Assignment 2 Functions
Due Date: Wednesday, January 19

January, 2022

1 Preamble

This assignment covers everything taught so far. The solutions that you hand in should be **neat** and **legible**, this is an assignment, not a quiz, so I expect you to take your time and present thorough and detailed solutions.

2 Name and Date:

Print your name and todays date below;

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Name	Date

Question 1. There exists a function that allows us to determine the length of a binary string. We call this function len. Here a few examples to understand how it works,

- If $\mathbf{S} = 1001$, then $\text{len}(\mathbf{S}) = 4$.
- If $\mathbf{R} = 110001$, then $\text{len}(\mathbf{R}) = 6$.
- If $\mathbf{T} = \epsilon$, then $\text{len}(\mathbf{T}) = 0$.

We can also define the operation of *multiplication by a scalar* for binary strings. So suppose $n \in \mathbb{N}$ and $\mathbf{S}$ is some binary string, then,

$$n \cdot \mathbf{S} = \underbrace{\mathbf{S} + \cdots + \mathbf{S}}_{\text{n times}}$$

Again we resort to a few examples to demonstrate how multiplication by a scalar works,

- If $\mathbf{S} = 1001$, then $2 \cdot \mathbf{S} = \mathbf{S} + \mathbf{S} = 10011001$.
- If $\mathbf{R} = 0$, then $4 \cdot \mathbf{R} = \mathbf{R} + \mathbf{R} + \mathbf{R} + \mathbf{R} = 0000$.
- If $\mathbf{T} = 01$, then $3 \cdot \mathbf{T} = \mathbf{T} + \mathbf{T} + \mathbf{T} = 010101$.

Let $\mathbf{S} = 001$ and $\mathbf{T} = 11$, answer the following,

(a) Let $\mathbb{S}$ represent the set of all binary strings. Define the length function using mapping notation.

(b) Compute $\text{len}(\mathbf{S})$.

(c) Compute $\text{len}(\mathbf{T})$.

(d) Compute $\text{len}(\mathbf{S} + \mathbf{T})$.

(e) Compute $\text{len}(3 \cdot \mathbf{S})$.

(f) Compute $3 \cdot \text{len}(\mathbf{S})$.

(g) Compute $\text{len}(4 \cdot \mathbf{T})$.

(h) Compute $4 \cdot \text{len}(\mathbf{T})$.

Question 2. Let F be a function. We call F linear if both of the following conditions are satisfied,

1. For all inputs x and y,
$$F(x + y) = F(x) + F(y).$$
2. For all $c \in \mathbb{F}$, and all inputs x,
$$F(c \cdot x) = c \cdot F(x).$$

   If $\mathbb{F} = \mathbb{N}$, then based on your results from Question 1, do you think that the length function, len, is linear? Explain your answer.

Question 3. Sometimes in math we would like a function that simply gets rid of trailing decimals and returns a whole number, aka an integer. This function is known as the floor function. We define it with mapping notation as $\text{floor}\colon \mathbb{R} \to \mathbb{Z}$, and it works as follows, if $x \in \mathbb{R}$, then $\text{floor}(x)$ is the smallest integer that is less than or equal to x. Lets see how it works in the following examples,

- If $x = 4.2$, then $\text{floor}(x) = \text{floor}(4.2) = 4$.
- If $x = -7.4$, then $\text{floor}(x) = \text{floor}(-7.4) = -8$.
- If $x = 5$, then $\text{floor}(x) = \text{floor}(5) = 5$.
- If $x = 0.4$, then $\text{floor}(x) = \text{floor}(0.4) = 0$.

(a) Compute $\text{floor}(2.5)$.

(b) Compute $\text{floor}(6/3)$.

(c) Compute $\text{floor}(19/4)$.

(d) Let $f(x) = (x+1)/2$ and $g(x) = \sqrt{x-1}$, compute $\text{floor}(f(g(5)))$.

(e) Is the floor function linear? If it is, then justify your claim. If it is not, then provide a counter example to show that it fails to be linear.

(f) Is the floor function invertible? If it is, then justify your claim. If it is not, then provide a counter example to show that it fails to be surjective or injective.

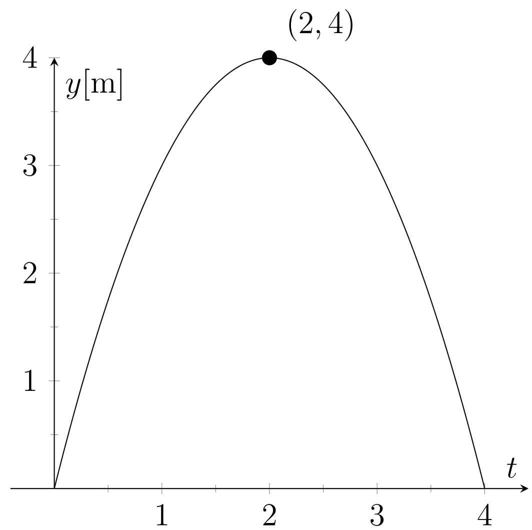
Question 4. Let $\mathcal{S} = \{1, 010, 00100, 0001000\}$, where each element is a binary string, and let $\mathcal{R} = \{4, 2, 6, 8\}$, where each element is a natural number.

(a) Come up with an invertible function Ψ between $\mathcal{S}$ and $\mathcal{R}$ and prove that your function is invertible. (**Hint:** Try using the length function)

(b) Come up with the correct formula for the inverse function Ψ^{-1} and prove that your formula is correct using mapping tables. (**Hint:** The correct formula uses the floor function)

Question 5. If you have ever kicked a soccer ball, you will have probably noticed that its trajectory closely imitates that of a parabola. This happens to be true under what we call ideal conditions, or in other words when the environment in which we kick the soccer ball is a vacuum. The primary motive behind this relationship is that gravitational acceleration does not effect motion in the horizontal direction, you'll learn more about this if you take a physics course. In this problem, we'll attempt to model different scenarios using transformations of functions.

Suppose I kick a ball from ground level at $t = 0$ seconds, I can model its trajectory as,



We plot the time elapsed on the x-axis, and the height of the ball (in meters) on the y-axis. From the figure above we see that at $t = 2$ seconds, the ball reached a height of 4 meters.

(A) We can model the equation of the trajectory as a transformation of $f(t) = -t^2$,

$$h(t) = f(t + A) + B.$$

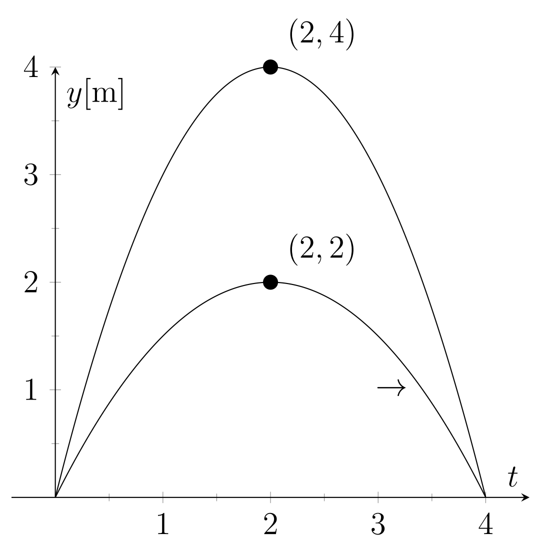
(a) What are the correct values for A and B?

(b) Determine the height of the ball at $t = 3$ seconds.

(c) Determine the height of the ball at $t = 4$ seconds.

(d) Determine the height of the ball at $t = 1$ seconds.

(B) I kick the ball a second time, this time not as high, and get the following trajectory (Graph indicated with arrow),



We can model the equation of the trajectory as a vertical scaling of $h(t)$,

$$g(t) = K \cdot h(t).$$

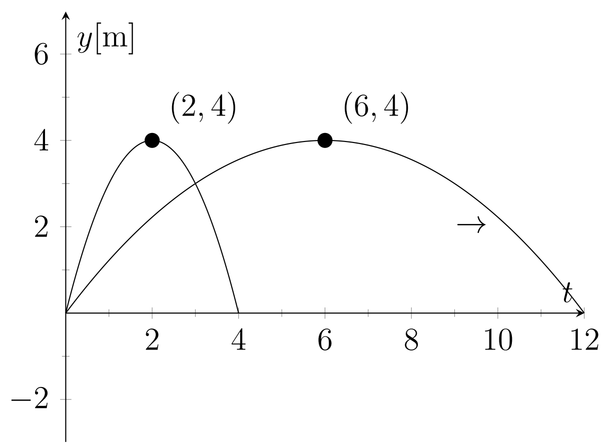
(a) What is the correct value for K? Also describe the precise vertical scaling that occurred.

(b) What was the height of the ball at $t = 1$ seconds?

(c) What was the height of the ball at $t = 3$ seconds?

(d) What was the height of the ball at $t = 4$ seconds?

(C) I then told my sister to kick the ball, and modelled her trajectory as (Graph indicated with arrow),



We can model the equation of her trajectory as a horizontal scaling of $h(t)$,

$$s(t) = h(D \cdot t).$$

(a) What is the correct value for D? Also describe the precise horizontal scaling that occurred.

(b) What was the height of the ball at $t = 3$ seconds?

(c) What was the height of the ball at $t = 9$ seconds?

(d) What was the height of the ball at $t = 12$ seconds?

(e) Based on her trajectory, did the ball travel farther horizontally for her kick? Explain your answer.