Binomial Trees – II

3.1 Real World vs. Risk-Neutral World

It should be emphasized that p is the probability of an up movement in a risk-neutral world. In general this is not the same as the probability of an up movement in the real world. In our example p = 0.6523. When the probability of an up movement is 0.6523, the expected return on both the stock and the option is the risk-free rate of 12%. Suppose that, in the real world, the expected return on the stock is 16% and $p*$ is the probability of an up movement. It follows that

$$22p^* + 18(1 - p^*) = 20e^{0.16\times 3/12}$$

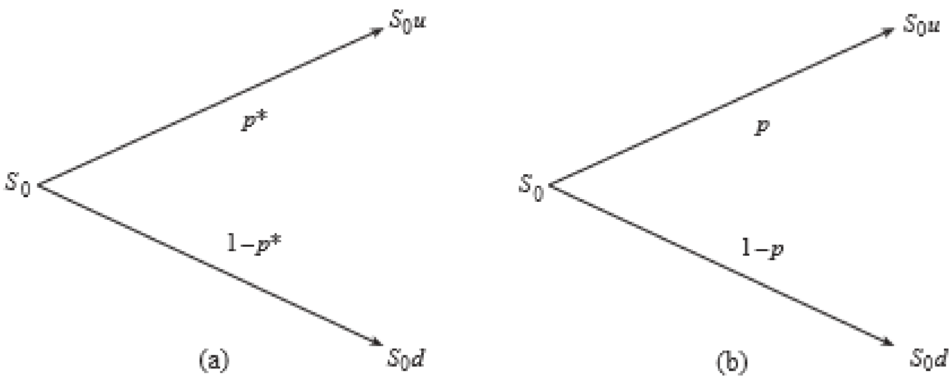
so that $p^* = 0.7041$.

The expected payoff from the option in the real world is then given by

$$p^* \times 1 + (1 - p^*) \times 0$$

This is 0.7041. Unfortunately it is not easy to know the correct discount rate to apply to the expected payoff in the real world. A position in a call option is riskier than a position in the stock. As a result the discount rate to be applied to the payoff from a call option is greater than 16%. Without knowing the option's value, but we do not know how much greater than 16% it should be. Using risk-neutral valuation solves this problem because we know that in a risk-neutral world the expected return on all assets (and therefore the discount rate to use for all expected payoffs) is the risk-free rate.

Figure 3.1. Change in stock price in time Δt in (a) the real world and (b) the risk neutral world



3.2 Continuously Compounded Returns

Continuously compounded returns are mathematically convenient and widely used in practice, both in pricing models and when computing volatility. Here we briefly summarize the important properties of continuously compounded returns.

- **The logarithmic function computes continuously compounded returns from prices.**

  Let S_t and S_{t+h} be stock prices at times t and $t+h$. The continuously compounded return between t and $t+h$, $r_{t,t+h}$ is then

$$r_{t,t+h} = ln\left(\frac{S_{t+h}}{S_t}\right)$$

- **The exponential function computes prices from continuously compounded returns.**

  If we know the continuously compounded return, $r_{t,t+h}$, we can obtain S_{t+h}.

$$S_{t+h} = S_t e^{r_{t,t+h}}$$

- **Continuously compounded returns are additive.**

  Suppose we have continuously compounded returns over consecutive periods—for example, $r_{t,t+h}, r_{t+h,t+2h}$, etc. The continuously compounded return over a long period is the *sum* of continuously compounded returns over the shorter periods, i.e.

$$r_{t,t+nh} = \sum_{i=1}^{n} r_{t+(i-1)h,t+ih}$$

Example 3.2.1. The stock price on four consecutive days is \$100, \$103, \$97, and \$98. The daily continuously compounded returns are,

ln (103/100) = 0.02956; ln (97/103)=−0.06002; ln (98/97) = 0.01026

The continuously compounded return from day 1 to day 4 is ln (98/100) = −0.0202. This is also the sum of the daily continuously compounded returns:

$r_{1,2} + r_{2,3} + r_{3,4}$ = 0.02956 + (−0.06002) + 0.01026 = −0.0202

3.3. Volatility

The **volatility** of an asset, defined as the standard deviation of continuously compounded returns, is a key input for any option pricing calculation. We can express volatility over different periods. For example, we could compute monthly volatility (the standard deviation of the monthly return) or annual volatility (the standard deviation of the annual return).
Suppose that the continuously compounded return over month i is $r_{monthly,i}$. Thus, the annual continuously compounded return is

$$r_{annual} = \sum_{i=1}^{12} r_{monthly,i}$$

The variance of the annual continuously compounded return is therefore

$$\mathrm{Var}(r_{annual}) = \mathrm{Var}\left(\sum_{i=1}^{12} r_{monthly,i}\right)$$

It is common to assume that returns are uncorrelated over time; i.e., the realization of the return in one period does not affect the expected returns in subsequent periods. With this assumption, the variance of a sum is the sum of the variances. Also suppose that each month has the same variance of returns.
If we let σ^2 denote the annual variance, then,

$$\sigma^2 = 12 \times \sigma_{monthly}^2$$

Taking the square root of both sides and rearranging, we can express the monthly standard deviation in terms of the annual standard deviation, σ:

$$\sigma_{monthly} = \frac{\sigma}{\sqrt{12}}$$

To generalize this formula, if we split the year into n periods of length h (so that $h = 1/n$), the standard deviation over the period of length h, σ_h, is

$$\sigma_h = \sigma\sqrt{h}$$

The standard deviation thus scales with the square root of time.

$$\boxed{\sigma = \frac{\sigma_h}{\sqrt{h}}}$$

3.3.1. Estimating Historical Volatility

	S&P 500		IBM	
Date	Price	$\ln(S_t/S_{t-1})$	Price	$\ln(S_t/S_{t-1})$
7/7/2010	1060.27		127	
7/14/2010	1095.17	0.03239	130.72	0.02887
7/21/2010	1069.59	−0.02363	125.27	−0.04259
7/28/2010	1106.13	0.03359	128.43	0.02491
8/4/2010	1127.24	0.01890	131.27	0.02187
8/11/2010	1089.47	−0.03408	129.83	−0.01103
8/18/2010	1094.16	0.00430	129.39	−0.00338
8/25/2010	1055.33	−0.03613	125.27	−0.03238
9/1/2010	1080.29	0.02338	125.77	0.00398
9/8/2010	1098.87	0.01705	126.08	0.00246
Standard deviation	0.02800		0.02486	
Standard deviation $\times \sqrt{52}$	0.20194		0.17926	

The table above lists 10 weeks of Wednesday closing prices for the S&P 500 composite index and for IBM, along with the standard deviation of the continuously compounded returns. Based on the historical returns in the table, the weekly standard deviation of returns was 0.02800 and 0.02486 for the S&P 500 index and IBM, respectively. These standard deviations measure the variability in weekly returns.

To compute the annualized standard deviations multiply the weekly standard deviations by $\sqrt{52}$ (because $h = {}^{1}/_{\sqrt{52}}$), giving annualized historical standard deviations of 20.19% for the S&P 500 index and 17.93% for IBM. We can use the estimated annualized standard deviation as σ in constructing a binomial tree.

3.4 Constructing a Forward tree

As a starting point in constructing u and d, we can ask: What if there were no uncertainty about the future stock price? With certainty, the stock price next period must equal to the forward price. Formula for the forward price is,

$$F_0 = S_0 e^{r\Delta t}$$

Thus, without uncertainty we must have $S_T = F_0$. The rate of return on the stock must be the risk free interest rate.

We incorporate uncertainty into the stock return using volatility, which measures how sure we are that the stock rate of return will be close to the expected rate of return. Stocks with a larger σ will have a greater chance of a return far from the expected return. We model the stock price evolution by adding uncertainty to the forward price:

$$S_0 u = F_0 e^{+\sigma\sqrt{\Delta t}}$$

$$S_0 d = F_0 e^{-\sigma\sqrt{\Delta t}}$$

We will model the stock returns u and d using the equations

$$u = e^{r\Delta t + \sigma\sqrt{\Delta t}}$$

$$d = e^{r\Delta t - \sigma\sqrt{\Delta t}}$$

3.5 Pricing an option using real probabilities

Consider two portfolios:

Portfolio A: Buy one call option. The cost of this is the call premium, which we are trying to determine.
Portfolio B: Buy 2/3 of a share of XYZ and borrow \$18.462 at the risk-free rate 8%.

This position costs,

2/3× \$41− \$18.462 = \$8.871

Now we compare the payoffs to the two portfolios 1 year from now.

For Portfolio A:

	Stock Price in 1 Year (S_1)	
	\$30	**\$60**
Payoff	0	\$20

For Portfolio B:

	Stock Price in 1 Year (S_1)	
	\$30	\$60
2/3 purchased shares	\$20	\$40
Repay loan of \$18.462	−\$20	−\$20
Total payoff	0	\$20

Note that Portfolios A and B have the same payoff: Zero if the stock price goes down, in which case the option is out-of-the-money, and \$20 if the stock price goes up.
Therefore, both portfolios should have the same cost. Since Portfolio B costs \$8.871, then given our assumptions, the price of one option must be \$8.871. Portfolio B is a synthetic call, mimicking the payoff to a call by buying shares and borrowing.

<u>The Binomial Solution</u>

In the preceding example, how did we know that buying 2/3 of a share of stock and borrowing \$18.462 would replicate a call option?

We have two instruments to use in replicating a call option: shares of stock and a position in bonds (i.e., borrowing or lending). To find the replicating portfolio, we need to find a combination of stock and bonds such that the portfolio mimics the option.

To be specific, we wish to find a portfolio consisting of Δ shares of stock and a dollar amount B in lending, such that the portfolio imitates the option whether the stock rises or falls.

Let S_0 be the stock price today. We can write the stock price as $S_0 u$ when the stock goes up and as $S_0 d$ when the price goes down. Let f_u and f_d represent the value of the option when the stock goes up or down, respectively. If the length of a period is Δt, the interest factor per period is $e^{r\Delta t}$.

The value of the replicating portfolio at time Δt,

$$(S_0\Delta + B)e^{r\Delta t}$$

The value of the replicating portfolio when the stock goes up,

$$S_0 u\Delta + Be^{r\Delta t} = f_u$$

The value of the replicating portfolio when the stock goes down,

$$S_0 d\Delta + Be^{r\Delta t} = f_d$$

Solving these two equations for Δ and B gives,

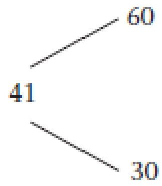
$$\boxed{\Delta = \frac{f_u - f_d}{S_0 u - S_0 d}}$$

$$\boxed{B = e^{-r\Delta t}\left(\frac{uf_d - df_u}{u - d}\right)}$$

Thus, the cost of the option is $\Delta S_0 + B$. We have,

$$\Delta S_0 + B = e^{-r\Delta t}\left[\left(\frac{e^{r\Delta t} - d}{u-d}\right) f_u + \left(\frac{u - e^{r\Delta t}}{u-d}\right) f_d\right]$$

Example 3.5.1.



Binomial tree depicting the movement of XYZ stock over 1 year. The current stock price is \$41. In addition, the call option have a strike price of \$40 and the risk free interest rate is 8%. Find Δ and B. Hence find the option price.

Example 3.5.2.

Construct a one-period forward tree for pricing a European call option assuming: S = \$41.00, K = \$40.00, σ = 0.30, r =0.08, T = 1.00 years.

$$uS = \$41e^{(0.08-0)\times 1+0.3\times\sqrt{1}} = \$59.954$$
$$dS = \$41e^{(0.08-0)\times 1-0.3\times\sqrt{1}} = \$32.903$$

$$\Delta = \frac{\$19.954 - 0}{\$41 \times (1.4623 - 0.8025)} = 0.7376$$
$$B = e^{-0.08}\frac{1.4623 \times \$0 - 0.8025 \times \$19.954}{1.4623 - 0.8025} = -\$22.405$$

Hence, the option price is given by

$$\Delta S + B = 0.7376 \times \$41 - \$22.405 = \$7.839$$

In practice, when constructing a binomial tree to represent the movements in a stock price, we choose the parameters u and d to match the volatility of the stock price. To see how this is done, we suppose that the expected return-on a stock (in-the.-real world) is μ and its volatility is σ. Figure 3.1a shows stock price movements over one step of a binomial tree in the real world and Figure 3.1b shows these movements in a risk-neutral world. The step is of length Δt. The stock price starts at S_0 and moves either up to S_0u or down to S_0d. These are the only two possible outcomes in both the real world and the risk-neutral world. The probability of an up movement in the real world is denoted by p* and, consistent with our earlier notation, in the risk-neutral world this probability is p. The expected stock price at the end of the first time step in the real world is $S_0e^{\mu\Delta t}$, where μ is the expected return. On the tree the expected stock price at this time is

$$p^* S_0 u + (1 - p^*) S_0 d$$

In order to match the expected return on the stock with the tree's parameters, we must therefore have

$$p^{*}S_0u + (1 - p^{*})S_0d = S_0e^{\mu\Delta t}$$

Or

$$p^{*} = \frac{e^{\mu\Delta t} - d}{u - d}$$

Using p^{*} the actual expected payoff to the option one period hence is

$$p^{*}f_u + (1 - p^{*})f_d = \frac{e^{\mu\Delta t} - d}{u - d}f_u + \frac{u - e^{\mu\Delta t}}{u - d}f_d$$

Now we face the problem with using real as opposed to risk-neutral probabilities: At what rate do we discount this expected payoff? It is not correct to discount the option at the expected return on the stock, μ, because the option is equivalent to a leveraged investment in the stock and, hence, is riskier than the stock.

Denote the appropriate per-period discount rate for the option as γ. To compute γ , we can use the fact that the required return on any portfolio is the weighted average of the returns on the assets in the portfolio.

An option is equivalent to holding a portfolio consisting of Δ shares of stock and B bonds. The expected return on this portfolio is

$$e^{\gamma\Delta t} = \frac{S_0\Delta}{S_0\Delta + B}e^{\mu\Delta t} + \frac{B}{S_0\Delta + B}e^{r\Delta t}$$

We can now compute the option price as the expected option payoff, discounted at the appropriate discount rate.

$$\boxed{f = e^{-\gamma\Delta t}\left[\frac{e^{\mu\Delta t} - d}{u - d}f_u + \frac{u - e^{\mu\Delta t}}{u - d}f_d\right]}$$

Example 3.5.3.

To see how to value an option using true probabilities, compute example 3.5.2 assuming that the continuously compounded expected return on XYZ is μ = 15%.

$$p^{*} = \frac{e^{0.15} - 0.8025}{1.4623 - 0.8025} = 0.5446$$

The expected payoff to the option in one period,

0.5446 × \$19.954 + (1− 0.5446) × \$0 = \$10.867

In this example, Δ = 0.738 and B =−\$22.405.

$$e^{\gamma h} = \frac{0.738 \times \$41}{0.738 \times \$41 - \$22.405}e^{0.15} + \frac{-\$22.405}{0.738 \times \$41 - \$22.405}e^{0.08}$$
$$= 1.386$$

The option price is then given by

$e^{-0.3264} \times \$10.867 = \$7.839$

3.6 The Cox-Ross-Rubinstein Binomial Tree

The best-known way to construct a binomial tree is that in Cox et al. (1979), in which the tree is constructed as

$$u = e^{\sigma\sqrt{\Delta t}}$$
$$d = e^{-\sigma\sqrt{\Delta t}}$$

The Cox-Ross-Rubinstein approach is often used in practice. A problem with this approach, however, is that if Δt is large or σ is small, it is possible that $e^{r\Delta t} > e^{\sigma\sqrt{\Delta t}}$, in which case the binomial tree violates the restriction $u > e^{r\Delta t} > d$. In real applications Δt would be small, so this problem does not occur. In any event, the tree based on the forward price never violates equation $u > e^{r\Delta t} > d$.

The Binomial Tree Formulas

The analysis in the previous section shows that, when the length of the time step on a binomial tree is Δt, we should match volatility by setting

$$u = e^{\sigma\sqrt{\Delta t}}$$
$$d = e^{-\sigma\sqrt{\Delta t}}$$

Also from equation

$$p = \frac{a - d}{u - d}$$

Where $a = e^{r\Delta t}$.

3.6.1 Options on other assets

The binomial model can be modified easily to price options on underlying assets other than non-dividend-paying stocks. In this section we present examples of options on stock indexes, currencies, and futures contracts. In every case the general procedure is the same: The difference for different underlying assets will be the construction of the binomial tree and the risk-neutral probability. As in the case of options on stocks, the value at a node (before the possibility of early exercise is considered) is p times the value if there is an up moment plus 1-p times the value if there is down moment, discounted at the risk free rate.

(a) For options on an index, we set $a = e^{(r-q)\Delta t}$, where q is the average dividend yield on the index during the life of the option.

Example 3.6.1: A stock index is currently 810 and has a volatility of 20% and a dividend yield of 2%. The risk-free rate is 5%. Value a European 6-month call option with a strike price of 800 using a two-step tree.

$$\Delta t = 0.25, \qquad u = e^{0.20\times\sqrt{0.25}} = 1.1052,$$

$$d = 1/u = 0.9048, \qquad a = e^{(0.05-0.02)\times 0.25} = 1.0075$$

$$p = (1.0075 - 0.9048)/(1.1052 - 0.9048) = 0.5126$$

The value of the option is 53.39.

(b) For options on a currency, we set $a = e^{(r-r_f)\Delta t}$, where r_f is the risk free interest rate in the currency.

Example 3.6.2: The Australian dollar is currently worth 0.6100 US dollars and this exchange rate has a volatility of 12%. The Australian risk-free rate is 7% and the US risk-free rate is 5%. Value a 3-month American call option with a strike price of 0.6000 using a three-step tree.

$$\Delta t = 0.08333, \quad u = e^{0.12\times\sqrt{0.08333}} = 1.0352$$

$$d = 1/u = 0.9660, \quad a = e^{(0.05-0.07)\times 0.08333} = 0.9983$$

$$p = (0.9983 - 0.9660)/(1.0352 - 0.9660) = 0.4673$$

The value of the option is 0.019.

(c) For options on a futures contract, we set a=1.

Example 3.6.3: A futures price is currently 31 and has a volatility of 30%. The risk-free rate is 5%. Value a 9-month American put option with a strike price of 30 using a three-step tree.

$$\Delta t = 0.25, \quad u = e^{0.3\sqrt{0.25}} = 1.1618$$

$$d = 1/u = 1/1.1618 = 0.8607, \quad a = 1,$$

$$p = (1 - 0.8607)/(1.1618 - 0.8607) = 0.4626$$

The value of the option is 2.84.

3.7 Jarrow-Rudd binomial model

This procedure for generating a tree was proposed by Jarrow and Rudd (1983) and is sometimes called the **Lognormal Tree** to construct the tree using

$$u = e^{(r-0.5\sigma^2)\Delta t + \sigma\sqrt{\Delta t}}$$

$$d = e^{(r-0.5\sigma^2)\Delta t - \sigma\sqrt{\Delta t}}$$

3.8 Why does risk neutral pricing work?

Utility-Based Valuation

The starting point is that the well-being of investors is not measured in dollars, but in *utility*, which is a measure of satisfaction. Economists typically assume that investors exhibit *declining marginal utility*: Starting from a given level of wealth, the utility gained if we add $1 to wealth is less than the utility lost if we remove $1 from wealth. Thus, we expect that more dollars will make an investor happier, but that if we keep adding dollars, each additional dollar will make the investor less happy than the previous dollars.

Declining marginal utility implies that investors are risk-averse, which means that an investor will prefer a safer investment to a riskier investment that has the same expected return. Since losses are more costly than gains are beneficial, a risk-averse investor will avoid a fair bet, which by definition has equal expected gains and losses.

To illustrate risk-neutral pricing, we imagine a world where there are two assets, a risky stock and a risk-free bond. Investors are risk-averse. Suppose the economy in one period will be in one of two states, a high state and a low state. How do we value assets in such a world? We need to know three things:

1. What utility value, expressed in terms of dollars today, does an investor attach to the marginal dollar received in each state in the future? Denote the values today of $1 received in the high and low states as X_H and X_L, respectively. Because the investor is risk-averse, $1 received in the high state is worth less than $1 received in the low state; hence, $X_H < X_L$.
2. How many dollars will an asset pay in each state? The bond pays $1 in each state, while the risky stock pays S_H in the high state and S_L in the low state.
3. What is the probability of each state occurring? Denote the probability of the high state as p^*. Another name for p^* is the **physical probability** of an up move.

We begin by defining a **state price** as the price of a security that pays $1 only when a particular state occurs. Let Q_H be the price of a security that pays $1 when the high state occurs, and Q_L the price of a security paying $1 when the low state occurs. Since X_H and X_L are the value today of $1 in each state, the price we would pay is just the value times the probability that state is reached:

$$Q_H = p^* \times X_H$$

$$Q_L = (1 - p^*) \times X_L$$

Since there are only two possible states, we can value any future cash flow using these state prices.

The price of the risky stock, S_0 is,

$$S_0 = (Q_H \times S_H) + (Q_L \times S_L)$$

The risk-free bond, with price B, pays $1 in each state. Thus, we have

$$B = (Q_H \times 1) + (Q_L \times 1)$$

We can calculate rates of return by dividing expected cash flows by the price. Thus, the risk-free rate is

$$1 + r = \frac{1}{Q_H + Q_L} = \frac{1}{B}$$

The expected return on the stock is

$$1 + \mu = \frac{p^* \times S_H + (1 - p^*) \times S_L}{(Q_H \times S_H) + (Q_L \times S_L)} = \frac{p^* \times S_H + (1 - p^*) \times S_L}{S_0}$$

Risk Neutral Pricing:

Instead of utility-weighting the cash flows and computing expectations, *we utility-weight the probabilities*, creating new “risk-neutral” probabilities.

Use the state prices in equation to define the risk-neutral probability of the high state p as,

$$p = \frac{p^* X_H}{p^* X_H + (1 - p^*) X_L} = \frac{Q_H}{Q_H + Q_L}$$

Now we compute the stock price by using the risk-neutral probabilities to compute expected cash flow, and then discounting at the risk-free rate. We have

$$\frac{pS_H + (1 - p)S_L}{1 + r} = \frac{\frac{Q_H}{Q_H + Q_L} S_H + \frac{Q_L}{Q_H + Q_L} S_L}{1 + r} = Q_H S_H + Q_L S_L$$

Example 3.8: Probabilities, utility weights, and equity cash flows in high and low states of the economy.

	High State	Low State
Cash flow to risk-free bond	$B_H = \$1$	$B_L = \$1$
Cash flow to stock	$S_H = \$180$	$S_L = \$30$
Probability	$p = 0.52$	$p = 0.48$
Value of \$1	$\chi_H = \$0.87$	$\chi_L = \$0.98$

1. Calculate the state prices.
2. Value the risk-free bond.
3. Find the risk-free rate.
4. Value the risky stock using real probabilities.
5. Find the expected cash flow on the stock in one period.
6. Calculate the expected return on the stock.
7. Calculate the risk risk-neutral probability.
8. Hence value the stock.

The state prices are *QH* = 0.52 × \$0.87= \$0.4524, and *QL*= 0.48 × \$0.98 = \$0.4704.
The risk-free bond pays \$1 in each state. Thus, the risk-free bond price, *B*, is
B = *QH* + *QL* = \$0.4524 + \$0.4704 = \$0.9228
The risk-free rate is 8.366%
the price of the stock is

$$S_0 = 0.4524 \times \$180 + 0.4704 \times \$30 = \$95.544$$

The expected cash flow on the stock in one period is
E(*S1)* = 0.52 × \$180 + 0.48 × \$30 = \$108
The expected return on the stock is therefore

$$\frac{\$108}{\$95.544} - 1 = 13.037\%$$

The risk-neutral probability is

$$p^* = \frac{\$0.4524}{\$0.4524 + \$0.4704}$$
$$= 49.025\%$$

Now we can value the stock using p^* instead of the true probabilities and discount at the risk-free rate:

$$S_0 = \frac{0.49025 \times \$180 + (1 - 0.49025) \times \$30}{1.08366}$$
$$= \$95.544$$