(P-1-2 MATH 239, Winter 2021) Let n be a positive integer. How many binary strings of length $2n+1$ have more $1's$ than $0's$?

Solution: We show that there exists a bijection between the set $\mathscr{P}$, the set of all binary strings that have more $1's$ than $0's$, and the set $\mathscr{H}$, the set of all subsets of $\{1, \ldots, 2n\}$. Let $f \colon \mathscr{H} \to \mathscr{P}$ be the function which assings to some $k-$element subset $\mathcal{S} \in \mathscr{H}$, the string $f(\mathcal{S}) = a_1 \cdots a_{2n+1}$, where $a_i = 0 \iff i \in \mathcal{S}$. Clearly $f(\mathcal{S}) \in \mathscr{P}$ since $f(\mathcal{S})$ will have at most $2n$ zero bits. It remains to show that $\mathscr{P}$ is the image of f. We define the inverse function $f^{-1} \colon \mathscr{P} \to \mathscr{H}$ by each mapping each binary string $\alpha = a_1 \cdots a_{2n+1} \in \mathscr{P}$ to the subset $\mathcal{S}$, where if $a_i = 0$ then $i \in S$. Clearly there must be at most $2n$ zero bits in α, hence S is at most a $2n$-element set. Hence we have proven that $\mathscr{P} \leftrightarrow \mathscr{H}$, which implies that $|\mathscr{P}| = |\mathscr{H}| = 2^{2n} = 4^n$.

$\blacksquare$