Solutions - Lecture 4 - Homework

Question 1.

Solution. Yes it is true that the inverse function for the identity function $\mathrm{id}_{\mathbb{R}}$ is $\mathrm{id}_{\mathbb{R}}$. We can provide proof for this claim quite simply by confirming that both conditions from Definition 4.1 hold,

1. For all $x \in \mathbb{R}$, $\mathrm{id}_{\mathbb{R}}(\mathrm{id}_{\mathbb{R}}(x)) = \mathrm{id}_{\mathbb{R}}(x) = x$.
2. For all $x \in \mathbb{R}$, $\mathrm{id}_{\mathbb{R}}(\mathrm{id}_{\mathbb{R}}(x)) = \mathrm{id}_{\mathbb{R}}(x) = x$.

Question 2.

Proof: Proceeding with mapping tables,

$\mathcal{V}$	$\mathcal{L}^{-1}(\mathcal{L}(v))$
1	$\mathcal{L}^{-1}(\mathcal{L}(1)) = \mathcal{L}^{-1}(3) = (3-2)^2 = 1$
4	$\mathcal{L}^{-1}(\mathcal{L}(4)) = \mathcal{L}^{-1}(4) = (4-2)^2 = 4$
9	$\mathcal{L}^{-1}(\mathcal{L}(9)) = \mathcal{L}^{-1}(5) = (5-2)^2 = 9$
25	$\mathcal{L}^{-1}(\mathcal{L}(25)) = \mathcal{L}^{-1}(7) = (7-2)^2 = 25$
49	$\mathcal{L}^{-1}(\mathcal{L}(49)) = \mathcal{L}^{-1}(9) = (9-2)^2 = 49$

$\mathcal{W}$	$\mathcal{L}(\mathcal{L}^{-1}(w))$
5	$\mathcal{L}(\mathcal{L}^{-1}(5)) = \mathcal{L}(9) = \sqrt{9} + 2 = 5$
4	$\mathcal{L}(\mathcal{L}^{-1}(4)) = \mathcal{L}(4) = \sqrt{4} + 2 = 4$
7	$\mathcal{L}(\mathcal{L}^{-1}(7)) = \mathcal{L}(25) = \sqrt{25} + 2 = 7$
9	$\mathcal{L}(\mathcal{L}^{-1}(9)) = \mathcal{L}(49) = \sqrt{49} + 2 = 9$
3	$\mathcal{L}(\mathcal{L}^{-1}(3)) = \mathcal{L}(1) = \sqrt{1} + 2 = 3$

By our results from the mapping tables, we conclude that $\mathcal{L}^{-1}(w) = (w-2)^2$ is indeed the inverse function of $\mathcal{L}$.

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Question 3.

Proof: Proceeding with mapping tables,

$\mathcal{V}$	$\mathcal{T}^{-1}(\mathcal{T}(v))$
-1	$\mathcal{T}^{-1}(\mathcal{T}(-1)) = \mathcal{T}^{-1}(1) = 2(1)/(1-3) = -1$
0	$\mathcal{T}^{-1}(\mathcal{T}(0)) = \mathcal{T}^{-1}(0) = 2(0)/(0-3) = 0$
1	$\mathcal{T}^{-1}(\mathcal{T}(1)) = \mathcal{T}^{-1}(-3) = 2(-3)/(-3-3) = 1$
5	$\mathcal{T}^{-1}(\mathcal{T}(5)) = \mathcal{T}^{-1}(5) = 2(5)/(5-3) = 5$
8	$\mathcal{T}^{-1}(\mathcal{T}(8)) = \mathcal{T}^{-1}(4) = 2(4)/(4-3) = 8$

$\mathcal{W}$	$\mathcal{T}(\mathcal{T}^{-1}(w))$
-3	$\mathcal{T}(\mathcal{T}^{-1}(-3)) = \mathcal{T}(1) = 3(1)/(1-2) = -3$
0	$\mathcal{T}(\mathcal{T}^{-1}(0)) = \mathcal{T}(0) = 3(0)/(0-2) = 0$
1	$\mathcal{T}(\mathcal{T}^{-1}(1)) = \mathcal{T}(-1) = 3(-1)/(-1-2) = 1$
4	$\mathcal{T}(\mathcal{T}^{-1}(4)) = \mathcal{T}(8) = 3(8)/(8-2) = 4$
5	$\mathcal{T}(\mathcal{T}^{-1}(5)) = \mathcal{T}(5) = 3(5)/(5-2) = 5$

By our results from the mapping tables, we conclude that $\mathcal{T}^{-1}(w) = \frac{2w}{w-4}$ is indeed the inverse function of $\mathcal{T}$.

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Question 4.

Solution. I suspect that the formula is $F(a) = 2a + 4$. In order to check that it this is true, it is sufficient at check that the mapping table for $F(a) = 2a + 4$ matches our mapping diagram.

$\mathcal{A}$	$F(a) = 2a + 4$
-8	$F(-8) = 2(-8) + 4 = -16 + 4 = -12$
-2	$F(-2) = 2(-2) + 4 = -4 + 4 = 0$
4	$F(4) = 2(4) + 4 = 8 + 4 = 12$
6	$F(6) = 2(6) + 4 = 12 + 4 = 16$

Since each element in $\mathcal{A}$ is mapped to the correct element in $\mathcal{B}$ with respect to the mapping diagram, we conclude that $F(a) = 2a + 4$ is indeed the correct formula for the function.

I claim that $F^{-1}(b) = -2b$ is the inverse function of F.

Proof: Its sufficient to check that $F^{-1}(b) = (b - 4)/2$ satisfies the conditions in Definition 4.1. In order to check that it does, we can use mapping tables.

$\mathcal{A}$	$F^{-1}(F(a))$
-8	$F^{-1}(F(-8)) = F^{-1}(-12) = (-12 - 4)/2 = -8$
-2	$F^{-1}(F(-2)) = F^{-1}(0) = (0 - 4)/2 = -2$
4	$F^{-1}(F(4)) = F^{-1}(12) = (12 - 4)/2 = 4$
6	$F^{-1}(F(6)) = F^{-1}(16) = (16 - 4)/2 = 6$

$\mathcal{B}$	$F(F^{-1}(b))$
0	$F(F^{-1}(0)) = F(-2) = 2(-2) + 4 = 0$
-12	$F(F^{-1}(-12)) = F(-8) = 2(-8) + 4 = -12$
16	$F(F^{-1}(16)) = F(6) = 2(6) + 4 = 16$
12	$F(F^{-1}(12)) = F(4) = 2(4) + 4 = 12$

By our results from the mapping tables, we conclude that the claim was correct and that $F^{-1}(b) = (b - 4)/2$ is indeed the inverse function of F.

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Question 5.

Solution. Well first we can talk about how we gain the ability to determine the domain of a function strictly based on knowledge of the inverse function and the range of the function. For a more more applicable example, suppose we have a function that goes from Celsius to Fahrenheit. Then of course at times we would also be interested in going from Fahrenheit to Celsius, this would require knowledge of the inverse function. (This would make a great test question. . .).

Question 6.

Solution. f is **not** invertible.

Proof: Its easy to show that f fails to be injective, for example note that,

$$f(-2) = |-2| = 2$$
$$f(2) = |2| = 2.$$

Since f fails to be injective, it fails to be invertible. ■

Question 7.

Solution. Since G is invertible, we know that its both surjective and injective. Both of these properties assert that each element in $\mathcal{B}$ is mapped to from a **unique** element in $\mathcal{A}$. Since we are given the inverse function G^{-1}, we can simply determine the output of each element $b \in \mathcal{B}$ to reconstruct $\mathcal{A}$. We can use mapping tables to help with this,

$\mathcal{B}$	$G^{-1}(b)$
-2	$G^{-1}(-2) = -2 - 2 = -4$
-1	$G^{-1}(-1) = -1 - 2 = -3$
0	$G^{-1}(0) = 0 - 2 = -2$
3	$G^{-1}(3) = 3 - 2 = 1$
7	$G^{-1}(7) = 7 - 2 = 5$

And hence $\mathcal{A} = \{-4, -3, -2, 1, 5\}$.

Question 8.

Solution.

(a) $\mathbf{S} + \mathbf{T} + 0 = 00101 + 1101 + 0 = 0010111010.$

(b) $1 + \mathbf{T} + 1 = 1 + 1101 + 1 = 111011.$

(c) $\epsilon + \mathbf{T} + \mathbf{S} = \epsilon + 1101 + 00101 = 110100101.$

(d) $\mathbf{T} + \mathbf{T} = 1101 + 1101 = 11011101.$

Question 9. This is **false**. An easy counter example is to let $\mathbf{S} = 1$ and $T = 0$, then

$$\mathbf{S} + \mathbf{T} = 1 + 0 = 10$$
$$\mathbf{T} + \mathbf{S} = 0 + 1 = 01.$$

Clearly $\mathbf{S} + \mathbf{T} \neq \mathbf{T} + \mathbf{S}$, hence this is not always true. (Weird right?)

Question 10.

Solution.

(a) $\mathbf{R} = \mathbf{a}_1\mathbf{a}_3\mathbf{a}_5 = 101.$

(b) $\mathbf{Z} = \mathbf{a}_1\mathbf{a}_2 = 11.$

(c) $\mathbf{X} = \mathbf{a}_2\mathbf{a}_4 = 11.$

Question 11.

Solution. I claim that $\phi^{-1}(\mathbf{R}) = \mathbf{r}_2\mathbf{r}_3$ is the inverse function of ϕ.

Proof: Its sufficient to check that $\phi^{-1}(\mathbf{R}) = \mathbf{r}_2\mathbf{r}_3$ satisfies the conditions in Definition 4.1. In order to check that it does, we can use mapping tables.

$\mathcal{S}$	$\phi^{-1}(\phi(\mathbf{S}))$
11	$\phi^{-1}(\phi(11)) = \phi^{-1}(11101) = 11$
01	$\phi^{-1}(\phi(01)) = \phi^{-1}(10101) = 01$
10	$\phi^{-1}(\phi(10)) = \phi^{-1}(11001) = 10$
00	$\phi^{-1}(\phi(00)) = \phi^{-1}(10001) = 00$

$\mathcal{R}$	$\phi(\phi^{-1}(\mathbf{R}))$
10001	$\phi(\phi^{-1}(10001)) = \phi(00) = 1 + 00 + 0 + 1 = 10001$
10101	$\phi(\phi^{-1}(10101)) = \phi(01) = 1 + 01 + 0 + 1 = 10101$
11001	$\phi(\phi^{-1}(11001)) = \phi(10) = 1 + 10 + 0 + 1 = 11001$
11101	$\phi(\phi^{-1}(11101)) = \phi(11) = 1 + 11 + 0 + 1 = 11101$

By our results from the mapping tables, we conclude that the claim was correct and that $\phi^{-1}(\mathbf{R}) = \mathbf{r}_2\mathbf{r}_3$ is indeed the inverse function of ϕ.

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