(A-6-2b) MATH 239, Winter 2021) Let G be connect a graph such that for every distinct vertices $x, y, z \in V(G)$, at least one of the edges xy, yz, xz is present. Prove that there is a path in G that contains all vertices of G.

Solution: Assume G is a connected graph and for every three vertices $a, b, c \in V(G)$, at least one of $ab, bc, ac \in E(G)$. Assume that $W = (v_1, \ldots, v_k)$ is the path of maximal length for some $k \leq n$, where $n = |V(G)|$. We prove that $k = n$. If W denotes the longest path, then $\deg(v_1) = \deg(v_k) = 1$. Let $w \in V(G) \setminus V(W)$, then $wv_1, wv_k \notin E(G)$ however this would imply that the triple $v_1, v_k, w \in V(G)$ form no edge with each other, a violation of a property we assumed G to have. Hence $v_k w \in E(G)$, and $W = (v_1, \ldots, v_k, w)$ is the longest path. We can continue with induction for the rest of $w \in V(G) \setminus V(W)$ to show that the longest path is $W' = (v_1, \ldots, v_n)$.

$\blacksquare$