

(A-1-1, MATH 239 Winter 2021)

1. Let  $k \geq 1$  be an integer. Let  $T$  denote the set of multisets of size  $n$  with  $t$  types such that for all  $i \in \{1, \dots, t\}$ , the number of elements of the  $i$ -th type is a multiple of  $k$ , or in other words  $k \mid i$  for each  $i$ . What is  $|T|$ ? (Assume  $k \mid n$ )
2. Let  $k \geq 0$  be an integer. Let  $H$  denote the set of multisets of size  $n$  with  $t$  types such that for each  $i \in \{1, \dots, t\}$ , the number of elements of the  $i$ -th type is at least  $k$ . What is  $|U|$ ? (Assume  $n \geq kt$ )

**Solution:**

1. Let the set  $H$  be the set of all multisets of size  $n/k$  and  $t$  types. We claim a bijection between  $T$  and  $H$ . Note that each element in  $T$  may be written as  $(kc_1, \dots, kc_t)$ , where each  $c_i \in \mathbb{Z}^+$ . Let  $f: T \rightarrow H$  be defined by mapping each  $\alpha = (kc_1, \dots, kc_t) \in T$  to  $f(\alpha) = (c_1, \dots, c_t)$ . By the construction of  $T$ ,  $c_i \in \mathbb{Z}^+$  and  $kc_1 + \dots + kc_t = n$  which implies that  $c_1 + \dots + c_t = n/k$  hence  $f(\alpha)$  is a multiset of size  $n/k$  and  $t$  types, hence  $f(\alpha) \in H$ . We define the inverse function  $g: H \rightarrow T$  by simply assigning each  $f(\alpha) = (c_1, \dots, c_t)$ , as constructed previously, to  $g(f(\alpha)) = (kc_1, \dots, kc_t)$ , which simply multiplies each part of the multiset by  $k$ .