Let $\mathcal{S}$ denote the set of all compositions (x, y, z) satisfying $|x - y| = |y - z| = 1$. Determine $\Phi_{\mathcal{S}}$ with respect to the weight function given by the size of the composition.

<u>Solution:</u> The generating series is given as,

$$\Phi_{\mathcal{S}}(x) = \sum_{\alpha \in \mathcal{S}} x^{\omega(\alpha)} = \sum_{\alpha \in \mathcal{S}} x^{x+y+z}$$

If we fix y and manipulate the other variables, we have the following possible composition arrangements; $(y+1, y, y-1), (y+1, y, y+1), (y-1, y, y+1), (y-1, y, y-1)$, which implies that the weight function has the following possible values; $3y-2, 3y, 3y+2, 3y$, for all $y \geq 2$. For $y = 1$, the only allowed composition would be $(2, 1, 2)$. Hence we compute the generating series as follows,

$$\begin{aligned}
\Phi_{\mathcal{S}}(x) &= x^5 + \sum_{y \geq 2} x^{3y-2} + x^{3y+2} + 2x^{3y} \\
&= x^5 + \sum_{y \geq 2} x^{3y}(x^{-2} + x^2 + 2) \\
&= x^5 + (x^{-2} + x^2 + 2) \sum_{y \geq 2} x^{3y} \\
&= x^5 + (x^{-2} + x^2 + 2) \sum_{y \geq 0} x^{3(y+2)} \\
&= x^5 + x^6(x^{-2} + x^2 + 2) \sum_{y \geq 0} x^{3y} \\
&= x^5 + \frac{x^4 + x^8 + 2x^6}{1 - x^3} \\
&= \frac{x^5 + x^4 + 2x^6}{1 - x^3}
\end{aligned}$$