Lecture 4 – Inverse Functions Part 2

Recall that in definition 3.4, we stated that **if** f is invertible, then an inverse function for f exists. The question remains, **what is this inverse function?**

Definition 4.1: *Let $\mathcal{A}$ and $\mathcal{B}$ be sets. Let $f\colon \mathcal{A} \to \mathcal{B}$ be some function. Let $g\colon \mathcal{B} \to \mathcal{A}$ be the function such that,*

1. *For all $a \in \mathcal{A}$, $g(f(a)) = \mathrm{id}_{\mathcal{A}}(a) = a$.*
2. *For all $b \in \mathcal{B}$, $f(g(b)) = \mathrm{id}_{\mathcal{B}}(b) = b$.*

*Then we say that g is the **inverse function** of f.*

Example 4.1: Let $\mathcal{A} = \{1, 2, 3, 5\}$ and $\mathcal{B} = \{2, 4, 6, 10\}$ be sets, lets define the following function,

- $f\colon \mathcal{A} \to \mathcal{B}$.
- $f(a) = 2a$.

I claim that the function,

- $g\colon \mathcal{B} \to \mathcal{A}$.
- $g(b) = \frac{1}{2}b$.

is the inverse function for f. **(Explanation in class)**

Notation 4.1: If a function f has an inverse, then we normally label this function as f^{-1}. This is meant to clear up confusion between functions, but ironically it causes more confusion amongst student. Basically whatever letter the function is labelled by, just put a '-1' on top to indicate the inverse function if it exists.

Example 4.2: Let $\mathcal{A} = \{1, 3, 5, 9\}$ and $\mathcal{B} = \{1, 2, 3, 5\}$ be sets, lets define the following function,

- $\phi\colon \mathcal{A} \to \mathcal{B}$.
- $\phi(a) = \frac{1}{2}(a + 1)$.

I claim that the function,

- $\phi^{-1}\colon \mathcal{B} \to \mathcal{A}$.
- $\phi^{-1}(b) = 2b - 1$.

is the inverse function. **(Explanation in class)**

Lets take a look at a more interesting function and its inverse, but before doing so, I will briefly introduce you to a new object.

4.1 Binary Strings

Definition 4.2: *A **binary string** is sequence of $1's$ and $0's$, formally referred to as bits. The string with no bits is denoted as ϵ (similar to the empty set).*

Example 4.3: The following are binary strings,

- $\mathbf{S} = 101011$.
- $\mathbf{T} = 001010$.
- $\mathbf{K} = 1$.

Definition 4.3: *Let $\mathbf{S}$ and $\mathbf{T}$ be binary strings. We define $\mathbf{S}+\mathbf{T}$ by gluing $\mathbf{S}$ and $\mathbf{T}$ together.*

Example 4.4: Let $\mathbf{S} = 101$ and $\mathbf{T} = 0001$, then $\mathbf{S} + \mathbf{T} = 1010001$.

Example 4.5: Let $\mathbf{S} = 111$ and $\mathbf{T} = \epsilon$, then $\mathbf{S} + \mathbf{T} = 111$.

Notation 4.2: Lets say we have a binary string $\mathbf{S}$, then $\mathbf{s}_i$ refers to the i^{th} bit of $\mathbf{S}$.

Example 4.6: Let $\mathbf{S} = 10100$, then $\mathbf{s}_1 = 1$, $\mathbf{s}_3 = 1$, $\mathbf{s}_5 = 0$, etc.

Example 4.7: Let $\mathbf{T} = 00100$, then $\mathbf{t}_1 = 0$, $\mathbf{t}_3 = 1$, $\mathbf{t}_4 = 0$ etc.

This notation allows us to build substrings of other strings, lets see how in the following example,

Example 4.8: Let $\mathbf{S} = 10100$, then $\mathbf{R} = \mathbf{s}_1\mathbf{s}_3\mathbf{s}_4 =$ **(In class)**.

Example 4.9: Let $\mathbf{T} = 000111$, then $\mathbf{B} = \mathbf{t}_2\mathbf{t}_3\mathbf{t}_6 =$ **(In class)**.

Example 4.10: Let $\mathcal{S} = \{111, 010, 110\}$ and $\mathcal{R} = \{0101, 1101, 1111\}$. Define the following function,

- $f\colon \mathcal{S} \to \mathcal{R}$.
- $f(\mathbf{S}) = \mathbf{S} + 1$.

We can draw a mapping diagram to understand how the function behaves.

Question: What is the inverse function f^{-1}? **(In class)**