(A-7-2a) MATH 239, Winter 2021) Let G be a graph, let $u, v \in V(G)$, $u \neq v$. Let P and Q be paths from u to v with no edges in common (that is, no edge of G appears in both P and Q). Prove that there is no edge of P that is a bridge.

<u>Solution:</u> Let $G = (V, E)$ denote a graph, let $u, v \in V(G)$. Let $P = (w_0, \dots, w_k)$ and $Q = (z_0, \dots, z_l)$ be distinct paths form vertices's $u = w_0 = z_0$ to $v = w_k = z_l$, such that no edges are shared between P and Q. We let $0 < b \leq k$ denote the smallest index for which w_b is on Q. Let $0 < c \leq l$ denote the index such that $w_b = v_c$. Note that the path $P' = (w_0, \dots, w_b)$ and $Q' = (z_0, \dots, z_c)$ are pairwise distinct since P, Q were paths in G. Hence we denote the closed walk

$$C = (w_0 \dots w_b z_{c-1} \dots z_0)$$

Since all vertices's are distinct expect for $z_0 = w_0 = u$, and the cycle contains at least a cycle of length 3 ($w_0 \neq w_b$) it follows that C is a cycle. If $b \neq k$, then we may simply preform induction on the next smallest index b' up to k to show that each vertex of P is contained within some cycle and hence has no bridges. (Proof used similar ideas used in the proof to proposition 6.7, New Notes). ∎