Modelling an Image of Rotating Fan Blades Captured By a Camera

Abdullah Saquib (Independent Researcher)

ABSTRACT

In this article, I have developed a framework for modelling photo of a rotating fan blade captured by a camera. I have used rotational kinematics, concept of exposure time of camera and introduced a mathematical definition of optical opacity to develop the framework. I have simulated a model of 4 blades fan, but the framework can be used to simulate any number of blades, with a freedom in choosing angular blade width, initial position of blade, blade's angular speed and exposure time of the camera. The model can be used in refining simulation of Stroboscopic Effect.

INTRODUCTION

In this article I make an attempt to introduce the effect of exposure time (shutter speed) of camera (or human eye) on the image of a rotating blade. The following are images of a fan captured by a camera at different exposure time:

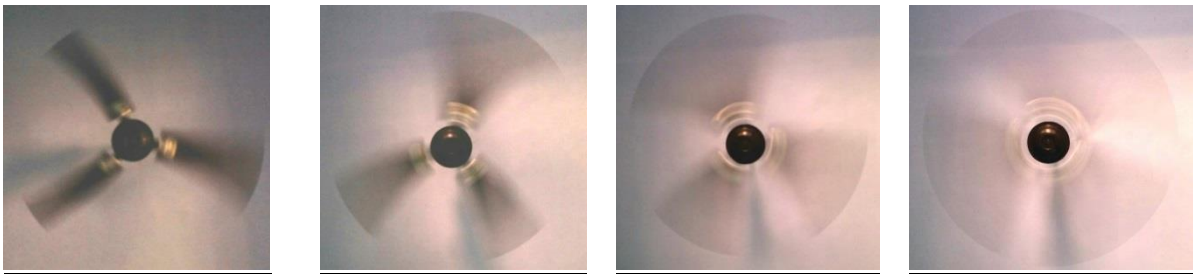


Figure 1 *The images of my fan at increasing exposure time (from left to right) of camera at constant angular speed of the fan.*

Several effects can be observed from the above images:

- Transparency of the rotating blades or the visibility of the background in the rotating blade's image
- The blade appears wider as the angular speed or the exposure time is increased.
- The presence of saturation opacity region, increasing opacity and decreasing opacity regions. These regions are unambiguously noticeable in the first two images, where the middle of the blade's image is more opaque than the edges.
- The contrast of the opacity of blade relative to the background decreases for faster rotating blades or longer exposure time. The opacity becomes almost uniform for very fast rotating blades.

In this article, I will develop a mathematical framework that can be used to model these effects. It can be helpful in developing fine simulations of Stroboscopic Effect. The framework can also be used to simulate videos of a rotating fan or wheels captured by a video camera, thus may find applications in animation industry.

THEORY

When the button of the camera is clicked to capture image of an object, the aperture of the camera opens up for some small time interval, the photons scattered by the object in the direction of the aperture enters the body of the camera where the photons are detected thereby storing the optical information about the object. The key idea is the picture of the object manifest information of all the photons detected during the time interval in which the aperture was open.

Shutter speed or exposure time (t_s) is the length of time when the film or digital sensor inside the camera is exposed to light.

In figure 1, observe how the blades are widened as the exposure time is increased. The reason is simple, the blades have moved more in longer exposure time than in shorter exposure time. If the exposure time is long enough such that the images of two adjacent blades partially overlaps, the image become lesser transparent (or more opaque) at these overlaps, which appear as fringes. The fringes are essentially the lesser (more) transparent regions surrounded by the more (less) transparent regions.

The next observation is the visibility of the background or transparency of blades. The reason is, like the blades, the background is also exposed to the camera leaving its effect on the image. Since, the effect of both the blades and the background has to be manifested in the resulting image, the blades appear transparent.

MATHEMATICAL FORMULATION

The opacity (P) is a measure of how opaque the blade appear relative to the background in the image. The opacity varies between 0 and 1, with 0 representing completely transparent and 1 representing completely opaque.

$$P(\phi) = \frac{t_c(\phi)}{t_s}$$

where $t_c(\phi)$ which can be termed as **Blade Occupation Time(BOT)** is the time for which the angular position ϕ is occupied by the blade. If the angular position ϕ is occupied by the blade for exposure time then $t_c(\phi) = t_s$ implying $P(\phi) = 1$, this is the case when the blade is stationary. If blade never occupies angular position ϕ, then $t_c(\phi) = 0$, implying $P(\phi) = 0$.

The maximum time (t_o) for which the blade of angular width w moving with angular speed ω can occupy angular position ϕ within exposure time t_s is

$$t_o = \begin{cases} \frac{w}{\omega} & if \quad \frac{w}{\omega} < t_s \\ t_s & otherwise \end{cases}$$

Consider a blade of angular width w, moving with uniform angular speed ω, initial angular position of the blade's edge is ϕ_i, and $\phi_f = \phi_i + \omega t_s$ is the final angular position of the blade's edge after exposure time t_s. Closely considering the rotational kinematics of blade, I have formulated blade occupation time $t_c(\phi)$ for the following two scenarios:

Case I: $\boldsymbol{\phi_f - \phi_i \geq w}$ (distance moved by the blade's edge is greater than or equal to the angular width of the blade)

$$t_c(\phi) = \begin{cases} \frac{(\phi - \phi_i)}{\omega} & \phi \in (\phi_i, \phi_i + w) \\ \frac{w}{\omega} & \phi \in (\phi_i + w, \phi_f) \\ \frac{(\phi_f + w - \phi)}{\omega} & \phi \in (\phi_f, \phi_f + w) \\ 0 & otherwise \end{cases}$$

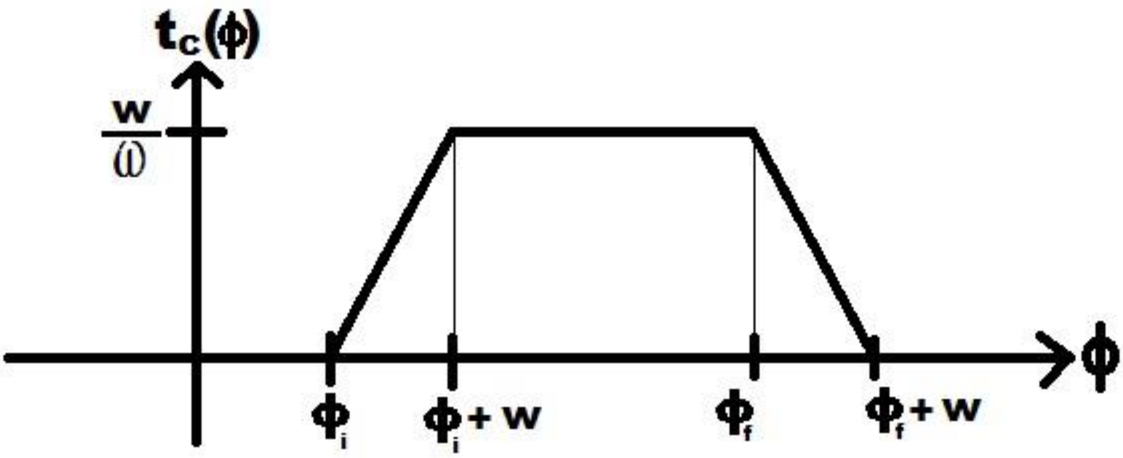


Figure 2 *Blade Occupation of Time of a blade satisfying case I.*

Case II: $\boldsymbol{\phi_f - \phi_i \leq w}$ (distance moved by the blade's edge is lesser than or equal to the angular width of the blade)

$$t_c(\phi) = \begin{cases} \dfrac{(\phi - \phi_i)}{\omega} & \phi \,\epsilon\, (\phi_i, \phi_f) \\ t_s & \phi \,\epsilon\, (\phi_f, \phi_i + w) \\ \dfrac{(\phi_i + w + \omega t_s - \phi)}{\omega} & \phi \,\epsilon\, (\phi_i + w, \phi_f + w) \\ 0 & otherwise \end{cases}$$

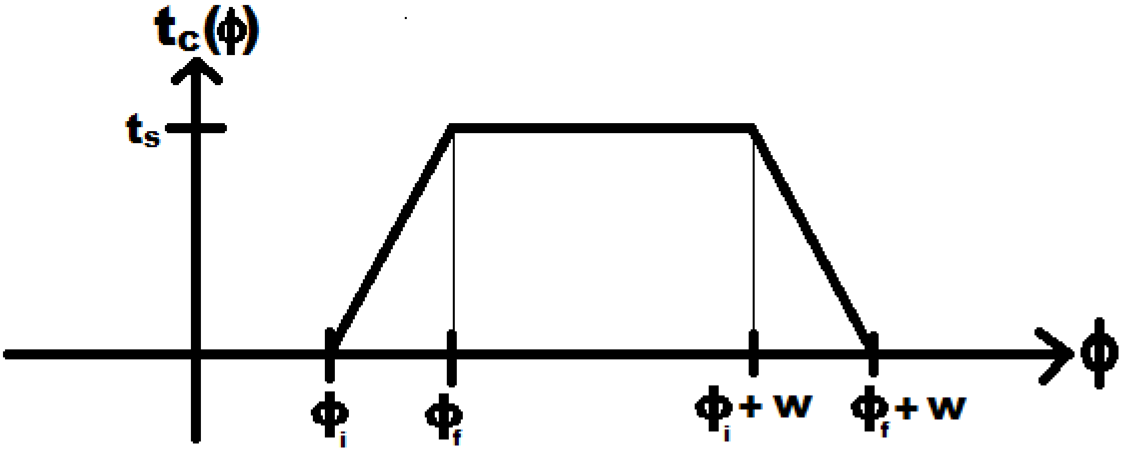


Figure 3 *Blade Occupation of Time of a blade satisfying case II.*

The blade's image can be divided into three parts:

- *The increasing opacity region* ($\phi \,\epsilon\, (\phi_i, \phi_i + w)$ for case I and $\phi \,\epsilon\, (\phi_i, \phi_f)$ for case II): In this region, the opacity increases from zero to its saturation value because the BOT is increasing.
- *The saturated opacity region* ($\phi \,\epsilon\, (\phi_i + w, \phi_f)$ for case I and $\phi \,\epsilon\, (\phi_i, \phi_f)$for case II): In this region, the opacity value is constant at $w/(\omega t_s)$ for case I and the opacity value is 1 for case II.
- *The decreasing opacity region* ($\phi \,\epsilon\, (\phi_f, \phi_f + w)$ for case I and $\phi \,\epsilon\, (\phi_i + w, \phi_f + w)$ for case II): In this region, the opacity decreases from its saturation value to zero because the BOT is decreasing.

Both the cases give correct blade occupation time for the case $\phi_f - \phi_i = w$

Using the above definition of blade occupation time in calculating opacity, we find that opacity is a function of ωt_s, that is the product of angular speed of the blade and the exposure time of camera. This suggests that we can obtain the same image of a blade for different pair of values of ω and t_s that keeps the product ωt_s constant. Intuitively, we do expect this result, as ωt_s is the angular displacement of the blade during the exposure time.

Finally, we should take care of the fact that $\phi = \phi_o, \phi_o + 2\pi, \phi_o + 4\pi, \dots, \phi_o + 2n\pi$ (where $\phi_o \epsilon (0, 2\pi)$ and n is a whole number) occupies the same points in space. We have to restrict blade occupation time within domain $(0, 2\pi)$. We define new **restricted blade occupation time (RBOT)** whose domain is $(0, 2\pi)$. It can be written in terms of blade occupation time as

$$t_r(\phi) = \sum_{n=0}^{n_f} t_c(\phi + 2n\pi)$$

where $n_f = \left[\frac{\phi_f}{2\pi}\right]$, the square bracket represents the least integer function. For example, if $\frac{\phi_f}{2\pi} = 1.1$ then $n_f = 2$.

We have to redefine opacity as the ratio of restricted blade occupation time to the exposure time.

$$P(\phi) = \frac{t_r(\phi)}{t_s}$$

Now the domain of opacity is $(0, 2\pi)$.

MULTIPLE BLADES MATHEMATICAL FORMULATION

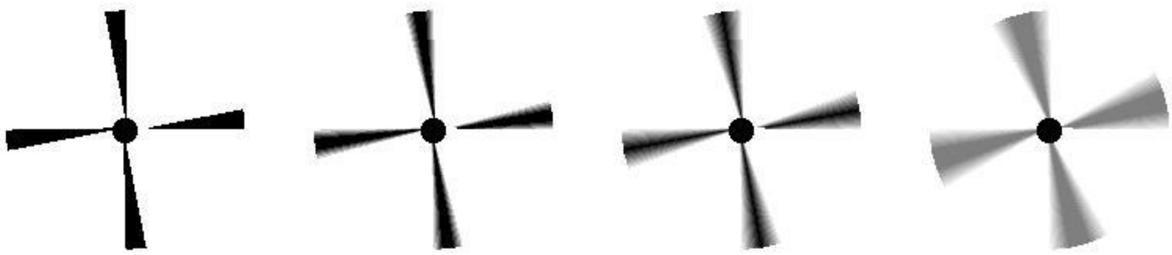
Consider n blades with restricted blade occupation times $t_{r1}(\phi)$, $t_{r2}(\phi)$, …, $t_{rn}(\phi)$. The net RBOT of the fan will simply be the sum of RBOTs of individual blades.

$$t_r(\phi) = \sum_{i=1}^{i=n} t_{rn}(\phi)$$

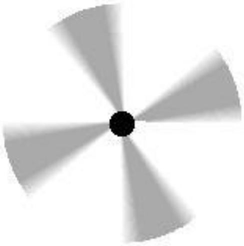
SIMULATION & RESULTS

I have simulated a fan with 4 blades separated by angular distance of 90^o from the adjacent blades, each blade is of width 10^o. The opacity is plotted on domain $(0, 360^o)$. The black colour represents opacity equal to 1, while white colour corresponds to opacity equal to 0.

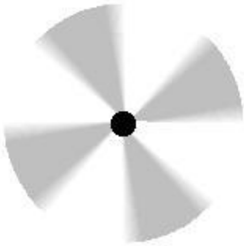
The following are the images obtained for fixed exposure time $t_s = 1\ second$:



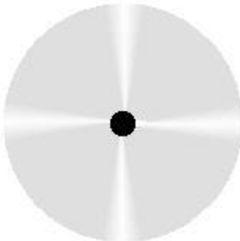
$\omega = 0$ $\omega = 5^o s^{-1}$ $\omega = 10^o s^{-1}$ $\omega = 20^o s^{-1}$



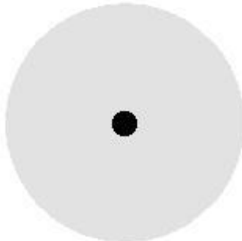
$\omega = 30^o s^{-1}$



$\omega = 40^o s^{-1}$

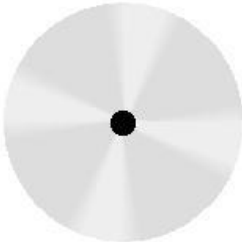


$\omega = 80^o s^{-1}$

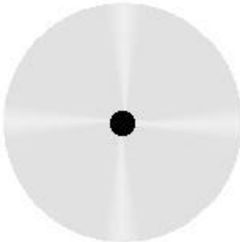


$\omega = 90^o s^{-1}$

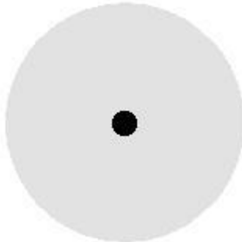
$\omega = 100^o s^{-1}$



$\omega = 150^o s^{-1}$

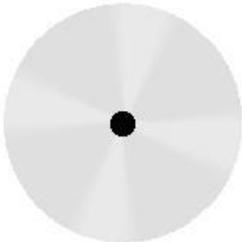


$\omega = 170^o s^{-1}$

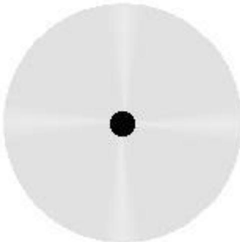


$\omega = 180^o s^{-1}$

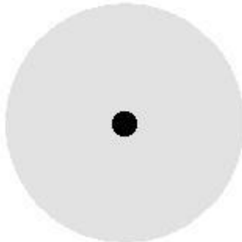
$\omega = 190^o s^{-1}$



$\omega = 240^o s^{-1}$

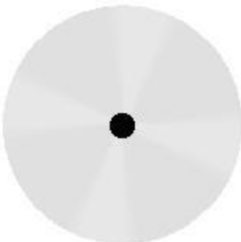


$\omega = 260^o s^{-1}$

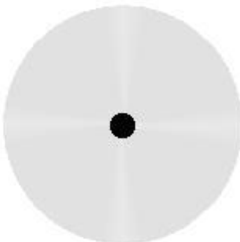


$\omega = 270^o s^{-1}$

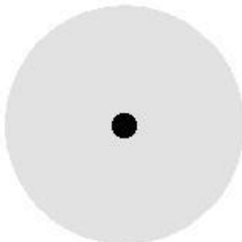
$\omega = 280^o s^{-1}$



$\omega = 330^o s^{-1}$



$\omega = 350^o s^{-1}$



$\omega = 360^o s^{-1}$

At $\omega = 0^o s^{-1}$ there is no motion of blades as a result they appear completely opaque. Angular speed in range $0^o s^{-1} \leq \omega \leq 10^o s^{-1}$ satisfies **Case II:** $\boldsymbol{\phi_f - \phi_i \leq w}$ (distance moved by the blade's edge is lesser than or equal to the angular width of the blade), thus there will be a completely opaque region centred at the centre of the blades' image.

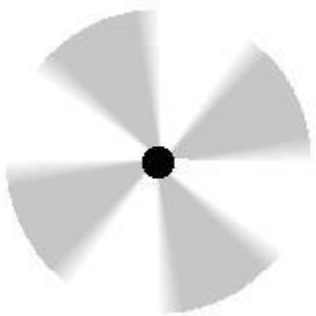
For $\omega > 10^o s^{-1}$ there will no completely opaque region in blades' image, opacity will be always lesser than 1 for any angular position. The condition satisfies **Case I:** $\boldsymbol{\phi_f - \phi_i \geq w}$

(distance moved by the blade's edge is greater than or equal to the angular width of the blade). The background will always be visible. The three regions of the blade's image can also be recognised in these images: the increasing opacity region, the saturated opacity regions and the decreasing opacity region.

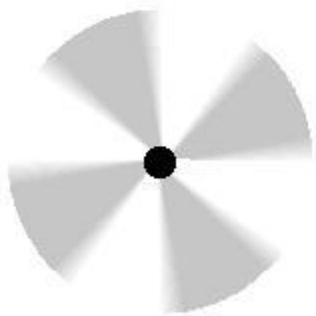
At $\omega = 90^o s^{-1}$ an interesting effect is observed, the opacity becomes uniform as a function of angular position. Note that the angular distance between the blades is 90^o, and the exposure time is $1s$. At the end of one second each blade's edge occupies the initial position of the adjacent blade's edge. The decreasing opacity region of the blade is overlapped by the increasing opacity region of the adjacent blade, the opacity of overlap adds up to the saturation opacity value. Similar effect can be observed at $\omega = 180^o s^{-1}$, $270^o s^{-1}$, $360^o s^{-1}$ and so on. Also note that the saturation opacity value is the same for all such angular speeds. This effect can be helpful in eliminating the stroboscopic effects.

For $\omega > 90^o s^{-1}$, there will be no region of zero opacity as each angular position had been occupied for some time by at least one of the blade. As ω increases, the contrast of the blade's image relative to the background decreases, at very large angular speed the opacity almost becomes uniform for all angular positions.

Now we will verify that opacity depends on the product ωt_s. The following two images are obtained with different pairs of ω and t_s. In both the cases the blade traversed angular distance of 45^o.



$\omega = 45^o s^{-1}$, $t_s = 1s$, $\omega t_s = 45^o$



$\omega = 90^o s^{-1}$, $t_s = 0.5s$, $\omega t_s = 45^o$

CONCLUSION

All the observed effects related to the rotating blade's image can be explained with the help of the aforementioned framework. The effects that were explained are:

- Transparency of the rotating blades and the visibility of the background in the rotating blade's image. The reason is when a certain condition is satisfied, the background is also exposed to the camera, and the image will manifest the effect of both blade and the background.

- The blade appears wider as the angular speed or the exposure time is increased. The reason being that the blade traverses more angular distance within the exposure time of the camera.
- The presence of saturation opacity region, increasing opacity and decreasing opacity regions.
- The interesting effect of uniform opacity when the angular speed is such that the increasing opacity and decreasing opacity regions of different blades overlaps completely.
- The contrast of the opacity of blade relative to the background decreases for faster rotating blades. The opacity becomes almost uniform for very fast rotating blades.
- The opacity of blades depends on the product ωt_s, that is the angular displacement of the blades during exposure time.

In future, I have planned to simulate and study stroboscopic effect using this framework.