Functions Test 1 - SOLUTIONS

December 14, 2021

1 Preamble

This is a test covering what we have learnt so far in lecture. Student's must show all work to receive full marks.

2 Allowed Aids

The following aids are allowed on the Test

- Pencil, Pen, Eraser, Highlighter, Ruler, Protractor, Spare sheets of **blank** paper.
- Reference sheet **(Double sided paper preprepared by student)**

3 Restrictions:

- **NO** calculator's.

4 Remarks:

- $\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$.
- $\text{rem}(x, y)$ is the remainder when you divide x by y.

5 Name and Date:

Print your name and todays date below;

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Name Date

Question 1. (10 marks) Answer the following True/False questions,

1. Let $\mathcal{R} = \{4, 5, 6, 7, 8\}$ and $\mathcal{H} = \emptyset$, then $\mathcal{R} + \mathcal{H} = \emptyset$.
   Answer: **False**, $\mathcal{R} + \mathcal{H} = \{4, 5, 6, 7, 8\}$.

2. Let $S = \{3, 4, 5\}$, then $S + S = S$.
   Answer: **True**.

3. $(\sqrt{4} + \pi) \in \mathbb{Z}$.
   Answer: **False**, $(4 + \pi) \in \mathbb{R}$.

4. The vertex of
   $$f(x) = 3(x + \pi)^2 - \sqrt{16}$$
   is $(-\pi, -8)$.
   Answer: **False**, The vertex is $(-\pi, -4)$.

5. The centre of the circle,
   $$(x - 1)^2 + (y - 2)^2 = 4$$
   is $(-1, -2)$.
   Answer: **False**, The centre of the circle is $(1, 2)$.

6. The vertex of,
   $$f(x) = -(x - 3)^2 - 4.$$
   represents a maximum.
   Answer: **True**.

7. The Domain and Range of,
   $$f(x) = -\frac{4}{2x + 1} + 8.$$
   is $\mathcal{D} = \{x \in \mathbb{R} \mid x \neq \frac{1}{2}\}$, $\mathcal{R} = \{y \in \mathbb{R} \mid y \neq 8\}$.
   Answer: **False**, $\mathcal{D} = \{x \in \mathbb{R} \mid x \neq -\frac{1}{2}\}$, $\mathcal{R} = \{y \in \mathbb{R} \mid y \neq 8\}$.

8. If $\mathcal{V} = \{v \in \mathbb{N} \mid v^2 = -1\}$, then $\mathcal{V}$ is the empty set.
   Answer: **True**.

9. The x-intercepts of $f(x) = x^2 - 5x + 6$ are $x_1 = -2$ and $x_2 = -3$.
   Answer: **False**, The x-intercepts are $x_1 = 2$ and $x_2 = 3$.

10. The vertex of $f(x) = x^2 + 6x + 5$ is $(-3, -4)$.
    Answer: **True**.

Question 2. (4 marks) Write down the elements of the following sets.
(**Recall:** $\mathbb{N} = \{1, 2, 3, 4, \dots\}$)

(a) $\mathcal{T} = \{a \in \mathbb{Z} \mid 3 < a < 7\}$.

Solution. $\mathcal{T} = \{4, 5, 6\}$.

(b) $\mathcal{X} = \{x \in \mathbb{N} \mid x \neq 1\}$.

Solution. $\mathcal{X} = \{2, 3, 4, 5, 6, \dots\}$.

(c) $\mathcal{Z} = \{y \in \mathbb{Z} \mid -3 \leq y \leq 0\} + \{i \in \mathbb{Z} \mid -2 \leq i \leq 1\}$
Hint: Add the two sets first.

Solution.

$$\begin{aligned} \mathcal{Z} &= \{-3, -2, -1, 0\} + \{-2, -1, 0, 1\} \\ &= \{-3, -2, -1, 0, -2, -1, 0, 1\} && \text{(Merging)} \\ &= \{-3, -2, -1, 0, 1\}. && \text{(Removing duplicates)} \end{aligned}$$

(d) $\mathcal{B} = \{x \in \mathbb{N} \mid \mathrm{rem}(x, 2) = 0\}$.

Solution. If you look carefully, all the numbers that satisfy the condition that $\mathrm{rem}(x, 2) = 0$ are all of the even numbers! Hence,

$$\mathcal{B} = \{2, 4, 6, 8, 10, \dots\}.$$

Question 3. (8 marks) Determine the Domain and Range of the following functions,

(a) $\mathcal{Y}(x) = -2\sqrt{5x - 10} - 8$.

Solution.

$$\mathcal{D} = \{x \in \mathbb{R} \mid x \geq 2\}$$
$$\mathcal{R} = \{y \in \mathbb{R} \mid y \leq -8\}.$$

(b) $x^2 + (y + 4)^2 = 16$.

Solution.

$$\mathcal{D} = \{x \in \mathbb{R} \mid -4 \leq x \leq 4\}$$
$$\mathcal{R} = \{y \in \mathbb{R} \mid -8 \leq y \leq 0\}.$$

(c) $\mathcal{L}(x) = -5|x + 1| - 3$.

Solution.

$$\mathcal{D} = \mathbb{R}$$
$$\mathcal{R} = \{y \in \mathbb{R} \mid y \leq -3\}.$$

(d) $\mathcal{E}(x) = -\frac{5}{2x-10} + 5$.

Solution.

$$\mathcal{D} = \{x \in \mathbb{R} \mid x \neq 5\}$$
$$\mathcal{R} = \{y \in \mathbb{R} \mid y \neq 5\}.$$

Question 4. Lets define the following function,

$$f : \mathbb{R} \to \mathbb{R}$$
$$f(x) = -x^2 + 4x + 3$$

(a) (4 marks) Convert $f(x)$ into vertex form by completing the square.

Solution.
Step 1.

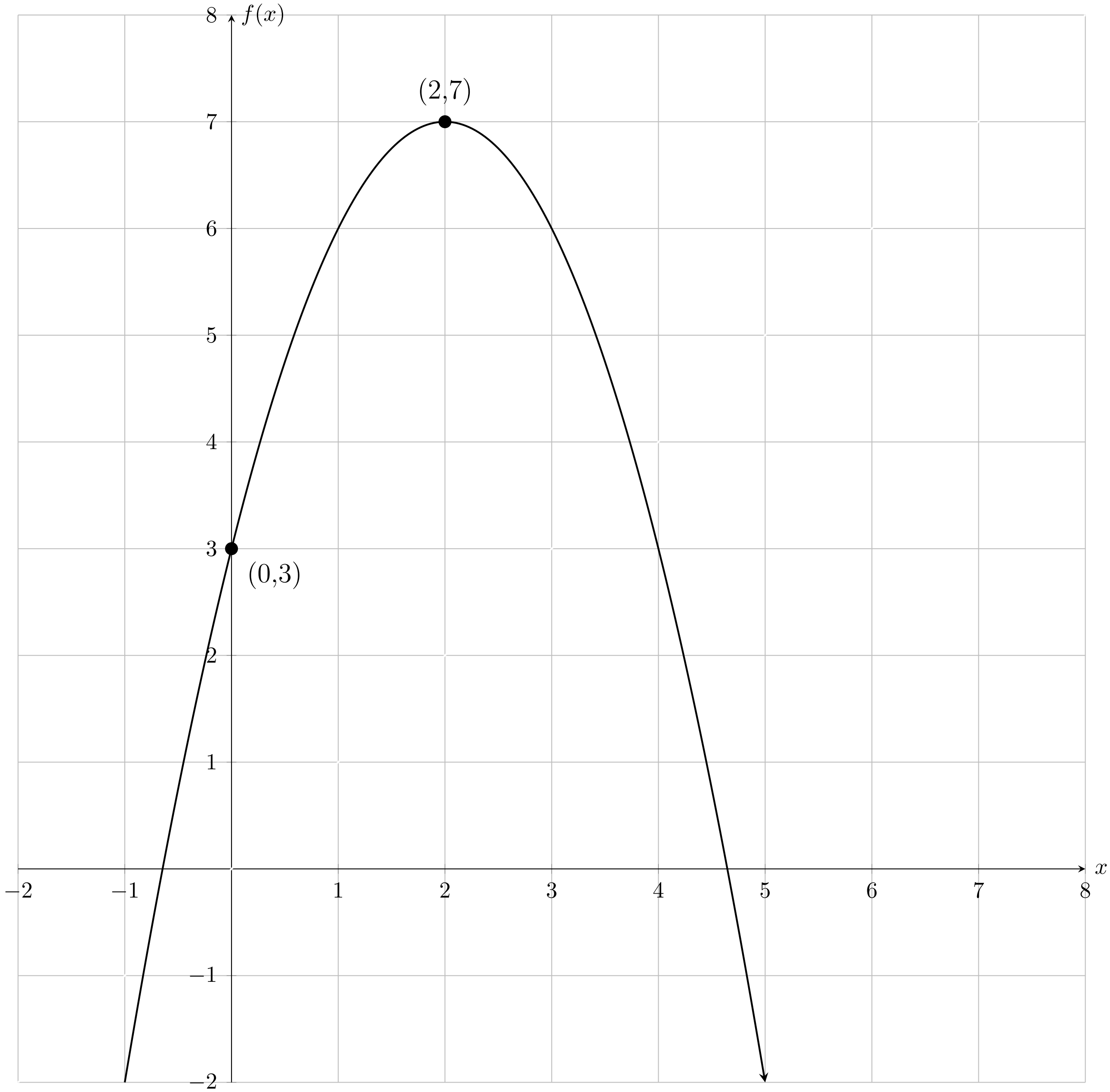
$$\begin{aligned}
0 &= b^2 - 4at \\
0 &= 4^2 - 4(-1)(t) \\
0 &= 16 + 4t \\
t &= -4.
\end{aligned}$$

Step 2. Since $a \cdot b = -1 \cdot 4 = -4$ we conclude that $a \cdot b < 0$ and hence,

$$\begin{aligned}
f(x) &= -\left(x - \sqrt{\frac{t}{a}}\right)^2 + [c - t] \\
&= -\left(x - \sqrt{\frac{-4}{-1}}\right)^2 + [3 - (-4)] \\
&= -(x - 2)^2 + 7.
\end{aligned}$$

(b) (3 marks) Sketch the function on the plot below. **(Label y-intercept and vertex)**

Solution. **(NEXT PAGE)**



$f(x)$
x
(2,7)
(0,3)
8
7
6
5
4
3
2
1
−1
−2
−2 −1 1 2 3 4 5 6 7 8

Question 5. (2 marks) We call a function idempotent if $f(f(x)) = f(x)$. Is the function $f(x) = x$ idempotent? Justify your answer.

Solution. Yes it is, $f(x) = x$ is the function that returns whatever is input, and hence

$$f(f(x)) = f(x).$$

Question 6. (7 marks) Factor the following quadratic functions,

(a) $f(x) = -x^2 + 9x - 20$.

Solution.

$$\begin{aligned} f(x) &= -\left(x^2 - 9x + 20\right) \\ &= -(x-4)(x-5). \end{aligned}$$

(b) $f(x) = 4x^2 - 1$.

Solution.

$$\begin{aligned} f(x) &= \left(\sqrt{4}x + \sqrt{|-1|}\right)\left(\sqrt{4}x - \sqrt{|-1|}\right) \\ &= (2x+1)(2x-1). \end{aligned}$$

(c) $f(x) = 2x^2 - 4x - 16$.
Hint: This can be factored simply.

Solution.

$$\begin{aligned} f(x) &= 2\left(x^2 - 2x - 8\right) \\ &= 2(x-4)(x+2). \end{aligned}$$

Question 7. (6 marks) Let $g(x) = 5x^2 + 14x - 3$,

(a) How many solutions will $g(x)$ have? (**Note:** $14^2 = 196$)

Solution. We turn to the discriminant formula to help answer this inquiry,

$$\begin{aligned} d &= b^2 - 4ac \\ &= (14)^2 - 4(5)(-3) \\ &= 196 + 60 \\ &= 256. \end{aligned}$$

Hence since $d > 0$, $g(x)$ will have two <u>distinct</u> solutions.

(b) Factor $g(x)$.

Solution.
Step 1. Find p, q such that,

$$\begin{aligned} p + q &= 14 \\ p \cdot q &= -15. \end{aligned}$$

After some thinking you should arrive at $p = 15$, $q = -1$. (Again how you labelled them was up to you).

Step 2. Compute t and k,

- $t = \gcd(|a|, |p|) = \gcd(5, 15) = 5$.
- $k = \gcd(|q|, |c|) = \gcd(|-1|, |-3|) = \gcd(1, 3) = 1$.

Step 3. Since $a \cdot q = 5 \cdot (-1) = -5$, we conclude that $a \cdot q < 0$ and hence,

$$g(x) = (tx - k)\left(\frac{a}{t}x + \frac{p}{t}\right) = (5x - 1)(x + 3).$$

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Question 8. (6 marks) Let $T(x) = -\frac{1}{2}(2x - 22)(x + 1)$. Convert $T(x)$ into vertex form.

Solution.
Step 1. Set both factors equal to null and solve,

$$\begin{aligned} 2x - 22 &= 0 \\ 2x &= 22 \\ x_1 &= 11 \end{aligned}$$

$$\begin{aligned} x + 1 &= 0 \\ x_2 &= -1 \end{aligned}$$

Step 2. Calculate h,

$$h = \frac{x_1 + x_2}{2} = \frac{11 + (-1)}{2} = \frac{10}{2} = 5.$$

Step 3. Calculate k,

$$\begin{aligned} k &= T(h) \\ &= T(5) \\ &= -\frac{1}{2}(2(5) - 22)(5 + 1) \\ &= -\frac{1}{2}(10 - 22)(6) \\ &= -\frac{1}{2}(-12)(6) \\ &= (6)(6) \\ &= 36. \end{aligned}$$

Step 4. Calculate a,

$$a = b \cdot m \cdot n = \left(-\frac{1}{2}\right)(2)(1) = -1.$$

Step 5. Finalize,

$$T(x) = a(x - h)^2 + k = -(x - 5)^2 + 36.$$

Question 9. (6 marks) A function is <u>nilpotent</u> if there exists some number t such that $f(f(t)) = 0$. Let $T(x) = x^2 - 1$.

(a) Determine $T(T(1))$.

Solution.
Step 1. Compute $T(1)$,

$$T(1) = (1)^2 - 1 = 1 - 1 = 0.$$

Step 2. Compute $T(T(1))$,

$$T(T(1)) = T(0) = 0^2 - 1 = -1.$$

(b) Determine $T(T(0))$.

Solution.
Step 1. We need to compute $T(0)$, however we did that in part (a) and got $T(0) = -1$.

Step 2. Compute $T(T(0))$,

$$T(T(0)) = T(-1) = (-1)^2 - 1 = 1 - 1 = 0.$$

(c) Is $T(x)$ <u>nilpotent</u>? Justify your answer.

Solution. Yes it is because $T(T(0)) = 0$. If you want to be thorough, then let $t = 0$, this would imply that

$$T(T(t)) = T(T(0)) = 0 = t.$$

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Question 10. (6 marks) Let $f(x) = x - 1$ and $g(x) = x + 1$. We call f and g <u>mutual inverses</u> of each other if **both** of the following conditions hold,

- For every number b, $f(g(b)) = b$.
- For every number a, $g(f(a)) = a$.

(a) Determine $f(g(1))$.

Solution.
Step 1. Compute $g(1)$,

$$g(1) = 1 + 1 = 2.$$

Step 2. Compute $f(g(1))$,

$$f(g(1)) = f(2) = 2 - 1 = 1.$$

(b) Determine $g(f(2))$.

Solution.
Step 1. Compute $f(2)$,
$$f(2) = 2 - 1 = 1.$$

Step 2. Compute $g(f(2))$,
$$g(f(2)) = g(1) = 1 + 1 = 2.$$

(c) Based on your answers from part (a) and (b), do you think f and g are inverses of each other?

Solution. I would say so because they are satisfying the conditions of mutual inverses so far. (BTW, you can give a formal proof that they are indeed mutual inverses of each other, ill leave that as a challenge for however wants to try it)

Question 11. (6 marks) Let $S = \{x \in \mathbb{R} \mid 8x^2 + 2x - 3 = 0\}$. Write down the elements of S.
Hint: Try factoring first.

Solution. Lets follow the hint by factoring the quadratic given in the set.
Step 1. Since $a \neq 1$, lets try factoring $f(x)$ by the non simple technique. Thus lets find the integers p, q such that,

$$p + q = 2$$
$$p \cdot q = -24.$$

After thinking long enough you should arrive at $p = 6$ and $q = -4$ (How you label them is up to you).

Step 2. Compute k and t,

- $t = \gcd(|a|, |p|) = \gcd(8, 6) = 2.$
- $k = \gcd(|q|, |c|) = \gcd(|-4|, |-3|) = \gcd(4, 3) = 1.$

Step 3. Since $a \cdot q = 8 \cdot (-4) = -32$, we conclude that $a \cdot q < 0$ and hence,

$$8x^2 + 2x - 3 = (tx - k)\left(\frac{a}{t}x + \frac{p}{t}\right) = (2x - 1)(4x + 3).$$

Step 4. At this point we have reduced the problem to determining the elements of ,

$$S = \{x \in \mathbb{R} \mid (2x - 1)(4x + 3) = 0\}.$$

The question remains, when is $(2x-1)(4x+3)=0$? This requires a subtle observation. When is the product of two numbers zero? In other words, when is $a \cdot b = 0$? If you think long enough, you'll conclude that this only happens when $a=0$ or $b=0$ (or both). Therefore we are left with the following two questions,

- For which x is $2x-1=0$?
- For which x is $4x+3=0$?

We proceed by answering our inquires,

$$\begin{aligned} 2x-1 &= 0 \\ 2x &= 1 \\ x &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} 4x+3 &= 0 \\ 4x &= -3 \\ x &= -\frac{3}{4} \end{aligned}$$

Hence the elements of S are,

$$S = \left\{ \frac{1}{2}, -\frac{3}{4} \right\}.$$