Functions Quiz 2 - SOLUTIONS

January, 2022

1 Name and Date:

Print your name and todays date below;

__________________________________ __________________________________

Name Date

Question 1: (8 points)

Answer the following True/False questions,

1. Let $\mathrm{id}_{\mathbb{R}} \colon \mathbb{R} \to \mathbb{R}$ be the identity function on $\mathbb{R}$, then

$$\mathrm{id}_{\mathbb{R}}(\mathrm{id}_{\mathbb{R}}(\mathrm{id}_{\mathbb{R}}(\mathrm{id}_{\mathbb{R}}(\mathrm{id}_{\mathbb{R}}(\mathrm{id}_{\mathbb{R}}(2)))))) = 2.$$

   Answer: **True**.

2. Let $\phi \colon \mathcal{A} \to \mathcal{B}$ be an *injective* function, then every element in $\mathcal{B}$ is mapped to.
   Answer: **False**.

3. Let $\mathcal{L} \colon \mathcal{A} \to \mathcal{B}$ be a *surjective* function, then $|\mathcal{R}_{\mathcal{L}}| = |\mathcal{B}|$, where $\mathcal{R}_{\mathcal{L}}$ is the range of $\mathcal{L}$.
   Answer: **True**, Surjectivness implies that $\mathcal{R}_{\mathcal{L}} = \mathcal{B}$, which would then imply that $|\mathcal{R}_{\mathcal{L}}| = |\mathcal{B}|$ (Ask me in class if you are still confused).

4. Let $\lambda \colon \mathcal{A} \to \mathcal{B}$ be a function, suppose λ is one of surjective or injective, then λ is invertible.
   Answer: **False**, If λ is **both** injective and surjective, then its invertible.

5. Let $\mathcal{X} = \{-1, 0, 1\}$ and $\mathcal{Y} = \{-1, 1, 2\}$ be sets, lets define the following function,
   - $\eta \colon \mathcal{X} \to \mathcal{Y}$.
   - $\eta(x) = 2x^2 - 1$.

   Then η is an invertible function.
   Answer: **False**, Since $\eta(-1) = \eta(1) = 1$, we conclude that η fails to be to be injective. Since it fails to be injective, it fails to invertible.

6. Let $\mathcal{V} = \{-2, 0\}$ and $\mathcal{W} = \{0, 4\}$ be sets, define the following function,
   - $\mathcal{T} \colon \mathcal{V} \to \mathcal{W}$.
   - $\mathcal{T}(v) = v^2$.

   Then the function,
   - $\mathcal{T}^{-1} \colon \mathcal{W} \to \mathcal{V}$.
   - $\mathcal{T}^{-1}(w) = \sqrt{w}$.

   is the inverse function for $\mathcal{T}$.
   Answer: **False**, Note that $\mathcal{T}^{-1}(\mathcal{T}(-2)) = \mathcal{T}^{-1}(4) = 2$, and hence we conclude that $\mathcal{T}^{-1}$ fails the first condition of invertible functions with this element, therefore this is not the inverse function for $\mathcal{T}$.

7. Let $\mathbf{A}$ and $\mathbf{B}$ be binary strings, then $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.
   Answer: **False**, Suppose that $\mathbf{A} = 1$ and $\mathbf{B} = 0$, then $\mathbf{A} + \mathbf{B} = 10$ wheras $\mathbf{B} + \mathbf{A} = 01$, and clearly $10 \neq 01$.

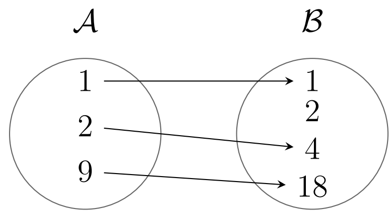
8. Let $G\colon \mathcal{H} \to \mathcal{T}$ be a function. If $|\mathcal{H}| = |\mathcal{T}|$, then G is invertible.
   Answer: **False**, This was a tricky question, it requires you to think of a function that fails to satisfy invertibilty even though the sizes of the sets are the same, so ill take this up in class. **(In class explanation)**

Question 2: (8 points)

For each of the following, you are given a function and its definition. For each question,

- Prove that the function is invertible **or** prove that the function is not invertible.
- Determine the range of the function.

(a) $g\colon \mathcal{A} \to \mathcal{B}$,



Solution: Since $2 \in \mathcal{B}$ is not mapped to, we conclude that g is not surjective. Since g fails to be surjective, we conclude that it fails to be invertible.

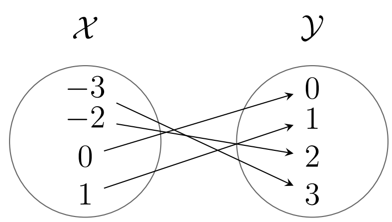
Based on the outputs given in the mapping diagram, we conclude that the range of the function is,

$$\mathcal{R}_g = \{1, 4, 18\}.$$

(b) Let $\mathcal{X} = \{-3 - 2, 0, 1\}$, $\mathcal{Y} = \{0, 1, 2, 3\}$ be sets and define,

- $H\colon \mathcal{X} \to \mathcal{Y}$.
- $H(x) = |x|$.

Solution:



From the mapping diagram we see that no two elements in $\mathcal{X}$ map to the same element in $\mathcal{Y}$, hence H is injective. Also note that every element in $\mathcal{Y}$ is mapped to, and hence H is surjective as well. Since H is both surjective and injective, it follows that H is invertible.

Based on the outputs given in the mapping diagram, we conclude that the range of the function is,

$$\mathcal{R}_H = \{0, 1, 2, 3\}.$$

Question 3: (7 marks)

Let $\mathcal{X} = \{3, 7, 4\}$ and $\mathcal{Y} = \{2, 0, 1\}$ be sets, define the following function,

- $A\colon \mathcal{X} \to \mathcal{Y}$.
- $A(x) = \sqrt{x-3}$.

Prove that the function,

- $A^{-1}\colon \mathcal{Y} \to \mathcal{X}$.
- $A^{-1}(y) = y^2 + 3$.

is the inverse function for $\mathcal{L}$.
Hint: Use mapping tables.

Solution: We confirm that both conditions of Definition 4.1 hold with mapping tables,

$\mathcal{X}$	$A^{-1}(A(x))$
3	$A^{-1}(A(3)) = A^{-1}(0) = (0)^2 + 3 = 3$
7	$A^{-1}(A(7)) = A^{-1}(2) = (2)^2 + 3 = 7$
4	$A^{-1}(A(4)) = A^{-1}(1) = (1)^2 + 3 = 4$

$\mathcal{Y}$	$A(A^{-1}(x))$
2	$A(A^{-1}(2)) = A(7) = \sqrt{7-3} = 2$
0	$A(A^{-1}(0)) = A(3) = \sqrt{3-3} = 0$
1	$A(A^{-1}(1)) = A(4) = \sqrt{4-3} = 1$

By our results from the mapping tables, we conclude that $A^{-1}(y) = y^2 + 3$ is indeed the inverse function for A.

Question 4: (3 marks)

Let $\mathcal{S} = \{101, 010, 111\}$ and $\mathcal{R} = \{11, 01, 10\}$ be sets of binary strings, define the following function,

- $Q\colon \mathcal{S} \to \mathcal{R}$.
- $Q(\mathbf{S}) = \mathbf{s}_1\mathbf{s}_2$.

Prove that the function,

- $Q^{-1}\colon \mathcal{R} \to \mathcal{S}$.
- $Q^{-1}(\mathbf{R}) = \mathbf{R} + 1$.

is **NOT** the inverse function for Q.
Hint: Which of the two conditions in Definition 4.1 does it fail to preserve?

Solution: Note that $Q^{-1}(Q(010)) = Q^{-1}(01) = 011$, clearly $011 \neq 010$ and hence $Q^{-1}(\mathbf{R}) = \mathbf{R} + 1$ fails to satisfy the first condition of Definition 4.1. We conclude that $Q^{-1}(\mathbf{R}) = \mathbf{R} + 1$ is not the inverse function for Q.