Functions Final Exam - SOLUTIONS

February 3, 2021

1 Preamble

This final exam covers everything we have learned in this course, with emphasize towards material after test 2. Student's **must show all work** to receive full marks.

2 Allowed Aids

The following aids are allowed on the Test

- Pencil, Pen, Eraser, Highlighter, Ruler, Protractor, Spare sheets of **blank** paper.
- Reference sheet **(Double sided paper preprepared by student)**

3 Restrictions:

- **NO** calculator's.

4 Name and Date:

Name ____________________ Date ____________________

Part A - Multiple Choice

Question 1. Answer the following True/False questions,

1. Let $\mathcal{S} = \{1, 2, 3\}$, then $\mathcal{S} + \mathcal{S} = \mathcal{S} + \mathcal{S} + \mathcal{S}$.
   Answer: $\boxed{\textbf{True}}$ **False** , $\mathcal{S} + \mathcal{S} = \mathcal{S}$ and $\mathcal{S} + \mathcal{S} + \mathcal{S} = \mathcal{S}$.

2. $\left(\sqrt{4} + \sqrt{64}\right) \notin \mathbb{N}$.
   Answer: **True** $\boxed{\textbf{False}}$, Note that $\sqrt{4} + \sqrt{64} = 2 + 8 = 10$ and $10 \in \mathbb{N}$.

3. The number 29 is a prime number.
   Answer: $\boxed{\textbf{True}}$ **False**

4. Let $T = \{x \in \mathbb{Z} \mid |x| = -1\}$, then T is **not** empty.
   Answer: **True** $\boxed{\textbf{False}}$, The absolute value of any number is always a positive number by definition, hence this set is empty.

5. The vertex of
   $$g(x) = 3\left(x + \sqrt{4}\right)^2 - 4^2$$
   is $(-4, -4)$.
   Answer: **True** $\boxed{\textbf{False}}$, The vertex here is $(-2, -16)$.

6. The vertex of,
   $$H(x) = -(x + 2)^2 + 1.$$
   represents a minimum.
   Answer: **True** $\boxed{\textbf{False}}$, The vertex here represents a maximum.

7. Let $f(x) = \sqrt{x}$. Suppose we apply the following transformations to f,
   - Reflection across the x-axis.
   - Vertical stretch by a factor of 2.
   - Horizontal compression by a factor of 2.
   - Horizontal shift, right by 2 units.
   - Vertical shift, down by 4 units.

   Then the corresponding transformation equation is $h(x) = -2f(2x - 4) - 4$.
   Answer: $\boxed{\textbf{True}}$ **False**

8. Let $f(x) = |x|$, and let $h(x) = -f(2x+4) - 5$ be a transformation of $f(x)$, then the corresponding coordinate transformation of f is,
   $$(x, f(x)) \longrightarrow \left(\frac{x+4}{2}, -f(x) - 5\right).$$
   Answer: **True** $\boxed{\textbf{False}}$, The correct coordinate transformation is $\left(\frac{x-4}{2}, -f(x) - 5\right)$.

9. Let $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 2\sqrt{x} + 1$ be a function. Then f is not invertible.
   Hint: Try using the Horizontal line test.
   Answer: **True** $\boxed{\textbf{False}}$, Try using the horizontal line test to see why.

10. Let $\mathcal{X} = \{45°, 60°, 240°\}$ and $\mathcal{Y} = \left\{0, 1, \sqrt{3}\right\}$ be sets, define the following function,

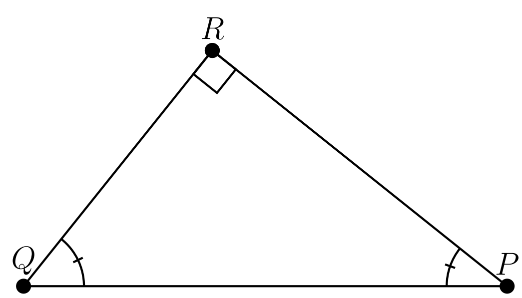
   - $\omega\colon \mathcal{X} \to \mathcal{Y}$.
   - $\omega(x) = \tan(x)$.

   Then ω is an invertible function.
   Answer: **True** $\boxed{\textbf{False}}$, Note that $\tan(60°) = \tan(240°) = \sqrt{3}$ and hence ω fails to be injective which asserts that it fails to be invertible as well.

11. Let $\triangle PQR$ be a **right triangle** with angle $\angle PQR = 60°$ and *hypotenuse* $PQ = 8$. Then $QR = 4$.
   Answer: $\boxed{\textbf{True}}$ **False**

12. The exact value of
   $$\sin 150° \cdot \sec 240° + \tan 240° \tan 30° = 0$$
   Answer: $\boxed{\textbf{True}}$ **False**

13. Let $\triangle PQR$ be a **right triangle** with $PQ = 6$ and $QR = 3\sqrt{3}$. Then, $\angle PQR = 60°$.



   Answer: **True** $\boxed{\textbf{False}}$, $\angle PQR = 30°$.

14. $\sin 330° = -0.5$.
   Answer: $\boxed{\textbf{True}}$ **False**

15. Suppose we have the standard coordinates $\mathbf{P}\left(2, -\sqrt{12}\right)$, then the corresponding polar coordinates are $\mathbf{P}(4, 60°)$.
   Answer: **True** $\boxed{\textbf{False}}$, The corresponding polar coordinates are $\mathbf{P}(4, 300°)$.

16. $\sqrt{4^4 \cdot 3^2 \cdot 2} = 48\sqrt{2}$.
   Answer: $\boxed{\textbf{True}}$ **False**

Part B

Question 2. Explain in your own words, what is a function?

Solution. Suppose we have two sets $\mathcal{S}, \mathcal{T}$, a function from the set $\mathcal{S}$ to $\mathcal{T}$, is a rule which assigns to each element in $\mathcal{S}$ a corresponding element in $\mathcal{T}$.

Question 3. Given a function $f \colon A \to B$, explain in your own words, what is the definition of the range of f, $\mathcal{R}_f$, what does it contain? Is it necessarily true that $\mathcal{R}_f = B$?

Solution. The range $\mathcal{R}_f$ is a <u>set</u> which contains all of the output values of the function after processing each element from the domain. It is <u>not</u> necessarily true that $\mathcal{R}_f = B$ but rather this a special case of surjectiveness.

Question 4. Given a function $f \colon A \to B$, explain in your own words what do we mean when we say that f is invertible?

Solution. We say that f is invertible if it is both surjective and injective. By surjective, we mean that each element in the co-domain B is mapped to and by injective we mean that no two elements in the domain A map to the same element in the co-domain B.

Question 5. Explain what the horizontal line test is as well as the vertical line test.

Solution. For a function f, the horizontal line test is used to provide a certificate of invertibility while the vertical line test asserts wether or not f is a proper function.

Part C

Question 6. Let $F(x) = x^3 + 1$, and $G(x) = 2x^2 + x - 1$ be functions,

(a) Compute $F(-1)$.

Solution.

$$F(-1) = (-1)^3 + 1 = -1 + 1 = 0.$$

(b) Compute $G(2)$.

Solution.

$$G(2) = 2(2)^2 + 2 - 1 = 8 + 2 - 1 = 9.$$

(c) Compute $F(G(1))$.

Solution.

$$\begin{aligned} G(1) &= 2(1)^2 + 1 - 1 = 2 + 1 - 1 = 2 \\ F(G(1)) &= F(2) = (2)^3 + 1 = 8 + 1 = 9. \end{aligned}$$

(d) Compute $G(F(F(0)))$.

Solution.

$$\begin{aligned} F(0) &= (0)^3 + 1 = 1 \\ F(F(0)) &= F(1) = (1)^3 + 1 = 1 + 1 = 2 \\ G(F(F(0))) &= G(2) = 9. \end{aligned}$$

Question 7. Let $g(x) = 2x^2 - 4x + 4$,

(a) How many solutions will $g(x)$ have?

Solution. Using the discriminant formula,

$$\begin{aligned} d &= b^2 - 4ac \\ &= (4)^2 - 4(2)(4) \\ &= 16 - 32 \\ &= -16. \end{aligned}$$

Since $d < 0$, $g(x)$ will have <u>no</u> solutions.

(b) Convert $g(x)$ into vertex form by completing the square.

Solution.
Step 1.

$$\begin{aligned} 0 &= b^2 - 4at \\ 0 &= (-4)^2 - 4(2)t \\ 0 &= 16 - 8t \\ t &= 2. \end{aligned}$$

Step 2. Since $a \cdot b = 2 \cdot -4 = -8$, we conclude that $a \cdot b < 0$, hence,

$$\begin{aligned} g(x) &= a\left(x - \sqrt{\frac{t}{a}}\right)^2 + [c - t] \\ &= 2\left(x - \sqrt{\frac{2}{2}}\right)^2 + [4 - 2] \\ &= 2(x - 1)^2 + 2. \end{aligned}$$

(c) Does the vertex of $g(x)$ represent a minimum or maximum, justify your answer.

Solution. Since $a = 2$ and $2 > 0$, the vertex of $g(x)$ represents a <u>minimum</u>.

Question 8. Determine the inverse function for the following functions,

(a) $F(x) = -8x + 16$.

Solution. Proceeding with the inverse algorithm,

$$\begin{aligned}
F(x) &= -8x + 16 \\
y &= -8x + 16 \\
y - 16 &= -8x \\
x &= -\frac{1}{8}(y - 16) \\
F^{-1}(x) &= -\frac{1}{8}(x - 16).
\end{aligned}$$

(b) $G(x) = 2\sqrt{x+8} - 4$.

Solution.

$$\begin{aligned}
G(x) &= 2\sqrt{x+8} - 4 \\
y &= 2\sqrt{x+8} - 4 \\
y + 4 &= 2\sqrt{x+8} \\
\frac{y+4}{2} &= \sqrt{x+8} \\
\left(\frac{y+4}{2}\right)^2 &= x + 8 \\
x &= \left(\frac{y+4}{2}\right)^2 - 8 \\
G^{-1}(x) &= \left(\frac{x+4}{2}\right)^2 - 8.
\end{aligned}$$

Question 9. Simplify the following exponential expression, **leave your answer with positive exponents**.

$$\frac{\left(2x^2x^4y^{-3}z^{-4}\right)^2}{\left(8x^{-2}y^{-5}z^2\right)^2}$$

Solution.

$$\begin{aligned}\frac{\left(2x^2x^4y^{-3}z^{-4}\right)^2}{\left(8x^{-2}y^{-5}z^2\right)^2} &= \frac{2^2x^{12}y^{-6}z^{-8}}{8^2x^{-4}y^{-10}z^4}\\ &= \frac{4x^{16}y^{10}}{64y^6z^{12}}\\ &= \frac{4x^{16}y^{10-6}}{64z^{12}}\\ &= \frac{x^{16}y^4}{16z^{12}}\end{aligned}$$

Question 10. Evaluate the following,

$$\left(16^{\frac{4}{4}}\right)\left(9^{\frac{3}{2}}\right)\left(4^{\frac{1}{2}}\right)\left(2^{-3}\right)$$

Solution.

$$\begin{aligned}\left(16^{\frac{4}{4}}\right)\left(9^{\frac{3}{2}}\right)\left(4^{\frac{1}{2}}\right)\left(2^{-3}\right) &= \frac{\left(16^1\right)\left(9^{\frac{1}{2}}\right)^3\left(4^{\frac{1}{2}}\right)}{2^3}\\ &= \frac{16(3)^3 2}{8}\\ &= 2(3)^3 2\\ &= 4(27)\\ &= 108\end{aligned}$$

Question 11. Simply the following radical expressions.

(a)

$$2\sqrt{27} + 3\sqrt{3} - 2\sqrt{12} + \sqrt{48}$$

Solution.

$$\begin{aligned} 2\sqrt{27} + 3\sqrt{3} - 2\sqrt{12} + \sqrt{48} &= 2\sqrt{9\cdot 3} + 3\sqrt{3} - 2\sqrt{4\cdot 3} + \sqrt{16\cdot 3} \\ &= 2\sqrt{9}\sqrt{3} + 3\sqrt{3} - 2\sqrt{4}\sqrt{3} + \sqrt{16}\sqrt{3} \\ &= 2(3)\sqrt{3} + 3\sqrt{3} - 2(2)\sqrt{3} + 4\sqrt{3} \\ &= 6\sqrt{3} + 3\sqrt{3} - 4\sqrt{3} + 4\sqrt{3} \\ &= 9\sqrt{3}. \end{aligned}$$

(b)

$$\left(2\sqrt{2} + \sqrt{3}\right)\left(5\sqrt{3} + 3\sqrt{2}\right)$$

Solution.

$$\begin{aligned} \left(2\sqrt{2} + \sqrt{3}\right)\left(5\sqrt{3} + 3\sqrt{2}\right) &= \left(2\sqrt{2}\right)\left(5\sqrt{3}\right) + \left(2\sqrt{2}\right)\left(3\sqrt{2}\right) + \left(\sqrt{3}\right)\left(5\sqrt{3}\right) + \left(\sqrt{3}\right)\left(3\sqrt{2}\right) \\ &= 10\sqrt{2}\sqrt{3} + 6\sqrt{2}\sqrt{2} + 5\sqrt{3}\sqrt{3} + 3\sqrt{3}\sqrt{2} \\ &= 10\sqrt{2\cdot 3} + 6\sqrt{2\cdot 2} + 5\sqrt{3\cdot 3} + 3\sqrt{3\cdot 2} \\ &= 10\sqrt{6} + 6\sqrt{4} + 5\sqrt{9} + 3\sqrt{6} \\ &= 13\sqrt{6} + 6(2) + 5(3) \\ &= 12 + 15 + 13\sqrt{6} \\ &= 27 + 13\sqrt{6} \end{aligned}$$

Question 12. Simplify the following,

(a)

$$\frac{2x^2-8x}{x^2-11x+18} \times \frac{2x^2-7x+6}{x^2-5x+4}$$

Solution.

$$\begin{aligned}
\frac{2x^2-8x}{x^2-11x+18} \times \frac{2x^2-7x+6}{x^2-5x+4} &= \frac{2x(x-4)}{(x-9)(x-2)} \times \frac{(2x-3)(x-2)}{(x-4)(x-1)} \\
&= \frac{2x(x-4)(2x-3)(x-2)}{(x-9)(x-2)(x-4)(x-1)} \\
&= \frac{2x\cancel{(x-4)}(2x-3)\cancel{(x-2)}}{(x-9)\cancel{(x-2)}\cancel{(x-4)}(x-1)} \\
&= \frac{2x(2x-3)}{(x-9)(x-1)}.
\end{aligned}$$

(b)

$$\frac{x}{x^2-5x+6}-\frac{3}{x^2-4x+4}$$

Solution. Let $A(x)=x$, $B(x)=x^2-5x+6$, $C(x)=3$, $D(x)=x^2-4x+4$.

$$\frac{x}{x^2-5x+6}-\frac{3}{x^2-4x+4}=\frac{x}{(x-3)(x-2)}-\frac{3}{(x-2)(x-2)}$$

Notice that the LCD here is,

$$\begin{aligned}
L(x)&=\frac{B(x)\cdot D(x)}{\gcd(B(x),D(x))}\\
&=\frac{(x-3)(x-2)\cdot(x-2)(x-2)}{\gcd((x-3)(x-2),(x-2)(x-2))}\\
&=\frac{(x-3)(x-2)(x-2)(x-2)}{(x-2)}\\
&=\frac{(x-3)(x-2)\cancel{(x-2)}(x-2)}{\cancel{(x-2)}}\\
&=(x-3)(x-2)(x-2)
\end{aligned}$$

The missing factor for $B(x)$ is $R(x)=(x-2)$ and the missing factor for $D(x)$ is $Q(x)=(x-3)$, and hence,

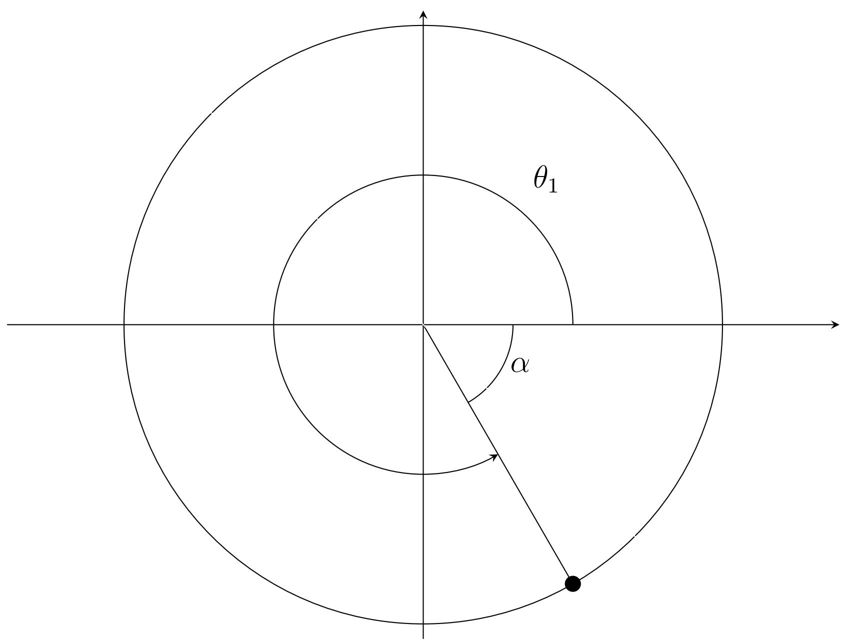
$$\begin{aligned}
\frac{x}{x^2-5x+6}-\frac{3}{x^2-4x+4}&=\frac{x}{(x-3)(x-2)}-\frac{3}{(x-2)(x-2)}\\
&=\frac{x(x-2)-3(x-3)}{(x-3)(x-2)(x-2)}\\
&=\frac{x^2-2x-3x^2+9}{(x-3)(x-2)(x-2)}\\
&=\frac{-2x^2-2x+9}{(x-3)(x-2)(x-2)}
\end{aligned}$$

Question 13. For each of the following, you are given a trigonometric ratio, solve for θ. Assume that each angle θ lies in the **fourth** quadrant. **(You can use the circle below if it helps)**.

(a)

$$\cos\theta_1 = \frac{1}{2}.$$

Solution.
Step 1.



Step 2.

$$\alpha = \cos^{-1}\left(\left|\frac{1}{2}\right|\right) = \cos^{-1}\left(\frac{1}{2}\right) = 60°.$$

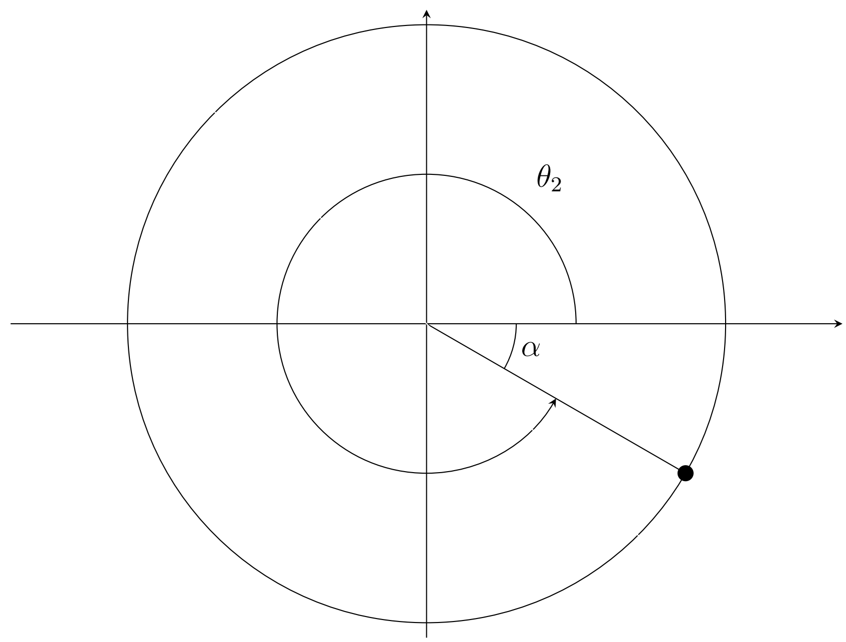
Step 3.

$$\theta_1 = 360° - \alpha = 360° - 60° = 300°.$$

(b)

$$\tan\theta_2 = -\frac{1}{\sqrt{3}}.$$

Solution.
Step 1.



Step 2.

$$\alpha = \tan^{-1}\left(\left|-\frac{1}{\sqrt{3}}\right|\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^\circ.$$

Step 3.

$$\theta_1 = 360^\circ - \alpha = 360^\circ - 30^\circ = 330^\circ.$$

Part D - Solve one of the two problems

Question 14. Suppose we have two standard coordinates $\mathbf{P}(x_1, y_1)$ and $\mathbf{Q}(x_2, y_2)$. Recall that the distance between these two points is given by the following formula,

$$\text{dist} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Given polar coordinates $\mathbf{R}(2, 150°)$ and $\mathbf{T}(2, 330°)$, compute the distance between them.

Solution. The strategy here is to convert each coordinate to standard coordinates and to use the distance formula from there,
Step 1. Convert $\mathbf{R}(2, 150°)$ to standard coordinates.

$$\begin{aligned} x &= r\cos\theta \\ &= 2\cos 150° \\ &= 2(\pm\cos\alpha) \\ &= 2(-\cos 30°) \\ &= 2\left(-\frac{\sqrt{3}}{2}\right) \\ &= -\sqrt{3} \end{aligned} \qquad \begin{aligned} y &= r\sin\theta \\ &= 2\sin 150° \\ &= 2(\pm\sin\alpha) \\ &= 2(\sin 30°) \\ &= 2\left(\frac{1}{2}\right) \\ &= 1 \end{aligned}$$

Hence $\mathbf{R}\left(-\sqrt{3}, 1\right)$.

Step 2. Convert $\mathbf{T}(2, 330°)$ to standard coordinates.

$$\begin{aligned} x &= r\cos\theta \\ &= 2\cos 330° \\ &= 2(\pm\cos\alpha) \\ &= 2(\cos 30°) \\ &= 2\left(\frac{\sqrt{3}}{2}\right) \\ &= \sqrt{3} \end{aligned} \qquad \begin{aligned} y &= r\sin\theta \\ &= 2\sin 330° \\ &= 2(\pm\sin\alpha) \\ &= 2(-\sin 30°) \\ &= 2\left(-\frac{1}{2}\right) \\ &= -1 \end{aligned}$$

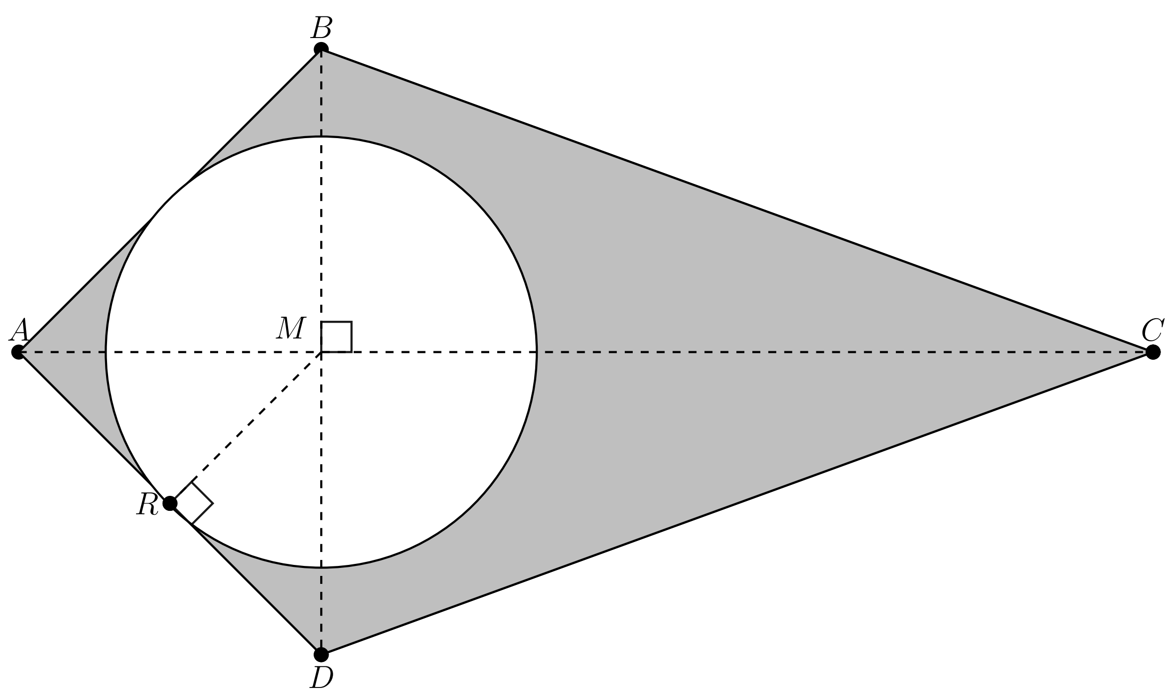
Hence $\mathbf{T}\left(\sqrt{3}, -1\right)$.

Step 3. Using the distance formula,

$$\begin{aligned}
\text{dist} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= \sqrt{\left(\sqrt{3} - \left(-\sqrt{3}\right)\right)^2 + (-1 - 1)^2} \\
&= \sqrt{\left(2\sqrt{3}\right)^2 + (-2)^2} \\
&= \sqrt{(2)^2\left(\sqrt{3}\right)^2 + (-2)^2} \\
&= \sqrt{4(3) + 4} \\
&= \sqrt{16} \\
&= 4.
\end{aligned}$$

Hence the distance between the two points is dist $= 4$.

Question 15. The figure below is composed of a kite and a circle. The radius of the circle is 4 and the point M represents the midpoint of the circle. $\angle ADM = 30°$ and $MC = 20$. If the area of the circle is $A_c = 48$, then determine the area of the shaded region in exact form.
(**Hint:** A kite has unqiue symmetric relationships which assert that $\angle ADM = \angle ABM$, $\angle MDC = \angle MBC$, $DM = MB$, $AD = AB$, $BC = DC$).



Solution. Since point M corresponds to the midpoint of the circle and the radius of the circle is 4, we conclude that $RM = 4$. If we analyse triangle MRD, then we can use the fact that $\angle ADM = 30°$ to conclude that $\angle RMD = 60°$. We can solve for side MD by,

$$\begin{aligned}
\cos \angle RMD &= \frac{RM}{MD} \\
\cos 60° &= \frac{4}{MD} \\
\frac{1}{2} &= \frac{4}{MD} \\
MD &= 8.
\end{aligned}$$

Notice that triangle AMD is a right triangle at vertex M, hence we can use a tangent ratio to solve for side AM,

$$\begin{aligned}
\tan \angle ADM &= \frac{AM}{MD} \\
\tan 30° &= \frac{AM}{8} \\
\frac{1}{\sqrt{3}} &= \frac{AM}{8} \\
AM &= \frac{8}{\sqrt{3}}.
\end{aligned}$$

At this point we can solve for the area of triangle ABD, note that side $BD = BM + MD = MD + MD = 8 + 8 = 16$, where the last line of equality follows from the symmetric relations on kites. Hence,

$$A_{\triangle ABD} = \frac{1}{2}(BD \cdot AM) = \frac{1}{2}\left(16 \cdot \frac{8}{\sqrt{3}}\right) = \frac{64}{\sqrt{3}}.$$

We can also solve for the area of triangle BDC,

$$A_{\triangle BDC} = \frac{1}{2}(BD \cdot MC) = \frac{1}{2}(16 \cdot 20) = 160.$$

Let the area of the shaded region be A_S, we obtain the area by first determining the area of the kite, we can do so by taking the sum of the two areas we have previously determined, and then subtracting the area of the circle from the result of the summation.

$$\begin{aligned}
A_S &= A_{kite} - A_c \\
&= A_{\triangle ABD} + A_{\triangle BDC} - A_c \\
&= \frac{64}{\sqrt{3}} + 160 - 48 \\
&= \frac{64}{\sqrt{3}} + 112.
\end{aligned}$$