(A-10-2 MATH 239, Winter 2021) Let G be a bipartite graph with bipartition (A, B) with $\deg(x) \geq 1$ for all $x \in A$, and with $\deg(x) \geq \deg(y)$ for every edge xy with $x \in A$. Prove that G has a matching that saturates A.

Solution: We prove by induction on the vertices's $V(A) = \{v_1, \ldots, v_n\}$ that some matching M saturates A. For the base case, since $\forall v \in V(G)$, $\deg(v) \geq 1$, it follows that v_1 has some neighbor in $w_1 \in V(B)$, such that $v_1 w_1 \in M$. We proceed by assuming that the vertices's $V' = \{v_1, \ldots, v_k\} \in V(A)$, $k \geq 1$, are matched correctly to some neighbor's $N = \{w_1, \ldots, w_k\} \in V(B)$. We would like to show that v_{k+1} can be matched correctly as well such that $v_{k+1} w_{k+1} \in M$, for some $w_{k+1} \in V(B)$. Since $\deg(v_{k+1}) \geq 1$, v_{k+1} has at least one neighbor w_{k+1}. If $w_{k+1} \notin N$ then we simply construct the match $v_{k+1} w_{k+1} \in M$. Else $w_{k+1} = w_i$ for some $w_i \in N$, if $v_i w_{k+1} \in E(G)$, then we match $v_{k+1} w_i, v_i w_{k+1} \in M$. If not, since $\deg(v_{k+1}) \geq \deg(w_i) \geq 2$, another neighbor $w_j \in N$ exists, then we simply repeat this algorithm until some vertex $v_r \in V'$ has w_{k+1} as its neighbor, this is guaranteed to happen since $\forall vw \in E(G)$, $\deg(v) \geq \deg(w)$. Hence by induction G saturates A. ■