Motivation: 3D Rotations

- 3x3 matrices can be used to describe rotations, but it's very hard to make sense of them intuitively.

- They require 9 bytes while Euler angles only require 3.
  - too many conventions
  - hard to understand when combining multiple rotations
  - problems such as gimbal lock.

3D complex Numbers - Quaternions:

1D imaginary numbers - already know them

2D imaginary numbers - don't work

3D imaginary numbers - Quaternions

- multiplication isn't commutative.

Quaternions just work nicely with 3D rotations → beautiful math?

- coincidence?

set of quaternions is denoted by $\mathbb{H}$ (honour of Hamilton).

How do we define quaternions?

There are many equivalent definitions:

· 4D algebra with certain multiplication law.

simplest: Quaternions are 4D vectors, with real coordinates, together with some special multiplication rule.

4-tuples of real numbers (a, b, c, d)

$$(a, b, c, d) \pm (w, x, y, z) = (a \pm w, b \pm x, c \pm y, d \pm z)$$

$$(a, b, c, d) \cdot (w, x, y, z) = (e, f, g, h)$$

$$p \cdot q \neq q \cdot p$$

this makes sense since 3D rotations don't commute either.

$$k \cdot (a, b, c, d) = (ka, kb, kc, kd)$$

↳ scalar-vector product is a special kind of quaternion-quaternion product.

$i = (0, 1, 0, 0)$

$j = (0, 0, 1, 0)$

$k = (0, 0, 0, 1)$

$(1, 0, 0, 0) \rightarrow$ simply called 1, because it behaves exactly like the real number 1.

in general any quaternion of they $(a, 0, 0, 0)$ behaves exactly like the real number a.

$(a,b,c,d) = a\cdot(1,0,0,0) + b\cdot(0,1,0,0) + c\cdot(0,0,1,0) + d\cdot(0,0,0,1)$

$= a\mathbb{I} + bi + cj + dk = a + bi + cj + dk$

Quaternion Multiplication:

$(a,b,c,d)\cdot(w,x,y,z) =$

$(aw - bx - cy - dz,$

$ax + bw + cz - dy,$ (consequence of a few simpler rules)

$ay - bz + cw + dx,$

$az + by - cx + dw)$

$p\cdot(q+s) = p\cdot q + p\cdot s$
$(q+s)\cdot p = q\cdot p + s\cdot p$
} p, q, r and s are some quaternions, so multiplication is not commutative.

$s\cdot(q\cdot p) = (s\cdot q)\cdot p = q\cdot(s\cdot p)$

p and q are quaternions and s is a real number.

$(a,b,c,d)\cdot(w,x,y,z) = (a\mathbb{I} + bi + cj + dk)\cdot(w\mathbb{I} + xi + yj + dk)$

$= aw\cdot\mathbb{I}^2 + bx\cdot i^2 + cy\cdot j^2 + dz\cdot k^2 +$

$+ ax\cdot\mathbb{I}i + bw\cdot i\mathbb{I} + cz\cdot jk + dy\cdot kj +$

$+ ay\cdot\mathbb{I}j + bz\cdot ik + cw\cdot j\mathbb{I} + dx\cdot ki +$

$+ az\cdot\mathbb{I}k + by\cdot ij + cx\cdot ji + dw\cdot k\mathbb{I}$

a, b, c, d, w, x, y and $z \in \mathbb{R}$

so we only need to decide on how to multiply basis quaternions.

↳ $4\times4 = 16$ different products.

As stated earlier $\mathbb{I}$ behaves exactly like the $\mathbb{R}$ number 1, so:

$$\mathbb{I}^2 = \mathbb{I}$$

$$\mathbb{I} \cdot i = i \cdot \mathbb{I} = i$$

$$\mathbb{I} \cdot j = j \cdot \mathbb{I} = j$$

$$\mathbb{I} \cdot k = k \cdot \mathbb{I} = k$$

$$a\mathbb{I} = (a, 0, 0, 0)$$

$$(a, 0, 0, 0) \cdot (b, 0, 0, 0) = (a\mathbb{I}) \cdot (b\mathbb{I}) = ab \cdot \mathbb{I}^2 = ab \cdot \mathbb{I} =$$

$$(ab, 0, 0, 0)$$

$$(a, 0, 0, 0) \cdot (w, x, y, z) = (a\mathbb{I}) \cdot (w\mathbb{I} + xi + yj + zk)$$

$$= aw \cdot \mathbb{I}^2 + ax \cdot \mathbb{I}i + ay \cdot \mathbb{I}j + az \cdot \mathbb{I}k$$

$$= aw + axi + ayj + azk = a \cdot (w, x, y, z)$$

$$i^2 = j^2 = k^2 = -1$$

$$ij = -ji = k$$

$$jk = -kj = i$$

$$ki = -ik = j$$

$\Rightarrow$ multiplication table of quaternions.

$$(a + bi + cj + dk) \cdot (w + xi + yj + zk) = (aw - bx - cy - dz) +$$
$$(ax + bw + cz - dy)i +$$
$$(ay - bz + cw + dx)j +$$
$$(az + by - cx + dw)k$$

Why exactly these multiplication rules?

- these don't contradict each other
- nice algebraic structure with useful properties.

Associativity: $p \cdot (q \cdot r) = (p \cdot q) \cdot r$

Given a quaternion $q = \underbrace{a}_{\text{scalar}} + \underbrace{bi + cj + dk}_{\text{vector } (u)}$

$q = (a, u) = a + u$ – why is this representation useful?

if we take 2 quaternions in the scalar-vector form:

$a+u$, $b+v$ and multiply them:

$$(a+u)\cdot(b+v) = (ab - \underbrace{\langle u, v \rangle}_{\text{dot product}}) + (av + bu + \underbrace{u \times v}_{\text{cross product}})$$

- compute quaternion multiplication in familiar vector operations.

Conjugate and Length:

First let's compute the square: q^2 of the quaternion,

$q = a + u$

$$q^2 = (a+u)\cdot(a+u) = (a^2 - \langle u, -u \rangle) + (au + au + u \times (-u))$$
$$= (a^2 - \langle u, u \rangle) + 2au$$

$u \times u = 0$ for any 3D vector u.

$\langle u, u \rangle = |u|^2 \rightarrow$ squared length of vector.

$(a+u)(a-u) = (a^2 - \langle u,u \rangle) + (au - au + u \times (-u))$

$= (a^2 - \langle u,u \rangle) + 0$

scalar quantity

$u = (x, y, z)$

$q = a + xi + yj + zk$, $a - u = a - xi - yj - zk$

$(a + xi + yj + zk) \cdot (a - xi - yj - zk) = a^2 + |u|^2$

$= a^2 + x^2 + y^2 + z^2$

$|q| = \sqrt{a^2 + x^2 + y^2 + z^2} = \sqrt{a^2 + \langle u,u \rangle}$

if $q = a + xi + yj + zk$ then conjugate of q.

$q^* = a - xi - yj - zk$

$|q|^2 = q \cdot q^*$

(conjugate also denoted by $\bar{q}$).

$(q^*)^* = q$

$|q^*|^2 = a^2 + \langle -u, -u \rangle = a^2 + (-x)^2 + (-y)^2 + (-z)^2$

$= a^2 + x^2 + y^2 + z^2$

$= |q|^2$

any q has the same length as its conjugate.

$q \cdot q^* = |q|^2 = |q^*|^2 = q^* \cdot q$

$\rightarrow$ multiplication of a quaternion and its conjugate is commutative.

conjugate of a product:

$(p \cdot q)^* = q^* \cdot p^*$

$(p_1 \cdot p_2 \cdot p_3 \cdot \ldots \cdot p_n)^* = p_n^* \cdot \ldots \cdot p_3^* \cdot p_2^* \cdot p_1^*$

$$|p \cdot q|^2 = (p \cdot q) \cdot (p \cdot q)^* = (p \cdot q) \cdot (q^* \cdot p^*)$$
$$= p \cdot (q \cdot q^*) \cdot p^* = p \cdot |q|^2 \cdot p^* = |q|^2 \cdot p \cdot p^*$$
$$= |q|^2 |p|^2$$

length of product is just the product of lengths.

$$|p_1 \cdot p_2 \cdot \ldots \cdot p_n| = |p_1| \cdot |p_2| \cdot \ldots \cdot |p_n|$$

Inverse and division:

Given any quaternion q, we call any quaternion p such that $q \cdot p = 1$ or $p \cdot q = 1$

conjugate multiplication is commutative.

$q \cdot q^* = |q|^2$

$\frac{q \cdot q^*}{|q|^2} = 1$

$q \cdot \frac{q^*}{|q|^2} = 1$

$\frac{q^*}{|q|^2} = q^{-1}$ $\rightarrow$ inverse of q.

$$q^{-1} = \frac{q^*}{|q|^2} = \frac{a - xi - yj - zk}{a^2 + x^2 + y^2 + z^2} \quad ; \quad a^2 + x^2 + y^2 + z^2 \neq 0$$

$q \cdot x = p$, $x = q^{-1} \cdot p$ → division is the solution to an equation.

Unit Quaternions (* important for 3D rotation)

if $|q| = 1$ then $q^{-1} = q^*$

$q + q^* = (a + u) + (a - u) = 2a$

$q - q^* = (a + u) - (a - u) = 2u$

scalar part of a quaternion is: $\frac{q + q^*}{2}$

vector part of a quaternion is: $\frac{q - q^*}{2}$

$q - q^* = 0 \iff q = q^*$; quaternion is purely scalar.

$q + q^* = 0 \iff q = -q^*$; quaternion is purely vector.

$v^* = -v$

$v^2 = -v \cdot (-v) = -v \cdot v^* = -|v|^2$

- square of a vector quaternion is minus its squared length.

Dot product via conjugation:

$p = a + u \quad , \quad q = w + v$

$$p \cdot q^* = (a+u)\cdot(w-v) = (aw + \langle u, v \rangle) + (-av + wu - u \times v)$$

$$\underbrace{aw + \langle u, v \rangle} = aw + bx + cy + dz$$

- scalar part is the dot product of p and q as 4-dimensional vectors. We'll denote this by $\langle p, q \rangle$.

- Now we know how to extract the scalar part.

$$\langle p, q \rangle = \frac{1}{2}\left(p \cdot q^* + (p \cdot q^*)^*\right) = \frac{1}{2}\left(p \cdot q^* + q \cdot p^*\right)$$

- Now we can calculate dot product using quaternion algebra.

if $p = 0 + u$ and $q = 0 + v$

$v^* = -v$ and $u^* = -u$

$$bx + cy + dz = \langle v, u \rangle = \frac{1}{2}\left(v \cdot (-u) + u \cdot (-v)\right)$$
$$= -\frac{1}{2}\left(v \cdot u + u \cdot v\right)$$

Cross product using conjugation:

$p = v$ and $q = u \rightarrow$ vector quaternions

$$v \cdot u = (-\langle v, u \rangle) + (v \times u)$$

$$v \times u = \frac{1}{2}\left(v \cdot u - (v \cdot u)^*\right) = \frac{1}{2}\left(v \cdot u - u^* \cdot v^*\right)$$

$v \times u = \frac{1}{2}(v \cdot u - (-u) \cdot (-v)) = \frac{1}{2}(v \cdot u - u \cdot v)$

Towards Rotation:

- magic formula (don't worry we saw visual representation 3Blue 1Brown GOAT!)

q be a quaternion and v be a purely vector quaternion.

consider:

$$q \cdot v \cdot q^*$$

$u = q \cdot v \cdot q^*$ is a purely vector quaternion. we can easily prove this by showing $u^* = -u$. compute:

$$u^* = (q \cdot v \cdot q^*)^* = (q^*)^* \cdot v^* \cdot q^* = q \cdot (-v) \cdot q^* = - q \cdot v \cdot q^* = -u$$

what is the length?

$$|u| = |q| \cdot |v| \cdot |q^*| = |q| \cdot |v| \cdot |q| = |q|^2 \cdot |v| .$$

- If q is a unit quaternion i.e. $|q| = 1$ then $|u| = |v|$.

Meaning the operation $v \mapsto q \cdot v \cdot q^*$ preserves lengths!

Maybe it even preserves arbibrary dot products?

$$\langle pvp^*, pup^* \rangle = -\frac{1}{2}((pvp^*) \cdot (pup^*) + (pup^*) \cdot (pvp^*))$$

$$= -\frac{1}{2}(pv(p^* \cdot p) up^* + pu(p^* \cdot p) vp^*)$$

$$\langle pvp^*, pup^* \rangle = -\frac{1}{2}\left(pv \cdot |q|^2 \cdot up^* + pu\,|q|^2 \cdot vp^* \right) \; ; \; |q| = 1$$

$$= -\frac{1}{2}\left(pvup^* + puvp^* \right) = -\frac{1}{2}\left(p(vu)p^* + p(uv)p^* \right)$$

$$= -p\cdot\frac{1}{2}(vu+uv)\,p^* = p\cdot\langle v,u\rangle\cdot p^* = \langle v,u\rangle\cdot p\cdot p^*$$

$$= \langle v,u\rangle\cdot|q|^2 = \langle v,u\rangle$$

$v \mapsto pvp^*$ preserves arbitrary dot products which means that it preserves angles as well. This means that it is a rotation.

- It could still be reflection, since that also preserves dot product.
- Reflection reverses orientation so we can check if our operation reverses orientation.

$$(pip^*) \times (pjp^*) = \frac{1}{2}\left(pip^*pjp^* - pjp^*pip^* \right)$$

$$= \frac{1}{2}\left(pijp^* - pjip^* \right) = \frac{1}{2}\left(pkp^* - p(-k)p^* \right)$$

$$= \frac{1}{2}\left(pkp^* + pkp^* \right) = pkp^*$$

- Hence, orientation is preserved.

composing Rotations:

say we have 2 quaternions p and q. We want to apply rotation q followed by rotation p.

$$p(qvq^*)p^* = (pq)v(q^*p^*) = (pq)v(pq)^*$$

same as rotation by quaternion pq.

Analyzing Rotations:

Unit quaternion $p = a + v$ and we want to understand what happens when we apply this rotation to some vector.

Let's first try with v.

$(a+v)\cdot v\cdot(a-v) = (a+v)(a-v)\cdot v = q\cdot q^* v = |q|^2 v = v$

so v remains unchanged. This means that v must be parallel to the axis of rotation and rotation happens in the orthogonal plane to v (as long as $v \neq 0$).

Constructing specific Rotations:

Let's try rotation about XY axis.

$p = a + bk$; k is a quaternion

$|p|^2 = a^2 + b^2 = 1$

what does this rotation do to the X-axis?

$$pip^* = (a+bk)i(a-bk) = (ai + bki)(a-bk) = (ai+bj)(a-bk)$$
$$= a^2 i - abik + abj - b^2 jk = a^2 i + abj + abj - b^2 i$$
$$= (a^2 - b^2)i + 2abj$$

We know that a general rotation by an angle θ should map the X axis to something like $\cos\theta i + \sin\theta j$

$$\begin{aligned} a^2 - b^2 &= \cos\theta \\ 2ab &= \sin\theta \end{aligned} \Rightarrow \begin{aligned} a &= \cos\theta/2 \\ b &= \sin\theta/2 \end{aligned}$$

our rotation looks like: $p = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\cdot k$

constructing Generic Rotations:

$$p = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\cdot n$$

rotation of θ degrees about the axis n.

$p = a + v \rightarrow \theta = 2\cos^{-1}(a)$

$\quad \hookrightarrow n = \dfrac{v}{|v|}$

Relation between rotations and unit quaternions:

single rotation can be represented by more than one quaternion. In fact p and $-p$ always represent the same rotation:

$$(-p)v(-p)^* = (-1)^2\cdot pvp^* = pvp^*$$

Rotating by angle θ around n is the same as rotating by $-\theta$ around $-n$.

Turns out that there are no quaternions other than $\pm p$ that represent the same rotation.