MATH*661 ASSIGNMENT

Differential Geometry

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Differential Geometry Assignment

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DIFFERENTIAL GEOMETRY

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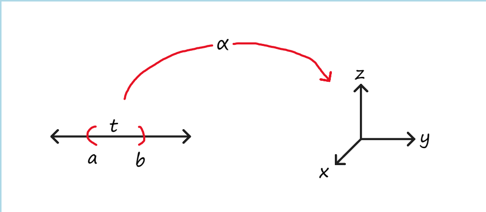
Contents

1 Curves in $\mathbb{R}^3$

1.1 Curves

Definition 1.1. A curve in $\mathbb{R}^3$ is a differentiable function $\alpha : I = (a, b) \subset \mathbb{R} \to \mathbb{R}^3$.



$$\alpha(t) = (x(t), y(t), z(t)) \text{ Parametric form}$$
$$t: \text{ is called a parameter}$$

Definition 1.2. (Regular Curve) A regular curve in $\mathbb{R}^3$ is a differentiable function $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$ such that

$$\frac{d\alpha}{dt} \neq 0 \quad \forall t \in I$$

Example 1.1. Show that the curve $\alpha(t) = (a\cos t, a\sin t, bt)$ is regular for $t \in \mathbb{R}$.
Solution:

$$\frac{d\alpha}{dt} = (-a\sin t, a\cos t, b) \neq 0$$
$$\therefore \alpha(t) \text{ is regular}$$

Example 1.2. Is $\alpha(t) = (t^3 - 4t, t^2 - 4)$ a regular curve for $t \in \mathbb{R}$?
Solution:

$$\frac{d\alpha}{dt} = (3t^2 - 4, 2t) \neq 0 \quad \forall t \in \mathbb{R}$$
$$\therefore \alpha(t) \text{ is regular}$$

Example 1.3. Is $\alpha(t) = (t, \cosh t)$ a regular curve for $t \in \mathbb{R}$?
Solution:

$$\frac{d\alpha}{dt} = (1, \sinh t) \neq 0 \quad \forall t \in \mathbb{R}$$
$$\therefore \alpha(t) \text{ is regular}$$

Example 1.4. Is $\alpha(t) = (\cos^2 t, \sin^2 t)$ a regular curve for $t \in \mathbb{R}$?
Solution:

$$\begin{aligned} \frac{d\alpha}{dt} &= (-2\sin t\cos t, 2\sin t\cos t) \\ &= (-\sin 2t, \sin 2t) = 0 \quad \text{at } t = 0 \\ &\therefore \alpha(t) \text{ is not regular} \end{aligned}$$

Example 1.5. Is $\alpha(t) = (\cos^2 t, \sin^2 t)$ a regular curve for $t \in (0, \frac{\pi}{2})$?
Solution:

$$\begin{aligned} \frac{d\alpha}{dt} &= (-2\sin t\cos t, 2\sin t\cos t) \\ &= (-\sin 2t, \sin 2t) \neq 0 \quad \forall t \in (0, \frac{\pi}{2}) \\ &\therefore \alpha(t) \text{ is regular} \end{aligned}$$

Example 1.6. Let $\alpha(t) = (e^t, e^{-t}, t^2)$ be a curve for $t \in \mathbb{R}$. Determine if $\alpha(t)$ is a regular curve.
Solution:

$$\begin{aligned} \frac{d\alpha}{dt} &= (e^t, -e^{-t}, 2t) \\ &\neq 0 \quad \forall t \in \mathbb{R} \\ &\therefore \alpha(t) \text{ is regular.} \end{aligned}$$

1.2 Arc Length

Definition 1.3. (Arc Length) Consider the curve $\alpha : I = (a, b) \subset \mathbb{R} \to \mathbb{R}^3$. The arc length of this curve from a to b is given by:

$$s = \int_a^b \left| \frac{d\alpha}{dt} \right| dt$$

Remark 1.1. In case $(a.b)$ was not given use $(0.t)$

Example 1.7. Find the arc length of the curve $\alpha(t) = (6t, 3t^2, t^3)$ for $0 \leq t \leq 6$.
Solution:

$$\begin{aligned}
\frac{d\alpha}{dt} &= (6, 6t, 3t^2) \\
\left|\frac{d\alpha}{dt}\right| &= \sqrt{36 + 36t^2 + 9t^4} \\
&= \sqrt{9(4 + 4t^2 + t^4)} \\
&= 3\sqrt{(2 + t^2)^2} \\
&= 3(2 + t^2) \\
&= 3t^2 + 6 \\
\therefore s &= \int_0^6 3(t^2 + 2)dt \\
&= 3\left[\frac{t^3}{3} + 2t\right]_0^6 \\
&= 3\left[\left(\frac{216}{3} + 12\right) - 0\right] \\
&= 3\left[\frac{216}{3} + \frac{36}{3}\right] \\
&= 252
\end{aligned}$$

Example 1.8. Calculate the arc length of the curve $\alpha(t) = (\cos t, \sin t, t)$ starting at the point $(0, t)$.
Solution:

$$\begin{aligned}
\frac{d\alpha}{dt} &= (-\sin t, \cos t, 1) \\
\left|\frac{d\alpha}{dt}\right| &= \sqrt{\sin^2 t + \cos^2 t + 1} \\
&= \sqrt{1 + 1} = \sqrt{2} \\
\therefore s &= \int_0^t \sqrt{2}dt = \sqrt{2}\int_0^t dt \\
&= \sqrt{2}\,[t]_0^t = t\sqrt{2}
\end{aligned}$$

Example 1.9. calculate the arc length of the curve $\alpha(t) = (e^t \cos t, e^t \sin t)$
Solution:

$$\frac{d\alpha}{dt} = \left(\frac{d}{dt}(e^t \cos t), \frac{d}{dt}(e^t \sin t)\right)$$
$$= \left(e^t \cos t - e^t \sin t, e^t \sin t + e^t \cos t\right)$$

$$\left|\frac{d\alpha}{dt}\right| = \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2}$$
$$= \sqrt{e^{2t}\cos^2 t - 2e^{2t}\cos t \sin t + e^{2t}\sin^2 t + e^{2t}\sin^2 t + 2e^{2t}\cos t \sin t + e^{2t}\cos^2 t}$$
$$= \sqrt{2e^{2t}(\sin^2 t + \cos^2 t)} = \sqrt{2}e^t$$

$$s = \int_0^t \left|\frac{d\alpha}{dt}\right| dt = \int_0^t \sqrt{2}e^t dt$$
$$= \sqrt{2}\int_0^t e^t dt = \sqrt{2}(e^t - e^0) = \sqrt{2}(e^t - 1)$$

So, the arc length of the curve $\alpha(t) = (e^t \cos t, e^t \sin t)$ is $\sqrt{2}(e^t - 1)$.

Example 1.10. Find the arc length of the curve

$$\alpha(t) = (\cosh t, \sinh t, t) \text{ for } 0 \leq t \leq 5$$

Solution:

$$\frac{d\alpha}{dt} = (\sinh t, cosht, 1)$$
$$\left|\frac{d\alpha}{dt}\right| = \sqrt{\sinh^2 t + cosh^2 t + 1}$$

$$\boxed{\cosh^2 t - \sinh^2 t = 1} \Rightarrow 1 + \sinh^2 t = \cosh^2 t$$

$$\left|\frac{d\alpha}{dt}\right| = \sqrt{2\cosh^2 t} = \sqrt{2}\cosh t$$
$$\therefore s = \int_0^5 \sqrt{2}\cosh t dt$$
$$= \sqrt{2}\,[\sinh]_0^5$$
$$= \sqrt{2}\,[\sinh 5 - \sinh 0]$$
$$s = \sqrt{2}\sinh 5$$

Remark 1.2.

$$\sinh x = \frac{e^x - e^{-x}}{2}$$
$$\sinh 0 = \frac{e^0 - e^0}{2} = \frac{1-1}{2} = \boxed{0}$$

Example 1.11. Find the arc length of the curve

$$\alpha(t) = (4\cos t, 4\sin t, 3t) \text{ for } 0 \leq t \leq 2\pi$$

Solution:

$$\frac{d\alpha}{dt} = (-4\sin t, 4\cos t, 3)$$
$$\left|\frac{d\alpha}{dt}\right| = \sqrt{16\sin^2 t + 16\cos^2 t + 9} = 5$$
$$\therefore s = \int_0^{2\pi} 5dt = 10\pi$$

Example 1.12. Find the parametric form of the curve

$$x^2 + y^2 = a^2 \quad , x^2 + z^2 = a^2$$

Solution:

$$x^2 + y^2 = a^2$$
$$-$$
$$x^2 + z^2 = a^2$$

$$= y^2 - z^2 = 0 \Rightarrow y^2 = z^2$$

$$\Rightarrow \text{put } y = z = a\sin t$$

$$\therefore x^2 + y^2 = a^2 \Rightarrow x = a\cos t$$

$$\alpha(t) = (a\cos t, a\sin t, a\sin t)$$

or

$$\alpha(t) = (a\sin t, a\cos t, a\cos t)$$

Example 1.13. Find the parametric form of the following curve:

$$x^2 + y^2 = 1 \quad \underline{curve} \tag{1}$$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1 \quad Elipse \tag{2}$$

$$y^2 - x^2 = 1 \quad hyperbola \tag{3}$$

$$y = (\ln x)^2 \tag{4}$$

$$y = x^2 \quad Parabola \tag{5}$$

sol: (1)

$$x^2 + y^2 = 1$$

put

$$x = \sin t \quad , y = \cos t$$

$$\Rightarrow \alpha(t) = (\sin t, \cos t)$$

or put

$$x = \cos t \quad y = \sin t \Rightarrow \alpha(t) = (\cos t, \sin t)$$

(2)

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

put

$$x = 2\sin t \quad , t = 3\cos t$$

$$\Rightarrow \alpha(t) = (2\sin t, 3\cos t)$$

(3)

$$y^2 - x^2 = 1$$

put

$$y = \sec t \quad x = \tan t$$

$$\Rightarrow \alpha(t) = (\tan t, \sec t)$$

$$\boxed{\sec^2 t - \tan^2 t = 1}$$

(4)

$$y = (\ln x)^2$$

put

$$\ln x = t \Leftrightarrow x = e^t$$

$$\therefore y = t^2 \Rightarrow \alpha(t) = (e^t, t^2)$$

(5)$y = x^2$put

$$x = t \Rightarrow y = t^2$$

$$\alpha(t) = (t, t^2)$$

Definition 1.4. A curve in $\mathbb{R}^3$can be defined as the intersection of two surface

$$F_1(x, y, z) = 0 \quad , F_2(x, y, z) = 0$$

Example 1.14. Find the parametric form of the curve $y^2 = x \quad z^2 = 1 - x$ sol:

$$\begin{aligned} y^2 &= x \\ &+ \\ z^2 &= 1 - x \end{aligned}$$

$$= y^2 + z^2 = 1$$

put $y = \sin t \quad , z = \cos t, x = \sin^2 t$

$$\Rightarrow \alpha(t) = (\sin^2 t, \sin t, \cos t)$$

Example 1.15. Find the parametric form of the curve

$$y = 2x^2 - 4$$

sol:
put$x = t \quad y = 2t^2 - 4$

$$\Rightarrow \alpha(t) = (t, 2t^2 - 4)$$

Definition 1.5. Unit-speed curve

Consider a curve $\alpha(t) = (x(t), y(t), z(t))$
speed of a curve :

$$\left|\frac{d\alpha}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

A curve $\alpha(t)$ is called unit speed curve if $\left|\frac{d\alpha}{dt}\right| = 1$

Example 1.16. Show that the curve $\alpha(t) = \left(\frac{4}{5}\cos t, 1 - \sin t, -\frac{3}{5}\cos t\right)$ is a unit speed curve.
Solution:

$$\begin{aligned} \frac{d\alpha}{dt} &= \left(-\frac{4}{5}\sin t, -\cos t, \frac{3}{5}\sin t\right) \\ \left|\frac{d\alpha}{dt}\right| &= \sqrt{\left(\frac{16}{25}\sin^2 t + \cos^2 t + \frac{9}{25}\sin^2 t\right)} \\ &= 1 \end{aligned}$$

$\therefore \alpha(t)$ is a unit speed curve.

Example 1.17. Show that the curve $\alpha(t) = \left(\frac{1}{3}(1+t)^{\frac{3}{2}}, \frac{1}{3}(1-t)^{\frac{3}{2}}, \frac{t}{\sqrt{2}}\right)$ is a unit speed curve.

Solution:

$$\begin{aligned}
\frac{d\alpha}{dt} &= \left(\frac{1}{3}\frac{3}{2}(1+t)^{\frac{1}{2}}, -\frac{1}{3}\frac{3}{2}(1-t)^{\frac{1}{2}}, \frac{1}{\sqrt{2}}\right) \\
&= \left(\frac{1}{2}(1+t)^{\frac{1}{2}}, -\frac{1}{2}(1-t)^{\frac{1}{2}}, \frac{1}{\sqrt{2}}\right) \\
\left|\frac{d\alpha}{dt}\right| &= \sqrt{\left(\frac{1}{4}(1+t) + \frac{1}{4}(1-t) + \frac{1}{2}\right)} \\
&= \sqrt{\left(\frac{1}{4} + \frac{1}{4}t + \frac{1}{4} - \frac{1}{4}t + \frac{1}{2}\right)} \\
&= 1
\end{aligned}$$

$\therefore \alpha(t)$ is a unit speed curve.

Example 1.18. Show that the curve $\alpha(t) = \left(\frac{2}{5}, \frac{3t}{5}, \frac{4t}{5}\right)$ is a unit speed curve.

Solution:

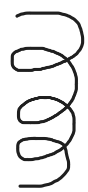
$$\begin{aligned}
\frac{d\alpha}{dt} &= \left(0, \frac{3}{5}, \frac{4}{5}\right) \\
\left|\frac{d\alpha}{dt}\right| &= \sqrt{\left(0 + \frac{9}{25} + \frac{16}{25}\right)} \\
&= \sqrt{\frac{25}{25}} \\
&= 1
\end{aligned}$$

$\therefore \alpha(t)$ is a unit speed curve.

1.3 Helix

$$\alpha(t) = (a\cos t, a\sin t, bt)$$

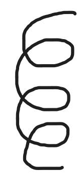
Right Circular Helix



$a > 0$

$b > 0$

Left Circular Helix



$a > 0$

$b < 0$

Remark 1.3. (i) If $b = 0$, then $\alpha(t) = (a\cos t, a\sin t, 0)$ is a circle.

(ii) If $a = 0$, then $\alpha(t) = (0, 0, bt)$ is a straight line.

1.4 Reparametrization

Example 1.19. Reparamertrize the helix $\alpha(t) = (a\cos t, a\sin t, bt)$ by the arc length.
Solution:

$$\begin{aligned}
\frac{d\alpha}{dt} &= (-a\sin t, a\cos t, b) \\
|\frac{d\alpha}{dt}| &= \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2+b^2} \\
s &= \int_0^t |\frac{d\alpha}{dt}|dt = \int_0^t \sqrt{a^2+b^2}dt = \sqrt{a^2+b^2}[t]_0^t \\
\Rightarrow s &= t\sqrt{a^2+b^2} \\
\Rightarrow t &= \frac{s}{\sqrt{a^2+b^2}} \\
\therefore \alpha(s) &= \left(a\cos\frac{s}{\sqrt{a^2+b^2}}, a\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{bs}{\sqrt{a^2+b^2}}\right).
\end{aligned}$$

Example 1.20. Reparameterize the curve $\alpha(t) = (e^t \cos t, e^t \sin t)$ by the arc length.
Solution:

$$\begin{aligned}
\frac{d\alpha}{dt} &= (e^t \cos t - e^t \sin t, e^t \sin t + e^t \cos t) \\
\left|\frac{d\alpha}{dt}\right| &= \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2} \\
&= \sqrt{e^{2t}(\cos^2 t - 2\cos t \sin t + \sin^2 t + \sin^2 t + 2\cos t \sin t + \cos^2 t)} \\
&= \sqrt{e^{2t}(2(\cos^2 t + \sin^2 t))} \\
&= \sqrt{2e^{2t}} \\
&= e^t\sqrt{2} \\
s &= \int_0^t \left|\frac{d\alpha}{dt}\right| dt = \int_0^t e^t\sqrt{2}dt = \sqrt{2}[e^t]_0^t \\
&= \sqrt{2}(e^t - e^0) = \sqrt{2}(e^t - 1) \\
\Rightarrow s &= e^t\sqrt{2} - \sqrt{2}
\end{aligned}$$

To find t in terms of s, we'll need to solve $s = e^t\sqrt{2} - \sqrt{2}$ for t:

$$\begin{gathered}
s + \sqrt{2} = e^t\sqrt{2} \\
\frac{s}{\sqrt{2}} + 1 = e^t \\
\ln\left(\frac{s}{\sqrt{2}} + 1\right) = t
\end{gathered}$$

Therefore, the reparameterized curve is:

$$\alpha(s) = \left(\left(\frac{s}{\sqrt{2}} + 1\right)\cos\ln\left(\frac{s}{\sqrt{2}} + 1\right), \left(\frac{s}{\sqrt{2}} + 1\right)\sin\ln\left(\frac{s}{\sqrt{2}} + 1\right)\right).$$

Example 1.21. Reparametrize the curve $\alpha(t) = (\cosh t, \sinh t, t)$ by the arc length.
Solution:

$$\begin{aligned}
\frac{d\alpha}{dt} &= (\sinh t, \cosh t, 1) \\
\left|\frac{d\alpha}{dt}\right| &= \sqrt{\sinh^2 t + \cosh^2 t + 1} \\
&= \sqrt{2\cosh^2 t} \\
&= \sqrt{2}\cosh t \\
s &= \int_0^t \left|\frac{d\alpha}{dt}\right| dt = \int_0^t \cosh t\sqrt{2}dt \\
&= \sqrt{2}\int_0^t \cosh t dt \\
&= \sqrt{2}[\sinh t]_0^t \\
&= \sqrt{2}(\sinh t - \sinh 0) \\
&= \sqrt{2}\sinh t \\
\Rightarrow s &= \sqrt{2}\sinh t
\end{aligned}$$

To find t in terms of s, we'll need to solve $s = \sqrt{2}\sinh t$ for t:

$$\begin{aligned}
\frac{s}{\sqrt{2}} &= \sinh t \\
\sinh^{-1}\left(\frac{s}{\sqrt{2}}\right) &= t
\end{aligned}$$

Therefore, the reparametrized curve is:

$$\alpha(s) = \left(\cosh\left(\sinh^{-1}\left(\frac{s}{\sqrt{2}}\right)\right), \sinh\left(\sinh^{-1}\left(\frac{s}{\sqrt{2}}\right)\right), \sinh^{-1}\left(\frac{s}{\sqrt{2}}\right)\right).$$

Example 1.22. Reparametrize the curve $\alpha(t) = (t, \frac{t^2}{2})$ by the arc length if possible.
Solution:

$$\begin{aligned}
\frac{d\alpha}{dt} &= (1, t) \\
\left|\frac{d\alpha}{dt}\right| &= \sqrt{1^2 + t^2} \\
&= \sqrt{1 + t^2} \\
s &= \int_0^t \left|\frac{d\alpha}{dt}\right| dt = \int_0^t \sqrt{1 + t^2}dt \\
s &= \sqrt{t^2 + 1} + \ln(t + \sqrt{t^2 + 1})
\end{aligned}$$

Since it is difficult to separate the variables, reparametrizing the curve becomes impossible.

Example 1.23. Reparametrize the curve $\alpha(t) = (-2\sin t, 3\cos t)$ by the arc length if possible.
Solution:

$$\begin{aligned}
\frac{d\alpha}{dt} &= (-2\cos t, -3\sin t) \\
\left|\frac{d\alpha}{dt}\right| &= \sqrt{(-2\cos t)^2 + (-3\sin t)^2} \\
&= \sqrt{4\cos^2 t + 9\sin^2 t} \\
&= \sqrt{4 + 5\sin^2 t} \\
s &= \int_0^t \left|\frac{d\alpha}{dt}\right| dt = \int_0^t \sqrt{4 + 5\sin^2 t} dt
\end{aligned}$$

Since it is difficult to evaluate the integral $\int_0^t \sqrt{4 + 5\sin^2 t} dt$, it is impossible to reparametrize the curve by arc length.

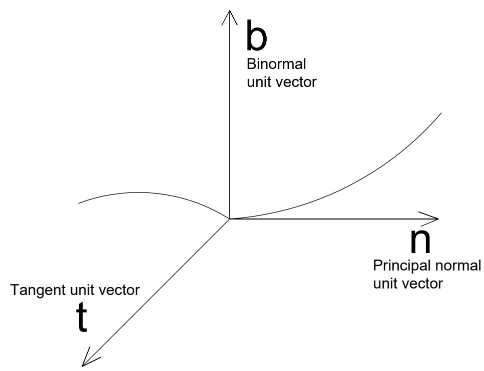
Example 1.24. Reparametrize the curve $\alpha(t) = (t, t^2, t^3)$ by the arc length. **Solution:**

$$\begin{aligned}
\frac{d\alpha}{dt} &= (1, 2t, 3t^2) \\
\left|\frac{d\alpha}{dt}\right| &= \sqrt{1^2 + (2t)^2 + (3t^2)^2} \\
&= \sqrt{1 + 4t^2 + 9t^4} \\
s &= \int_0^t \sqrt{1 + 4t^2 + 9t^4}\, dt
\end{aligned}$$

Since the integral $\int \sqrt{1 + 4t^2 + 9t^4}\, dt$ does not have a closed form solution, it is impossible to compute the arc length parameter s explicitly. Therefore, we cannot reparametrize the curve using arc length.

2 Unit Vectors, Lines and Planes Associated to The Curve

2.1 Unit Vectors Associated to The Curve



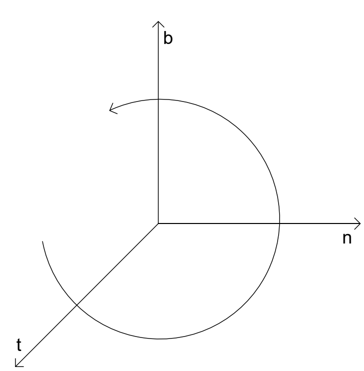
$$\alpha' = \frac{d\alpha}{ds} \qquad \alpha^{\cdot} = \frac{d\alpha}{dt}$$

$$\alpha(s) \qquad \alpha(t)$$

$$\mathbf{t} = \alpha'(s) \qquad \mathbf{t} = \frac{\alpha^{\cdot}}{|\alpha^{\cdot}|}$$

$$\mathbf{n} = \frac{\alpha''(s)}{|\alpha''(s)|} \qquad \mathbf{b} = \frac{\alpha^{\cdot} \times \alpha^{\cdot\cdot}}{|\alpha^{\cdot} \times \alpha^{\cdot\cdot}|}$$

$$\mathbf{b} = \mathbf{t} \times \mathbf{n} \qquad \mathbf{n} = \mathbf{b} \times \mathbf{t}$$



Example 2.1. Find **t**,**n**,**b** for the helix

$$\alpha(s) = \left(a\cos\frac{s}{\sqrt{a^2+b^2}}, a\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{bs}{\sqrt{a^2+b^2}}\right)$$

Solution:

$$\begin{aligned}
\mathbf{t} = \alpha'(s) &= \left(\frac{-a}{\sqrt{a^2+b^2}}\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}}\cos\frac{s}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right) \\
\alpha''(s) &= \left(\frac{-a}{a^2+b^2}\cos\frac{s}{\sqrt{a^2+b^2}}, \frac{-a}{a^2+b^2}\sin\frac{s}{\sqrt{a^2+b^2}}, 0\right) \\
|\alpha''(s)| &= \sqrt{\frac{a^2}{(a^2+b^2)^2}\cos^2\frac{s}{\sqrt{a^2+b^2}} + \frac{a^2}{(a^2+b^2)^2}\sin^2\frac{s}{\sqrt{a^2+b^2}}} \\
&= \sqrt{\frac{a^2}{(a^2+b^2)^2}} = \frac{a}{a^2+b^2}
\end{aligned}$$

$$\mathbf{n} = \frac{\alpha''(s)}{|\alpha''(s)|} = \left(-\cos\frac{s}{\sqrt{a^2+b^2}}, -\sin\frac{s}{\sqrt{a^2+b^2}}, 0\right)$$

$$\begin{aligned}
\mathbf{b} &= \mathbf{t}\times\mathbf{n} \\
&= \begin{vmatrix} \mathbf{t} & \mathbf{n} & \mathbf{b} \\ \frac{-a}{\sqrt{a^2+b^2}}\sin\frac{s}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}}\cos\frac{s}{\sqrt{a^2+b^2}} & \frac{b}{\sqrt{a^2+b^2}} \\ -\cos\frac{s}{\sqrt{a^2+b^2}} & -\sin\frac{s}{\sqrt{a^2+b^2}} & 0 \end{vmatrix} \\
&= \frac{b}{\sqrt{a^2+b^2}}\sin\frac{s}{\sqrt{a^2+b^2}}t - \frac{b}{\sqrt{a^2+b^2}}\cos\frac{s}{\sqrt{a^2+b^2}}n \\
&+ \left(\frac{a}{\sqrt{a^2+b^2}}\sin^2\frac{s}{\sqrt{a^2+b^2}} + \frac{a}{\sqrt{a^2+b^2}}\cos^2\frac{s}{\sqrt{a^2+b^2}}\right)b \\
\therefore \mathbf{b} &= \left(\frac{b}{\sqrt{a^2+b^2}}\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{-b}{\sqrt{a^2+b^2}}\cos\frac{s}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}}\right)
\end{aligned}$$

Example 2.2. Find **t**,**n**,**b**

$$\alpha(s) = (6\sinh s, 6\cosh s, s)$$

Solution:

$$\begin{aligned}
\mathbf{t} = \alpha'(s) &= (6\cosh s, 6\sinh s, 1) \\
\alpha''(s) &= (6\sinh s, 6\cosh s, 0) \\
|\alpha''(s)| &= \sqrt{6^2(\sinh^2 s + \cosh^2 s)} \\
&= 6\sqrt{\cosh 2s}
\end{aligned}$$

$$\boxed{\cosh^2 s + \sinh^2 s = \cosh 2s}$$

$$\mathbf{n} = \frac{\alpha''(s)}{|\alpha''(s)|} = \left(\frac{\sinh s}{\sqrt{\cosh 2s}}, \frac{\cosh s}{\sqrt{\cosh 2s}}, 0\right)$$

$$\mathbf{b} = \mathbf{t} \times \mathbf{n}$$

$$= \begin{vmatrix} \mathbf{t} & \mathbf{n} & \mathbf{b} \\ 6\cosh s & 6\sinh s & 1 \\ \frac{\sinh s}{\sqrt{\cosh 2s}} & \frac{\cosh s}{\sqrt{\cosh 2s}} & 0 \end{vmatrix}$$

$$= (\frac{-\cosh s}{\sqrt{\cosh 2s}})\mathbf{t} + (\frac{\sinh s}{\sqrt{\cosh 2s}})\mathbf{n} + (\frac{6\cosh^2 s}{\sqrt{\cosh 2s}} - \frac{6\sinh^2 s}{\sqrt{\cosh 2s}})\mathbf{b}$$

$$\therefore \mathbf{b} = \left(\frac{-\cosh s}{\sqrt{\cosh 2s}}, \frac{\sinh s}{\sqrt{\cosh 2s}}, \frac{6}{\sqrt{\cosh 2s}}\right)$$

Example 2.3. Find **t**,**n**,**b** for the parabola

$$\alpha(t) = (t, t^2/2, 0)$$

solution:

$$\alpha^{\cdot}(t) = (1, t, 0) \quad |\alpha^{\cdot}(t)| = \sqrt{1+t^2}$$

$$\Rightarrow \mathbf{t} = \frac{\alpha^{\cdot}}{|\alpha^{\cdot}|} = \frac{(1,t,0)}{\sqrt{1+t^2}} = \left(\frac{1}{\sqrt{1+t^2}}, \frac{t}{\sqrt{1+t^2}}, 0\right)$$

$\alpha^{\cdot\cdot} = (0, 1, 0)$

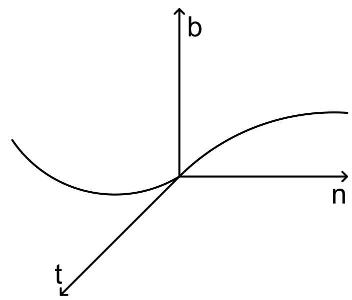
$$\alpha^{\cdot} \times \alpha^{\cdot\cdot} = \begin{vmatrix} \mathbf{t} & \mathbf{n} & \mathbf{b} \\ 1 & t & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0\mathbf{t} - 0\mathbf{n} + 1\mathbf{b}$$

$\alpha^{\cdot} \times \alpha^{\cdot\cdot} = (0,0,1) \quad \Rightarrow |\alpha^{\cdot} \times \alpha^{\cdot\cdot}| = 1 \quad \therefore \mathbf{b} = \frac{\alpha^{\cdot} \times \alpha^{\cdot\cdot}}{|\alpha^{\cdot} \times \alpha^{\cdot\cdot}|} = (0,0,1)$

$$\mathbf{n} = \mathbf{b} \times \mathbf{t} = \begin{vmatrix} \mathbf{t} & \mathbf{n} & \mathbf{b} \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{1+t^2}} & \frac{t}{\sqrt{1+t^2}} & 0 \end{vmatrix} = \frac{-t}{\sqrt{1+t^2}}\mathbf{t} + \frac{1}{\sqrt{1+t^2}}\mathbf{n} + 0\mathbf{b}$$

$$\therefore \mathbf{n} = \left(\frac{-t}{\sqrt{1+t^2}}, \frac{1}{\sqrt{1+t^2}}, 0\right)$$

2.2 Lines Associated to the Curve at the Point $\alpha(s_\circ)$ or $\alpha(t_\circ)$



Tangent line → passing through $\alpha(s_\circ)$

Tangent line → parallel $\mathbf{t}$

Tangent equation of this line is

$$\alpha_T = \alpha(s_\circ) + \lambda t(s_\circ)$$

$$(x, y, z) = (a, b, c) + \lambda(A_1, A_2, A_3)$$

$$\left.\begin{array}{l} \text{x=a+}\lambda\text{ A}_1 \\ \text{y=b+}\lambda\text{ A}_2 \\ \text{z=c+}\lambda\text{ A}_3 \end{array}\right\} \quad \text{Parametric equations of the line}$$

$$\frac{x-a}{A_1} = \lambda \qquad , \frac{y-b}{A_2} = \lambda \qquad , \frac{z-c}{A_3}\lambda$$

$$\boxed{\frac{x-a}{A_1} = \frac{y-b}{A_2} = \frac{z-c}{A_3}}$$ Parametric equations of straight lines(standard form)

principle normal line → passing through the point $\alpha(s_\circ)$=(a,h,c)

principle normal line → parallel $\mathbf{n}(s_\circ) = (B_1, B_2, B_3)$

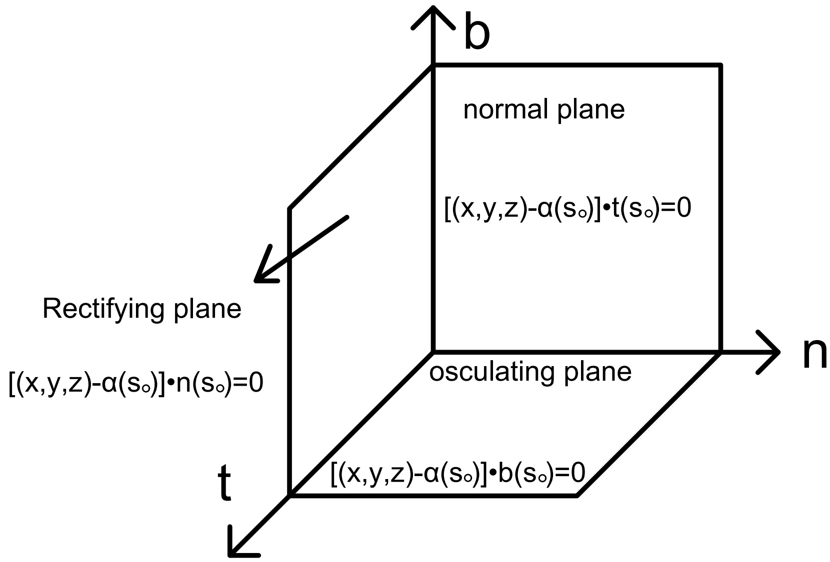
$$\boxed{\frac{x-a}{B_1} = \frac{y-b}{B_2} = \frac{z-c}{B_3}}$$

binormal line → passing through the point $\alpha(s_\circ)$=(a,h,c)

binormal line → parallel $\mathbf{b}(s_\circ) = (C_1, C_2, C_3)$

$$\boxed{\frac{x-a}{C_1} = \frac{y-b}{C_2} = \frac{z-c}{C_3}}$$

2.3 Planes Associated to the Curve



Example 2.4. Consider the helix $\alpha(s) = (a\cos\frac{s}{\sqrt{a^2+b^2}}, a\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{bs}{\sqrt{a^2+b^2}})$. Find the tangent line, principal normal line, binormal line, osculating plane, normal plane, and rectifying plane at $\alpha(0)$ or $s = 0$.

Solution:

$$\begin{aligned}
\alpha(0) &= (a, 0, 0) \\
\mathbf{t} = \alpha'(s) &= \left(-\frac{a}{\sqrt{a^2+b^2}}\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}}\cos\frac{s}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right) \\
\mathbf{t}(0) &= \left(0, \frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right)
\end{aligned}$$

Tangent line:

$$\boxed{\frac{x-a}{0} = \frac{y-0}{\frac{a}{\sqrt{a^2+b^2}}} = \frac{z-0}{\frac{b}{\sqrt{a^2+b^2}}} \quad \text{or} \quad \frac{x-a}{0} = \frac{y}{a} = \frac{z}{b}}$$

Normal plane:

$$\begin{aligned}
&[(x,y,z) - \alpha(0)] \cdot t(0) = 0 \\
&[(x,y,z) - (a,0,0)] \cdot \left(0, \frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right) = 0 \\
&(x-a, y, z) \cdot \left(0, \frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right) = 0 \\
&\frac{a}{\sqrt{a^2+b^2}}y, \frac{b}{\sqrt{a^2+b^2}}z = 0 \Rightarrow ay = bz = 0
\end{aligned}$$

Example 2.5. Find the tangent line, principal normal line, binormal line, osculating plane, normal plane, and rectifying plane to the parabola: $\alpha(t) = (t, \frac{t^2}{2}, 0)$ at $t = 1$.
Solution:

$$\begin{aligned}
\dot{\alpha} &= (1, t, 0) \\
|\dot{\alpha}| &= \sqrt{1+t^2} \\
\mathbf{t} &= \frac{\dot{\alpha}}{|\dot{\alpha}|} = \frac{(1,t,0)}{\sqrt{1+t^2}} = \left(\frac{1}{\sqrt{1+t^2}}, \frac{t}{\sqrt{1+t^2}}, 0\right) \\
\mathbf{t}(1) &= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)
\end{aligned}$$

Tangent line:

$$\frac{x-1}{\frac{1}{\sqrt{2}}} = \frac{y-\frac{1}{2}}{\frac{1}{\sqrt{2}}} = \frac{z-0}{\frac{0}{\sqrt{2}}} \Rightarrow \frac{x-1}{1} = \frac{y-\frac{1}{2}}{1} = \frac{z}{0}$$

Normal plane:

$$\begin{aligned}
&[(x,y,z) - \alpha(1)] \cdot t(1) = 0 \\
&(x-1, y-\frac{1}{2}, z) \cdot \left(\frac{1}{2}, \frac{1}{2}, 0\right) = 0 \\
&\frac{1}{2}(x-1) + \frac{1}{2}\left(y - \frac{1}{2}\right) = 0 \\
&\Rightarrow x - 1 + y - \frac{1}{2} = 0 \\
&\Rightarrow x + y - \frac{3}{2} = 0
\end{aligned}$$

2.4 Frenet-Serret Equations

Along a curve $\alpha(s)$, the vectors $\mathbf{t}$, $\mathbf{n}$, and $\mathbf{b}$ satisfy the following equations:

1. $\mathbf{t}' = K\mathbf{n}$

2. $\mathbf{n}' = K\mathbf{t} - \tau\mathbf{b}$

3. $\mathbf{b}' = \tau\mathbf{n}$

$$\text{or } \begin{bmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \end{bmatrix} = \begin{bmatrix} 0 & K & 0 \\ -K & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}$$

Proof:
1. $\because \mathbf{t} = \alpha' \Rightarrow \mathbf{t}' = \alpha''$
$\because \mathbf{n} = \frac{\alpha''}{|\alpha''|} = \frac{\alpha''}{K} \Rightarrow \alpha'' = K\mathbf{n}$
$\Rightarrow \boxed{\mathbf{t}' = K\mathbf{n}}$
2. $\because \mathbf{n} = \mathbf{b} \times \mathbf{t}$
Differentiate both sides w.r.t. s
$\mathbf{n}' = \mathbf{b} \times \mathbf{t}' + \mathbf{b}' \times \mathbf{t}$
$= \mathbf{b} \times (K\mathbf{n}) + (\tau\mathbf{n}) \times \mathbf{t}$
$= K(\mathbf{b} \times \mathbf{n}) + \tau(\mathbf{n} \times \mathbf{t})$
$= K(-\mathbf{t}) + \tau(-\mathbf{b})$
$= \boxed{-K\mathbf{t} - \tau\mathbf{b}}$ 3. $\because \tau = \mathbf{b}'\mathbf{n}$
$\tau\mathbf{n} = \mathbf{b}' \cdot \mathbf{n} \cdot \mathbf{n} = \mathbf{b}'(\mathbf{n} \cdot \ \mathbf{n})$
$\Rightarrow \boxed{\mathbf{b}' = \tau\mathbf{n}}$

Example 2.6. A curve $\alpha(s)$ is a straight line iff $K = 0$

Proof:

(i) Suppose $\alpha(s)$ is a straight line.
$\Rightarrow \alpha(s) = as + b$
$\Rightarrow \alpha'(s) = a \Rightarrow \alpha''(s) = 0 \Rightarrow |\alpha''| = 0 \Rightarrow K = 0$

(ii) Suppose $K = 0$.
$\Rightarrow |\alpha''| = 0 \Rightarrow \alpha'' = 0$
$\xrightarrow{\text{integrating}} \alpha' = a \xrightarrow{\text{integrating}} \alpha(s) = as + b$
$\Rightarrow \alpha(s)$ is a straight line.

3 Plane Curves, Involutes, Evolutes, and Bertrand Curves

3.1 Plane Curves $\tau = 0$

Remark 3.1. If one of the components of a curve is found to be constant, the curve can be classified as a plane curve. However, the converse is not true in general.
e.g. $\alpha(t) = (t, \frac{t^2}{2}, 0)$ 0 is a constant, then the curve is a plane curve.

Example 3.1. Show that the curve $\alpha(t) = (t, \frac{1+t}{t}, \frac{1-t^2}{t})$ is a plane curve
Solution:
$\alpha(t) = (t, \frac{1}{t} + 1, \frac{1}{t} - t)$
$\dot{\alpha}(t) = \left(1, -\frac{1}{t^2}, -\frac{1}{t^2} - 1\right)$
$\ddot{\alpha}(t) = \left(0, \frac{2}{t^3}, \frac{2}{t^3}\right)$
$\dddot{\alpha}(t) = \left(0, -\frac{6}{t^4}, -\frac{6}{t^4}\right)$
$\begin{vmatrix} \mathbf{t} & \mathbf{n} & \mathbf{b} \\ 1 & -\frac{1}{t^2} & -\frac{1}{t^2} - 1 \\ 0 & \frac{2}{t^3} & \frac{2}{t^3} \end{vmatrix} = \frac{2}{t^3}t - \frac{2}{t^3}n + \frac{2}{t^3}b$

3.2 Curvature and Torsion

Definition 3.1. (Curvature K)

$$K = |\alpha''(s)| \longrightarrow \text{in the parameter } (s)$$

$$K = \frac{\dot{\alpha}(t) \times \ddot{\alpha}(t)}{|\dot{\alpha}(t)|^3} \longrightarrow \text{in the parameter } (t)$$

Definition 3.2. (Torsion τ)

$$\tau = \mathbf{b}' \cdot \mathbf{n} \longrightarrow \text{in the parameter } (s)$$

$$\tau = \frac{-(\dot{\alpha} \times \ddot{\alpha}) \cdot \dddot{\alpha}}{|\dot{\alpha} \times \ddot{\alpha}|^2} \longrightarrow \text{in the parameter } (t)$$

Example 3.2. Find K and τ for the helix

$$\alpha(s) = \left(a\cos\frac{s}{\sqrt{a^2+b^2}}, a\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{bs}{\sqrt{a^2+b^2}}\right)$$

Solution:
$\alpha'(s) = \left(-\frac{a}{\sqrt{a^2+b^2}}\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}}\cos\frac{s}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right)$
$\alpha''(s) = \left(-\frac{a}{a^2+b^2}\cos\frac{s}{\sqrt{a^2+b^2}}, -\frac{a}{a^2+b^2}\sin\frac{s}{\sqrt{a^2+b^2}}, 0\right)$

$$K = \sqrt{\frac{a^2}{(a^2+b^2)^2}\cos^2\left(\frac{s}{\sqrt{a^2+b^2}}\right) + \frac{a^2}{(a^2+b^2)^2}\sin^2\left(\frac{s}{\sqrt{a^2+b^2}}\right)} = \frac{a}{\sqrt{a^2+b^2}}$$

Now, we previously found that $\mathbf{b}$ and $\mathbf{n}$ are:
$\because \mathbf{b} = \left(\frac{b}{\sqrt{a^2+b^2}}\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{-b}{\sqrt{a^2+b^2}}\cos\frac{s}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}}\right)$
$\mathbf{b}' = \left(\frac{b}{a^2+b^2}\cos\frac{s}{\sqrt{a^2+b^2}}, \frac{-b}{a^2+b^2}\sin\frac{s}{\sqrt{a^2+b^2}}, 0\right)$
$\because \mathbf{n} = \left(-\cos\frac{s}{\sqrt{a^2+b^2}}, -\sin\frac{s}{\sqrt{a^2+b^2}}, 0\right)$
$\therefore \tau = \ \mathbf{b}'\cdot\mathbf{n} = \left(-\cos\frac{s}{\sqrt{a^2+b^2}}\right)\left(-\frac{a}{a^2+b^2}\right) + \left(-\sin\frac{s}{\sqrt{a^2+b^2}}\right)\left(-\frac{a}{a^2+b^2}\right) + 0$
$= \frac{a}{a^2+b^2}\cos\frac{s}{\sqrt{a^2+b^2}} + \frac{a}{a^2+b^2}\sin\frac{s}{\sqrt{a^2+b^2}}$
$= \frac{-b}{\sqrt{a^2+b^2}}$

Example 3.3. Find K and τ for the parabola $\alpha(t) = (t, \frac{t^2}{2}, 0)$.
Solution:

$$\dot{\alpha}(t) = (1, t, 0) \quad \Rightarrow \quad |\dot{\alpha}| = \sqrt{1+t^2}$$

$$\ddot{\alpha}(t) = (0, 1, 0)$$

$$\dddot{\alpha}(t) = (0, 0, 0)$$

$$\begin{vmatrix} t & n & b \\ 1 & t & 0 \\ 0 & 1 & 0 \end{vmatrix} = (0, 0, 1)$$

$$|\dot{\alpha} \times \ddot{\alpha}| = 1 \quad \Rightarrow \quad K = \frac{|\dot{\alpha} \times \ddot{\alpha}|}{|\dot{\alpha}|^3} = \frac{1}{(\sqrt{1+t^2})^3} = \frac{1}{(1+t^2)^{\frac{3}{2}}}$$

$$\because (\dot{\alpha} \times \ddot{\alpha}) \cdot \dddot{\alpha} = 0 \quad \Rightarrow \quad \tau = 0$$

$$\therefore \dot{\alpha} \times \ddot{\alpha} = \left(\frac{2}{t^3}, \frac{-2}{t^3}, \frac{2}{t^3}\right)$$

$$(\dot{\alpha} \times \ddot{\alpha}) \cdot \dddot{\alpha} = 0 + \frac{1^2}{t^7} - \frac{1^2}{t^7} = 0$$

$$\therefore \tau = \frac{-(\dot{\alpha} \times \ddot{\alpha}) \cdot \dddot{\alpha}}{|\dot{\alpha} \times \ddot{\alpha}|^2} = \frac{0}{|\dot{\alpha} \times \ddot{\alpha}|^2} = 0$$

$$\therefore \alpha(t) \text{ is a plane curve.}$$

Let $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$ be a regular parametrized curve. We need to prove the following:

(I) $\frac{dt}{ds} = \frac{1}{|\dot{\alpha}|}$

(II) $\frac{d^2t}{ds^2} = -\frac{\dot{\alpha} \cdot \ddot{\alpha}}{|\dot{\alpha}|^4}$

(III) $\mathbf{b}(t) = \frac{\dot{\alpha} \times \ddot{\alpha}}{|\dot{\alpha} \times \ddot{\alpha}|}$

(IV) $K(t) = \frac{|\dot{\alpha} \times \ddot{\alpha}|}{|\dot{\alpha}|^3}$

Proof (I):

$$\because s = \int_0^t |\frac{d\alpha}{dt}| dt = \int_0^t |\dot{\alpha}| dt$$

Differentiating w.r.t. t :

$$\frac{ds}{dt} = |\dot{\alpha}| \Rightarrow \frac{dt}{ds} = \frac{1}{\dot{\alpha}}$$

Proof (II):

$$\because \frac{dt}{ds} = \frac{1}{\dot{\alpha}}$$

Differentiating:

$$\begin{aligned}
\frac{d}{ds}\left(\frac{dt}{ds}\right) &= \frac{d}{ds}\left(\frac{1}{\dot{\alpha}}\right) \\
\Rightarrow \frac{d^2t}{ds^2} &= \frac{d}{dt}\left(\frac{1}{\dot{\alpha}}\right)\frac{dt}{ds} \\
&= \frac{d}{dt}(|\dot{\alpha}|)^{-1} \cdot \frac{1}{\dot{\alpha}} \\
&= -(|\dot{\alpha}|)^{-2}\frac{d}{dt}(|\dot{\alpha}|)\frac{1}{\dot{\alpha}} \\
\Rightarrow \frac{d^2t}{ds^2} &= -\frac{1}{|\dot{\alpha}|^3}\frac{d}{dt}(|\dot{\alpha}|)
\end{aligned}$$

Proof (III) & (IV):

$$\begin{aligned}
\because \dot{\alpha} &= \frac{d\alpha}{dt} = \frac{d\alpha}{ds}\frac{ds}{dt} = \alpha'|\dot{\alpha}| = |\dot{\alpha}|\mathbf{t} \\
\ddot{\alpha} &= \frac{d}{dt}(\dot{\alpha}) \\
&= \frac{d}{dt}(\alpha'|\dot{\alpha}|) \\
&= \frac{d}{ds}(\alpha'|\dot{\alpha}|)\frac{ds}{dt} \\
&= \frac{d}{ds}(\alpha'|\dot{\alpha}|) \cdot |\dot{\alpha}| \\
&= \left(\alpha'\frac{d}{ds}(|\dot{\alpha}|) + \alpha''|\dot{\alpha}|\right)|\dot{\alpha}| \\
&= \alpha'|\dot{\alpha}|\frac{d}{ds}(|\dot{\alpha}|) + \alpha''|\dot{\alpha}|^2 \\
&= \alpha'|\dot{\alpha}|\frac{d}{dt}(|\dot{\alpha}|)\frac{dt}{ds} + \alpha''|\dot{\alpha}|^2 \\
&= \alpha'|\dot{\alpha}|\frac{d}{dt}(|\dot{\alpha}|)\frac{1}{|\dot{\alpha}|} + \alpha''|\dot{\alpha}|^2 \\
\alpha'' &= \alpha'\frac{d}{dt}(|\dot{\alpha}|) + \alpha''|\dot{\alpha}|^2 (*)
\end{aligned}$$

Now, differentiate $\dot{\alpha} \cdot \dot{\alpha} = |\dot{\alpha}|^2$ w.r.t. t:

$$\begin{aligned}
\dot{\alpha} \cdot \ddot{\alpha} + \ddot{\alpha} \cdot \dot{\alpha} &= 2|\dot{\alpha}|\frac{d}{dt}(|\dot{\alpha}|) \\
2(\dot{\alpha} \cdot \ddot{\alpha}) &= 2|\dot{\alpha}|\frac{d}{dt}(|\dot{\alpha}|) \\
\Rightarrow \frac{d}{dt}(|\dot{\alpha}|) &= \frac{\dot{\alpha} \cdot \ddot{\alpha}}{|\dot{\alpha}|} (**)
\end{aligned}$$

Substituting from (∗∗) into (∗):

$$\ddot{\alpha} = \alpha' \frac{\dot{\alpha}\cdot\ddot{\alpha}}{|\dot{\alpha}|} + \alpha''|\dot{\alpha}|^2$$
$$= \boxed{\frac{\dot{\alpha}\cdot\ddot{\alpha}}{|\dot{\alpha}|}\mathbf{t} + |\dot{\alpha}|^2 K\mathbf{n}}$$

$$\dot{\alpha}\times\ddot{\alpha} = \begin{vmatrix} \mathbf{t} & \mathbf{n} & \mathbf{b} \\ |\dot{\alpha}| & 0 & 0 \\ \frac{\dot{\alpha}\cdot\ddot{\alpha}}{|\dot{\alpha}|} & |\dot{\alpha}|K & 0 \end{vmatrix}$$

$$\dot{\alpha}\times\ddot{\alpha} = 0\mathbf{t} - 0\mathbf{n} + |\dot{\alpha}|^3 K\mathbf{b}$$

$$\dot{\alpha}\times\ddot{\alpha} = (0, 0, |\dot{\alpha}|^3 K)$$

$$|\dot{\alpha}\times\ddot{\alpha}| = \sqrt{|\dot{\alpha}|^6 K^2} = |\dot{\alpha}|^3 K$$

$$\boxed{(IV) \quad K = \frac{|\dot{\alpha}\times\ddot{\alpha}|}{|\dot{\alpha}|^3}}$$

And for **b**:

$$\because \dot{\alpha}\times\ddot{\alpha} = |\dot{\alpha}|^3 K\mathbf{b}$$

$$\mathbf{b} = \frac{\dot{\alpha}\times\ddot{\alpha}}{k|\dot{\alpha}|^3} = \frac{\dot{\alpha}\times\ddot{\alpha}}{\frac{|\dot{\alpha}\times\ddot{\alpha}|}{|\dot{\alpha}|^3}}|\dot{\alpha}|^3$$

$$\Rightarrow \mathbf{b} = \frac{\dot{\alpha}\times\ddot{\alpha}}{|\dot{\alpha}\times\ddot{\alpha}|}$$

And this completes the proof of (III) .

3.3 Involutes and Evolutes

Consider a curve $\alpha(s)$ where s is the arc length parameter. By drawing tangents at every point along this curve, we generate a surface known as the tangent surface. An **involute** $\alpha^*(s)$ is a curve that lies on this tangent surface and is orthogonal to all the tangents drawn at every point along the original curve.

Equation of The Involute:

Consider a point $\alpha(s)$ on the curve α and the point $\alpha^*(s)$ on the curve α^* (involute).
Thus, $\alpha^*(s) - \alpha(s)$ is a tangent vector of α, but $\mathbf{t}$ is the unit tangent vector of α.
$\therefore (\alpha^*(s) - \alpha(s)) \propto \mathbf{t} \Rightarrow \alpha^*(s) - \alpha(s) = \lambda(s)\mathbf{t}(s) \longrightarrow (1)$
Differentiate Eq. (1) with respect to s:

$$\begin{aligned}\frac{d\alpha^*}{ds} - \frac{d\alpha}{ds} &= \lambda(s)\mathbf{t}'(s) + \mathbf{t}(s)\lambda'(s)\\ \frac{d\alpha^*}{ds} &= \frac{d\alpha}{ds} + \lambda(s)\mathbf{t}'(s) + \mathbf{t}(s)\lambda'(s)\\ &= \mathbf{t}(s) + \lambda(s)\mathbf{t}'(s) + \mathbf{t}(s)\lambda'(s)\\ \therefore \frac{d\alpha^*}{ds} &= (1+\lambda'(s))\mathbf{t}(s) + \lambda(s)k\mathbf{n}(s) \longrightarrow (2)\end{aligned}$$

$\because \frac{d\alpha^*}{ds}$ is a tangent of α^*:
$\Rightarrow \frac{d\alpha^*}{ds} \perp \mathbf{t} \Rightarrow \frac{d\alpha^*}{ds} \cdot \mathbf{t} = 0$
Multiplying (2) (scalar product) by $\mathbf{t}$ we get:

$$0 = \frac{d\alpha^*}{ds} \cdot \mathbf{t} = (1+\lambda'(s))\mathbf{t}\cdot\mathbf{t} + \lambda(s)k\mathbf{n}\cdot\mathbf{t}$$

$$0 = 1 + \lambda'(s) \Rightarrow \lambda'(s) = -1$$

$$\xrightarrow{\text{Integration}} \lambda(s) = -s + c \text{ or } \lambda(s) = c - s$$

Substituting into (1)
$\alpha^*(s) - \alpha(s) = (c-s)\mathbf{t}(s)$ or
$\alpha^*(s) = \alpha(s) + (c-s)\mathbf{t}(s)$

Definition 3.3. (Involutes)
$\alpha^*(s) = \alpha(s) + (c-s)\mathbf{t}(s)$

Remark 3.2. The curve α has an infinite number of involutes since c is an arbitrary constant.

Example 3.4. Find the involute of the helix.
$\alpha(s) = \left(a\cos\frac{s}{\sqrt{a^2+b^2}}, a\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{bs}{\sqrt{a^2+b^2}}\right)$
Solution:

$$\begin{aligned}
&\alpha^*(s) = \alpha(s) + (c-s)\mathbf{t}(s) \\
&\text{where} \quad \mathbf{t} = \alpha'(s) = \left(\frac{-a}{\sqrt{a^2+b^2}}\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}}\cos\frac{s}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right) \\
&\Rightarrow \alpha^*(s) = \left(a\cos\frac{s}{\sqrt{a^2+b^2}}, a\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{bs}{\sqrt{a^2+b^2}}\right) \\
&\quad + (c-s)\left(\frac{-a}{\sqrt{a^2+b^2}}\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}}\cos\frac{s}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right) \\
&= \left(a\cos\frac{s}{\sqrt{a^2+b^2}} - \frac{(c-s)a}{\sqrt{a^2+b^2}}\sin\frac{s}{\sqrt{a^2+b^2}},\right. \\
&\quad \left. a\sin\frac{s}{\sqrt{a^2+b^2}} + \frac{(c-s)a}{\sqrt{a^2+b^2}}\cos\frac{s}{\sqrt{a^2+b^2}}, \frac{bs}{\sqrt{a^2+b^2}} + \frac{(c-s)b}{\sqrt{a^2+b^2}}\right) \\
&= \left(a\cos\frac{s}{\sqrt{a^2+b^2}} - \frac{(c-s)a}{\sqrt{a^2+b^2}}\sin\frac{s}{\sqrt{a^2+b^2}},\right. \\
&\quad \left. a\sin\frac{s}{\sqrt{a^2+b^2}} + \frac{(c-s)a}{\sqrt{a^2+b^2}}\cos\frac{s}{\sqrt{a^2+b^2}}, \frac{bc}{\sqrt{a^2+b^2}}\right)
\end{aligned}$$

Example 3.5. Find the involute of the curve.
$\alpha(s) = (s, s^2, \frac{s^3}{3})$, where s is the arc length parameter.
Solution:

$$\begin{aligned}
\alpha^*(s) &= \alpha(s) + (c-s)\mathbf{t}(s) \\
\text{where} \quad \mathbf{t}(s) &= \alpha'(s) = (1, 2s, s^2) \\
&\Rightarrow \alpha^*(s) = (s, s^2, \frac{s^3}{3}) \\
&\quad + (c-s)(1, 2s, s^2) \\
&= (s, s^2, \frac{s^3}{3}) + (c-s)(1, 2s, s^2) \\
&= (s + (c-s), s^2 + 2s(c-s), \frac{s^3}{3} + s^2(c-s)) \\
&= (c, s^2 + 2sc - 2s^2, \frac{s^3}{3} + s^2c - s^3) \\
&= (c, s^2(1-2), s^2(\frac{1}{3} - 1)) \\
&= (c, -s^2, -\frac{2}{3}s^2)
\end{aligned}$$

Example 3.6. Find the involute of the unit circle $\alpha(t) = (\cos t, \sin t, 0)$.
Solution:

$$\begin{aligned}
\dot{\alpha} &= (-\sin t, \cos t, 0) \\
|\dot{\alpha}| &= \sqrt{\sin^2 t + \cos^2 t + 0^2} = \sqrt{1} = 1 \\
\mathbf{t} &= \frac{\dot{\alpha}}{|\dot{\alpha}|} = (-\sin t, \cos t, 0) \\
&= (-\sin t, \cos t, 0) \\
\text{Since } s &= \int_0^t |\dot{\alpha}| dt = \int_0^t 1 dt \\
&= [t]_0^t = t \\
\Rightarrow s &= t \\
\alpha^*(s) &= (\cos t, \sin t, 0) + (c - t)(-\sin t, \cos t, 0) \\
&= (\cos t - (c - t)\sin t, \sin t + (c - t)\cos t, 0)
\end{aligned}$$

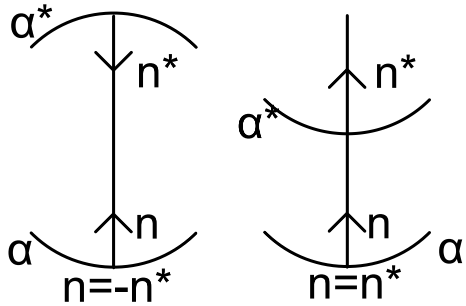
Example 3.7. Find the involute of the curve $\alpha(t) = (\cosh t, \sinh t, t)$.
Solution:

$$\begin{aligned}
\dot{\alpha} &= (\sinh t, \cosh t, 1) \\
|\dot{\alpha}| &= \sqrt{\sinh^2 t + \cosh^2 t + 1^2} = \sqrt{2\cosh^2 t} = \sqrt{2}\cosh t \\
\mathbf{t} &= \frac{\dot{\alpha}}{|\dot{\alpha}|} = \frac{(\sinh t, \cosh t, 1)}{\sqrt{2}\cosh t} \\
&= \left(\frac{1}{\sqrt{2}}\tanh t, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\operatorname{sech} t\right) \\
\text{Since } s &= \int_0^t |\dot{\alpha}| dt = \int_0^t \sqrt{2}\cosh t dt \\
&= \sqrt{2}\left[\sinh t\right]_0^t = \sqrt{2}\left[\sinh t - \sinh 0\right] \\
\Rightarrow s &= \sqrt{2}\sinh t \\
\alpha^*(s) &= (\cosh t, \sinh t, t) + (c - \sqrt{2}\sinh t)\left(\frac{1}{\sqrt{2}}\tanh t, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\operatorname{sech} t\right) \\
&= \left(\cosh t + \frac{c - \sqrt{2}}{\sqrt{2}}\sinh t \tanh t, \sinh t + \frac{c - \sqrt{2}}{\sqrt{2}}\sinh t, t + \frac{c - \sqrt{2}}{\sqrt{2}}\sinh t \operatorname{sech} t\right)
\end{aligned}$$

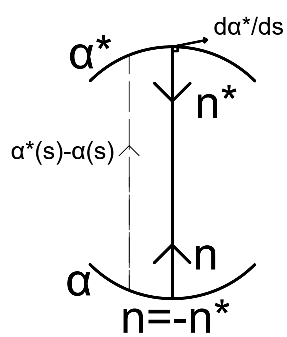
3.4 Bertrand Curves

Definition 3.4. Two curves α(s) and α^*(s) are called Bertrand curves if $n = \pm n^*$

where that n is principle normal unite of α and n^* is principle normal unite of α^*



Example 3.8. Prove that the distance between corresponding points on two Bertrand curves is constant
Proof: consider α and $\alpha*$ are two Bertrand curves($n = \pm n^*$)
take a point $\alpha(s)$ on α and $\alpha*(s)$ on $\alpha*$ thus $\alpha*(s) - \alpha(s)$ is a vector



Example 3.9. From the figure

$$\alpha^*(s)-\alpha(s)//\ n$$

$$\Rightarrow \alpha^*(s)-\alpha(s)=\lambda(s)\ n(s)\ \rightarrow 1$$

Diff Eq(1) with regards of s

$$\Rightarrow d\alpha^*/ds-d\alpha/ds=\lambda\ n'(s)+n(s)\ \lambda'(s)$$

$$d\alpha^*/ds=d\alpha/ds+\lambda(s)\ n'(s)+n(s)\ \lambda'(s)$$

$$d\alpha^*/ds=t+\lambda(s)(-k\ t-\tau\ b)+n(s)\ \lambda'(s) \qquad \boxed{\text{n}'\text{=-k t-}\tau\text{ b}}$$

$$\Rightarrow \frac{\alpha^*(s)}{ds}=(1-k\ \lambda(s))t+\lambda'(s)\ n-\tau\lambda(s)b\ \rightarrow 2$$

$\therefore \frac{d\alpha^*}{ds}$ is atangent of α^*

$$\therefore \frac{d\alpha^*}{ds}\perp n^* \Rightarrow \frac{d\alpha^*}{ds}\perp n \Rightarrow \boxed{\frac{d\alpha^*}{ds}*n=0}$$

multiplying Eq (2)by n

$$\frac{\alpha^*(s)}{ds}*n=(1-k\ \lambda(s))(t*n)+\lambda'(s)\ (n*n)-\tau\lambda(s)(b*n)$$

as

$$t\ n=b\ n=0\ \ \&\ \ n\ n=1$$

$0=\lambda'(s)$ integrate both sides

$$\Rightarrow \lambda(s)=c$$

substituting into 1 we get

$$\alpha^*(s)-\alpha(s)=c\ n(s)$$

$$|\alpha^*(s)-\alpha(s)|=|c\ n(s)|=c|n(s)|=c$$

$$\Rightarrow |\alpha^*(s)-\alpha(s)|=c\ \ (constant)$$

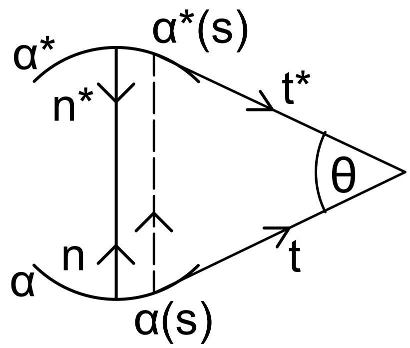
Example 3.10. Show that two curves:

$$\alpha(s) = 1/2(\cos^{-1} s - s\sqrt{1-s^2}, 1-s^2, 0)$$
$$\alpha^*(s) = 1/2(\cos^{-1} s - s\sqrt{1-s^2} - s, 1-s^2 + \sqrt{1-s^2}, 0)$$

are Bertrand curves
Sol:

$$\alpha(s) - alpha^*(s) = 1/2(-s, \sqrt{1-s^2}, 0)$$
$$|\alpha(s) - alpha^*(s)| = 1/2\sqrt{s^2 + 1 - s^2 + 0} = 1/2 \ \ (constant)$$

$\therefore \alpha^*(s)$ and$\alpha(s)$are Bertrand curves



Example 3.11. show that the angle between corresponding tangent lines on two Bertrand curves is constant
sol:
consider α and $\alpha^*$are two Bertrand curves $(n = \pm n^*)$
take a point $\alpha(s)$ on α and a point $\alpha^*(s)$ on α^*
take t is the tangent unit vector of α at $\alpha(s)$ and t^* is the tangent unite vector of α at $\alpha^*(s)$
now we prove that θ is const or $\cos\theta$=const

$$\because \cos\theta = \frac{t \cdot t^*}{|t|\ |t^*|} = \frac{t \cdot t^*}{1 \cdot 1} = t \cdot t^*$$

now we prove that $t \cdot t^*$ =const

$$\begin{aligned}
(t \cdot t^*)' &= t \cdot t^{*'} + t^* \cdot t' \\
&= t \cdot \frac{dt^*}{ds} + t^* k \cdot n \\
&= t \cdot \left(\frac{dt^*}{ds^*} \cdot \frac{ds^*}{ds} \right) + k(t^* \cdot n) \\
&= \frac{ds^*}{ds}(t \cdot k^* n^*) + k(t^* \cdot n) \\
&= k^* \frac{ds^*}{ds}(t \cdot n^*) + k(t^* \cdot n) \\
&= \pm k^* \frac{ds^*}{ds}(t \cdot n) \pm k(t^* \cdot n^*) \\
&= \pm k^* \frac{ds^*}{ds}(0) \pm k(0)
\end{aligned}$$

$\Rightarrow (t \cdot t^*)' = 0 \qquad \Rightarrow t \cdot t^*$ =const

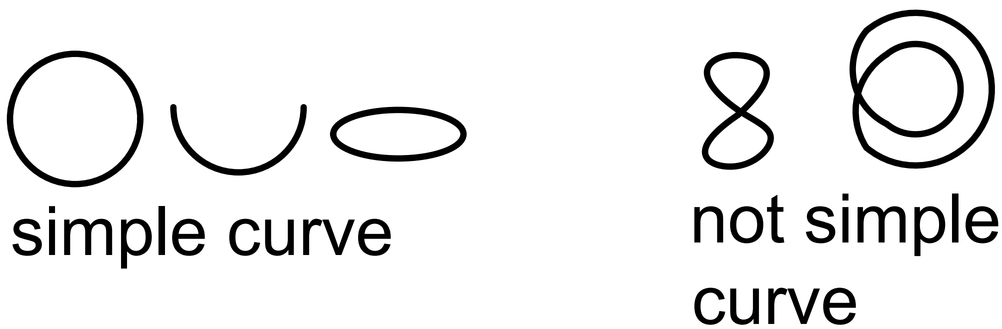
$$\Rightarrow \cos\theta = const$$

$$\theta = const$$

3.5 Global Properties of Plane Curves

Definition 3.5.
1- A closed plane curve is aregular parametrized curve α(a):[a,b]$rightarrow\mathbb{R}^2$ sucsk that $\alpha(a) = \alpha(b), \alpha'(a) = \alpha'(b), ...$
2-the curve α is simple if it has no further self-intersection

EX:



that is

the curve $\alpha\ :\ I \to \mathbb{R}^2$ is simple if $\alpha(1-1)\ i.e,\ \ t_1, t_2 \in I \quad \alpha(t_1) = \alpha(t_2) \Rightarrow t_1 = t_2$

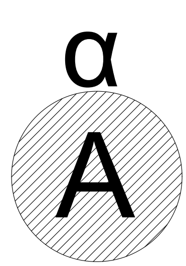
Theorem 3.1. consider the simple closed plane curve

$$\alpha(t) = (x(t), y(t)),\ \ t \in [a, b]$$

$$A = -\int_a^b y(t)x^{\cdot}(t)dt$$

$$= \int_a^b x(t)y^{\cdot}(t)dt$$

$$A = 1/2\int_a^b [x(t)y^{\cdot}(t) - y(t)x^{\cdot}(t)]dt$$



Example 3.12. Circle $x^2 + y^2 = \alpha^2$
Parametric form :

$$\alpha(t) = (\alpha \cos t, \alpha sint)$$
$$x = a \cos t \qquad , y = a \sin t$$

$$\begin{aligned}
A &= \int_a^b x(t) y^{\cdot}(t) dt \\
A &= \int_0^{2\pi} \alpha \cos t \cdot \alpha \cos t dt \\
&= \alpha^2 \int_0^{2\pi} \cos^2 t dt = \alpha^2 \int_0^{2\pi} \frac{1}{2}(1 + \cos 2t) dt \\
&= \frac{\alpha^2}{2} \left[t \cdot \frac{\sin 2t}{2} \right]_0^{2\pi} \\
&= \frac{\alpha^2}{2} \left[(2\pi + \frac{\sin 4\pi}{2}) - (0) \right] = \pi \alpha^2
\end{aligned}$$

Example 3.13.

$$\alpha(t) = (\ln(\sec t + \tan t) - \sin t, \cos t)$$

$$\begin{aligned}
A &= -\int_a^b x^{\cdot}(t) y(t) dt \\
&= -\int_0^{\pi/3} \frac{\sin^2 t}{\cos t} \cdot \cos t dt = -\int_0^{\pi/3} \sin^2 t \, dt \\
&= -\int_0^{\pi/3} \frac{1 - \cos 2t}{2} dt = -\left[\frac{x}{2} - \frac{\sin 2t}{4} \right]_0^{\pi/3} \\
&= \frac{\sqrt{3}}{8} - \frac{\pi}{6}
\end{aligned}$$

Theorem 3.2. let α be a simple closed plane curve with length L, and let A be the area of the region bounded by α.then

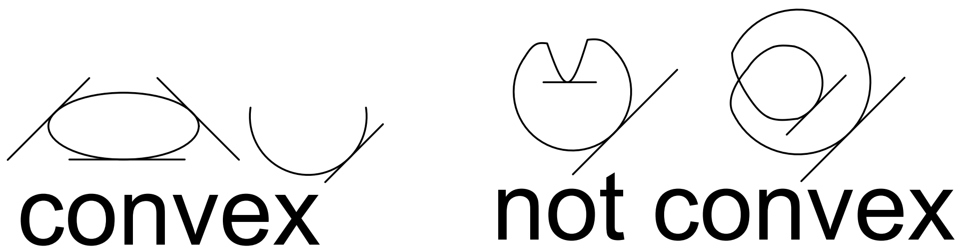
$$L^2 - 4\pi A \geq 0$$

the equality hold if and only if α is a circle

since the area of circle A=πr^2 and the length L=$2\pi r$

$$L^2 - 4\pi A = 4\pi^2 r^2 - 4\pi(\pi r^2) = 4\pi^2 r^2 - 4\pi^2 r^2 = 0$$

Definition 3.6. .
A regular plane curve $\alpha : I \to \mathbb{R}^2$ is convex if for all $t \in I$ the trace $\alpha(t)$ lies entirely on one side of any tangent line at t



Definition 3.7. Vertex of a curve The vertex of a curve $\alpha(t)$ is a point at which $K^{\cdot}(t) = 0$

Example 3.14. prove that the parabola $\alpha(t) = (t, t^2/2, 0)$ has only one vertex
sol:

$$K = \frac{|\alpha^{\cdot} \times \alpha^{\cdot\cdot}|}{|\alpha^{\cdot}|^3}$$

$$\alpha^{\cdot} = (1, t, 0) \quad |\alpha^{\cdot}| = \sqrt{1 + t^2}$$

$$\alpha^{\cdot\cdot} = (0, 1, 0)$$

$$\alpha^{\cdot} \times \alpha^{\cdot\cdot} = \begin{vmatrix} t & n & b \\ 1 & t & 0 \\ 0 & 1 & 0 \end{vmatrix} = (0, 0, 1)$$

$$\Rightarrow |\alpha^{\cdot} \times \alpha^{\cdot\cdot}| = 1 \qquad K = \frac{1}{(\sqrt{1 + t^2})^3} =$$

$$K^{\cdot} = \frac{-3}{2}(1 + t^2)^{\frac{-5}{2}} \cdot (2t) = -3t(1 + t^2)^{\frac{-5}{2}}$$

put$K^{\cdot} = 0$

$$0 = -3t(1 + t^2)^{\frac{-5}{2}} \qquad \Rightarrow t = 0$$

$$\Rightarrow \alpha(t) = (t, t^2/2, 0) \quad \text{has only one vertex}$$

Example 3.15. prove that the ellipse

$$\frac{x^2}{4} + \frac{y^2}{1} = 1 \qquad \alpha(t) = (2\cos t, \sin t)$$

has four vertices
sol:

$$k = \frac{x'y'' - x''y'}{(x'^2 + y'^2)^{\frac{3}{2}}}$$

$\alpha^{\cdot}(x) = -2\sin t \quad \alpha^{\cdot\cdot}(x) = -2\cos t$
$\alpha^{\cdot}(y) = \cos t \quad \alpha^{\cdot\cdot}(x) = -\sin t$

$$K = \frac{2\sin t \cdot \sin t + 2\cos t \cdot \cos t}{(4\sin^2 t + \cos^2 t)^{\frac{2}{3}}} = \frac{2}{(3\sin^2 t + 1)^{3/2}}$$

$$K^{\cdot} = \frac{-9\sin 2t}{(3\sin^2 t + 1)^{\frac{3}{2}}}$$

put $K^{\cdot} = 0$

$$0 = \frac{-9\sin 2t}{(3\sin^2 t + 1)^{\frac{3}{2}}}$$

$$\Rightarrow t = \{0, \pi, \frac{\pi}{2}, \frac{3\pi}{2}\}$$

Example 3.16. prove that $\aleph(t) = (t, t^2, t^2 + t)$
sol:

$$K = \frac{|\alpha^{\cdot} \times \alpha^{\cdot\cdot}|}{|\alpha^{\cdot}|^3}$$

$\alpha^{\cdot}(t) = (1, 2t, 2t + 1) \qquad \alpha^{\cdot\cdot} = (0, 2, 2) \qquad |\alpha^{\cdot}| = \sqrt{8t^2 + 4t + 2}$

$$\alpha^{\cdot} \times \alpha^{\cdot\cdot} = \begin{vmatrix} t & n & b \\ 1 & 2t & 2t+1 \\ 0 & 2 & 2 \end{vmatrix} = (-2, -2, 2)$$

$$\Rightarrow |\alpha^{\cdot} \times \alpha^{\cdot\cdot}| = \sqrt{12} \qquad K = \frac{\sqrt{12}}{(\sqrt{8t^2 + 4t + 2})^3}$$

$$K^{\cdot} = -\frac{24t\sqrt{3} + 6\sqrt{3}}{(8t^2 + 4t + 2)^{3/2}(4t^2 + 2t + 1)}$$

put $K^{\cdot} = 0$

$$0 = -\frac{24t\sqrt{3} + 6\sqrt{3}}{(8t^2 + 4t + 2)^{3/2}(4t^2 + 2t + 1)} \qquad \Rightarrow t = \frac{-1}{4}$$

$$\Rightarrow \alpha(t) = (t, t^2, t^2 + t) \quad \text{has only one vertex}$$

Theorem 3.3. Four vertex Theorem
A simple closed convex plane curve has at least four vertices

4 Exercises

Example 4.1. Find all the function $f(t)$ that make the curve flat

$$x_1 = \cos t \quad x_2 = \sin t \quad x_3 = f(t)$$

sol:

$$\alpha(t) = (\cos t, \sin t, f(t))$$

to make the curve flat $\Rightarrow \tau = 0$

$$\Rightarrow \because \tau = \frac{[\alpha'(t), \alpha''(t), \alpha'''(t)]}{|\alpha'(t) \times \alpha''(t)|^2} = 0$$

$$\Rightarrow [\alpha'(t), \alpha''(t), \alpha'''(t)] = 0$$

$$\alpha'(t) = (-\sin t, \cos t, f')$$

$$\alpha''(t) = (-\cos t, -\sin t, f'')$$

$$\alpha'''(t) = (\sin t, -\cos t, f''')$$

$$\begin{vmatrix} -\sin t & \cos t & f' \\ -\cos t & -\sin t & f'' \\ \sin t & -\cos & f''' \end{vmatrix} = 0$$

$$\Rightarrow -\sin t(-f''' \sin t + f'' \cos t) - \cos t(-f''' \cos t - f'' \sin t) + f'(\cos^2 t + \sin^2 t) = 0$$

$$\Rightarrow f''' \sin^2 t - f'' \sin t \cos t + f''' \cos^2 t + f'' \sin t \cos t + f' = 0$$

$$\Rightarrow f''' \sin^2 t + f''' \cos^2 t + f' = 0$$

$$\Rightarrow f''' + f' = 0 \qquad \text{put } f'(x) = g(t)$$

$$\therefore g'' + g = 0 \quad \rightarrow g'' = -g$$

$$g = A \cos t + B \sin t$$

$$f'(t) = A \cos t + B \sin t$$

$$f(t) = A \sin t - B \cos t + c$$

Example 4.2. prove that if $\alpha = \alpha(s)$ then $K^2\tau = [\alpha', \alpha'', \alpha''']$

$$\begin{aligned}
\because \alpha &= \alpha(s) \\
\Rightarrow \alpha' &= T \\
\alpha'' &= T' = KN \\
\alpha''' &= (KN)' = KN' + K'N \\
&= K(-KT + \tau B) + K'N \\
\alpha''' &= -K^2T + K'N + K\tau B
\end{aligned}$$

$$[\alpha', \alpha'', \alpha'''] = \begin{vmatrix} 1 & 0 & 0 \\ 0 & K & 0 \\ -K^2 & K' & K\tau' \end{vmatrix} = (K^2\tau)$$

Example 4.3. if $|\alpha'(s)| = 1$ and $K > 0$ Prove that

$$1 - [\alpha'', \alpha''', \alpha^{\text{iv}}] = K^5(\tau/K)'$$
$$2 - \alpha \text{ is general spiral } \leftrightarrow [\alpha'', \alpha''', \alpha^{\text{iv}}] = 0$$

sol:

$$\begin{aligned}
\because \alpha''' &= -K^2T + K'N + K\tau B \quad \text{from previous Example} \\
\alpha^{\text{iv}} &= -K^2T' + 2KK'T + K'N' + K''N + K'\tau B + K\tau B + K\tau B' \\
&= -K^3N - 2KK'T + K'(-KT + \tau B) + K''N + K'\tau B + K\tau' B - K\tau^2 N \\
\alpha^{\text{iv}} &= (-3KK')T + (-K^3 + K'' - K\tau^2)N + (2K'\tau + K\tau')B
\end{aligned}$$

$$[\alpha'', \alpha''', \alpha^{\text{iv}}] = \begin{vmatrix} 0 & K & 0 \\ -K^2 & K' & K\tau' \\ -3KK' & -K^3 + K'' - K\tau^2 & 2K'\tau + K\tau' \end{vmatrix}$$

$$\begin{aligned}
&= -k\left[-2K^2K'\tau - K^3\tau' + 3K^2K'\tau\right] \\
&= -K^3\left[K'\tau - K\tau'\right] \\
&= K^5\left[\frac{K\tau' - K'\tau}{K^2}\right] = K^5\left(\frac{\tau}{K}\right)'
\end{aligned}$$

2- $(\Leftarrow)$ assume that $[\alpha'', \alpha''', \alpha^{\text{iv}}] = 0$

$$\Rightarrow K^5\left(\frac{\tau}{K}\right)' = 0 \qquad \because K > 0$$

$$\left(\frac{\tau}{K}\right)' = 0 \quad \Rightarrow \left(\frac{\tau}{K}\right) = const \Rightarrow \text{ is general spiral}$$

$(\Rightarrow)$assume that α is general spiral $\therefore \left(\frac{\tau}{K}\right)' = \text{const} \Rightarrow \left(\frac{\tau}{K}\right)' = 0 \Rightarrow [\alpha'', \alpha''', \alpha^{\text{iv}}] = 0$

Example:

$$\alpha(t) = (t, \cosh t, 0)$$

Find T,N,B -Tangent line - Principal normal line - Binormal line
and compute arc length - Osculating Plane - normal plane - Rectifying plane at t=0
Reparametrization
and find Involutes and Evolutes
and compute Curvature and Torsion

solution:
1 - Tangent line

$$\alpha(0) = (0, 1, 0)$$

$$\alpha^{\cdot}(t) = (1, \sinh t, 0) \qquad |\alpha^{\cdot}| = \cosh t$$

$$T = \left(\frac{1}{\cosh t}, \tanh t, 0\right) \qquad T(0) = (1, 0, 0)$$

$$\text{Tangent line:} \qquad \frac{x-a}{A_1} = \frac{y-b}{A_2} = \frac{z-c}{A_3}$$

$$\frac{x-0}{1} = \frac{y-1}{0} = \frac{z-0}{0} \Rightarrow \frac{x}{1} = \frac{y-1}{0} = \frac{z}{0}$$

2 - Binormal line

$$B = \frac{\alpha^{\cdot} \times \alpha^{\cdot\cdot}}{|\alpha^{\cdot} \times \alpha^{\cdot\cdot}|}$$

$$\alpha^{\cdot\cdot} = (0, \cosh t, 0)$$

$$\alpha^{\cdot} \times \alpha^{\cdot\cdot} = \begin{vmatrix} t & n & b \\ 1 & \sinh t & 0 \\ 0 & \cosh t & 0 \end{vmatrix} = (0, 0, \cosh t)$$

$$|\alpha^{\cdot} \times \alpha^{\cdot\cdot}| = \cosh t$$

$$B = \frac{(0, 0, \cosh t)}{\cosh t} = (0, 0, 1)$$

$$B(0) = (0, 0, 1)$$

$$\text{Binormal line:} \qquad \frac{x-a}{C_1} = \frac{y-b}{C_2} = \frac{z-c}{C_3}$$

$$\frac{x}{0} = \frac{y-1}{0} = \frac{z}{1}$$

3 - Principal normal unit vector

$$N = B \times T$$

$$B \times T = \begin{vmatrix} t & n & b \\ 0 & 0 & 1 \\ 1/\cosh t & \tanh t & 0 \end{vmatrix} = \left(-\tanh t\ ,\ \frac{1}{\cosh t}\ ,\ 0\right)$$

$$N(0) = (0, 1, 0)$$

$$\text{Principle normal line:} \qquad \frac{x-a}{B_1} = \frac{y-b}{B_2} = \frac{z-c}{B_3}$$

$$\frac{x}{0} = \frac{y-1}{1} = \frac{z}{0}$$

4 - Arc length

$$\alpha^{\cdot}(t) = (1, \sinh t, 0) \qquad |\alpha^{\cdot}| = \cosh t$$

$$s = \int_a^b \left|\frac{d\alpha}{dt}\right| dt = \int_a^b \cosh t = \sinh b - \sinh a$$

5 - Normal plane

$$[(x,y,z) - \alpha(t_0)] \cdot T(t_0) = 0$$

$$[(x,y,z) - \alpha(0)] \cdot T(0) = 0$$

$$\Rightarrow [(x,y,z) - (0,1,0)] \cdot (1,0,0) = 0 \qquad (x, y-1, z) \cdot (1,0,0) = x$$

$$\Rightarrow x = 0$$

6 - Osculating plane

$$[(x,y,z) - \alpha(t_0)] \cdot B(t_0) = 0$$

$$[(x,y,z) - (0,1,0)] \cdot (0,0,1) = 0$$

7 - Rectifying plane

$$[(x,y,z) - \alpha(t_0)] \cdot N(t_0) = 0$$

$$[(x,y,z) - (0,1,0)] \cdot (0,1,0) = 0$$

$$\Rightarrow y = 1$$

8 - Reparametrize by the arc length

$$s = \int_0^t \left|\frac{d\alpha}{dt}\right| dt = \sinh t$$

$$\Rightarrow t = \sinh^{-1} s$$

$$\therefore \alpha(s) = (\sinh^{-} 1s, \cosh \sinh^{-1} s, 0)$$

9 - Find the involute

$$\alpha^*(t) = \alpha(t) + (c-s)T(t)$$

$$\text{since} \Rightarrow \quad s = \sinh t \qquad \text{and} \quad T = \left(\frac{1}{\cosh t}, \tanh t, 0\right)$$

$$\alpha^*(t) = (t, \cosh t, 0) + (c - \sinh t) \cdot \left(\frac{1}{\cosh t}, \tanh t, 0\right)$$

$$\alpha^*(t) = (t, \cosh t, 0) + \left(\frac{c - \sinh t}{\cosh t}, (c - \sinh t) \cdot \tanh t, 0\right)$$

$$\alpha^*(t) = \left(t + \frac{c - \sinh t}{\cosh t}, \cosh t + (c - \sinh t) \cdot \tanh t, 0\right)$$

10 - Curvature

$$K = \frac{|\alpha^{\cdot} \times \alpha^{\cdot\cdot}|}{|\alpha^{\cdot}|^3}$$

$$K = \frac{\cosh t}{\sinh^3 t}$$

11 - Torsion

$$\tau = \frac{-(\alpha^{\cdot} \times \alpha^{\cdot\cdot}) \cdot \alpha^{\cdot\cdot\cdot}}{|\alpha^{\cdot} \times \alpha^{\cdot\cdot}|^2}$$

$$\alpha^{\cdot\cdot\cdot} = (0, \sinh t, 0)$$

$$\tau = \frac{-(0, 0, \cosh t) \cdot (0, \sinh t, 0)}{\cosh^2 t} = 0$$

Example 4.4. Explain with an example that the four-vertices theory is not true if the curve is not simple

sol:

we take

$$\alpha(t) = (\cos t - 2\sin t \cos t, \sin t - 2\sin^2 t, 0)$$

$$\alpha'(t) = (-\sin t + 2\sin^2 t - 2\cos^2 t, \cos t - 4\sin t \cos t, 0)$$

$$\alpha''(t) = (-\cos t + 4\sin t \cos t + 4\cos t \sin t, -\sin t + 4\sin^2 t - 4\cos^2 t, 0)$$

$$|\alpha'(t) \times \alpha''(t)| = 9 - 6\sin t \qquad |\alpha'(t)| = (5 - 4\sin t)^{1/2}$$

$$K = \frac{|\alpha'(t) \times \alpha''(t)|}{|\alpha'(t)|^3} \qquad \Rightarrow K(t) = \frac{9 - 6\sin t}{(5 - 4\sin t)^{3/2}} = (9 - 6\sin t)(5 - 4\sin t)^{-3/2}$$

$$\begin{aligned}
K'(t) &= (9 - 6\sin t) \cdot (3/2)(5 - 4\sin t)^{-5/2} \cdot (-4\cos t) - 6\cos t((5 - 4\sin t)^{-3/2}) \\
&= 6\cos t(9 - 6\sin t)(5 - 4\sin t)^{-5/2} - 6\cos t(5 - 4\sin t)^{-3/2} \\
&= 6\cos t(5 - 4\sin t)^{-5/2}\left[9 - 6\sin t - (5 - 4\sin t)\right] \\
&= 6\cos t(5 - 4\sin t)^{-5/2}(4 - 2\sin t)
\end{aligned}$$

$$K'(s) = \frac{dK}{ds} = \frac{dK}{dt} \cdot \frac{dt}{ds} = 0 \qquad \Rightarrow K'(s) \cdot \frac{dt}{ds} = 0$$

$$\Rightarrow 6\cos t(5 - 4\sin t)^{-5/2}(4 - 2\sin t) \cdot \frac{dt}{ds} = 0 \qquad \Rightarrow \frac{dt}{ds} \neq 0$$

$$\therefore \Rightarrow \cos t(4 - 2\sin t) = 0$$

$$\cos t = 0 \qquad \& \quad 2 - \sin t = 0$$

$$\Rightarrow t = \frac{\pi}{2}, 3\frac{3\pi}{2} \qquad \Rightarrow \qquad \sin t = 2$$

but we know that $-1 \leq \sin t \leq 1$

$$\therefore t = \frac{\pi}{2} \quad , \quad \frac{3\pi}{2}$$

$\therefore$ The given curve has only two vertices

therefore The four-vertices theory must not be true if the curve is not simple

Example 4.5. Explain with an example that the four-vertices theory is not true if the curve is not closed
sol:
We take the parabola

$$\alpha(t) = (t, t^2) \qquad \alpha'(t) = (1, 2t) \qquad \alpha''(t) = (0, 2)$$

$$|\alpha'(t) \times \alpha''(t)| = 2 \qquad\qquad |\alpha'(t)| = \left(1 + 4t^2\right)^{\frac{1}{2}}$$

$$K(t) = \frac{2}{(1 + 4t^2)^{\frac{3}{2}}} = 2(1 + 4t^2)^{\frac{-3}{2}}$$

$$K'(t) = 2(\frac{-3}{2})(1 + 4t^2)^{\frac{-5}{2}} \cdot 8t = -24t(1 + 4t^2)^{\frac{-5}{2}}$$

$$\Rightarrow K'(s) = \frac{dK}{ds} = \frac{dK}{dt} \cdot \frac{dt}{ds} = 0$$

$$\Rightarrow K'(s) \cdot \frac{dt}{ds} = \qquad \Rightarrow -24t(1 + 4t^2)^{\frac{-5}{2}} \cdot \frac{dt}{ds} = 0$$

$$\because (1 + 4t^2)^{\frac{-5}{2}} \neq 0 \;\&\; \frac{dt}{ds} \neq 0 \qquad\qquad \Rightarrow \therefore t = 0$$

$\therefore$ the parabola has only one-vertex
$\therefore$ therefore The four-vertices theory must not be true if the curve is not closed

Example 4.6. Prove that the ellipse has four vertices

$$\alpha(t) = (2\cos t, \sin t)$$

sol:

$$\dot{\alpha} = (-2\sin t, \cos t) \qquad \ddot{\alpha} = (-2\cos t, -\sin t)$$

$$\dot{\alpha} \times \ddot{\alpha} = \begin{vmatrix} i & j & k \\ -2\sin t & \cos t & 0 \\ -2\cos t & -\sin t & 0 \end{vmatrix} = (0, 0, 2)$$

$$|\dot{\alpha} \times \ddot{\alpha}| = \sqrt{0 + 0 + 4} = 2$$

$$\begin{aligned} |\dot{\alpha}| &= \sqrt{4\sin^2 t + \cos^2 t} \\ &= \sqrt{4\sin^2 t + 1 - \sin^2 t} \\ &= \sqrt{3\sin^2 + 1} \\ \therefore K &= \frac{2}{(3\sin^2 + 1)^{3/2}} = 2(3\sin^2 + 1)^{-3/2} \\ \dot{K(t)} &= 2(\frac{-3}{2})(3\sin^2 + 1)^{-5/2}(\sin t \cos t) \\ \text{Put} \quad \dot{K} &= 0 \end{aligned}$$

$$0 = -18(3\sin^2 + 1)^{-5/2}(\sin t \cos t)$$

$$\because -18(3\sin^2 + 1)^{-5/2} \neq 0$$

$$\therefore \boxed{\sin t \cos t = 0}$$

$$\begin{array}{ccc} \sin t = 0 & \text{or} & \cos t = 0 \\ \Rightarrow t = 0, \pi & & t = \frac{\pi}{2}, \frac{3\pi}{2} \end{array}$$

$$\therefore \ \alpha(t) \text{ ellipse has four vertex}$$