Functions Test 2 - SOLUTIONS

January 17, 2021

1 Preamble

This is a test covering what we have learnt so far in lecture. Student's **must show all work** to receive full marks.

2 Allowed Aids

The following aids are allowed on the Test

- **Open Book Test** (Your entire binder is allowed).

3 Restrictions:

- **NO** calculator's.

4 Remarks:

- $n \cdot \mathbf{S} = \underbrace{\mathbf{S} + \cdots + \mathbf{S}}_{\text{n times}}$. $\qquad (n \in \mathbb{N})$
- len($\mathbf{S}$) is number of bits in the binary string $\mathbf{S}$.
- floor(x) is the smallest integer less than or equal to x.

5 Name and Date:

Print your name and todays date below;

Name $\qquad\qquad$ Date

Part A - Multiple Choice

Question 1. Answer the following True/False questions,

1. Let $\mathrm{id}_{\mathbb{R}}\colon \mathbb{R} \to \mathbb{R}$ be the identity function on $\mathbb{R}$, then
$$\mathrm{id}_{\mathbb{R}}\Big(\mathrm{id}_{\mathbb{R}}^{-1}\Big(\mathrm{id}_{\mathbb{R}}\Big(\mathrm{id}_{\mathbb{R}}^{-1}\Big(\mathrm{id}_{\mathbb{R}}\Big(\mathrm{id}_{\mathbb{R}}^{-1}(-4)\Big)\Big)\Big)\Big)\Big) = 4.$$
<u>**Answer**</u>: **True** $\boxed{\textbf{False}}$, Note that $\mathrm{id}_{\mathbb{R}}^{-1} = \mathrm{id}_{\mathbb{R}}$.

2. Let $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 2(x-1)^2$ be a function. Then f is not invertible.
Hint: Try using the Horizontal line test.
<u>**Answer**</u>: $\boxed{\textbf{True}}$ **False** , Follow the hint and try using the horizontal line test to see why.

3. Let $\mathcal{X} = \{0.5, \pi, 3.8\}$ and $\mathcal{Y} = \{1, 4, 4.8\}$ be sets, define the following function,
   - $\psi\colon \mathcal{X} \to \mathcal{Y}$.
   - $\psi(x) = \mathrm{floor}(x) + 1$.

   Then ψ is an invertible function.
<u>**Answer**</u>: **True** $\boxed{\textbf{False}}$, Draw a mapping diagram to see why.

4. Let $\mathcal{S} = \{10, 1100, 111000\}$ be a set of *binary strings* and $Y = \{5, 7, 3\}$ be a set of natural numbers, define the following function,
   - $\Delta\colon \mathcal{S} \to Y$.
   - $\Delta(\mathbf{S}) = \mathrm{len}(\mathbf{S}) + 1$.

   Then the function,
   - $\Delta^{-1}\colon Y \to \mathcal{S}$.
   - $\Delta^{-1}(y) = \mathrm{floor}(y/2) \cdot \mathbf{1} + \mathrm{floor}(y/2) \cdot \mathbf{0}$, (where $\mathbf{1}$ and $\mathbf{0}$ are *binary strings*),

   is the inverse function for Δ.
<u>**Answer**</u>: $\boxed{\textbf{True}}$ **False**

5. Let $g(x) = \sqrt{x-4} - 1$ be a function, then $g^{-1}(x) = (x-1)^2 + 4$ is the inverse of g.
<u>**Answer**</u>: **True** $\boxed{\textbf{False}}$, $g^{-1}(x) = (x+1)^2 + 4$

6. Let $f(x) = x^2$. Suppose we apply the following transformations to f,
   - Reflection across the y-axis.
   - Vertical compression by a factor of 3.
   - Horizontal compression by a factor of 2.
   - Horizontal shift, left by 2 units.
   - Vertical shift, down by 2 units.

   Then the corresponding transformation equation is $h(x) = \frac{1}{3}f(-2x-4) - 2$.
<u>**Answer**</u>: $\boxed{\textbf{True}}$ **False**

7. Let $f(x) = |x|$, and let $h(x) = -2f(5x-3)+9$ be a transformation of $f(x)$, then the corresponding coordinate transformation of f is,

$$(x, f(x)) \longrightarrow \left(\frac{x-3}{5}, -2f(x)+9\right).$$

Answer: **True** $\boxed{\textbf{False}}$, The coordinate transformation is $((x+3)/5, -2f(x)+9)$

8. Let $\Omega\colon \mathcal{H} \to \mathcal{T}$ be a *surjective function*, then $|\mathcal{H}| = |\mathcal{T}|$.
Answer: **True** $\boxed{\textbf{False}}$

9. Let $f(x) = x^2$, let $h(x) = -f(x)$ be a transformation of f, and let $r(x) = -h(-x)$ be a transformation of h, then $r(x) = f(x)$.
Answer: $\boxed{\textbf{True}}$ **False** , Note that $h(x)$ is a reflection of f across the x-axis. $r(x)$ reflects h back through the x-axis, and hence f returns to its original position, $r(x)$ then reflects h through the x-axis, however this has no effect on x^2 due to its symmetry.

10. Let $f\colon \mathbb{N} \to \mathbb{R}$, $f(x) = x^2$ be a function. Then f is not invertible.
Answer: **True** $\boxed{\textbf{False}}$, This one was tricky, notice that the domain is $\mathbb{N}$ so this function will look different from a standard parabola. We'll take this up in class.

Part B - Solve all problems

Question 2. For each of the following, you are given a function and its definition. For each question,

(i) Prove that the function is invertible **or** prove that the function is not invertible.

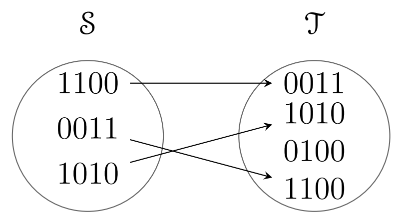
(ii) Determine the range of the function.

(a) Let $\mathcal{S} = \{1100, 0011, 1010\}$, $\mathcal{T} = \{0011, 1010, 0100, 1100\}$ be sets of binary strings and define,

- $\lambda \colon \mathcal{S} \to \mathcal{T}$.
- $\lambda(\mathbf{S}) = \mathbf{s}_3\mathbf{s}_4\mathbf{s}_1\mathbf{s}_2$.

Solution.

(i)



From the mapping diagram, we conclude that since $0100 \in \mathcal{T}$ is not mapped to, λ fails to be surjective, since λ fails to be surjective, it fails to be invertible.

(ii) Based on the outputs given in the mapping diagram, we conclude that the range of the function is,

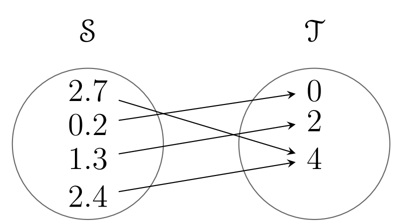
$$\mathcal{R}_\lambda = \{0011, 1010, 1100\}.$$

(b) Let $\mathcal{N} = \{2.7, 0.2, 1.3, 2.4\}$, $\mathcal{M} = \{0, 2, 4\}$ be sets and define,

- $\omega \colon \mathcal{N} \to \mathcal{M}$.
- $\omega(n) = 2 \cdot \text{floor}(n)$.

Solution.

(i)



From the mapping diagram, we conclude that since $\omega(2.7) = \omega(2.4) = 4$, ω fails to be injective, since ω fails to be injective, it fails to be invertible.

(ii) Based on the outputs given in the mapping diagram, we conclude that the range of the function is,

$$\mathcal{R}_\omega = \{0, 2, 4\}.$$

Question 3. Let $\beta\colon V \to W$ be an invertible function. Suppose that the formula for the invertible function is,

$$\beta^{-1}(w) = 2w - 4.$$

(a) Given the co-domain $W = \{-2, -4, 0, 2\}$ of β, recover the domain V.

Solution. Since β is invertible, each element in the domain V corresponds to a unique element in the co-domain W, the inverse function β^{-1} allows us to remap each element in W to its unique corresponding element in the domain V. Hence, it suffices to determine the output of $\beta^{-1}(w)$ for each $w \in W$. **(Ask me in class if you are still confused)**

W	$\beta^{-1}(w)$
-2	$\beta^{-1}(-2) = 2(-2) - 4 = -8$
-4	$\beta^{-1}(-4) = 2(-4) - 4 = -12$
0	$\beta^{-1}(0) = 2(0) - 4 = -4$
2	$\beta^{-1}(2) = 2(2) - 4 = 0$

And hence $V = \{-12, -8, -4, 0\}$.

(b) Determine the formula for $\beta(v)$.
Hint: Use the same algorithm for determining the inverse.

Solution. Proceeding with the inverse algorithm,

$$\begin{aligned}
\beta^{-1}(w) &= 2w - 4 \\
y &= 2w - 4 \\
y + 4 &= 2w \\
w &= \frac{1}{2}(w + 4) \\
\beta(v) &= \frac{1}{2}(v + 4).
\end{aligned}$$

(c) Confirm that your formula for $\beta(v)$ is correct by checking that each element in V correctly maps back to the corresponding elements in W.

Solution. To do so we proceed with a table of outputs,

V	$\beta(v)$
-12	$\beta(-12) = (-12 + 4)/2 = -4$
-8	$\beta(-8) = (-8 + 4)/2 = -2$
-4	$\beta(-4) = (-4 + 4)/2 = 0$
0	$\beta(0) = (0 + 4)/2 = 2$

By the table of outputs, we conclude that each element in V maps to the correct corresponding element in W. And hence we have certificate of correctness for our formula for β.

Question 4. Let $X = \{-3, 0, -5\}$ and $Y = \{5, 3, 0\}$ be sets, define the following function,

- $\Phi \colon X \to Y$.
- $\Phi(x) = |x|$.

Prove that the function,

- $\Phi^{-1} \colon Y \to X$.
- $\Phi^{-1}(y) = -\operatorname{id}_Y(y)$.

is the inverse function for Φ.

Solution. We confirm that both conditions of Definition 4.1 hold with mapping tables,

X	$\Phi^{-1}(\Phi(x))$
-3	$\Phi^{-1}(\Phi(-3)) = \Phi^{-1}(3) = -\operatorname{id}_Y(3) = -3$
0	$\Phi^{-1}(\Phi(0)) = \Phi^{-1}(0) = -\operatorname{id}_Y(0) = 0$
-5	$\Phi^{-1}(\Phi(-5)) = \Phi^{-1}(5) = -\operatorname{id}_Y(5) = -5$

Y	$\Phi(\Phi^{-1}(y))$
5	$\Phi(\Phi^{-1}(5)) = \Phi(-5) = \lvert -5 \rvert = 5$
3	$\Phi(\Phi^{-1}(3)) = \Phi(-3) = \lvert -3 \rvert = 3$
0	$\Phi(\Phi^{-1}(0)) = \Phi(0) = \lvert 0 \rvert = 0$

By our results from the mapping tables, we conclude that $\Phi^{-1}(y) = -\operatorname{id}_Y$ is indeed the inverse function for Φ.

Question 5. Determine the inverse function for the following functions,

(a) $f(x) = 4x + 8$.

Solution. Proceeding with the inverse algorithm,

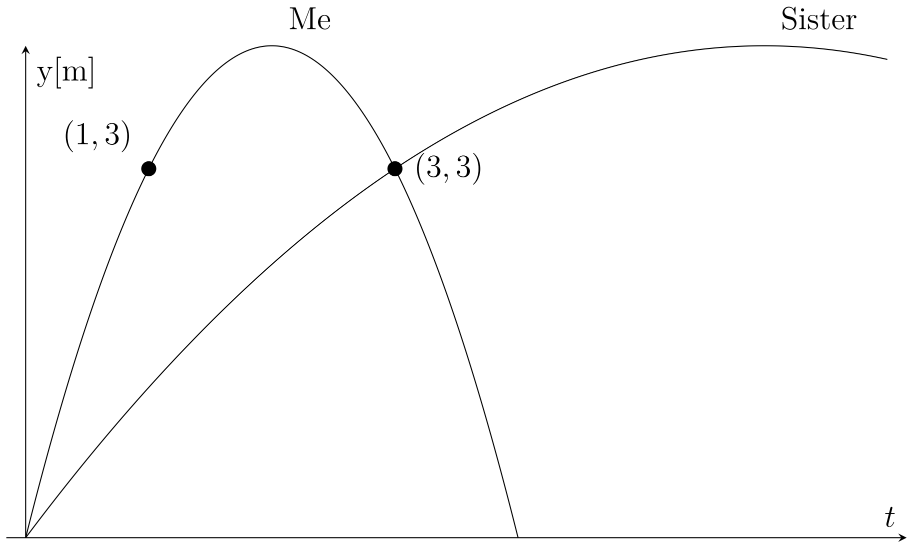
$$\begin{aligned} f(x) &= 4x + 8 \\ y &= 4x + 8 \\ y - 8 &= 4x \\ x &= \frac{1}{4}(y - 8) \\ f^{-1}(x) &= \frac{1}{4}(x - 8). \end{aligned}$$

(b) $H(x) = \sqrt{x - 16} + 2$.

Solution. Proceeding with the inverse algorithm,

$$\begin{aligned} H(x) &= \sqrt{x - 16} + 2 \\ y &= \sqrt{x - 16} + 2 \\ y - 2 &= \sqrt{x - 16} \\ (y - 2)^2 &= x - 16 \\ x &= (y - 2)^2 + 16 \\ f^{-1}(x) &= (x - 2)^2 + 16. \end{aligned}$$

Question 6. Suppose that I throw a ball from ground level and my sister simultaneously throws a rock. She manages to hit the ball at exactly $t = 3$ seconds.



Let $M(x)$ denote my graph and $S(x)$ denote the graph of my sister. We can represent the graph of my sister as a horizontal scaling of my graph,

$$S(x) = M(B \cdot x) \qquad (B \in \mathbb{R}, B \neq 1)$$

Using the data given in the plot, determine the correct value for B.

Solution (1). Since my sister hits the ball at exactly $t = 3$ seconds, it follows that $S(3) = 3$. Since $M(1) = 3$, we can horizontally stretch my graph $M(x)$ by a factor 3 to obtain,

$$S(3) = M\left(\frac{1}{3} \cdot 3\right) = M(1) = 3.$$

And hence,

$$B = \frac{1}{3}.$$

Solution (2). Another solution is to think about how to transform,

$$(1, 3) \longrightarrow (3, 3).$$

After which you could conclude that a horizontal stretch by a factor of 3 should do the job,

$$(1, 3) \longrightarrow \left(\frac{1}{1/3}, 3\right) = (3, 3).$$

Question 7. Let $f(x) = |x|$, and let $R(x) = -\frac{1}{2}f(2x+4)+1$ be a transformation of f.

(a) Describe the transformation.

Solution. Let $A = -1/2, B = 2, H = 4, K = 1$.We first we factor $R(x)$ to obtain,

$$R(x) = -\frac{1}{2}f(2x+4)+1 = -\frac{1}{2}f(2(x+2))+1.$$

From which we can describe the transformations,

- Since $A < 0$, f is reflected across the x-axis.
- f is horizontally shifted left by 2 units.
- f is vertically shifted up by 1 unit.
- Since $|A| = \frac{1}{2}$ and $0 < \frac{1}{2} < 1$ we conclude that f has been vertically compressed by a factor of 2.
- Since $|B| = 2$ and $2 > 1$ we conclude that f has been horizontally compressed by a factor of 2.

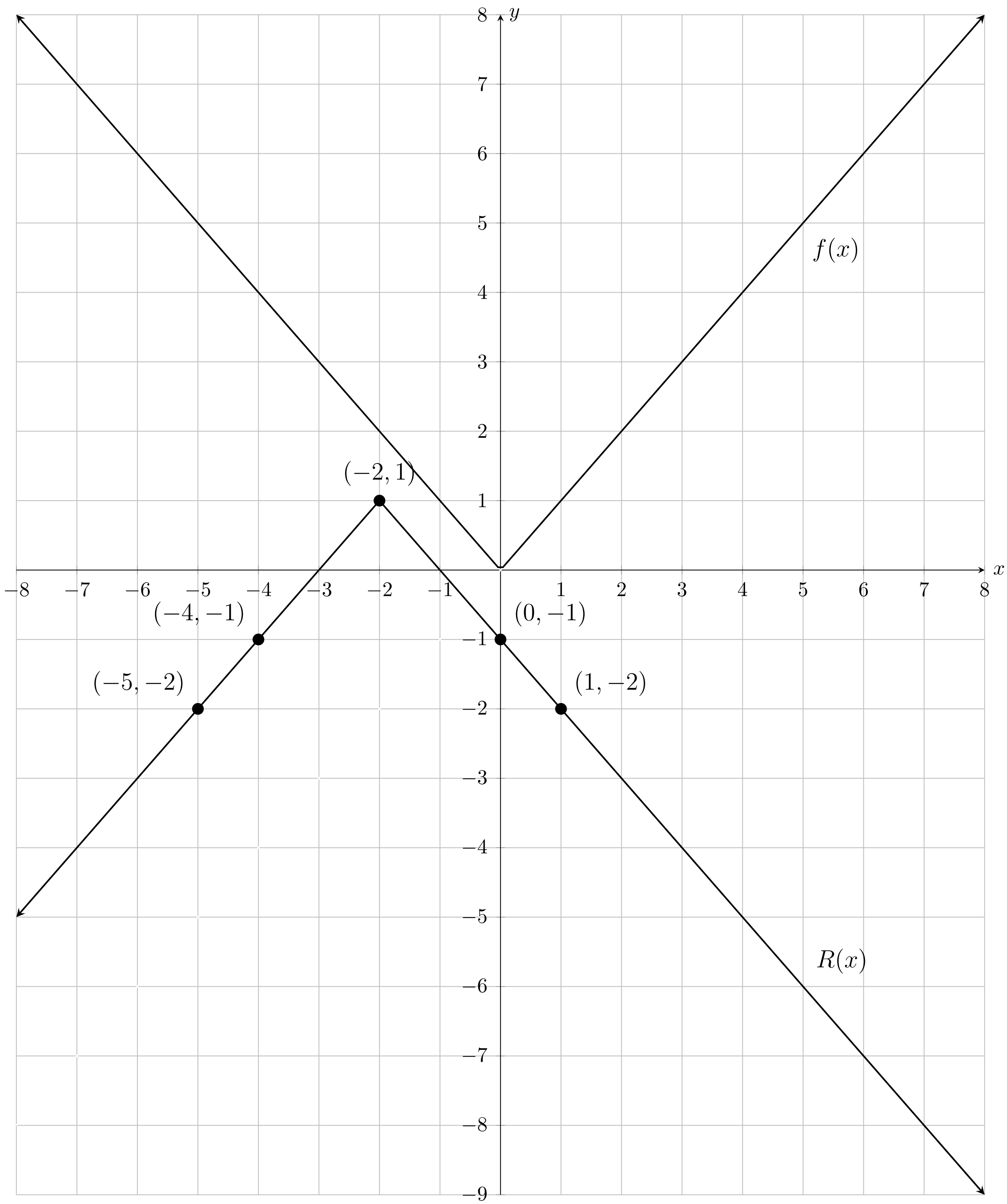
(b) Determine the expression for the coordinate transformation,

$$\left(\frac{x-H}{B}, Af(x)+K\right) = \left(\frac{x-4}{2}, -\frac{1}{2}f(x)+1\right)$$

(c) Complete the following coordinate table to determine the corresponding transformed coordinates.

$(x, f(x))$	$\left((x-4)/2, -\frac{1}{2}f(x)+1\right)$
$(0,0)$	$(-2,1)$
$(-6,6)$	$(-5,-2)$
$(6,6)$	$(1,-2)$
$(-4,4)$	$(-4,-1)$
$(4,4)$	$(0,-1)$

(d) Using your results from the coordinate table, sketch the transformation $R(x)$. Be sure to **label** the transformed coordinates as well as the function.



Part C - Solve exactly one of the three problems.

Question 8. Let $A = \{a, b, c\}$, $B = \{x, y, z\}$ be sets. Let $\mathcal{L}$ be the set of all functions from $A \to B$. Let $\mathcal{M} = \{f \in \mathcal{L} \mid f \text{ is invertible}\}$. Determine $|\mathcal{M}|$ and justify that your answer is correct.
Note: Try counting all possible mapping diagrams between A and B. Two invertible functions are the same if their mapping diagrams are equivalent.

Solution (1). Each function $f \in \mathcal{M}$ is an invertible function between A and B with a unique mapping diagram where each element in A is mapped to a unique element in B. Hence we can count all possible ways to construct such a mapping diagram. For $a \in A$, we can map it to either x, y, z, this gives us 3 choices. For each of those choices, we can map $b \in A$ to either of the 2 remaining choices. And lastly, for $c \in A$, we can map it to the remaining single choice. This gives us a total of 6 choices, and hence $|\mathcal{M}| = 6$.

Solution (2). Another approach to count all possible invertible mapping diagrams between A and B is to draw all of them.

Question 9. Let A, B be sets, and let $F \colon A \to B$ be a function between the sets. We define the **nullset** of F to be,

$$\mathrm{Null}(F) = \{a \in A \mid F(a) = 0\}.$$

Let $G \colon \mathbb{R} \to \mathbb{R}$, $G(x) = 2x - 4$,

(a) Determine $\mathrm{Null}(G)$.

(b) What do you think $\mathrm{Null}(G^{-1})$ contains and why?

(c) Determine $\mathrm{Null}(G^{-1})$.

Solution.

(a) Note that,

$$\begin{aligned} \mathrm{Null}(G) &= \{x \in \mathbb{R} \mid G(x) = 0\} \\ &= \{x \in \mathbb{R} \mid 2x - 4 = 0\}. \end{aligned}$$

At this point we conclude that $\mathrm{Null}(G)$ contains the solution to the equation $2x - 4 = 0$. If we solve the equation by isolating for x, we get that $x = 2$. Hence,

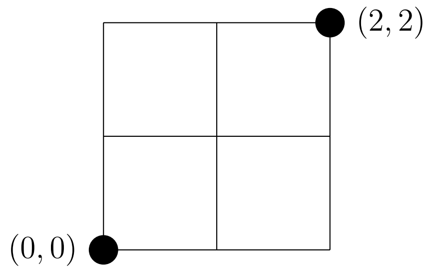
$$\mathrm{Null}(G) = \{2\}.$$

(b) $\mathrm{Null}(G^{-1})$ contains the y-intercept of G. This is because $G(0) = 4$ is the y-intercept of G, therefore the inverse function G^{-1} will map 4 back to 0. Hence, $G^{-1}(4) = 0$. Because G is invertible, G^{-1} can at most map a single element back to 0, or else it would fail to be injective. Hence G^{-1} contains only the y-intercept of G.

(c) From our previous argument, we conclude that,

$$\mathrm{Null}(G^{-1}) = \{4\}.$$

Question 10. Consider the following grid below,



Let `R` denote a rightward move and `U` denote an upward move. We define a path from $(0,0)$ to $(2,2)$ to be a sequence of rightward and upward moves. For example `RRUU` is a path from $(0,0)$ to $(2,2)$, and so is `UURR`.

(a) Determine all paths from $(0,0)$ to $(2,2)$. Collect all of these paths into the set $\mathcal{P}$.
Hint: $|\mathcal{P}| = 6$.

(b) Let $Z = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$.

(i) Describe a function $\Psi\colon \mathcal{P} \to Z$ that assigns a relationship between the two sets.
Note: The function description can be in words.

(ii) Draw a mapping diagram for Ψ based on your description of the function.
Hint: In my description : $\Psi(\texttt{UURR}) = \{3,4\}, \Psi(\texttt{RUUR}) = \{1,4\}$ (Yours could be different)

(c) Describe the inverse function $\Psi^{-1}\colon Z \to \mathcal{P}$.

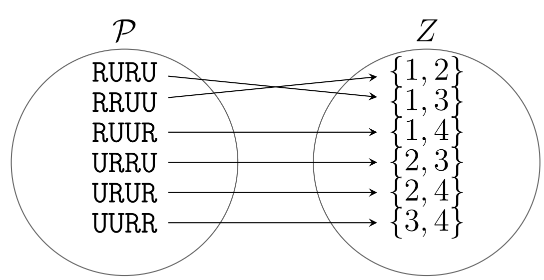
Solution.

(a) After counting all the paths, you should obtain,

$$\mathcal{P} = \{\texttt{RURU}, \texttt{RRUU}, \texttt{RUUR}, \texttt{URRU}, \texttt{URUR}, \texttt{UURR}\}.$$

(b) (i) The function can be described as follows, for every path $p \in \mathcal{P}$, $\Psi(p)$ assigns to the path p the set in Z which contains the integers which correspond to the positions of the character `R` in the path p. So for example, $\Psi(\texttt{URRU}) = \{2,3\}$.

(ii) By our description of Ψ from (i), we can deduce the mapping diagram,



(c) The inverse function can be described as follows, for every set $S \in Z$, $\Psi^{-1}(S)$ assigns to the set S the path in $\mathcal{P}$ which contains the character `R` at the positions indicated by the integers in the set S. So for example, $\Psi^{-1}(\{2,4\}) = \texttt{URUR}$.