Assignment 2 Functions - SOLUTIONS
Due Date: Wednesday, January 19

January, 2022

1 Preamble

This assignment covers everything taught so far. The solutions that you hand in should be **neat** and **legible**, this is an assignment, not a quiz, so I expect you to take your time and present thorough and detailed solutions.

2 Name and Date:

Print your name and todays date below;

Name	Date

Question 1. There exists a function that allows us to determine the length of a binary string. We call this function len. Here a few examples to understand how it works,

- If $\mathbf{S} = 1001$, then $\text{len}(\mathbf{S}) = 4$.
- If $\mathbf{R} = 110001$, then $\text{len}(\mathbf{R}) = 6$.
- If $\mathbf{T} = \epsilon$, then $\text{len}(\mathbf{T}) = 0$.

We can also define the operation of *multiplication by a scalar* for binary strings. So suppose $n \in \mathbb{N}$ and $\mathbf{S}$ is some binary string, then,

$$n \cdot \mathbf{S} = \underbrace{\mathbf{S} + \cdots + \mathbf{S}}_{\text{n times}}$$

Again we resort to a few examples to demonstrate how multiplication by a scalar works,

- If $\mathbf{S} = 1001$, then $2 \cdot \mathbf{S} = \mathbf{S} + \mathbf{S} = 10011001$.
- If $\mathbf{R} = 0$, then $4 \cdot \mathbf{R} = \mathbf{R} + \mathbf{R} + \mathbf{R} + \mathbf{R} = 0000$.
- If $\mathbf{T} = 01$, then $3 \cdot \mathbf{T} = \mathbf{T} + \mathbf{T} + \mathbf{T} = 010101$.

Let $\mathbf{S} = 001$ and $\mathbf{T} = 11$, answer the following,

(a) Let $\mathbb{S}$ represent the set of all binary strings. Define the length function using mapping notation.

Solution. $\text{len} \colon \mathbb{S} \to \mathbb{N}$.

(b) Compute $\text{len}(\mathbf{S})$.

Solution. $\text{len}(\mathbf{S}) = \text{len}(001) = 3$.

(c) Compute $\text{len}(\mathbf{T})$.

Solution. $\text{len}(\mathbf{T}) = \text{len}(11) = 2$.

(d) Compute $\text{len}(\mathbf{S} + \mathbf{T})$.

Solution. $\text{len}(\mathbf{S} + \mathbf{T}) = \text{len}(001 + 11) = \text{len}(00111) = 5$.

(e) Compute $\text{len}(3 \cdot \mathbf{S})$.

Solution. $\text{len}(3 \cdot \mathbf{S}) = \text{len}(3 \cdot 001) = \text{len}(001001001) = 9$.

(f) Compute $3 \cdot \text{len}(\mathbf{S})$.

Solution. $3 \cdot \text{len}(\mathbf{S}) = 3 \cdot 3 = 9$.

(g) Compute $\text{len}(4 \cdot \mathbf{T})$.

Solution. $\text{len}(4 \cdot \mathbf{T}) = \text{len}(4 \cdot 11) = \text{len}(11111111) = 8$.

(h) Compute $4 \cdot \text{len}(\mathbf{T})$.

Solution. $4 \cdot \text{len}(\mathbf{T}) = 4 \cdot 2 = 8$.

Question 2. Let F be a function. We call F linear if both of the following conditions are satisfied,

1. For all inputs x and y,
$$F(x+y) = F(x) + F(y).$$

2. For all $c \in \mathbb{F}$, and all inputs x,
$$F(c \cdot x) = c \cdot F(x).$$

   If $\mathbb{F} = \mathbb{N}$, then based on your results from Question 1, do you think that the length function, len, is linear? Explain your answer.

Solution. From our results in parts (b),(c) we see that $\text{len}(\mathbf{S}) + \text{len}(\mathbf{T}) = 3 + 2 = 5$, and from part (d) we see that $\text{len}(\mathbf{S} + \mathbf{T}) = 5$, hence it looks like the first condition of linearity is satisfied so far.

From our result in part (e), we see that $\text{len}(3 \cdot \mathbf{S}) = 9$ and from part (f) we see that $3 \cdot \text{len}(\mathbf{S}) = 9$, hence it looks like the second condition of linearity is satisfied as well.

Combining our two hypothesis, we conclude that it appears like the length function is indeed linear.

Question 3. Sometimes in math we would like a function that simply gets rid of trailing decimals and returns a whole number, aka an integer. This function is known as the floor function. We define it with mapping notation as $\text{floor}\colon \mathbb{R} \to \mathbb{Z}$, and it works as follows, if $x \in \mathbb{R}$, then $\text{floor}(x)$ is the smallest integer that is less than or equal to x. Lets see how it works in the following examples,

- If $x = 4.2$, then $\text{floor}(x) = \text{floor}(4.2) = 4$.
- If $x = -7.4$, then $\text{floor}(x) = \text{floor}(-7.4) = -8$.
- If $x = 5$, then $\text{floor}(x) = \text{floor}(5) = 5$.
- If $x = 0.4$, then $\text{floor}(x) = \text{floor}(0.4) = 0$.

(a) Compute $\text{floor}(2.5)$.

   Solution. $\text{floor}(2.5) = 2$.

(b) Compute $\text{floor}(6/3)$.

   Solution. $\text{floor}(6/3) = \text{floor}(2) = 2$.

(c) Compute $\text{floor}(19/4)$.

   Solution. $\text{floor}(19/4) = \text{floor}(4.75) = 4$.

(d) Let $f(x) = (x+1)/2$ and $g(x) = \sqrt{x-1}$, compute $\text{floor}(f(g(5))$.

   Solution. We first compute $g(5)$,
$$g(5) = \sqrt{5-1} = \sqrt{4} = 2.$$
   Next we compute $f(g(5))$,
$$f(g(5)) = f(2) = \frac{2+1}{2} = \frac{3}{2}.$$
   Finally we compute $\text{floor}(f(g(5)))$,
$$\text{floor}(f(g(5))) = \text{floor}\left(\frac{3}{2}\right) = \text{floor}(1.5) = 1.$$

(e) Is the floor function linear? If it is, then justify your claim. If it is not, then provide a counter example to show that it fails to be linear.

Solution. The floor function is **not** linear, we draw an easy counter example to show why. Notice that,

$$\text{floor}(2.5 + 1.5) = \text{floor}(4) = 4.$$

On the other hand,

$$\text{floor}(2.5) + \text{floor}(1.5) = 2 + 1 = 3.$$

Clearly $\text{floor}(2.5 + 1.5) \neq \text{floor}(2.5) + \text{floor}(1.5)$, and hence, the floor function is not linear.

(f) Is the floor function invertible? If it is, then justify your claim. If it is not, then provide a counter example to show that it fails to be surjective or injective.

Solution. The floor function is **not** invertible, notice that since $\text{floor}(2.5) = \text{floor}(2.4) = 2$, the floor function fails to be injective. Since it fails to be injective, it fails to be invertible.

Question 4. Let $\mathcal{S} = \{1, 010, 00100, 0001000\}$, where each element is a binary string, and let $\mathcal{R} = \{4, 2, 6, 8\}$, where each element is a natural number.

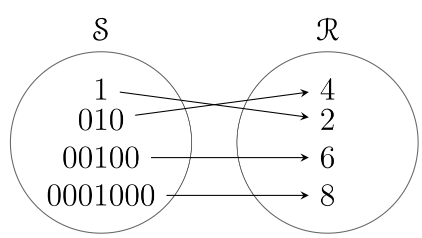
(a) Come up with an invertible function Ψ between $\mathcal{S}$ and $\mathcal{R}$ and prove that your function is invertible. (**Hint:** Try using the length function)

Solution. I claim that,

$$\Psi\colon \mathcal{S} \to \mathcal{R}\,, \quad \Psi(\mathbf{S}) = \text{len}(\mathbf{S}) + 1.$$

is an invertible function between $\mathcal{S}$ and $\mathcal{R}$.

Proof: To prove the claim from above, it is sufficient to show that the mapping diagram preserves both injectivity and surjectivity,



From the mapping diagram, we conclude that $\Psi(\mathbf{S}) = \text{len}(\mathbf{S}) + 1$ is both injective and surjective, and hence its invertible. ■

(b) Come up with the correct formula for the inverse function Ψ^{-1} and prove that your formula is correct using mapping tables. (**Hint:** The correct formula uses the floor function)

Solution. I claim that,

$$\Psi^{-1}\colon \mathcal{R} \to \mathcal{S}\,, \quad \Psi^{-1}(r) = \text{floor}\left(\frac{r-1}{2}\right) \cdot \mathbf{0} + \mathbf{1} + \text{floor}\left(\frac{r-1}{2}\right) \cdot \mathbf{1}.$$

is the correct formula for the inverse function, where $\mathbf{0}, \mathbf{1}$ are binary bits. (They are being bolded just so we don't confuse them for integers $0, 1$)

Proof: To prove our claim, it is sufficient to show that both conditions of Definition 4.1 hold using mapping tables,

$\mathcal{S}$	$\Psi^{-1}(\Psi(\mathbf{S}))$
1	$\Psi^{-1}(\Psi(1)) = \Psi^{-1}(2) = \text{floor}(0.5)\cdot\mathbf{0}+1+\text{floor}(0.5)\cdot 0 = 0\cdot\mathbf{0}+1+0\cdot\mathbf{0} = 1$
010	$\Psi^{-1}(\Psi(010)) = \Psi^{-1}(4) = \text{floor}(1.5)\cdot\mathbf{0}+1+\text{floor}(1.5)\cdot 0 = 1\cdot\mathbf{0}+1+1\cdot\mathbf{0} = 010$
00100	$\Psi^{-1}(\Psi(00100)) = \Psi^{-1}(6) = \text{floor}(2.5)\cdot\mathbf{0}+1+\text{floor}(2.5)\cdot 0 = 2\cdot\mathbf{0}+1+2\cdot\mathbf{0} = 00100$
0001000	$\Psi^{-1}(\Psi(0001000)) = \Psi^{-1}(8) = \text{floor}(3.5)\cdot\mathbf{0}+1+\text{floor}(3.5)\cdot 0 = 3\cdot\mathbf{0}+1+3\cdot\mathbf{0} = 0001000$

$\mathcal{R}$	$\Psi(\Psi^{-1}(r))$
4	$\Psi(\Psi^{-1}(4)) = \Psi(010) = \text{len}(010)+1 = 3+1 = 4$
2	$\Psi(\Psi^{-1}(2)) = \Psi(1) = \text{len}(1)+1 = 1+1 = 2$
6	$\Psi(\Psi^{-1}(6)) = \Psi(00100) = \text{len}(00100)+1 = 5+1 = 6$
8	$\Psi(\Psi^{-1}(8)) = \Psi(0001000) = \text{len}(0001000)+1 = 7+1 = 8$

By our results from the mapping tables, we conclude that

$$\Psi^{-1}(r) = \text{floor}\left(\frac{r-1}{2}\right)\cdot\mathbf{0}+\mathbf{1}+\text{floor}\left(\frac{r-1}{2}\right)\cdot\mathbf{1}.$$

is indeed the inverse function for Ψ.

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