Lecture 5 – Transformations of Functions

In this lecture we will study the notion of transformations of functions. By *transformation* we mean how the graph of a function transforms from one state to the other, either by a translation, stretch, compression, and so on. This topic involves a lot of sketching and graphing, therefore I have some tips to share before we proceed.

Tip 1. Try using a variety of colours to differentiate between the original function and the various transformed functions in your sketches.

Tip 2. Try your best to use graph paper. **(Ask me if you need some)**

5.1 Vertical Shifting

Let $f(x)$ be a function and $c \in \mathbb{R}$. Then the function $h(x) = f(x) + c$ is is a **vertical shift** of $f(x)$, where each coordinate is transformed to : $(x, f(x)) \to (x, f(x) + c)$.

- **If** $c > 0$, then we say $f(x)$ has been **shifted upwards** by $|c|$ units.
- **Else If** $c < 0$, then we say $f(x)$ has been **shifted downwards** by $|c|$ units.

Example 5.1: Let $f(x) = x^2$. Sketch and describe each transformation below. **(In class)**

(a) $h(x) = f(x) + 3$.

(b) $r(x) = f(x) - 5$ **(** With Class)**

5.2 Horizontal Shifting

Let $f(x)$ be a function and $c \in \mathbb{R}$. Then the function $h(x) = f(x + c)$ is is a **horizontal shift** of $f(x)$, where each coordinate is transformed to : $(x, f(x)) \to (x - c, f(x))$.

- **If** $c > 0$, then we say $f(x)$ has been **shifted left** by $|c|$ units.
- **Else If** $c < 0$, then we say $f(x)$ has been **shifted right** by $|c|$ units.

Example 5.2: Let $f(x) = \sqrt{x}$. Sketch and describe each transformation below. **(In class)**

(a) $h(x) = f(x - 1)$.

(b) $r(x) = f(x + 4)$ **(** With Class)**.

5.3 Reflecting Graphs

Let $f(x)$ be a function. Then,

- The function $h(x) = f(-x)$ is a *reflection* of $f(x)$ across the **y-axis**, where each coordinate is transformed to : $(x, f(x)) \to (-x, f(x))$.
- The function $h(x) = -f(x)$ is a *reflection* of $f(x)$ across the **x-axis**, where each coordinate is transformed to : $(x, f(x)) \to (x, -f(x))$.

Example 5.3: Let $f(x) = \sqrt{x}$. Sketch and describe each transformation below. **(In class)**

(a) $h(x) = f(-x)$.

(b) $r(x) = -f(x)$ **(** With Class)**.

5.4 Vertical Compressions & Stretches

Let $f(x)$ be a function and $c \in \mathbb{R}$. Then the function $h(x) = cf(x)$ is is a vertical scaling of $f(x)$, where each coordinate is transformed to : $(x, f(x)) \to (x, |c| \cdot f(x))$.

- **If** $|c| > 1$, then we say $f(x)$ has been **vertically stretched** by a factor of $|c|$. ([1])
- **If** $0 < |c| < 1$, then we say $f(x)$ has been **vertically compressed** by a factor of $1/|c|$. ([1])

Example 5.4: Let $g(x) = |x|$. Sketch and describe each transformation below. **(In class)**

(a) $h(x) = 2g(x)$.

(b) $r(x) = \frac{2}{3}g(x)$ **(** With Class)**.

5.5 Horizontal Compressions & Stretches

Let $f(x)$ be a function and $c \in \mathbb{R}$. Then the function $h(x) = f(cx)$ is is a horizontal scaling of $f(x)$, where each coordinate is transformed to : $(x, f(x)) \to (x/|c|, f(x))$.

- **If** $|c| > 1$, then we say $f(x)$ has been **horizontally compressed** by a factor of $|c|$. ([1])
- **If** $0 < |c| < 1$, then we say $f(x)$ has been **horizontally stretched** by a factor of $1/|c|$. ([1])

Example 5.5: Let $f(x) = \sqrt{x}$. Sketch and describe each transformation below. **(In class)**

(a) $h(x) = f(3x)$.

(b) $r(x) = f(\frac{1}{2}x)$ **(** With Class)**.

At this point we can analyse combinations of transformations, involving both horizontal and vertical shifts.

Theorem 5.1: *(Transformations [1]) Let $f(x)$ be a function and let $A, B, H, K \in \mathbb{R}$. Then the function,*

$$h(x) = Af(Bx + H) + K.$$

is a transformation of $f(x)$, where each coordinate is transformed to :

$$(x, f(x)) \longrightarrow \left(\frac{x - H}{B}, Af(x) + K\right).$$

Note: In order to describe such a transformation, you **must** factor the transformed function,

$$h(x) = Af\left(B\left(x + \frac{H}{B}\right)\right) + K.$$

Where,

- A is the vertical scaling factor, which could include a reflection.
- B is the horizontal scaling factor, which could include a reflection.
- H/B is the horizontal shifting factor.
- K is the vertical shifting factor.

Example 5.6: Sketch and describe each transformation below. **(In class)**

(a) $f(x) = |x|$, Transformation : $r(x) = -f\left(\frac{1}{2}x - 3\right) + 1$.

(b) $f(x) = \sqrt{x}$, Transformation : $h(x) = \frac{1}{2}f(-4x + 16) - 2$ **(** With Class)**.

Bibliography

[1] Carl Stitz and Jeff Zeager. College Algebra. 3 edition, July 4, 2013.