MATH 235, Class 2 Practice Problems

Andrej Vukovic

September 2021

1 Problems

1. Which of the following are subspaces of the $\mathbb{R}$–vector space $M_{n\times n}(\mathbb{R})$?

(a) The set of $n \times n$ real matrices whose top-left entry is 0.

(b) $\mathfrak{sl}_n(\mathbb{R})$, the set of $n \times n$ matrices with real entries whose trace is 0, i.e., the sum of whose diagonal entries is 0.

(c) $M_{n\times n}(2\mathbb{Z})$, the set of $n \times n$ integer matrices whose entries are all even.

2. Recall that $C([0,1]) = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$ can be viewed as a vector space over $\mathbb{R}$. (The notation $f : [0,1] \to \mathbb{R}$ means f is a function from the closed interval $[0,1]$ to the real numbers $\mathbb{R}$.) For $r \in [0,1]$, let $S(r) := \{f \in C([0,1]) \mid f(r) = 0\}$. For which $r \in [0,1]$ is $S(r)$ a subspace of $C([0,1])$?

3. Let V be a vector space over $\mathbb{R}$ and S_1, S_2 two subspaces of V. Prove that the union $S_1 \cup S_2$ is a subspace of V if and only if $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$. (Here $\subseteq$ means "is a subset of", *not* "is a subspace of".)

2 Solutions

1. (a) This is a subspace since it satisfies all conditions of the subspace test. Sums and scalar multiples of matrices with top-left entry 0 still have top-left entry 0, and the zero matrix has top-left entry 0.

(b) The zero matrix has trace 0, and sums and scalar multiples of trace 0 matrices have trace 0, so this is a subspace.

(c) This is not a subspace because it's not closed under scalar multiplication, as you can see by, for example, multiplying the $n \times n$ matrix all of whose entries are 2 by the scalar 0.5. It fails the subspace test.

2. Note that the zero vector in $C([0, 1])$ is just given by the zero function $z(x) = 0$ for all $x \in [0, 1]$. For $S(r)$ to be a subspace, we need $z \in S(r)$. Since $z(r) = 0$ for any $r \in [0, 1]$, $z \in S(r)$ for all r.

By the subspace test, it remains to check that $S(r)$ is closed under addition and scalar multiplication for each $r \in [0, 1]$. If $f, g \in C([0, 1])$ are such that $f(r) = g(r) = 0$, then $(f + g)(r) = f(r) + g(r) = 0$, so $S(r)$ is closed under addition. If $c \in \mathbb{R}$ and $f \in S(r)$, then $cf(r) = (c)(0) = 0$, so $cf \in S(r)$. It follows that for every $r \in [0, 1]$, $S(r)$ is a subspace of $C([0, 1])$.

3. Suppose $S_1 \cup S_2$ is a subspace of V. For the sake of contradiction, assume that $S_1 \not\subseteq S_2$ and $S_2 \not\subseteq S_1$. Since subspaces must contain the zero vector, they can't be empty, so S_1 and S_2 are not empty. Thus there exist $v \in S_1 \setminus S_2$ and $w \in S_2 \setminus S_1$.

Since $S_1 \cup S_2$ is a subspace by assumption, $v + w \in S_1 \cup S_2$, i.e., $v + w \in S_1$ or $v + w \in S_2$. Suppose $v + w \in S_1$. Then since $v \in S_1$, we have $-v \in S_1$, so $v + w + (-v) \in S_1$. But $v + w + (-v) = w$, so $w \in S_1$, contradicting the assumption that $w \in S_2 \setminus S_1$. Similarly, if we had assumed that $v + w \in S_2$ instead, we would have concluded that $v \in S_2$, contradicting the assumption that $v \in S_1 \setminus S_2$. Since we have reached a contradiction in either case, we conclude that $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$.

Conversely, suppose that $S_1 \subseteq S_2$. Then $S_1 \cup S_2 = S_2$ is a subspace by assumption, so we are done. The case where $S_2 \subseteq S_1$ is similar.$\square$