General Approach for a bijection:

1. Setup functions $f\colon X \to Y$ and $g\colon Y \to X$.
2. Show that f is well defined (Properly maps to Y).
3. Show that g is well defined (Properly maps to X).
4. Show that $\forall x \in X$, $g(f(x)) = x$.
5. Show that $\forall y \in Y$, $f(g(y)) = y$.
6. Clarify that steps $4, 5$ imply that the pair of functions f, g are mutually inverse bijections and hence X and Y are in bijection.

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1. Let $k \geq 1$ be an integer. Let T denote the set of multisets of size n with t types such that for all $i \in \{1, \ldots, t\}$, the number of elements of the i-th type is a multiple of k. What is $|T|$? (Assume $k \mid n$)
2. Let $k \geq 0$ be an integer. Let U denote the set of multisets of size n with t types such that for each $i \in \{1, \ldots, t\}$, the number of elements of the $i-$th type is at least k. What is $|U|$? (Assume $n \geq kt$)

Solution:

1. Let H be the set of all multisets of size n/k and t types. We claim a bijection between T and H. Note that each element $(a_1, \ldots, a_t) \in T$ may be written as $(kc_1, \ldots, kc_t)$, where each $c_i \in \mathbb{Z}^+$. Let $f\colon T \to H$ be defined by mapping each $\alpha = (kc_1, \ldots, kc_t) \in T$ to $f(\alpha) = (c_1, \ldots, c_t)$. By construction of T, each element c_i is in $\mathbb{Z}^+$ and $kc_1 + \cdots + kc_n = n$ which implies that $c_1 + \cdots + c_n = n/k$ hence $f(\alpha)$ is a multieset of size n/k and t types, hence $f(\alpha) \in H$ (Recall we assume that $k \mid n$ and hence n/k is nonnegative integer). We define the inverse function $g\colon H \to T$ by simply assigning each $\gamma = (c_1, \cdots, c_t)$, to $g(\alpha) = (kc_1, \ldots, kc_t)$, which simply multiplies each element of the multiset by k. Clearly $(kc_1, \ldots, kc_t)$ is a mutiset of size n and t types, hence $g(\gamma) \in T$. Clearly by inspection, the constructions $T \mapsto H$ and $H \mapsto T$ are mutually inverse bijections and hence,
$$|T| = |H| = \binom{n/k + t - 1}{t - 1}$$
2. Let X be the set of all multisets of size $n - kt$ at t types. We claim a bijecton between X and U. Note that each element $(m_1, \ldots, m_t) \in U$ may be written as $(c_1 + k, \ldots, c_t + k)$, where each $c_i \in \mathbb{Z}^+$. Let $f\colon U \to X$ be defined by mapping each $\alpha = (c_1 + k, \ldots, c_t + k) \in T$ to $f(\alpha) = (c_1, \ldots, c_t)$. By construction of U, each element c_i is in $\mathbb{Z}^+$ and $(c_1+k)+\cdots+(c_t+k) = n$ which implies that $c_1 + \cdots + c_t = n - kt$, hence $f(\alpha)$ is a multiset of size $n - kt$ and t types, hence $f(\alpha) \in X$. (Recall we assume that $n \geq kt$). We define $g\colon X \to U$ by assigning each