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The Variational Iteration Method

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Abstract

The Variational Iteration Method (VIM) is an analytical technique used to handle a wide variety of linear and nonlinear , homogeneous and non-homogeneous differential equations. It is an iterative approach that refines an initial approximation to the solution of a problem by incorporating corrections derived from a functional constructed on the original equation.

The VIM established by Professor Jihuan He (1999a),is an improvement on Lagrange's multiplication. This method does not require the presence of small parameters in the differential equation and provides the solution (or an approximation to it) as a sequence of iterations The earliest variational problem was put forward by the famous scientist Pierre de Fermat (1601–1665). At the beginning of the calculus formation, Newton, Bernoulli, and many scientists proposed questions about the functional extremum, called the variational problem. Methods to deal with these problems are called the variational method (He, 1997, 1998)

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Introduction

Integral equations are essential tools in mathematical analysis, used to formulate and solve a wide range of problems in theoretical and applied fields. This project examines both Volterra and Fredholm integral equations, with a focus on solving problems using the Variational Iteration Method (VIM). We explore both linear and nonlinear integral equations, as well as integro-differential equations and systems of integral equations.

Our work highlights the effectiveness of VIM in obtaining approximate solutions to these types of equations. Throughout the project, we present several problems and demonstrate how VIM can be applied to systematically solve them. This study provides a clear understanding of how VIM can be utilized for both Volterra and Fredholm integral equations in different contexts.

Chapter 1

The Variation Method

Variational Iteration Method (VIM) Overview

The Variational Iteration Method (VIM) is an efficient technique for solving both linear and nonlinear equations, particularly integral equations such as Volterra equations. It constructs a correction functional that iteratively improves the solution approximation.

Problem Formulation

We start with the general differential equation:

$$Lu + Nu = g(t)$$

or written differently:

$$Lu + Nu - g(t) = 0$$

where:

- L is a linear operator,
- N is a nonlinear operator,
- $g(t)$ is the inhomogeneous term.

Correction Functional

The correction functional is defined as:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)(Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi))d\xi$$

Where:

- $u_n(x)$ is the n-th approximation of the solution,
- $\lambda(\xi)$ is the **Lagrange multiplier**, determined using variational principles,
- $\tilde{u}_n(\xi)$ is a restricted value (variation $\delta\tilde{u}_n = 0$).

Lagrange Multiplier

For Volterra equations, the Lagrange multiplier $\lambda(\xi)$ is typically given by:

$$\lambda = (-1)^n \frac{1}{(n-1)!}(\xi - x)^{n-1}$$

This is derived through variational theory and ensures that the correction functional improves convergence.

Iteration Formula

Using the Lagrange multiplier, the iteration process becomes:

$$u_{n+1}(x) = u_n(x) + (-1)^n \int_0^x \frac{1}{(n-1)!}(\xi - x)^{n-1} \left[u_n^{(n)} + f(u_n)\right] d\xi$$

Where:

- $u_n^{(n)}$ is the n-th derivative of the current approximation,
- $f(u_n)$ represents any nonlinear terms.

Initial Approximation $u_0(x)$ Selection via Taylor Series

The initial approximation $u_0(x)$ is chosen based on the derivatives present in the problem:

- **If $u'(x)$ appears,** set $u_0(x) = u(0)$.
- **If $u''(x)$ appears,** use $u_0(x) = u(0) + xu'(0)$.
- **If $u'''(x)$ appears,** use $u_0(x) = u(0) + xu'(0) + \frac{1}{2!}x^2u''(0)$.
- **For higher-order derivatives,** continue using the general Taylor series expansion:

$$u_0(x) = u(0) + xu'(0) + \frac{1}{2!}x^2u''(0) + \frac{1}{3!}x^3u'''(0) + \frac{1}{4!}x^4u''''(0) + \dots$$

This approximation ensures faster convergence, and the final solution is obtained by:

$$u(x) = \lim_{n\to\infty} u_n(x)$$

Exact Solution Possibility

As $n \to \infty$, the VIM often converges to the exact solution. In many cases, we take the limit of $u_n(x)$:

$$u(x) = \lim_{n\to\infty} u_n(x)$$

If the iterative process converges, this limit represents the exact solution of the problem.

For instance, in certain examples like:

$$u(x) = 1 + \int_0^x u(t)dt$$

The VIM provides an exact solution:

$$u(x) = e^x$$

after summing the infinite iterations. Therefore, the method can yield exact solutions when the series converges.

Definition 1. ***Leibniz Rule for Differentiation of Integrals.***

Let $f(x,t)$ be continuous and $\frac{\partial f}{\partial t}$ be continuous in a domain

$$F(x) = \int_{g(x)}^{h(x)} f(x,t)\,dt \quad \forall\, a \le x \le b,\ t_0 \le t \le t_1.$$

Then the differentiation of the integral exists and is given by

$$F'(x) = \frac{dF}{dx} = f(x,h(x))\frac{dh(x)}{dx} - f(x,g(x))\frac{dg(x)}{dx} + \int_{g(x)}^{h(x)} \frac{\partial f}{\partial x}(x,t)\,dt.$$

Example 1. *Solve the Volterra integral equation using the Variational Iteration Method (VIM):*

$$u(x) = 1 + \int_0^x u(t)\,dt.$$

Solution:

Using Leibniz's rule to differentiate both sides of this equation gives:

$$\begin{aligned} u'(x) &= \frac{d}{dx}\left(1+\int_0^x u(t)\,dt\right) \\ &= 0 + u(x)\cdot\frac{d}{dx}x - u(0)\cdot\frac{d}{dx}0 + \int_0^x \frac{\partial u(t)}{\partial x}\,dt \\ &= u(x). \end{aligned}$$

Thus, we obtain the differential equation

$$u'(x) - u(x) = 0.$$

Substituting $x = 0$ into the original integral equation:

$$u(0) = 1 + \int_0^0 u(t)\,dt = 1,$$

which provides the initial condition $u(0) = 1$.

Using the VIM, the correction functional for the equation $u'(x) - u(x) = 0$ is

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\left(u_n'(\xi) - u_n(\xi)\right)d\xi.$$

From the correction functional, we determine the Lagrange multiplier:

$$\lambda = -\frac{(\xi - x)^0}{0!} = -1.$$

Substituting this value of the Lagrange multiplier $\lambda = -1$ into the functional gives the iteration formula:

$$u_{n+1}(x) = u_n(x) + \int_0^x \left(u_n'(\xi) - u_n(\xi)\right)d\xi.$$

Using the initial condition $u_0(x) = u(0) = 1$, we proceed iteratively:

$$\begin{aligned} &u_0(x) = 1, \\ &u_1(x) = 1 - \int_0^x \left(u_0'(\xi) - u_0(\xi)\right)d\xi = 1 + x, \\ &u_2(x) = 1 + x - \int_0^x \left(u_1'(\xi) - u_1(\xi)\right)d\xi = 1 + x + \frac{x^2}{2!}, \\ &u_3(x) = 1 + x + \frac{x^2}{2!} - \int_0^x \left(u_2'(\xi) - u_2(\xi)\right)d\xi = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}, \end{aligned}$$

and so on. The VIM leads to the series solution

$$u(x) = \lim_{n\to\infty} u_n(x) = \lim_{n\to\infty}\left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\right),$$

which converges to the exact solution

$$u(x) = e^x.$$

Example 2. *Solve the Volterra integral equation by using the variational iteration method:*

$$u(x) = x + \int_0^x (x-t)u(t)\,dt.$$

Solution:

Differentiating both sides of the equation once with respect to x gives the following integro-differential equation:

$$\frac{d}{dx}u(x) = \frac{d}{dx}\left(x + \int_0^x (x-t)u(t)\,dt\right)$$
$$u'(x) = 1 + \frac{d}{dx}\left(\int_0^x (x-t)u(t)\,dt\right).$$

and by using the Leibnitz rule

$$u'(x) = 1 + \int_0^x u(t)dt$$

so by differentiating the equation with respect to x we obtain the differential equation:

$$\frac{d}{dx}\left(\frac{d}{dx}u(x)\right) = \frac{d}{dx}\left(1 + \int_0^x u(t)dt\right)$$

$$\Rightarrow u''(x) = u(x)$$
$$u''(x) - u(x) = 0$$

To determine the initial condition, we substitute $x = 0$ into both the original equation and the first derivative equation.

Substituting into the original equation:

$$u(0) = 0 + \int_0^0 (0-t)u(t)\,dt = 0,$$

which gives $u(0) = 0$.

Substituting into the first derivative equation:

$$u'(0) = 1 + \int_0^0 u(t)\,dt = 1.$$

Thus, the initial condition is $u(0) = 0$ and $u'(0) = 1$.

using the variational iteration method

In this solution, we apply the variational iteration method twice. The first iteration gives an initial approximation of $u(x)$, but additional refinement is necessary to *improve the accuracy of the solution.*

(i) We begin by applying the variational iteration method (VIM) to the first derivative of the integro-differential equation. The first derivative equation is:

$$u'(x) = 1 + \int_0^x u(t)\,dt,\ u(0) = 0.$$

The correction functional :

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\left(u_n'(\xi) - 1 - \int_0^\xi \tilde{u}_n(r)dr\right)d\xi$$

Next, we determine the Lagrange multiplier lambda,

$$\lambda = (-1)^n \frac{1}{(n-1)!}(\xi - x)^{n-1}$$

where the $n = 1 \ \therefore\ \lambda = -1$ *Substituting this value into the correction functional*

$$u_{n+1}(x) = u_n(x) + \int_0^x (-1)\left(u_n'(\xi) - 1 - \int_0^\xi u_n(r)dr\right)d\xi$$

Using the initial condition $u_0(x) = u(0) = 0$*, we proceed iteratively:*

$$\begin{aligned}
&u_0(x) = 0,\\
&u_1(x) = 0 - \int_0^x \left(u_0'(\xi) - 1 - \int_0^\xi u_0(r)dr\right)d\xi = x,\\
&u_2(x) = x - \int_0^x \left(u_1'(\xi) - 1 - \int_0^\xi u_1(r)dr\right)d\xi = x + \frac{x^3}{3!},\\
&u_3(x) = x + \frac{x^3}{3!} - \int_0^x \left(u_2'(\xi) - 1 - \int_0^\xi u_2(r)dr\right)d\xi = x + \frac{x^3}{3!} + \frac{x^5}{5!},
\end{aligned}$$

and so on. The VIM leads to the series solution

$$u(x) = \lim_{n\to\infty} u_n(x) = \lim_{n\to\infty}\left(x + \frac{x^3}{3!} + \cdots + \frac{x^{2n+1}}{(2n+1)!}\right),$$

which converges to the exact solution

$$u(x) = \sinh x.$$

(ii) We can obtain the same result by applying the variational iteration method to handle the initial value proemm given by

$$u''(x) - u(x) = 0,$$

first, we determine the Lagrange multiplier lambda where n=2,$\Rightarrow \lambda = \xi - x$ *Substituting this value into the correction functional*

$$u_{n+1}(x) = u_n(x) + \int_0^x (\xi - x)(u_n''(\xi) - u_n(\xi))d\xi$$

Using the initial condition $u_0(x) = u(0) + xu'(0) = x$, we proceed iteratively:

$$\begin{aligned}
u_0(x) &= x, \\
u_1(x) &= x + \int_0^x (\xi - x)(u_0''(\xi) - u_0(\xi))d\xi = x + \frac{x^3}{3!}, \\
u_2(x) &= x + \frac{x^3}{3!} + \int_0^x (\xi - x)(u_1''(\xi) - u_1(\xi))d\xi = x + \frac{x^3}{3!} + \frac{x^5}{5!}, \\
u_3(x) &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \int_0^x (\xi - x)(u_2''(\xi) - u_2(\xi)) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!},
\end{aligned}$$

and so on. The VIM leads to the series solution

$$u(x) = \lim_{n\to\infty} u_n(x) = \lim_{n\to\infty} \left(x + \frac{x^3}{3!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} \right),$$

which converges to the exact solution

$$u(x) = \sinh x.$$

Example 3. *Solve the Volterra integral equation by using the variational iteration method:*

$$u(x) = 1 + x + \frac{x^3}{3!} - \int_0^x (x - t)u(t)dt$$

Solution:

Differentiating both sides of the equation once with respect to x gives the following integro-differential equation:

$$\begin{aligned}
\frac{d}{dx}u(x) &= \frac{d}{dx}\left(1 + x + \frac{x^3}{3!} - \int_0^x (x - t)u(t)\,dt\right) \\
u'(x) &= 1 + \frac{x^2}{2} - \frac{d}{dx}\left(\int_0^x (x - t)u(t)\,dt\right).
\end{aligned}$$

Using the Leibnitz rule for the derivative of an integral, we have:

$$u'(x) = 1 + \frac{x^2}{2} - \int_0^x u(t)\,dt.$$

Differentiating the equation again with respect to x yields the second-order integro-differential equation:

$$\frac{d}{dx}u'(x) = \frac{d}{dx}\left(1 + \frac{x^2}{2} - \int_0^x u(t)\,dt\right),$$

giving us the second derivative:

$$u''(x) = x - u(x).$$

So the integro-differential equation becomes:

$$u''(x) + u(x) = x.$$

To determine the initial condition, we substitute $x = 0$ *into both the original equation and the first derivative equation.*

Substituting into the original equation:

$$u(0) = 1 + 0 + \frac{0^3}{3!} - \int_0^0 (0 - t)u(t)\,dt = 1,$$

which gives $u(0) = 1$.

Substituting into the first derivative equation:

$$u'(0) = 1 + \frac{0^2}{2} - \int_0^0 u(t)\,dt = 1.$$

Thus, the initial conditions are $u(0) = 1$ *and* $u'(0) = 1$.

Using the variational iteration method:

(i) We begin by applying the variational iteration method (VIM) to the integro-differential equation. The first derivative equation is:

$$u'(x) = 1 + \frac{x^2}{2} - \int_0^x u(t)\,dt.$$

The correction functional is:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\left(u_n'(\xi) - 1 - \frac{\xi^2}{2} + \int_0^\xi \tilde{u}_n(r)\,dr\right)d\xi.$$

Next, we determine the Lagrange multiplier $\lambda(\xi)$*. For* $n = 1$*, we have:*

$$\lambda(\xi) = -1.$$

Substituting this value into the correction functional gives:

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(\xi) - 1 - \frac{\xi^2}{2} + \int_0^\xi u_n(r)\,dr\right)d\xi.$$

Using the initial conditions to select $u_0(x) = u(0) = 1$*, we proceed iteratively:*

$$u_0(x) = 1,$$
$$u_1(x) = 1 - \int_0^x \left(u_0'(\xi) - 1 - \frac{\xi^2}{2} + \int_0^\xi u_0(r)dr\right)d\xi = 1 + x - \frac{x^2}{2!} + \frac{x^3}{3!},$$
$$u_2(x) = 1 + x - \frac{x^2}{2!} + \frac{x^3}{3!} - \int_0^x \left(u_1'(\xi) - 1 - \frac{\xi^2}{2} + \int_0^\xi u_1(r)dr\right)d\xi = 1 + x - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^5}{5!},$$
$$u_3(x) = u_2(x) - \int_0^x \left(u_2'(\xi) - 1 - \frac{\xi^2}{2} + \int_0^\xi u_2(r)dr\right)d\xi = 1 + x - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^7}{7!},$$

and so on. The VIM leads to the series solution:

$$u(x) = \lim_{n\to\infty} u_n(x) = x + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n}{(2n)!}x^{2n}\right),$$

which converges to the exact solution:

$$u(x) = x + \cos x.$$

(ii) Alternatively, we can apply the VIM to handle the second-order integro-differential equation:

$$u''(x) + u(x) = x.$$

First, we determine the Lagrange multiplier for $n = 2$, which gives $\lambda = \xi - x$. Substituting this into the correction functional:

$$u_{n+1}(x) = u_n(x) + \int_0^x (\xi - x)(u_n''(\xi) + u_n(\xi) - \xi)\, d\xi.$$

Using the initial guess $u_0(x) = 1 + x$, we proceed iteratively:

$$\begin{aligned}
&u_0(x) = 1 + x,\\
&u_1(x) = 1 + x + \int_0^x (\xi - x)(u_0''(\xi) + u_0(\xi) - \xi)d\xi = 1 + x - \frac{x^2}{2!},\\
&u_2(x) = 1 + x - \frac{x^2}{2!} + \int_0^x (\xi - x)(u_1''(\xi) + u_1(\xi) - \xi)d\xi = 1 + x - \frac{x^2}{2!} + \frac{x^4}{4!},\\
&u_3(x) = 1 + x - \frac{x^2}{2!} + \frac{x^4}{4!} + \int_0^x (\xi - x)(u_2''(\xi) + u_2(\xi) - \xi)d\xi = 1 + x - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!},
\end{aligned}$$

and so on. The VIM leads to the series solution:

$$u(x) = \lim_{n\to\infty} u_n(x) = x + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n}{(2n)!}x^{2n}\right),$$

which converges to the exact solution:

$$u(x) = x + \cos(x).$$

1.1 Application of the Variational Iteration Method to Fredholm Integral Equations

Previously, we applied the variational iteration method (VIM) to Volterra integral equations by converting them into initial value problems or integro-differential equations. In this section, we extend the method to Fredholm integral equations, the method works effectively if the kernel $K(x,t)$ is separable and can be written in the form $K(x,t) = g(x)h(t)$.

Our approach here mirrors the one used earlier: we differentiate both sides of the Fredholm integral equation, converting it into an integro-differential form. This transformation allows us to employ VIM effectively, but it's crucial to note that an integro-differential equation requires an initial condition to be fully defined. In this section, we focus on cases where $g(x) = x^n, n \geq 1$.

The general form of a Fredholm integral equation of the second kind is:

$$u(x) = f(x) + \int_a^b K(x,t)u(t)dt$$

, Since the kernel is separable, the integral simplifies:

$$u(x) = f(x) + g(t)\int_a^b h(t)u(t)dt$$

To apply VIM, the Fredholm integral equation is first converted into an integro-differential form. By differentiating both sides of the equation with respect to x,

$$u'(x) = f'(x) + g'(x)\int_a^b h(t)u(t)dt$$

The correction functional for the integro-differential equation is

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\left(u_n'(\xi) - f'(\xi) - g'(\xi)\int_a^b h(r)\tilde{u}_n(r)dr\right)$$

Example 4. *Use the variational iteration method to solve the Fredholm integral equation*

$$u(x) = e^x - x + x\int_0^1 tu(t)dt$$

Solution:

Differentiating both sides of the equation once with respect to x gives the following integro-differential equation:

$$u'(x) = e^x - 1 + \int_0^1 tu(t)dt$$

. Next, we determine the Lagrange multiplier $\lambda(\xi)$. For $n = 1$, we have:

$$\lambda(\xi) = -1.$$

The correction functional is:

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(\xi) - e^\xi + 1 - \int_0^1 ru_n(r)dr\right) d\xi$$

, and the initial condition $u(0) = 1$ is obtained by substituting $x = 0$ into the original equation
using the initial condition to select $u_0(x) = u(0) = 1$

$$\begin{aligned}
&u_0(x) = 1,\\
&u_1(x) = u_0(x) - \int_0^x \left(u_0'(\xi) - e^\xi + 1 - \int_0^1 ru_0(r)dr\right) d\xi = e^x - \frac{x}{2},\\
&u_2(x) = u_1(x) - \int_0^x \left(u_1'(\xi) - e^\xi + 1 - \int_0^1 ru_1(r)dr\right) d\xi = e^x - \frac{x}{2\times 3},\\
&u_3(x) = u_2(x) - \int_0^x \left(u_2'(\xi) - e^\xi + 1 - \int_0^1 ru_2(r)dr\right) d\xi = e^x - \frac{x}{2\times 3^2},\\
&\quad\vdots\\
&u_{n+1} = e^x - \frac{x}{2\times 3^n}, n \geq 0.
\end{aligned}$$

The VIM admits the use of:

$$u(x) = \lim_{n\to\infty} u_n(x) = e^x$$

Example 5. *Use the variational iteration method to solve the Fredholm integral equation*

$$u(x) = \sin x - x + x \int_0^{\frac{\pi}{2}} u(t)\, dt.$$

Solution:

Differentiating both sides of the equation with respect to x gives the following integro-differential equation:

$$u'(x) = \cos x - 1 + \int_0^{\frac{\pi}{2}} u(t)\, dt.$$

Next, we determine the Lagrange multiplier $\lambda(\xi)$. For $n = 1$, we have:

$$\lambda(\xi) = -1.$$

The correction functional is:

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(\xi) - \cos\xi + 1 - \int_0^{\frac{\pi}{2}} u_n(r)\, dr \right) d\xi,$$

and the initial condition $u(0) = 0$ is obtained by substituting $x = 0$ into the original equation.

Using this initial condition, we choose the initial guess $u_0(x) = 0$:

$$\begin{aligned}
u_0(x) &= 0,\\
u_1(x) &= u_0(x) - \int_0^x \left(u_0'(\xi) - \cos\xi + 1 - \int_0^{\frac{\pi}{2}} u_0(r)\, dr \right) d\xi = (\sin x - x),\\
u_2(x) &= u_1(x) - \int_0^x \left(u_1'(\xi) - \cos\xi + 1 - \int_0^{\frac{\pi}{2}} u_1(r)\, dr \right) d\xi = (\sin x - x) + \left(x - \frac{\pi^2}{8} x \right),\\
u_3(x) &= u_2(x) - \int_0^x \left(u_2'(\xi) - \cos\xi + 1 - \int_0^{\frac{\pi}{2}} u_2(r)\, dr \right) d\xi\\
&= (\sin x - x) + \left(x - \frac{\pi^2}{8} x \right) + \left(\frac{\pi^2}{8} x - \frac{\pi^4}{64} x \right),\\
&\vdots\\
u_{n+1}(x) &= \sin x - \frac{\pi^{2*n}}{8^n} x, \quad n \geq 0.
\end{aligned}$$

Thus, the VIM gives the solution:

$$u(x) = \lim_{n\to\infty} u_n(x) = \sin x.$$

Example 6. *Use the variational iteration method to solve the Fredholm integral equation*

$$u(x) = -2x + \sin x + \cos x + \int_0^{\pi} xu(t)\,dt$$

Solution:

Differentiating both sides of the equation once with respect to x gives the following integro-differential equation:

$$u'(x) = -2 + \cos x - \sin x + \int_0^{\pi} u(t)\,dt$$

Next, we determine the Lagrange multiplier $\lambda(\xi)$. For $n = 1$, we have:

$$\lambda(\xi) = -1.$$

The correction functional is:

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(\xi) + 2 - \cos\xi + \sin\xi - \int_0^{\pi} u_n(r)\,dr\right) d\xi$$

the initial condition $u(0) = 1$ is obtained by substituting $x = 0$

we can start with $u_0(x) = 0$. Now we proceed with the iterations:

$$\begin{aligned}
u_0(x) &= 1,\\
u_1(x) &= u_0(x) - \int_0^x \left(u_0'(\xi) + 2 - \cos\xi + \sin\xi - \int_0^{\pi} u_0(r)dr\right) d\xi,\\
&= \sin x + \cos x + (\pi x - 2x),\\
u_2(x) &= u_1(x) - \int_0^x \left(u_1'(\xi) + 2 - \cos\xi + \sin\xi - \int_0^{\pi} u_1(r)dr\right) d\xi,\\
&= \sin x + \cos x + (\pi x - 2x) + \left(-\pi x + 2x - \pi^2 x + \frac{\pi^3}{2}x\right),\\
u_3(x) &= u_2(x) - \int_0^x \left(u_2'(\xi) + 2 - \cos\xi + \sin\xi - \int_0^{\pi} u_2(r)dr\right) d\xi,\\
&= \sin x + \cos x + (\pi x - 2x) + \left(-\pi x + 2x - \pi^2 x + \frac{\pi^3}{2}x\right) + \left(\pi^2 x - \frac{\pi^3}{2}x + \cdots\right),
\end{aligned}$$

and so on.Canceling the noise terms, the exact solution is given by

$$u(x) = \sin x + \cos x.$$

Example 7. *Use the variational iteration method to solve the Fredholm integral equation*

$$u(x) = -x^3 + \cos x + \int_0^{\frac{\pi}{2}} x^3 u(t)\, dt$$

Solution:

Differentiating both sides of the equation once with respect to x gives the following integro-differential equation:

$$u'(x) = -3x^2 - \sin x + 3x^2 \int_0^{\frac{\pi}{2}} u(t)\, dt$$

Next, we determine the Lagrange multiplier $\lambda(\xi)$. For $n = 1$, we have:

$$\lambda(\xi) = -1.$$

The correction functional is:

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(\xi) + 3\xi^2 + \sin \xi - 3\xi^2 \int_0^{\frac{\pi}{2}} u_n(t)\, dt \right) d\xi$$

the initial condition $u(0) = 1$, we can start with $u_0(x) = 1$.

Now we proceed with the iterations:

$$\begin{aligned}
u_0(x) &= 1, \\
u_1(x) &= u_0(x) - \int_0^x \left(u_0'(\xi) + 3\xi^2 + \sin \xi - 3\xi^2 \int_0^{\frac{\pi}{2}} u_0(t)\, dt \right) d\xi \\
&= \cos x + \left(\frac{\pi}{2} x^3 - x^3 \right), \\
u_2(x) &= u_1(x) - \int_0^x \left(u_1'(\xi) + 3\xi^2 + \sin \xi - 3\xi^2 \int_0^{\frac{\pi}{2}} u_1(t)\, dt \right) d\xi \\
&= \cos x + \left(\frac{\pi}{2} x^3 - x^3 \right) + \left(-\frac{\pi}{2} x^3 + x^3 - \frac{\pi^4}{64} x^3 + \frac{\pi^5}{128} x^3 \right).
\end{aligned}$$

and so on.Canceling the noise terms, the exact solution is given by:

$$u(x) = \cos x.$$

Chapter 2

Utilizing VIM in Solving Volterra Integro-Differential Equations

Solution Steps Using the Variational Iteration Method (VIM)

We consider the following Volterra integro-differential equation:

$$u^{(i)}(x) = f(x) + \int_0^x K(x,t)u(t)\,dt$$

Step 1: Initial Approximation via Taylor Series

The initial approximation $u_0(x)$ is obtained using the Taylor series expansion:

$$u_0(x) = u(0) + u'(0)x + \frac{u''(0)}{2!}x^2 + \cdots$$

where the derivatives are evaluated at $x = 0$.

Step 2: Correction Functional

The correction functional for the $(n+1)$-th iteration is given by:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\left[u_n^{(i)}(\xi) - f(\xi) - \int_0^\xi K(\xi,r)\tilde{u}_n(r)\,dr\right]d\xi$$

Step 3: Lagrange Multiplier

The Lagrange multiplier $\lambda(\xi)$ is determined by the following formula:

$$\lambda(\xi) = \frac{(-1)^n}{(n-1)!}(\xi - x)^{n-1}$$

Step 4: Iteration Process

The process is repeated iteratively to improve the solution, starting from $u_0(x)$ and generating subsequent approximations $u_1(x), u_2(x), \ldots$, until a desired level of accuracy is achieved. The Variational Iteration Method (VIM) will be illustrated by studying the following examples:

Example 8. *Use the Variational Iteration Method to Solve the Volterra Integro-Differential Equation*

$$u'(x) = 1 + \int_0^x u(t)\,dt$$

with the initial condition $u(0) = 1$.

Solution Steps

1. Initial Approximation: *From the Taylor series expansion, since we have a first-order integro-differential equation, we take the first term:* $u_0(x) = u(0) = 1$.

2. Correction Functional: *The correction functional for this equation is given by:*

$$u_{n+1}(x) = u_n(x) - \int_0^x \left[u_n'(\xi) - 1 - \int_0^\xi u_n(r)\,dr\right] d\xi.$$

3. Lagrange Multiplier: *Since we have a first-order integro-differential equation, we set* $n = 1$. *Substituting this value into the Lagrange multiplier formula:*

$$\lambda = (-1)^n \frac{1}{(n-1)!}(\xi - x)^{n-1} \quad \Rightarrow \quad \lambda = (-1)^1 \frac{1}{(1-1)!}(\xi - x)^{1-1} = -1.$$

4. Iterative Process: *We can use the initial condition to select* $u_0(x) = u(0) = 1$. *Substituting this selection into the correction functional yields the following successive approximations:*

- *First Approximation:*

$$u_1(x) = u_0(x) - \int_0^x \left[u_0'(\xi) - 1 - \int_0^\xi u_0(r)\,dr\right] d\xi = 1 + x + \frac{1}{2!}x^2.$$

- *Second Approximation:*

$$u_2(x) = u_1(x) - \int_0^x \left[u_1'(\xi) - 1 - \int_0^\xi u_1(r)\,dr\right] d\xi = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4.$$

- *Third Approximation:*

$$u_3(x) = u_2(x) - \int_0^x \left[u_2'(\xi) - 1 - \int_0^\xi u_2(r)\,dr\right] d\xi = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6.$$

Continuing this process, we have:

$$u(x) = \lim_{n\to\infty} u_n(x),$$

which converges to the exact solution:

$$u(x) = e^x.$$

Example 9. *Use the Variational Iteration Method to Solve the Volterra Integro-Differential Equation.*

$$u''(x) = 1 + \int_0^x (x-t)u(t)\,dt,$$

with the initial conditions $u(0) = 1$ and $u'(0) = 0$.

Solution Steps

1. Initial Approximation: *Since we are dealing with a second-order integro-differential equation, we take the first two terms of the Taylor expansion at $x = 0$. Using the initial conditions $u(0) = 1$ and $u'(0) = 0$, we obtain:*

$$u_0(x) = u(0) + xu'(0) = 1 + 0 = 1.$$

2. Correction Functional: *The correction functional for this equation is given by:*

$$u_{n+1}(x) = u_n(x) + \int_0^x (\xi - x)\left[u_n''(\xi) - 1 - \int_0^\xi (\xi - r)u_n(r)\,dr\right]d\xi.$$

3. Lagrange Multiplier: *For second-order integro-differential equations, we set $n = 2$ in the Lagrange multiplier formula, which yields:*

$$\lambda = (-1)^n \frac{1}{(n-1)!}(\xi - x)^{n-1} = (-1)^2 \frac{1}{(2-1)!}(\xi - x)^{2-1} = 1 \cdot (\xi - x) = \xi - x.$$

4. Iterative Process: *We start with the initial approximation $u_0(x) = 1$ and apply the correction functional to compute successive approximations:*

- First Approximation:

$$\begin{aligned} u_1(x) &= u_0(x) + \int_0^x (\xi - x)\left[u_0''(\xi) - 1 - \int_0^\xi (\xi - r)u_0(r)\,dr\right]d\xi \\ &= 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4. \end{aligned}$$

- *Second Approximation:*

$$\begin{aligned} u_2(x) &= u_1(x) + \int_0^x (\xi - x)\left[u_1''(\xi) - 1 - \int_0^\xi (\xi - r)u_1(r)\,dr\right] d\xi \\ &= 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \frac{1}{8!}x^8. \end{aligned}$$

- *Third Approximation:*

$$\begin{aligned} u_3(x) &= u_2(x) + \int_0^x (\xi - x)\left[u_2''(\xi) - 1 - \int_0^\xi (\xi - r)u_2(r)\,dr\right] d\xi \\ &= 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \frac{1}{10!}x^{10} + \frac{1}{12!}x^{12}. \end{aligned}$$

Continuing this process, we obtain the exact solution:

$$u(x) = \cosh(x).$$

Example 10. *Use the Variational Iteration Method to Solve the Volterra Integro-Differential Equation:*

$$u'''(x) = 1 + x + \frac{1}{3!}x^3 + \int_0^x (x - t)u(t)\,dt,$$

with the initial conditions $u(0) = 1$*,* $u'(0) = 0$*, and* $u''(0) = 1$*.*

Solution Steps

1. Initial Approximation: *Since we are dealing with a third-order integro-differential equation, we take the first three terms of the Taylor expansion at* $x = 0$*. Using the initial conditions* $u(0) = 1$*,* $u'(0) = 0$*, and* $u''(0) = 1$*, we obtain:*

$$u_0(x) = 1 + 0 + \frac{1}{2}x^2 = 1 + \frac{1}{2}x^2.$$

2. Correction Functional: *The correction functional for this equation is given by:*

$$u_{n+1}(x) = u_n(x) - \frac{1}{2}\int_0^x (\xi - x)^2\left[u_n'''(\xi) - 1 - \xi - \frac{1}{3!}\xi^3 - \int_0^\xi (\xi - r)u_n(r)\,dr\right] d\xi.$$

3. Lagrange Multiplier: *For third-order integro-differential equations, we set* $n = 3$ *in the Lagrange multiplier formula. Since we have a third-order integro-differential equation, we use:*

$$\lambda = (-1)^n \frac{1}{(n-1)!}(\xi - x)^{n-1}.$$

Substituting $n = 3$:

$$\lambda = (-1)^3 \frac{1}{(3-1)!}(\xi - x)^{3-1} = -1 \cdot \frac{1}{2!}(\xi - x)^2 = -\frac{1}{2}(\xi - x)^2.$$

4. Iterative Process: *We start with the initial approximation:*

$$u_0(x) = 1 + \frac{1}{2}x^2$$

and apply the correction functional to compute successive approximations:

First Approximation:

$$u_1(x) = u_0(x) - \frac{1}{2}\int_0^x (\xi - x)^2 \left[u_0'''(\xi) - 1 - \xi - \frac{1}{3!}\xi^3 - \int_0^\xi (\xi - r)u_0(r)\,dr \right] d\xi,$$

$$= 1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7.$$

Second Approximation:

$$u_2(x) = u_1(x) - \frac{1}{2}\int_0^x (\xi - x)^2 \left[u_1'''(\xi) - 1 - \xi - \frac{1}{3!}\xi^3 - \int_0^\xi (\xi - r)u_1(r)\,dr \right] d\xi,$$

$$= 1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 + \frac{1}{8!}x^8 + \cdots.$$

Final Answer: *To obtain a familiar series, we observe that the approximations $u_n(x)$ are essentially the Taylor series expansion of e^x but shifted. Specifically, we can add and subtract x to the series as follows:*

$$u_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \cdots - x.$$

Taking the limit as $n \to \infty$, we find:

$$\lim_{n\to\infty} u_n(x) = e^x - x.$$

Thus, the exact solution to the Volterra integro-differential equation is:

$$u(x) = e^x - x.$$

2.1 Converting Volterra Equations of the First Kind to the Second Kind

The standard form of a Volterra integro-differential equation of the first kind is:

$$\int_0^x K_1(x,t)u(t)\,dt + \int_0^x K_2(x,t)u^{(n)}(t)\,dt = f(x), \quad K_2(x,t) \neq 0,$$

where the initial conditions are specified. To convert this equation into a Volterra integro-differential equation of the second kind, we differentiate both sides. In this process, we apply Leibniz's rule to differentiate the integrals on the left side.

As stated before, to use the Variational Iteration Method (VIM), we follow these steps:

- **Initial Approximation via Taylor Series**: The initial approximation $u_0(x)$ is obtained using the Taylor series expansion:

$$u_0(x) = u(0) + u'(0)x + \frac{u''(0)}{2!}x^2 + \cdots,$$

where the derivatives are evaluated at $x = 0$.

- **Lagrange Multiplier**: The Lagrange multiplier $\lambda(\xi)$ is determined by the formula:

$$\lambda(\xi) = (-1)^n \frac{(\xi - x)^{n-1}}{(n-1)!}.$$

- **Correction Functional**: The correction functional for the $(n+1)$-th iteration is given by:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi) \left(u_n^{(i)}(\xi) - f(\xi) - \int_0^\xi K(\xi, r)\tilde{u}_n(r)\,dr \right) d\xi.$$

- **Iteration Process**: This process is repeated iteratively to improve the solution, starting from $u_0(x)$ and generating subsequent approximations $u_1(x), u_2(x), \ldots$ until the desired level of accuracy is achieved.

Solving a Volterra Integro-Differential Equation of the First Kind

Example 11. *We begin with the Volterra integro-differential equation of the first kind:*

$$\int_0^x (x-t+1)u'(t)\,dt = e^x + \frac{1}{2}x^2 - 1, \quad u(0) = 1.$$

we first differentiate both sides with respect to x:

$$\frac{d}{dx}\left(\int_0^x (x-t+1)u'(t)\,dt\right) = \frac{d}{dx}\left(e^x + \frac{1}{2}x^2 - 1\right).$$

By applying Leibniz's rule to the left-hand side, we get:

$$\frac{d}{dx}\left(\int_0^x (x-t+1)u'(t)\,dt\right) = (x-x+1)u'(x) + \int_0^x u'(t)\,dt.$$

This simplifies to:

$$u'(x) + \int_0^x u'(t)\,dt.$$

Next, we differentiate the right-hand side:

$$\frac{d}{dx}\left(e^x + \frac{1}{2}x^2 - 1\right) = e^x + x.$$

Equating both sides, we get:

$$u'(x) + \int_0^x u'(t)\,dt = e^x + x.$$

Rearranging everything to the left-hand side:

$$u'(x) - e^x - x + \int_0^x u'(t)\,dt = 0.$$

1. Initial Approximation:

Since we are dealing with a first-order integro-differential equation, we take the first term of the Taylor expansion at $x = 0$. Using the initial condition $u(0) = 1$, we obtain:

$$u_0(x) = 1.$$

2. ***Lagrange Multiplier:*** *For first-order integro-differential equations, we set $n = 1$ in the Lagrange multiplier formula, which yields:*

$$\lambda(t) = (-1)^n \frac{1}{(n-1)!}(t-x)^{n-1} = (-1)^1 \frac{1}{(1-1)!}(t-x)^{1-1} = -1 \cdot \frac{1}{1} \cdot 1 = -1.$$

3. ***Correction Functional:***

The correction functional is given by:

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(t) - e^t - t + \int_0^t u_n'(r)\,dr \right) dt.$$

4. ***Iterative Process:***

$$u_0(x) = 1$$

$$u_1(x) = u_0(x) - \int_0^x \left(u_0'(t) - e^t - t + \int_0^t u_0'(r)\,dr \right) dt$$

$$u_1(x) = 1 - \int_0^x \left(0 - e^t - t + \int_0^t 0\,dr \right) dt = e^x + \frac{1}{2!}x^2$$

$$u_2(x) = u_1(x) - \int_0^x \left(u_1'(t) - e^t - t + \int_0^t u_1'(r)\,dr \right) dt$$

$$u_2(x) = \left(e^x + \frac{1}{2!}x^2 \right) - \int_0^x \left(e^t + x - e^t - t + \int_0^t e^r\,dr \right) dt = 1 + x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3$$

$$u_3(x) = e^x - \frac{1}{3!}x^3 + \frac{1}{4!}x^4,$$

$$u_4(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5,$$

$$u_5(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6.$$

This gives

$$u_n(x) = x + \left(1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \cdots \right).$$

The exact solution is therefore given by

$$u(x) = x + \cosh x.$$

Example 12. *Use the variational iteration method to solve the Volterra integro-differential equation of the first kind*

$$\int_0^x (x-t)u(t)\,dt - \frac{1}{2}\int_0^x (x-t+1)u'(t)\,dt = 2x^2 - \frac{1}{2}x, \quad u(0) = 5.$$

Differentiate both sides

Left-Hand Side:

$$\frac{d}{dx}\left(\int_0^x (x-t)u(t)\,dt\right) = \int_0^x u(t)\,dt,$$

$$\frac{d}{dx}\left(-\frac{1}{2}\int_0^x (x-t+1)u'(t)\,dt\right) = -\frac{1}{2}\left(u'(x) + \int_0^x u'(t)\,dt\right).$$

Combining gives:

$$\int_0^x u(t)\,dt - \frac{1}{2}\left(u'(x) + \int_0^x u'(t)\,dt\right).$$

Right-Hand Side:

$$\frac{d}{dx}\left(2x^2 - \frac{1}{2}x\right) = 4x - \frac{1}{2}.$$

Setting the two sides equal:

$$\int_0^x u(t)\,dt - \frac{1}{2}\left(u'(x) + \int_0^x u'(t)\,dt\right) = 4x - \frac{1}{2}.$$

Multiplying by 2:

$$2\int_0^x u(t)\,dt - 8x + 1 = u'(x) + \int_0^x u'(t)\,dt.$$

Thus, we find:

$$u'(x) = -8x + 1 + \int_0^x (2u(t) - u'(t))\,dt.$$

1. Initial Approximation:

Since we are dealing with a first-order integro-differential equation, we take the first term of the Taylor expansion at $x = 0$. Using the initial condition $u(0) = 5$, we obtain:

$$u_0(x) = 5.$$

2. Lagrange Multiplier:

For first-order integro-differential equations, we set $n = 1$ in the Lagrange multiplier formula, which yields:

$$\lambda(t) = (-1)^n \frac{1}{(n-1)!}(t-x)^{n-1} = (-1)^1 \frac{1}{(1-1)!}(t-x)^{1-1} = -1 \cdot \frac{1}{1 \cdot 1} = -1.$$

3. The Correction Functional:

The correction functional is given by:

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(t) + 8t - 1 - \int_0^t (2u_n(r) - u_n'(r))\, dr \right) dt.$$

4. Iterative Process

$$\begin{aligned}
u_0(x) &= 5,\\
u_1(x) &= 5 + x + x^2,\\
u_2(x) &= 5 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^4,\\
u_3(x) &= 5 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{12}x^4 - \frac{1}{30}x^5 + \cdots,\\
u_4(x) &= 5 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{90}x^6 + \cdots,\\
u_5(x) &= 5 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots.
\end{aligned}$$

This gives

$$u_n(x) = 4 + \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots \right).$$

Thus, the exact solution is therefore given by

$$u(x) = 4 + e^x.$$

Example 13. *Use the variational iteration method to solve the Volterra integro-differential equation of the first kind:*

$$\int_0^x (x-t)u(t)\, dt + \int_0^x (x-t+1)u''(t)\, dt = \sin x + \frac{1}{3!}x^3, \quad u(0) = -1, \quad u'(0) = 1.$$

Differentiating both sides of this equation once with respect to x gives the Volterra integro-differential equation of the second kind:

$$u''(x) = \cos x + \frac{1}{2}x^2 - \int_0^x u(t)\,dt - \int_0^x u''(t)\,dt.$$

1. Initial Approximation: *Since we are dealing with a second-order integro-differential equation, we take the first two terms of the Taylor expansion at $x = 0$. Using the initial conditions $u(0) = -1$ and $u' = 1$, we obtain:*

$$u_0(x) = -1 + x.$$

2. Lagrange Multiplier: *For second-order integro-differential equations, we set $n = 2$ in the Lagrange multiplier formula, which yields:*

$$\lambda(t) = (-1)^n \frac{1}{(n-1)!}(t-x)^{n-1} = (-1)^2 \frac{1}{(2-1)!}(t-x)^{2-1} = 1 \cdot \frac{1}{1!}(t-x) = (t-x).$$

3. The Correction Functional:

The correction functional for this equation is given by

$$u_{n+1}(x) = u_n(x) + \int_0^x ((t-x)\Gamma(t))\,dt.$$

where

$$\Gamma(t) = \left(u_n''(t) - \cos t - \frac{1}{2}t^2 + \int_0^t (u_n(r) + u_n''(r))\,dr\right).$$

4. Iterative Process

$$\begin{aligned}
u_0(x) &= -1 + x,\\
u_1(x) &= x + \frac{1}{3!}x^3 - \cos x,\\
u_2(x) &= -1 + x + \frac{1}{2!}x^2 - \frac{1}{12}x^4 + \cdots,\\
u_3(x) &= -1 + x + \frac{1}{2!}x^2 - \frac{1}{4!}x^4 + \cdots,\\
u_4(x) &= -1 + x + \frac{1}{2!}x^2 - \frac{1}{4!}x^4 + \cdots,\\
u_5(x) &= -1 + x + \frac{1}{2!}x^2 - \frac{1}{4!}x^4 + \frac{1}{6!}x^6 - \frac{1}{8!}x^8 + \cdots,
\end{aligned}$$

This gives

$$u_n(x) = x - \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \cdots\right)$$

The exact solution is therefore given by

$$u(x) = x - \cos x.$$

Chapter 3

The Variational Iteration Method for Fredholm and Mixed Volterra-Fredholm Integro-Differential Equations

The method generates quickly converging iterative approximations to the exact solution, provided that a closed-form solution exists.
The standard ith order Fredholm integro-differential equation is of the form

$$\frac{d^i u}{dx^i} = u^{(i)}(x) = f(x) + \int_a^b K(x,t)u(t)dt,$$

where $u(0), u'(0), \cdots, u^{(i-1)}(0)$ are the intial conditions,

Correction functional:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi) \left(u_n^{(i)}(\xi) - f(\xi) - \int_a^b K(\xi, r)u_n(r)dr \right) d\xi.$$

and The Lagrange multiplier used in this method is given by

$$\lambda(\xi) = (-1)^{(i)} \frac{(\xi - x)^{(i-1)}}{(i-1)!}$$

using the given inital values $u(0), u'(0), \cdots$ for the selective $u_0(x)$.
The solution is obtained by taking the limit of the successive approximations, expressed as

$$u(x) = \lim_{n\to\infty} u_n(x)$$

This limit represents the exact solution as the number of iterations approaches infinity, provided that the approximations converge.

Example 14. *Use the variational iteration method to solve the Fredholm integro-differential equation*

$$u'(x) = -1 + \cos x + \int_0^{\frac{\pi}{2}} tu(t)dt, \quad u(0) = 0.$$

solution:

The correction functional for this equation is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(\xi) + 1 - \cos\xi - \int_0^{\frac{\pi}{2}} tu(t)dt \right) d\xi.$$

for the first order integro-differential equations, the Lagrange multiplier is defined as $\lambda(\xi) = -1$

Using the initial condition to select $u_0(x) = u(0) = 0$*, substituting this selection into the correction functional yields the following approximations:*

$$\begin{aligned}
u_0(x) &= 0, \\
u_1(x) &= u_0(x) - \int_0^x \left(u_0'(\xi) + 1 - \cos\xi - \int_0^{\frac{\pi}{2}} tu_0(t)dt \right) d\xi = \sin x - x, \\
u_2(x) &= u_1(x) - \int_0^x \left(u_1'(\xi) + 1 - \cos\xi - \int_0^{\frac{\pi}{2}} tu_1(t)dt \right) d\xi = (\sin x - x) + \left(x - \frac{\pi^3}{24}x \right), \\
u_3(x) &= u_2(x) - \int_0^x \left(u_2'(\xi) + 1 - \cos\xi - \int_0^{\frac{\pi}{2}} tu_2(t)dt \right) d\xi \\
&= (\sin x - x) + \left(x - \frac{\pi^3}{24}x \right) + \left(\frac{\pi^3}{24}x - \frac{\pi^6}{576}x \right), \\
&\vdots \\
u_{n+1} &= (\sin x - x) + \left(x - \frac{\pi^3}{24}x \right) + \left(\frac{\pi^3}{24}x - \frac{\pi^6}{576}x \right) + \left(\frac{\pi^6}{576}x + \cdots \right) + \cdots, n \geq 0.
\end{aligned}$$

The VIM admits the use of:

$$u(x) = \lim_{n\to\infty} u_n(x) = \sin x$$

Example 15. *Use the variational iteration method to solve the Fredholm integro-differential equation*

$$u''(x) = 2 - \cos x + \int_0^{\pi} tu(t)dt, \quad u(0) = 1, \quad u'(0) = 0$$

solution:

The correction functional for this equation is given by

$$u_{n+1}(x) = u_n(x) + \int_0^x (\xi - x)\left(u_n''(\xi) - 2 + \cos\xi - \int_0^{\pi} tu(t)dt\right)d\xi.$$

for the second-order integro-differential equations, the Lagrange multiplier is defined as $\lambda(\xi) = (\xi - x)$

Using the initial condition to select $u_0(x) = u(0) + xu'(0) = 1$*, substituting this selection into the correction functional yields the following approximations:*

$$\begin{aligned}
u_0(x) &= 1,\\
u_1(x) &= u_0(x) + \int_0^x (\xi - x)\left(u_0''(\xi) - 2 + \cos\xi - \int_0^{\pi} tu_0(t)dt\right)d\xi = \cos x + x^2\left(1 + \frac{\pi^2}{4}\right),\\
u_2(x) &= u_1(x) + \int_0^x (\xi - x)\left(u_1''(\xi) - 2 + \cos\xi - \int_0^{\pi} tu_1(t)dt\right)d\xi\\
&= \cos x + x^2\left(1 + \frac{\pi^2}{4}\right) - x^2\left(1 + \frac{\pi^2}{4}\right) + x^2\left(\frac{\pi^4}{8} + \frac{\pi^6}{32}\right),\\
u_3(x) &= u_2(x) + \int_0^x (\xi - x)\left(u_2''(\xi) - 2 + \cos\xi - \int_0^{\pi} tu_2(t)dt\right)d\xi\\
&= \cos x + x^2\left(1 + \frac{\pi^2}{4}\right) - x^2\left(1 + \frac{\pi^2}{4}\right) + x^2\left(\frac{\pi^4}{8} + \frac{\pi^6}{32}\right) - x^2\left(\frac{\pi^4}{8} + \frac{\pi^6}{32}\right) + \cdots,
\end{aligned}$$

and so on.The VIM admits the use of:

$$u(x) = \lim_{n\to\infty} u_n(x) = \cos x$$

Example 16. *Use the Variational Iteration Method to solve the Fredholm integro-differential equation:*

$$u'''(x) = e^x - 1 + \int_0^1 tu(t)\,dt, \quad u(0) = 1, \quad u\prime(0) = 1, \quad u\prime\prime(0) = 1.$$

solutions:

The correction functional for this equation is given by:

$$u_{n+1}(x) = u_n(x) - \frac{1}{2}\int_0^x (\xi - x)^2 \left(u_n'''(\xi) - e^\xi + 1 - \int_0^1 tu(t)\,dt \right) d\xi.$$

For the third-order integro-differential equations, the Lagrange multiplier is defined as:

$$\lambda(\xi) = \frac{-(\xi - x)^2}{2!}.$$

Using the initial conditions to select

$$u_0(x) = u(0) + xu'(0) + \frac{x^2}{2}u''(0) = 1 + x + \frac{x^2}{2},$$

substituting this selection into the correction functional yields the following approximations:

$$\begin{aligned}
u_0(x) &= 1 + x + \frac{x^2}{2}, \\
u_1(x) &= u_0(x) - \frac{1}{2}\int_0^x (\xi - x)^2 \left(u_0'''(\xi) - e^\xi + 1 - \int_0^1 tu_0(t)\,dt \right) d\xi = e^x - \frac{x^3}{144} \\
u_2(x) &= u_1(x) - \frac{1}{2}\int_0^x (\xi - x)^2 \left(u_1'''(\xi) - e^\xi + 1 - \int_0^1 tu_1(t)\,dt \right) d\xi \\
&= \left(e^x - \frac{x^3}{144} \right) + \left(\frac{x^3}{144} - \frac{29x^3}{4320} \right) \\
u_3(x) &= u_2(x) - \frac{1}{2}\int_0^x (\xi - x)^2 \left(u_2'''(\xi) - e^\xi + 1 - \int_0^1 tu_2(t)\,dt \right) d\xi \\
&= \left(e^x - \frac{x^3}{144} \right) + \left(\frac{x^3}{144} - \frac{29x^3}{4320} \right) \left(\frac{29x^3}{4320} + \cdots \right)
\end{aligned}$$

The process continues iteratively, yielding higher-order approximations.

The VIM admits the use of:

$$u(x) = \lim_{n \to \infty} u_n(x) = e^x.$$

3.1 VIT for Volterra-Fredholm Integro-Differential Equations

The standard i-th order Volterra-Fredholm integro-differential equation is of the form:

$$u^{(i)}(x) = f(x) + \int_0^x K_1(x,t)u(t)\,dt + \int_a^b K_2(x,t)u(t)\,dt,$$

where $u^{(i)}(x) = \frac{d^i u}{dx^i}$, and the initial conditions are $u(0), u'(0), \ldots, u^{(i-1)}(0)$.

The right-hand side contains two disjoint integrals. Regarding the kernel $K_2(x,t)$, we discuss two cases:

Case 1: Separable Kernel

If $K_2(x,t)$ is separable, i.e., $K_2(x,t) = g(x)h(t)$, the second integral on the right-hand side becomes:

$$\int_a^b K_2(x,t)u(t)\,dt = \alpha g(x),$$

where $\alpha = \int_a^b h(t)u(t)\,dt$.

The corresponding correction functional for this case is:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\left[u_n^{(i)}(\xi) - f(\xi) - \int_0^t K_1(\xi,r)\tilde{u}_n(r)\,dr - \alpha g(\xi)\right] d\xi.$$

Case 2: Difference Kernel

Alternatively, if $K_2(x,t)$ depends on the difference $x - t$, we define $K_2(x,t) = K_2(x-t) = g(x) - h(t)$. Consequently, the second integral becomes:

$$\int_a^b K_2(x,t)u(t)\,dt = \beta g(x) - \alpha,$$

where $\beta = \int_a^b u(t)\,dt$ and $\alpha = \int_a^b h(t)u(t)\,dt$.

The corresponding correction functional for this case is:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\left[u_n^{(i)}(\xi) - f(\xi) - \int_0^t K_1(\xi,r)\tilde{u}_n(r)\,dr - \beta g(\xi) + \alpha\right] d\xi.$$

Example 17. *Solve the following Volterra-Fredholm integro-differential equation using the variational iteration method:*

$$u'(x) = 1 + \int_0^x (x-t)u(t)\,dt + \int_0^1 xtu(t)\,dt, \quad u(0) = 1.$$

We identify Fredholm kernel as follows:

$$K_2(x,t) = g(x)h(t).$$

In this case, we have $K_2(x,t) = xt$, which is separable. Therefore, we can express the integral as:

$$\int_0^1 xtu(t)\,dt = x\int_0^1 tu(t)\,dt = x\alpha,$$

where

$$\alpha = \int_0^1 tu(t)\,dt.$$

The correction functional for this equation is given by:

$$u_{n+1}(x) = u_n(x) - \int_0^x \left[u_n'(t) - 1 - \int_0^t (t-r)u_n(r)\,dr - \alpha t\right] dt,$$

where we used $\lambda = -1$ for the first-order integro-differential equation.

$$u_0(x) = u(0) = 1$$

by Taylor expansion

Iterative Solutions

$$\begin{aligned}
u_0(x) &= 1,\\
u_1(x) &= u_0(x) - \int_0^x \left[u_0'(t) - 1 - \int_0^t (t-r)u_0(r)\,dr - \alpha t\right] dt\\
&= 1 + x + \frac{1}{2}\alpha x^2 + \frac{1}{3!}x^3,\\
u_2(x) &= u_1(x) - \int_0^x \left[u_1'(t) - 1 - \int_0^t (t-r)u_1(r)\,dr - \alpha t\right] dt\\
&= 1 + x + \frac{1}{2}\alpha x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}\alpha x^5 + \frac{1}{6!}x^6,\\
u_3(x) &= u_2(x) - \int_0^x \left[u_2'(t) - 1 - \int_0^t (t-r)u_2(r)\,dr - \alpha t\right] dt\\
&= 1 + x + \frac{1}{2}\alpha x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}\alpha x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 + \frac{1}{8!}\alpha x^8 + \frac{1}{9!}x^9,\\
u_4(x) &= u_3(x) - \int_0^x \left[u_3'(t) - 1 - \int_0^t (t-r)u_3(r)\,dr - \alpha t\right] dt\\
&= 1 + x + \frac{1}{2}\alpha x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}\alpha x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 + \frac{1}{8!}\alpha x^8 + \frac{1}{9!}x^9 + \frac{1}{10!}x^{10} + \frac{1}{11!}\alpha x^{11}
\end{aligned}$$

To determine α, we substitute $u_4(x)$ into the expression for α:

$$\alpha = \int_0^1 tu(t)\,dt.$$

Finding that:

$$\alpha = 1.$$

Substituting $\alpha = 1$ into $u_4(x)$, and using

$$u(x) = \lim_{n\to\infty} u_n(x),$$

we find that the exact solution is

$$u(x) = e^x,$$

obtained upon using the Taylor series for e^x.

Example 18. *Solve the following Volterra-Fredholm integro-differential equation by using the variational iteration method:*

$$u'(x) = 9 - 5x - x^2 - x^3 + \int_0^x (x-t)u(t)\,dt + \int_0^1 (x-t)u(t)\,dt, \quad u(0) = 2,$$

which can be written as:

$$u'(x) = 9 - 5x - x^2 - x^3 + \int_0^x (x-t)u(t)\,dt + \alpha x - \beta, \quad u(0) = 2,$$

where

$$\alpha = \int_0^1 u(t)\,dt, \quad \beta = \int_0^1 tu(t)\,dt.$$

The correction functional for the Volterra-Fredholm integro-differential equation is given by:

$$u_{n+1}(x) = u_n(x) - \int_0^x \left[u_n'(t) - 9 + 5t + t^2 + t^3 - \int_0^t (t-r)u_n(r)\,dr - \alpha t + \beta \right] dt,$$

where we used $\lambda = -1$ for first-order integro-differential equations and the initial guess $u_0(x) = u(0) = 2$. Using the Taylor expansion, the solution is built iteratively.

$$u_0(x) = 2,$$

$$u_1(x) = u_0(x) - \int_0^x \left(u_0'(t) - 9 + 5t + t^2 + t^3 - \int_0^t (t-r)u_0(r)dr - \alpha t + \beta \right) dt$$
$$= 2 + (9-\beta)x - \frac{5-\alpha}{2}x^2 - \frac{1}{4}x^4,$$
$$u_2(x) = u_1(x) - \int_0^x \left(u_1'(t) - 9 + 5t + t^2 + t^3 - \int_0^t (t-r)u_1(r)dr - \alpha t + \beta \right) dt$$
$$= 2 + (9-\beta)x - \frac{5-\alpha}{2!}x^2 + \frac{3-\beta}{4!}x^4 - \frac{5-\alpha}{5!}x^5 - \frac{1}{840}x^7,$$
$$u_3(x) = u_2(x) - \int_0^x \left(u_2'(t) - 9 + 5t + t^2 + t^3 - \int_0^t (t-r)u_2(r)dr - \alpha t + \beta \right) dt,$$
$$= 2 + (9-\beta)x - \frac{5-\alpha}{2!}x^2 + \frac{3-\beta}{4!}x^4 - \frac{5-\alpha}{5!}x^5 - \frac{3-\beta}{7!}x^7.$$

$$-\frac{5-\alpha}{5!}x^5 + \frac{3-\beta}{7!}x^7 - \frac{5-\alpha}{8!}x^8 + \dots$$

To determine α and β, we substitute $u_3(x)$ into the equations:

$$\alpha = \int_0^1 u(t)\,dt, \quad \beta = \int_0^1 tu(t)\,dt,$$

and solve the resulting equations to find that

$$\alpha = 5, \quad \beta = 3.$$

This, in turn, gives the exact solution

$$u(x) = 2 + 6x,$$

Example 19. *Solve the following Volterra-Fredholm integro-differential equation by using the variational iteration method:*

$$u''(x) = -8 + 6x - 3x^2 + x^3 + \int_0^x u(t)\,dt + \int_{-1}^1 (1-2xt)u(t)\,dt, \quad u(0) = 2, \quad u'(0) = 6,$$

that can be written as

$$u''(x) = -8 + 6x - 3x^2 + x^3 + \int_0^x u(t)\,dt + \alpha - \beta x,$$

where

$$\alpha = \int_0^1 u(t)\,dt, \quad \beta = \int_0^1 2tu(t)\,dt.$$

The correction functional for the Volterra-Fredholm integro-differential equation is given by

$$u_{n+1}(x) = u_n(x) + \int_0^x (t-x)\left[u_n''(t) + 8 - 6t + 3t^2 - t^3 - \int_0^t u_n(r)\,dr - \alpha + \beta t\right] dt,$$

where we used $\lambda = (t-x)$ *for second-order integro-differential equations.*

We can select

$$u_0(x) = u(0) + u'(0)x = 2 + 6x$$

to determine the successive approximations:

$$\begin{aligned}
u_0(x) &= 2 + 6x,\\
u_1(x) &= 2 + 6x + \frac{1}{2}(\alpha - 4)x^2 + \left(\frac{4}{3} - \frac{1}{6}\beta\right)x^3 + \frac{1}{20}x^5,\\
u_2(x) &= 2 + 6x + \frac{1}{2}(\alpha - 4)x^2 + \left(\frac{4}{3} - \frac{1}{6}\beta\right)x^3\\
&\quad + \frac{1}{120}\alpha x^5 - \frac{1}{60}x^5 + \left(\frac{1}{90} - \frac{1}{720}\beta\right)x^6 + \frac{1}{6720}x^8,\\
u_3(x) &= 2 + 6x + \frac{1}{2}(\alpha - 4)x^2 + \left(\frac{4}{3} - \frac{1}{6}\beta\right)x^3\\
&\quad + \frac{1}{120}\alpha x^5 - \frac{1}{60}x^5 + \left(\frac{1}{90} - \frac{1}{720}\beta\right)x^6\\
&\quad + \frac{1}{40320}\alpha x^8 - \frac{1}{20160}x^8 + \left(\frac{1}{45360} - \frac{1}{362880}\beta\right)x^9 + \cdots
\end{aligned}$$

And so on. To determine α *and* β*, we substitute* $u_3(x)$ *into*

$$\alpha = \int_0^1 u(t)\,dt, \quad \beta = \int_0^1 2tu(t)\,dt,$$

and solve the resulting equations to find that

$$\alpha = 2, \quad \beta = 8.$$

This in turn gives the exact solution

$$u(x) = 2 + 6x - 3x^2.$$

Example 20. *Solve the following Volterra-Fredholm integro-differential equation by using the variational iteration method*

$$u^{(3)}(x) = -\frac{1}{2}x^2 + \int_0^x u(t)\,dt + \int_{-\pi}^{\pi} xu(t)\,dt, \quad u(0) = u'(0) = -u''(0) = 1,$$

that can be written as

$$u^{(3)}(x) = -\frac{1}{2}x^2 + \int_0^x u(t)\,dt + \alpha x, \quad u(0) = u'(0) = -u''(0) = 1,$$

where

$$\alpha = \int_{-\pi}^{\pi} u(t)\,dt.$$

The correction functional for the Volterra-Fredholm integro-differential equation is given by

$$u_{n+1}(x) = u_n(x) - \frac{1}{2}\int_0^x (t-x)^2 \left(u_n^{(3)}(t) + \frac{1}{2}t^2 - \int_0^t u_n(r)\,dr - \alpha t\right) dt,$$

where we used

$$\lambda = -\frac{1}{2}(t-x)^2,$$

for third-order integro-differential equations. We can use the initial conditions to select

$$u_0(x) = u(0) + u'(0)x + \frac{1}{2}u''(0)x^2 = 1 + x - \frac{1}{2}x^2.$$

Using this selection into the correction functional gives the following successive approximations:

$$u_0(x) = 1 + x - \frac{1}{2}x^2,$$

$$u_1(x) = 1 + x - \frac{1}{2}x^2 + \frac{1}{4!}(1+\alpha)x^4 - \frac{1}{6!}x^6,$$

$$u_2(x) = 1 + x - \frac{1}{2}x^2 + \frac{1}{4!}(1+\alpha)x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}(1+\alpha)x^8 - \frac{1}{10!}x^{10},$$

$$u_3(x) = 1+x-\frac{1}{2}x^2+\frac{1}{4!}(1+\alpha)x^4-\frac{1}{6!}x^6+\frac{1}{8!}(1+\alpha)x^8-\frac{1}{10!}x^{10}+\frac{1}{12!}(1+\alpha)x^{12}-\frac{1}{14!}x^{14},$$

where other approximations are obtained up to $u_8(x)$, but not listed. To determine α, we substitute $u_8(x)$ into $\alpha = \int_{-\pi}^{\pi} u(t)\,dt$ and solve the resulting equations to find that

$$\alpha = 0.$$

This in turn gives the series solution

$$u(x) = x + \left(1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \frac{1}{10!}x^{10} + \cdots\right),$$

that converges to the exact solution

$$u(x) = x + \cos x.$$

3.2 System of Integral Equations using The Variational Iteration Method

System of Volterra Integro-Differential Equations (Second Kind):

$$u^{(i)}(x) = f_1(x) + \int_0^x (K_1(x,t)u(t) + K_1(x,t)v(t) + \ldots)\, dt,$$

$$v^{(i)}(x) = f_2(x) + \int_0^x (K_2(x,t)u(t) + K_2(x,t)v(t) + \ldots)\, dt.$$

Key Components:

- $u^{(i)}(x)$ and $v^{(i)}(x)$ are the unknown functions.
- $f_1(x)$ and $f_2(x)$ are the given functions.
- $K_1(x,t)$ and $K_2(x,t)$ are the kernel functions.

The Correction Functionals for the Volterra System of Integro-Differential Equations:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t) \left(u_n^{(i)}(t) - f_1(t) - \int_0^t K(t,r)\tilde{u}_n(r)\, dr \right) dt,$$

$$v_{n+1}(x) = v_n(x) + \int_0^x \lambda(t) \left(v_n^{(i)}(t) - f_2(t) - \int_0^t K(t,r)\tilde{v}_n(r)\, dr \right) dt.$$

As presented before, the Lagrange multiplier λ is obtained through integration by parts and is defined by the formula:

$$\lambda = \frac{(-1)^n}{(n-1)!}(\xi - x)^{n-1},$$

The initial approximations $u_0(x)$ and $v_0(x)$ can be derived using the Taylor series expansion:

$$u_0(x) = u(0) + u'(0)x + \frac{u''(0)}{2!}x^2 + \cdots,$$

where the derivatives are evaluated at $x = 0$. This provides a starting point for the iterative process.

Once λ is determined, an iteration formula, without restricted variation, is used to find the successive approximations $u_{n+1}(x)$ and $v_{n+1}(x)$ for the solutions $u(x)$ and $v(x)$.

Finally, the solutions can be expressed as:

$$u(x) = \lim_{n\to\infty} u_n(x), \quad v(x) = \lim_{n\to\infty} v_n(x).$$

The Variational Iteration Method (VIM) will be illustrated through the study of the following examples.

Example 21. *Use the VIM to solve the system of Volterra integro-differential equations:*

$$u'(x) = 1 + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \int_0^x ((x-t)u(t) + (x-t+1)v(t))\,dt,$$

$$v'(x) = -1 - 3x - \frac{3}{2}x^2 - \frac{1}{3}x^3 + \int_0^x ((x-t+1)u(t) + (x-t)v(t))\,dt,$$

with initial conditions:

$$u(0) = 1, \quad v(0) = 1.$$

The correction functionals for this system are given by:

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(t) - 1 - t + \frac{1}{2}t^2 - \frac{1}{3}t^3 - I_1(t)\right) dt,$$

$$v_{n+1}(x) = v_n(x) - \int_0^x \left(v_n'(t) + 1 + 3t + \frac{3}{2}t^2 + \frac{1}{3}t^3 - I_2(t)\right) dt,$$

where

$$I_1(t) = \int_0^t ((t-r)u_n(r) + (t-r+1)v_n(r))\,dr,$$

$$I_2(t) = \int_0^t ((t-r+1)u_n(r) + (t-r)v_n(r))\,dr,$$

and

$$\lambda = -1 \quad \textit{for the first-order integro-differential equation.}$$

The initial approximations are given by:

$$u_0(x) = u(0) = 1,$$

$$v_0(x) = v(0) = 1$$

via Taylor series expansion.

Using this selection into the correction functionals gives the following successive approximations.

$$\begin{cases} u_0(x) = 1, \\ v_0(x) = 1, \end{cases}$$

$$\begin{cases} u_1(x) = 1 + x + x^2 + \frac{1}{6}x^3 + \frac{1}{12}x^4, \\ v_1(x) = 1 - x - x^2 - \frac{1}{6}x^3 - \frac{1}{12}x^4, \end{cases}$$

$$\begin{cases} u_2(x) = 1 + x + x^2 + \left(\frac{1}{6}x^3 - \frac{1}{6}x^3\right) + \left(\frac{1}{12}x^4 - \frac{1}{12}x^4\right) - \frac{1}{120}x^5 - \frac{1}{360}x^6, \\ v_2(x) = 1 - x - x^2 + \left(\frac{1}{6}x^3 - \frac{1}{6}x^3\right) + \left(\frac{1}{12}x^4 - \frac{1}{12}x^4\right) + \frac{1}{120}x^5 + \frac{1}{360}x^6, \end{cases}$$

$$\begin{cases} u_3(x) = 1 + x + x^2 + \left(\frac{1}{120}x^5 - \frac{1}{120}x^5\right) + \left(\frac{1}{360}x^6 - \frac{1}{360}x^6\right) + \cdots, \\ v_3(x) = 1 - x - x^2 + \left(\frac{1}{120}x^5 - \frac{1}{120}x^5\right) + \left(\frac{1}{360}x^6 - \frac{1}{360}x^6\right) + \cdots, \end{cases}$$

and so on. By canceling the noise terms, the exact solutions are given by

$$(u(x), v(x)) = \left(1 + x + x^2, 1 - x - x^2\right).$$

Example 22. *Use the VIM to solve the system of Volterra integro-differential equations:*

$$u'(x) = 1 - x^2 + e^x + \int_0^x (u(t) + v(t))\, dt,$$

$$v'(x) = 3 - 3e^x + \int_0^x (u(t) - v(t))\, dt.$$

$$u(0) = 1, \quad v(0) = -1,$$

The correction functionals for this system are given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(t) - 1 + t^2 - e^t - \int_0^t (u_n(r) + v_n(r))\, dr \right) dt,$$

$$v_{n+1}(x) = v_n(x) - \int_0^x \left(v_n'(t) - 3 + 3e^t - \int_0^t (u_n(r) - v_n(r))\, dr \right) dt.$$

where

$$\lambda = -1, \quad u_0(x) = u(0) = 1, \quad v_0(x) = v(0) = -1.$$

$$\begin{cases} u_0(x) = 1 \\ v_0(x) = -1 \end{cases}$$

$$\begin{cases} u_1(x) = u_0(x) - \int_0^x \left(u_0'(t) + 1 + t^2 - e^t - \int_0^t (u_0(r) + v_0(r))\, dr \right)\, dt \\ = 1 - \frac{1}{3}x^3 \\ v_1(x) = v_0(x) - \int_0^x \left(v_0'(t) - 3 + 3e^t - \int_0^t (u_0(r) - v_0(r))\, dr \right)\, dt \\ = 2 + 3x - 3e^x + x^2 \end{cases}$$

$$\begin{cases} u_2(x) = u_1(x) - \int_0^x \left(u_1'(t) + 1 + t^2 - e^t - \int_0^t (u_1(r) + v_1(r))\, dr \right)\, dt \\ = x + \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots\right) \\ v_2(x) = v_1(x) - \int_0^x \left(v_1'(t) - 3 + 3e^t - \int_0^t (u_1(r) - v_1(r))\, dr \right)\, dt \\ = x - \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots\right) \end{cases}$$

$$\begin{cases} u_3(x) = x + \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots\right) \\ v_3(x) = x - \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \cdots\right) \end{cases}$$

and so on. The exact solutions are therefore given by

$$(u(x), v(x)) = (x + e^x, x - e^x).$$

Example 23. *Use the VIM to solve the system of Volterra integro-differential equations*

$$u''(x) = -1 - x^2 - \sin x + \int_0^x (u(t) + v(t))\, dt, \quad u(0) = 1,\ u'(0) = 1$$

$$v''(x) = 1 - 2\sin x - \cos x + \int_0^x (u(t) - v(t))\, dt, \quad v(0) = 0,\ v'(0) = 2$$

The correction functionals for this system are given by:

$$u_{n+1}(x) = u_n(x) + \int_0^x (t - x) \left(u_n''(t) + 1 + t^2 + \sin t - \int_0^t (u_n(r) + v_n(r))\, dr \right) dt,$$
$$v_{n+1}(x) = v_n(x) + \int_0^x (t - x) \left(v_n''(t) - 1 + 2\sin t + \cos t - \int_0^t (u_n(r) - v_n(r))\, dr \right) dt,$$

where we used $\lambda = t - x$ *for the second-order integro-differential equation.*

$$\begin{cases} u_0(x) = 1 + x, \\ v_0(x) = 2x, \end{cases}$$

$$\begin{cases} u_1(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \sin x, \\ v_1(x) = \cos x + 2\sin x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 - \frac{1}{4!}x^4 - 1, \end{cases}$$

$$\begin{cases} u_2(x) = x + \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right), \\ v_2(x) = x + \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \cdots\right), \end{cases}$$

and so on. The exact solutions are therefore given by

$(u(x), v(x)) = (x + \cos x, x + \sin x)$.

Example 24. *Using the Variational Iteration Method to Solve the System of Volterra Integro-Differential Equations*

We consider the following system:

$$u'(x) = 2 + e^x - 3e^{2x} + e^{3x} + \int_0^x (6v(t) - 3w(t))\, dt, \quad u(0) = 1,$$
$$v'(x) = e^x + 2e^{2x} - e^{3x} + \int_0^x (3w(t) - u(t))\, dt, \quad v(0) = 1,$$
$$w'(x) = -e^x + e^{2x} + 3e^{3x} + \int_0^x (u(t) - 2v(t))\, dt, \quad w(0) = 1.$$

The correction functionals for this system are given by:

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(t) - 2 - e^t + 3e^{2t} - e^{3t} - \int_0^t (6v_n(r) - 3w_n(r))\, dr \right) dt,$$
$$v_{n+1}(x) = v_n(x) - \int_0^x \left(v_n'(t) - e^t - 2e^{2t} + e^{3t} - \int_0^t (3w_n(r) - u_n(r))\, dr \right) dt,$$
$$w_{n+1}(x) = w_n(x) - \int_0^x \left(w_n'(t) + e^t - e^{2t} - 3e^{3t} - \int_0^t (u_n(r) - 2v_n(r))\, dr \right) dt.$$

The initial conditions can be used to select the zeroth approximations as

$$\begin{cases} u_0(x) = 1 \\ v_0(x) = 1 \\ w_0(x) = 1 \end{cases}$$

Using this selection in the correction functionals gives the following successive approximations:

$$\begin{cases} u_1(x) = \frac{7}{6} + 2x + \frac{3}{2}x^2 + e^x - \frac{3}{2}e^{2x} + \frac{1}{3}e^{3x} \\ v_1(x) = -\frac{2}{3} + x^2 + e^x + e^{2x} - \frac{1}{3}e^{3x} \\ w_1(x) = \frac{1}{2} - \frac{1}{2}x^2 - e^x + \frac{1}{2}e^{2x} + e^{3x} \end{cases}$$

$$\begin{cases} u_2(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots \\ v_2(x) = 1 + 2x + \frac{1}{2!}(2x)^2 + \frac{1}{3!}(2x)^3 + \frac{1}{4!}(2x)^4 + \cdots \\ w_2(x) = 1 + 3x + \frac{1}{2!}(3x)^2 + \frac{1}{3!}(3x)^3 + \frac{1}{4!}(3x)^4 + \cdots \end{cases}$$

and so on. The exact solutions are therefore given by

$$(u(x), v(x), w(x)) = (e^x, e^{2x}, e^{3x}).$$

3.3 Variational Iteration Method for Solving Coupled Fredholm Integro-Differential Equations

The Variational Iteration Method (VIM) is a versatile tool for solving linear and nonlinear equations in a unified way, without requiring specific restrictions or transformations. By constructing correction functionals iteratively, VIM can be applied directly to integro-differential equations such as:

$$u^{(i)}(x) = f(x) + \int_a^b \Big(K_1(x,t)u(t) + \tilde{K}_1(x,t)v(t)\Big)\,dt,$$

$$v^{(i)}(x) = g(x) + \int_a^b \Big(K_2(x,t)u(t) + \tilde{K}_2(x,t)v(t)\Big)\,dt,$$

where $K_1(x,t), K_2(x,t), \tilde{K}_1(x,t), \tilde{K}_2(x,t)$represent kernel functions of the integro-differential equation, and$f(x), g(x)$are source terms. In each iteration i ,corrections are made to approximate the functions $u(x), v(x)$converging to an accurate solution.

The correction functionals for the system integro-differential equations are given by

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t)\left(u_n^{(i)}(t) - f(t) - \int_a^b \Big(K_1(t,s)\tilde{u}_n(s)ds + \tilde{K}_1(t,s)\tilde{v}_n(s)ds\Big)\right)dt,$$

$$v_{n+1}(x) = v_n(x) + \int_0^x \lambda(t)\left(v_n^{(i)}(t) - g(t) - \int_a^b \Big(K_2(t,s)\tilde{u}_n(s)ds + \tilde{K}_2(t,s)\tilde{v}_n(s)ds\Big)\right)dt,$$

we first determine the Lagrange multiplier, denoted by λ, which is identified optimally. Once λ is determined, an iteration formula, without restricted variation, is applied to obtain the successive approximations $\{u_{n+1}(x), v_{n+1}(x)\}_{n\geq 0}$ for the solutions $u(x)$ and $v(x)$. The zeroth approximations, $u_0(x)$ and $v_0(x)$, can be any selected functions. However, it is advantageous to use the initial conditions to determine these zeroth approximations, ensuring they align with the problem's boundary or initial requirements.

Thus, the VIM iteratively refines these approximations, providing an efficient pathway to the solutions of integro-differential equations.

This process will be further illustrated through the following examples.

Example 25. *Solve the system of Fredholm integro-differential equations by using the variational iteration method*

$$u'(x) = -2 - \sin x + \int_0^{\pi} (u(t)) + v(t))dt, \quad u(0) = 1,$$

$$v'(x) = 2 - \pi + \cos x + \int_0^{\pi} (tu(t) + tv(t))dt, \quad v(0) = 0,$$

The correction functionals for this system are given by

$$\begin{aligned} u_{n+1}(x) &= u_n(x) - \int_0^x \left(u_n'(t) + 2 + \sin t - \int_0^{\pi} (u_n(s) + v_n(s))ds \right) dt, \\ v_{n+1}(x) &= v_n(x) - \int_0^x \left(v_n'(t) + \pi - 2 - \cos t - \int_0^{\pi} (su_n(s) + sv_n(s))ds \right) dt, \end{aligned}$$

Selecting $u_0(x) = 1$ *and* $v_0(x) = 0$, *the correction functionals gives the follwing successive approximations*

$$\begin{aligned} u_0(x) &= 1, \\ v_0(x) &= 0, \\ u_1(x) &= u_0(x) - \int_0^x \left(u_0'(t) + 2 + \sin t - \int_0^{\pi} (u_0(s) + v_0(s))ds \right) dt, \\ &= \cos x - 2x + \pi x, \\ v_1(x) &= v_0(x) - \int_0^{\pi} \left(v_0'(t) + \pi - 2 - \cos t - \int_0^{\pi} (su_0(s) + sv_0(s))dr \right) dt, \\ &= \sin x + 2x - \pi x + \frac{\pi^2}{2} x, \\ u_2(x) &= u_1(x) - \int_0^x \left(u_1'(t) + 2 + \sin t - \int_0^{\pi} (u_1(s) + v_1(s))ds \right) dt, \\ &= \cos x + \frac{\pi^4}{4} x, \\ v_2(x) &= v_1(x) - \int_0^{\pi} \left(v_1'(t) + \pi - 2 - \cos t - \int_0^{\pi} (su_1(s) + sv_1(s))ds \right) dt, \\ &= \sin x + \frac{\pi^5}{6} x, \end{aligned}$$

and so on. it is obvious that noise terms appear in each component. By canceling the noise terms, the exact solutions are therefore given by

$$(u(x), v(x)) = (\cos x, \sin x).$$

Example 26. *Solve the system of Fredholm integro-differential equations by using the variational iteration method*

$$u'(x) = -1 + \cosh x + \int_0^{\ln 2} (u(t) + v(t))\, dt, \quad u(0) = 0,$$
$$v'(x) = 1 - 2\ln 2 + \sinh x + \int_0^{\ln 2} (tu(t) + tv(t))\, dt, \quad v(0) = 1.$$

The correction functionals for this system are given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(t) + 1 - \cosh t - \int_0^{\ln 2} (u_n(s) + v_n(s))\, ds \right) dt,$$
$$v_{n+1}(x) = v_n(x) - \int_0^x \left(v_n'(t) + 2\ln 2 - 1 - \sinh t - \int_0^{\ln 2} (su_n(s) + sv_n(s)) \right) dt,$$

We can use the initial conditions to select $u_0(x) = 0$ and $v_0(x) = 1$.
Using this selection into the correction functionals gives the following successive approximations

$$
\begin{aligned}
u_0(x) &= 0,\\
v_0(x) &= 1,\\
u_1(x) &= u_0(x) - \int_0^x \left(u_0'(t) + 1 - \cosh t - \int_0^{\ln 2} (u_0(s) + v_0(s)) ds \right) dt\\
&= \sinh x + x\ln 2 - x\\
v_1(x) &= v_0(x) - \int_0^x \left(v_0'(t) + 2\ln 2 - 1 - \sinh t - \int_0^{\ln 2} (su_0(s) + sv_0(s)) ds \right) dt,\\
&= \cosh x + x - 2x\ln 2 + \frac{(\ln 2)^2}{2} x,\\
u_2(x) &= u_1(x) - \int_0^x \left(u_1'(t) + 1 - \cosh t - \int_0^{\ln 2} (u_1(s) + v_1(s)) ds \right) dt\\
&= \sinh x + \frac{(\ln 2)^4}{4} x\\
v_2(x) &= v_1(x) - \int_0^x \left(v_1'(t) + 2\ln 2 - 1 - \sinh t - \int_0^{\ln 2} (su_1(s) + sv_1(s)) ds \right) dt,\\
&= \cosh x - \frac{(\ln 2)^4}{3} x + \frac{(\ln 2)^5}{6} x,
\end{aligned}
$$

and so on. It is obvious that noise terms appear in each component. By canceling the noise terms, the exact solutions are therefore given by

$$(u(x), v(x)) = (\sinh x, \cosh x).$$

Example 27. *Solve the system of Fredholm integro-differential equations by using the variational iteration method*

$$
\begin{aligned}
u'(x) &= e^x - 6 + \int_0^{\ln 3} (u(t) + v(t))\, dt, \quad u(0) = 1, \\
v'(x) &= 2e^{2x} + 2 + \int_0^{\ln 3} (u(t) - v(t))\, dt, \quad v(0) = 1.
\end{aligned}
$$

The correction functionals for this system are given by

$$
\begin{aligned}
u_{n+1}(x) &= u_n(x) - \int_0^x \left(u_n'(t) - e^t + 6 - \int_0^{\ln 3} (u_n(s) + v_n(s))\, ds \right) dt, \\
v_{n+1}(x) &= v_n(x) - \int_0^x \left(v_n'(t) - 2e^{2t} - 2 - \int_0^{\ln 3} (u_n(s) - v_n(s))\, ds \right) dt.
\end{aligned}
$$

selecting $u_0(x) = 1$ and $v_0(x) = 1$ gives the following approximations

$$
\begin{aligned}
u_0(x) &= 1, \\
v_0(x) &= 1, \\
u_1(x) &= u_0(x) - \int_0^x \left(u_0'(t) - e^t + 6 - \int_0^{\ln 3} (u_0(s) + v_0(s)) ds \right) dt, \\
&= e^x - 6x + 2x \ln 3 \\
v_1(x) &= v_0(x) - \int_0^x \left(v_0'(t) - 2e^{2t} - 2 - \int_0^{\ln 3} (u_0(s) - v_0(s))\, ds \right) dt, \\
&= e^{2x} + 2x, \\
u_2(x) &= u_1(x) - \int_0^x \left(u_1'(t) - e^t + 6 - \int_0^{\ln 3} (u_1(s) + v_1(s)) ds \right) dt, \\
&= e^x - 2x(\ln 3)^2 + x(\ln 3)^3 \\
v_2(x) &= v_1(x) - \int_0^x \left(v_1'(t) - 2e^{2t} - 2 - \int_0^{\ln 3} (u_1(s) - v_1(s))\, ds \right) dt, \\
&= e^{2x} - 4x(\ln 3)^2 + x(\ln 3)^3,
\end{aligned}
$$

and so on. It is obvious that noise terms appear in each component. By canceling the noise terms, the exact solutions are therefore given by

$$
(u(x), v(x)) = (e^x, e^{2x}).
$$

Example 28. *Solve the system of Fredholm integro-differential equations by using the variational iteration method*

$$u''(x) = 2\cos 2x - \frac{3\pi}{4}(2+\pi) + \int_0^x (u(t) + tv(t))dt, \quad u(0) = 1 \quad u'(0) = 0$$
$$v''(x) = -2\cos 2x + \frac{3\pi}{4}(2-\pi) + \int_0^\pi (tu(t) - v(t))dt, \quad v(0) = 2, \quad v'(0) = 0$$

The correction functionals for this system are given by

$$u_{n+1}(x) = u_n(x) + \int_0^x (t-x)\left(u_n''(t) - 2\cos 2t + \frac{3\pi}{4}(2+\pi) - \int_0^\pi (u_n(s) + sv_n(s))ds\right)dt,$$

$$v_{n+1}(x) = v_n(x) + \int_0^x (t-x)\left(v_n''(t) + 2\cos 2t - \frac{3\pi}{4}(2-\pi) - \int_0^\pi (su_n(s) + v_n(s))ds\right)dt,$$

because the equation is of second order we havethe lagrange multiplier $\lambda(t) = (t-x)$
We can use the initial conditions to select
$u_0(x) = u(0) + xu'(0) = 1$ *and* $v_0(x) = v(0) + xv'(0) = 2.$
Using this selection into the correction functionals and proceeding as before we obtain the following successive approximations

$$\begin{aligned}
u_0(x) &= 1,\\
v_0(x) &= 2,\\
u_1(x) &= u_0(x) + \int_0^x (t-x)\left(u_0''(t) - 2\cos 2t + \frac{3\pi}{4}(2+\pi) - \int_0^\pi (u_0(s) + sv_0(s))ds\right)dt,\\
&= 1 + \sin^2 x - \frac{\pi}{8}(2-\pi)x^2,\\
v_1(x) &= v_0(x) + \int_0^x (t-x)\left(v_0''(t) + 2\cos 2t - \frac{3\pi}{4}(2-\pi) - \int_0^\pi (su_0(s) + v_0(s))ds\right)dt\\
&= 1 + \cos^2 x - \frac{\pi}{8}(2+\pi)x^2,\\
u_2(x) &= 1 + \sin^2 x + \left(\frac{\pi}{8}(2-\pi)x^2 - \frac{\pi}{8}(2-\pi)x^2\right) - x^2\left(\frac{3\pi^6 + 2\pi^5 + 4\pi^4}{192}\right),\\
v_2(x) &= 1 + \cos^2 x + \left(\frac{\pi}{8}(2+\pi)x^2 - \frac{\pi}{8}(2+\pi)x^2\right) + x^2\left(\frac{3\pi^6 - 10\pi^5 - 8\pi^4}{192}\right).
\end{aligned}$$

and so on. It is obvious that noise terms appear as explained in the previous examples. By canceling the noise terms, the exact solutions are therefore given by

$$(u(x), v(x)) = (1 + \sin^2 x, 1 + \cos^2 x).$$

Chapter 4

Nonlinear Integro-Differential Equations

4.1 Variational Iteration Method for Solving Nonlinear Equations

The variational iteration method has proven effective for solving linear problems in a straightforward manner. In this chapter, we focus on extending its application to nonlinear problems, examining how it generates rapidly convergent successive approximations to the exact solution when a closed-form solution exists.

The standard ith order nonlinear Fredholm integro-differential equation is of the form

$$u^{(i)}(x) = f(x) + \int_a^b K(x,t)F(u(t))dt,$$

where $F(u(x))$is nonlinear function of $u(x)$.
The initial conditions should be prescribed for the complete determination of the exact solution.
The correction function for the nonlinear integro-differential equation is

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t)\left(u_n^{(i)}(t) - f(t) - \int_a^b K(t,s)F(\tilde{u}_n(s))ds\right)dt.$$

we first determine the Lagrange multiplier, denoted by λ, which is identified optimally. Once λ is determined, an iteration formula, without restricted variation, is applied to obtain the successive approximations $u_{n+1}(x), n \geq 0$ of the soluotion

$u(x)$,The zeroth approximation $u_0(x)$ can be selective function.However, it is advantageous to use the initial conditions to determine these zeroth approximations, ensuring they align with the problem's boundary or initial requirements.
The solution is give by

$$u(x) = \lim_{n\to\infty} u_n(x).$$

The VIM will be illustrated by studying the following examples.

Example 29. *Use the variational iteration method to solve the nonlinear Fredholm integrodifferential equation*

$$u'(x) = \cos x - \frac{\pi}{48}x + \frac{1}{24}\int_0^\pi xu^2(t)dt, \quad u(0) = 0.$$

The correction functional for this equation is given by

$$u_{n+1} = u_n(x) - \int_0^x \left(u_n'(t) - \cos t + \frac{\pi}{24}t - \frac{1}{24}\int_0^\pi tu_n^2(s)ds \right) dt$$

, where we used $\lambda = -1$ *for first-order integro-differential equation.*
We can use the initial condition to select $u_0(x) = u(0) = 0$.
Using this selection into the correction functional gives the following successive approximations.

$$u_0(x) = 0,$$
$$u_1(x) = u_0(x) - \int_0^x \left(u_0'(t) - \cos t + \frac{\pi}{48}t - \frac{1}{24}\int_0^\pi tu_0^2(s)ds \right) dt = \sin x - \frac{\pi}{96}x^2,$$
$$u_2(x) = u_1(x) - \int_0^x \left(u_1'(t) - \cos t + \frac{\pi}{48}t - \frac{1}{24}\int_0^\pi tu_1^2(s)ds \right) dt = \sin x - \frac{4\pi - \pi^3}{2304} - \frac{\pi^7}{2211840},$$
$$u_3(x) = u_2(x) - \int_0^x \left(u_2'(t) - \cos t + \frac{\pi}{48}t - \frac{1}{24}\int_0^\pi tu_2^2(s)ds \right) dt = \sin x - 0.00156723251x^2,$$
$$u_4(x) = u_32(x) - \int_0^x \left(u_3'(t) - \cos t + \frac{\pi}{48}t - \frac{1}{24}\int_0^\pi tu_3^2(s)ds \right) dt = \sin x - 0.00038016125x^2,$$

and so on.this gives the exact solution by

$$u(x) = \sin x.$$

Example 30. *Use the variational iteration method to solve the nonlinear Fredholm integro-differential equation*

$$u''(x) = -\cos x - \frac{3\pi}{128}x + \frac{1}{64}\int_0^{\pi} xu^2(t)dt, \quad u(0) = 2, u'(0) = 0.$$

The correction functional for this equation is given by

$$u_{n+1}(x) = u_n(x) + \int_0^x (t-x)\left(u_n''(t) + \cos t + \frac{3\pi}{128}t - \frac{1}{64}\int_0^{\pi} tu_n^2(s)ds\right)dt.$$

the Lagrange multiplier $\lambda(t) = (t-x)$ *for second-order integro-differential equation. We can use the initial condition to select* $u_0(x) = u(0) + xu'(0) = 2$.
Using this selection into the correction functional gives the following successive approximations

$$\begin{aligned}
u_0(x) &= 2,\\
u_1(x) &= u_0(x) + \int_0^x (t-x)\left(u_0''(t) + \cos t + \frac{3\pi}{128}t - \frac{1}{64}\int_0^{\pi} tu_0^2(s)ds\right)dt\\
&= 1 + \cos x + \frac{5\pi}{768}x^3,\\
u_2(x) &= u_1(x) + \int_0^x (t-x)\left(u_1''(t) + \cos t + \frac{3\pi}{128}t - \frac{1}{64}\int_0^{\pi} tu_1^2(s)ds\right)dt\\
&= 1 + \cos x + 0.00118839931x^3,\\
u_3(x) &= u_2(x) + \int_0^x (t-x)\left(u_2''(t) + \cos t + \frac{3\pi}{128}t - \frac{1}{64}\int_0^{\pi} tu_2^2(s)ds\right)dt\\
&= 1 + \cos x + 0.00004332607x^3
\end{aligned}$$

and so on. The VIM gives the exact solution by

$$u(x) = 1 + \cos x.$$

Example 31. *Use the variational iteration method to solve the nonlinear Fredholm integro-differential equation*

$$u'(x) = \cos x - x\sin x - \frac{\pi(\pi^2+3)}{512}x + \frac{1}{64\pi}\int_0^{\pi} xtu^2(t)dt, \quad u(0) = 0.$$

The correction functional for this equation is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n' - \cos t + t\sin t + \frac{\pi(\pi^2+3)}{512}t - \frac{1}{64\pi}\int_0^{\pi} tsu_n^2(s)ds \right) dt.$$

the Lagrange multiplier $\lambda(t) = -1$ for first-order integro-differential equation. We can use the initial condition to select $u_0(x) = u(0) = 0$

Using this selection into the correction functional gives the following successive approximations

$$\begin{aligned}
u_0(x) &= 0,\\
u_1(x) &= u_0(x) - \int_0^x \left(u_0' - \cos t + t\sin t + \frac{\pi(\pi^2+3)}{512}t - \frac{1}{64\pi}\int_0^{\pi} tsu_0^2(s)ds \right) dt.\\
&= x\cos x - 0.039483x^2\\
u_2(x) &= u_1(x) - \int_0^x \left(u_1' - \cos t + t\sin t + \frac{\pi(\pi^2+3)}{512}t - \frac{1}{64\pi}\int_0^{\pi} tsu_1^2(s)ds \right) dt.\\
&= x\cos x + 0.01017x^2\\
u_3(x) &= u_2(x) - \int_0^x \left(u_2' - \cos t + t\sin t + \frac{\pi(\pi^2+3)}{512}t - \frac{1}{64\pi}\int_0^{\pi} tsu_2^2(s)ds \right) dt.\\
&= x\cos x - 0.0024184x^2
\end{aligned}$$

and so on. Proceeding as before, the exact solution is given by

$$u(x) = x\cos x.$$

4.2 Variational Iteration Method for Solving Systems of Nonlinear Equations

The variational iteration method effectively addresses Fredholm integral and integro-differential equations, including nonlinear cases. It produces rapidly converging successive approximations of the exact solution, if a closed-form solution exists. This method is versatile, handling both linear and nonlinear problems consistently without requiring specific restrictions.

The standard ith order nonlinear Fredholm integro-differential equation is of the form

$$u^{(i)}(x) = f(x) + \int_a^b \Big(K_1(x,t)F_1(u(t)) + \tilde{K}_1(x,t)\tilde{F}_1(v(t))\Big)\,dt,$$

$$v^{(i)}(x) = g(x) + \int_a^b \Big(K_2(x,t)F_2(u(t)) + \tilde{K}_2(x,t)\tilde{F}_2(v(t))\Big)\,dt,$$

The correction function for the nonlinear integro-differential equation are given by

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t)\left(u_n^{(i)}(t) - f(t) - \int_a^b \Big(K_1(t,s)F_1(\tilde{u}_n(s)) + \tilde{K}_1(t,s)\tilde{F}_1(\tilde{v}_n(s))\Big)\,ds\right)dt$$

$$v_{n+1}(x) = v_n(x) + \int_0^x \lambda(t)\left(v_n^{(i)}(t) - g(t) - \int_a^b \Big(K_2(t,s)F_2(\tilde{u}_n(s)) + \tilde{K}_2(t,s)\tilde{F}_2(\tilde{v}_n(s))\Big)\,ds\right)dt$$

we first determine the Lagrange multiplier λ, which is identified optimally.
Once λ is determined, an iteration formula, without restricted variation,
is applied to obtain the successive approximations $u_{n+1}(x), v_{n+1}(x), n \geq 0$ of the soluotion $u(x)$, $v(x)$,The zeroth approximation $u_0(x)$, $v_0(x)$ can be selective function. However, it is advantageous to use the initial conditions to determine these zeroth approximations, ensuring they align with the problem's boundary or initial requirements.
The solution are give by

$$u(x) = \lim_{n\to\infty} u_n(x), \quad v(x) = \lim_{n\to\infty} v_n(x).$$

The VIM will be illustrated by studying the following examples.

Example 32. *Use the VIM to solve the system of nonlinear Fredholm integro-differential equations*

$$u'(x) = \cos x - 5\sin x + \int_0^\pi \cos(x-t)\left(u^2(t)+v^2(t)\right)dt, \quad u(0)=1,$$
$$v'(x) = 3\cos x - \sin x + \int_0^\pi \sin(x-t)\left(u^2(t)+v^2(t)\right)dt, \quad v(0)=1.$$

The correction functionals for this system are given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(t) - \cos t + 5\sin t - \int_0^\pi \cos(t-s)\left(u_n^2(s)+v_n^2(s)\right)ds\right)dt,$$
$$v_{n+1}(x) = v_n(x) - \int_0^x \left(v_n'(t) - 3\cos t + \sin t - \int_0^\pi \sin(t-s)\left(u_n^2(s)+v_n^2(s)\right)ds\right)dt,$$

We can select the zeroth approximations $u_0(x) = u(0) = 1$ *and* $v_0(x) = v(0) = 1$. *Using this selection, the correction functionals give the following successive approximations.*

$$u_0(x) = 1, \qquad v_0(x) = 1,$$
$$\begin{aligned}
u_1(x) &= u_0(x) - \int_0^x \left(u_0'(t) - \cos t + 5\sin t - \int_0^\pi \cos(t-s)\left(u_0^2(s)+v_0^2(s)\right)ds\right)dt,\\
&= u_0(x) - \int_0^x \left(-\cos t + 5\sin t - 2\int_0^\pi \cos(t-s)ds\right)dt,\\
&= \cos x + \sin x\\
v_1(x) &= v_0(x) - \int_0^x \left(v_0'(t) - 3\cos t + \sin t - \int_0^\pi \sin(t-s)\left(u_0^2(s)+v_0^2(s)\right)ds\right)dt,\\
&= v_0(x) - \int_0^x \left(-3\cos t + \sin t - 2\int_0^\pi \sin(t-s)ds\right)dt,\\
&= \cos x - \sin x\\
u_2(x) &= u_1(x) - \int_0^x \left(u_1'(t) - \cos t + 5\sin t - \int_0^\pi \cos(t-s)\left(u_1^2(s)+v_1^2(s)\right)ds\right)dt,\\
&= u_1(x) - \int_0^x \left(4\sin t - 2\int_0^\pi \cos(t-s)ds\right)dt\\
&= \cos x + \sin x\\
v_2(x) &= v_1(x) - \int_0^x \left(v_0'(t) - 3\cos t + \sin t - \int_0^\pi \sin(t-s)\left(u_1^2(s)+v_1^2(s)\right)ds\right)dt,\\
&= v_1(x) - \int_0^x \left(-4\cos t - 2\int_0^\pi \sin(t-s)ds\right)dt,\\
&= \cos x - \sin x
\end{aligned}$$

where we obtained the same approximations for $u_i(x)$ *and* $v_i(x) \forall i \geq 2$. *The exact solutions are therefore given by*

$$(u(x), v(x)) = (\cos x + \sin x, \cos x - \sin x).$$

Example 33. *Use the variational iteration method to solve the system of nonlinear Fredholm integro-differential equations*

$$u'(x) = 2x + \frac{149}{64} + \frac{1}{64}\int_0^1 \left(u^2(t) + v^2(t)\right) dt, \quad u(0) = 1,$$
$$v'(x) = 2x - \frac{67}{64} + \frac{1}{64}\int_0^1 \left(u^2(t) - v^2(t)\right) dt, \quad v(0) = 1.$$

The correction functionals for this system are given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(t) - 2t - \frac{149}{64} - \frac{1}{64}\int_0^1 \left(u_n^2(s) + v_n^2(s)\right) ds\right) dt,$$
$$v_{n+1}(x) = v_n(x) - \int_0^x \left(v_n'(t) - 2t + \frac{67}{64} - \frac{1}{64}\int_0^1 \left(u_n^2(s) - v_n^2(s)\right) ds\right) dt.$$

We can select the zeroth approximations $u_0(x) = u(0) = 1$ and $v_0(x) = v(0) = 1$. Using this selection, the correction functionals give the following successive approximations.

$$\begin{aligned}
u_0(x) &= 0,\\
v_0(x) &= 0,\\
u_1(x) &= u_0(x) - \int_0^x \left(u_0'(t) - 2t - \frac{149}{64} - \frac{1}{64}\int_0^1 \left(u_0^2(s) + v_0^2(s)\right) ds\right) dt,\\
&= 1 + 0.9625x + x^2\\
v_1(x) &= v_0(x) - \int_0^x \left(v_0'(t) - 2t + \frac{67}{64} - \frac{1}{64}\int_0^1 (u_0^2(s) - v_0(s)) ds\right) dt\\
&= 1 - 1.046875x + x^2\\
u_2(x) &= 1 + 0.9981388855x + x^2\\
v_2(x) &= 1 - 1.0006633x + x^2
\end{aligned}$$

and so on. Consequently, the exact solutions are given by

$$(u(x), v(x)) = (1 + x + x^2, 1 - x + x^2)$$

Example 34. *Use the variational iteration method to solve the system of nonlinear Fredholm integro-differential equations*

$$u''(x) = -\cos x - \frac{3\pi}{128} + \frac{1}{64}\int_0^{\frac{\pi}{2}} (u^2(t) + v^2(t))dt, \quad u(0) = 2\ \ , u'(0) = 0,$$

$$v''(x) = \sin x - \frac{1}{16} + \frac{1}{64}\int_0^{\frac{\pi}{2}} (u^2(t) - v^2(t))dt, \quad v(0) = 1\ \ , v'(0) = -1,$$

The correction functionals for this system are given by

$$u_{n+1}(x) = u_n(x) + \int_0^x \left((t-x)(u_n''(t) + \cos t + \frac{3\pi}{128} - \frac{1}{64}\int_0^{\frac{\pi}{2}} (u_n^2(s) + v_n^2(s))ds \right) dt,$$

$$v_{n+1}(x) = v_n(x) + \int_0^x \left((t-x)(v_n''(t) - \sin t + \frac{1}{16} - \frac{1}{64}\int_0^{\frac{\pi}{2}} (u_n^2(s) - v_n^2(s))ds \right) dt,$$

We can select the zeroth approximations $u_0(x) = u(0) + xu'(0) = 2$ and $v_0(x) = v(0) + xv'(0) = 1 - x$. Using this selection, the correction functionals give the following successive approximations.

$$\begin{aligned}
u_0(x) &= 2,\\
v_0(x) &= 1 - x,\\
u_1(x) &= u_0(x) + \int_0^x \left((t-x)(u_0''(t) + \cos t + \frac{3\pi}{128} - \frac{1}{64}\int_0^{\frac{\pi}{2}} (u_0^2(s) + v_0^2(s))ds \right) dt,\\
&= 1 + \cos x + 0.0153603x^2\\
v_1(x) &= v_0(x) + \int_0^x \left((t-x)(v_0''(t) - \sin t + \frac{1}{16} - \frac{1}{64}\int_0^{\frac{\pi}{2}} (u_0^2(s) - v_0^2(s))ds \right) dt,\\
&= 1 - \sin x - 0.0147489x^2,\\
u_2(x) &= 1 + \cos x + 0.0004636686x^2,\\
v_2(x) &= 1 - +0.0003878775x^2,\\
u_3(x) &= 1 + \cos x + 0.0000136626x^2,\\
v_3(x) &= 1 - \sin x + 0.000763919x^2,\\
u_4(x) &= 1 + \cos x + 0.00000217874186x^2,\\
v_4(x) &= 1 - \sin x - 0.00000142757975x^2,
\end{aligned}$$

and so on. Consequently, the exact solutions are given by

$$(u(x), v(x)) = (1 + \cos x, 1 - \sin x).$$

4.3 Nonlinear Volterra Integro-Differential Equations

VIM handles both linear and nonlinear problems in the same manner without the need for specific restrictions such as the Adomian polynomials.

The standard ith order nonlinear **Volterra integro-differential equation** is given by:

$$u^{(i)}(x) = f(x) + \int_0^x K(x,t)F(u(t))\,dt$$

The function $F(u(x))$ is a nonlinear function of $u(x)$ such as $u^2(x)$, $\sin(u(x))$, and $e^{u(x)}$.

The correction functional for the nonlinear integro-differential equation is:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\left(u_n^{(i)}(\xi) - f(\xi) - \int_0^\xi K(\xi,r)F(\tilde{u}_n(r))\,dr\right)d\xi$$

Example 35. *Use the* ***Variational Iteration Method (VIM)*** *to solve the nonlinear Volterra integro-differential equation:*

$$u'(x) = 1 + e^x - 2xe^x - e^{2x} + \int_0^x e^{x-t}u^2(t)\,dt,$$

with the initial condition $u(0) = 2$.

The correction functional for this equation is given by:

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(t) - 1 - e^t + 2te^t + e^{2t} - \int_0^t e^{t-r}u_n^2(r)\,dr\right)dt,$$

where we used $\lambda = -1$ *for the first-order integro-differential equation.*

We can use the initial condition to select $u_0(x) = u(0) = 2$*. Using this selection in the correction functional gives the following successive approximations:*

$$\begin{aligned}
u_0(x) &= 2\\
u_1(x) &= 2 + x + \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{3}{8}x^4 - \frac{19}{120}x^5 + \cdots,\\
u_2(x) &= 2 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{8}x^5 + \cdots,\\
u_3(x) &= 2 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots,
\end{aligned}$$

and so on for other approximations.

The VIM admits the use of:

$$u(x) = \lim_{n\to\infty} u_n(x)$$

which gives the exact solution:

$$u(x) = 1 + e^x$$

Example 36. *Use the* ***Variational Iteration Method (VIM)*** *to solve the nonlinear Volterra integro-differential equation:*

$$u'(x) = -x + \sec x \tan x - \tan x + \int_0^x (1 + u^2(t))\, dt,$$

with the initial condition $u(0) = 1$*.*

The correction functional for this equation is given by:

$$\begin{aligned} u_{n+1}(x) &= u_n(x) \\ - \int_0^x \left(u_n'(t) + t - \sec t \tan t + \tan t - \int_0^t (1 + u_n^2(r))\, dr \right) dt, \end{aligned}$$

where we used $\lambda = -1$ *for the first-order integro-differential equation.*

We can use the initial condition to select $u_0(x) = u(0) = 1$*. Using this selection in the correction functional gives the following successive approximations:*

$$\begin{aligned} u_0(x) &= 1 \\ u_1(x) &= 1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{16}x^6 + \cdots, \\ u_2(x) &= 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{19}{240}x^6 + \cdots, \\ u_3(x) &= 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \cdots, \end{aligned}$$

and so on. The VIM admits the use of:

$$u(x) = \lim_{n\to\infty} u_n(x)$$

which gives the exact solution:

$$u(x) = \sec x$$

Example 37. *Use the **Variational Iteration Method (VIM)** to solve the nonlinear Volterra integro-differential equation:*

$$u'(x) = x + \cos x - \tan x + \tan^2 x + \int_0^x (\sin t + u^2(t))\, dt,$$

with the initial condition $u(0) = 0$.

The correction functional for this equation is given by:

$$\begin{gathered} u_{n+1}(x) = u_n(x) \\ -\int_0^x \left(u_n'(t) - t - \cos t + \tan t - \tan^2 t - \int_0^t (\sin t + u_n^2(r))\, dr \right) dt. \end{gathered}$$

We can use the initial condition to select $u_0(x) = u(0) = 0$. Using this selection in the correction functional gives the following successive approximations:

$$\begin{gathered} u_0(x) = 0 \\ u_1(x) = x + \frac{1}{3}x^3 - \frac{1}{12}x^4 + \frac{2}{15}x^5 - \frac{1}{45}x^6 + \frac{17}{315}x^7 + \cdots, \\ u_2(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots, \end{gathered}$$

and so on. The VIM admits the use of:

$$u(x) = \lim_{n\to\infty} u_n(x)$$

which gives the exact solution:

$$u(x) = \tan x$$

Example 38. *Use the* ***Variational Iteration Method (VIM)*** *to solve the non-linear Volterra integro-differential equation:*

$$u''(x) = -\frac{5}{3}\sin x + \frac{1}{3}\sin(2x) + \int_0^x \cos(x-t)u^2(t)\,dt,$$

with the initial conditions $u(0) = 0$ and $u'(0) = 1$.

The correction functional for this equation is given by:

$$u_{n+1}(x) = u_n(x)$$
$$+ \int_0^x (t-x)\left(u_n''(t) + \frac{5}{3}\sin t - \frac{1}{3}\sin(2t) - \int_0^t \cos(t-r)u_n^2(r)\,dr\right)dt.$$

We can use the initial condition to select $u_0(x) = u(0) + xu'(0) = x$. Using this selection in the correction functional gives the following successive approximations:

$$\begin{aligned}
u_0(x) &= x \\
u_1(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{720}x^7 + \cdots, \\
u_2(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots,
\end{aligned}$$

and so on. The VIM gives the exact solution by:

$$u(x) = \sin x.$$

4.4 Nonlinear System of Volterra Integro-Differential Equations

The correction functionals for the Volterra system of integro-differential equations are given by:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t) \left(u_n'(t) - f_1(t) - \int_0^t \gamma_1(t,r)\,dr \right) dt,$$
$$v_{n+1}(x) = v_n(x) + \int_0^x \lambda(t) \left(v_n'(t) - f_2(t) - \int_0^t \gamma_2(t,r)\,dr \right) dt,$$

where

$$\gamma_1(t,r) = K_1(t,r)F_1(\tilde{u}_n(r)) + \tilde{K}_1(t,r)\tilde{F}_1(\tilde{v}_n(r)),$$
$$\gamma_2(t,r) = K_2(t,r)F_2(\tilde{u}_n(r)) + \tilde{K}_2(t,r)\tilde{F}_2(\tilde{v}_n(r)).$$

Example 39. *Use the VIM to solve the system of nonlinear Volterra integro-differential equations*

$$u'(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{12}x^4 + \int_0^x \left[(x-t)u^2(t) + v^2(t)\right] dt,$$
$$v'(x) = -1 - x - \frac{3}{2}x^2 - \frac{1}{12}x^4 + \int_0^x \left[u^2(t) + (x-t)v^2(t)\right] dt,$$

where $u(0) = 1$, $v(0) = 1$. The correction functionals for this system are

$$u_{n+1}(x) = u_n(x) - \int_0^x \left[u_n'(t) - 1 + t - \frac{1}{2}t^2 + \frac{1}{12}t^4 - I_1(t)\right] dt,$$
$$v_{n+1}(x) = v_n(x) - \int_0^x \left[v_n'(t) + 1 + t + \frac{3}{2}t^2 + \frac{1}{12}t^4 - I_2(t)\right] dt,$$

where

$$I_1(t) = \int_0^t \left[(t-r)u_n^2(r) + v_n^2(r)\right] dr,$$
$$I_2(t) = \int_0^t \left[u_n^2(r) + (t-r)v_n^2(r)\right] dr,$$

and $\lambda = -1$ for first-order integro-differential equations.

Selecting $u_0(x) = u(0) = 1$ and $v_0(x) = v(0) = 1$ gives the successive approximations

$$u_0(x) = 1,$$

$$v_0(x) = 1,$$

$$u_1(x) = 1 + x + \frac{1}{3}x^3 - \frac{1}{6}x^4 - \frac{1}{60}x^5,$$

$$v_1(x) = 1 - x - \frac{1}{3}x^3 - \frac{1}{60}x^5,$$

$$u_2(x) = 1 + x + \left(\frac{1}{3}x^3 - \frac{1}{3}x^3\right) + \left(\frac{1}{6}x^4 - \frac{1}{6}x^4\right) - \frac{1}{30}x^5 + \cdots,$$

$$v_2(x) = 1 - x + \left(\frac{1}{3}x^3 - \frac{1}{3}x^3\right) + \left(\frac{1}{6}x^4 - \frac{1}{6}x^4\right) + \frac{1}{30}x^5 + \cdots,$$

$$u_3(x) = 1 + x + \left(\frac{1}{30}x^5 - \frac{1}{30}x^5\right) + \cdots,$$

$$v_3(x) = 1 - x + \left(\frac{1}{30}x^5 - \frac{1}{30}x^5\right) + \cdots,$$

and so on. It is obvious that the noise terms appear in each approximation, hence the exact solutions are given by

$$(u(x), v(x)) = (1 + x, 1 - x).$$

Example 40. *Use the VIM to solve the system of Volterra integro-differential equations*

$$u'(x) = e^x - \frac{1}{2}e^{2x} - \frac{1}{6}x^4 + x + \frac{3}{2} + \int_0^x [(x-t)u^2(t) + (x-t)v^2(t)]\,dt,$$

$$v'(x) = 7e^x - 4xe^x - 4x - 7 + \int_0^x [(x-t)u^2(t) - (x-t)v^2(t)]\,dt,$$

where $u(0) = 1$, $v(0) = -1$. The correction functionals for this system are given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left[u_n'(t) - e^t + \frac{1}{2}e^{2t} + \frac{1}{6}t^4 - t - \frac{3}{2} - I_1(t)\right]\,dt,$$

$$v_{n+1}(x) = v_n(x) - \int_0^x \left[v_n'(t) - 7e^t + 4te^t + 4t + 7 - I_2(t)\right]\,dt,$$

where

$$I_1(t) = \int_0^t \left[(t-r)u_n^2(r) + (t-r)v_n^2(r)\right]\,dr,$$

$$I_2(t) = \int_0^t \left[(t-r)u_n^2(r) - (t-r)v_n^2(r)\right] \, dr,$$

and $\lambda = -1$ *for first-order integro-differential equations.*

We can use the initial conditions to select $u_0(x) = u(0) = 1$ *and* $v_0(x) = v(0) = -1$*. Using this selection into the correction functionals gives the following successive approximations*

$$\begin{aligned}
u_0(x) &= 1, \\
v_0(x) &= -1, \\
u_1(x) &= 1 + 2x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 - \frac{1}{8}x^4 - \frac{11}{120}x^5 + \cdots, \\
v_1(x) &= -1 - \frac{1}{2!}x^2 - \frac{1}{3!}x^3 - \frac{5}{24}x^4 - \frac{3}{40}x^5 + \cdots, \\
u_2(x) &= 1 + 2x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots, \\
v_2(x) &= -(1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots),
\end{aligned}$$

and so on. The exact solutions are therefore given by

$$(u(x), v(x)) = (x + e^x, x - e^x).$$

Example 41. *Use the VIM to solve the system of nonlinear Volterra integro-differential equations:*

$$\begin{aligned}
u''(x) &= \cosh x - \frac{1}{2}\sinh^2 x - \frac{1}{6}x^4 - \frac{1}{2}x^2 \\
&\quad + \int_0^x \left[(x-t)u^2(t) + (x-t)v^2(t)\right] dt, \\
u(0) &= 1,\ u'(0) = 1,
\end{aligned}$$

$$\begin{aligned}
v''(x) &= -(1+4x)\cosh x + 8\sinh x - 4x \\
&\quad + \int_0^x \left[(x-t)u^2(t) - (x-t)v^2(t)\right] dt, \\
v(0) &= -1,\ v'(0) = 1.
\end{aligned}$$

The correction functionals for this system are given by

$$
\begin{aligned}
u_{n+1}(x) = u_n(x) + \int_0^x & \left[(t-x)\left(u_n''(t) - \cosh t + \frac{1}{2}\sinh^2 t\right.\right. \\
& \left.\left. + \frac{1}{6}t^4 + \frac{1}{2}t^2 - \int_0^t I_1(t,r)dr\right)\right] dt, \\
v_{n+1}(x) = v_n(x) + \int_0^x & \left[(t-x)\left(v_n''(t) + (1+4t)\cosh t\right.\right. \\
& \left.\left. - 8\sinh t + 4t - \int_0^t I_2(t,r)dr\right)\right] dt,
\end{aligned}
$$

where

$$I_1(t,r) = (t-r)u^2(r) + (t-r)v^2(r), \quad I_2(t,r) = (t-r)u^2(r) - (t-r)v^2(r),$$

and $\lambda = t - x$ *for second-order integro-differential equations.*

Using $u_0(x) = 1 + x$ *and* $v_0(x) = -1 + x$*, we get the successive approximations:*

$$
\begin{aligned}
u_0(x) &= 1 + x, \quad v_0(x) = -1 + x, \\
u_1(x) &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{240}x^6 + \cdots, \\
v_1(x) &= -1 + x - \frac{1}{2!}x^2 - \frac{1}{4!}x^4 - \frac{1}{720}x^6 + \cdots, \\
u_2(x) &= x + 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \cdots, \\
v_2(x) &= x - 1 - \frac{1}{2!}x^2 - \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots,
\end{aligned}
$$

and so on. The exact solutions are therefore given by

$$(u(x), v(x)) = (1 + x + \cosh x, -1 + x - \cosh x).$$

Example 42. *Use the variational iteration method to solve the following system:*

$$
\begin{aligned}
u''(x) &= e^x + \frac{e^{2x}}{2}(x-1) + \frac{e^{4x}}{4}(3x-1) + \frac{3}{4}(x+1) \\
&\quad + \int_0^x \left[(x-2t)u^2(t) + (x-4t)v^2(t)\right] dt, \\
u(0) &= 1,\ u'(0) = 1,
\end{aligned}
$$

$$
\begin{aligned}
v''(x) &= 4e^{2x} + \frac{e^{4x}}{4}(3x-1) + \frac{e^{6x}}{6}(5x-1) + \frac{5}{12}(x+1) \\
&\quad + \int_0^x \left[(x-4t)v^2(t) + (x-6t)u^2(t)\right] dt, \\
v(0) &= 1,\ v'(0) = 2,
\end{aligned}
$$

$$
\begin{aligned}
w''(x) &= 9e^{3x} + \frac{e^{2x}}{2}(x-1) + \frac{e^{6x}}{6}(5x-1) + \frac{2}{3}(x+1) \\
&\quad + \int_0^x \left[(x-6t)u^2(t) + (x-2t)v^2(t)\right] dt, \\
w(0) &= 1,\ w'(0) = 3.
\end{aligned}
$$

We select the zeroth approximations as $u_0(x) = 1 + x$, $v_0(x) = 1 + 2x$, and $w_0(x) = 1 + 3x$. Proceeding as before, we obtain:

$$
\begin{aligned}
u_0(x) &= 1 + x, \quad v_0(x) = 1 + 2x, \quad w_0(x) = 1 + 3x, \\
u_1(x) &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots, \\
v_1(x) &= 1 + 2x + \frac{1}{2!}(2x)^2 + \frac{1}{3!}(2x)^3 + \frac{1}{4!}(2x)^4 + \frac{1}{5!}(2x)^5 + \cdots, \\
w_1(x) &= 1 + 3x + \frac{1}{2!}(3x)^2 + \frac{1}{3!}(3x)^3 + \frac{1}{4!}(3x)^4 + \frac{1}{5!}(3x)^5 + \cdots,
\end{aligned}
$$

The exact solutions are therefore given by:

$$
(u(x), v(x), w(x)) = (e^x, e^{2x}, e^{3x}).
$$

Bibliography

[1] A. Wazwaz, *Linear and Nonlinear Integral Equations: Methods and Applications*, Higher Education Press, Beijing and Springer Heidelberg Dordrecht London New York, 2011.

[2] Ji-Huan He, "Variational iteration method – a kind of non-linear analytical technique: some examples," *International Journal of Non-Linear Mechanics*, Vol. 34, Issue 4, July 1999, pp. 699-708.

[3] Ji-Huan He, "Variational iteration method—Some recent results and new interpretations," *Journal of Computational and Applied Mathematics*, Vol. 207, Issue 1, 1 October 2007, pp. 3-17.

[4] E. Rama, K. Somaiah, and K. Sambiah, "A study of variational iteration method for solving various types of problems," *Malaya Journal of Matematik*, Vol. 9, No. 1, 2021, pp. 701-708, `https://doi.org/10.26637/MJM0901/0123`.