MATH 115 HANDBOOK

Abdullah Zubair
a6zubair@uwaterloo.ca

In the name of Allah, the compassionate, the merciful.

Introduction

Hello, my name is Abdullah Zubair, I am currently an undergraduate student in mathematics at the University of Waterloo. This book covers some of the fundamental concepts covered in MATH 116, a course on precalculus taught to first year engineering students at the University of Waterloo. Of course this text assumes that the reader is well versed in high school algebra, and hence will refrain from any preface or review on concepts that students should be familiar with.

Some students will find linear algebra to be tricky, this is normal. It introduces the abstract side of mathematics and builds upon it. This in itself is important for students to become accustomed to, in upper years you will slowly deviate from repeated computational, or algorithmic style questions and slowly dwell into the world of abstraction, trying to grasp spacial geometry beyond our 3 dimensions, for example. (A subtle foreshadow). Thinking abstractly will not only allow you to understand the artistic elegance of mathematics, but will also make you more proficient at solving problems. To draw upon an analogy, CS students are often always required to implement some algorithm to solve some problem within a given time and space constraint. For example, given a list of distinct numbers, sort the list in ascending order in linear time (Appendix (**Insert Number**)). Such problems are great examples of the flavor of mathematics I am alluding to.

Throughout this book you will encounter the following pattern, "Here is a new idea (definition), here are some things that are true about it (Theorems), here are some things that are true as a consequence (Corollary's, Lemmas)", this is something you will slowly begin to realize throughout the remainder of your studies in mathematics (or maybe something you have already noticed). A **statement** is a sentence with a definite truth value (take my example from the previous paragraph). A **definition** defines a new idea/concept, or explains an abstract object, for example I will eventually define the *function* as a mapping operation. A **Theorem** is almost always related to the corresponding definition, it is a statement that asserts something true or false about the object defined in the definition. **Lemmas** are "mini" theorems that are often used as building blocks for theorems, and hence precede them. **Corollary's** are statements that happen to be true or false as a consequence of a theorem, and hence follow them.

In this course, you will not only encounter, but will also formulate **proof's** for ideas, theorems and propositions. This is often a hurdle for most students, and I can at most empathize with your grievance, however, as the old saying goes, *Whatever doesn't kill you, makes you stronger - Friedrich Nietzsche*. Going forward, reading and solving is going to be crucial skill to develop, not only will it enhance your problem solving skills, it often tests your comprehension of the material and weather or not you really understand it *like the back of your hand*. For example, after the first (**INSERT NUMBER**) chapters, I would like you to be able to prove the following statement,

$$\forall x \in C, \exists y \in C, xy \in \mathbb{R}$$

Note: Software engineering students may feel free to skip chapters **INSERT NUMBER:**.

Chapter 1

Sets

1.1 Introduction

Definition 1

Sets are defined to be a collection of objects in a pair curly braces $\{\}$.

To define what we mean by an *object* can be complicated, and hence I will refer to objects as anything that has been previously defined or "tangible" (although even this can get a little philosophical). For example if the object is an integer, then we can build a set with some integers, take $\{2, 3, 44, 5\}$ as an example or take the following tangible objects $\clubsuit, \heartsuit, \triangle$ and build a set with them $\{\clubsuit, \heartsuit, \triangle\}$. There are some key properties of sets to note. **Order does not matter**, meaning any rearrangement of the objects in a set yields the same set, for example we say that $\{1, 2, 3, 4, 5\} = \{2, 3, 4, 5, 1\} = \{1, 2, 3, 5, 4\}$, etc. Also, **duplicates are not allowed** so whenever we observe a duplicate object, we immediately remove it and yield an equivalent set, so $\{1, 2, 2, 3, 4\} = \{1, 2, 3, 4\}$. We say that the **cardinality** of a set $\mathcal{S}$ is the number of elements (or objects) in the set, and denote the quantity as $|\mathcal{S}|$, for example if $\mathcal{S} = \{1, 2, 3, 4, 5\}$ then $|\mathcal{S}| = 5$.

Definition 2: Empty set

We denote $\emptyset$ as the set with no elements, and call it the **empty set**. This implies $|\emptyset| = 0$.

Notation: If some element x is contained within a set $\mathcal{S}$, then we say that x is an element of $\mathcal{S}$ and write $x \in S$. Consequently, if some element y is **not** an element of $\mathcal{S}$, then we say that y is not an element of $\mathcal{S}$ and write $y \notin \mathcal{S}$.

Take for example the set $S = \{3, 4, \heartsuit, 54\}$, the cardinality is $|S| = 4$ and we observe that $\heartsuit \in S$, however $-3 \notin S$.

1.2 Common Sets

There are a few common recurring sets that are the building blocks for the objects we will manipulate throughout this book. We list them here, (note that the $\ldots$ notation indicates a continuation following the logical pattern)

1. $\mathbb{Z}$ denotes the set of all integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
2. $\mathbb{R}$ denotes the set of all real numbers (rational or irrational).
3. $\mathbb{Q}$ denotes the set of all rational numbers.
4. $\mathbb{Z}^+$ denotes the set of all non-negative integers $\mathbb{Z}^+ = \{0, 1, 2, 3, \ldots\}$
   Remark: Some texts will not allow 0 to be apart of $\mathbb{Z}^+$.
5. $\mathbb{R}^+$ denotes the set of all non-negative positive real numbers.
6. $\mathbb{Z}^-$ denotes the set of all negative integers $\mathbb{Z}^+ = \{-1, -2, -3, \ldots\}$
7. $\mathbb{R}^-$ denotes the set of all negative real numbers.

The **universe of discourse** denoted $\mathcal{U}$, is the set of all objects we may be interested in a given scenario. In this book, we are mostly always working with the set $\mathbb{R}$, and hence the universe of discourse will almost always be $\mathcal{U} = \mathbb{R}$. (There may be a few special cases were we explicitly differentiate).

Definition 3: Set-Builder

We say that the set $\{x \in \mathcal{U} \colon \textbf{statement}\}$ is the set of all elements x in $\mathcal{U}$ such that the **statement** is true for x. (The semicolon means "such that", some texts will use a $|$ instead)

Example 1

1. Write out all of the elements of set $S = \{n \in \mathbb{Z} \colon 0 \leq n \leq 4\}$.
2. Write out all of the elements of the set $H = \{x \in \mathbb{Z} \colon -x > 0\}$

Solution 1

1. This is the set of all integers n that are between $0, 4$ inclusive, hence $S = \{1, 2, 3, 4\}$.
2. Careful observation reveals that this set is precisely $\mathbb{Z}^-$. This is the set of all x such that the product $(-1)x$ is positive, hence x must have be negative.

Definition 4: Subset

We say that a set B is a subset of a set A, and denote $B \subseteq A$, to mean that all of the elements of B are in A, that is, if $x \in B$ then $x \in A$.

Remark: If $B \subseteq A$, then this allows for the possibility that $B = A$. The notation $B \subset A$, guarantees that $B \neq A$, in this case we call B a proper-subset of A. Consequently if B is *not* a subset of A, we right $B \nsubseteq A$.

Definition 5: Union of sets

Let A and B be sets, we define the **union** of A and B to be,

$$A \cup B = \{x \in \mathcal{U} \colon x \in A \text{ OR } x \in B\}$$

.

Remark: Another interpretation is that the union of two sets is the set of all elements from the set A and the set B placed into a single set, however if a given element in $\mathcal{U}$ appears in both A and B, we include it once in the union. For example, if $A = \{1, 2, 3\}$ and $B = \{2, 3, 4, 5\}$, then $A \cup B = \{1, 2, 3, 2, 3, 4, 5\}$, after removing duplicates we obtain $A \cup B = \{1, 2, 3, 4, 5\}$.

Definition 6: Intersection of sets

Let A and B be sets, we define the **intersection** of A and B to be,

$$A \cap B = \{x \in \mathcal{U} \colon x \in A \text{ AND } x \in B\}$$

.

Remark: Essentially the intersection of two sets is the set of all elements that belong to *both* A and B. In our previous example, $A \cap B = \{2, 3\}$.

Definition 7: Set difference

Let A and B be sets, we define the **set-difference** of A and B to be

$$A - B = \{x \in \mathcal{U} \colon x \in A \text{ AND } x \notin B\}$$

Remark: Informally, $A - B$ is the set of all elements that belong to A but *do not* belong to B. In our example from earlier, $A - B = \{1\}$, however $B - A = \{4, 5\}$.

Definition 8: Compliment

Let A be a set, we define the **compliment** of A, and denote it $\overline{A}$, to be,

$$\overline{A} = \{x \in \mathcal{U} \colon x \notin A\}$$

We may also define it as $\overline{A} = \mathcal{U} - A$

Remark: Again more informally, $\overline{A}$ is the set of all elements in the universe of discourse, that are *not* in A.

Definition 9: Disjoint

Let A and B be sets, if $A \cap B = \emptyset$, then we say that A and B are **disjoint**.

Remark: This essentially means that A and B have *no* common elements.

Definition 10: Set Equality

Let A and B be sets, if $A = B$, then $A \subseteq B$ and $B \subseteq A$.

Remark: This makes sense, if $A = B$, then all of the elements contained in A are also contained in B, or more concisely, $A \subseteq B$. Since an equality can be read both ways, if $A = B$ then $B = A$, this implies that all of the elements contained in B are also contained in A, or in other words $B \subseteq A$.

Chapter 2

Logic and Proofs

Chapter 3

Functions

3.1 Introduction

Definition 11: Function

Let X, Y be sets. A function f is a rule which assigns to each element $a \in A$ a unique element $b \in B$. Formally, we say that $f\colon X \to Y$, and $f(a) = b$.

A function f is an abstract mapping operation, it takes elements from set a A for which the function is well defined, called the **domain**, and maps it to elements in a set B, called the **co-domain** (I will formalize these definitions below). This mapping operation is more concisely written as $f\colon A \to B$, where if $x \in A$, then $f(x) = y \in B$, where we call y the **image point** and x the **preimage**. You may refer to the example below to help illustrate the concept, here we let $f(x) = x^2$, $A = \{-2, -1, 1, 2\}$ and $B = \{1, 2, 4, 6\}$

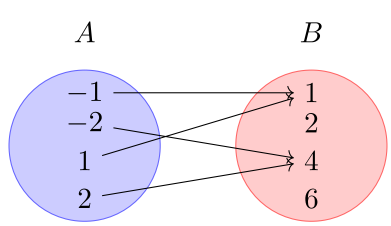


Figure 3.1: Mapping diagram of f

At this point you may be inquiring whether or not this definition of a function has any correspondence to functions you may have dealt with from highschool, the answer is yes. If we take a classical example, let $f(x) = x^2$, then we say that $f\colon \mathbb{R} \to \mathbb{R}$, clearly we are free to take x from $\mathbb{R}$, and x^2 also produces an element in $\mathbb{R}$, hence $\mathbb{R}$ *to* $\mathbb{R}$. Therefore this definition, albeit a more intricate one, is much more of an intrinsic definition of a function. You will find this definition *especially* convenient during your studies in linear algebra.

Definition 12: Range (Image) and Domain

Let f be a function,

1. The **domain** D of f is the set of all x such that $f(x)$ is well defined.
2. The **range** (or image) R of f is defined as

$$R = \{f(x) \colon x \in D\}$$

The range of a function will always be a subset of the co-domain, in other words $R \subseteq B$.

Example 2

State the domain and range of f in Figure 1.1

Solution 2

Clearly the domain $D = A$, the range can be computed by determining all image points of f. We list them, $f(-1) = (-1)^2 = 1, f(-2) = (-2)^2 = 4, f(1) = 1^2 = 1, f(2) = 2^2 = 4$. Hence, $R = \{1, 4\}$

Example 3

State the domain for $f(x) = \sqrt{-3x - 9}$

Solution 3

We determine the set D of all possible values of x such that $f(x)$ is well defined. This occurs precisely when $-3x - 9$ is a non-negative value, hence we require $-3x - 9 \geq 0$ which would imply that $x \leq -3$. Hence the domain is any real number x such that $x \leq -3$, or to put more concisely using set-builder notation $D = \{x \in R \colon x \leq -3\}$ (Try to determine what the universe of discourse was here and the **statement** used).

Example 4

State the domain and range for $f(x) = \frac{-3}{4}(x + 3)^2 - 5$.

Solution 4

Clearly f is well defined for any $x \in \mathbb{R}$, hence $D = \{x \in \mathbb{R}\}$. Determining the range of f can be tricky, I find the best way to determine the range is to quickly sketch the function, this is a parabola that has $y-$intercept $= -5$ and points *downwards*, hence $R = \{y \in \mathbb{R} \colon y \leq -5\}$.

3.2 Invertibility

Definition 13: Identity function

The **identity function** corresponding to the set A, is defined to be:

$$i_A\colon A \to A,\ \ i_A(x) = x$$

Remark: Essentially the identity function takes all elements from the domain A, and maps it to themselves. So each image point of f corresponds to the preimage of f.

Definition 14: Composition

Let $f\colon A \to B$ and $g\colon C \to A$ be functions. The **composition** of f and g, denoted $f \circ g$, is the function defined as

$$f \circ g\colon C \to B,\ \ (f \circ g)(x) = f(g(x))$$

Remark: Since the domain f is the set of all elements for which f is well defined, we require $g(x)$ to be well defined, this occurs precisely when $x \in C$, hence the domain must be C. The co-domain is of course B, since f maps elements to the set B, however the range R_C of the composition is $R_C = \{f(g(x))\colon x \in C\}$. Note that since we first determine $g(x)$ then $f(g(x))$, it is *not* always true that $f \circ g = g \circ f$.

Definition 15: Injective

Let $f\colon A \to B$ be a function, then we say that f is **injective** to mean that for every pair of elements $x, y \in A$, if $x \neq y$ then $f(x) \neq f(y)$.

Informally, if f is injective, then no two distinct elements in the domain can be mapped to the same element in the co-domain.

Remark: Given a graph of function, we can preform the so called "horizontal line test" on the graph by determining whether or not there exists some line segment we can draw through the function such that the segment intersects the function at least twice. If it does, then the function is *not* injective.

Definition 16: Surjective

Let $f\colon A \to B$ be a function, then we say that f is **surjective** to mean that for every element $y \in B$, there exists some $x \in A$ such that $f(x) = y$.

Again, more informally this definition says that if f is surjective then every element in the co-domain can be mapped by choosing some appropriate element from the domain as the independent variable.

Lemma 1

Let $f\colon A \to B$ be a function and let R denote the range of f. If f is surjective, then $R = B$.

Proof. Recall that we must show that $R \subseteq B$, and $B \subseteq R$. We start with the former, by the definition of range, $R \subseteq B$. Now with the latter, assume that f is indeed surjective, now let $y \in B$, then by the definition of surjective, there must exist some element $x \in A$ such that $f(x) = y$. But then y is an image point contained within the range of f, hence $y \in R$. ■

Example 5

True or false, are the following functions injective? (Provide justification if your choice is false)

1. $f(x) = (x-2)^2$
2. $f(x) = \sqrt{x-4}$
3. $f(x) = x^3$
4. $f(x) = |x|$

Solution 5

1. *False* We provide a counterexample, clearly $1 \neq 3$ however $f(1) = (1-2)^2 = 1 = f(3) = (3-2)^2 = 1$, hence f is not injective.
2. *True*
3. *True*
4. *False* Notice that $f(-1) = |-1| = 1 = f(1) = |1| = 1$.

Recommendation: Repeat Example 5 using the graphs of the functions.

Definition 17: Invertibility

Let $f\colon A \to B$ be a function. If f is both injective and surjective, then we say that f is **invertible** and an inverse function $f^{-1}\colon B \to A$ exists such that:

$$f(f^{-1}(x)) = i_B(x) = x$$

$$f^{-1}(f(x)) = i_A(x) = x$$

.

The inverse function provides us with a remapping operation for f, that is, for $x \in A$ if $f(x) = y \in B$, then $f^{-1}(y) = x \in A$, a remap. The definition expresses this idea more succinctly, if $f(x)$ is mapped to some element, then the remapping f^{-1} should reproduce x. more Notice that a consequence of invertibility is that the co-domain and domain will have the same cardinality, or in other words, if $f: A \to B$, then $|A| = |B|$. (You should convince yourself why this is true). If f is invertible, then sometimes we may say that f is a *one-to-one correspondence*, to mean that each element in the domain can be paired uniquely to a corresponding element in the co-domain.

Algorithm 1: Determining the inverse function

Let $f(x)$ be a function, assume that f is invertible. We determine $f^{-1}(x)$ as follows,

1. Let $f(x) = y$
2. Solve for x on the LHS of the expression in (1)
3. Replace $x \to f^{-1}(x)$ and $y \to x$, from the expression deduced in step (2).

Remark: If $f(x)$ is a function of x, then by the way we understand inverses, the inverse function f^{-1} is a remapping operation, hence it takes elements from the co-domain and remaps them to domain, so why is the inverse function also defined in terms of x, where x represents elements from the domain?. The answer is, technically it doesn't, it is merely a notation preference. It is true that f^{-1} will take elements from the co-domain and remap them to the domain, however mathematicians like to use x as an independent variable for functions, and clearly f^{-1} is a function, so we would like to denote $f^{-1}(x)$ instead of $f^{-1}(y)$. If you think this adds confusion to the definition, then I would concur.

Example 6

True or False, are the following functions invertible? If true, provide the inverse function.

1. $f(x) = x^2 + 2x$
2. $f(x) = \frac{x+1}{x-1}$

Solution 6

1. *False* (Check out its graph, or convince yourself why it fails to be injective)
2. *True*, let $f(x) = y$,

$$\begin{aligned} \frac{x+1}{x-1} &= y \\ x+1 &= y(x-1) \\ x - yx &= -y - 1 \\ x(1-y) &= -y - 1 \\ x &= \frac{y+1}{y-1} \\ f^{-1}(x) &= \frac{x+1}{x-1} \end{aligned}$$

3.3 Properties of functions

Definition 18: Even and Odd functions

Let $f(x)$ be a function, then we say that f is **even** if for each element x in the domain,

$$f(-x) = f(x)$$

We say that f is **odd** to mean that for each element x in the domain,

$$f(-x) = -f(x)$$

This property of a function tells us more about its properties as a graph rather than an abstract mapping operation, in fact aside from applied mathematics, we aren't really concerned with the parity of a function. However for engineers this property is quite important, and useful. If f is *even*, then each independent variable $x < 0$ has the same image as $x \geq 0$, hence the function will be symmetric about the vertical axis. In the *odd* case, we say that the function is symmetric about the origin.
Application: If an engineer would like to design a shell like structure, he/she would like it to be symmetric about the center, hence a depiction of $y = cx^2$ in your preferred design software should get the job done. (Of course a revolution afterwards for a 3D illustration)
Remark: A function is *either*, even, odd or neither.

Properties 1: Transformations

Let $f(x)$ be a function, let $c \in \mathbb{R}$, $c > 0$ be a constant. Let $y'(x)$ denote the function obtained from a **transformation** of $f(x)$ by c. The following is a list of possible transformations preformed on f,

1. $y'(x) = f(x) + c$ corresponds to a upward shift.
2. $y'(x) = f(x) - c$ corresponds to a downward shift.
3. $y'(x) = f(x - c)$ corresponds to a rightward shift.
4. $y'(x) = f(x + c)$ corresponds to a leftward shift.
5. $y'(x) = f(-x)$ corresponds to a reflection about the vertical axis.
6. $y'(x) = -f(x)$ corresponds to a reflection about the horizontal axis.
7. $y'(x) = f(x/c)$ corresponds to a horizontal stretch.
8. $y'(x) = f(cx)$ corresponds to horizontal compression.
9. $y'(x) = cf(x)$ corresponds to a vertical stretch.
10. $y'(x) = \frac{f(x)}{c}$ corresponds to a vertical compression.

Again, these properties tell us more about the properties of the graph of the function rather than its algebraic properties. You should convince yourself why such correspondences make sense. Ill provide a sketch of a proof for property (3). *Sketch of proof,* If $y'(x)$ differs relative to $f'(x)$ by only a rightward shift, then they should have the same image points, for $y'(x)$ and $f(x)$ to have the same image, we would take the image of y' at $y'(x+c) = f((x+c)-c) = f(x)$. Hence each element from the domain of $f(x)$ is shifted by $+c$ to obtain y'.

Example 7

Determine whether the following functions are even, odd or neither.

1. $f(x) = 5x^6 - 2x^4 + x^2 + 1$
2. $f(x) = x^5$
3. $f(x) = \sqrt{|x| + x}$
4. $f(x) = \frac{x|x|}{3+2x^2}$

Solution 7

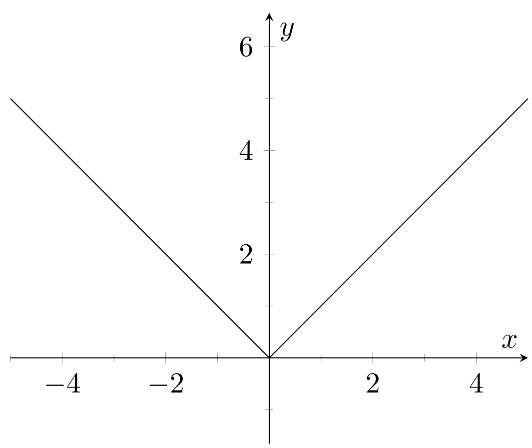
1. $f(-x) = 5(-x)^6 - 2(-x)^4 + (-x)^2 + 1 = 5x^6 - 2x^4 + x^2 + 1 = f(x)$, hence f is even.
2. $f(-x) = (-x)^5 = -x^5 = -f(x)$, hence f is odd.
3. $f(-x) = \sqrt{|-x| - x} = \sqrt{|x| - x}$, hence f is neither odd or even.
4. $f(-x) = \frac{-x|-x|}{3+2(-x)^2} = \frac{-x|x|}{3+2x^2} = -f(x)$, hence f is odd.

3.4 Piecewise Functions

Sometimes we are interested in defining a function f on given collection of intervals $I_1, \ldots, I_n$, and we say that for each interval I_k, $f(x) = g_{I_k}(x)$, or in other words, on the interval I_k, f takes on the corresponding function specified on that interval. Engineers find the piecewise form to be quite convenient for modeling series of synchronized applications (****try a bettter word****) of some piece of machinery or physical apparatus. Therefore we stress the importance of these functions in this chapter. We start with the **absolute value function** which is piecewise defined,

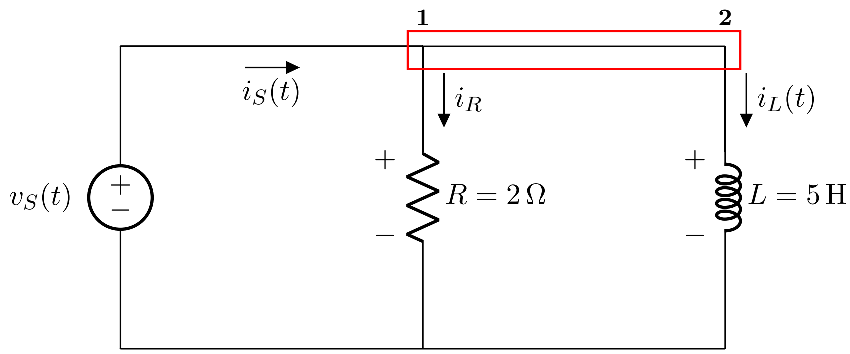
$$|x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$$

This is a satisfying definition for the absolute value function from a both geometric and algebraic perspective. From an algebraic perspective, for all $x < 0$, we would like $|x|$ to return x positive, the function $g_1(x) = -x$ satisfies this requirement. If $x \geq 0$, then we just return x, the latter interval satisfies this as well. From a geometric perspective, we refer to the figure below and observe that the piecewise definition of $|x|$ is a linear function on both intervals, the reflected version of $y = x$ for $x < 0$.



3.4.1 Circuit Applications

As engineering students, you may be inclined to an application of piecewise functions, perhaps for you to satisfy the following inquiry most engineering students have towards mathematics, "Where will I ever need this?". Let us consider the example of the circuit in the diagram below, with a single voltage source, resistor and inductor. We give the current flow through the inductor modeled as a piecewise function,



$$i_L(t) = \begin{cases} \frac{t}{5} & 0 \leq t \leq 1 \\ \frac{1}{5} & 1 \leq t \leq 3 \\ \frac{-t}{5} + \frac{4}{5} & 3 \leq t \leq 4 \\ 0 & 4 \geq 0 \end{cases}$$

Suppose we wanted to model the voltage of the source $v_S(t)$ as a function of time. We could apply Kirchoff's Voltage Law to deduce that $v_S(t) = v_L(t)$, where $v_L(t)$ is the time derivative of $L(i_L(t))$ (L denotes the inductance of the coil). Hence,

$$v_S(t) = v_L(t) = L\frac{di}{dt} = \begin{cases} 1 & 0 \leq t \leq 1 \\ 0 & 1 \leq t \leq 3 \\ -1 & 3 \leq t \leq 4 \\ 0 & 4 \geq 0 \end{cases}$$

Similarly, we may be interested in modeling the current flow of the source as a function of time. To do so we apply Kirchoff's Current Law at the node colored in red to deduce that $i_S(t) = i_R + i_L(t)$. The current through the resistor is given by $v = iR$, where the voltage is equivalent to the voltage of the source, since they are connected in parallel. Hence,

$$i_S(t) = i_R + i_L(t) = \frac{v_S(t)}{R} + i_L(t) = \begin{cases} \frac{t}{5} + \frac{1}{2} & 0 \leq t \leq 1 \\ \frac{1}{5} & 1 \leq t \leq 3 \\ \frac{-t}{5} + \frac{3}{5} & 3 \leq t \leq 4 \\ 0 & 4 \geq 0 \end{cases}$$

Remark: A subtle point to mention, this example assumes you have the ability to differentiate linear functions.

3.5 Heaviside Functions

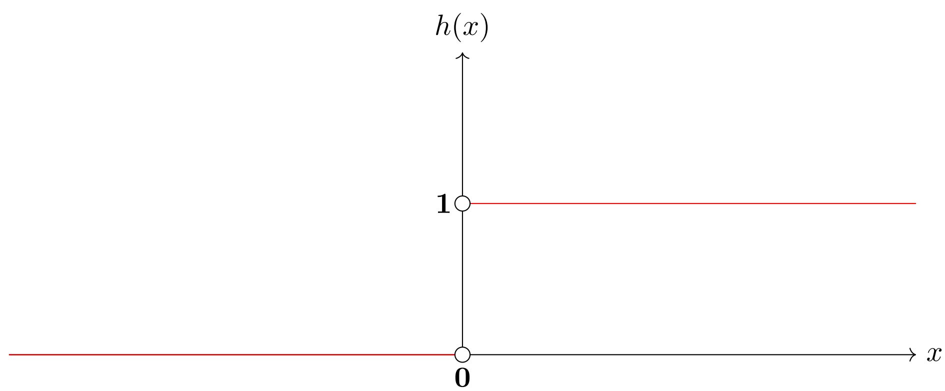
The Heaviside function is an important tool in engineering and physics for modeling functions, specifically piecewise functions. Recall the importance of piecewise functions discussed in the previous section, we would like a concise way to write a piecewise function that may perhaps tell us more information about the function. We define the heaviside function below,

Definition 19: Heaviside Function

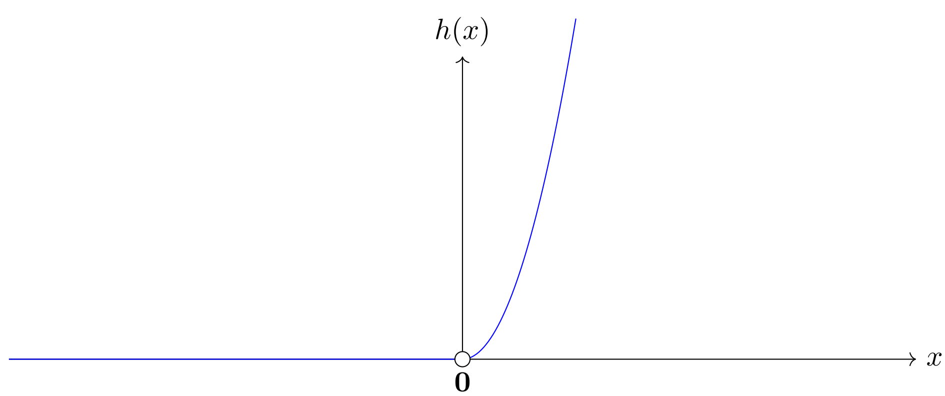
The **Heaviside function** is defined as,

$$h(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

Sometimes it may be convenient to give the graph of the heaviside function,



We interpret the heaviside function as an "on/off" function, a light bulb for functions so to speak. It toggles a function to some non-zero value if the argument of the heaviside function is positive and toggles a function to zero if the argument is negative. For example, consider the function $g(x) = x^2 h(x)$, we give the graph of the function below,



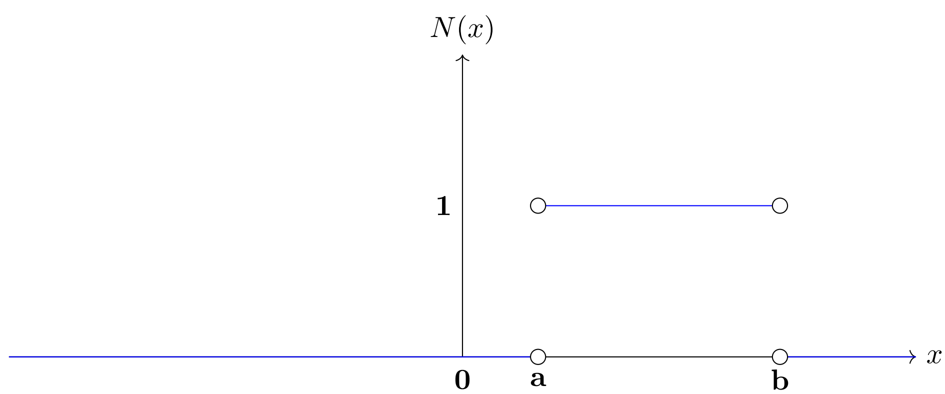
We observe that $g(x) = x^2$ when $h(x)$ is "toggled on", which occurs precisely when $x > 0$, similarly $g(x) = 0$ when $h(x)$ is "toggled off". $(x < 0)$. We could also describe the piecewise form of $g(x)$,

$$g(x) = \begin{cases} x^2 & x > 0 \\ 0 & x < 0 \end{cases}$$

Sometimes we may be interested in "toggling" functions "on" over some interval (a, b). We could do so by introducing an offset to the argument contained within the heaviside function. For example, if we would like to model x^2 over $(2, \infty)$, we would require the argument of the heaviside function to be positive over $(2, \infty)$, this occurs precisely when the argument is $x - 2$, hence $f(x) = x^2 h(x - 2)$ would be an accurate model for the desired function (One could instead argue that a rightward shift of the heaviside function is required, hence we use $x - 2$).

The problem becomes slightly complicated when we would like to model functions over some finite (non-infinite) interval, we would have to introduce some destructive superimposition to the heaviside function. Essentially, we would like the heaviside function to be toggled on over some interval (a, b), where $a < b$ are finite values. We know from before that $h(x - a)$ is toggled on for all $x > a$, similarly, $h(x - b)$ is toggled on for all $x > b$, hence if we subtract the latter from the former, we would obtain a model of the heaviside function that allows us to toggle functions over the interval (a, b). We formalize this below,

$$N(x) = h(x - a) - h(x - b) = \begin{cases} 0 & x < a \\ 1 & a < x < b \\ 0 & x > b \end{cases}$$



Reacll that we claimed that this model of the heaviside function should toggle any function f over (a, b), we describe this systematically below,

$$f(x)N(x) = \begin{cases} 0 & x < a \\ f(x) & a < x < b \\ 0 & x > b \end{cases}$$

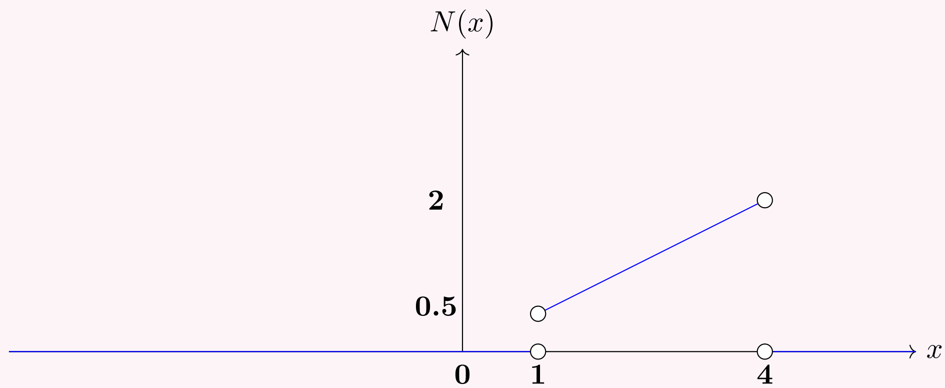
Example 8

Provide the piecewise form of $f(x) = x/2$ over $(1, 4)$ as well as its graph.

Solution 8

We take the product $f(x)\,[h(x-1) - h(x-4)]$ to obtain,

$$f(x)\,[h(x-1) - h(x-4)] = x\,[h(x-1) - h(x-4)] = \begin{cases} 0 & x < 1 \\ x & 1 < x < 4 \\ 0 & x > 4 \end{cases}$$



Example 9

Let $f(x)$ be piecewise defined,

$$f(x) = \begin{cases} 2x - 1 & x < 1 \\ \ln(x) & 1 < x < 4 \\ x^3 & x > 4 \end{cases}$$

Denote the corresponding equivalent algebraic form of $f(x)$ using the heaviside functions.

Solution 9

$$\begin{aligned} f(x) &= (2x-1)h(1-x) + \ln(x)\left[h(x-1) - h(x-4)\right] + x^3 h(x-4)\right] \\ &= (2x-1)h(1-x) + \ln(x)h(x-1) - \ln(x)h(x-4) + x^3 h(x-4) \\ &= (2x-1)h(1-x) + \ln(x)h(x-1) + (x^3 - \ln(x))h(x-4) \end{aligned}$$

Remark: Note that $2x - 1$ was toggled for all $x < 1$, or for all $-x > -1$, hence we would require the argument of the corresponding heaviside function to be $-x + 1$.(Remember the heaviside function is always toggled on whenever the argument is positive!)