1. Let $k \geq 1$ be an integer. Let T denote the set of multisets of size n with t types such that for all $i \in \{1, \ldots, t\}$, the number of elements of the i-th type is a multiple of k, or in other words $k \mid i$ for each i. What is $|T|$? (Assume $k \mid n$)

2. Let $k \geq 0$ be an integer. Let H denote the set of multisets of size n with t types such that for each $i \in \{1, \ldots, t\}$, the number of elements of the $i-$th type is at least k. What is $|U|$? (Assume $n \geq kt$)

Solution:

1. Let the set H be the set of all multisets of size n/k and t types. We claim a bijection between T and H. Note that each element in T may be written as $(kc_1, \ldots, kc_t)$, where each $c_i \in \mathbb{Z}^+$. Let $f\colon T \to H$ be defined by mapping each $\alpha = (kc_1, \ldots, kc_t) \in T$ to $f(\alpha) = (c_1, \ldots, c_t)$. By the construction of T, $c_i \in \mathbb{Z}^+$ and $kc_1 + \cdots + kc_n = n$ which implies that $c_1 + \cdots + c_n = n/k$ hence $f(\alpha)$ is a multieset of size n/k and t types, hence $f(\alpha) \in H$. We define the inverse function $g\colon H \to T$ by simply assinging each $f(\alpha) = (c_1, \cdots, c_t)$, as constructed previously, to $g(f(\alpha)) = (kc_1, \ldots, kc_t)$, which simply multiplies each part of the multiset by k.