Introduction to Generalized Linear Models

Generalized Linear Models

Three components:

- Response variable y
- Explanatory variables $x_1, x_2, x_3, \ldots$
- Link Function g

1. Response variable y

- a.k.a. random component
- Assume independent observations $y_1, \ldots, y_n$ from a particular distribution
  - Model: $\mu_X = E(Y_X)$
  - i.e. how response depends on explanatory variables

2. Explanatory variables (the x's)

- a.k.a. systematic component
- "covariates," "explanatory variables," "features"
- Linear combination of your covariates: $x_1, \ldots, x_p$

$$\eta = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_p x_p$$

2. The link function

- The relationship between your response variable, y, and the linear combination of your covariates: x_1, …, x_p

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- So if you have:
    1. Response variable Y:

       Model: $\mu_X = E(Y_X)$

    2. Explanatory variables $x_1, \ldots, x_p$:

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GLM for Linear Regression

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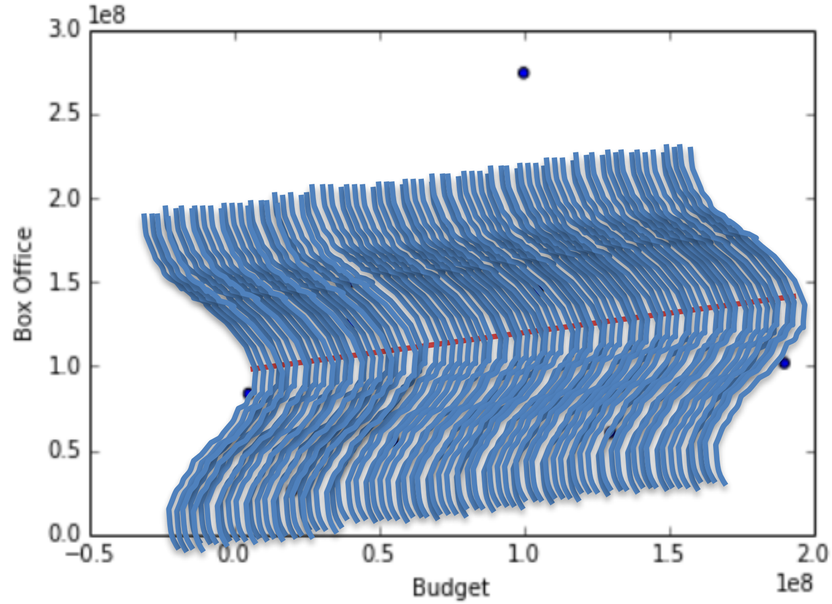
1. Response variable, Y
2. The covariates: $x_1, \ldots, x_p$
3. The link function between the Y and the x's

GLM for Linear Regression

1. Response variable

Y is normally distributed (conditioned on the covariates) with $E(Y_X) = \mu_X$ and constant variance $E(Y) = \sigma^2$:

→ $\varepsilon \sim N(0, \sigma^2)$
→ $Y \sim N(\mu_X, \sigma^2)$



GLM for Linear Regression

2. The covariates: $x_1, \ldots, x_p$

The covariates produce a linear predictor η given by:

$$\eta = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_p x_p$$

GLM for Linear Regression

3. Link function g:

   How do we connect $E(Y_X)$ to $\beta_0 + \beta_1 x_1 + \ldots + \beta_p x_p$?

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   With the identity function: $g(x) = x$

$$g(E(Y_X)) = \eta$$

$$E(Y_X) = \beta_0 + \beta_1 x_1 + \ldots + \beta_p x_p$$

In this formulation of classical linear models in the GLM framework, the random component Y_X has a normal distribution, and the link function is the identify function.

GLM allows for 2 extensions of OLS

1. The distribution of Y can come from any exponential family distribution
2. The link function g can be any (monotonic differentiable) function.

Note: the same Maximum Likelihood fitting procedure applies to all GLMs.

GLMs for binary data

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Notice: $Var\ (Y)$ changes as π changes!

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Once again, it's just a linear combination of the covariates $x_1, \ldots, x_p$

$$\eta = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_p x_p$$

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The input of this function is

$$E(Y_X) = \mu_X = \pi \; ; 0 \leq \pi \leq 1$$

The output of this function is

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The link function for logistic regression is the logit function.

GLMs for everything!