Engineering Calculus III (GEG217)

A Comprehensive Introduction

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November 6, 2023

Outline

Course Overview

- This course will introduce students to the basic concepts of matrices, complex numbers, complex functions, and their applications.
- The course will cover topics such as linear transformations, conformal mapping, the Cauchy-Riemann equations, complex line integrals, Laurent series and numerical methods for complex analysis.

Course Schedule

- Week 1: Matrices and Linear Transformation
- Week 2: Elementary Complex Analysis
- Week 3: Logarithmic, Exponential and Circular Complex function
- Week 4: Mapping by complex functions
- Week 5: Limit, Continuity and Differentiability of Complex function
- Week 6: Cauchy-Riemann's Equations
- Week 7: Complex Line Intergrals
- Week 8: Integration of functions of Complex Variables
- Week 9: Applications of Complex Analysis
- Week 10: Numerical Methods for Complex Analysis

Course Objectives

- Define matrices and complex numbers.
- Perform basic matrix operations and complex number operations.
- Represent complex numbers graphically.
- Define complex functions and their basic properties.
- Identify analytic functions.
- Apply the Cauchy-Riemann equations to analyze the behavior of complex functions.
- Define complex line integrals.
- Apply Cauchy's integral theorem to evaluate complex line integrals.
- Apply Cauchy's integral formula to solve problems in complex analysis.

Course Objectives Cont'd

- Integrate complex functions using Cauchy's residue theorem.
- Expand complex functions in Laurent series.
- Apply Laurent series to solve problems in complex analysis.
- Apply conformal mapping to solve geometric problems.
- Apply potential theory to solve problems in fluid dynamics.
- Apply Fourier analysis to solve problems in signal processing.
- Implement numerical methods for complex functions.
- Integrate complex functions numerically.
- Solve complex differential equations numerically.

Course Requirements

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  - Mathematical Analysis with Applications by Richard Courant and Herbert Robbins

Week 1: Matrices and Linear Transformation
Topics

- Introduction to matrices
- Matrix operations
- Linear transformations
- Applications of linear transformations

Week 2: Elementary Complex Analysis
Topics

- Complex numbers
- Complex functions
- Analytic functions
- Cauchy-Riemann equations

Week 3: Logarithmic, Exponential and Circular Complex function

Topics

- Logarithmic functions
- Exponential functions
- Circular functions
- Trigonometric functions

Week 4: Mapping by complex functions

Topics

- Conformal mapping
- Riemann mapping theorem
- Applications of conformal mapping

Week 5: Limit, Continuity and Differentiability of Complex function

Topics

- Limits of complex functions
- Continuity of complex functions
- Differentiability of complex functions

Week 6: Cauchy-Riemann's Equations

Topics

- Cauchy-Riemann equations
- Analytic functions
- Cauchy-Riemann equations in polar coordinates

Topics

- Complex line integrals
- Cauchy's integral theorem
- Cauchy's integral formula

Week 8: Integration of functions of Complex Variables

Topics

- Integration of complex functions
- Cauchy's residue theorem
- Laurent series

Week 9: Applications of Complex Analysis

Topics

- Conformal mapping
- Potential theory
- Fourier analysis

Week 10: Numerical Methods for Complex Analysis
Topics

- Numerical methods for complex functions
- Numerical integration of complex functions
- Numerical solution of complex differential equations

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Introduction to matrices

A matrix is a two-dimensional array of numbers, arranged in rows and columns, whereas a vector is an array of scalars.

Definition

A matrix is a rectangular arrangement of uv numbers enclosed within a bracket, denoted by capital letters such as A, B, or C.
Given that:

$$A = \begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ . & . & .... & . \\ . & . & .... & . \\ . & . & .... & . \\ a_{m1} & a_{m2} & .... & a_{mn} \end{bmatrix} = (a_{ij})_{u\times v}$$

A is a matrix of order $u \times v$ i^{th} row j^{th} column element of the matrix denoted by a_{ij}.

Matrix Operation

The operations of matrices primarily involve three algebraic operations: addition, subtraction, and scalar matrix multiplication.

Addition & Subtraction of matrices

Given two matrices A and B, such that

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ . & . & .... & . \\ . & . & .... & . \\ . & . & .... & . \\ a_{m1} & a_{m2} & .... & a_{mn} \end{bmatrix} \tag{1}$$

and

$$B = [b_{ij}] = \begin{bmatrix} b_{11} & b_{12} & .... & b_{1n} \\ . & . & .... & . \\ . & . & .... & . \\ . & . & .... & . \\ b_{m1} & b_{m2} & .... & b_{mn} \end{bmatrix} \tag{2}$$

Matrix Operation Cont'd

Then, their addition A + B is defined as

$$[a_{ij} + b_{ij}] = \begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ . & . & .... & . \\ . & . & .... & . \\ . & . & .... & . \\ a_{m1} & a_{m2} & .... & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & .... & b_{1n} \\ . & . & .... & . \\ . & . & .... & . \\ . & . & .... & . \\ b_{m1} & b_{m2} & .... & b_{mn} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & .... & a_{1n} + b_{1n} \\ . & . & .... & . \\ . & . & .... & . \\ . & . & .... & . \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & .... & a_{mn} + b_{mn} \end{bmatrix}$$

Matrix Operation Cont'd

Likewise, A - B is defined as

$$[a_{ij} - b_{ij}] = \begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ . & . & .... & . \\ . & . & .... & . \\ . & . & .... & . \\ a_{m1} & a_{m2} & .... & a_{mn} \end{bmatrix} - \begin{bmatrix} b_{11} & b_{12} & .... & b_{1n} \\ . & . & .... & . \\ . & . & .... & . \\ . & . & .... & . \\ b_{m1} & b_{m2} & .... & b_{mn} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11} - b_{11} & a_{12} - b_{12} & .... & a_{1n} - b_{1n} \\ . & . & .... & . \\ . & . & .... & . \\ . & . & .... & . \\ a_{m1} - b_{m1} & a_{m2} - b_{m2} & .... & a_{mn} - b_{mn} \end{bmatrix}$$

where ij represents the element in i^{th} row and j^{th} column.

Example

Perform addition and subtraction operations on the following sets of matrices

(a) $\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$

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(c) $\begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & -1 \\ 3 & -2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 3 \\ 2 & 4 \end{bmatrix}$

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Example

$$(e)\ \begin{bmatrix} 0 & 4 \\ 7 & 0 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 0 & 4 \end{bmatrix}$$

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$$(f)\ \begin{bmatrix} 1 & 4 & 7 \\ 8 & 6 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$(g)\ \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

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$$(h)\ \begin{bmatrix} 5 & 9 & -2 \\ 1 & 2 & 3 \\ 4 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 1 \\ 5 & 7 & -1 \\ 9 & 7 & 8 \end{bmatrix}$$

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$$(j)\ \begin{bmatrix} 3 & 0 & -2 \\ 1 & 6 & 3 \end{bmatrix} - \begin{bmatrix} 9 & 0 & 1 \\ 5 & 1 & 1 \end{bmatrix}$$

Solution

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$$(a)\ \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1+1 & 2+3 \\ 0+-1 & 1+2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$$

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Solution Cont'd

$$(b)\ \begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 1+0 & -2+0 \\ 1+0 & 2+0 & 3+0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix}$$

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$$(b)\ \begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 1+0 & -2+0 \\ 1+0 & 2+0 & 3+0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix}$$

Solution Cont'd

$$= \begin{bmatrix} -6 & 0 & -3 \\ -4 & 5 & 1 \end{bmatrix}$$

Matrix Scalar Multiplication & Matrix Multiplication

Matrix Scalar Multiplication

Some examples of scalar matrix multiplication are shown below:

- (-2) $\begin{bmatrix} 1 & 6 \\ 9 & 3 \\ 6 & 0 \end{bmatrix} = \begin{bmatrix} -2 & -12 \\ -18 & -6 \\ -12 & 0 \end{bmatrix}$
- $(\alpha + \beta)A = \alpha A + \beta A; (\alpha\beta)A$
- $\alpha(A + B) = \alpha A + \alpha B$
- $0.A = 0; 1.A = A$

The scalar quantities are respectively -2, $(\alpha + \beta$ & $\alpha\beta)$, α, and (0 & 1).

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Matrix Scalar Multiplication & Matrix Multiplication (Cont'd)

Matrix Multiplication

Definition

Two matrices A and B are said to be confirmable for product AB if number of columns in A equals to the number of rows in matrix B.

Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{n\times r}$ be two matrices. The product matrix $C = AB$, is matrix of order $m \times r$

$$\text{where} \quad c_{ij} = \sum_{k=1}^{n} a_{ij} b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots\ldots + a_{in}b_{nj} \qquad (3)$$

Matrix Scalar Multiplication & Matrix Multiplication (Cont'd)

Rules for multiplying matrices:

- $(AB)C = A(BC)$
- $k(AB) = (kA)B = A(kB)$, k is scalar (number)
- $A(B \pm C) = AB \pm AC$ and $(B \pm C)A = BA \pm CA$
- $AB \neq BA$
- $OA = AO = O$, O is zero matrix
- $IA = AI = A$, I is Identity matrix

Example

Find the product of

$$(a) \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix} \begin{bmatrix} -2 & 4 & 2 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix}$$

Example

Find the product of

$$(a) \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix} \begin{bmatrix} -2 & 4 & 2 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix}$$

$$(b) \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -5 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -5 & 0 \\ 6 & -2 \\ -1 & -3 \end{bmatrix}$$

Example

Find the product of

$$(a)\begin{bmatrix}1 & 0 & 3\\ 2 & -1 & -2\end{bmatrix}\begin{bmatrix}-2 & 4 & 2\\ 1 & 0 & 0\\ -1 & 1 & -1\end{bmatrix}$$

$$(b)\begin{bmatrix}1 & 2 & -3 & 4\\ 0 & -5 & -1 & 1\end{bmatrix}\begin{bmatrix}2 & 3\\ -5 & 0\\ 6 & -2\\ -1 & -3\end{bmatrix}$$

$$(c)\begin{bmatrix}1 & 2 & 4\\ 2 & 6 & 0\end{bmatrix}\begin{bmatrix}4 & 1 & 4 & 3\\ 0 & -1 & 3 & 1\\ 2 & 7 & 5 & 2\end{bmatrix}$$

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$$(b)\begin{bmatrix}1 & 2 & -3 & 4\\ 0 & -5 & -1 & 1\end{bmatrix}\begin{bmatrix}2 & 3\\ -5 & 0\\ 6 & -2\\ -1 & -3\end{bmatrix}$$

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$$(d)\begin{bmatrix}1 & 6\\ 9 & 3\end{bmatrix}\begin{bmatrix}0 & -1\\ -1 & 2\end{bmatrix}$$

Solution (a)

Step 1: Analysis.

Since the number of columns of the first matrix is the same number as the rows of the second, the two matrices are able to be multiplied resulting in a 2×3 matrix.

Step 2: Perform the row-by-column operations.

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix} \times \begin{bmatrix} -2 & 4 & 2 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

$c_{11} = (1)(-2) + (0)(1) + (3)(-1) = -2 + 0 + (-3) = -5$

$c_{12} = (1)(4) + (0)(0) + (3)(1) = 4 + 0 + 3 = 7$

$c_{13} = (1)(2) + (0)(0) + (3)(-1) = 2 + 0 + (-3) = -1$

$c_{21} = (2)(-2) + (-1)(1) + (-2)(-1) = -4 + (-1) + 2 = -3$

$c_{22} = (2)(4) + (-1)(0) + (-2)(1) = 8 + 0 + (-2) = 6$

$c_{23} = (2)(2) + (-1)(0) + (-2)(-1) = 4 + 0 + 2 = 6$

Step 3: Write the products into a matrix form. $\begin{bmatrix} -5 & 7 & -1 \\ -3 & 6 & 6 \end{bmatrix}$

Transpose of matrices

Definition

Transpose of $m \times n$ matrix A, denoted A^T or A', is $n \times m$ matrix with rows and columns of A transposed in A^T

$$(A^T)_{ij} = A_{ji} \tag{4}$$

Properties of Transpose:

(i) $(A+B)^T = A^T + B^T$

(ii) $(A^T)^T = A$

(iii) $(kA)^T = kA^T$ for scalar k

(iv) $(AB)^T = B^T A^T$

For example, the transpose of $\begin{bmatrix} 0 & 4 \\ 7 & 0 \\ 3 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 7 & 3 \\ 4 & 0 & 1 \end{bmatrix}$

Definition

Let $A = (a_{ij})_{n \times n}$ be a square matrix of order n, then $|A|$ is called the determinant of matrix A.

Case 1: Determinant of a 2×2 matrix

Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, then |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \times a_{22} - a_{12} \times a_{21}$$

Case 2: Determinant of a 3×3 matrix

Let

$$A = B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

Definition Cont'd

Then,

$$|B| = b_{11}\begin{vmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{vmatrix} - b_{12}\begin{vmatrix} b_{21} & b_{23} \\ b_{31} & b_{33} \end{vmatrix} + b_{13}\begin{vmatrix} b_{21} & b_{22} \\ b_{31} & b_{32} \end{vmatrix}$$

$$|B| = b_{11}(b_{22} \times b_{33} - (b_{23} \times b_{32})) - b_{12}(b_{21} \times b_{33} - (b_{23} \times b_{31}))$$

$$+b_{13}(b_{21} \times b_{32} - (b_{22} \times b_{31}))$$

Properties of determinant

- The determinant of a matrix A and its transpose A^T are equal $|A| = |A^T|$
  Let A be a square matrix
  (i) If A has a row (or column) of zeros then $|A| = 0$
  (ii) If A has two identical rows (or columns) then $|A| = 0$

Properties of determinant Cont'd

- If A is triangular matrix then $|A| = 0$ is product of the diagonal elements.
- If A is a square matrix of order n and k is a scalar then $|kA| = k^n|A|$

Example

Calculate the determinant of the matrix $\begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \\ 1 & 9 & 5 \end{bmatrix}$

Solution

$\begin{vmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \\ 1 & 9 & 5 \end{vmatrix} = \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix}$

$= b_{11}(b_{22} \times b_{33} - (b_{23} \times b_{32})) - b_{12}(b_{21} \times b_{33} - (b_{23} \times b_{31})) + b_{13}(b_{21} \times b_{32} - (b_{22} \times b_{31}))$

$b_{11} = 1, b_{12} = 3, b_{13} = 4, b_{21} = 2, b_{22} = 6, b_{23} = 8, b_{31} = 1, b_{32} = 9, b_{33} = 5$

$= 1(6 \times 5 - (8 \times 9)) - 3(2 \times 5 - (8 \times 1)) + 4(2 \times 9 - (6 \times 1))$

$= 1(30 - 72) - 3(10 - 8) + 4(18 - 6) = -42 - 6 + 48 = 0$

$Therefore, \begin{vmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \\ 1 & 9 & 5 \end{vmatrix} = 0$

Minor and Cofactors

Definition

Let $A = (a_{ij})_{n \times n}$ be a square matrix. Then M_{ij} denotes a sub matrix of A with order $(n-1) \times (n-1)$ obtained by deleting its i^{th} row and j^{th} column. The determinant $|M_{ij}|$ is called the minor of the element a_{ij} of A.

The cofactor of a_{ij} is denoted by A_{ij} and is equal to $(-1)^{i+j}|M_{ij}|$.

Example

Find the minors of the following matrix and use the results to determine its cofactors

$$\begin{bmatrix} 5 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & -2 & -1 \end{bmatrix}$$

Solution

Step 1: Determine the minors

$$M_{11} = \text{Minor of } a_{11} = \begin{vmatrix} 3 & 1 \\ -2 & -1 \end{vmatrix} = 3 \times -1 - 1 \times -2 = -3 + 2 = -1$$

$$M_{12} = \text{Minor of } a_{12} = \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} = 2 \times -1 - 1 \times 3 = -2 - 3 = -5$$

$$M_{13} = \text{Minor of } a_{13} = \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} = 2 \times -2 - 3 \times 3 = -4 - 9 = -13$$

$$M_{21} = \text{Minor of } a_{21} = \begin{vmatrix} 4 & 2 \\ -2 & -1 \end{vmatrix} = 4 \times -1 - 2 \times -2 = -4 + 4 = 0$$

$$M_{22} = \text{Minor of } a_{22} = \begin{vmatrix} 5 & 2 \\ 3 & -1 \end{vmatrix} = 5 \times -1 - 2 \times 3 = -5 - 6 = -11$$

$$M_{23} = \text{Minor of } a_{23} = \begin{vmatrix} 5 & 4 \\ 3 & -2 \end{vmatrix} = 5 \times -2 - 4 \times 3 = -10 - 12 = -22$$

$$M_{31} = \text{Minor of } a_{31} = \begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix} = 4 \times 1 - 2 \times 3 = 4 - 6 = -2$$

Solution Cont'd

$$M_{32} = \text{Minor of } a_{32} = \begin{vmatrix} 5 & 2 \\ 2 & 1 \end{vmatrix} = 5 \times 1 - 2 \times 2 = 5 - 4 = 1$$

$$M_{33} = \text{Minor of } a_{33} = \begin{vmatrix} 5 & 4 \\ 2 & 3 \end{vmatrix} = 5 \times 3 - 4 \times 2 = 15 - 8 = 7$$

Step 2: Determine the cofactors using the results of the minors

$c_{11} = \text{Cofactor of } M_{11} = (-1)^{1+1} \times -1 = -1 \times -1 = -1$

$c_{12} = \text{Cofactor of } M_{12} = (-1)^{1+2} \times -5 = -1 \times -5 = 5$

$c_{13} = \text{Cofactor of } M_{13} = (-1)^{1+3} \times -13 = 1 \times -13 = -13$

$c_{21} = \text{Cofactor of } M_{21} = (-1)^{2+1} \times 0 = -1 \times 0 = 0$

$c_{22} = \text{Cofactor of } M_{22} = (-1)^{2+2} \times -11 = 1 \times -11 = -11$

$c_{23} = \text{Cofactor of } M_{23} = (-1)^{2+3} \times -22 = -1 \times -22 = 22$

Solution Cont'd

$$c_{31} = \text{Cofactor of } M_{31} = (-1)^{3+1} \times -2 = 1 \times -2 = -2$$
$$c_{32} = \text{Cofactor of } M_{32} = (-1)^{3+2} \times 1 = -1 \times 1 = -1$$
$$c_{33} = \text{Cofactor of } M_{33} = (-1)^{3+3} \times 7 = 1 \times 7 = 7$$

Adjoint

Definition

The transpose of the cofactor matrix with element a_{ij} of A denoted by adj(A) is called adjoint of matrix A.

Theorem

For any square matrix A,
$A(adjA) = (adjA)A = |A|I$ *where I is the identity matrix of same order.*

Adjoint Cont'd

Proof.

Let $A = (a_{ij})_{n\times n}$

Since A is a square matrix of order n, then adj A is also in the same order.

Consider $A = \begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ . & . & .... & . \\ . & . & .... & . \\ a_{n1} & a_{n2} & .... & a_{nn} \end{bmatrix}$

Then, $AdjA = \begin{bmatrix} A_{11} & A_{12} & .... & A_{1n} \\ A_{21} & A_{22} & .... & A_{2n} \\ . & . & .... & . \\ . & . & .... & . \\ A_{n1} & A_{n2} & .... & A_{nn} \end{bmatrix}$

Now consider the product $A(adjA)$ $\square$

Adjoint Cont'd

$$A(adjA) = \begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ . & . & .... & . \\ . & . & .... & . \\ a_{n1} & a_{n2} & .... & a_{nn} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & .... & A_{1n} \\ A_{21} & A_{22} & .... & A_{2n} \\ . & . & .... & . \\ . & . & .... & . \\ A_{n1} & A_{n2} & .... & A_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{j=1}^{n} a_{1j}A_{1j} & \sum_{j=1}^{n} a_{1j}A_{2j} & .... & \sum_{j=1}^{n} a_{1j}A_{nj} \\ \sum_{j=1}^{n} a_{2j}A_{1j} & \sum_{j=1}^{n} a_{2j}A_{2j} & .... & \sum_{j=1}^{n} a_{2j}A_{nj} \\ . & . & .... & . \\ . & . & .... & . \\ \sum_{j=1}^{n} a_{nj}A_{1j} & \sum_{j=1}^{n} a_{nj}A_{2j} & .... & \sum_{j=1}^{n} a_{nj}A_{nj} \end{bmatrix}$$

$$= \begin{bmatrix} |A| & 0 & .... & 0 \\ 0 & |A| & .... & 0 \\ 0 & 0 & .... & 0 \\ 0 & 0 & .... & |A| \end{bmatrix}$$

Adjoint Cont'd

Since $\sum_{j=1}^{n} a_{ij}A_{ij} = |A|$ and $\sum_{j=1}^{n} a_{ij}A_{kj} = 0$ when $i \neq k$.

$$= |A| \begin{bmatrix} 1 & 0 & .... & 0 \\ 0 & 1 & .... & 0 \\ 0 & 0 & .... & 0 \\ 0 & 0 & .... & 1 \end{bmatrix}$$

$= |A|I_n$

where I_n is unit matrix of order n.

Theorem

If A is a non-singular matrix of order n, then $|adjA| = |A|^{n-1}$

Also show that $(AB)adj(AB) = adj(AB)AB = |AB|I$ and $(adjB.adjA)AB = adjB(adjAA)B = |AB|I$

Some results of adjoint

(i) For any square matrix A, $adj(A)^T = adjA^T$
(ii) The adjoint of an identity matrix is the identity matrix.
(iii) The adjoint of a symmetric matrix is a symmetric matrix.

Inverse of a matrix

Definition

If A and B are square matrices of the same size, such that $AB = BA = I$, then each is said to be inverse of the other. The inverse of A is A^{-1}, while that of B is B^{-1}.

Theorem (Existence of the Inverse)

The necessary and sufficient condition for a square matrix A to have an inverse is that $|A| \neq 0$(That is A is non singular).
If A does not have an inverse, it is called singular or non-invertible.

Proof.

(i) The necessary condition
Let A be a square matrix of order n and B is inverse of it, then

$$AB = I$$

Inverse of a matrix Cont'd

$$|AB| = |A||B|$$

Therefore, $|A| \neq 0$

(ii) The sufficient condition:

If $|A| \neq 0$, then we define the matrix B such that

$$B = \frac{1}{|A|}(adjA)$$

$$Then, AB = A\frac{1}{|A|}(adjA) = \frac{1}{|A|}A(adjA)$$

$$= \frac{1}{|A|}|A|I = I$$

$$Similarly, BA = \frac{1}{|A|}(adjA)A = \frac{1}{|A|}A(adjA) = \frac{1}{|A|}|A|I = I$$

Thus $AB = BA = I$, and B is inverse of A given by $A^{-1} = \frac{1}{|A|}(adjA)$

Inverse of a matrix Cont'd

Theorem (Uniqueness of the Inverse)

If the inverse of a matrix exists, it is unique.

Proof.

Let B and C be inverses of the matrix A then, $AB = BA = I$ and $AC = CA = I$

$$B(AC) = BI$$

$$BA(C) = B$$

$$C = B$$

□

Inverse of a matrix Cont'd

Properties of inverse

- $(A^{-1})^{-1} = A$, i.e., inverse of inverse is original matrix (assuming A is invertible)
- $(AB)^{-1} = B^{-1}A^{-1}$(assuming A, B are invertible)
- $(A^{-T})^{-1} = (A^{-1})^{-T}$ (assuming A is invertible)
- $I^{-1} = I$
- $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$ (assuming A is invertible, $\alpha \neq 0$)
- If $y = Ax$, where $x \in \mathbf{R}^n$ and A is invertible, then $x = A^{-1}y : A^{-1}y = A^{-1}Ax = Ix = x$

Example

Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$ find A^{-1}

Solution $A^{-1} = \frac{1}{|A|} adj(A) = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ since

$|A| = 1 \times 2 - (-1 \times 1) = 3$ and $adj(A) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

Linear transformations

The main focus of linear algebra is the analysis of linear functions, represented on a finite and dimensional vector space. A typical example of this sort includes the analysis of shear transformation. In order to fully understand the concept of linear function or transformation, let us consider the following:

Definition

Given two vectors $\hat{u}$ and $\hat{v}$ (not necessarily having the same dimension), and a scalar C. If T is a linear transformation, then it follows that:

$$T(\hat{u} + \hat{v}) = T(\hat{u}) + T(\hat{v}) \tag{5}$$

$$T(C\hat{u}) = CT(\hat{u}) \quad C \in \mathbf{R} \tag{6}$$

Linear transformations Cont'd

The principle of additivity applies to T in (5), while in (6) homogeneity is attributed to T . The same definition is true if $\hat{u}$ and $\hat{v}$ are complex vectors, except that in (6), $C \in \mathbf{C}$.

Example

Determine whether the following transformations are linear or not.

(1) $\left(\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right) \rightarrow \begin{bmatrix} w_1 - w_2 \\ w_1 + w_2 \\ 2 \times w_1 \end{bmatrix}$ (2) $\left(\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right) \rightarrow \begin{bmatrix} w_1 + w_2 \\ w_2 + 2 \end{bmatrix}$

Solution 1

$$T(\hat{u} + \hat{v}) = T(\hat{u}) + T(\hat{v})$$

Let

$$\hat{u} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \; and \;\; \hat{v} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Linear transformations Cont'd

Solution 1 Cont'd

$$T(\hat{u}+\hat{v}) = \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right)$$

$$= \left(\begin{bmatrix} x_1 + y_1 - (x_2 + y_2) \\ x_1 + y_1 + (x_2 + y_2) \\ 2 \times (x_1 + y_1) \end{bmatrix}\right)$$

$$T(\hat{u}+\hat{v}) = \left(\begin{bmatrix} x_1 + y_1 - x_2 - y_2 \\ x_1 + y_1 + x_2 + y_2 \\ 2x_1 + 2y_1 \end{bmatrix}\right) = LHS$$

$$T(\hat{u}) + T(\hat{v}) = T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) + T\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right)$$

Solution 1 Cont'd

$$= \left(\begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \\ 2x_1 \end{bmatrix} \right) + \left(\begin{bmatrix} y_1 - y_2 \\ y_1 + y_2 \\ 2y_1 \end{bmatrix} \right)$$

$$T(\hat{u}) + T(\hat{v}) = \left(\begin{bmatrix} x_1 - x_2 + y_1 - y_2 \\ x_1 + x_2 + y_1 + y_2 \\ 2x_1 + 2y_1 \end{bmatrix} \right) = RHS$$

Therefore, $T(\hat{u} + \hat{v}) = T(\hat{u}) + T(\hat{v})$ is satisfied.

Applying definition 6

$$T(C\hat{u}) = CT(\hat{u}) \quad C \in \mathbf{R}$$

$$T(C\hat{u}) = T\left(C \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = T\left(\begin{bmatrix} Cx_1 \\ Cx_2 \end{bmatrix} \right)$$

Linear transformations Cont'd

Solution 1 Cont'd

$$T = \left(\begin{bmatrix} Cx_1 - Cx_2 \\ Cx_1 + Cx_2 \\ 2Cx_1 \end{bmatrix} \right) = LHS$$

$$CT(\hat{u}) = C \left(T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \left(C \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \\ 2x_1 \end{bmatrix} \right)$$

$$= \left(\begin{bmatrix} C(x_1 - x_2) \\ C(x_1 + x_2) \\ C \times 2x_1 \end{bmatrix} \right) = RHS$$

Since definitions 5 and 6 are satisfied, this transformation is linear.
Can you try the second example?

-**Reflection with respect to *x*-axis:**

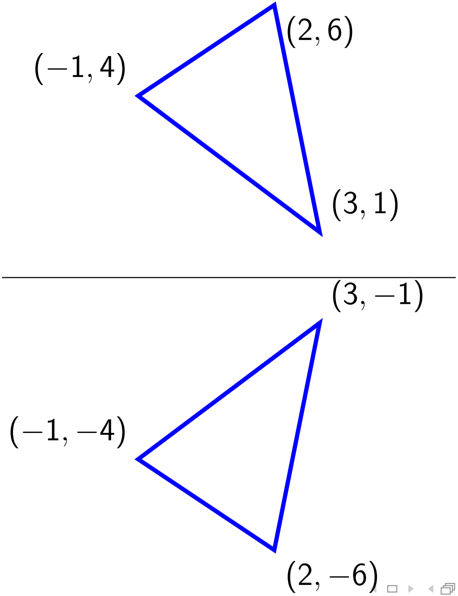
$$L : R^2 \rightarrow R^2, L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = A\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ -u_2 \end{bmatrix}$$

For example, the reflection for the triangle with vertices (-1,4), (3,1), (2,6) is

$$L\left(\begin{bmatrix} -1 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ -4 \end{bmatrix}, L\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, L\left(\begin{bmatrix} 2 \\ 6 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$$

The plot is given below.

Reflection with respect to x-axis



-**Reflection with respect to** $y = -x$**:**

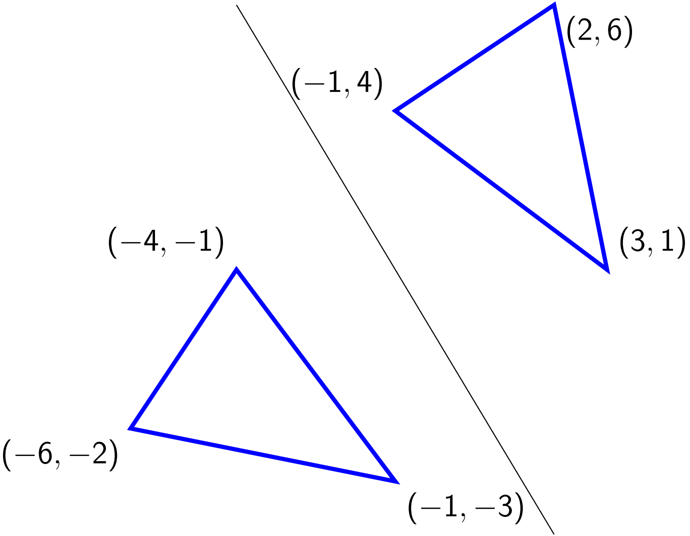
$$L : R^2 \to R^2, L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = A\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -u_1 \\ -u_2 \end{bmatrix}$$

Thus, the reflection for the triangle with vertices (-1,4), (3,1), (2,6) is

$$L\left(\begin{bmatrix} -1 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ 1 \end{bmatrix}, L\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ -3 \end{bmatrix}, L\left(\begin{bmatrix} 2 \\ 6 \end{bmatrix}\right) = \begin{bmatrix} -6 \\ -2 \end{bmatrix}$$

The plot is given below.

-Reflection with respect to $y = -x$:



-Rotation:

$L: R^2 \to R^2, L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = A\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

For example, as $\theta = \pi/2$,

$A = \begin{bmatrix} cos(\pi/2) & -sin(\pi/2) \\ sin(\pi/2) & cos(\pi/2) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Thus, the rotation for the triangle with vertices (0,0), (1,0), (1,1) is

$$L\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

,

$$L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

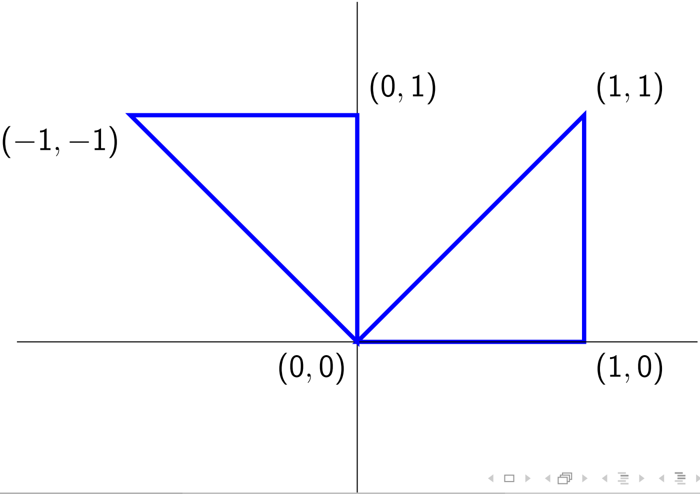
,

-Rotation: Cont'd

$$L\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix}$$

,

The plot is given below.



-Shear in the x-direction:

$L : R^2 \to R^2, L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 + ku_2 \\ u_2 \end{bmatrix}, k \in \mathbf{R}$

For example, as $k = 2$,

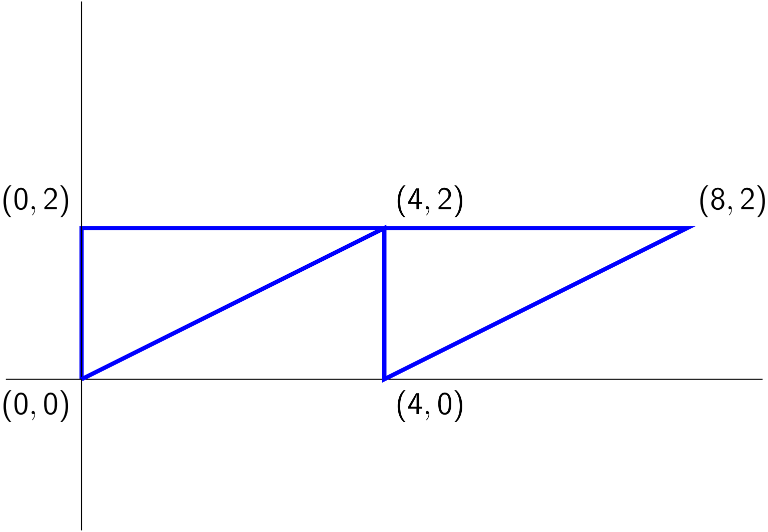
$$L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 + 2u_2 \\ u_2 \end{bmatrix}$$

Thus, the shear for the rectangle with vertices (0,0), (0,2), (4,0), (4,2) in the x-direction is

$$L\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, L\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, L\left(\begin{bmatrix} 4 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 0 \end{bmatrix}, L\left(\begin{bmatrix} 4 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$

The plot is given below.

-Shear in the x-direction:



Cryptography

Suppose we are interested in setting up a meeting with our friend, for security purpose, we first code the alphabet.

Classwork

Determine whether or not this transformation is linear.

$$\left(\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right) \rightarrow \begin{bmatrix} w_1 + w_2 \\ w_2 + 2 \end{bmatrix}$$

Thank you for your attention!