$\frac{m}{s} \times \frac{1}{s} = \frac{m}{s^2} = \frac{m/s}{s}$ (m/s)/s "rate of change of your motion"

$\vec{\Delta d}$, $\vec{v_{av}}$, $\vec{a_{av}}$

# 1 Acceleration

**Definition 1.0.1**
**Average Acceleration**, $\vec{a}_{av}$, refers to the rate of change of velocity, or in other words the ratio of the change of velocity to the time elapsed. (**Units:** $m/s^2$)

$$\vec{a}_{av} = \frac{\Delta \vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i}$$

First we note that acceleration is a vector quantity because $\Delta \vec{v}$ is a vector quantity. Acceleration is experienced any time an object is increasing or decreasing its velocity, *any* change in velocity results in acceleration. For example, you must initially accelerate your vehicle in order for it to reach the desired velocity, similarly you must first *accelerate* your vehicle in order to come to a stop and change your velocity to $(+0\,\mathrm{m/s})$. In this course we will consider only uniform acceleration of a moving body and avoid situations where the acceleration a given body is non-uniform.

**Remark :** It is common to hear the term *de-accelerate*, however this term is rather redundant because the term acceleration refers to any change in velocity, regardless of weather you would like to increase your velocity or bring yourself to a holt ($\vec{v} = +0\,\mathrm{m/s}$).

**Remark :** If you are wondering why we are no longer working with $\vec{v}_{av}$, it is because when we were working with average velocity, we were not concerned with the precise velocity of the moving body at a given point in time but rather the "most common" velocity over a time interval. Average acceleration is concerned with changes in *exact* velocities, we will discuss these differences in a latter subsection.

**Example 1.0.1**
A vehicle on the highway changes his velocity from $\vec{v}_i = 500\,\mathrm{m/s}$[East] to $\vec{v}_f = 612\,\mathrm{m/s}$ [West] in $\Delta t = 2\,\mathrm{min}$. Compute his average acceleration,

**Solution**
$\Longrightarrow$

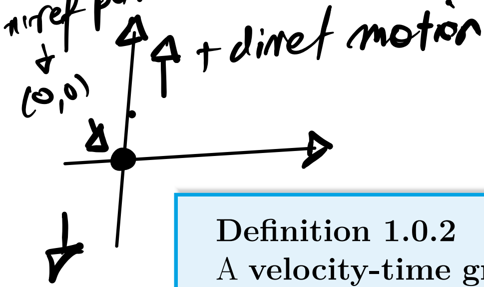
[E] +  $\Delta t = 2\,\mathrm{min} = 120\,\mathrm{s}$

$$\vec{a_{av}} = \frac{\vec{v_f} - \vec{v_i}}{\Delta t} = \frac{-612 - 500}{120} = -9.27\ \mathrm{m/s^2}$$

or $= 9.27\ \mathrm{m/s^2}$ [West]



a point




**Definition 1.0.2**
A **velocity-time graph** is a plot of velocity versus time, with velocity on the vertical axis and time on the horizontal axes.

Similar to the analogy of how a Pos V. Time plot helps us understand velocity better, a Velocity v. Time plot will help us understand acceleration better. Again we mention some basic properties, again we take the reference point to be $(0,0)$. Also we take the positive direction of motion to be above the vertical axes. We now mention a proposition similar to a one we have seen earlier.

**Proposition 1.0.2**
Given a **Linear** Velocity v. Time plot of a moving body, the slope $m$ of the plot represents the average acceleration, $\vec{a}_{av}$, of the body.

**Proof**
We prove the result similar to the method we used in the previous section. Let the slope of the Velocity v. Time graph be $m$, let us compute this slope by using the slope formula, namely,

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Since the $y-$coordinates on a Velocity v. Time graph are velocity vectors $\vec{v}$, and the $x-$coordinates are time points, $t$, we can translate this slope formula to the equivalent,

$$m = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1}$$

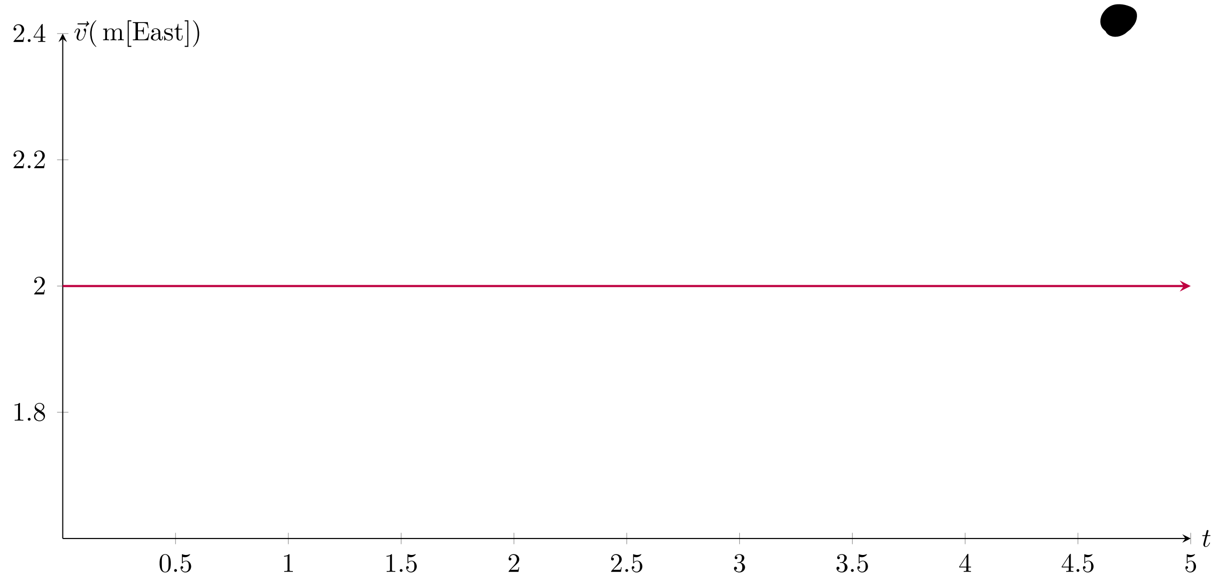
At this point we are free to choose any two coordinate pairs $(\vec{v}_1, t_1)$, $(\vec{v}_2, t_2)$, let us choose $(\vec{v}_f, t_f)$, $(\vec{v}_i, t_i)$, the final and initial coordinate pairs of the moving body. This gives,

$$m = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i} = \vec{a}_{av}$$

## 1.1 Types of motion from Velocity v. Time Plots

Similar to before, we will encounter common types of motion and hence it would be useful to make mention of their plots and what they look like.

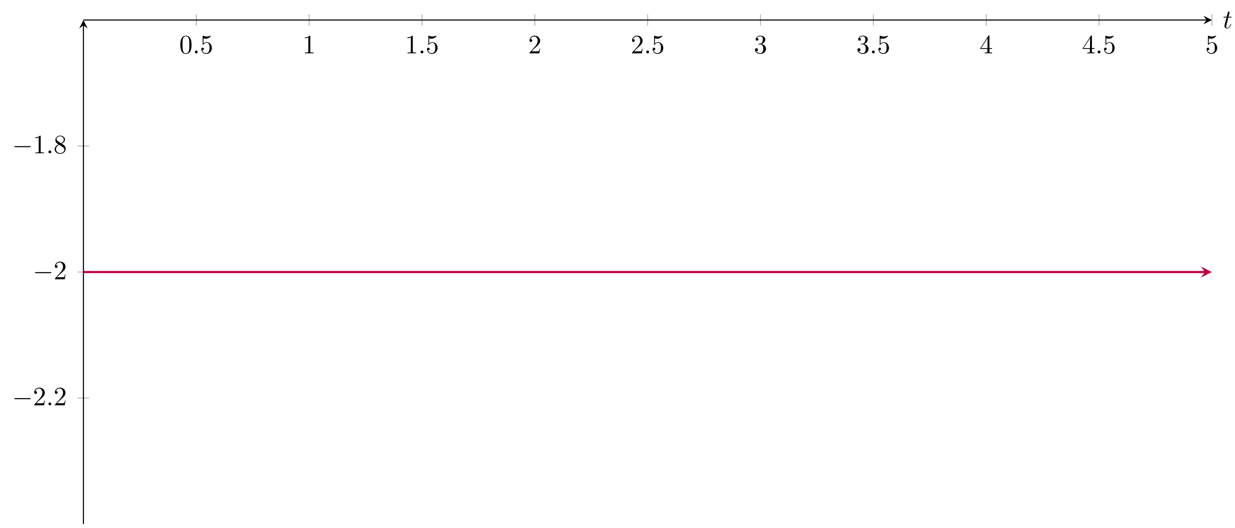
East




(a)

## Properties of type (a):

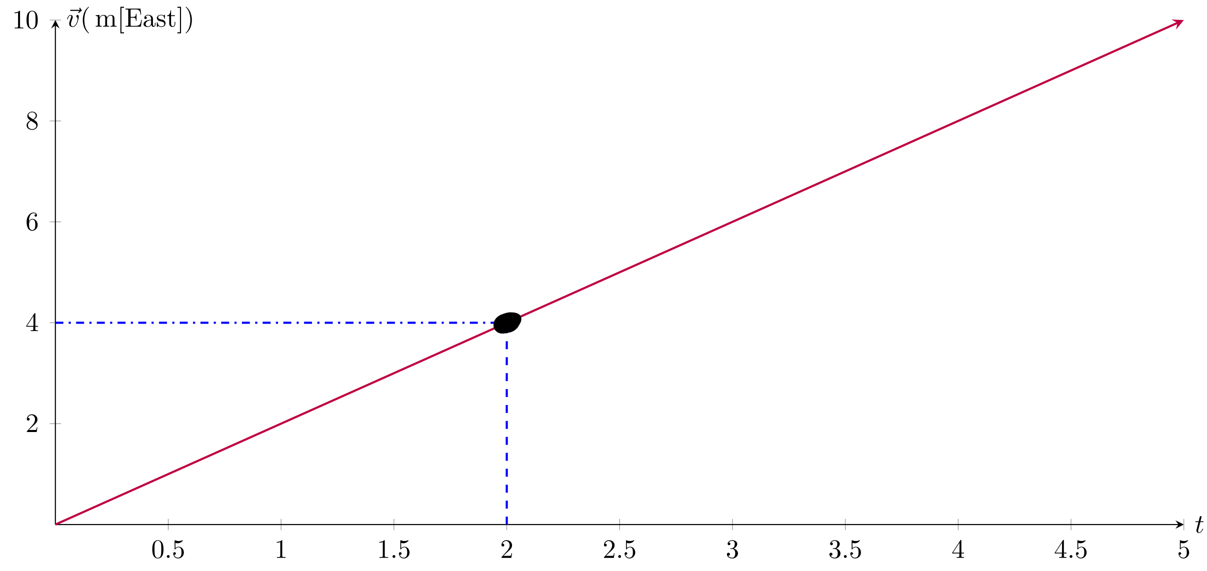
- The slope of the graph is zero, hence $\vec{a}_{av} = +0\,\mathrm{m/\,s^2}$.
- The object is experiencing **uniform motion**.
- The object is moving [East] relative to the reference point (0,0).




(b)

## Properties of type (b):

- The slope of the graph is zero, hence $\vec{a}_{av} = +0\,\mathrm{m/\,s^2}$.
- The object is experiencing **uniform motion**.
- The object is moving [West] relative to the reference point (0,0).

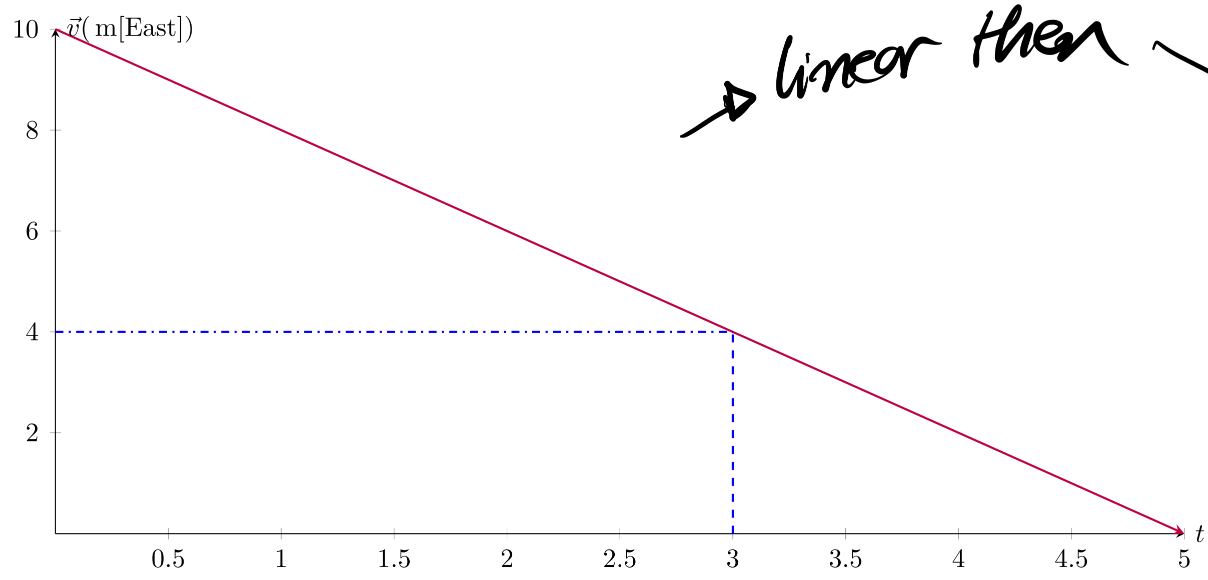



(c)

## Properties of type (c):

- The slope of the graph is $m = +2$, hence $\vec{a}_{av} = +2\,\mathrm{m/\,s^2}$.
- The object experiencing **uniform acceleration**.
- The object is traveling in the [East] direction.

linear then




(d)

## Properties of type (d):

- The slope of the graph is $m = -2$, hence $\vec{a}_{av} = -2\,\mathrm{m/\,s^2}$.
- The object experiencing **uniform acceleration**.
- The object is traveling in the [West] direction.

$\vec{v}_{av}$ ($t_1 \to t_2$)

$\Rightarrow$ " most average vel "

## 1.2 Instantaneous and Average Velocity

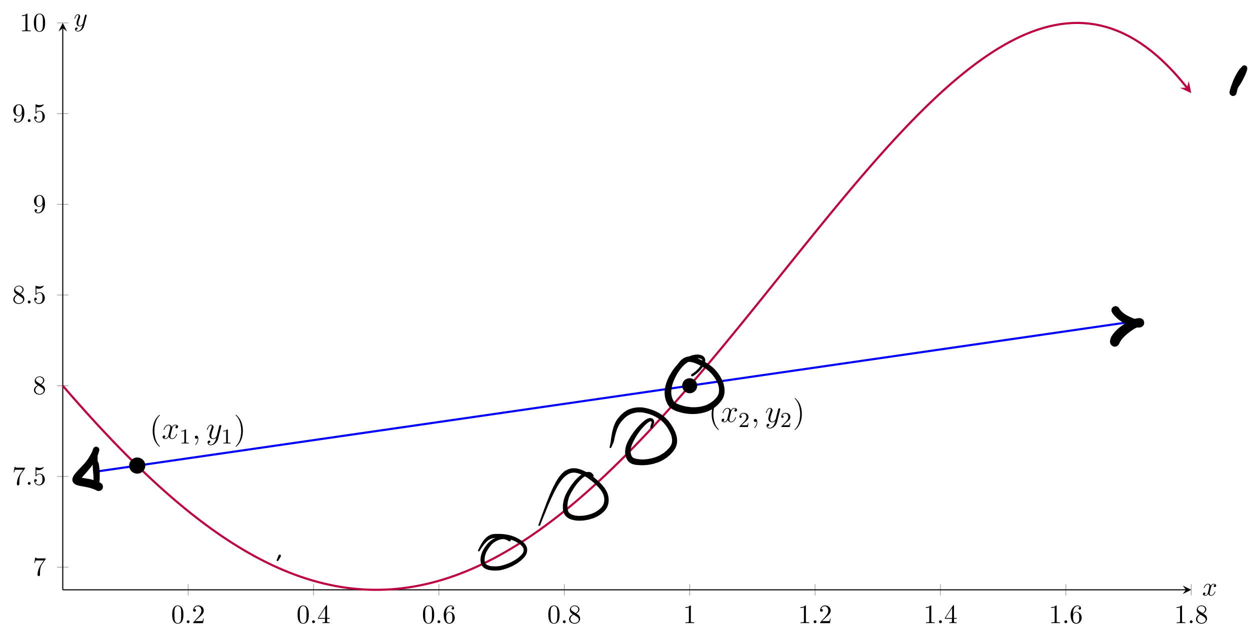
**Definition 1.2.1**
**Uniform acceleration** is motion where the acceleration of the body is fixed.

**Definition 1.2.2**
The **instantaneous velocity**, $\vec{v}$, of an object is the *exact* velocity of an object at a given time $t$

It remains to ask how we can compute the instantaneous velocity of an object at a given time $t$. To answer this question requires knowledge of basic Calculus, however we can still introduce the idea of secant and tangent lines.

**Definition 1.2.3**
A **secant line** is a line segment connecting two points, $(x_1, y_1)$ and $(x_2, y_2)$, on a graph.

The Slope of the Secant line helps us understand the "average" *slope* of a graph over an interval $[x_1, x_2]$. This is why we say that the slope of the secant line corresponds to average velocity. It helps to understand the idea of a secant line using an illustration,




The slope of the secant line would be computed as,

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

I get $\vec{v}_{av}$

If we replaced the $y-$coordinates with position vectors and the $x-$coordinates with time points, the slope of the secant line would represent the average velocity within the interval $[t_1, t_2]$.

**Theorem 1.2.1**
Given a Pos v. Time graph of a moving body, the slope of a secant line in the interval $[t_1, t_2]$ represents the average velocity between $[t_1, t_2]$.

$\vec{d_2} = \vec{d_f}$, $\vec{d_1} = \vec{d_i}$

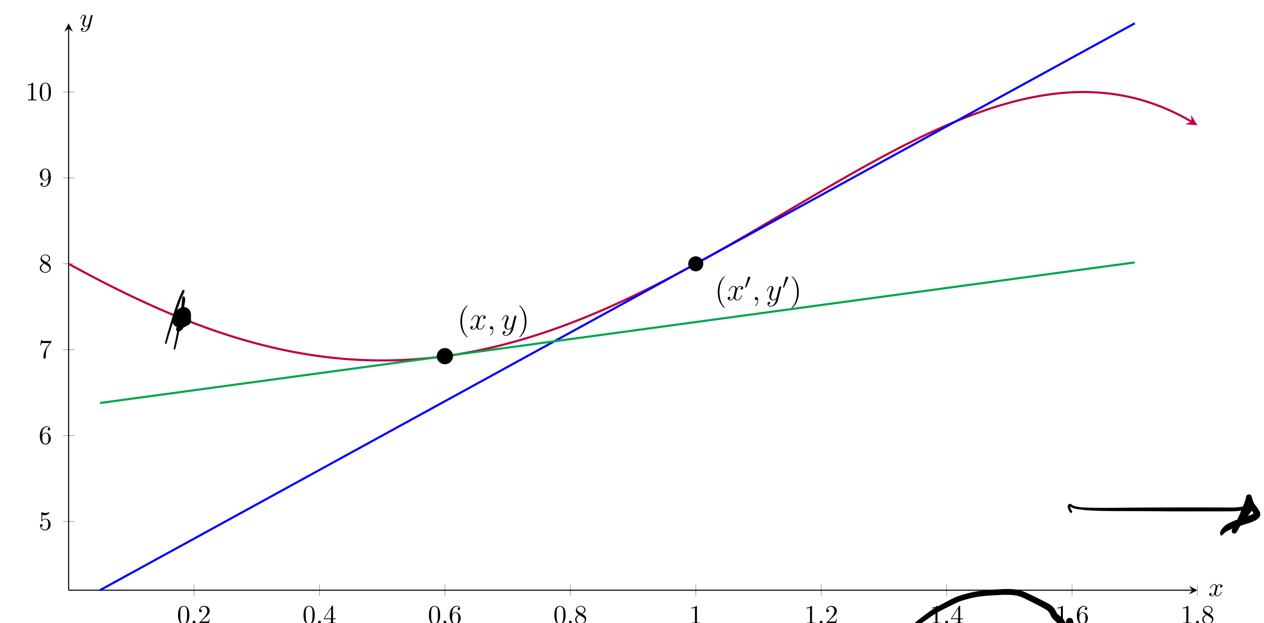
**Proof**
We proceed with similar reasoning from above, the slope of the secant line would be given as,

$$m = \frac{\vec{d_2} - \vec{d_1}}{t_2 - t_1} = \vec{v}_{av}$$

$t_2 = t_f$, $t_1 = t_i$

**Definition 1.2.4**
A **tangent line** is a secant line, with the property that $x_2$ and $x_1$ are infinitesimally close apart.

If $x_2$ and $x_1$ are infinitesimally close apart, then of course $y_2$ and $y_1$ must also be infinestesmly apart as well (How $y$ changes with respect to $x$ wouldn't matter at the infinitesimal level). As a result of this restriction, the tangent line to a graph would look as if it merely touches the graph at a single point, but in fact it does not, if we were to zoom in infinestesmly we would indeed observe that the tangent line touches the graph at two distinct points. Again it helps to provide an illustration,




$(2,1)$, $(\underbrace{2+dx}_{\text{Basialy } 2}, \underbrace{1+dy}_{\text{Basialy}})$



$(x,y)$, $(x+dx$

dx

Smaller number along x-axes thats NOT zero

This illustration provides **two** tangent lines, one at point $(x, y)$ and another at point $(x', y')$. Now if tangent lines are secant lines with property that the two intersected points are infinitesimally close apart, then we like to say that the *slope* of the tangent line provides us with the **exact** slope of the graph at any given point.

**Theorem 1.2.2**
Let the slope of a graph at some point $(x, y)$ be $m$. Let the tangent line to this graph at $(x, y)$ be $L$, then the slope of the tangent line $m_L = m$.

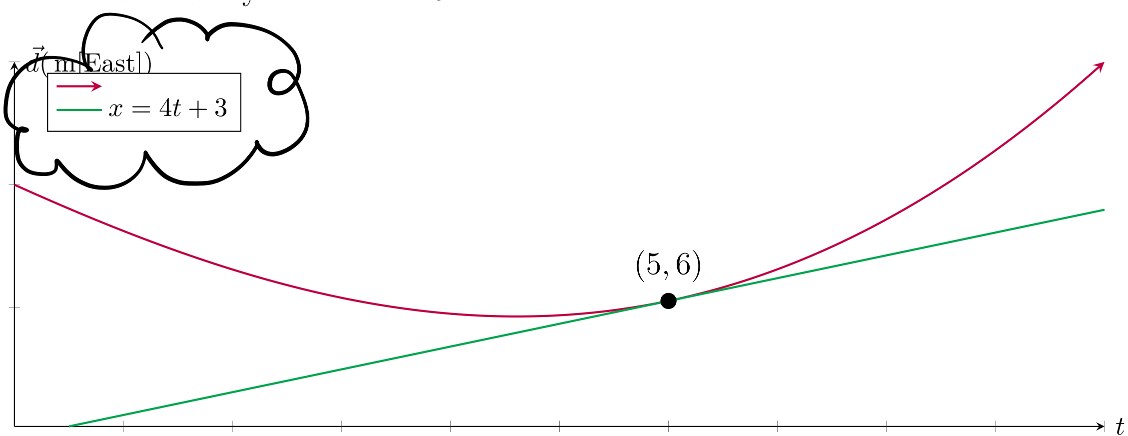
**Proof**
At this stage, I am unable to prove this theorem as it would require knowledge of Calculus.

At this point we can bridge the connection between the instantaneous velocity, $\vec{v}$ , and the idea of the tangent line. Since the slope of the tangent line provides us with the exact slope of the graph at a given point $(x, y)$, then we say that on a Pos v. Time graph, the slope of the tangent line at a given time $t$ provides us with the *instantaneous* velocity.

**Theorem 1.2.3**
Given a Pos v. Time graph, let the tangent line at some point $(t, \vec{d})$ be $L$. Then the slope of the tangent line $m_L$, is the instantaneous velocity $\vec{v}$ at that time $t$.

**Proof**
Again, at this stage, I am unable to prove this theorem as it would require knowledge of Calculus.

**Example 1.2.4**
Given the Pos v. Time graph below as well as a tangent line at time $t = 5$, compute the instantaneous velocity at time $t = 5$.






"Velocity @ time t"

↳ $\vec{v}(t)$ = Velocity @ time $t$

↳ $\vec{v}(1)$ = Velocity @ time = 1

**Solution**

$\Longrightarrow$ Thereom 1.2.3 says

$m_L = \vec{v}(t)$

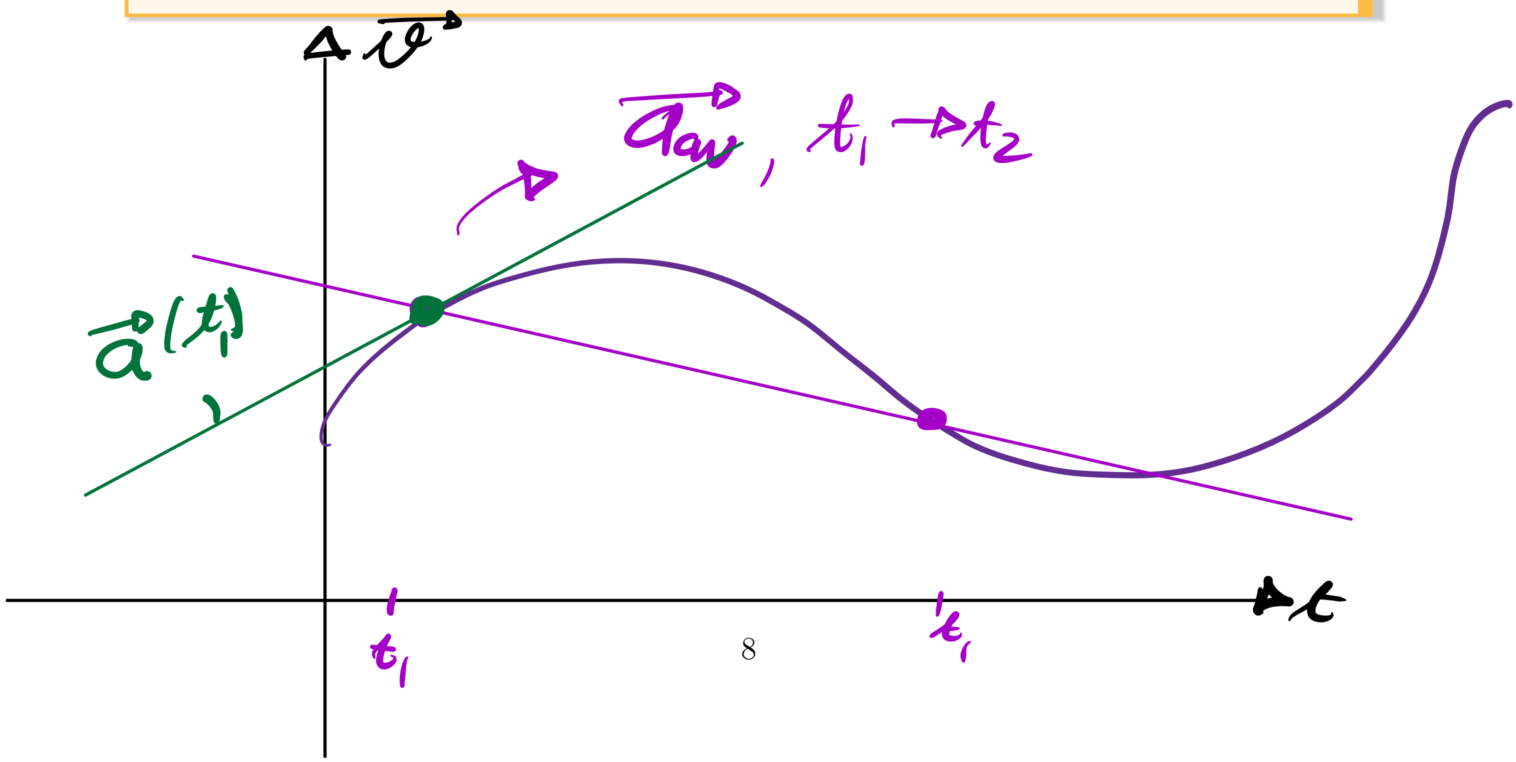
$m_L(5) = \vec{v}(5) = +4 \, m/s$

## 1.3 Extending concept to Acceleration

Given a Pos v. Time plot, we have stated that the slope of the secant line over the interval $[t_1, t_2]$ gives us the average velocity over that time interval and that the slope of the tangent line at some time $t$ gives us the instantaneous velocity at that time $t$. However this concept can be extended over to acceleration as well, that is, if we are given the *Velocity v. Time* plot, we can extract the average and instantaneous acceleration. We state their theorems below (Without proofs).

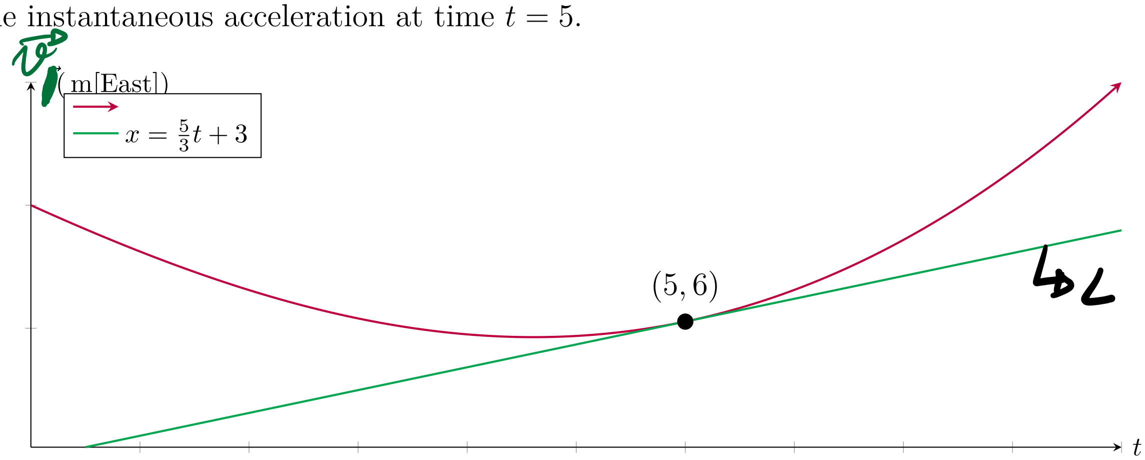
**Theorem 1.3.1**
Given a Velocity v. Time graph of a moving body, the slope of a secant line in the interval $[t_1, t_2]$ represents the average acceleration between $[t_1, t_2]$.

**Theorem 1.3.2**
Given a Velocity v. Time graph, let the tangent line at some point $(t, \vec{v})$ be $L$. Then the slope of the tangent line $m_L$, is the instantaneous acceleration $\vec{a}$ at that time $t$.



**Example 1.3.3**

Given the Velocity v. Time graph below as well as a tangent line at time $t = 5$, compute the instantaneous acceleration at time $t = 5$.




**Solution**

$\Longrightarrow$ Using the Thereom $m_L(5) = \vec{a}(5)$

$$= +\frac{5}{3}\ \mathrm{m/s^2}$$

**Example 1.3.4**

A horse had a forward velocity of $\vec{v}_i = 500\,\mathrm{m/s}$[East] and then began to accelerate at an average acceleration of $\vec{a}_{av} = 54\,\mathrm{m/s^2}$[West] over a duration of $\Delta t = 24\,\mathrm{s}$. Compute his final velocity.

**Solution**

$\longrightarrow +$ [E]

$\Longrightarrow$

$$\vec{a}_{av} = \frac{\vec{v}_f - v_i}{t_f - t_i}$$

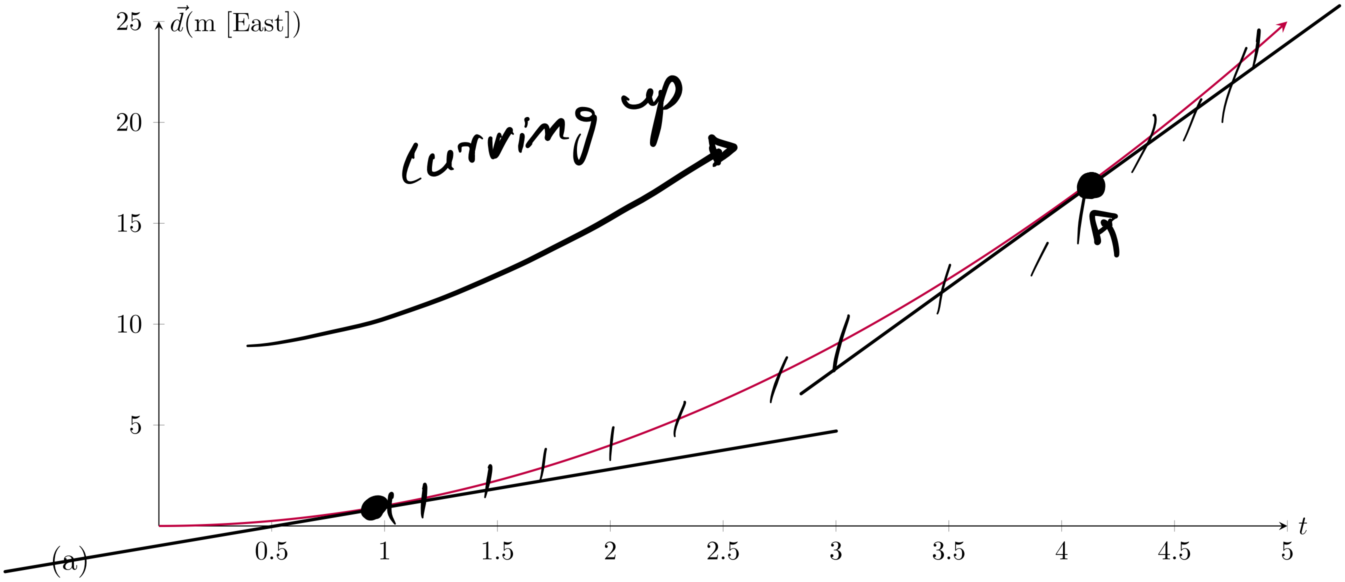
$$-54 = \frac{\vec{v}_f - 500}{24}$$

$$\boxed{\vec{v}_f = 796\ \mathrm{m/s}\ [\mathrm{West}]}$$

$$(-54)(24) = \vec{v}_f - 500$$

$$\vec{v}_f = (-54)(24) + 500 = -796\ \mathrm{m/s}$$

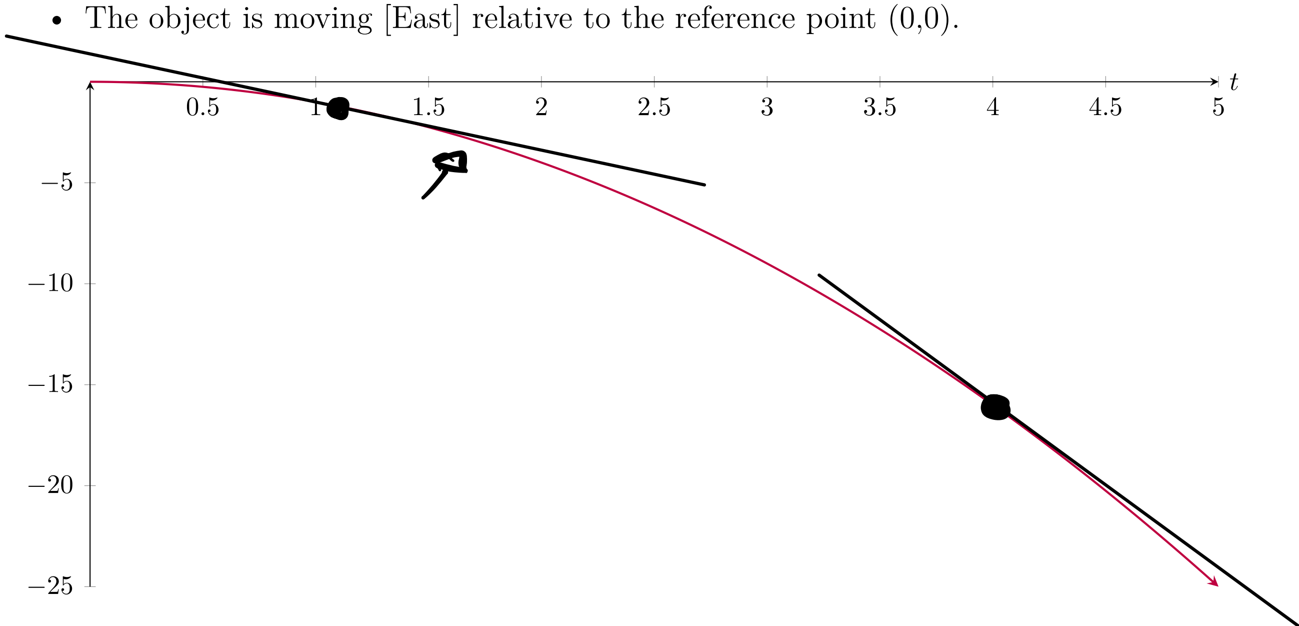
## 1.4 Non-Linear Position v. Time plots




(a)

### Properties of type (a):

- The object is experiencing **non-uniform motion**.
- The object is **speeding up**
- The object is moving [East] relative to the reference point (0,0).




(b)

### Properties of type (b):

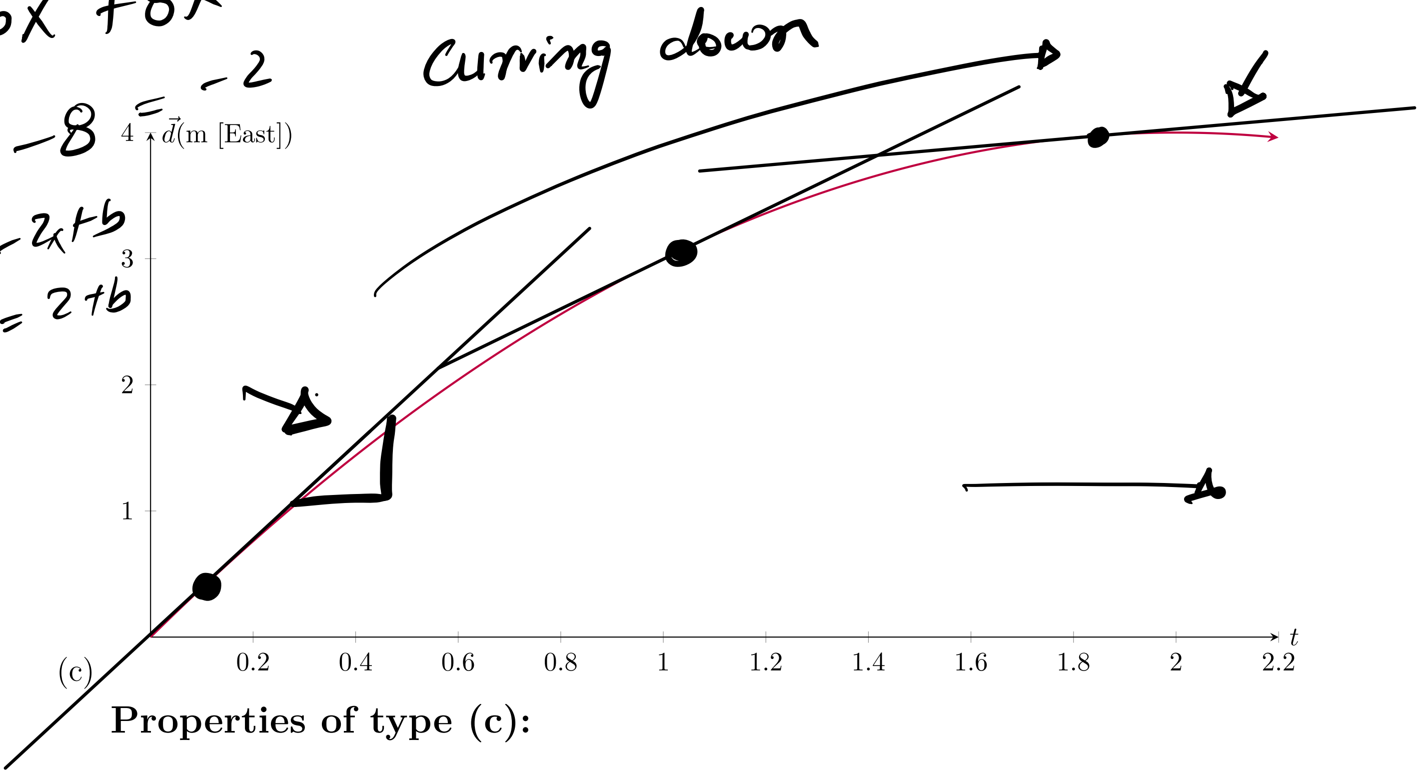
- The object is experiencing **non-uniform motion**.
- The object is **speeding up**
- The object is moving [West] relative to the reference point (0,0).

$6x^2+8x$

$6-8=-2$

$y=-2x+b$

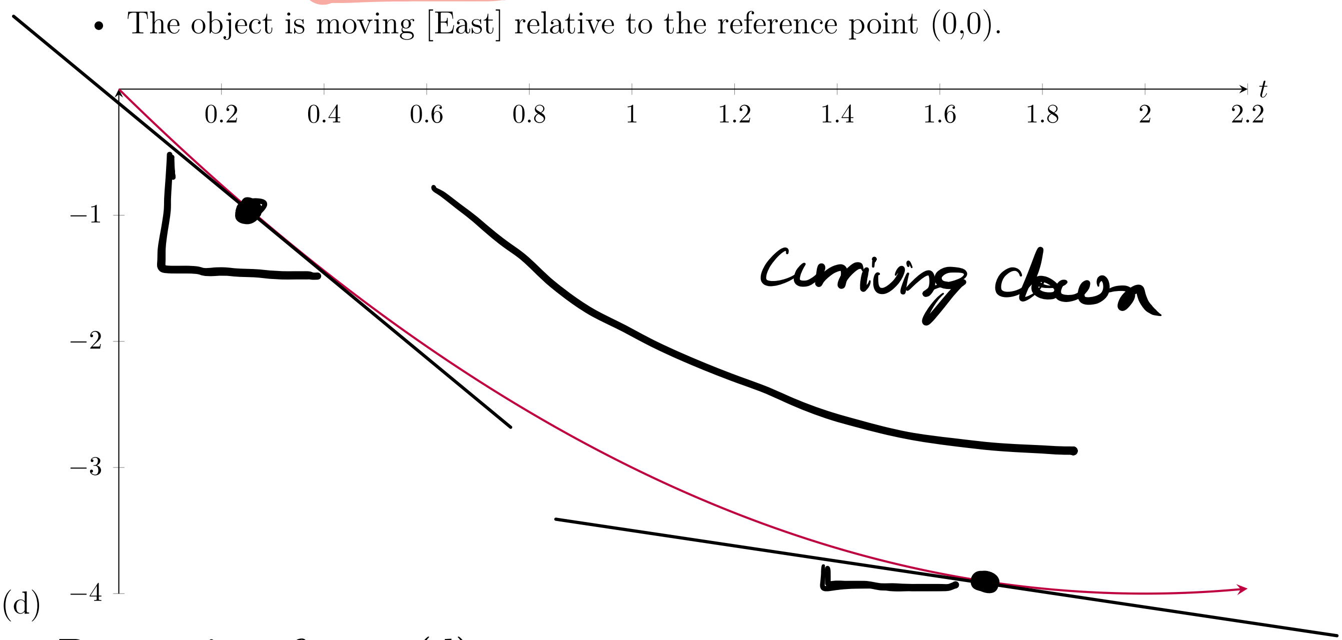
$+2=2+b$




(c)

### Properties of type (c):

- The object is experiencing **non-uniform motion**.
- The object is **slowing down**
- The object is moving [East] relative to the reference point (0,0).




(d)

### Properties of type (d):

- The object is experiencing **non-uniform motion**.
- The object is **slowing down**
- The object is moving [West] relative to the reference point (0,0).

## 1.5 Going from Velocity to displacement

If I give you a Velocity v. Time plot, could you reconstruct the Position V. Time plot for that corresponding body in motion? The answer is yes you can, the way we do so is by taking the area under of the Velocity V. Time plot at different instances in time. Suppose I wanted to know by displacement from time $[t_1, t_2]$, then I would compute the are of the Velocity V. Time graph from $[t_1, t_2]$ and then result would give me the corresponding displacement. We state this in a theorem,

**Theorem 1.5.1**
Given a Velocity V. Time graph, the area under the graph from $[t_1, t_2]$ is the displacement of the body from $[t_1, t_2]$. Or in other words,

$$\Delta \vec{d} = \text{Area}[t_1, t_2]$$ of vel v Time plot

**Remark :** Any portion of the graph below the vertical axes has **NEGATIVE AREA**. (This makes sense due to the definition of displacement).

Now using this theorem we can construct an accurate reconstruction of the corresponding Position V. Time plot by taking the area over some interval $[a, b]$. Now in the case where taking the are over the interval $[a, b]$ is not so trivial, then we may break up the interval and take the area form $a$ to some simpler end point $c$, and then take the area of $c$ to $b$, and finally sum the two pieces. This is just a simplification of a more general theorem,

**Theorem 1.5.2**
Let $a, b, c$ $x-$coordinates, then → $a \le c \le b$

$$\text{Area}[a, b] = \text{Area}[a, c] + \text{Area}[c, b]$$

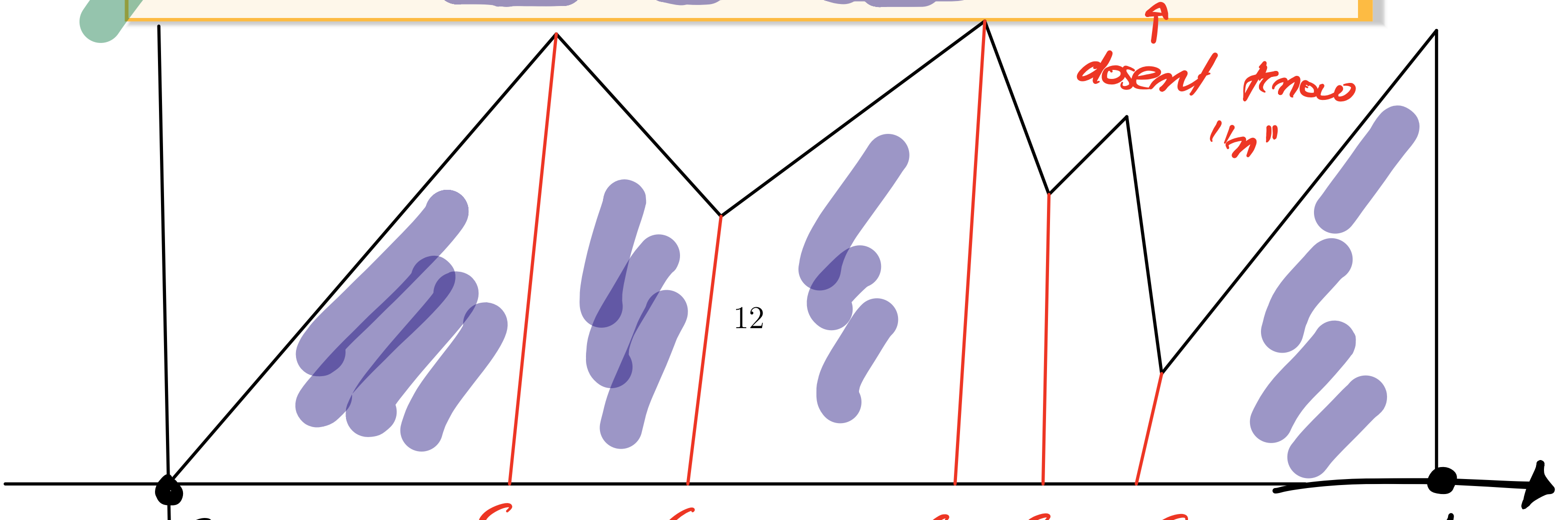
"c is in middle"

This theorem could be redefined recursively to extend the idea to multiple "bridge points" instead of just one,

**Theorem 1.5.3**
Let $a, b, c_1, c_2, \ldots, c_n$ be $x-$coordinates, then $c_1 \ldots c_n$

$$\text{Area}[a, b] = \text{Area}[a, c_1] + \text{Area}[c_1, c_2] + \text{Area}[c_2, c_3] + \cdots + \text{Area}[c_n, b]$$

dosent know "n"



$a \qquad c_1 \qquad c_2 \qquad c_3 \quad c_4 \quad c_5 \qquad b$

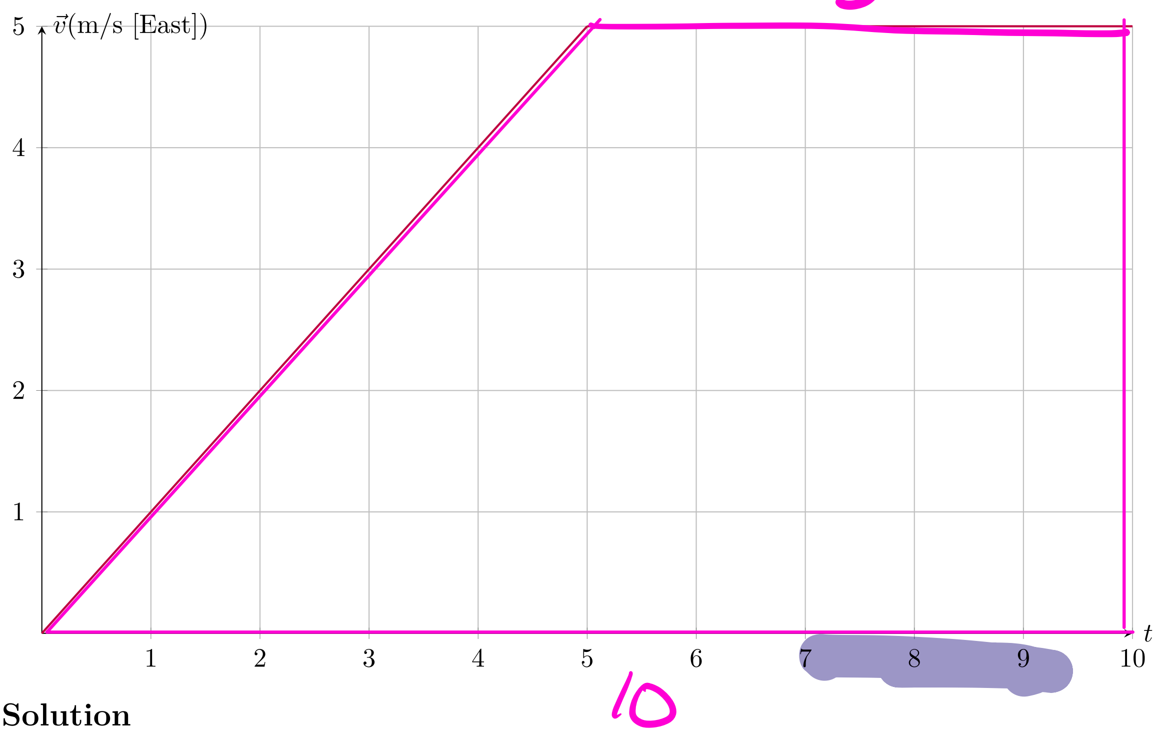
$n=5$

$c_n = c_5$

**Example 1.5.4**
Given the Velocity V. Time graph below, determine the following,

(a) The displacement from time $t = 2 \to t = 4$.

(b) The displacement from time $t = 0 \to t = 10$




**Solution**

$\Longrightarrow$

a) $\vec{\Delta d} = \text{Area}[2,4]$

$$= \left(\frac{2+4}{2}\right)2$$

$$= (3)2 = +6$$

$$\boxed{\vec{\Delta d} = 6m\ [\text{East}]}$$

b) $\vec{\Delta d} = \text{Area}[0,10]$

$$= \left(\frac{5+10}{2}\right)5$$

$$= \left(\frac{15}{2}\right)5 = \frac{75}{2}$$

$= +37.5$

$\vec{\Delta d} = 37.5\text{m [East]}$

"Area is BETWEEN curve and the x-axis"