Lecture 7 – Rational Expressions

Definition 7.1: *Let $a, b, c \in \mathbb{R}$. A* ***polynomial*** *$P(x)$ is an expression that has form,*

$$P(x) = ax^2 + bx + c.$$

Remark 7.1: Really polynomials are not limited to second degree, however they will be for this course.

Example 7.1: Examples of polynomials,

- $Q(x) = 2x^2 + 3x + 1$.
- $P(x) = 3$.
- $R(x) = 5x - 3$.

Definition 7.2: *A* ***rational expression*** *is a ratio of two polynomials, $P(x)$ and $Q(x)$,*

$$\frac{P(x)}{Q(x)}.$$

7.1 Multiplication and Division.

Suppose we are given a product of two rational expressions,

$$\frac{A(x)}{B(x)} \times \frac{C(x)}{D(x)}.$$

Follow these steps to simply,

Step 1. Factor all of $A(x), B(x), C(x), D(x)$.

Step 2. Rewrite as a single fraction **in factored form**,

$$\frac{A(x)C(x)}{B(x)D(x)}.$$

Step 2. Cancel all common factors, reduce any coefficients with common factors.

If instead we are faced with a division of two rational expressions, then simply use the reciprocal to change to the equivalent product,

$$\frac{A(x)}{B(x)} \div \frac{C(x)}{D(x)} = \frac{A(x)}{B(x)} \times \frac{D(x)}{C(x)}.$$

Example 7.2: Simply the following,

a)

$$\frac{x^2+7x+12}{x^2+5x+4}.$$

b)

$$\frac{x^2-3x-18}{2x^2+5x-3} \qquad (**)$$

Example 7.3: Simply the following,

a)

$$\frac{x^2+10x+21}{x+3} \times \frac{x+3}{x^2+9x+14}.$$

b)

$$\frac{2x^2-8x}{x^2-3x-10} \times \frac{4x^2}{x^2-9x+20} \qquad (**)$$

Example 7.4: Simply the following,

a)

$$\frac{16x^5}{x^2-2x+1} \div \frac{4x^3}{x^2-1}.$$

b)

$$\frac{x^2-x}{x^2+x-2} \div \frac{4x}{x^2+3x+2} \qquad (**)$$

7.2 GCD's of polynomials

We can extend the concept of the gcd to polynomials as well. Generally speaking, given polynomials $P(x)$ and $Q(x)$ in **factored form**, the value of $\gcd(P(x), Q(x))$ can be obtained trivially by analysing the greatest common product of factors.

Example 7.5: Let $P(x) = x^2-5x+6$ and $Q(x) = 2x^2-14x+24$. Determine $\gcd(P(x), Q(x))$.

Example 7.6: Let $P(x) = 5x-10$ and $R(x) = 10x$. Determine $\gcd(P(x), R(x))$. (**)

Example 7.7: Let $A(x) = x^2-4x+3$ and $B(x) = 2x^2-7x+3$. Determine $\gcd(A(x), B(x))$. (**)

7.3 Addition and Subtraction

Suppose we are given a sum or difference of two rational expressions,

$$\frac{A(x)}{B(x)} \pm \frac{C(x)}{D(x)}.$$

The arithmetic here is actually **very similar** to addition of regular fractions. Follow these steps to simplify, *Step 1.* Factor both $B(x)$ and $D(x)$.

Step 2. Determine the LCD,

$$L(x) = \frac{B(x) \cdot D(x)}{\gcd(B(x), D(x))}.$$

Step 3. Determine the *missing factors*,

$$R(x) \qquad Q(x).$$

Step 4. Rewrite the fraction with the LCD and simplify,

$$\frac{A(x) \cdot R(x) \pm C(x) \cdot Q(x)}{L(x)}.$$

Example 7.8: Simply the following,

(a)
$$\frac{1}{5x} + \frac{1}{2x}.$$

(b)
$$\frac{x}{x-1} - \frac{x+1}{x+2}.$$

(c)
$$\frac{x}{x^2+2x+1} + \frac{2}{x+1}. \qquad (**)$$

(d)
$$\frac{x+9}{x^2+2x-48} - \frac{x-9}{x^2-x-30} \qquad (**)$$