Interference Channels: A Review

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1. Introduction

An Interference Channel (IC) models the situation where a number (N) of independent transmitters try to communicate their separate information to N different receivers via a common channel. There is a strict one-to-one correspondence between transmitters and receivers. Consequently, the transmission of information from each source to its corresponding receiver interferes with the communication between the other Transmitters and their receivers. The interference channel can therefore be viewed as being comprised of N principal links and $N(N-1)$ interference links. An example of an IC is when far end cross talk occurs between two twisted pair cables in the same binder in a DSL.

The earliest study of a channel similar to the Interference Channel from an Information Theoretic standpoint was initiated by Shannon [1] and pursued further by Ahlswede [2] with limited success. However, the problem of specifying the rate region for the general IC is still unsolved. A lot of research on this topic has focused on the two user-two receiver case.

In this summary, we shall review a few results for the two user-two receiver interference channel. This will include achievability and converse results for the case of strong interference in both the Discrete Memoryless and Gaussian cases. The general achievability result by Han and Kobayashi [3] will also be described. The Han-Kobayashi (HK) rate region, while being the best achievable rate region known to date, is also extremely hard to compute even for simple cases. Therefore, we shall also survey a few other achievability results which are special cases of the Han-Kobayashi region, but can be computed much more easily. This is followed by a review of an outer bound for the rate region. Finally, we shall look at recent results which make use of a game theoretic approach to the interference channel.

2. Preliminaries

We shall start by setting down a few important definitions. Fig. 1 shows the standard model of the interference channel. A discrete memoryless IC $(\mathcal{X}_1, \mathcal{X}_2, p(y_1, y_2|x_1, x_2), \mathcal{Y}_1, \mathcal{Y}_2)$ consists of two finite input sets $\mathcal{X}_1$ and $\mathcal{X}_2$, two finite output sets $\mathcal{Y}_1$ and $\mathcal{Y}_2$ and the conditional probability of $(y_1, y_2) \in \mathcal{Y}_1^n \times \mathcal{Y}_2^n$ given $(x_1, x_2) \in \mathcal{X}_1^n \times \mathcal{X}_2^n$. In the case of a Gaussian interference channel, the sets $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2$ are not finite and the conditional probability distribution $p(y_1, y_2|x_1, x_2)$ is a two dimensional Gaussian distribution.

A $(\lceil 2^{nR_1} \rceil, \lceil 2^{nR_2} \rceil, n)$ code for the interference channel consists of

- Two independent message sets $\mathcal{W}_1$ and $\mathcal{W}_2$, where $\mathcal{W}_i = \{1, 2, \ldots, M_i\}, i = 1, 2$. The message (W_1, W_2) is uniformly distributed in $\mathcal{W}_1 \times \mathcal{W}_2$.
- Two encoding functions $E_i : \mathcal{W}_i \to \mathcal{X}_i^n, i = 1, 2$. Here, E_i encodes the message $w_i \in \mathcal{W}_i$ into a codeword $x_i^n(w_i)$.

- Two decoding functions $D_i : \mathcal{Y}_i \rightarrow \mathcal{W}_i, i = 1, 2$. The function D_i maps the received sequence y_i^n into a message $\hat{w}_i \in \mathcal{W}_i$.

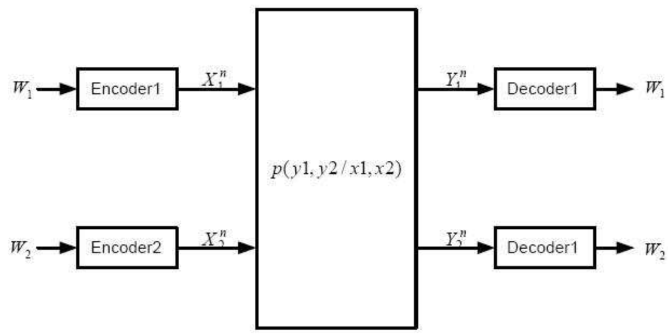

Fig. 1. Model of the Interference channel.

The error probabilities are defined as

$$\lambda_{1,n} = \frac{1}{M_1 M_2} \sum_{w_1, w_2} p(D_1(Y_1^n) \neq w_1 / W_1 = w_1, W_2 = w_2) \tag{2.1}$$

$$\lambda_{2,n} = \frac{1}{M_1 M_2} \sum_{w_1, w_2} p(D_2(Y_2^n) \neq w_2 / W_1 = w_1, W_2 = w_2) \tag{2.2}$$

$$\lambda_n = \max(\lambda_{1,n}, \lambda_{2,n}) \tag{2.3}$$

A rate pair (R_1, R_2) is said to be achievable on the interference channel if there exists a sequence of $(\lceil 2^{nR_1} \rceil, \lceil 2^{nR_2} \rceil, n)$ codes such that $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$. The capacity region of the interference channel is defined as the closure of the set of all achievable rate pairs.

A. Gaussian Interference Channel

In the case of the Gaussian Interference Channel, given channel outputs $Y_k^{'}$ and channel inputs $X_k^{'}$ with block power constraints

$$\sum_{t=1}^{n} |X_k^{'}(t)|^2 \leq n P_k^{'}, k = 1, 2 \tag{2.4}$$

for some positive $P_1^{'}$ and $P_2^{'}$. The channel outputs at time t are given by

$$Y_1^{'}(t) = c_{11} X_1^{'}(t) + c_{21} X_2^{'}(t) + Z_1^{'}(t) \tag{2.5}$$

$$Y_2^{'}(t) = c_{12} X_1^{'}(t) + c_{22} X_2^{'}(t) + Z_2^{'}(t) \tag{2.6}$$

Where $Z_k^{'}$ is the Gaussian noise term, with $E[|Z_k^{'}|^2] = N_k, N_k \geq 0, k = 1, 2$. One can use elementary scaling transformations to convert the above IC into the standard form as shown in Fig. 2 where,

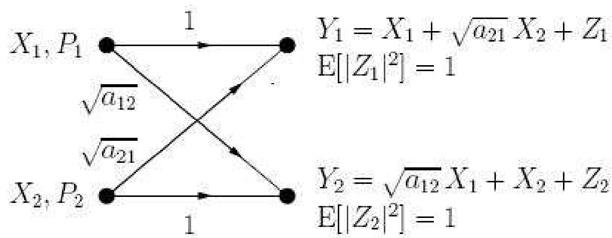

Fig. 2. Standard Model of the Gaussian Interference channel.

$$X_1 = \frac{c_{11}}{\sqrt{N_1}} X_1^{'}, Y_1 = \frac{1}{\sqrt{N_1}} Y_1^{'}, Z_1 = \frac{1}{\sqrt{N_1}} Z_1^{'}$$
$$X_2 = \frac{c_{22}}{\sqrt{N_2}} X_2^{'}, Y_2 = \frac{1}{\sqrt{N_2}} Y_2^{'}, Z_2 = \frac{1}{\sqrt{N_2}} Z_2^{'}$$

and the new power constraints and channel gains are

$$P_1 = \frac{c_{11}^2}{N_1} P_1^{'}, P_2 = \frac{c_{22}^2}{\sqrt{N_2}} P_2^{'}$$
$$a_{12} = \frac{c_{12}^2 N_1}{c_{11}^2 N_2}, a_{21} = \frac{c_{21}^2 N_2}{c_{22}^2 N_1}$$

It should be noted that in the standard form, the direct links have unit gains and the noise has unit variance.

3. Very Strong and Strong Interference.

Intuitively, we are accustomed to thinking of strong interference as being a more detrimental effect than weak interference. Very often, in practical communication systems, the approach is to try and suppress interference as much as possible to improve the quality of communication. From an information theoretic point of view however, Carleial [4], Sato [5],Costa and El Gamal [6] have shown that strong interference is less harmful than weak interference and very strong interference is as good as no interference at all.

A. The Discrete Memoryless Interference Channel

Definition 1: The Discrete Memoryless (DM) Interference Channel is said to have strong interference [5] if:

$$I(X_1; Y_1/X_2) \leq I(X_1; Y_2/X_2) \tag{3.1}$$
$$I(X_2; Y_2/X_1) \leq I(X_2; Y_1/X_1) \tag{3.2}$$

for all $(X_1, X_2) \sim p(x_1)p(x_2)$.

Definition 2: The Discrete Memoryless Interference Channel is said to have very strong interference [5] if:

$$I(X_1;Y_1/X_2) \leq I(X_1;Y_2) \tag{3.3}$$
$$I(X_2;Y_2/X_1) \leq I(X_2;Y_1) \tag{3.4}$$

for all $(X_1, X_2) \sim p(x_1)p(x_2)$.

It is easy to see that if a channel has very strong interference, it also satisfies the condition for strong interference, but not vice versa. The importance of strong and very strong interference is that under these conditions it is possible for the receiver to decode the interference and cancel it out perfectly. With this idea in mind, we first state a useful lemma [6] followed by the theorems for strong interference.

Lemma 3.1: Let a Discrete Memoryless Channel have inputs X_1, X_2 and outputs Y_1, Y_2. If $I(X_1;Y_1/X_2) \leq I(X_1;Y_2/X_2)$, for all product distributions on $\mathcal{X}_1 \times \mathcal{X}_2$, then $I(X_1^n;Y_1^n/X_2^n) \leq I(X_1^n;Y_2^n/X_2^n)$.

Theorem 3.2: [6] The capacity region of the DM Interference Channel with strong interference can be expressed as the closure of the convex hull of the rate pairs (R_1, R_2) which satisfy

$$0 \leq R_1 \leq I(X_1;Y_1/X_2,Q) \tag{3.5}$$
$$0 \leq R_2 \leq I(X_2;Y_2/X_1,Q) \tag{3.6}$$
$$R_1 + R_2 \leq \min\{I(X_1,X_2;Y_1/Q), I(X_1,X_2;Y_2/Q)\} \tag{3.7}$$

for some $p(q)p(x_1/q)p(x_2/q)$ where Q is a time sharing random variable of cardinality at most 4.

Proof: The proof of achievability is immediate as the capacity region is the region where both the messages W_1 and W_2 are required at both receivers [2]. Therefore, we shall focus on the proof of the converse.

For the converse, the first two inequalities are obvious upper bounds on the rates R_1 and R_2. Therefore, it is sufficient to prove the third inequality. By symmetry, it is sufficient to prove that $R_1 + R_2 \leq I(X_1, X_2; Y_2/Q)$

From Fano's Inequality, we have

$$H(W_1/Y_1^n) \leq nR_1\lambda_{1,n} + h(\lambda_{1,n}) \triangleq n\epsilon_{1,n} \tag{3.8}$$
$$H(W_2/Y_2^n) \leq nR_2\lambda_{2,n} + h(\lambda_{2,n}) \triangleq n\epsilon_{2,n} \tag{3.9}$$

where $h(\cdot)$ is the binary entropy function and $\epsilon_{1,n}, \epsilon_{2,n} \to 0$ as $\lambda_n \to 0$.

$$\begin{aligned} n(R_1+R_2) &= H(W_1) + H(W_2) \\ &= I(W_1;Y_1^n) + I(W_2;Y_2^n) + H(W_1/Y_1^n) + H(W_2/Y_2^n) \end{aligned}$$

By Fano's Inequality, with $\epsilon_n = \max\{\epsilon_{1,n}, \epsilon_{2,n}\}$,

$$\begin{aligned} n(R_1+R_2) &\leq I(W_1;Y_1^n) + I(W_2;Y_2^n) + 2n\epsilon_n \\ &\overset{(a)}{\leq} I(X_1^n;Y_1^n) + I(X_2^n;Y_2^n) + 2n\epsilon_n \\ &\overset{(b)}{\leq} I(X_1^n;Y_1^n/X_2^n) + I(X_2^n;Y_2^n) + 2n\epsilon_n \\ &\overset{(c)}{\leq} I(X_1^n;Y_2^n/X_2^n) + I(X_2^n;Y_2^n) + 2n\epsilon_n \end{aligned} \tag{3.10}$$

where (a) follows from the data processing inequality, (b) follows from the independence of X_1^n and X_2^n and (c) follows from lemma 3.1.

$$\begin{aligned} n(R_1+R_2) &\leq I(X_1^n, X_2^n; Y_1^n) + 2n\epsilon_n \\ &\leq \sum_{i=1}^{n} I(X_{1i}, X_{2i}; Y_{1i}) + 2n\epsilon_n \\ &\leq nI(X_1, X_2; Y_1/Q) + 2n\epsilon_n \end{aligned} \tag{3.11}$$

which completes the proof of the converse. ■

We must note that the capacity region in this case happens to be the intersection of the capacity regions of the two multiple access channels $(X_1, X_2) \to Y_1$ and $(X_1, X_2) \to Y_2$. This simplification is due to the presence of strong interference.

Theorem 3.3: The capacity region of the DM Interference Channel with very strong interference can be expressed as the union of the rate pairs (R_1, R_2) which satisfy

$$0 \leq R_1 \leq I(X_1; Y_1/X_2, Q) \tag{3.12}$$

$$0 \leq R_2 \leq I(X_2; Y_2/X_1, Q) \tag{3.13}$$

for some $p(q)p(x_1/q)p(x_2/q)$ where Q is a time sharing random variable of cardinality at most 4.

B. *The Gaussian Interference Channel.*

Definiton: The condition for Strong Interference in the standard Gaussian interference channel as given in Fig. 2 is $a_{21} \geq 1$ and $a_{12} \geq 1$. The condition for very strong interference is given by $a_{21} \geq 1 + P_1$ and $a_{12} \geq 1 + P_2$.

Theorem 3.4: [5] The capacity region for the Standard Gaussian Interference Channel with strong interference is the set of rate pairs (R_1, R_2), such that

$$R_1 \leq \frac{1}{2}\log_2(1 + P_1) \tag{3.14}$$

$$R_2 \leq \frac{1}{2}\log_2(1 + P_2) \tag{3.15}$$

$$R_1 + R_2 \leq \min\{\frac{1}{2}\log_2(1 + a_{21}P_2 + P_1), \frac{1}{2}\log_2(1 + a_{12}P_1 + P_2)\} \tag{3.16}$$

Theorem 3.5: [4] The capacity region for the Standard Gaussian Interference Channel with very strong interference is the set of rate pairs (R_1, R_2), such that

$$R_1 \leq \frac{1}{2}\log_2(1 + P_1) \tag{3.17}$$

$$R_2 \leq \frac{1}{2}\log_2(1 + P_2) \tag{3.18}$$

We can see that according to the above theorem, the presence of very strong interference is as good as having no interference at all.

4. The Han-Kobayashi Achievable Rate Region.

The best known achievable rate region for the general two user - two receiver Interference Channel is due to Han and Kobayashi [1] [3]. Their result is a generalization of the approach first used by Carleial [8]. The principle behind their approach is to split the data from each source into two parts, one (U_j with rate S_j) which is public and the other (V_j with rate T_j) which is private, for transmitters $j = 1, 2$. The private part of the message is to be decoded only by the receiver for which it is intended, while the public part can be decoded by both receivers.

Let $\mathcal{P}$ be the set of probability distributions $P(\cdot)$ such that

$$\begin{aligned} P(q, v_1, u_1, x_1, v_2, u_2, x_2, y_1, y_2) = P(q)P(v_1/q)P(u_1/q)P(x_1/v_1, u_1, q) \\ P(v_2/q)P(u_2/q)P(x_2/v_2, u_2, q)P(y_1, y_2/x_1, x_2) \end{aligned} \tag{4.1}$$

Suppose we fix $P(\cdot)$. Consider receiver 1 and the set $\mathcal{R}_{HK}^{(1)}(P)$ of rate tuples (S_1, T_1, S_2, T_2) that satisfy

$$\begin{aligned}
S_1 &\leq I(V_1; Y_1/U_1, U_2, Q) && (4.2)\\
T_1 &\leq I(U_1; Y_1/V_1, U_2, Q) && (4.3)\\
T_2 &\leq I(U_2; Y_1/V_1, U_1, Q) && (4.4)\\
S_1 + T_1 &\leq I(V_1, U_1; Y_1/U_2, Q) && (4.5)\\
S_1 + T_2 &\leq I(V_1, U_2; Y_1/U_1, Q) && (4.6)\\
T_1 + T_2 &\leq I(U_1, U_2; Y_1/V_1, Q) && (4.7)\\
S_1 + T_1 + T_2 &\leq I(V_1, U_1, U_2; Y_1/Q) && (4.8)
\end{aligned}$$

Similarly, let the set $\mathcal{R}_{HK}^{(2)}(P)$ be the set of (S_1, T_1, S_2, T_2) that satisfy the above equations with indices 1 and 2 swapped. For a set $\mathcal{S}$ of 4-tuples (S_1, T_1, S_2, T_2), let $\Pi(\mathcal{S})$ be the set of (R_1, R_2) such that $R_1 = S_1 + T_1$ and $R_2 = S_2 + T_2$ for some $(S_1, T_1, S_2, T_2) \in \mathcal{S}$. We have the following result [3].

Theorem 4.1: The set

$$\mathcal{R}_{HK} = \prod \left(\bigcup_{P \in \mathcal{P}} \left(\mathcal{R}_{HK}^{(1)}(P) \cap \mathcal{R}_{HK}^{(2)}(P) \right) \right) \tag{4.9}$$

is an achievable rate region for the Memoryless IC.

The proof for the above result is very tedious and will not be given here. However, the approach used is similar to that used in the achievability proof for the Multiple Access Channel. In fact, the above expressions are obtained if we consider the two MAC channels $(V_1, U_1, U_2) \rightarrow Y_1$ and $(V_2, U_1, U_2) \rightarrow Y_2$ and apply the standard results for the MAC channel's rate region .

[1]Chong, Garg and Motani recently derived a simplified version of the HK region which they conjecture is a better rate region [7]. However, they haven't been able to prove this conjecture and at least for the Gaussian Channel, simulations seem to indicate that their region appears to coincide with the HK region.

5. Recent Achievability Results and Outer Bounds.

While the Han-Kobayashi region is the best available rate region, it is extremely difficult to compute for all but the simplest of cases. Sason [9] recently derived the following achievability result for the Gaussian Interference Channel, which gives an easily computable rate region which is a particular case of the Han-Kobayashi region.

Theorem 5.1: The set of rate pairs (R_1, R_2) given by

$$D = \bigcup_{\alpha,\beta,\lambda \in [0,1]} \left\{ (R_1, R_2) : \begin{array}{c} R_1 \leq \lambda \cdot \gamma(\frac{\alpha P_1}{\lambda}) + (1-\lambda)\text{min}\{\gamma(\frac{(1-\alpha)P_1}{1-\lambda+a_{21}(1-\beta)P_2}), \gamma(\frac{a_{12}(1-\alpha)P_1}{1-\lambda+(1-\beta)P_2})\} \\ R_2 \leq (1-\lambda) \cdot \gamma(\frac{(1-\beta)P_2}{1-\lambda}) + \lambda \cdot \text{min}\{\gamma(\frac{\beta P_2}{\lambda+a_{12}\alpha P_1}), \gamma(\frac{a_{21}\beta P_2}{\lambda+\alpha P_1})\} \end{array} \right\} \tag{5.1}$$

where

$$\gamma(x) = \frac{1}{2}\text{log}(1+x) \tag{5.2}$$

For a review of other recent achievability results, refer [10].

There has also been a certain amount of research devoted to computing outer bounds on the capacity of the IC. Two outer bounds for the Gaussian case were recently derived by Kramer [11], one using a genie aided approach and the other using the results for the degraded Interference Channel. We state both these results here here without proof.

Theorem 5.2: The capacity region of the Gaussian Interference Channel is contained within the set (R_1, R_2) satisfying

$$R_1 \leq \frac{1}{2}\text{log}(1+P_1) \tag{5.3}$$

$$R_2 \leq \frac{1}{2}\text{log}(1+P_2) \tag{5.4}$$

$$R_1 + R_2 \leq \frac{1}{2}\text{log}\Big[(P_1 + a_{21}P_2 + 1)\Big(\frac{P_2+1}{\text{min}(a_{21},1)P_2+1}\Big)\Big] \tag{5.5}$$

for all values of a_{21}. Further, the sum rate bound obtained by swapping indices 1 and 2 is also valid by symmetry.

Theorem 5.3: If $0 < a_{21} \leq 1$ then the rate region of the Interference Channel is contained within the set (R_1, R_2) satisfying

$$R_1 \leq \frac{1}{2}\text{log}(1 + \frac{P_1^{'}}{P_2^{'} + 1/a_{21}}) \tag{5.6}$$

$$R_2 \leq \frac{1}{2}\text{log}(1 + P_2^{'}) \tag{5.7}$$

where $P_1^{'} + P_2^{'} = P_1/a_{21} + P_2$. If $0 < a_{12} \leq 1$, the inequalities obtained by swapping indices 1 and 2 would also be valid.

6. A Game Theoretic Approach to the Interference Channel

A recent approach to the Interference Channel is to view the two senders in the Interference Channel as two players in a game. The traditional view of the IC, while allowing the two senders to be independent, allows them to be cooperative in their coding strategies. If such cooperation cannot be assumed, the system becomes a non cooperative game. Yu and Cioffi [12] use game theory to prove the existence, uniqueness and stability of a pure Nash equilibrium under mild interference conditions for a Gaussian Interference Channel with memory when interference subtraction is not performed.

The model for the Interference channel that is used is

$$\mathbf{y_1} = \mathbf{x_1} + \mathcal{A}_2\mathbf{x_2} + \mathbf{n_1} \tag{6.1}$$

$$\mathbf{y_2} = \mathbf{x_2} + \mathcal{A}_1\mathbf{x_1} + \mathbf{n_2} \tag{6.2}$$

The square magnitude of the interference transfer functions $\mathcal{A}_1$ and $\mathcal{A}_2$ are denoted by $\alpha_1(f)$ and $\alpha_2(f)$ respectively. The noise power spectral densities are $N_1(f)$ and $N_2(f)$ and the players have power spectra $P_1(f)$ and $P_2(f)$ respectively. It is also assumed that the interference coupling functions and noise power spectra are known to both players and only deterministic strategy is used. Under all these assumptions for the Interference Channel, the optimal response of a player is the waterfilling of his power with respect to combined noise and interference. If the power distributions are such that the waterfilling solution is simultaneously achieved for both players, a Nash Equilibrium is reached.

Theorem 6.1: If $\sup_f \alpha_1(f) \cdot \sup_f \alpha_2(f) < 1$ then a pure strategy Nash equilibrium exists, is unique and is stable.

The above condition $\sup_f \alpha_1(f) \cdot \sup_f \alpha_2(f) < 1$ can be interpreted as the condition for weak interference in at least one of the two Interference links. The significance of the above result is that it guarantees the existence of a Nash Equilibrium where a simple water filling solution will be optimal in the case of weak interference. Therefore, the result guarantees that if each sender's sole objective is to maximize his data rate without any kind of cooperation with the other user, a unique and stable equilibrium can be achieved in this competitive environment.

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