3 Matrix Multiplication, Systems of Linear Equations

3.1 Matrix Multiplication

Definition 1: Matrix Multiplication

Let $A \in M_{m\times p}(\mathbb{F})$, $B \in M_{p\times n}$. We define the *matrix product* to be $C = AB \in M_{m\times n}$. The entries c_{ij} of the matrix C are determined by,

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

Where $1 \leq i \leq m$, $1 \leq j \leq n$.

Definition 2: Indentity Matrix

We define $I_n \in M_{n\times n}$ to be,

$$I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

Properties 1: Matrix Multiplication

1. For $A \in M_{m\times n}(\mathbb{F})$, $B \in M_{n\times p}(\mathbb{F})$, $C \in M_{p\times r}(\mathbb{F})$,

$$(AB)C = A(BC) \tag{Associative}$$

2. For $A \in M_{m\times n}(\mathbb{F})$, $B, C \in M_{n\times p}(\mathbb{F})$,

$$A(B + C) = AB + AC \tag{Left distributivity}$$

3. For $A, B \in M_{m\times n}(\mathbb{F})$, $C \in M_{n\times p}$

$$(A + B)C = AC + BC \tag{Right distributivity}$$

4. For $A \in M_{m\times n}(\mathbb{F})$,

$$\begin{aligned} I_m A &= A \\ A I_n &= A \end{aligned}$$

   For $Z_m = 0 \in M_{m\times m}, Z_n = 0 \in M_{n\times n}$,

$$\begin{aligned} Z_m A &= Z_m \\ A Z_n &= Z_n \end{aligned}$$

5. For $A \in M_{m\times n}(\mathbb{F}), B \in M_{n\times p})(\mathbb{F})$, and $c \in \mathbb{F}$,

$$c(AB) = (cA)B = A(cB)$$

3.2 Solving Systems of Linear Equations

Definition 3: Systems of Linear Equations, Homogenous and Inhomogenousand

A *system of linear equations* is a simultaneous set of equations of the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + x_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + x_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + x_{mn}x_n = b_m \end{cases}$$

Where the quantities $x_1, \ldots, x_n$ are the *unknowns* and the numbers a_{ij} and $b_i \in \mathbb{F}$. We would like to determine which values $x_1, \ldots, x_n \in \mathbb{F}$ satisfy all m equations. Such an assignment is known as a *solution* to the system.

If $b_1 = \cdots = b_n = 0$ then the system is refereed to as a *homogenous*. Otherwise, the system is called *inhomogeneous*

Definition 4: Elementary Row Operations

Suppose $A \in M_{m\times n}(\mathbb{F})$. An *elementary row operation* is an operation preformed on the entries of A. The three types of operations are;

1. Swap two distinct rows of A, $\text{Row}_i \leftrightarrow \text{Row}_j$
2. Multiply all entires of a row Row_i by a *non-zero* scalar $c \in \mathbb{F}$, $\text{Row}_i \to c\text{Row}_i$
3. Add a multiple of some row Row_j to row Row_i $(i \neq j)$, $\text{Row}_i \to \text{Row}_i + c\text{Row}_j$

Note that all of the operations are *invertible*. For example, if A_1 was obtained from A by preforming one of the three operations, then there exists an operation of the same type to reobtain A.

Definition 5: Row equiavalence

If B is a matrix obtained from A by preforming some finite number of operations, then we say that A and B are *row equivalent*.

The importance of row equivalency is the fact that the solutions to $A\mathbf{x} = \mathbf{0}$ is preserved in $B\mathbf{x} = \mathbf{0}$, this is primarily due to the fact that row operations are reversible and that A can be obtained from B by preforming some finite number of row operations.

Definition 6: REF, RREF

Suppose $R\ in M_{m \times n}(\mathbb{F})$. We say that R is in *row echelon form*(REF) if:

1. All zero rows are below all non-zero rows
2. In any non-zero row, the *pivot* (first non-zero entry in the row) has only zero entires below in the same column.

We say that R is in *reduced row echelon form*(RREF) if:

1. All zero rows are below all non-zero rows
2. In any non-zero row, the *pivot*(first non-zero entry in the row) is 1 and it is the only non-zero entry in the corresponding column.

Theorem 1

Suppose $A \in M_{m \times n}(\mathbb{F})$, and let $\mathbf{b}$ be a non-zero vector in $\mathbb{F}^m$.

1. The set of solutions $\mathbf{x} \in F^n$ to the homogenous system $A\mathbf{x} = \mathbf{0}$ forms a subspace of $\mathbb{F}^n$.
2. If $\mathbf{x}_p$ is some particular solution to the inhomogeneous system $A\mathbf{x} = \mathbf{b}$, then every solution to the inhomogeneous system has the form $\mathbf{y} + \mathbf{x}_p$, where $A\mathbf{y} = \mathbf{0}$. Conversely, every vector of the form $\mathbf{y} + \mathbf{x}_p$, where $A\mathbf{y} = \mathbf{0}$, a solution to $A\mathbf{x} = \mathbf{b}$