2 Properties of Vector Spaces, Intro to Subspaces

Theorem 1

In any vector space, the zero vector is unique. That is if $0, z \in V$ have property where $v + 0 = v$ and $v + z = v$, then $z = 0$.

Theorem 2

In any vector space, the additive inverse of a vector is unique. That is, $\forall v \in V$, if there exists vectors $y, x \in V$ such that $v + y = 0$ and $v + x = 0$, then $y = x$.

Definition 1: Subtraction of vectors

Let V be a vector space, suppose $v, w \in V$. The *difference* of these two vectors, denoted $v - w$, is simply that sum $v + (-w)$, that is the sum of v and the additive inverse of w.

Lemma 1

Let V be a vector space. The following properties hold for V:

1. $\forall \mathbf{v} \in V$, $0 \in \mathbb{F}$, $0 \cdot \mathbf{v} = \mathbf{0}$.
2. $\forall c \in \mathbb{F}$, $c \cdot \mathbf{0} = \mathbf{0}$.
3. Given $c \in \mathbb{F}$, $\mathbf{v} \in V$, if $c \cdot \mathbf{v} = \mathbf{0}$, then $c = 0$ or $\mathbf{v} = \mathbf{0}$.

Lemma 2

Let V be a vector space. The following properties hold for V:

1. $\forall v \in V$, $(-1) \cdot v = -v$. (The additive inverse of v)
2. $\forall c \in \mathbb{F}$, $v \in V$, $c \cdot (-v) = (-c) \cdot v = -(c \cdot v)$.
3. $\forall c \in \mathbb{F}$, $v, w \in V$, $c(v - w) = cv - cw$.

2.1 Subspaces

Definition 2: Subspace

Let V be a vector space over $\mathbb{F}$. A non-empty subset S of V is called a *subspace* of V if S itself forms a vector space with the addition and scalar multiplication operations equipped with V.

Lemma 3: The Subspace Test

Let V be a vector space over $\mathbb{F}$. A subset S of V is a subspace of V if and only if the following conditions hold:

1. The zero vector belongs to S
2. The set is *closed under addition*, that is for $v, w \in S$, $v + w \in S$.
3. The set is *closed under scalar multiplication*, that is for $c \in \mathbb{F}$, $v \in S$, $cv \in S$.

Because S itself is a subset of V (a vector space), there ends up being some redundancy in verifying certain properties of a vector space, hence we can reduce the subspace test to just these three conditions which need to checked independent of S being a subset of V.

Definition 3: Matrix Transpose

Let $A \in M_{m\times n}(\mathbb{F})$,

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

The *Transpose* of A is $A^T \in M_{n\times m}(\mathbb{F})$,

$$A^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix}$$

Definition 4: Symmetric Matrices

A matrix (square) $A \in M_{n\times n}(\mathbb{F})$ is called *symmetric* if $A^T = A$.

Properties 1: Matrix Transposes

Let $A, B \in M_{m\times n}(\mathbb{F})$, $c \in \mathbb{F}$. The following properties hold:

1. $(A+B)^T = A^T + B^T$
2. $(cA)^T = cA^T$

Definition 5: Evaluation of a Polynomial

Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n \in P(\mathbb{F})$, for $a_0, \ldots, a_n \in \mathbb{F}$. We define the *evaluation of p at c* to be

$$p(c) = a_0 + a_1 c + \cdots + a_n c \in \mathbb{F}$$

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