(A-10-1 MATH 239, Winter 2021) Let $G$ be a graph with a perfect matching, and such that all vertices in $G$ have degree at least 2. Prove that if $G$ is bipartite, then $G$ has at least two perfect matchings.

**<u>Solution:</u>** Assume that $G = (V, E)$ is a bipartite graph, let $(A, B)$ represent the bipartition, denote the perfect matching of $G$ as $M$. Since edges strictly connect between $A$ and $B$, the edge's of $M$ exist exclusivity between the two sets. Let $v_0 \in A$, let $v_0w_0 \in M$ represent the matching edge, we show that $E(G) \setminus \{v_0, w_0\}$ has a perfect matching. Let $S \subseteq V(G)$, since for each $v \in S, v \neq v_0$,$\exists w \in V$, $vw \in M$, each vertex $v$ has at least one unique neighbor, since $\deg(v) \geq 2$, it follows that for any pair of vertices's $v_i, v_j \in S$, $|N(v_i) \setminus N(v_j)| \geq 2$. Hence we conclude that since $G$ is bipartite, $|N(S)| = \bigcup_{v \in S} |N(v)| \geq |S|$, by Hall's Theorem there exists an alternative perfect matching $M'$. $\blacksquare$