Lecture 3 – Inverse Functions Part 1

In this lesson, we will explore the notion of an invertible function. This is of particular importance since invertible functions tell us more about the domain, the co-domain and correspondences between the two. In particular, we can reconstruct the entire domain based strictly on knowledge of the co-domain.

Definition 3.1: *Let $\mathcal{A}$ and $\mathcal{B}$ be sets. The **identity function** on the set $\mathcal{A}$ is the function defined as,*

$$\mathrm{id}_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{B}$$
$$\mathrm{id}_{\mathcal{A}}(a) = a$$

This is essentially the function that takes each element in $\mathcal{A}$ and returns it.

The identity function is essentially the most trivial function, lets see how it works in the following example.

Example 3.1: Let $\mathcal{H} = \{4, 6, 7, 10, 12\}$ and $\mathcal{T} = \{1, 4, 5, 7, 8, 10, 12, 6\}$ be sets. Draw the mapping diagram for the identity function on $\mathcal{H}$, i.e $\mathrm{id}_{\mathcal{H}} \colon \mathcal{H} \to \mathcal{T}$. **(In class)**

Definition 3.2: *Let $\mathcal{A}$ and $\mathcal{B}$ be sets. Let $f \colon \mathcal{A} \to \mathcal{B}$ be some function. We say that f is **surjective** if every element in $\mathcal{B}$ is mapped to.*

Example 3.2: Let $\mathcal{A} = \{2, 3, 4, 5, 6\}$ and $\mathcal{B} = \{0, 1\}$ be sets, lets define the following function,

- $\mathcal{R} \colon \mathcal{A} \to \mathcal{B}$.
- $\mathcal{R}(a) = \mathrm{rem}(a, 2)$.

I claim that $\mathcal{R}$ is surjective. **(Explanation in Class)**

Example 3.3: Notice that the function defined in Example 3.1 is **not** surjective. **(Explanation in Class)**

Definition 3.3: *Let $\mathcal{A}$ and $\mathcal{B}$ be sets. Let $f \colon \mathcal{A} \to \mathcal{B}$ be some function. We say that f is **injective** if no two elements in $\mathcal{A}$ map to a single element in $\mathcal{B}$.*

Example 3.4: Notice that the function defined in Example 3.2 is **not** injective. **(Explanation in Class)**

Example 3.5: Let $\mathcal{A} = \{0, 1, 2\}$ and $\mathcal{B} = \{1, 2, 3, 25, 36\}$ be sets, lets define the following function,

- $f\colon \mathcal{A} \to \mathcal{B}$.
- $f(a) = a + 1$.

I claim that the function f is injective . **(In class explanation)**

We are now ready to define invertible functions

Definition 3.4: *Let $\mathcal{A}$ and $\mathcal{B}$ be sets. Let $f\colon \mathcal{A} \to \mathcal{B}$ be some function. If f is both injective and surjective, then we say f is* ***invertible*** *and an inverse function for f exists.*

Example 3.6: Let $\mathcal{A} = \{1, 2, 3, 5, 18\}$ and $\mathcal{B} = \{2, 4, 6, 10, 36\}$ be sets, lets define the following function,

- $\mathcal{L}\colon \mathcal{A} \to \mathcal{B}$.
- $\mathcal{L}(a) = 2a$.

I claim that $\mathcal{L}$ is invertible. **(Explanation in Class)**

Example 3.7: Let $\mathcal{A} = \{-2, -1, 0, 1, 2, 3\}$ and $\mathcal{B} = \{0, 1, 4, 9, 25, 36\}$ be sets, lets define the following function,

- $f\colon \mathcal{A} \to \mathcal{B}$.
- $f(a) = a^2$.

I claim that the function f is **not** invertible. **(In class explanation)**

Question 3.1: *Let $\mathcal{A}$, $\mathcal{B}$ be sets, and let $f\colon \mathcal{A} \to \mathcal{B}$ be a function. Suppose that f is* ***surjective****, then is it true that the range is equal to $\mathcal{B}$? In other words, is $\mathcal{R}_f = \mathcal{B}$?*

Answer 3.1: Yes ! **(In class)**

Question 3.2: *Let $\mathcal{A}$, $\mathcal{B}$ be sets, and let $f\colon \mathcal{A} \to \mathcal{B}$ be an* ***invertible*** *function. Then is it true that $|\mathcal{A}| = |\mathcal{B}|$?* ***(Class Question)***

Answer 3.2: Yes ! **(In class)**