MATH 235, Class 5 Practice Problems

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1 Problems

1. The following problem is a special case of something called the "splitting lemma".

(a) Find an example of linear transformations $T : \mathbb{R} \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}$ such that $\operatorname{ran}(T) = \ker(S)$.

(b) Find a linear transformation $F_1 : \mathbb{R}^2 \to \mathbb{R}$ such that $F_1 \circ T(x) = x$ for all $x \in \mathbb{R}$.

(c) Find a linear transformation $F_2 : \mathbb{R} \to \mathbb{R}^2$ such that $S \circ F_2(x) = x$ for all $x \in \mathbb{R}$.

2. Consider the following 2×2 matrices:

$$A_1 := \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

$$A_2 := \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},$$

$$A_3 := \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},$$

$$A_4 := \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let $B := \{A_1, A_2, A_3, A_4\}$.

(a) Prove that B is a basis for $M_{2\times 2}(\mathbb{R})$.

(b) Consider the linear transformation $T : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}^2$ defined by setting

$$T(A_1) = T(A_3) := \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$T(A_2) = T(A_4) := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and extending by linearity to all of $M_{2\times 2}(\mathbb{R})$. (You do not need to check this is well-defined since that's shown in the proof of Theorem 5.3 in the notes.) Find $\operatorname{ran}(T)$ and $\ker(T)$ with proof.

2 Solutions

1. (a) For instance, take $T(x) := (x, 0)$ for all $x \in \mathbb{R}$ and $S(x, y) := y$ for all $(x, y) \in \mathbb{R}^2$. Then $\text{ran}(T) = \{(x, 0) \mid x \in \mathbb{R}\} = \ker(S)$.

(b) In the example from part (a), take $F_1(x, y) := x$ for all $(x, y) \in \mathbb{R}^2$. Then $F_1(T(x)) = F_1(x, 0) = x$, so it satisfies the desired property.

(c) In the example from part (a), take $F_2(x) := (0, x)$ for all $x \in \mathbb{R}$. Then $S(F_2(x)) = S(0, x) = x$, so it satisfies the desired property.

2. (a) To see that B spans $M_{2\times 2}(\mathbb{R})$, note that

$$c_1A_1 + c_2A_2 + c_3A_3 + c_4A_4 = \begin{pmatrix} c_1 + c_2 & c_1 + c_4 \\ c_2 + c_3 & c_2 + c_4 \end{pmatrix}.$$

Choosing $c_1 := 1/2$, $c_2 := 1/2$, $c_3 := -1/2$, $c_4 := -1/2$ gives $E_{11} \in \text{Span}(B)$. (Recall that E_{ij} means the matrix with a 1 in its (i, j)th entry and 0 elsewhere.) Choosing $c_1 := 1/2$, $c_2 := -1/2$, $c_3 := 1/2$, $c_4 := 1/2$ gives $E_{12} \in \text{Span}(B)$. Similarly, we get $E_{21}, E_{22} \in \text{Span}(B)$. Since we know E_{11}, E_{12}, E_{21}, and E_{22} span $M_{2\times 2}(\mathbb{R})$, this is enough to conclude that $\text{Span}(B) = M_{2\times 2}(\mathbb{R})$.

To prove linear independence, suppose

$$c_1A_1 + c_2A_2 + c_3A_3 + c_4A_4 = \begin{pmatrix} c_1 + c_2 & c_1 + c_4 \\ c_2 + c_3 & c_2 + c_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This gives the following system of equations:

$$c_1 + c_2 = 0,$$

$$c_1 + c_4 = 0,$$
$$c_2 + c_3 = 0,$$
$$c_2 + c_4 = 0.$$

This is simple enough that we can solve it directly without putting it into an augmented matrix. We get $c_2 = -c_1$, $c_4 = -c_1$, $c_3 = -c_2 = c_1$, and $c_4 = -c_2 = c_1$. Since $c_4 = -c_1$ and $c_4 = c_1$, we must have $c_1 = 0$. This implies that $c_2 = c_3 = c_4 = 0$. Thus, B is linearly independent, so since it's also a spanning set, B is a basis for $M_{2\times 2}(\mathbb{R})$.

(b) First, we find $\text{ran}(T)$. This is almost immediate. Since $e_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ form a basis for $\mathbb{R}^2$, given $v \in \mathbb{R}^2$, we can write

$$v = c_1e_1 + c_2e_2$$

for some unique $c_1, c_2 \in \mathbb{R}$. But then

$$T(c_1A_1 + c_2A_2) = c_1T(A_1) + c_2T(A_2) = c_1e_1 + c_2e_2 = v.$$

Therefore, $\operatorname{ran}(T) = \mathbb{R}^2$. Next, we find $\ker(T)$. We calculate

$$T(c_1A_1+c_2A_2+c_3A_3+c_4A_4) = c_1T(A_1)+c_2T(A_2)+c_3T(A_3)+c_4T(A_4) = (c_1+c_3)e_1+(c_2+c_4)e_2.$$

Since $\{e_1, e_2\}$ is a basis for $\mathbb{R}^2$, the unique way the zero vector in $\mathbb{R}^2$ can be expressed as a linear combination of e_1 and e_2 is if the coefficients of e_1 and e_2 are both zero. Thus, $T(c_1A_1 + c_2A_2 + c_3A_3 + c_4A_4) = 0$ if and only if $c_1 + c_3 = 0$ and $c_2 + c_4 = 0$, i.e., $c_3 = -c_1$ and $c_4 = -c_2$. From this it follows that

$$\begin{aligned}
\ker(T) &= \{c_1A_1 + c_2A_2 - c_1A_3 - c_2A_4 \mid c_1, c_2 \in \mathbb{R}\} \\
&= \{c_1(A_1 - A_3) + c_2(A_2 - A_4) \mid c_1, c_2 \in \mathbb{R}\} \\
&= \left\{ c_1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mid c_1, c_2 \in \mathbb{R} \right\} \\
&= \operatorname{Span} \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}.
\end{aligned}$$