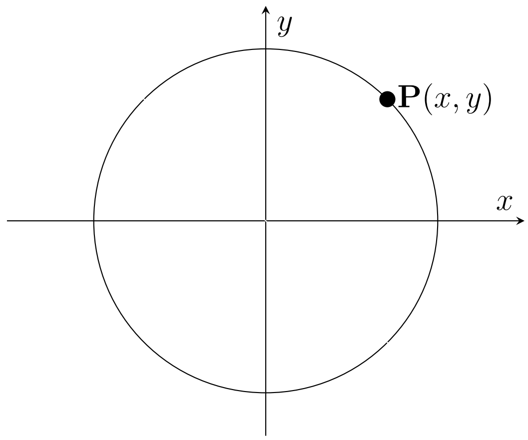
Lecture 9 – Trigonometry Part II

9.1 Motivation

Recall the equation of a circle with radius r,

$$x^2 + y^2 = r^2$$

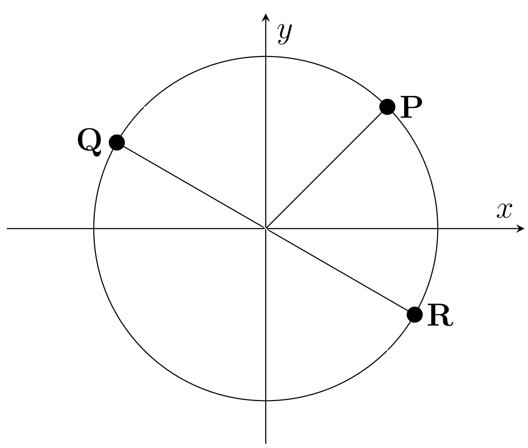


We would like a more efficient way of determining coordinates $\mathbf{P}(x, y)$ around the circle. To do so we employ some trigonometry to therefore introduce a different way to so called *parametrize* the circle. By *parametrize* we mean to define a new metric for measuring coordinates in standard euclidean geometry. You'll see why this ends up being more efficient.

9.2 Introduction

Definition 9.1: *Given a circle, the* ***terminal arm*** *at a point* ***P*** *is a ray from the origin to the point* $\mathbf{P}$.

Example 9.1: Suppose we have a circle $x^2 + y^2 = 4$ and points $\mathbf{P}, \mathbf{R}, \mathbf{Q}$. The following is a diagram of their corresponding terminal arms,



Definition 9.2: ***Polar coordinates*** *are a metric defined on circles where each coordinate is defined as* $\mathbf{P}(r, \theta)$*. Here,* r *corresponds to the radius of the circle and* θ *corresponds to the angle of the terminal arm at* $\mathbf{P}$*, known as the terminal angle.*

Example 9.2: Given a circle $x^2 + y^2 = 9$, plot the following polar coordinates,

(a) $\mathbf{P}(3, 60°)$

(b) $\mathbf{Q}(3, 120°)$ (**)

(c) $\mathbf{R}(3, 240°)$

(d) $\mathbf{T}(3, 330°)$ (**)

Negative angles

Remark 9.1: Given a polar coordinate $\mathbf{P}(r, \theta)$, if $\theta > 0°$ then this corresponds to a counter-clockwise rotation of our terminal arm to pivot it to its correct position on the circle. However if $\theta < 0°$, then we rotate clockwise.

Example 9.3: Given a circle $x^2 + y^2 = 4$, plot the following polar coordinates,

(a) $\mathbf{P}(2, 60°)$

(b) $\mathbf{Q}(2, -330°)$

(c) $\mathbf{K}(2, -240°)$ (**)

(d) $\mathbf{N}(2, -40°)$ (**)

Reference angles

Definition 9.3: *Given a polar coordinate* $\mathbf{P}(r, \theta)$*, the* ***reference angle*** α *is the <u>acute angle</u> between the terminal arm at* $\mathbf{P}$ *and the* x*-axis.*

Example 9.4: Given a circle $x^2 + y^2 = 4$, for each of the following polar coordinates determine the corresponding reference angle,

a) $\mathbf{P}(2, 60°)$ b) $\mathbf{R}(2, 150°)$ c) $\mathbf{Q}(2, 180°)$ (**) d) $\mathbf{T}(2, 240°)$ (**) e) $\mathbf{G}(2, 270°)$ f) $\mathbf{H}(2, 340°)$ (**)

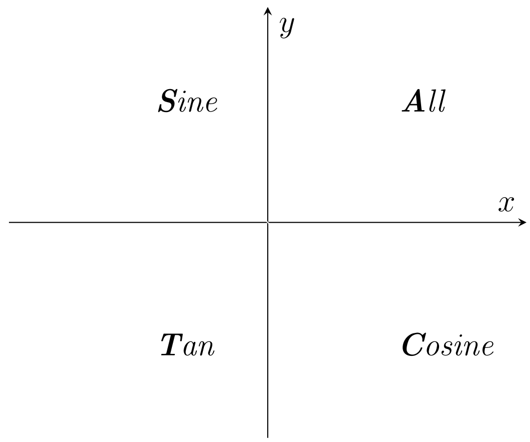
Example 9.5 (Negative Angles)**:** Given a circle $x^2 + y^2 = 4$, for each of the following polar coordinates determine the corresponding reference angle,

a) $\mathbf{P}(2, -45°)$ b) $\mathbf{R}(2, -90°)$ c) $\mathbf{Q}(2, -120°)$ (**) d) $\mathbf{T}(2, -230°)$ (**) e) $\mathbf{G}(2, -330°)$ (**)

Theorem 9.1 (CAST Rule)**:** *Given a polar coordinate* $\mathbf{P}(r, \theta)$*,*

- *If the terminal arm lies in quadrants I or IV, then* $\cos\theta$ *is positive.*
- *If the terminal arm lies in quadrants I or II, then* $\sin\theta$ *is positive .*
- *If the terminal arm lies in quadrants I or III, then* $\tan\theta$ *is positive.*

This can be summarized into the so called CAST rule,



Theorem 9.2 (Angle Symmetry)**:** *Given an angle* θ*, the following symmetric relationship holds*

$$\cos\theta = \pm\cos\alpha \qquad \sin\theta = \pm\sin\alpha \qquad \tan\theta = \pm\tan\alpha.$$

Where,

- α *is the corresponding reference angle.*
- *The* $(\pm)$ *sign can be determined by using the CAST rule.*

Example 9.6: Determine the exact value of the following trigonometric ratios,

a) $\sin 60°$ b) $\cos 120°$ c) $\tan 210°$ d) $\sin 270°$ e) $\cos 300°$

9.3 Converting : $\mathbf{P}(r, \theta) \to \mathbf{P}(x, y)$

Theorem 9.3: *Given a polar coordinate* $\mathbf{P}(r, \theta)$*, the corresponding standard coordinates are,*

$$x = r\cos\theta \qquad y = r\sin\theta.$$

Example 9.7: Convert the following polar coordinates to standard coordinates,

a) $\mathbf{P}(2, 60°)$ b) $\mathbf{Q}(2, 120°)$ (**) c) $\mathbf{T}(4, 210°)$ (**) d) $\mathbf{M}(1, 270°)$ e) $\mathbf{G}(6, 330°)$ (**)

Example 9.8 (Negative Angles)**:** Convert the following polar coordinates to standard coordinates,

a) $\mathbf{P}(2, -45°)$ b) $\mathbf{R}(3, -180°)$ c) $\mathbf{Q}(4, -150°)$ (**) d) $\mathbf{G}(8, -300°)$ (**)

9.4 Solving Trigonometric equations

Suppose we were given one of the following trigonometric ratios where the angle θ is unknown,

$$\cos\theta = \frac{x}{r} \qquad \sin\theta = \frac{y}{r} \qquad \tan\theta = \frac{y}{x}.$$

Suppose that we know the quadrant in which the angle θ lies within, then we can use the following procedure to solve for θ,

Step 1. Draw the terminal arm corresponding to the quadrant in which θ lies within.

Step 2. Compute the reference angle α,

$$\alpha = \cos^{-1}\left(\left|\frac{x}{r}\right|\right) \qquad \alpha = \sin^{-1}\left(\left|\frac{y}{r}\right|\right) \qquad \alpha = \tan^{-1}\left(\left|\frac{y}{x}\right|\right).$$

Step 3. Use α as well as your diagram to compute θ.

Example 9.9: For each of the following, you are given a trigonometric ratio, solve for θ. Assume that each angle θ lies in the **third** quadrant.

a) $\cos\theta_1 = -\frac{1}{2}$ b) $\sin\theta_2 = -\frac{\sqrt{3}}{2}$ (**) c) $\tan\theta_3 = 1$ (**)

Example 9.10: For each of the following, you are given a trigonometric ratio, solve for θ. Assume that each angle θ lies in the **second** quadrant.

a) $\cos\theta_1 = -\frac{1}{4}$ b) $\sin\theta_2 = \frac{1}{\sqrt{2}}$ (**)

9.5 Converting : $\mathbf{P}(x, y) \to \mathbf{P}(r, \theta)$

Theorem 9.4: *Given standard coordinates $\mathbf{P}(x, y)$, the corresponding polar coordinates are,*

$$r = \sqrt{x^2 + y^2} \qquad \tan\theta = \frac{y}{x}.$$

Where θ is the solution to the trigonometric equation.

Example 9.11: Convert the following standard coordinates to polar coordinates,

a) $\mathbf{P}\big(2, 2\sqrt{3}\big)$ b) $\mathbf{Q}(2, -5)$ c) $\mathbf{T}(-6, -8)$ (**) d) $\mathbf{M}\big(-3\sqrt{3}, 3\big)$ (**)