Assignment 1 Functions - SOLUTIONS
Due Date: Thursday, December 16

November, 2021

1 Preamble

This assignment covers everything most of Unit 1. The solutions that you hand in should be **neat** and **legible**, this is an assignment, not a quiz, so I expect you to take your time and present thorough and detailed solutions.

2 Name and Date:

Print your name and todays date below;

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Name Date

Question 1. We define the **cardinality** of sets to be the size of the set, in other words the cardinality of a set is the number of elements in the set. For example if $S = \{3, 2, 1, \triangle\}$, then we say the that cardinality of S is 4, because S consists of four elements. The notation we use to describe the cardinality of a set is a pair of bars ($||$) similar to absolute values. So going back to our example with S, instead of saying the sentence; The cardinality of S is 4, we could instead simply write $|S| = 4$. In this question we will discover that for sets A, B, C, it is **not** always true that $|A + B| = |A| + |B|$.

Lets say we have the following sets,

- $\mathcal{H} = \{x \in \mathbb{Z} \mid -1 \leq x \leq 5\}$
- $\mathcal{T} = \{y \in \mathbb{Z} \mid 2 \leq y < 7\}$

(a) Write down the elements of both sets.

Solution.

$$\begin{aligned} \mathcal{H} &= \{-1, 0, 1, 2, 3, 4, 5\} \\ \mathcal{T} &= \{2, 3, 4, 5, 6\}. \end{aligned}$$

(b) Determine $|\mathcal{H}|$ and $|\mathcal{T}|$.

Solution.

$$\begin{aligned} |\mathcal{H}| &= 7 \\ |\mathcal{T}| &= 5. \end{aligned}$$

(c) Determine $|\mathcal{H} + \mathcal{T}|$. (Go back to Homework-1 if you forgot how we preform set addition).

Solution. We first determine $\mathcal{H} + \mathcal{T}$,

$$\begin{aligned} \mathcal{H} + \mathcal{T} &= \{-1, 0, 1, 2, 3, 4, 5\} + \{2, 3, 4, 5, 6\} \\ &= \{-1, 0, 1, 2, 3, 4, 5, 2, 3, 4, 5, 6\} && \text{(Merge Step)} \\ &= \{-1, 0, 1, 2, 3, 4, 5, 6\}. && \text{(Remove duplicates)} \end{aligned}$$

Hence $|\mathcal{H} + \mathcal{T}| = 8$.

(d) Explain why $|\mathcal{H} + \mathcal{T}| \neq |\mathcal{H}| + |\mathcal{T}|$.

Solution. Since $|\mathcal{H} + \mathcal{T}| = 8$, and $|\mathcal{H}| + |\mathcal{T}| = 7 + 5 = 12$, then clearly $|\mathcal{H} + \mathcal{T}| \neq |\mathcal{H}| + |\mathcal{T}|$.

Question 2. Let

$$f(x) = -2x^2 - 4x + 5$$
$$g(x) = \frac{3}{4}x - 4$$

(a) Compute $f(-2)$ **and** $g(3)$.

Solution.

$$\begin{aligned} f(-2) &= -2(-2)^2 - 4(-2) + 5 \\ &= -2(4) + 8 + 5 \\ &= -8 + 8 + 5 \\ &= 5 \end{aligned}$$

$$\begin{aligned} g(3) &= \frac{3}{4}(3) - 4 \\ &= \frac{9}{4} - 4 \\ &= \frac{9 - 16}{4} \\ &= -\frac{7}{4}. \end{aligned}$$

(b) Compute $f(g(f(0)))$.

Solution.
Step 1. Compute $f(0)$,

$$f(0) = -2(0)^2 - 4(0) + 5 = 0 - 0 + 5 = 5.$$

Step 2. Compute $g(f(0))$,

$$g(f(0)) = g(5) = \frac{3}{4}(5) - 4 = \frac{15}{4} - 4 = -\frac{1}{4}.$$

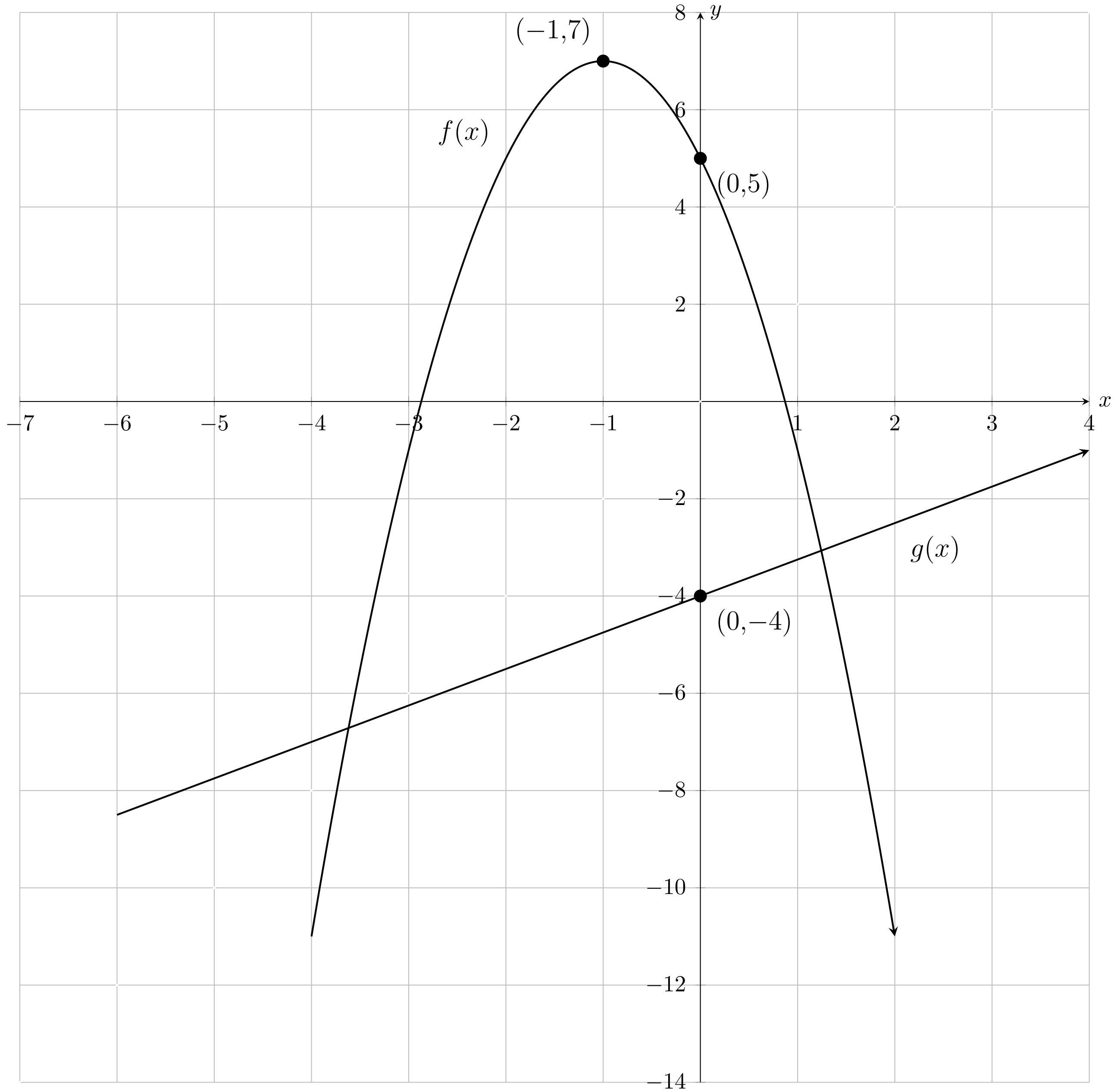
Step 3.

$$\begin{aligned} f(g(f(0))) &= f\left(-\frac{1}{4}\right) \\ &= -2\left(-\frac{1}{4}\right)^2 - 4\left(-\frac{1}{4}\right) + 5 \\ &= -\frac{2}{16} + 1 + 5 \\ &= -\frac{1}{8} + 6 \\ &= \frac{47}{8} \end{aligned}$$

(c) Provide a **sketch** of f on the axis sheet.

(d) Provide a **graph** of g on the axis sheet.

Solution. Solution:



(e) Does the vertex of f represent a minimum or maximum? Explain your answer.

Solution. The vertex **does** repersent a maximum since the direction of opening is downwards.

Question 3. Recall that when you divide two numbers, there will always be a remainder, sometimes the remainder is zero, other times it may not be zero, lets take a look at the following examples;

- $\frac{5}{3}$, the remainder is 2.
- $\frac{12}{2}$, the remainder is 0.
- $\frac{4}{7}$, the remainder is 4.

There is a better way to describe the remainder when dividing two numbers, that is to use the following notation,

$$\text{rem}(a, b).$$

The output of this is the remainder when dividing a by b, so if we repeat our previous examples again using this new notation, we would have,

- $\text{rem}(5, 3) = 2$
- $\text{rem}(12, 2) = 0$
- $\text{rem}(4, 7) = 4$

The set of all 'positive numbers' is written as,

$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, \dots\}.$$

Now lets define the following function,

$$f\colon \mathbb{N} \to \mathbb{N}$$
$$f(n) = \text{rem}\,(n, 2) + 1$$

Lets take a look at some examples to see how the function behaves,

- $f(4) = \text{rem}\,(4, 2) + 1 = 0 + 1 = 0$
- $f(11) = \text{rem}\,(11, 2) + 1 = 1 + 1 = 2$

Determine the following,

(a) $f(6)$

Solution.

$$f(6) = \text{rem}(6, 2) + 1 = 0 + 1 = 1.$$

(b) $f(15)$

Solution.

$$f(15) = \text{rem}(15, 2) + 1 = 1 + 1 = 2.$$

(c) $f(f(1))$

Solution.
Step 1. Compute $f(1)$,
$$f(1) = \text{rem}(1,2) + 1 = 1 + 1 = 2.$$

Step 2. Compute $f(f(1))$,
$$f(f(1)) = f(2) = \text{rem}(2,2) + 1 = 0 + 1 = 1.$$

(d) The range of f.
(Think carefully about this one, have you seen a pattern with the outputs so far?)

Solution. I claim that $\mathcal{R} = \{1, 2\}$ is the range.

Proof: If the input number n is even, then since the remainder of an even number divided by two is zero, we conclude that,
$$f(n) = \text{rem}(n,2) + 1 = 0 + 1 = 1.$$

Else if the input number n is odd, then since the remainder of an odd number divided by two is one, we conclude that,
$$f(n) = \text{rem}(n,2) + 1 = 1 + 1 = 2.$$

This covers every input case and hence we conlcude that all possible outputs (the range) is $\mathcal{R} = \{1, 2\}$.

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Question 4. Determine the Domain and Range of the following functions,

(a) $L(x) = -4x^2 + 8x + 4$

Solution.
$$\mathcal{D} = \mathbb{R}$$
$$\mathcal{R} = \{y \in \mathbb{R} \mid y \leq 8\}.$$

(b) $R(x) = -50|x-3| - 7$

Solution.
$$\mathcal{D} = \mathbb{R}$$
$$\mathcal{R} = \{y \in \mathbb{R} \mid y \leq -7\}.$$

(c) $Q(x) = \frac{-3}{2x-4} + \frac{4}{5}$

Solution.

$$\mathcal{D} = \{x \in \mathbb{R} \mid x \neq 2\}$$
$$\mathcal{R} = \left\{y \in \mathbb{R} \mid y \neq \frac{4}{5}\right\}.$$

(d) $g(x) = 11\sqrt{-3x+12} + 1$

Solution.

$$\mathcal{D} = \{x \in \mathbb{R} \mid x \leq 4\}$$
$$\mathcal{R} = \{y \in \mathbb{R} \mid y \geq 1\}.$$

(e) $x^2 + (y+11)^2 = 4.$

Solution.

$$\mathcal{D} = \{x \in \mathbb{R} \mid -2 \leq x \leq 2\}$$
$$\mathcal{R} = \{y \in \mathbb{R} \mid -13 \leq y \leq -9\}.$$