MATH 235, Class 3 Practice Problems

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1 Problems

1. Prove the other direction of Theorem 3.2. That is, prove that every vector of the form $y+x_p$, where $Ax_p = b$ and $Ay = 0$, is a solution to the system $Ax = b$.

2. Find a formula for $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^n$ for all $n \geq 1$ and prove it by induction.

3. Prove Theorem 3.1, Part (1).

2 Solutions

1. We have

$$A(y + x_p) = Ay + Ax_p = 0 + b = b. \quad \square$$

2. Calculating it for the first two or three values of $n \geq 1$ suggests that

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}.$$

To prove this, we use induction on n. We see it is true when $n = 1$. Suppose the claim holds for $k \geq 1$. We will prove it also holds for $k + 1$. By assumption,

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix}.$$

(Note that this is a statement about that specific value of k, not about *all* values of k which is what we're trying to prove.) Then,

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2k+2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2(k+1) \\ 0 & 1 \end{pmatrix}.$$

By induction, this completes the proof. $\square$

3. We calculate

$$((AB)C)_{ij} = \sum_{k=1}^{p} (AB)_{ik} c_{kj} = \sum_{k=1}^{p} \sum_{r=1}^{n} a_{ir} b_{rk} c_{kj}.$$

Also,

$$(A(BC))_{ij} = \sum_{r=1}^{n} a_{ir} (BC)_{rj} = \sum_{r=1}^{n} \sum_{k=1}^{p} a_{ir} b_{rk} c_{kj}.$$

Since these are finite sums, we can switch the order of summation of $\sum_{r=1}^{n}$ and $\sum_{k=1}^{p}$, so we get

$$((AB)C)_{ij} = (A(BC))_{ij}.$$

Since this holds for all (i, j), we conclude that $(AB)C = A(BC)$. $\square$