Functions Test 2

January 17, 2021

1 Preamble

This is a test covering what we have learnt so far in lecture. Student's **must show all work** to receive full marks.

2 Allowed Aids

The following aids are allowed on the Test

- **Open Book Test** (Your entire binder is allowed).

3 Restrictions:

- **NO** calculator's.

4 Remarks:

- $n \cdot \mathbf{S} = \underbrace{\mathbf{S} + \cdots + \mathbf{S}}_{\text{n times}}$. $(n \in \mathbb{N})$
- len($\mathbf{S}$) is number of bits in the binary string $\mathbf{S}$.
- floor(x) is the smallest integer less than or equal to x.

5 Name and Date:

Print your name and todays date below;

Name _____ Date _____

Part A - Multiple Choice

Question 1. Answer the following True/False questions,

1. Let $\mathrm{id}_{\mathbb{R}} \colon \mathbb{R} \to \mathbb{R}$ be the identity function on $\mathbb{R}$, then
$$\mathrm{id}_{\mathbb{R}}\Big(\mathrm{id}_{\mathbb{R}}^{-1}\Big(\mathrm{id}_{\mathbb{R}}\Big(\mathrm{id}_{\mathbb{R}}^{-1}\Big(\mathrm{id}_{\mathbb{R}}\Big(\mathrm{id}_{\mathbb{R}}^{-1}(-4)\Big)\Big)\Big)\Big)\Big) = 4.$$
Circle the correct answer: **True** **False**

2. Let $f \colon \mathbb{R} \to \mathbb{R}$, $f(x) = 2(x-1)^2$ be a function. Then f is not invertible.
Hint: Try using the Horizontal line test.
Circle the correct answer: **True** **False**

3. Let $\mathcal{X} = \{0.5, \pi, 3.8\}$ and $\mathcal{Y} = \{1, 4, 4.8\}$ be sets, define the following function,
   - $\psi \colon \mathcal{X} \to \mathcal{Y}$.
   - $\psi(x) = \mathrm{floor}(x) + 1$.

   Then ψ is an invertible function.
   Circle the correct answer: **True** **False**

4. Let $\mathcal{S} = \{10, 1100, 111000\}$ be a set of *binary strings* and $Y = \{5, 7, 3\}$ be a set of natural numbers, define the following function,
   - $\Delta \colon \mathcal{S} \to Y$.
   - $\Delta(\mathbf{S}) = \mathrm{len}(\mathbf{S}) + 1$.

   Then the function,
   - $\Delta^{-1} \colon Y \to \mathcal{S}$.
   - $\Delta^{-1}(y) = \mathrm{floor}(y/2) \cdot \mathbf{1} + \mathrm{floor}(y/2) \cdot \mathbf{0}$, (where $\mathbf{1}$ and $\mathbf{0}$ are *binary strings*),

   is the inverse function for Δ.
   Circle the correct answer: **True** **False**

5. Let $g(x) = \sqrt{x-4} - 1$ be a function, then $g^{-1}(x) = (x-1)^2 + 4$ is the inverse of g.
Circle the correct answer: **True** **False**

6. Let $f(x) = x^2$. Suppose we apply the following transformations to f,
   - Reflection across the y-axis.
   - Vertical compression by a factor of 3.
   - Horizontal compression by a factor of 2.
   - Horizontal shift, left by 2 units.
   - Vertical shift, down by 2 units.

   Then the corresponding transformation equation is $h(x) = \frac{1}{3} f(-2x - 4) - 2$.
   Circle the correct answer: **True** **False**

7. Let $f(x) = |x|$, and let $h(x) = -2f(5x - 3) + 9$ be a transformation of $f(x)$, then the corresponding coordinate transformation of f is,

$$(x, f(x)) \longrightarrow \left(\frac{x-3}{5}, -2f(x) + 9\right).$$

Circle the correct answer: **True** **False**

8. Let $\Omega \colon \mathcal{H} \to \mathcal{T}$ be a *surjective function*, then $|\mathcal{H}| = |\mathcal{T}|$.
Circle the correct answer: **True** **False**

9. Let $f(x) = x^2$, let $h(x) = -f(x)$ be a transformation of f, and let $r(x) = -h(-x)$ be a transformation of h, then $r(x) = f(x)$.
Circle the correct answer: **True** **False**

10. Let $f \colon \mathbb{N} \to \mathbb{R}$, $f(x) = x^2$ be a function. Then f is not invertible.
Circle the correct answer: **True** **False**

Part B - Solve all problems

Question 2. For each of the following, you are given a function and its definition. For each question,

(i) Prove that the function is invertible **or** prove that the function is not invertible.

(ii) Determine the range of the function.

(a) Let $\mathcal{S} = \{1100, 0011, 1010\}$, $\mathcal{T} = \{0011, 1010, 0100, 1100\}$ be sets of binary strings and define,

- $\lambda \colon \mathcal{S} \to \mathcal{T}$.
- $\lambda(\mathbf{S}) = \mathbf{s}_3\mathbf{s}_4\mathbf{s}_1\mathbf{s}_2$.

(b) Let $\mathcal{N} = \{2.7, 0.2, 1.3, 2.4\}$, $\mathcal{M} = \{0, 2, 4\}$ be sets and define,

- $\omega \colon \mathcal{N} \to \mathcal{M}$.
- $\omega(n) = 2 \cdot \text{floor}(n)$.

Question 3. Let $\beta\colon V \to W$ be an invertible function. Suppose that the formula for the invertible function is,

$$\beta^{-1}(w) = 2w - 4.$$

(a) Given the co-domain $W = \{-2, -4, 0, 2\}$ of β, recover the domain V.

(b) Determine the formula for $\beta(v)$.
Hint: Use the same algorithm for determining the inverse.

(c) Confirm that your formula for $\beta(v)$ is correct by checking that each element in V correctly maps back to the corresponding elements in W.

Question 4. Let $X = \{-3, 0, -5\}$ and $Y = \{5, 3, 0\}$ be sets, define the following function,

- $\Phi \colon X \to Y$.
- $\Phi(x) = |x|$.

Prove that the function,

- $\Phi^{-1} \colon Y \to X$.
- $\Phi^{-1}(y) = -\operatorname{id}_Y(y)$.

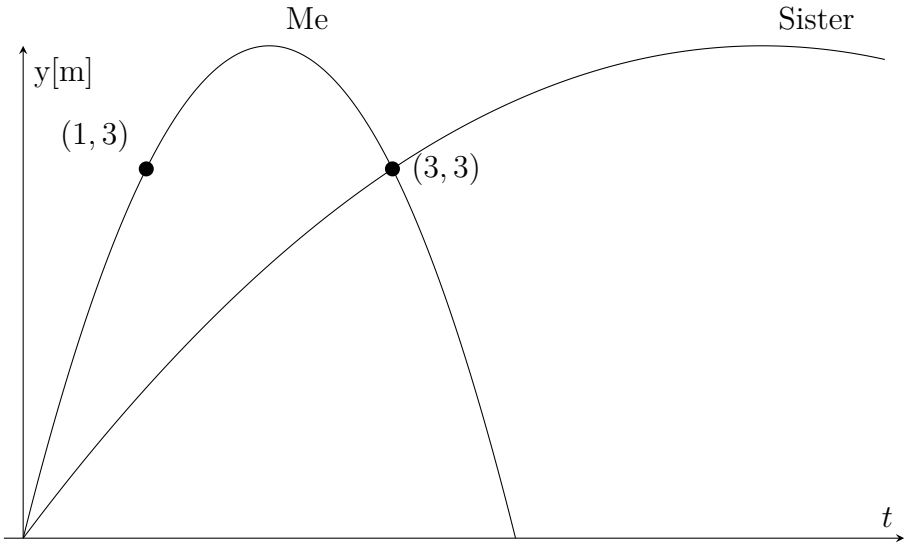
is the inverse function for Φ.

Question 5. Determine the inverse function for the following functions,

(a) $f(x) = 4x + 8$.

(b) $H(x) = \sqrt{x - 16} + 2$.

Question 6. Suppose that I throw a ball from ground level and my sister simultaneously throws a rock. She manages to hit the ball at exactly $t = 3$ seconds.



Let $M(x)$ denote my graph and $S(x)$ denote the graph of my sister. We can represent the graph of my sister as a horizontal scaling of my graph,

$$S(x) = M(B \cdot x) \qquad (B \in \mathbb{R}, B \neq 1)$$

Using the data given in the plot, determine the correct value for B.

Question 7. Let $f(x) = |x|$, and let $R(x) = -\frac{1}{2}f(2x+4)+1$ be a transformation of f.

(a) Describe the transformation.

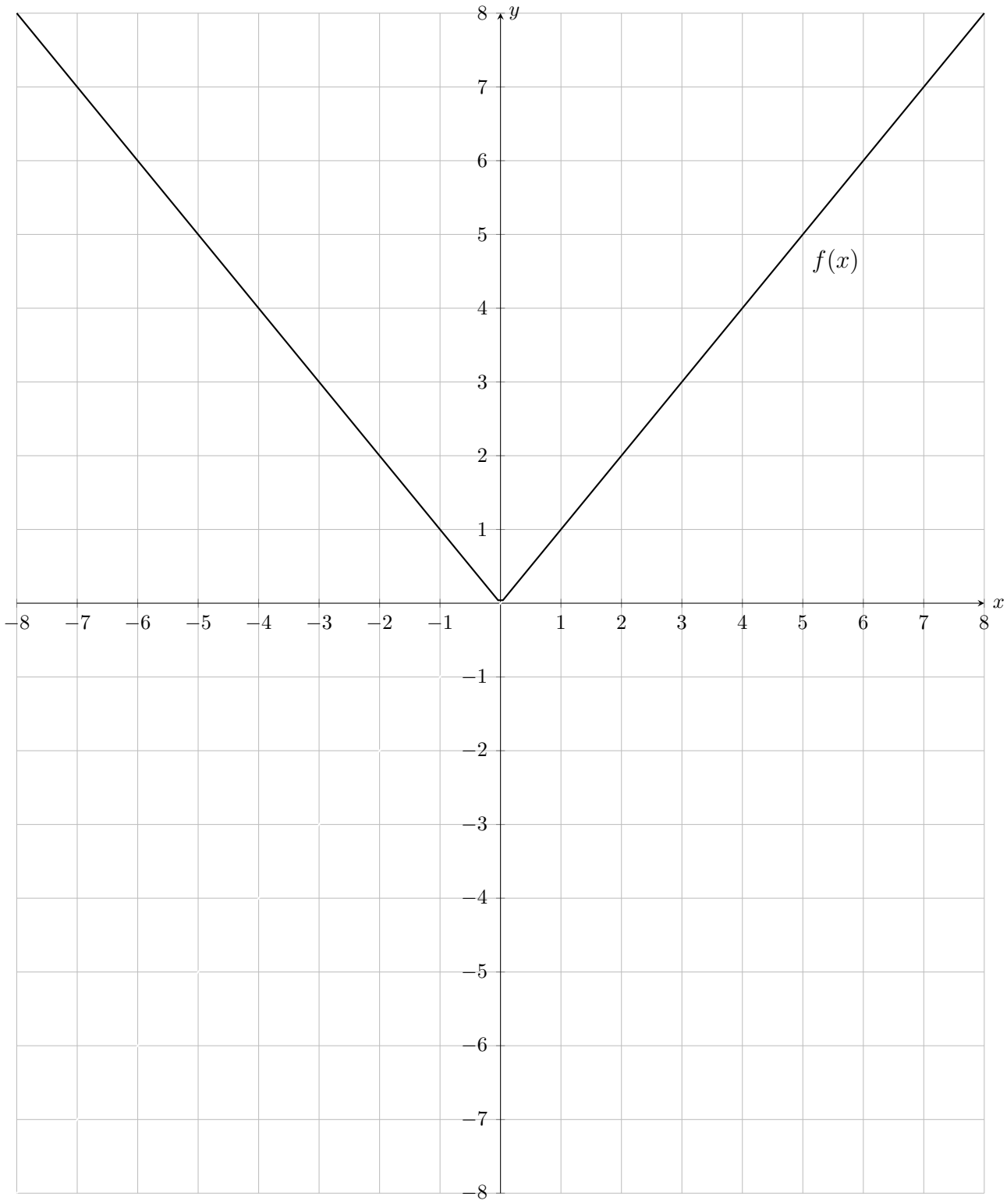
(b) Determine the expression for the coordinate transformation,

$$\left(\frac{x-H}{B}, Af(x)+K\right) = \Big(\qquad\qquad , \qquad\qquad \Big)$$

(c) Complete the following coordinate table to determine the corresponding transformed coordinates.

$(x, f(x))$	$\Big(\qquad\qquad , \qquad\qquad \Big)$
$(0,0)$	
$(-6,6)$	
$(6,6)$	
$(-4,4)$	
$(4,4)$	

(d) Using your results from the coordinate table, sketch the transformation $R(x)$. Be sure to **label** the transformed coordinates as well as the function.



Part C - Solve exactly one of the three problems.

Question 8. Let $A = \{a, b, c\}$, $B = \{x, y, z\}$ be sets. Let $\mathcal{L}$ be the set of all functions from $A \to B$. Let $\mathcal{M} = \{f \in \mathcal{L} \mid f \text{ is invertible}\}$. Determine $|\mathcal{M}|$ and justify that your answer is correct.
Note: Try counting all possible mapping diagrams between A and B. Two invertible functions are the same if their mapping diagrams are equivalent.

Question 9. Let A, B be sets, and let $F\colon A \to B$ be a function between the sets. We define the **nullset** of F to be,

$$\text{Null}(F) = \{a \in A \mid F(a) = 0\}.$$

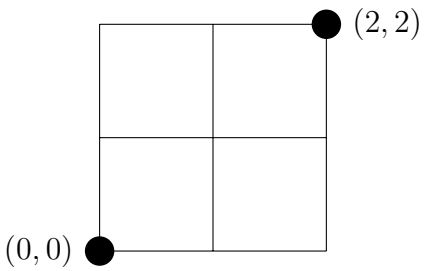
Let $G\colon \mathbb{R} \to \mathbb{R}$, $G(x) = 2x - 4$,

(a) Determine $\text{Null}(G)$.

(b) What do you think $\text{Null}(G^{-1})$ contains and why?

(c) Determine $\text{Null}(G^{-1})$.

Question 10. Consider the following grid below,



Let `R` denote a rightward move and `U` denote an upward move. We define a path from $(0, 0)$ to $(2, 2)$ to be a sequence of rightward and upward moves. For example `RRUU` is a path from $(0, 0)$ to $(2, 2)$, and so is `UURR`.

(a) Determine all paths from $(0, 0)$ to $(2, 2)$. Collect all of these paths into the set $\mathcal{P}$.
Hint: $|\mathcal{P}| = 6$.

(b) Let $Z = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$.

(i) Describe a function $\Psi\colon \mathcal{P} \to Z$ that assigns a relationship between the two sets.
Note: The function description can be in words.

(ii) Draw a mapping diagram for Ψ based on your description of the function.
Hint: In my description : $\Psi(\texttt{UURR}) = \{3, 4\}, \Psi(\texttt{RUUR}) = \{1, 4\}$ (Yours could be different)

(c) Describe the inverse function $\Psi^{-1}\colon Z \to \mathcal{P}$.

CHOOSE AND SOLVE ON NEXT PAGE

Question _____.