(A-5-4 MATH 239, Winter 2021) Let G be a graph such that the edge set does not cotain three distinct vertices x, y, z such that $xy, yz \in E(G)$ and $xz \notin E(G)$. Prove that there exists $k \in \mathbb{N}$ and sets $V_1, \ldots, V_k$ such that:

- $V_1 \cup \cdots \cup V_k = V(G)$
- $V_i \cap V_j = \emptyset$ for all $i \neq j$
- For all i, we have $xy \in E(G)$ for all distinct $x, y \in V_i$
- For all $i \neq j$, we have $xy \in E(G)$ for all $x \in V_i$ and $y \in V_j$.

<u>**Solution:**</u> Note that for any given subgraph G', $|V(G')| = 3$, the corrosponding edge set will have size $|E(G)| = 3, 1, 0$. If $|E(G)| = 2$, then for some i, j, k $\{v_i, v_j\}, \{v_j, v_k\} \in E(G')$ and $v_i v_k \notin E(G')$, which violates properties of G. We construct sets $V_1, \ldots, V_k$ using the following algorithm, let $|V(G)| = n$, for each vertex $v_i \in V(G)$, let $U = N(v_i) \cup v_i$, then preform $V_j = V_j \cup U$ for some set V_j. Repeat the algorithm for the next vertex v_r, $r \geq i$, such that $v_r \not N(v_i)$ and for the next set V_{j+1}. This will give us $k - subgraphs$, $V_1, \ldots, v_k$, $k \leq n$.

Clearly $V_1 \cup \cdots \cup V_k = V(G)$ since we consider all vertices $v_i \in V(G)$ and their corrosponding neighbours. Also, note that $V_i \cap V_j = \emptyset$ for all $i \neq j$, if we let V_i, V_j repersent the neighbourhood of v_i, v_j and if there existed some $v \in V(G)$ such that $v \in V_j$ and $v \in V_i$, $i \neq j$, then the subgraph G' conataing vertices v_i, v_j, v would contain an edge set $v_i v, v v_j$, however $v_i v_j \notin E(G')$, a violation. Observe that each distinct $x, y \in V_i$ implies that $xy \in E(G)$, this follows from the fact that V_i is the neighbourhood of some vertex v, hence $xv, vy \in E(G)$, however from properties of G, if $xv, vy \in E(G)$ then $xy \in E(G)$. Also for each $i \neq j$, if $x \in V_i$ and $y \in V_j$, then $xy \notin E(G)$. This follows from the previous argument, V_i repersents the neighbourhood of some vertex v, if $x = v$, then clearly $xy \notin E(G)$, since $V_i \neq V_j$. Else, if we assume that $xy \in E(G)$, the the triplet x, y, v would produce an edge set $vx, xy \in E(G)$ and $vy \notin E(G)$, a violation of properites of G. ■