(P-1-3 MATH 239, Winter 2021) For integers $n \geq 1$ define, A_n to be the set of all ordered pairs of subsets of $\{1, \ldots, n\}$, i.e.

$$A_n = \{(S_1, S_2) \mid S_1, S_2 \subseteq \{1, \ldots, n\}\}$$

By counting $|A_n|$ in two different ways, prove that

$$4^n = \sum_{k=0}^{n} \binom{n}{k} 3^{n-k}$$

Solution: We begin with the left hand side of the expression, we first determine $|A_n|$ by considering the number of ways we can choose a pair (S_1, S_2), since there are 2^n subsets of an $n-$element set, we can choose S_1 in 2^n ways and for each of these choices we can choose S_2 in 2^n ways, hence $|A_n| = 2^n \cdot 2^n = 4^n$.

Our second approach will consider the common subsets between (S_1, S_2). For each $k-$element subset B such that $B \subseteq S_1, S_2$, we have three possibilities for each element x from the remaining $n-k$ elements, either $x \in S_1$, $x \in S_2$ or x is not a member of either set. Hence we can construct an indicator vector for the remaining $n-k$ elements $\beta = (\alpha_1, \ldots, \alpha_{n-k})$, where each α_i has 3 possible values, $0, 1, 2$ (We like to think of these numbers as simple representations of the possible states that each of the remaining $n-k$ elements can have, they don't necessarily have to have some intrinsic meaning attached). The set of all associated indicator vectors is $\{0, 1, 2\}^{n-k}$, the size of which is $|\{0, 1, 2\}^{n-k}| = 3^{n-k}$. If we consider all $k-$element subsets, we obtain a summation over all k, this gives $|A_n| = \sum_{k=0}^{n} \binom{n}{k} 3^{n-k}$

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