Assignment 3 Functions - SOLUTIONS
Due Date: Tuesday, Feburary 1

1 Preamble

This assignment covers everything taught so far. The solutions that you hand in should be **neat** and **legible**, this is an assignment, not a quiz, so I expect you to take your time and present thorough and detailed solutions.

This assignment has **(REPLACE)** questions. **Start early**.

2 Replacement

If your mark on this assignment is better than the last assignment, then I will replace the old assignment with this one. This means that your final assignment mark will consist of two assignment marks, either A1, A3 or A1, A2.

3 Bonus

If you type this assignment in LaTeX, I will give you a bonus 7%.

4 Name and Date:

Print your name and todays date below;

______________________	______________________
Name	Date

Question 1. State in your own words, what does it mean for a number to be prime. What types of numbers can be primes? Why do we care about primes?

Solution. A number x is prime if its only positive divisors are 1 and itself. Only natural numbers can be primes, and we care about as they are the building blocks for the fundamental theorem of arithmetic.

Question 2. Let $x \in \mathbb{Z}$ and let p be a prime.

(a) Determine all possible values of $\gcd(x, p)$.

Solution. Since p is a prime, we have the following possible scenarios. Either p divides x perfectly, or it doesn't. In the former case, $\gcd(x, p) = p$, in the latter case, since the only other divisor of p is 1, we have that $\gcd(x, p) = 1$.

(b) Determine all possible values of $\gcd(x^2, p)$.

Solution. The key observation here is that nothing has really changed form part (a), we still have either of the two possibilities, either p divides x^2 or it doesn't, $\gcd(x^2, p) = p$, in the former case and $\gcd(x^2, p) = 1$, in the latter case by similar reasoning from before.

(c) Determine all possible values of $\gcd(x, p^2)$.

Solution. In this case we have an additional scenario, either p divides x, or p^2 divides x, or neither. By similar reasoning from before we conclude that the possible values for the gcd are, $\gcd(x, p^2) = p^2, p, 1$. (The commas here mean the value can be this **or** this **or** this).

(d) Determine all possible values of $\gcd(x^2, p^2)$.

Solution. Again note that nothing has really changed from before, hence $\gcd(x^2, p^2) = p^2, p, 1$.

(e) Determine all possible values of $\gcd(p^2, p)$.

Solution. Notice in this situation, there are no obscurities or ambiguities, albeit the state of p is still arbitrary, you can convince yourself that regardless of the prime number we substitute, p will **always divide** p^2. Hence $\gcd(p^2, p) = p$.

Question 3. Let p be a prime number. Determine the value of $\text{rem}(p, 2)$ **and** explain how you got your answer.

Solution. The key observation here is that a prime number p **cannot** be an even number. Hence p must be an odd number, from assignment 1 we now that the remainder of an odd number divided by 2 is 1, and hence $\text{rem}(p, 2) = 1$.

Question 4. This question is meant for review as it will definitely appear on the Final. Let $f(x) = \sqrt{x}$, and let $R(x) = -2f(\frac{1}{2}x + 1) + 2$ be a transformation of f.

(a) Describe the transformation. **(Remember to Factor First)**

Solution. Let $A = -2, B = 1/2, H = 1, K = 2$.We first we factor $R(x)$ to obtain,

$$R(x) = -2f\left(\frac{1}{2}x + 1\right) + 2 = -2f\left(\frac{1}{2}(x + 2)\right) + 2.$$

From which we can describe the transformations,

- Since $A < 0$, f is reflected across the x-axis.
- f is horizontally shifted left by 2 units.
- f is vertically shifted up by 2 units.
- Since $|A| = 2$ and $2 > 1$ we conclude that f has been vertically stretched by a factor of 2.
- Since $|B| = \frac{1}{2}$ and $0 < \frac{1}{2} < 1$ we conclude that f has been horizontally strethced by a factor of 2.

(b) Determine the expression for the coordinate transformation,

$$\left(\frac{x - H}{B}, Af(x) + K\right) = (2(x - 1), -2f(x) + 2)$$

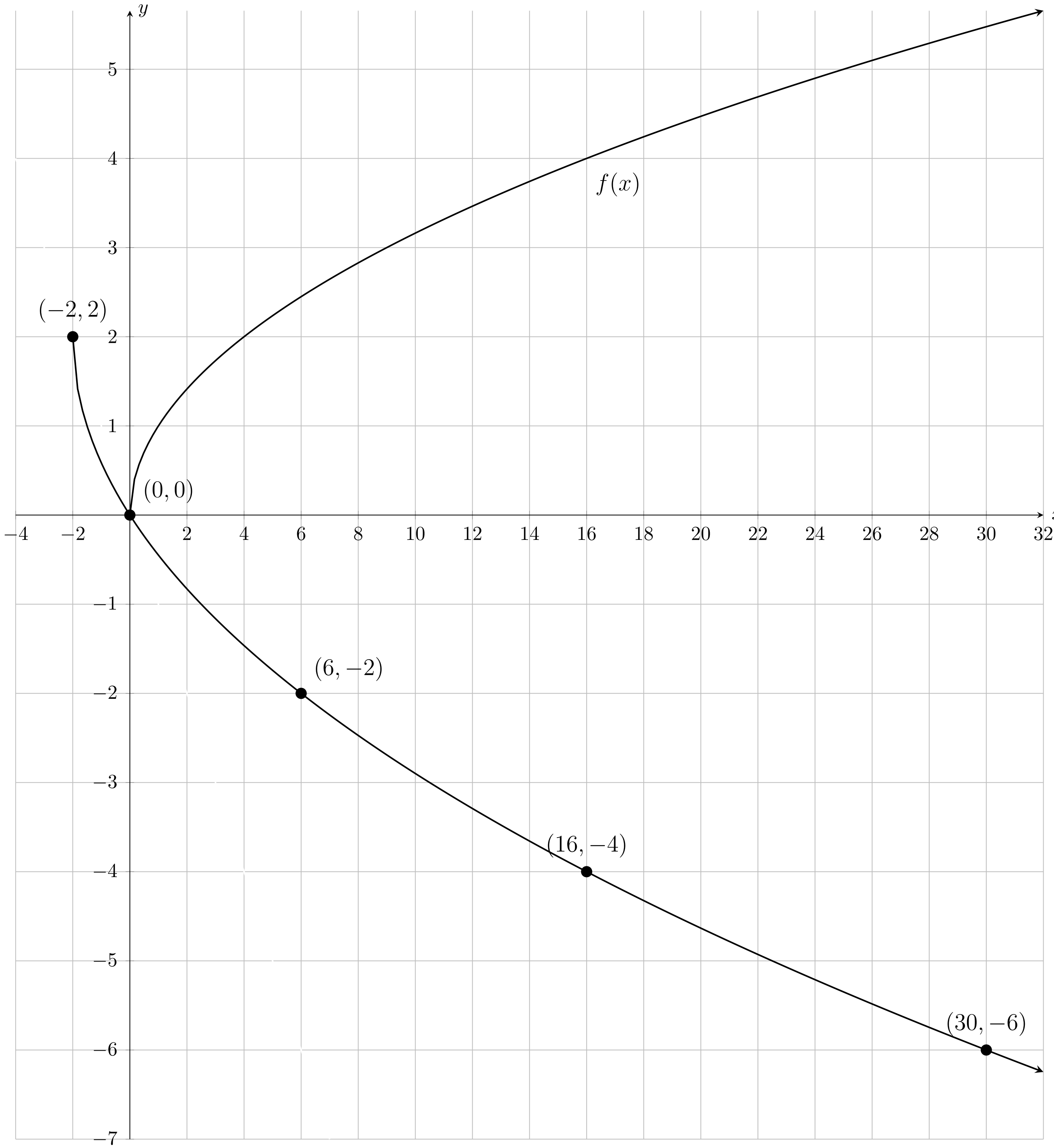
(c) Complete a coordinate table to determine the corresponding transformed coordinates using the following base coordinates,

$$(0, 0) \qquad (1, 1) \qquad (4, 2) \qquad (9, 3) \qquad (16, 4).$$

Solution.

$(x, f(x))$	$(2(x - 1), -2f(x) + 2)$
$(0, 0)$	$(-2, 2)$
$(1, 1)$	$(0, 0)$
$(4, 2)$	$(6, -2)$
$(9, 3)$	$(16, -4)$
$(16, 4)$	$(30, -6)$

(d) Using your results from the coordinate table, sketch the transformation $R(x)$ **on your axis sheet**. Be sure to **label** the transformed coordinates as well as the function.



Question 5. Preform a prime factorization of the following natural numbers.
Note: If the number is a prime itself, then state that it is.

(a) 3634.

Solution. $3634 = 2 \cdot 23 \cdot 79$.

(b) 555.

Solution. $555 = 3 \cdot 5 \cdot 37$.

(c) 663.

Solution. $663 = 3 \cdot 13 \cdot 17$.

(d) 991.

Solution. 991 is a prime number.

Question 6. **Fully** simplify the following exponential expressions.
(Leave answers with positive exponents)

(a) $-x^2(-x)^2x^{-3}$

Solution. $-x^2(-x)^2x^{-3} = -x$.

(b) $(x^{-4})/(y^2)^{-3}$.

Solution.

$$\left(x^{-4}\right)/\left(y^2\right)^{-3} = \frac{y^6}{x^4}.$$

(c) $(4^{-1}y^2z^{-3}x^8x^{-3})^{-3}y^{-6}z^9$.

Solution.

$$\left(4^{-1}y^2z^{-3}x^8x^{-3}\right)^{-3}y^{-6}z^9 = \frac{64z^{18}}{y^{12}x^{15}}.$$

(d) $(81a^3b^2z^{-6})^{-2}/(3a^9bz^{-4})^{-3}$.

Solution.

$$\left(81a^3b^2z^{-6}\right)^{-2}/\left(3a^9bz^{-4}\right)^{-3} = \frac{a^{21}}{243b}.$$

(e) $(16)^{\frac{3}{2}}(9)^{\frac{3}{2}}(4)^{\frac{-5}{2}}$.

Solution.

$$(16)^{\frac{3}{2}}(9)^{\frac{3}{2}}(4)^{\frac{-5}{2}} = 54.$$

Question 7. Textbook, Pg. 39 Q1. a),c),e)

Question 8. Textbook, Pg. 39 Q3. a),c),e)

Question 9. Textbook, Pg. 39 Q4. a),c),e)

Question 10. Textbook, Pg. 39 Q5. a),c),e)

Question 11. Textbook, Pg. 39 Q7. a),c),e)

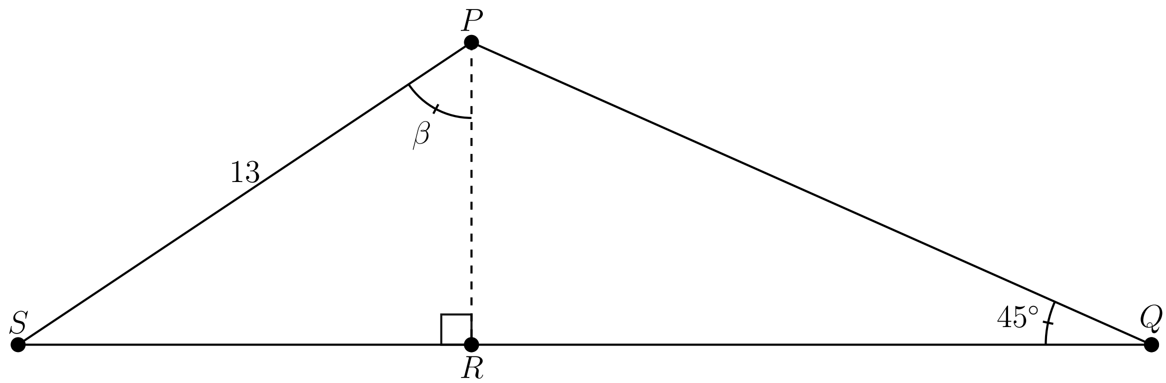
Question 12. Textbook, Pg. 94 Q4. a),c)

Question 13. Textbook, Pg. 94 Q6. c),d)

Question 14. Textbook, Pg. 94 Q6. b),c)

Solution (Q7-Q14)**. Refer to textbook solutions**

Question 15. You are given $\triangle SPQ$ where $SP = 13$, $\angle PQS = 45°$ and $\cos\beta = \sqrt{3}/2$. Determine the **exact** area of $\triangle SPQ$.



Solution.

$$A_{\triangle SPQ} = \frac{169}{8}\left(3 + \sqrt{3}\right).$$

Question 16. Describe in your own words what polar coordinates are? Why are they useful and what advantages do they provide as a metric?

Solution. Polar coordinates are a metric that parametrize euclidean space. They are useful because they provide an efficient and concise notation to describe coordinates, they are beneficial because they are isometries, or so called distance preserving metrics.

Question 17. Convert (a), (b) to polar coordinates and (c), (d) to standard coordinates.

(a) $\mathbf{P}(-3,4)$. (b) $\mathbf{R}(-1,-3)$. (c) $\mathbf{Q}(4,320°)$. (d) $\mathbf{T}(9,30°)$.

Solution.

(a) $\mathbf{P}(-3,4) = \mathbf{P}(5, 306.87°)$.

(b) $\mathbf{R}(-1,-3) = \mathbf{P}\left(\sqrt{10}, 251.565°\right)$.

(c) $\mathbf{Q}(4,320°) = \mathbf{Q}(3.064, -2.571)$.

(d) $\mathbf{T}(9,30°) = \mathbf{T}\left(\frac{9\sqrt{3}}{2}, \frac{9}{2}\right)$.

Question 18. Determine the six trigonometric ratios for the following angles,

(a) $\theta_1 = 60°$ (b) $\theta_2 = 220°$ (c) $\theta_3 = -240°$ (d) $\theta_4 = 330°$

Solution.

(a)

$$\sin 60° = \frac{\sqrt{3}}{2} \qquad \cos 60° = \frac{1}{2} \qquad \tan 60° = \sqrt{3}$$

$$\csc 60° = \frac{2}{\sqrt{3}} \qquad \sec 60° = 2 \qquad \cot 60° = \frac{1}{\sqrt{3}}$$

(b)

$$\sin 220° = -0.642787 \qquad \cos 220° = -0.76604 \qquad \tan 220° = -0.8390$$

$$\csc 220° = -1.55572 \qquad \sec 220° = -1.3054 \qquad \cot 220° = 1.1917$$

(c)

$$\sin 240° = -\frac{\sqrt{3}}{2} \qquad \cos 240° = -\frac{1}{2} \qquad \tan 240° = \sqrt{3}.$$

$$\csc 240° = -\frac{2}{\sqrt{3}} \qquad \sec 240° = -2 \qquad \cot 240° = \frac{1}{\sqrt{3}}.$$

(d)

$$\sin 330° = -\frac{1}{2} \qquad \cos 330° = \frac{\sqrt{3}}{2} \qquad \tan 330° = -\frac{1}{\sqrt{3}}.$$

$$\csc 330° = -2 \qquad \sec 330° = \frac{2}{\sqrt{3}} \qquad \cot 330° = -\sqrt{3}.$$