(A-7-3 MATH 239, Winter 2021) Let G be a graph with at least three vertices such that for all $v \in V(G)$, the graph arising from G by deleting v is a tree. Prove that G is a cycle.

Solution: Let $G = (V, E)$ denote the graph, since $|V(G)| \geq 3$, G is not-empty. Let $v \in V(G)$, we denote T as the tree arising from G by deleting v. Since T is a tree, it is minimally connected and hence has property $c(T) = 1$. Assume, for the purposes of a contradiction, that $e = vw \in E(G)$ is a bridge of G where $w \in V(G)$ denotes the neighbour of v, it follows that $c(G \setminus e) > c(G)$ by definition of a bridge. However since $c(T) = 1$ and $c(G) \neq 0$, the deletion of all $vw_i \in E(G)$, where each $w_i \in V(G)$ denotes the neighbour of v, has no effect on the connectivity of G, and hence $c(G) = 1$. This would also imply that each edge of G is not a bridge and is hence contained in a cycle.

We now prove that G is 2-regular. Clearly there cannot exists a $v \in V(G)$ such that $\deg(v) < 2$, else this would imply the existence of a bridge, contradicting our proof above. Suppose that there exists a vertex $v_0 \in V(G)$ such that $\deg(v_0) \geq 2$. We denote $C_1 = (v_0 \cdots v_j v_0)$ to be the cycle that v_0 is contained within. Since v_0 has some neighbour outside of C_1, there must exist some distinct cycle $C_2 = (v_0 \cdots v_k v_0)$ in which v_0 is also contained within, this follows from our proof above, that being that all edges must be contained within a cycle. Since these cycles are distinct, there is some $w \in C_1, w \notin C_2$, such that the deletion of w will preserve adjacency on C_1. However since C_1 is a cycle, a tree T does not arise. Hence G is a connected 2-regular graph, and is hence a cycle. $\blacksquare$