(P-1-2 MATH 239, Winter 2021) Let n be a positive integer. How many binary strings of length $2n+1$ have more $1's$ than $0's$?

<u>Solution:</u> We show that there exists a bijection between the set $\mathscr{P}$, the set of all binary strings that have more $1's$ than $0's$, and the set $\mathscr{H}$, the set of all binary strings that have more $0's$ than $1's$. Let $f\colon \mathscr{P} \to \mathscr{H}$ be the function which assings for some $\alpha = a_1 \cdots a_{2n+1} \in \mathscr{H}$, $f(\alpha) = b_1 \cdots b_{2n+1}$, the string such that $a_i = 1 \implies b_i = 0$. Clearly $f(\alpha) \in \mathscr{H}$, since α was a string with more $1's$ than $0's$, $f(\alpha)$ would be string with more $0's$ than $1's$. It remains to show that the image of f is $\mathscr{H}$. Let $f^{-1}\colon \mathscr{H} \to \mathscr{P}$ be the inverse function which assings for each $\gamma = b_1 \cdots b_{2n+1}$ the string $f(\gamma) = a_1 \cdots a_{2n+1}$, where $b_i = 1 \implies a_i = 0$. Applying similar reasoning from before, $f^{-1}(\gamma) \in \mathscr{P}$. Hence we have proven that $\mathscr{P} \rightleftharpoons \mathscr{H}$, which implies that $|\mathscr{H}| = |\mathscr{P}|$, since $|\mathscr{P}| + |\mathscr{H}| = |\{1,0\}^{2n+1}| = 2^{2n+1} \implies 2|\mathscr{P}| = 2^{2n+!} \implies |\mathscr{P}| = 2^{2n}$.

$\blacksquare$