Consider a simple numerical integration for scalar function $f: \mathbb{R} \to \mathbb{R}$

$I = \int_a^b f(x)\,dx$ using the trapezoid rule: $\frac{b-a}{2}\left[f(a)+f(b)\right]$.

This method can be interpreted as follows:

Step 1: Apply linear interpolation to data points $(a, f(a))$ and $(b, f(b))$

Step 2: Compute the area of the trapezium formed by four points $(a,0)$, $(b,0)$, $(a, f(a))$, $(b, f(b))$.

Now imagine $g: \mathbb{R} \to S^2$.

1. Is there any physical meaning to the corresponding integral for g: $J = \int_a^b g(t)\,dt$. Is there any application from physics, like electromagnetism, and it might also relate to the winding number or turning number in understanding plane curves.

2. Can we develop an integration formula similar to trapezoidal rule using SLERP.

3. Higher order methods for scalar functions construct higher order polynomials for reconstruction and then integrate the resulting polynomials. Can we extend the integration rule in (2) using SQUAD.

Need $g: \mathbb{R} \to S^2$

$S^2 = \{ (x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1 \}$

$g(t) = (\cos(t), \sin(t), 0)$

$g(t) = (\sin(2t), \cos(2t), 0)$

what is the meaning of integrating this function?

How do I even integrate this function?

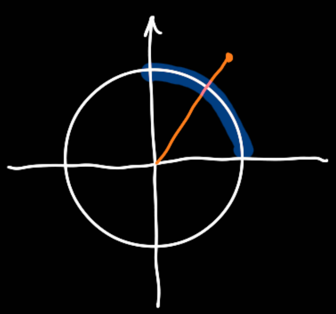
1. $g(t) = (\cos t, \sin t, 0)$; $t \in [0, 2\pi]$

$$\int_0^{2\pi} g(t)\, dt = \left(\int_0^{2\pi} \cos t \, dt , \int_0^{2\pi} \sin t \, dt , \int_0^{2\pi} 0 \, dt \right)$$

$$\int_0^{2\pi} g(t)\, dt = (0,0,0)$$

2. $\displaystyle\int_0^{\pi/2} g(t)\, dt = (1,1,0)$

This result points diagonally towards the direction of the quarter arc.



- A few things to note:
  - integration gives us a specific point, which doesn't necessarily $\in S^2$.

$$\frac{1}{b-a} \int_a^b g(t)\, dt \to \text{can normalize to project back to } S^2.$$

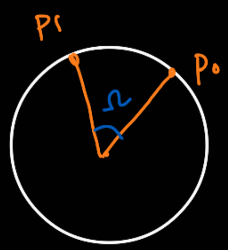
- gives the average direction of the path.

Application in any field of physics e.g. electromagnetism.

↳ polarization of light (circular polarization)

Formula similar to trapezoid rule:

- interpolate between two points on a circle.
- do some calculation.



compute Ω using dot product: $\cos\Omega = P_0 \cdot P_1$

$$\text{slerp}(P_0, P_1; t) = \frac{\sin[(1-t)\Omega]}{\sin\Omega} P_0 + \frac{\sin[t\Omega]}{\sin\Omega} P_1$$

Normal Trapezium Rule: $\int_a^b f(x)\,dx = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \dots + f(x_n)\right]$

$$\int_a^b f(x)\,dx = \sum_{i=0}^{N-1} \frac{f(x_i) + f(x_{i+1})}{2} \cdot \Delta x$$

Divide the interval $[a, b]$ into N equal subintervals of width $\Delta x = \frac{b-a}{N}$.

$x_i = a + i\Delta x$. For each interval we have two points $[x_i, x_{i+1}]$.

$P_i = f(x_i)$ and $P_{i+1} = f(x_{i+1})$.

SLERP between points P_i and P_{i+1}. What is the area under the SLERP curve.

$$\int_{x_i}^{x_{i+1}} f(x)\,dx \approx \int_{x_i}^{x_{i+1}} \text{SLERP}\left(P_i, P_{i+1}, \frac{x - x_i}{\Delta x}\right) dx$$

$$t = \frac{x - x_i}{\Delta x} \quad ; \quad dx = \Delta x \cdot dt$$

$$x = x_i \ , \ t = 0 \qquad \text{and} \qquad x = x_{i+1} \ , \ t = 1$$

$$\Delta x \int_0^1 SLERP(P_i \, , \, P_{i+1} \, , \, t) \, dt$$

$$\Delta x \int_0^1 \frac{\sin[(1-t)\theta]}{\sin\theta} P_i + \frac{\sin(t\theta)}{\sin\theta} P_{i+1} \quad dt$$

$$\frac{\Delta x}{\sin\theta} \left(P_i \int_0^1 \sin[(1-t)\theta] \, dt + P_{i+1} \int_0^1 \sin(t\theta) \, dt \right)$$

$$\frac{\Delta x}{\sin\theta} \left(P_i \left[\frac{\cos[(1-t)\theta]}{+\theta} \right]_0^1 + P_{i+1} \left[\frac{\cos(t\theta)}{-\theta} \right]_0^1 \right)$$

$$\frac{\Delta x}{\sin\theta} \left(\left[\frac{1}{\theta} - \frac{\cos\theta}{\theta} \right] P_i + \left[-\frac{\cos\theta}{\theta} + \frac{1}{\theta} \right] P_{i+1} \right)$$

$$\frac{\Delta x}{\sin\theta} \left(\frac{1 - \cos\theta}{\theta} \right) (P_i + P_{i+1})$$

one interval: $\dfrac{\Delta x}{\sin(\theta_i)} \left(\dfrac{1 - \cos\theta_i}{\theta_i} \right) (P_i + P_{i+1})$

$$\cos\theta_i = P_i \cdot P_{i+1} \quad \Rightarrow \quad \theta_i = \arccos(P_i \cdot P_{i+1})$$

$$\int_a^b f(x) \, dx \approx \sum_{i=0}^{N-1} \frac{\Delta x}{\sin\theta_i} \left(\frac{1 - \cos\theta_i}{\theta_i} \right) \left(f(x_i) + f(x_{i+1}) \right)$$