A binary string $a_1 \cdots a_n$ is a palindrome if it reads equivalently forward as backward, or in other words,

$$a_n \cdots a_1 = a_1 \cdots a_n$$

We consider $\epsilon$ to be a palindrome. Let the $S$ be the set of all binary strings that are plaindromes.

1. Find a formula for the number of $n$ length binary strings in the set $S$ with proof.
2. Let $\omega(a_1 \cdots a_n) = \sum_{i=1}^{n}(a_i + 1)$. Let $H$ be the set of strings in $S$ with even length. Determine $\Phi_H(x)$ with repsect to $\omega$

**Solution:**

1. Let $T$ be the set of all length $\lceil n/2 \rceil$ binary strings. Let $S_n$ be the set of all length $n$ binary strings. We claim a bijection $T \rightleftharpoons S_n$. Let $f\colon T \to S_n$ be a function by letting $\alpha = b_1 \cdots b_{\lceil n/2 \rceil} \in T$, be mapped to

$$f(\alpha) = b_1 \cdots b_{\lceil n/2 \rceil} b_{\lceil n/2 \rceil - 1} \cdots b_1$$

Clearly $l(f(\alpha)) = n$, also the string is symmetric about $b_{\lceil n/2 \rceil}$ and hence $f(\alpha) \in S_n$. Let $\gamma = b_1 \cdots b_n \in S_n$, the we choose $\beta = b_1 \cdots b_{\lceil n/2 \rceil} \in T$ to construct the map $f(\beta) = \gamma$, hence $f$ is surjective. If $f(\gamma) = f(\beta)$ then $a_1 \cdots a_{\lceil n/2 \rceil} = b_1 \cdots b_{\lceil n/2 \rceil}$, hence each bit is equal which implies $\gamma = \beta$, hence $f$ is injective. Since $f$ is both sujrective and injective, we conclude that there exists a bijection between $S_n$ and $T$, and hence $|T| = |S_n| = 2^{\lceil n/2 \rceil}$

2.

$$\begin{aligned}
\sum_{\alpha \in H} x^{\omega(\alpha)} &= \sum_{k \geq 0} \sum_{\beta \in T} x^{\left(2 \sum_{i=1}^{k}(b_i+1)\right)} \\
&= \sum_{k \geq 0} \sum_{\beta \in T} x^{(2(b_1+1+\cdots+b_k+1))} \\
&= \sum_{k \geq 0} \sum_{\beta \in T} x^{2(b_1+1)+\cdots+2(b_k+1)} \\
&= \sum_{k \geq 0} \sum_{b_1=0}^{1} x^{2(b_1+1)} \sum_{b_2=0}^{1} x^{2(b_2+1)} \cdots \sum_{b_k=0}^{1} x^{2(b_k+1)} \\
&= \sum_{k \geq 0} \sum_{b_1=0}^{1} x^{2(b_1+1)} \sum_{b_2=0}^{1} x^{2(b_2+1)} \cdots \sum_{b_k=0}^{1} x^{2(b_k+1)} \\
&= \sum_{k \geq 0} \left( \sum_{b=0}^{1} x^{2(b+1)} \right)^k \\
&= \sum_{k \geq 0} (x^2 + x^4)^k \\
&= \frac{1}{1 - x^2 - x^4}
\end{aligned}$$