MATH 235, Class 4 Practice Problems

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1 Problems

1. Write down a basis and prove it is one for the following $\mathbb{R}$–vector spaces from Practice Problems 2:

(a) The vector space V of $n \times n$ real matrices whose top-left entry is 0.

(b) $W := \mathfrak{sl}_n(\mathbb{R})$, the vector space of $n \times n$ matrices with real entries whose trace, i.e., sum of diagonal entries, is 0.

2. (a) Let V be a finite-dimensional vector space over a field F and S a subspace of V. Prove that S is also finite-dimensional.

(b) Let V be a finite-dimensional vector space over a field F. Prove there is no infinite descending chain $S_1 \supsetneq S_2 \supsetneq S_3 \supsetneq \cdots$ of subspaces of V.

2 Solutions

1. (a) Let E_{ij} denote the $n \times n$ matrix with 1 in the (i,j)th entry and 0 elsewhere. Then $B := \{E_{ij} \mid (i,j) \neq (1,1)\}$ is a basis for V. Given a matrix $A \in V$, $A = \sum_{(i,j)\neq(1,1)} a_{ij}E_{ij}$, so B spans V. Suppose $\sum_{(i,j)\neq(1,1)} c_{ij}E_{ij} = 0$. The LHS is the matrix C with (i,j)th entry c_{ij} for $(i,j) \neq (1,1)$ and $(1,1)$ entry 0. Since C is equal to the zero matrix, we must have $c_{ij} = 0$ for all $(i,j) \neq 1$, so B is linearly independent. Since it is a linearly independent spanning set, it is a basis for V.

(b) Suppose $A \in W$. Writing $A =: (a_{ij})$, the condition $\mathrm{tr}(A) = 0$ means $a_{11}+a_{22}+\cdots+a_{nn} = 0$, or $a_{11} = -a_{22} - a_{33} - \cdots - a_{nn}$. Thus,

$$\begin{aligned}
A &= (-a_{22} - a_{33} - \cdots - a_{nn})E_{11} + a_{12}E_{12} + a_{13}E_{13} + \cdots + a_{nn}E_{nn} \quad (1)\\
&= (a_{12}E_{12} + \cdots + a_{1n}E_{1n}) + (a_{21}E_{21} + a_{22}(-E_{11} + E_{22}) + \cdots + a_{2n}E_{2n}) \quad (2)\\
&+ \cdots + (a_{n1}E_{n1} + a_{n2}E_{n2} + \cdots + a_{nn}(-E_{11} + E_{nn})).
\end{aligned}$$

I claim $B := \{E_{12}, \ldots, E_{1n}, E_{21}, -E_{11} + E_{22}, \ldots, E_{2n}, \ldots, E_{n1}, E_{n2}, \ldots, -E_{11} + E_{nn}\}$. which has $(n-1) + n(n-1) = (n+1)(n-1) = n^2 - 1$ entries is a basis for W. The calculation above shows B spans W. To see B is linearly independent, observe that if the linear combination in line (2) above were equal to 0, then so would be the linear combination in line (1), so the matrix A would be the zero matrix. This would mean that $a_{ij} = 0$ for all $(i,j) \neq (1,1)$, which would imply that the $(1,1)$ entry also equals 0. So all the coefficients of the linear combination would equal 0, proving linear independence of B. Therefore, B is a basis for V, which proves that $\dim_{\mathbb{R}} \mathfrak{sl}_n(\mathbb{R}) = n^2 - 1$.

2. (a) Let $n := \dim(V)$. By Theorem 4.5 and Definition 4.5, every basis of V has n elements. Since a basis is a spanning set, by Lemma 4.1, any linearly independent subset of V has at most n elements.

Clearly there exists a set of linearly independent vectors in S: for instance, take $0 \neq w \in S$ and consider the set $\{w\}$. Suppose you have a finite linearly independent subset $\{w_1, w_2, \ldots, w_m\} \subseteq S$. If these vectors don't span S, then choose $w_{m+1} \in S \setminus \mathrm{Span}\{w_1, w_2, \ldots, w_m\}$. By Theorem 4.3, $\{w_1, w_2, \ldots, w_{m+1}\}$ is then linearly independent. We can repeat the process: if $\{w_1, w_2, \ldots, w_{m+1}\}$ doesn't span S, we can find a vector $w_{m=2} \in S \setminus \mathrm{Span}\{w_1, \ldots, w_{m+1}\}$ and add it to the set. We can continue this way, but we know by the previous paragraph that once our set of w_i's has more than n vectors, it can no longer be linearly independent. Thus, S must have a linearly independent spanning set with at most n elements, so S is finite-dimensional and $\dim(S) \leq \dim(V)$.

(b) By part (a), each of the S_i must be finite-dimensional. Since each S_i is a subspace of all of $S_{i-1}, S_{i-2}, \ldots, S_1$, we must have $\dim(V) \geq \dim(S_1) \geq \dim(S_2) \geq \cdots$. But $\dim(V)$ is finite and all the dimensions are non-negative integers, yet there does not exist an infinite descending sequence of non-negative integers. This completes the proof.