MATH 235, Class 1 Practice Problems

Andrej Vukovic

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1 Problems

1. Which of the following are vector spaces over $\mathbb{R}$ with the usual addition and multiplication?

(a) The real numbers, $\mathbb{R}$.
(b) The non-negative real numbers, $\mathbb{R}_{\geq 0} := \{x \in \mathbb{R} \mid x \geq 0\}$.
(c) The integers, $\mathbb{Z}$.

2. Can you think of an example of a vector space that wasn't in the notes or the previous question?

3. Is there a vector space V, whose underlying set is $\mathbb{Z}$, such that V is a vector space over $\mathbb{R}$ with the usual addition but not necessarily the usual scalar multiplication?

2 Solutions

1. (a) We check it against the definition of a vector space in the notes for Class 1. Observe that $\mathbb{R}$ is a set equipped with addition and multiplication. If $c, x, y \in \mathbb{R}$, then $x + y \in \mathbb{R}$ and $cx \in \mathbb{R}$, so $\mathbb{R}$ is closed under addition and scalar multiplication.

If $x, y, z \in \mathbb{R}$, then $(x+y)+z = x+(y+z)$ and $x+y = y+x$ since addition in $\mathbb{R}$ is associative and commutative. Also, if $0 \in \mathbb{R}$ the usual number 0, then for all $x \in \mathbb{R}$, $x + 0 = x$, so 0 is an additive identity. Also, if $x \in \mathbb{R}$, there exists $-x \in \mathbb{R}$ such that $x + (-x) = 0$. Here, $-x$ is just the usual negative of a number.

If $c_1, c_2, x, y \in \mathbb{R}$, then $(c_1 + c_2)x = c_1x + c_2x$ and $c_1(x + y) = c_1x + c_1y$ by left- and right-distributivity of multiplication in $\mathbb{R}$. Also, $c_1(c_2x) = (c_1c_2)x$ by associativity of multiplication in $\mathbb{R}$. Finally, if $1 \in \mathbb{R}$ is the usual number 1, then $1x = x$ for all $x \in \mathbb{R}$, so we have a multiplicative identity.

It follows that $\mathbb{R}$, equipped with the usual addition and scalar multiplication, forms a vector space over $\mathbb{R}$.

(b) This is similar to the example in part (a) except that there are no additive inverses in general. For instance, $1 \in \mathbb{R}_{\geq 0}$ has no additive inverse in that set since $-1 \notin \mathbb{R}_{\geq 0}$. It follows that, with the uusal addition and scalar multiplication, $\mathbb{R}_{\geq 0}$ is not an $\mathbb{R}$–vector space. ("$\mathbb{R}$–vector space" is another way of saying "vector space over $\mathbb{R}$".)

(c) The example of $\mathbb{Z}$ superficially satisfies all the axioms of a vector space. However, there is an issue. Our field of scalars is $\mathbb{R}$, and the definition of a vector space requires that for every scalar $c \in \mathbb{F}$ and every vector $v \in V$, cv is in V. This does not happen here. If you take $c := 0.5 \in \mathbb{R}$ and $v := 3 \in \mathbb{Z}$, then $cv = (0.5)(3) = 1.5 \notin \mathbb{Z}$. So $\mathbb{Z}$ does not form an $\mathbb{R}$–vector space with the usual addition and scalar multiplication.

2. There are many examples. For an easy example, take the set $\{0\}$ with addition defined by $0 + 0 = 0$ and scalar multiplication, say over $\mathbb{R}$, defined by $c \cdot 0 = 0$. Here the additive identity is 0 and the additive inverse of 0 is 0. Also, *any* element of $\mathbb{R}$ is a multiplicative identity since for any $c \in \mathbb{R}$, $c \cdot 0 = 0$. You can check that the other properties of a vector space are also satisfied for this example. This set also forms a vector space over $\mathbb{C}$ in the same way.

For a more interesting example, take the space $C([0, 1])$ of continuous functions $f : [0, 1] \to \mathbb{R}$. If $c \in \mathbb{R}$ is a scalar and $f, g \in C([0, 1])$, then $f + g$ is also continuous and defined on $[0, 1]$, as is cf. This set is therefore closed under addition and scalar multiplication. You can check that $C([0, 1])$ forms a vector space over $\mathbb{R}$ by pretty much following Example 1.4 in the notes for Class 1 and checking that every function involved is continuous and defined on $[0, 1]$. Notice that $C([0, 1])$ is *not* a $\mathbb{C}$–vector space. Indeed, the function defined by $f(x) = x$ for all $x \in [0, 1]$ is continuous and is a function from $[0, 1]$ to $\mathbb{R}$. But the function if, given by multiplying f by $i = \sqrt{-1}$, while still continuous, is no longer a function from $[0, 1]$ to $\mathbb{R}$. For instance, $if(1) = (i)(1) = i$. So $C([0, 1])$ is closed under scalar multiplication over $\mathbb{R}$ but not over $\mathbb{C}$.

3. To minimize confusion, I'm going to write e for the multiplicative identity in the field of scalars $\mathbb{R}$, which necessarily exists by vector space axiom 2 on the second page of the class notes, and $\mathbf{1}$ for the element of V corresponding to the number $1 \in \mathbb{Z}$. Notice that the multiplicative identity e does not have to actually equal the number $1 \in \mathbb{R}$ because we could have some unusual scalar multiplication.

The answer to the question is no. Suppose we had an $\mathbb{R}$–vector space V whose underlying set is $\mathbb{Z}$ equipped with the usual addition but not necessarily the usual scalar multiplication.

Since $e \in \mathbb{R}$, we can divide it by 2 to get $e/2 \in \mathbb{R}$. Then we have

$$\mathbf{1} = e \cdot \mathbf{1} = (\frac{e}{2} + \frac{e}{2}) \cdot \mathbf{1} = \frac{e}{2} \cdot \mathbf{1} + \frac{e}{2} \cdot \mathbf{1}$$

by left-distributivity (axiom 3 on page 2 of the Class 1 notes). But $\frac{e}{2} \cdot \mathbf{1} \in V$, so it has to be equal to some integer $n := \frac{e}{2} \cdot \mathbf{1}$. This means there is some integer n such that

$$\mathbf{1} = n + n.$$

But since we made the assumption that we're using the usual addition operation, this is impossible because there is no $n \in \mathbb{Z}$ such that $n + n = 1$. This completes the proof.