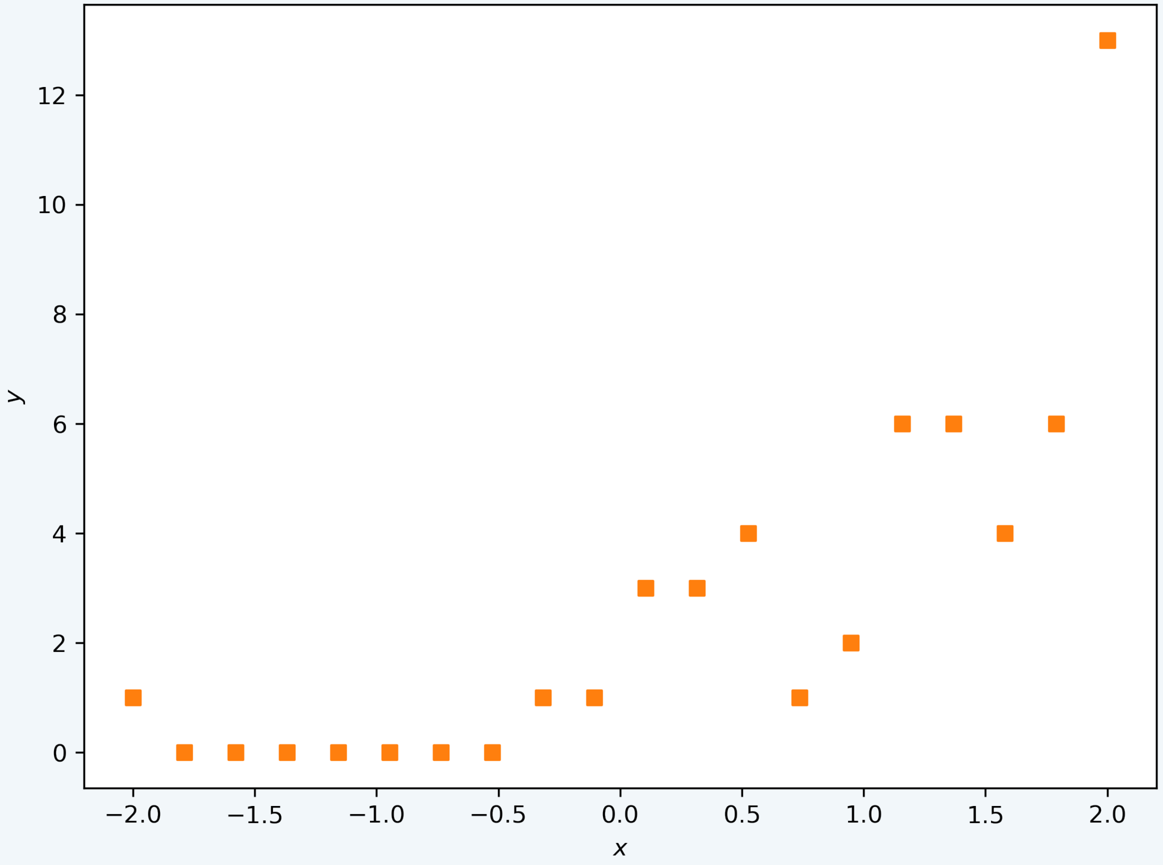

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The Poisson regression model is

$$y \sim \text{Poisson}\left(\exp(X\beta)\right)$$

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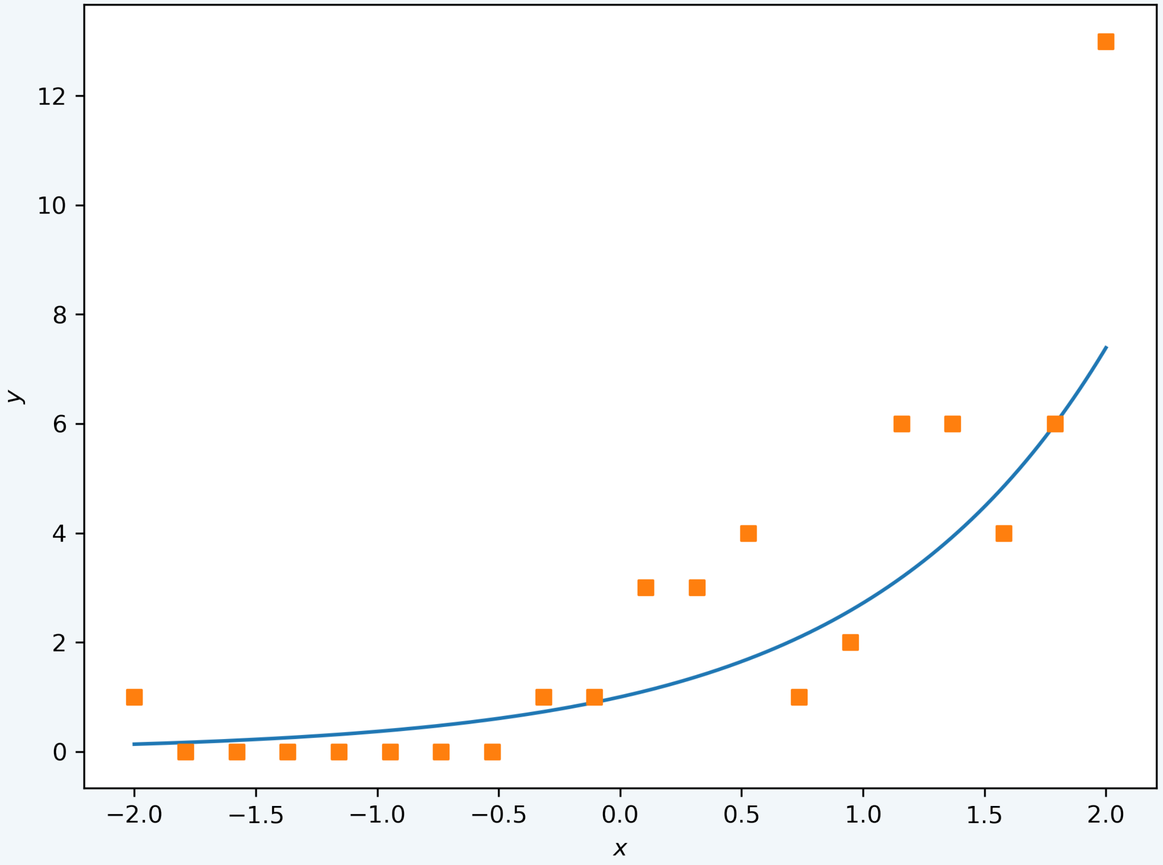

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Here's the conditional expectation

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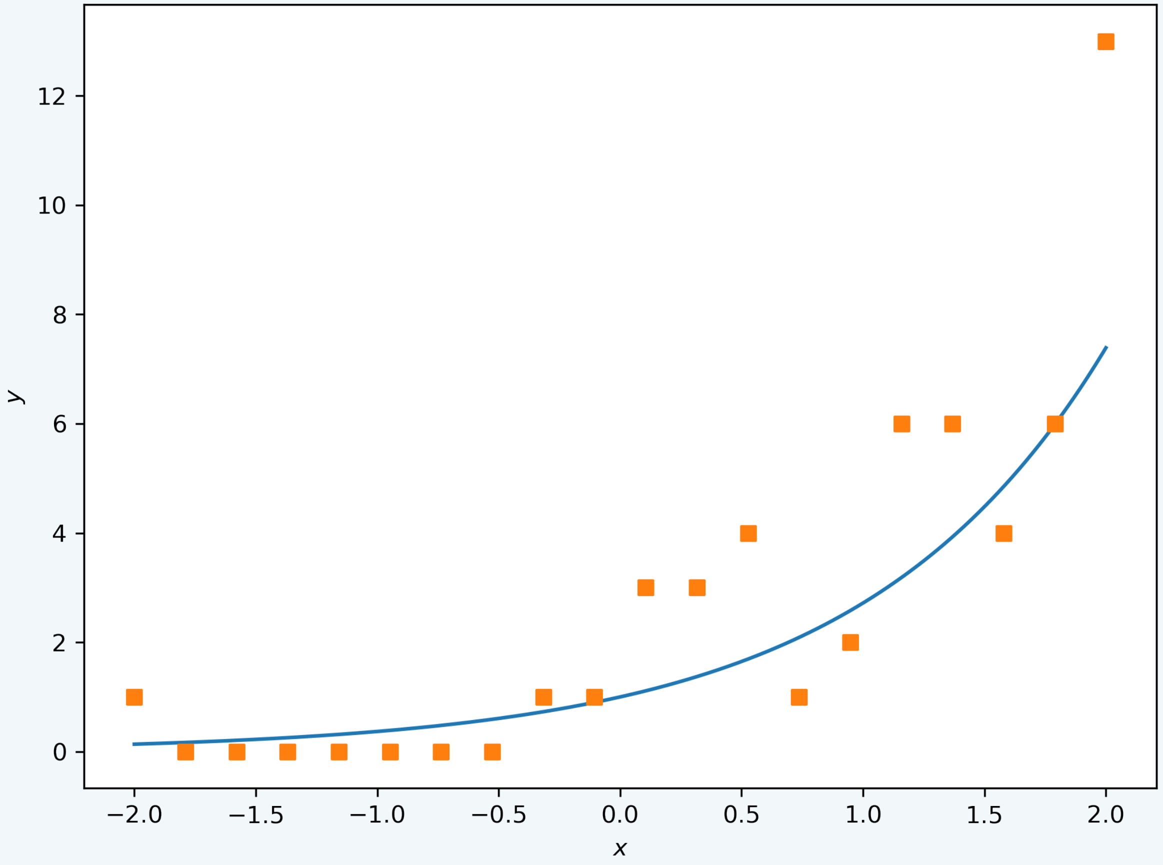

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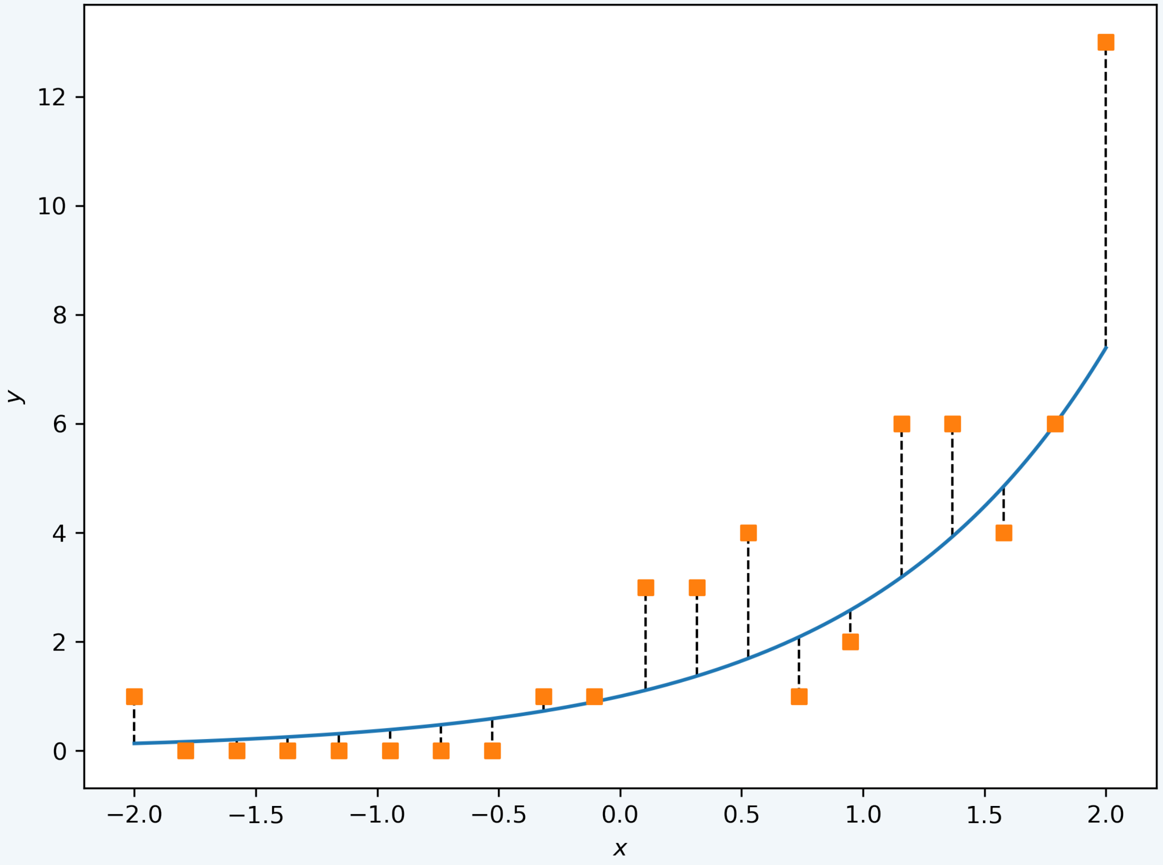
Here's the conditional expectation

What about residuals?

These Aren't the Residuals You're Looking For



Why do we usually take "observed minus predicted"?

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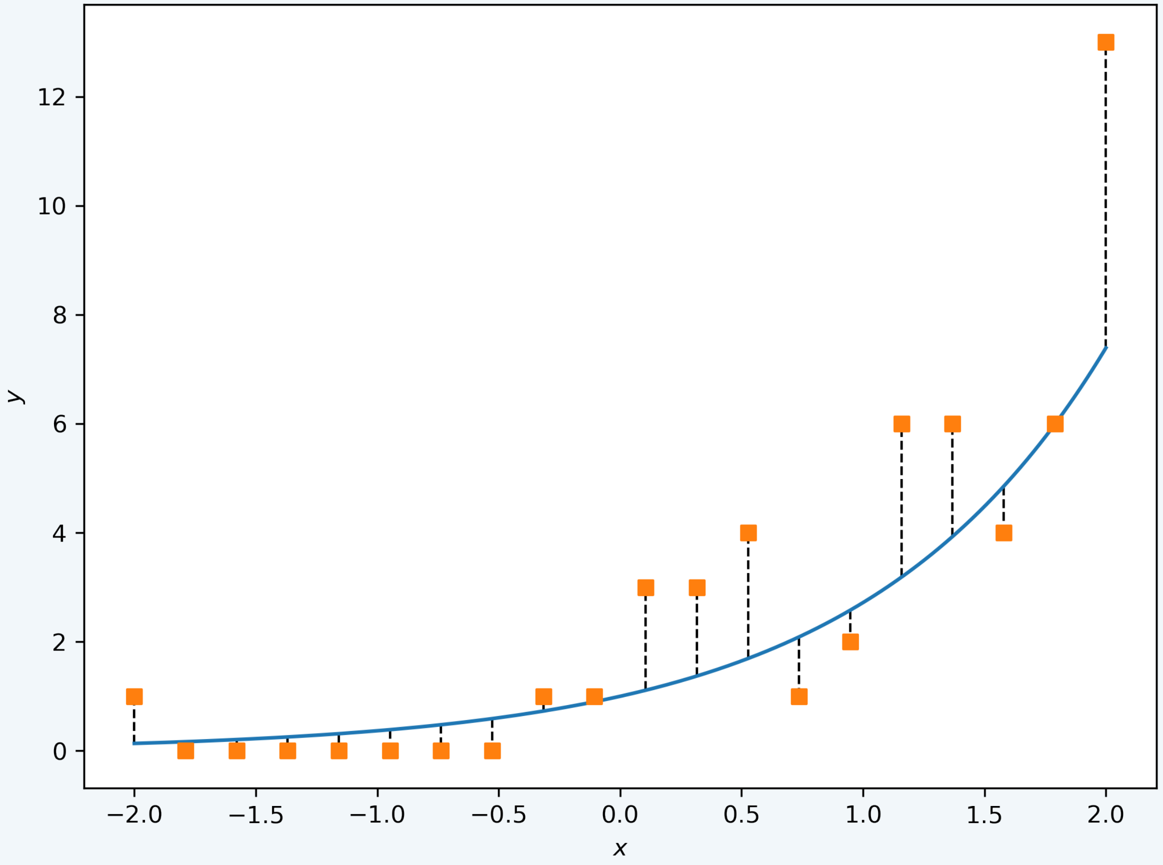

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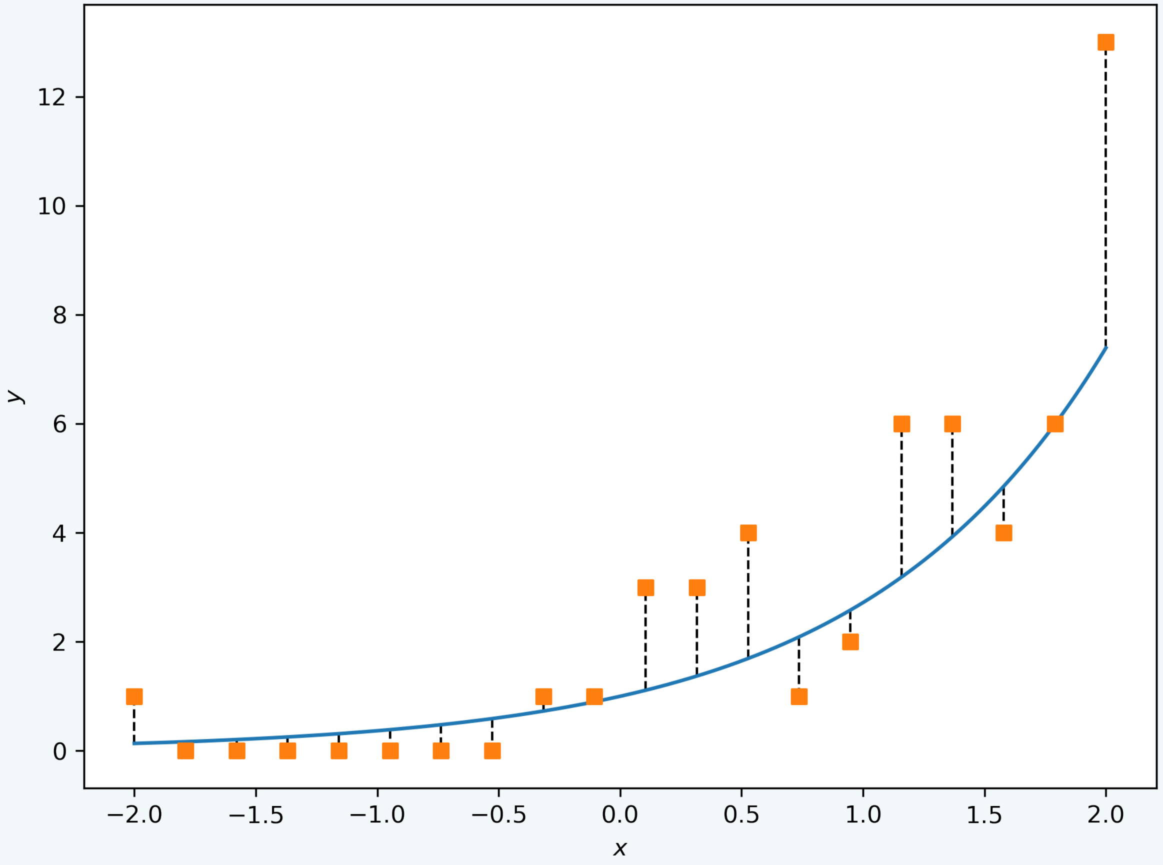

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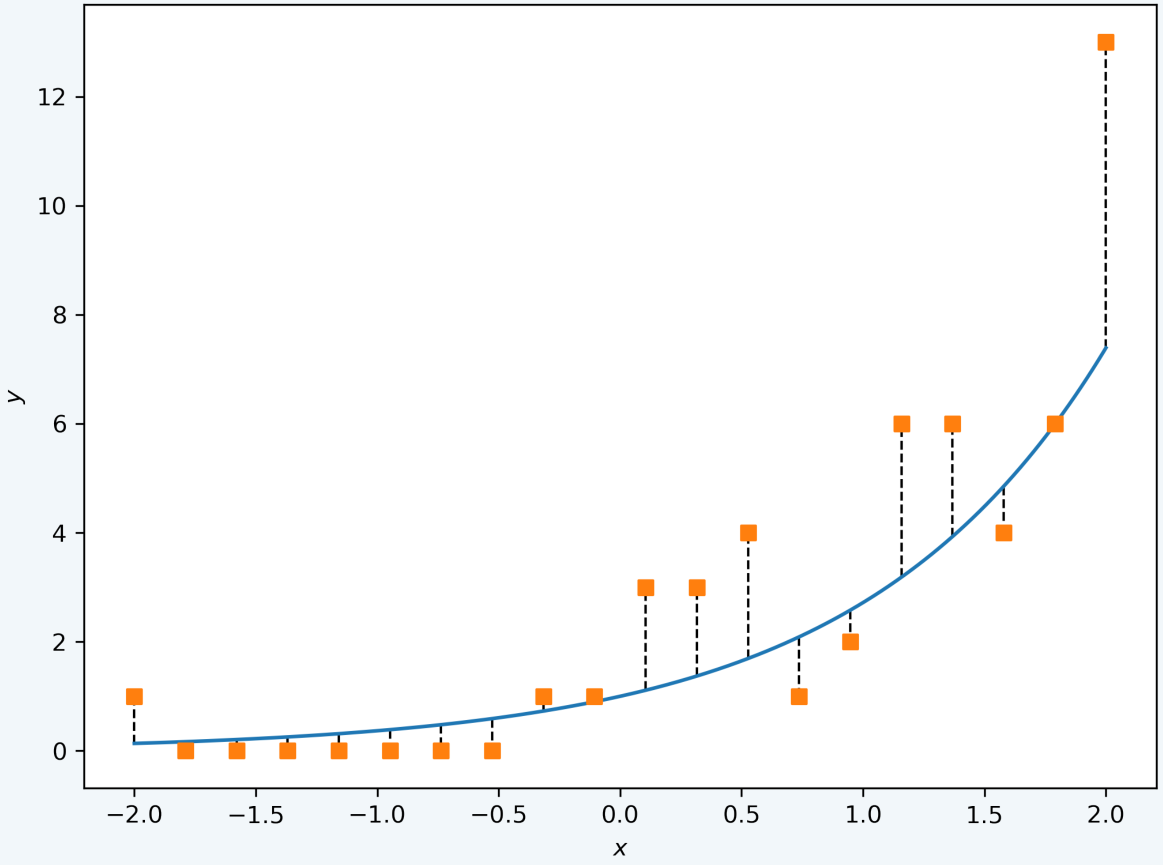

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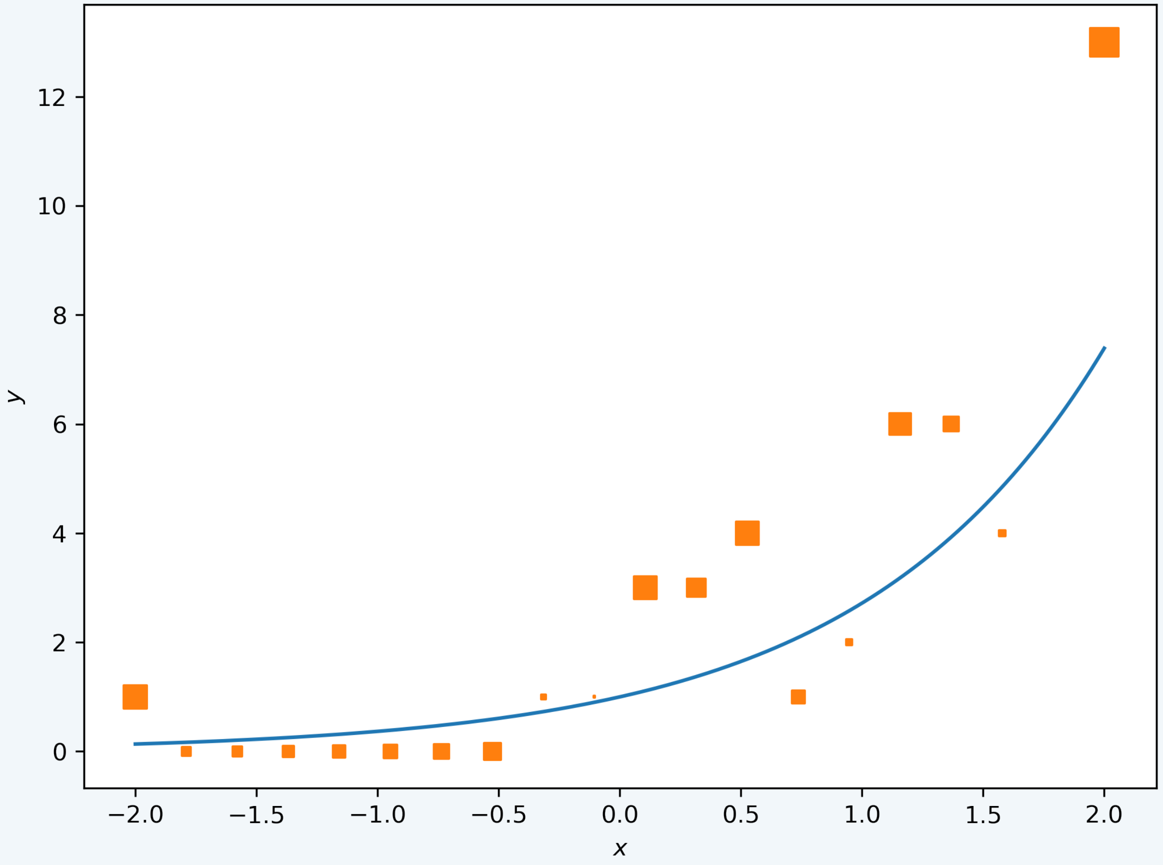
"Residuals" will be different, but we can still calculate the contribution of each point to the overall cost

Unit Deviance



Here are the contributions of each point to the total cost

For a linear model, a perfectly-fit point has zero contribution (squared residual)

Here, we subtract cost for a hypothetical “saturated model” to make this work (more on this soon)

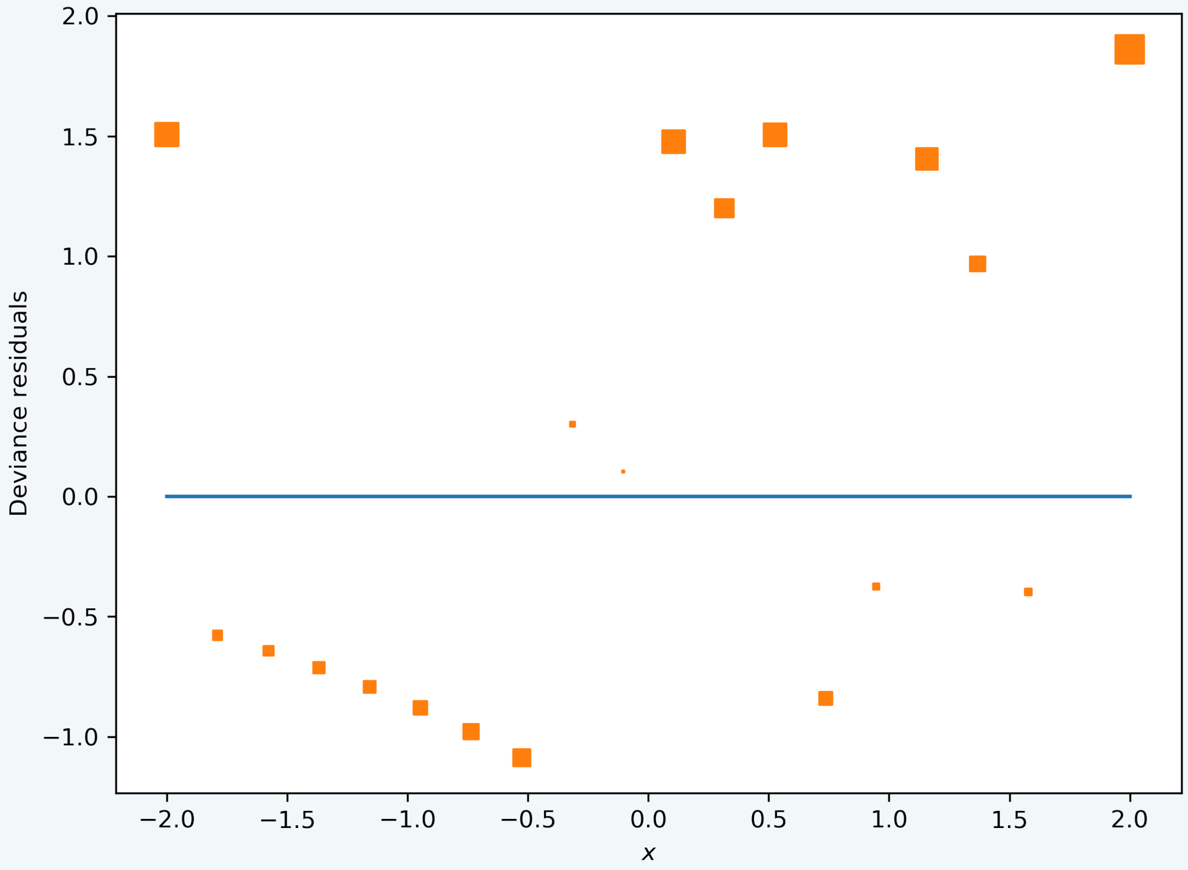
Resulting *unit deviance* values play the role of squared residuals

Remember the linear case:

- Squared residuals give contribution to cost
- Sign indicates data above or below prediction

Can we make that work out here? Spoiler: yes!

Deviance Residuals



Here's the result!

Use these *deviance residuals* just like you'd use residuals for a linear model

There can be some unavoidable patterns, like the "Hawaii" in the lower-left

Linear Models: Some more details

- For a normal linear model, the likelihood and log-likelihood for each data point are

$$L = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-\hat{y})^2}{2\sigma^2}\right)$$

$$\ell = -\frac{1}{2}\log(2\pi\sigma^2) - \frac{(y-\hat{y})^2}{2\sigma^2}$$

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- We've seen the sum of squared residuals, but what happened to the first term?

The Saturated Model, and Deviance

Imagine we could fit a model perfectly. Our log-likelihood

$$\ell = -\frac{N}{2}\log(2\pi\hat{\sigma}^2) - \frac{1}{2\hat{\sigma}^2}\sum_{j=1}^{N}(y_j - \hat{y}_j)^2$$

would become

$$\ell_S = -\frac{N}{2}\log(2\pi\hat{\sigma}^2)$$

Deviance measures how far we are from a hypothetical perfect fit:

$$D = 2(\ell_S - \ell) = \frac{1}{\hat{\sigma}^2}\sum_{j=1}^{N}(y_j - \hat{y}_j)^2 = \sum_{j=1}^{N}\left(\frac{y_j - \hat{y}_j}{\hat{\sigma}}\right)^2$$

For a linear model, deviance is the sum of *Studentized* residuals!

GLM Hypothesis Testing

- For a linear model, deviance is

$$D = \frac{1}{\sigma^2} \sum_{j=1}^{N} (y_j - \hat{y}_j)^2$$

- If $\mathscr{M}_0$ is a “submodel” of $\mathscr{M}$ (so $\mathscr{M}_0 \subset \mathscr{M}$), which will fit training data better? $\mathscr{M}$
- What does this mean about the deviance? $D_0 > D$

- How do we measure “how much better” D fits? With a χ^2 test!

$$D_0 - D \sim \chi^2_{\Delta p}$$

“Degrees of freedom” is difference in number of parameters