Introduction to Categorical Data Analysis

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Categorical Data Analysis

Methods for response *(y)* variable having scale that is categorical

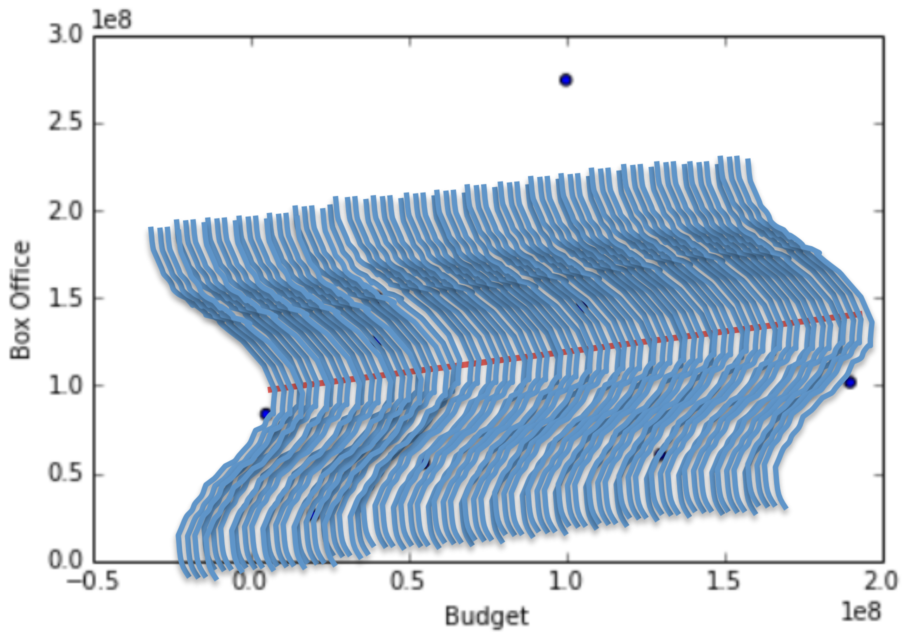
Examples:

- Binary (Democrat or Republican)
  (Success or Not Success)
- Nominal (Favorite music genre: rock, classical, jazz)
- Ordinal (Rank: 1^{st}, 2^{nd}, 3^{rd})

For these problems, linear regression is not the correct approach.

Probability Distributions for Categorical Data

In linear regression: response y is continuous, and the (conditional) response $y \mid x$ is normally distributed.



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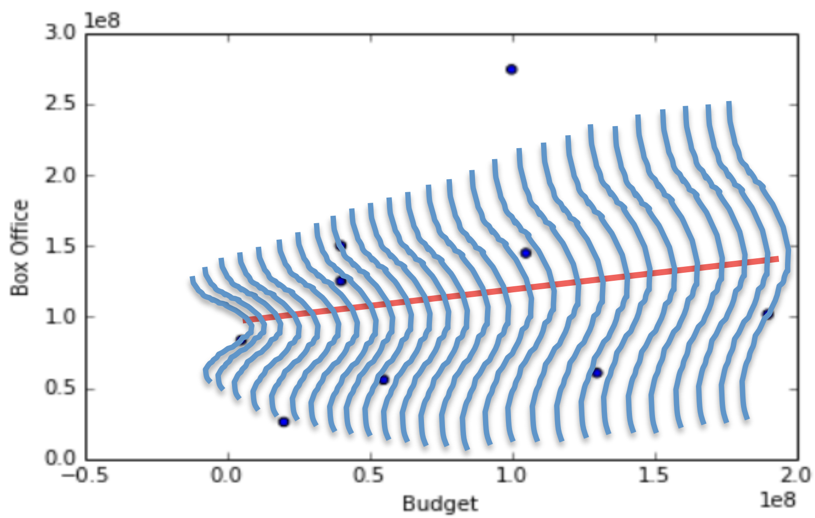
In models where y is categorical, the response $y \mid x$ can take other distributions.



Let’s first start simple.

Binomial Experiments

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Three conditions:

- n independent trials

- The outcome of a single trial is classified as 1 of 2 types:
    - *Success*
    - *Not Success*

- For each trial, the probability of success is the same:
    - $P(Success) = \pi$
    - Note: π is called the parameter of the experiment.

Binomial Experiments

Y: random variable, represents the # of successes out of n trials

Example:

- 200 voters
- Each person votes:

  Democrat – with $\mathrm{p} = 0.55$

  Republican – with $\mathrm{p} = 0.45$

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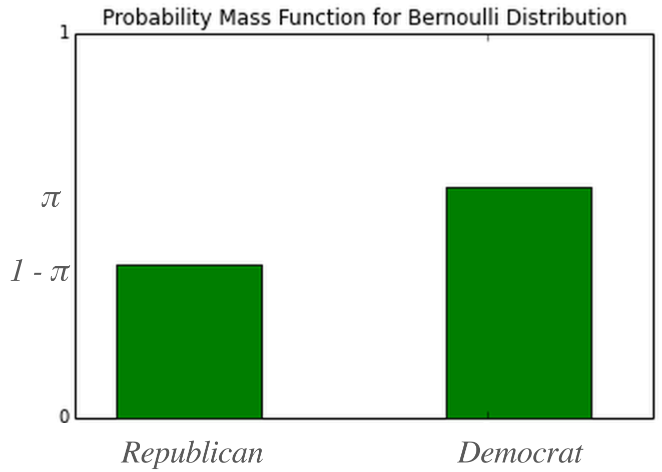

The Binomial Distribution

More generally, the random variable

$$Y \sim Binomial\ (n, p)$$

has probability mass function:

$$P(Y = y) = \frac{n!}{y!(n-y)!}\ \pi^{y} (1-\pi)^{n-y}$$

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In general, for a binomial random variable Y,

- The <u>Expected Value</u> is:

$$E(Y) = \mu_Y = n\pi$$

- The <u>Variance</u> is:

$$Var(Y) = \sigma_Y^2 = n\pi(1-\pi)$$

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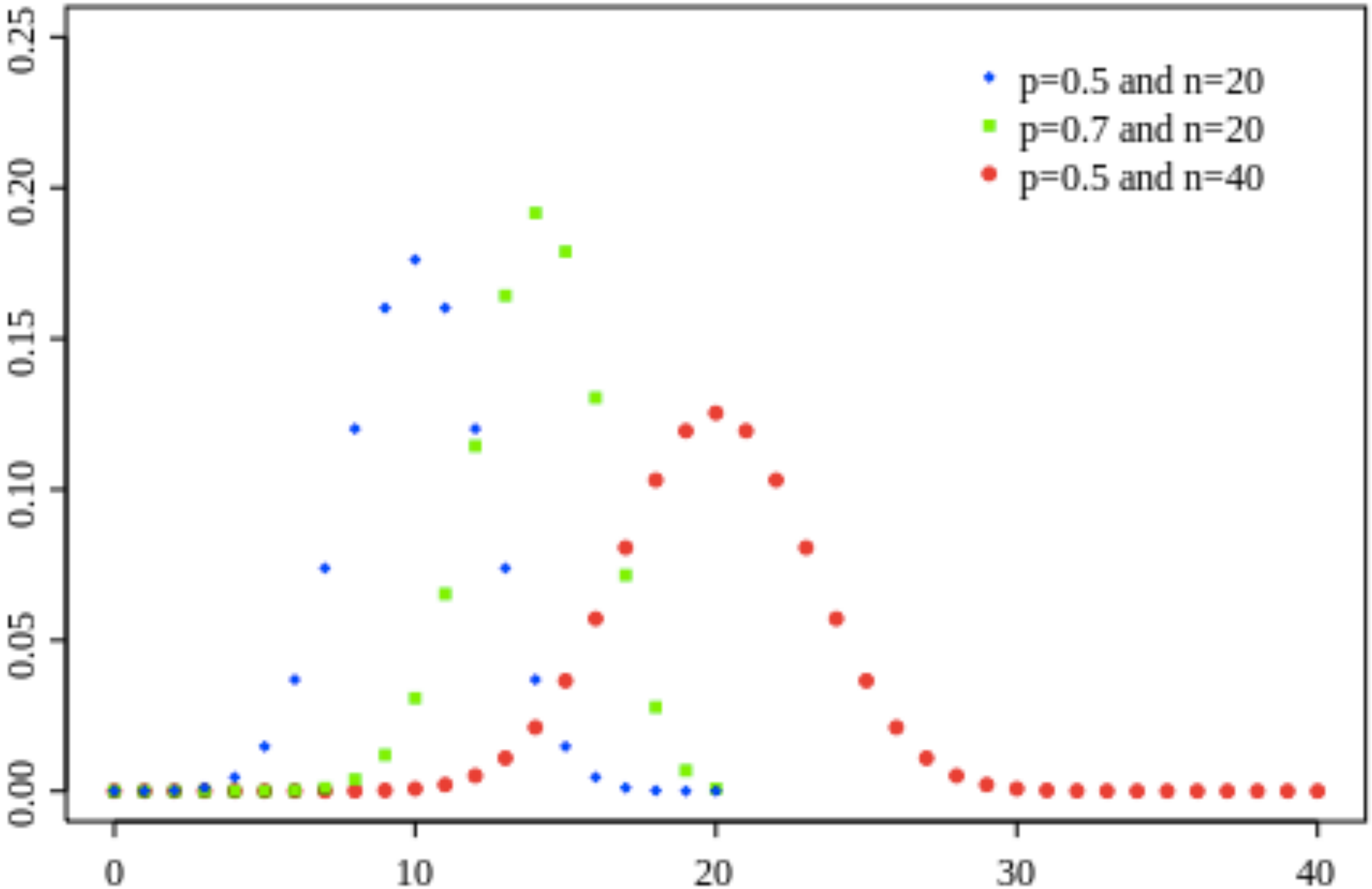

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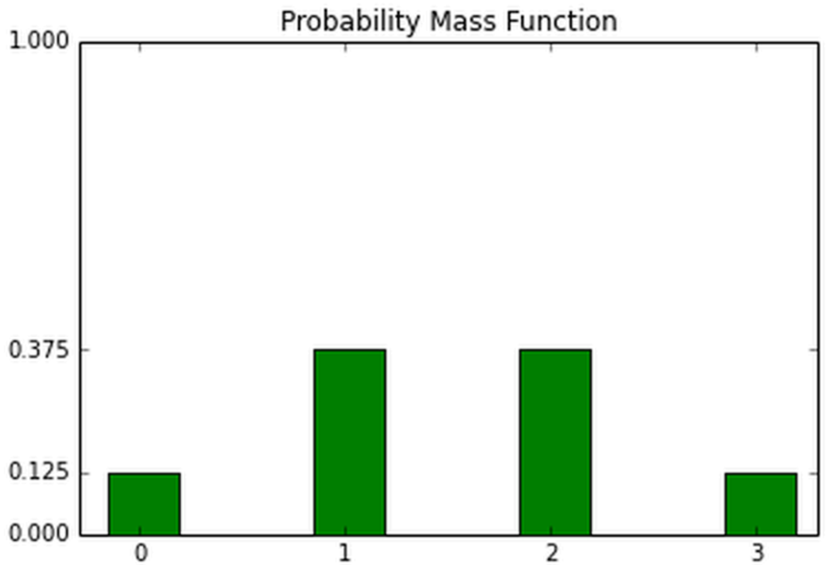

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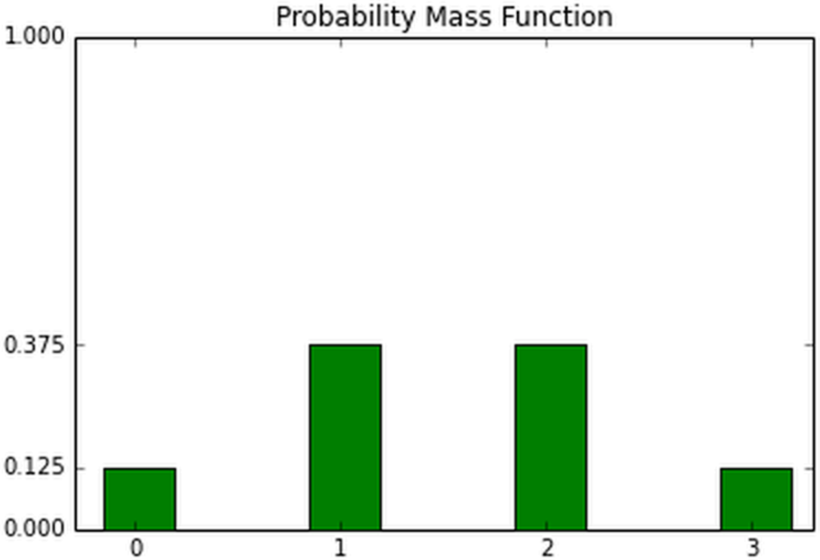
$$P(Y = y) = \frac{3!}{y!(3 - y)!}\ (1/2)^{y}\,(1/2)^{3-y}$$

$$= \frac{3!}{y!(3 - y)!}\ (1/8)$$



Thinking with Likelihoods

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Before, we were thinking about the probability of various possible outcomes of Y, given a parameter π (and n data points).

Given π, what is is the probability of seeing *y # successes*, $P(Y=y)$?

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Given $Y = y$ # *successes*, what is the underlying proportion π?

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Given $Y = y$ *# successes*, what is the underlying proportion π?

Example: on Wall St, I count 100 people, 15 of whom are women. What is the proportion of women on Wall St?

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- There is an unknown parameter π (like a "state of nature")
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In particular, we'd like to find the probability that is *maximal*.

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   Some controversy in getting the prior:
   - How to get it, what it means, subjective disagreements, subjective biases

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Definition. The <u>likelihood function</u> is the probability of the observed data occurring, expressed as a function of the parameter value.

Example.
Of the N = 2 people on Wall Street, 1 was a woman.
Then $Y = \#\ successes = 1$.
$Y \sim Binomial\ (n = 2, \pi = ?)$

Maximum Likelihood Example, continued.

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The likelihood function, below, is the probability of observing $y=1$, expressed as a function of parameter value π:

$$L(\pi) = 2\pi(1 - \pi)$$

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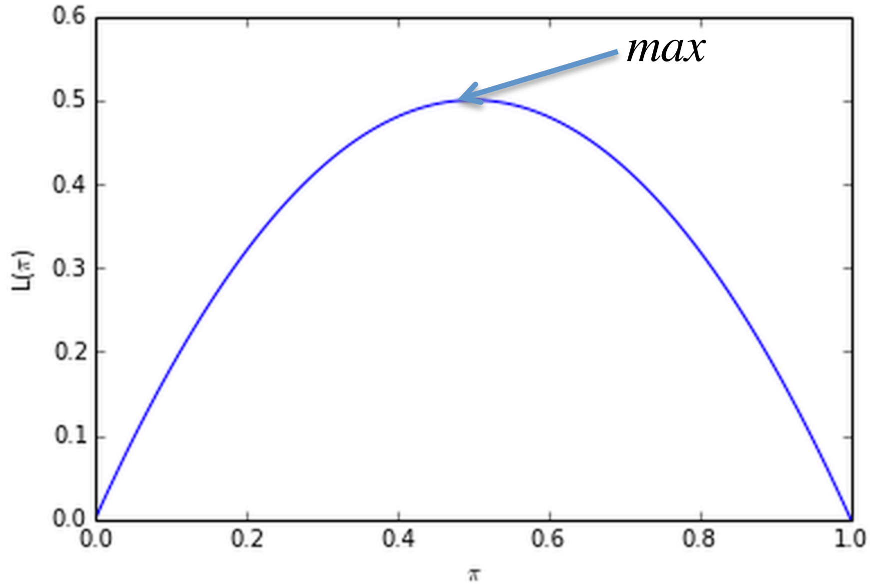

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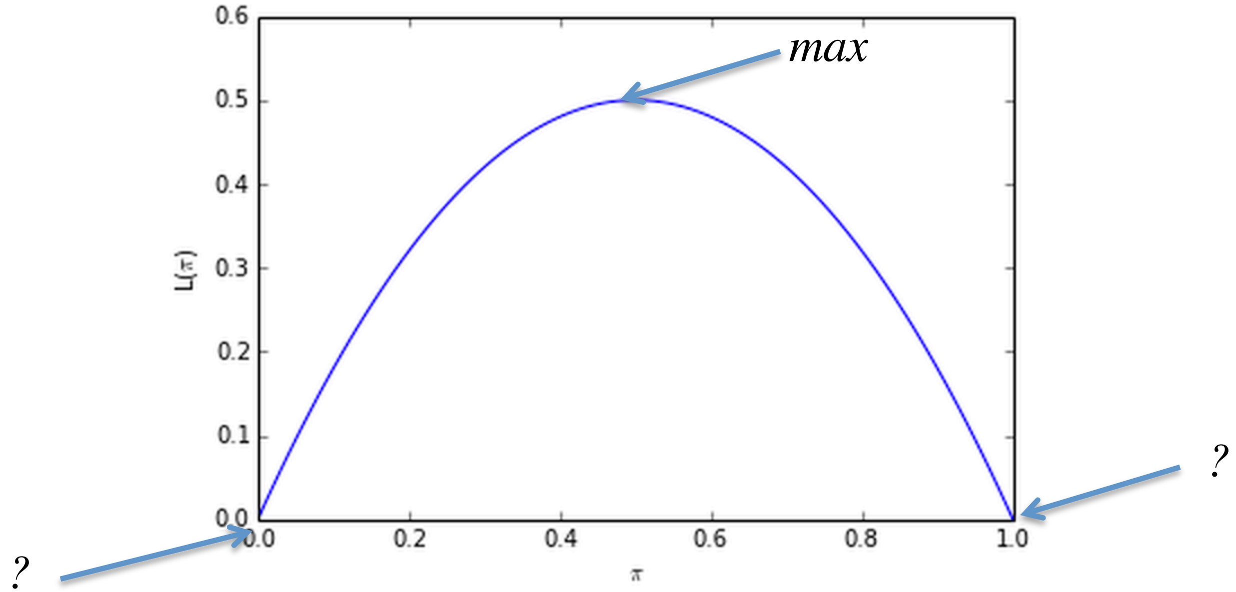

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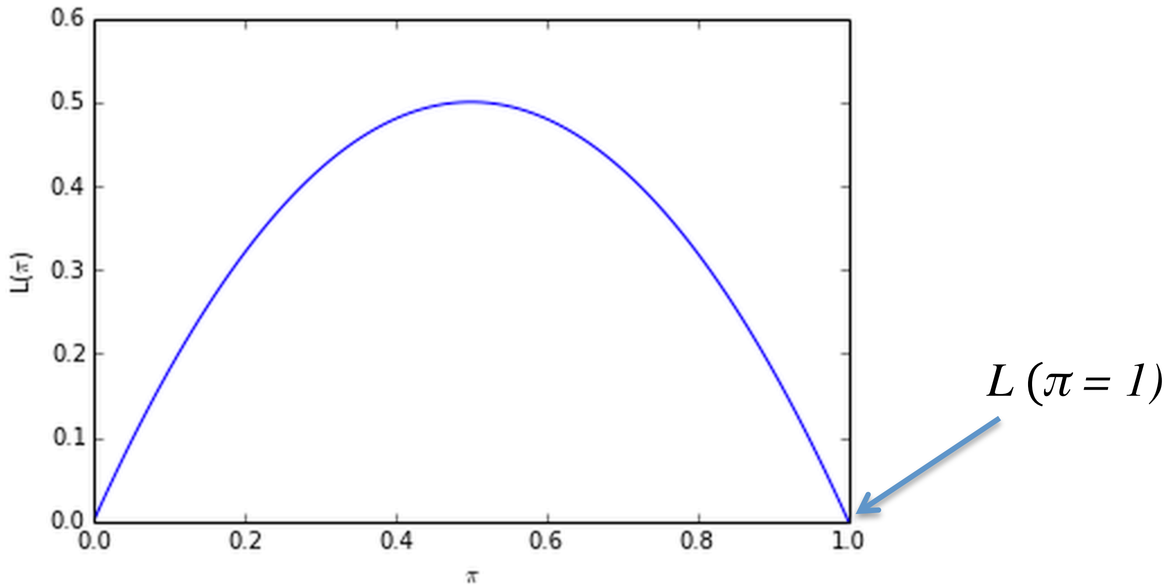
$$L(\pi) = 2\pi(1 - \pi)$$

If $\pi = 1$ *, the likelihood function represents the probability of observing* $y = 1$.

$$L\ (\pi = 1) = 2 \times 1 \times (1 - 1)$$
$$= 0$$



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$$L\ (\pi = 0) = 2 \times 0 \times (1 - 0)$$
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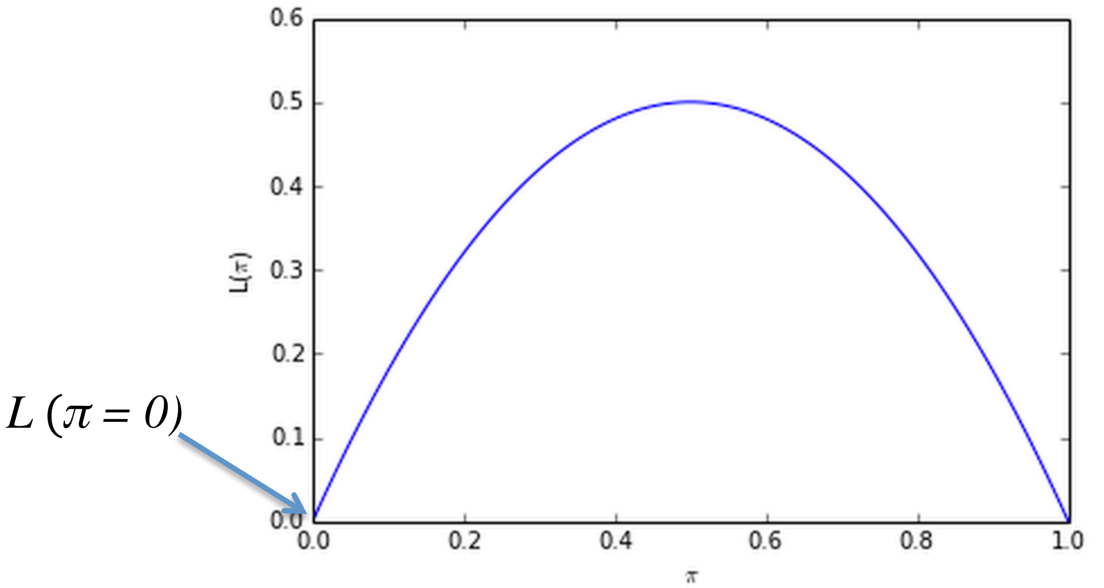

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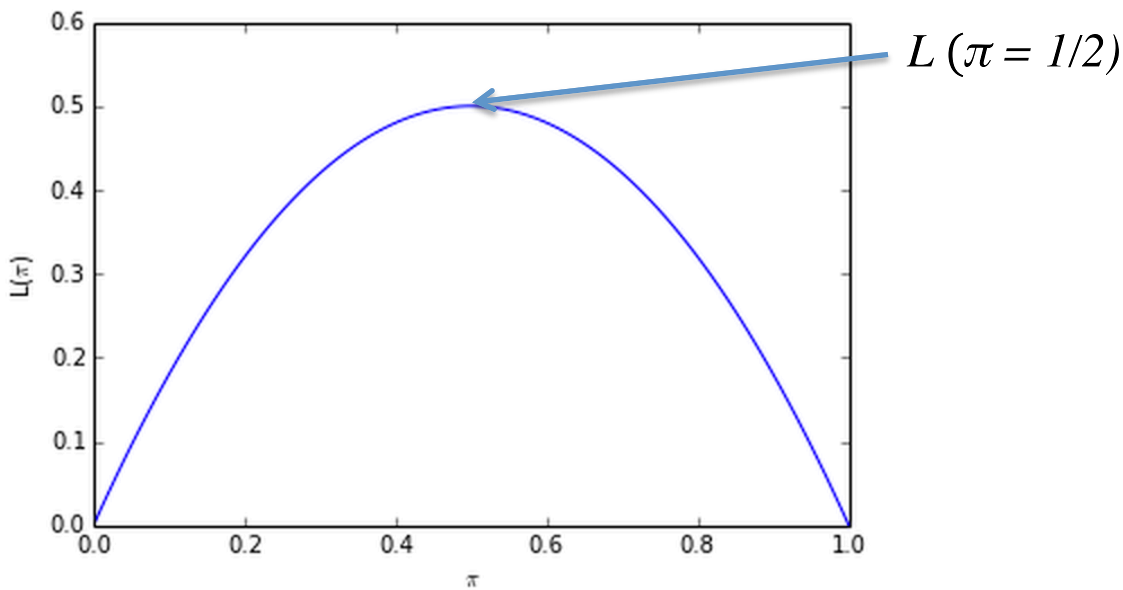
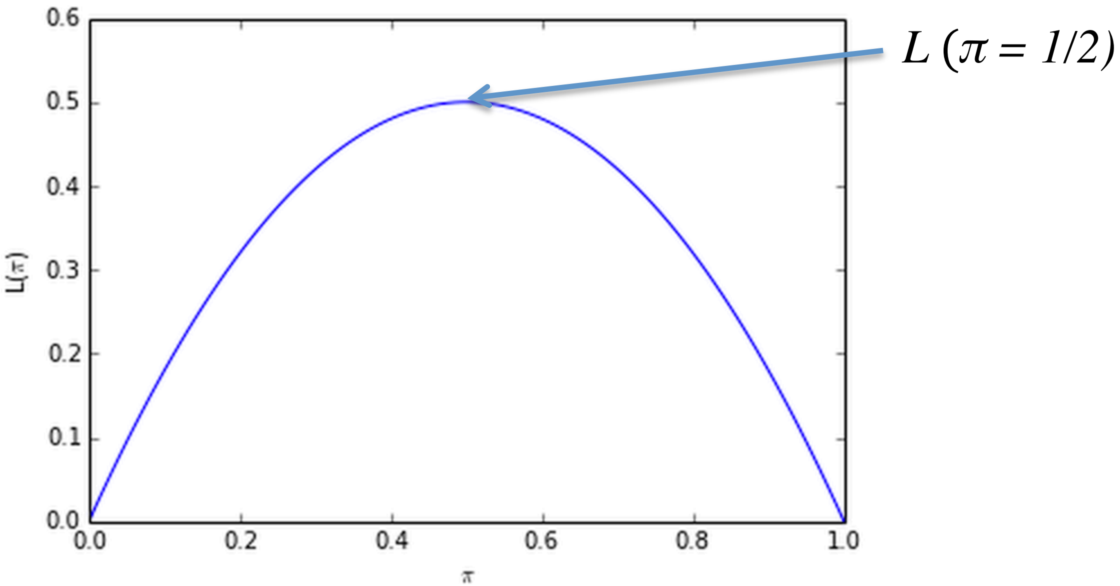
$$= 1/2$$



This is the maximum likelihood estimate.



Definition. The Maximum Likelihood (ML) estimate of a parameter π is the parameter value at which the likelihood function takes its maximum.

Intuitively, the Maximum Likelihood estimate $\hat{\pi}$ is your guess for the "state of nature" that best explains the data. It is the estimate for which the data is *most likely* to occur.

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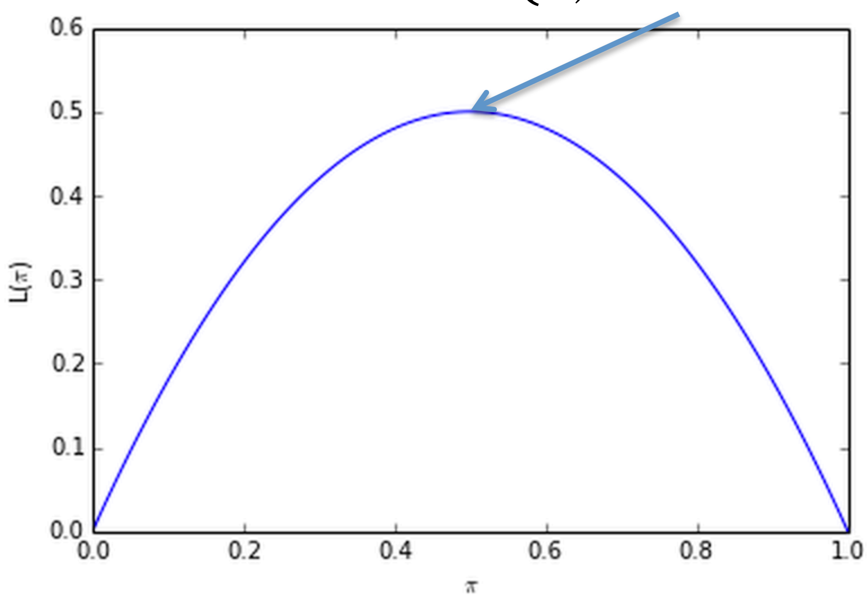
Example: $L(\pi) = 2\pi(1 - \pi)$ is maximized at $\pi = ½$.
Thus the Maximum Likelihood estimate for $L(\pi)$ is $\hat{\pi} = ½$.



Notes.

For Binomial random variable Y: $\hat{\pi} = y/n$

Mathematically, in Ordinary Least Squares ($Y|X \sim Normal$), the least squares estimates for the β parameters are the Maximum Likelihood estimates.