ABDULMOHSEN SABORRAMADAN

RESEARCH PROJECT

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Projective Modules and Their Relationship With Some Important Rings

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Abdulmohsen Mohammad Saleh Sabor Ramadan
ID: 4560406

Supervisor:

Professor Yasser AbdelWahab

Department of Mathematics
College of Science, Taibah University
Madinah, Saudi Arabia

Introduction

Module theory provides a powerful framework for studying the structure of rings, extending concepts from linear algebra over fields to the more general setting of modules over rings. Within this rich theory, the concept of projectivity stands out as particularly fundamental and fruitful. Projective modules, initially arising as generalizations of free modules, possess unique lifting properties that make them essential tools in homological algebra and representation theory. Their close relationship with exact sequences, particularly split exact sequences, highlights their structural significance.

This research project delves into the fascinating interplay between projective modules and the properties of specific classes of rings. While free modules are always projective, the converse is not true, leading to natural questions about which rings have the property that all projective modules are free, or how projectivity characterizes the ring itself. We aim to thoroughly investigate the notion of projectivity and explore how it illuminates the structure of several important types of rings, namely local, semilocal, semiperfect, and perfect rings.

The research project is structured as follows:

- **Chapter 1** lays the necessary groundwork. We begin by reviewing essential concepts from module theory, including homomorphisms, free and finitely generated modules, radicals (Jacobson and module radicals), idempotents and their lifting properties, small (superfluous) submodules, nilpotency, and the theory of exact sequences. This foundation leads into the formal definition and exploration of projective modules, establishing their key properties and equivalences, such as their relationship to direct summands of free modules and the exactness of the Hom functor.

- **Chapter 2** shifts focus to specific classes of rings whose structure is deeply connected to projectivity. We introduce local and semilocal rings, examining their defining characteristics, particularly concerning their Jacobson radicals and quotient structures (e.g., $R/J(R)$ being a division ring for local R, or semisimple for semilocal R). We prove a key result (Theorem 2.1) showing that finitely generated projective modules over local rings are necessarily free. This chapter also introduces the critical concepts of semiperfect and perfect rings, along with the closely related notion of projective covers – epimorphisms from projective modules with small kernels.

- **Chapter 3** presents the main results of this project, culminating in detailed characterizations of semiperfect and perfect rings. Building upon the concepts developed earlier, we introduce the notion of R-projectivity (projectivity relative to the ring R itself) and R-projective covers. We then prove theorems (Theorem 3.1 and Theorem 3.2) establishing the equivalence between a ring being semiperfect or perfect and various conditions involving projective covers. Notably, these theorems demonstrate that a ring R is semiperfect if and only if every simple right R-module has an R-projective cover, and R is right perfect if and only if every semisimple right R-module has an R-projective cover. These characterizations offer a potentially streamlined approach by focusing on specific module classes (simple or semisimple) and the refined concept of R-projectivity.

Through this structured exploration, this research project seeks to provide a clear and comprehensive account of projective modules and demonstrate their profound connection to the structure theory of local, semilocal, semiperfect, and perfect rings.

Abstract

Projective modules, characterized by their fundamental lifting property, serve as essential tools in understanding the structure of rings and modules. This research project investigates the deep connections between projectivity and the properties of specific classes of rings, particularly local, semilocal, semiperfect, and perfect rings. We explore how the existence and nature of projective modules and related concepts, such as projective covers, can be used to characterize these rings.

After establishing foundational concepts from module theory in Chapter 1, including radicals, idempotents, and exact sequences, Chapter 2 focuses on local and semilocal rings, proving that finitely generated projective modules over local rings are free (Theorem 2.1). The chapter then introduces semiperfect and perfect rings alongside the crucial notion of projective covers.

Chapter 3 presents the core results, providing characterizations for semiperfect and perfect rings based on projective covers. We introduce R-projectivity (projectivity relative to R) and R-projective covers. Key findings include Theorem 3.1, showing that a ring is semiperfect if and only if every simple right module possesses an R-projective cover, and Theorem 3.2, demonstrating that a ring is right perfect if and only if every semisimple right module has an R-projective cover. These results offer refined criteria for classifying these important ring types, emphasizing the roles of simple/semisimple modules and R-projectivity, thereby contributing to the broader structure theory of rings and modules.

Contents

List of Figures

List of Tables

Nomenclature

XYZ xxx yyy zzz

Chapter 1

Projective Modules

This chapter lays the essential groundwork for the study of projective modules. We begin by reviewing fundamental concepts from module theory, including homomorphisms, generation properties, radicals, idempotents, small submodules, and exact sequences. These concepts provide the necessary tools and context for the subsequent formal introduction and investigation of projective modules, their defining lifting property, and their key characterizations, setting the stage for exploring their deeper connections with ring structures in later chapters.

1.1 Essentials of Module Theory

1.1.1 Homomorphisms

Definition 1.1. Let R be a ring and let M and N be R-modules.

1. A map $\varphi : M \to N$ is an R-module homomorphism if it respects the R-module structures of M and N, i.e.,
   (a) $\varphi(x + y) = \varphi(x) + \varphi(y)$, for all $x, y \in M$, and
   (b) $\varphi(xr) = \varphi(x)r$, for all $r \in R$, $x \in M$.
2. An R-module homomorphism is an isomorphism (of R-modules) if it is both a monomorphism (injective) and an epimorphism (surjective). The modules M and N

are said to be isomorphic, denoted $M \cong N$, if there is some R-module isomorphism $\varphi : M \to N$.

3. If $\varphi : M \to N$ is an R-module homomorphism, let $\ker \varphi = \{m \in M \mid \varphi(m) = 0\}$ (the kernel of φ) and let $\varphi(M) = \{n \in N \mid n = \varphi(m) \text{ for some } m \in M\}$ (the image of φ, as usual).

4. Let M and N be R-modules and define $\operatorname{Hom}_R(M, N)$ to be the set of all R-module homomorphisms from M into N.

5. The ring $\operatorname{Hom}_R(M, M)$ is called the *endomorphism ring* of M and will often be denoted by $\operatorname{End}_R(M)$, or just $\operatorname{End}(M)$. Elements of $\operatorname{End}(M)$ are called *endomorphisms.*

Example 1.1. Let M be a module, N a submodule of M, and X, Y be modules. We now introduce several key maps related to these modules.

1. Natural Projection: The map $\pi : M \to M/N$, defined by

$$\pi(m) = m + N \quad \text{for all } m \in M,$$

is an epimorphism called the *natural projection* of M onto the quotient module M/N.

2. Canonical Projections: The maps $\pi_X : X \oplus Y \to X$ and $\pi_Y : X \oplus Y \to Y$, defined by

$$\pi_X(x, y) = x \quad \text{and} \quad \pi_Y(x, y) = y \quad \text{for all } (x, y) \in X \oplus Y,$$

are called the *canonical projections.* These maps are epimorphisms.

3. Natural Injection: The inclusion map $i : N \to M$, defined by

$$i(n) = n \quad \text{for all } n \in N,$$

is called the *natural injection.* This map is a monomorphism.

4. Canonical Injections: The maps $i_X : X \to X \oplus Y$ and $i_Y : Y \to X \oplus Y$, defined by

$$i_X(x) = (x, 0) \quad \text{and} \quad i_Y(y) = (0, y) \quad \text{for all } x \in X, y \in Y,$$

are called the *canonical injections*. These maps are monomorphisms.

1.1.2 Free & Finitely Generated Modules

Definition 1.2. Let R be a ring. A module F over R is called a **free module** if it satisfies one of the following equivalent conditions:

1. F has a **basis**, i.e., a linearly independent set that spans F.

2. F is isomorphic to a direct sum of copies of R, i.e., $F \cong \bigoplus_{i \in I} R$ for some set I.

Definition 1.3. Let M be a module over a ring R. A subset $S \subseteq M$ is called a **generating set** for M if every element of M can be written as a finite linear combination of elements of S with coefficients in R. That is, for every $m \in M$, there exist $s_1, s_2, \ldots, s_n \in S$ and $r_1, r_2, \ldots, r_n \in R$ such that

$$m = r_1 s_1 + r_2 s_2 + \ldots + r_n s_n.$$

Remark 1.1. Every module M over a ring R has a generating set. In particular, M itself is always a generating set.

Definition 1.4. Let R be a ring. A module M over R is called **finitely generated** if there exists a finite subset $\{m_1, m_2, \ldots, m_n\} \subseteq M$ such that every element of M can be written as a finite linear combination of these elements with coefficients in R.

Proposition 1.1. If M is an R-module and M can be generated by n elements, and N is a submodule of M, then M/N can be generated by n elements.

Proof. Suppose that M is a finitely generated R-module and let $m_1, m_2, \ldots, m_n$ be its generators. So, any element of M can be expressed as

$$m = \sum_{i=1}^{n} m_i r_i$$

with $r_i \in R$. Then

$$m + N = \sum_{i=1}^{n} (m_i + N) r_i$$

which shows that the n elements $m_1 + N, \ldots, m_n + N$ generate M/N. □

Proposition 1.2. Every R-module M is an epimorphic image of a free module.

Proof. Let $S = \{m_i\}_{i \in I}$ be a generating set for M. This means every element of M can be written as a finite linear combination of elements from S with coefficients in R.

Consider the free R-module F with basis $\{f_i\}_{i \in I}$, where each f_i corresponds to $m_i \in S$. Formally, $F = \bigoplus_{i \in I} f_i R$. This means every element of F can be uniquely written as a finite sum $\sum_{i \in I} f_i r_i$ with $r_i \in R$.

Define the homomorphism $\phi : F \to M$ by $\phi(\sum_{i \in I} f_i r_i) = \sum_{i \in I} m_i r_i$.

Since S generates M, for any $m \in M$, there exist $r_i \in R$ such that $m = \sum_{i \in I} m_i r_i$. Thus, ϕ maps $\sum_{i \in I} f_i r_i$ to m, showing that ϕ is surjective. □

Corollary 1.1. *Every R-module M is isomorphic to a quotient module of a free module.*

Proof. Let M be a right R-module. By Proposition 1.2, there exists a free module F and an epimorphism $\varphi : F \to M$.

Since φ is surjective, by the First Isomorphism Theorem for modules, we have:

$$M \cong F/\ker(\varphi).$$

This shows that $M \cong F/\ker(\varphi)$, proving that M is isomorphic to a quotient of the free module F. □

Lemma 1.1. *Let M be a finitely generated R-module. Then every proper submodule $N \subsetneq M$ is contained in some maximal submodule of M.*

Definition 1.5 (Simple Module)**.** A right module M over a ring R is called *simple* if it is nonzero and has no nontrivial submodules, i.e., its only submodules are 0 and M.

Definition 1.6 (Semisimple Module)**.** A right module M over a ring R is called *semisimple* if every submodule of M is a direct summand, or equivalently, if M is isomorphic to a direct sum of simple right modules.

Definition 1.7 (Semisimple Ring)**.** A ring R is called *semisimple* if it is semisimple as a right module over itself, that is, if the right regular module R_R is semisimple.

Proposition 1.3. Let M be a semisimple module over a ring R. If M is finitely generated, then M is a finite direct sum of simple modules.

Proof. Assume that M is a semisimple module over R. By definition of semisimplicity, M can be expressed as a direct sum of simple submodules:

$$M = \bigoplus_{i \in I} M_i,$$

where each M_i is a simple module and I is an index set.

Suppose M is finitely generated. Then there exists a finite set of generators $\{m_1, m_2, \ldots, m_n\} \subseteq M$ such that every element of M can be written as a finite linear combination of these generators with coefficients in R.

Since each generator m_j (for $j = 1, 2, \ldots, n$) lies in M, and M is the direct sum of the M_i's, each m_j can be expressed as a finite sum of elements from the M_i's. Consequently, there exists a finite subset $J \subseteq I$ such that:

$$\{m_1, m_2, \ldots, m_n\} \subseteq \bigoplus_{i \in J} M_i.$$

Because M is generated by $\{m_1, m_2, \ldots, m_n\}$, it follows that:

$$M \subseteq \bigoplus_{i \in J} M_i.$$

On the other hand, $\bigoplus_{i\in J} M_i$ is a submodule of M. Therefore, we have:

$$M = \bigoplus_{i\in J} M_i.$$

Since J is finite, M is a finite direct sum of simple modules. This completes the proof. □

Proposition 1.4. Let R be a semisimple ring. Then, M_R is also semisimple.

Proof. Assume that R is semisimple, and consider the right R-module M. By Corollary 1.1, we know that

$$M \cong F/K,$$

where F is a free R-module and K is a submodule of F. By definition, $F \cong \bigoplus_{i\in I} R$, so F is also semisimple.

Since any quotient of a semisimple module is also semisimple, it follows that the quotient module $M \cong F/K$ is semisimple. □

Definition 1.8 (Socle)**.** Let M be a right R-module. The socle of M, denoted by $\mathrm{Soc}(M)$, is the sum of all simple right submodules of M. If there are no such submodules, then $\mathrm{Soc}(M) = 0$.

Remark 1.2.

1. If $M = R_R$, then $\mathrm{Soc}(R_R)$ is the sum of all minimal right ideals of R and it is a right ideal of R.

2. If I is a minimal ideal in R, then for any $x \in R$ either $Ix = 0$ or Ix is a minimal right ideal, and in both cases $Ix \subseteq \mathrm{Soc}(R_R)$.

3. For a semisimple module M we have $\mathrm{Soc}(M) = M$.

Proposition 1.5. The following are equivalent for a ring R:

1. Every maximal right ideal of R is a direct summand.

2. R_R is semisimple.

Proof. (1) ⇒ (2):

Suppose that every maximal right ideal of R is a direct summand. If $\mathrm{Soc}(R) = R$, we are done. Assume for the sake of contradiction that $\mathrm{Soc}(R) \neq R$. Then $\mathrm{Soc}(R)$ is properly contained in some maximal right ideal A, whose existence is guaranteed by Lemma 1.1 since R_R is cyclic (generated by 1) and hence finitely generated. The hypothesis implies that A is a direct summand of R, so we can write $R = A \oplus T$ for some right ideal $T \subseteq R$.

By the second isomorphism theorem,

$$\frac{A+T}{A} \cong \frac{T}{A \cap T},$$

and since $R = A + T$, we have $R/A \cong T$.

Now, A is maximal, so R/A is a simple right R-module, and hence T is simple. This implies that $T \subseteq \mathrm{Soc}(R) \subseteq A$. Therefore, $T \subseteq A$, but $R = A \oplus T$ implies $A \cap T = 0$ and $T \neq 0$ (since A is proper). Thus $T \subseteq A \cap T = 0$, which forces $T = 0$, a contradiction. Therefore, $\mathrm{Soc}(R) = R$, and R is semisimple.

(2) ⇒ (1):

If R is semisimple, then every right ideal is a direct summand, and in particular, every maximal right ideal is a direct summand. □

Proposition 1.6. Every submodule of a semisimple module is semisimple.

Proof. Every submodule of a semisimple module is semisimple because semisimple modules are direct sums of simple submodules, and every submodule of a semisimple module is itself a direct summand. □

Lemma 1.2. *Let K be a simple module, and let $f\colon K \to M$ be a module homomorphism. Then exactly one of the following holds:*

1. *If $\ker(f) = 0$, then f is injective and $K \cong \mathrm{im}(f)$.*

2. *If $\ker(f) = K$, then f is the zero homomorphism and $f(K) = 0$.*

Proof. Since K is simple, its only submodules are 0 and K. The kernel of f, being a submodule of K, must therefore be either 0 or K.

- If $\ker(f) = 0$, then f is injective. By the First Isomorphism Theorem, it follows that $K \cong \mathrm{im}(f)$.
- If $\ker(f) = K$, then f maps every element of K to zero. Hence, f is the zero homomorphism and $f(K) = 0$.

□

Lemma 1.3. *Let M be a semisimple module and $f\colon M \to N$ a surjective homomorphism. Then N is semisimple.*

Proof. Since M is semisimple, it decomposes as a direct sum of simple modules:

$$M = \bigoplus_{i \in I} K_i, \quad \text{where each } K_i \text{ is simple.}$$

By surjectivity, $N = f(M) = \sum_{i \in I} f(K_i)$. By Lemma 1.2, each $f(K_i)$ is either:

- Isomorphic to K_i (if $\ker(f|_{K_i}) = 0$), or
- Zero (if $\ker(f|_{K_i}) = K_i$).

Thus, N is a sum of simple modules (the nonzero $f(K_i)$), and hence semisimple. □

1.1.3 Radicals

Definition 1.9 ($J(R)$ or $\operatorname{rad} R$)**.** For a ring R, the **Jacobson radical**, denoted by $J(R)$ or sometimes $\operatorname{rad} R$, is the intersection of all maximal right ideals of R.

Definition 1.10 (radM)**.** For any right R-module M_R, we define $\operatorname{rad} M$ to be the intersection of all the **maximal submodules** of M. If there are no maximal submodules in M, we define $\operatorname{rad} M$ to be M.

Remark 1.3. The module radical $\operatorname{rad} M$ generalizes the Jacobson radical $J(R)$ to modules. For the right regular module R_R, $\operatorname{rad} R_R = J(R)$. If $R \neq 0$, maximal right ideals exist, so $J(R) \neq R$. Similarly, for a finitely generated module M_R, maximal submodules exist, and $\operatorname{rad} M \neq M$. However, for non-finitely-generated modules, $\operatorname{rad} M = M$ may occur, as maximal submodules need not exist.

Proposition 1.7. Let M be a right R-module. For every submodule K of M, the quotient module $(K + \operatorname{rad} M)/K$ is contained in $\operatorname{rad}(M/K)$ as a submodule.

Proof. Let M be a right R-module and let K be a submodule of M. Since K and $\operatorname{rad} M$ are both submodules of M, their sum $K + \operatorname{rad} M$ is also a submodule of M.

Consider the quotient $(K+\operatorname{rad} M)/K$. This is a submodule of M/K because $K+\operatorname{rad} M$ contains K.

Let $x \in K + \operatorname{rad} M$. Then $x = k + m$ where $k \in K$ and $m \in \operatorname{rad} M$. In M/K we have $x + K = m + K$.

Since $m \in \operatorname{rad} M$, m is in every maximal submodule of M. Therefore, $m + K$ is in every maximal submodule of M/K, which means $m + K \in \operatorname{rad}(M/K)$.

Thus $(K + \operatorname{rad} M)/K \subseteq \operatorname{rad}(M/K)$.

Therefore, $(K + \operatorname{rad} M)/K$ is a submodule of M/K that is contained in $\operatorname{rad}(M/K)$, completing the proof. $\square$

Proposition 1.8. For $y \in R$, the following are equivalent:

1. $y \in J(R)$
2. $1 - yx$ is right-invertible for any $x \in R$
3. $My = 0$ for any simple right R-module M.

Proof. **(1)** $\implies$ **(2)**

Assume $y \in J(R)$. We must show that for any $x \in R$, $1 - yx$ is right-invertible.

Suppose for contradiction that for some $x \in R$, $1 - yx$ is not right-invertible. Then the right ideal $(1 - yx)R$ is proper, i.e., $(1 - yx)R \neq R$.

Since $(1 - yx)R$ is proper, it is contained in some maximal right ideal $\mathfrak{m}$ (by Zorn's lemma). But $1 - yx \in (1 - yx)R \subseteq \mathfrak{m}$.

Since $y \in J(R)$, y lies in every maximal right ideal, so $y \in \mathfrak{m}$. Thus, $yx \in \mathfrak{m}$.

Now, $1 - yx \in \mathfrak{m}$ and $yx \in \mathfrak{m}$ imply:

$$1 = (1 - yx) + yx \in \mathfrak{m}.$$

But $1 \in \mathfrak{m}$ forces $\mathfrak{m} = R$, contradicting that $\mathfrak{m}$ is maximal.

Therefore, $1 - yx$ must be right-invertible for all $x \in R$.

(2) $\implies$ (3)

Assume (2) holds, but (3) fails. That is, there exists a simple right R-module M and an element $m \in M$ with $m \neq 0$ such that $my \neq 0$.

Since M is simple, the submodule generated by my must be all of M (because M has no proper nonzero submodules). Thus, there exists $x \in R$ such that:

$$m = (my)x.$$

This can be rewritten as:

$$m - m(yx) = 0 \quad \implies \quad m(1 - yx) = 0.$$

By hypothesis (2), $1 - yx$ is right-invertible, so there exists $z \in R$ such that:

$$(1 - yx)z = 1.$$

Multiplying both sides of $m(1 - yx) = 0$ by z on the right gives:

$$m(1 - yx)z = 0 \implies m \cdot 1 = 0 \implies m = 0.$$

This contradicts our choice of $m \neq 0$. Therefore, (3) must hold.

(3) $\implies$ (1)

Assume (3) holds, i.e., y annihilates every simple right R-module.

For any maximal right ideal $\mathfrak{m}$, the quotient $R/\mathfrak{m}$ is a simple right R-module. By (3),

$$y \cdot (R/\mathfrak{m}) = 0,$$

which implies $y \in \mathfrak{m}$.

Since this holds for all maximal right ideals $\mathfrak{m}$, by definition we have $y \in J(R)$. □

Proposition 1.9. Let R be a ring with Jacobson radical $J(R)$, and let M be a simple right R-module. Then M is annihilated by $J(R)$, i.e., $M \cdot J(R) = 0$.

Proof. Let M be a simple right R-module, meaning $M \neq \{0\}$ and its only submodules are $\{0\}$ and M.

Fix $x \in J(R)$ and $m \in M$. We show that $m \cdot x = 0$. Suppose, for contradiction, that $m \cdot x \neq 0$.

Since M is simple, the submodule generated by $m \cdot x$ must be all of M. Thus, there exists $r \in R$ such that

$$m \cdot x \cdot r = m.$$

Rewriting this gives

$$m - m \cdot xr = 0 \quad \text{or equivalently} \quad m \cdot (1 - xr) = 0.$$

By Proposition 1.8, since $x \in J(R)$, the element $1 - xr$ is right-invertible. Hence, there exists $z \in R$ such that

$$(1 - xr)z = 1.$$

Multiplying $m \cdot (1 - xr) = 0$ by z on the right yields

$$m \cdot (1 - xr)z = m \cdot 1 = m = 0,$$

contradicting $m \neq 0$.

Therefore, $m \cdot x = 0$ must hold for all $x \in J(R)$ and $m \in M$, proving $M \cdot J(R) = 0$. □

1.1.4 Idempotents

Definition 1.11. Let R be a ring.

1. An element $e \in R$ is called an **idempotent** if $e^2 = e$.
2. Two idempotents e and f are called **orthogonal** if $ef = fe = 0$.
3. A set $\{e_1, e_2, \ldots, e_n\} \subseteq R$ is called **pairwise orthogonal idempotents** if $e_i^2 = e_i$ for all i and $e_i e_j = 0$ for all $i \neq j$.
4. An equality $1 = e_1 + e_2 + \cdots + e_n$, where $e_1, e_2, \ldots, e_n$ are pairwise orthogonal **idempotents**, is called a **decomposition of the identity** of the ring R.
5. An idempotent $e \in R$ is said to be **primitive** if e has no decomposition into a sum of nonzero orthogonal idempotents $e = e_1 + e_2$ in R with e_1, e_2 orthogonal and nonzero.

Theorem 1.1 (Right Peirce Decomposition). *Let R be a ring and $e \in R$ an idempotent. Then $R = eR \oplus (1 - e)R$, as right R-modules.*

Proof. For any $r \in R$, $r = (e + (1-e))r = er + (1-e)r$, so $R = eR + (1-e)R$. For directness, take $x \in eR \cap (1-e)R$. Then $x = er = (1-e)s$, so $ex = er = e(1-e)s = 0$, hence $x = 0$. □

Proposition 1.10. Let R be a ring such that $R = A \oplus B$, where A and B are ideals of R. Then there exists an idempotent $e \in R$ such that $A = eR$.

Proof. Since $R = A \oplus B$, the identity element decomposes uniquely as $1 = e + f$ with $e \in A$ and $f \in B$. Multiplying by e yields $e = e^2 + ef$. Because $ef \in A \cap B = \{0\}$, it follows that e is idempotent.

For any $a \in A$, we have $a = a \cdot 1 = a(e+f) = ae + af$. Since $af \in A \cap B = \{0\}$, this shows $a = ae \in eR$, proving $A \subseteq eR$. Conversely, any $er \in eR$ (with $r \in R$) decomposes as $er = e(a+b) = ea + eb$ where $a \in A$, $b \in B$. Since $eb \in A \cap B = \{0\}$, we get $er = ea \in A$, establishing $eR \subseteq A$.

Thus $A = eR$, where e is the constructed idempotent. □

Definition 1.12. An idempotent $e \in R$ is called **local** if the ring eRe is local. a local idempotent is always a primitive idempotent.

Definition 1.13. Let I be an ideal in a ring R and let $g + I$ be an idempotent element of R/I. We say that this idempotent can be *lifted (to e) modulo I* in case there is an idempotent $e \in R$ such that $g + I = e + I$. We say that *idempotents lift modulo I* in case every idempotent in R/I can be lifted to an idempotent in R.

Remark 1.4. Let R be a ring, and let I be an ideal of R. An idempotent $\overline{e} \in R/I$ is said to **lift modulo** I if there exists an idempotent $e \in R$ such that $\pi(e) = e + I = \overline{e}$, where $\pi : R \to R/I$ is the natural projection map.

Remark 1.5. The concept of lifting idempotents modulo an ideal refers to finding an idempotent e in R such that $e - g \in I$. This means that e and g are congruent modulo I, and thus $e + I = g + I$ in R/I. The idempotent e in R is a lift of the idempotent $g + I$ in R/I.

The idempotents in a ring R correspond to idempotents in every quotient ring of R.

However, idempotent cosets in a quotient ring of R do not necessarily have idempotent representatives in R.

For example, $\mathbb{Z}$ has only two idempotents, while $\mathbb{Z}_6$ has four. The proof of this statement will be provided in the following proposition, and a counterexample illustrating the distinction will be presented in the subsequent example.

Proposition 1.11. Let R be a ring and $I \subseteq R$ an ideal. Then idempotents in R always map to idempotents in R/I.

Proof. Let $e \in R$ be an idempotent, meaning that $e = e^2$.

Consider the coset $e + I$ in the quotient ring R/I. We need to show that $e + I$ is an idempotent in R/I, i.e., that $(e + I)^2 = e + I$.

Observe that

$$(e + I)^2 = (e + I)(e + I) = e^2 + I = e + I.$$

Thus, the coset $e + I$ is an idempotent in R/I.

This completes the proof. $\square$

Example 1.2. The converse of the previous proposition, namely that an idempotent in the quotient ring R/I can always be mapped to an idempotent in R, is not true in general. To illustrate this, consider the ring of integers $\mathbb{Z}$ and the quotient ring $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}_6$.

It is well-known that $\mathbb{Z}$ has exactly two idempotents: 0 and 1. However, $\mathbb{Z}/6\mathbb{Z}$ has four idempotents:

$$0 + 6\mathbb{Z} \quad \text{since } (0 + 6\mathbb{Z})^2 = 0 + 6\mathbb{Z},$$

$$1 + 6\mathbb{Z} \quad \text{since } (1 + 6\mathbb{Z})^2 = 1 + 6\mathbb{Z},$$

$$3 + 6\mathbb{Z} \quad \text{since } (3 + 6\mathbb{Z})^2 = 9 + 6\mathbb{Z} = 3 + 6\mathbb{Z},$$

$$4 + 6\mathbb{Z} \quad \text{since } (4 + 6\mathbb{Z})^2 = 16 + 6\mathbb{Z} = 4 + 6\mathbb{Z}.$$

Thus, the quotient ring $\mathbb{Z}/6\mathbb{Z}$ contains four distinct idempotents. However, only the elements $0+6\mathbb{Z}$ and $1+6\mathbb{Z}$ have corresponding idempotent representatives in $\mathbb{Z}$. Specifically, there do not exist any idempotents $e \in \mathbb{Z}$ such that $e + 6\mathbb{Z} = 3 + 6\mathbb{Z}$ or $e + 6\mathbb{Z} = 4 + 6\mathbb{Z}$.

From this example, we can conclude two important points: First, not every idempotent in R/I can be mapped to an idempotent in R. Second, not all idempotents in R/I can be lifted modulo $I = 6\mathbb{Z}$. Specifically, only the idempotents $0 + 6\mathbb{Z}$ and $1 + 6\mathbb{Z}$ have corresponding idempotents in $\mathbb{Z}$, while $3 + 6\mathbb{Z}$ and $4 + 6\mathbb{Z}$ do not. This demonstrates that idempotent lifting modulo I is not guaranteed. In fact, for idempotent lifting modulo I to be possible, every idempotent in R/I must be liftable to an idempotent in R.

Proposition 1.12. If I is a nil-ideal in a ring R, then an idempotent lifts modulo I.

Proof. Suppose I is a nil-ideal of R, and let $g + I$ be an idempotent in R/I. Then:

$$(g + I)^2 = g^2 + I = g + I,$$

which implies $g^2 - g \in I$.

We will prove the statement by constructing an idempotent e from the given element g. Set $r = g^2 - g$ and define:

$$g_1 = g + r - 2gr.$$

First, observe that:

$$gr = g(g^2 - g) = g^3 - g^2 \quad \text{and} \quad rg = (g^2 - g)g = g^3 - g^2.$$

Thus, $gr = rg$.

Next, calculate $g_1^2 - g_1$:

$$g_1^2 - g_1 = (g + r - 2gr)^2 - (g + r - 2gr).$$

Expanding this, we obtain:

$$g^2 + r^2 + 4g^2r^2 + 2gr - 4g^2r - 4gr^2 - g - r + 2gr.$$

Simplifying further:

$$g_1^2 - g_1 = r^2 + 4r^3 - 4r^2 = r^2(4r - 3).$$

Set $r_1 = g_1^2 - g_1 = r^2(4r-3)$. Since $r = g^2 - g \in I$, it follows that $r^2 \in I$, and thus $r_1 \in I$.

Now, define $g_2 = g_1 + r_1 - 2g_1r_1$. By a similar calculation, we obtain:

$$g_2^2 - g_2 = r_1^2(4r_1 - 3).$$

Set $r_2 = g_2^2 - g_2 = r_1^2(4r_1 - 3)$. Since $r_1 \in I$, we have $r_1^2 \in I$, and thus $r_2 \in I$.

Continuing this process, we define:

$$g_i = g_{i-1} + r_{i-1} - 2g_{i-1}r_{i-1},$$

and observe that:

$$r_i = g_i^2 - g_i = r_{i-1}^2(4r_{i-1} - 3).$$

Since I is a nil ideal, there exists an integer $k > 0$ such that $r^k = 0$. Thus, after a finite number of steps, say n, we have:

$$r_n = g_n^2 - g_n = 0.$$

This implies $g_n^2 = g_n$, so g_n is idempotent.

Now, observe that:

$$g_1 - g = (g + r - 2gr) - g = r(1 - 2g) = (1 - 2g)(g^2 - g).$$

Since $r = g^2 - g \in I$, it follows that $g_1 - g \in I$. Inductively, for each $i \in \{1, 2, \ldots, n\}$, we have:

$$g_i - g_{i-1} \in I.$$

Thus, the cumulative difference satisfies:

$$g_n - g = (g_n - g_{n-1}) + (g_{n-1} - g_{n-2}) + \cdots + (g_2 - g_1) + (g_1 - g).$$

Since each term $g_i - g_{i-1} \in I$ and I is closed under addition, we conclude that $g_n - g \in I$.

Finally, set $e = g_n$, the idempotent in R constructed through our iterative process. Since $e - g \in I$, e lifts g modulo I. □

Proposition 1.13 (C. Hopkins)**.** The Jacobson radical $J(R)$ of a right Artinian ring R is nilpotent.

Proof. □

Corollary 1.2. *Idempotents can be lifted modulo the Jacobson radical of an Artinian ring.*

Proof. Since every nilpotent ideal is a nil-ideal, the proof follows from Proposition 1.13 and Proposition 1.12. □

Remark 1.6. In general, if we have a pair of orthogonal idempotents $g_1 + I$ and $g_2 + I$ in R/I, which lift to idempotents $e_1, e_2 \in R$, there is no guarantee that e_1 and e_2 will be orthogonal. However, in some special cases (e.g., nil-ideals), orthogonality of idempotents can be preserved.

1.1.5 Small Submodules

Definition 1.14 (Small Submodule)**.** A submodule S of a module M is called **small** (or **superfluous**), denoted by $S \ll M$, if for any submodule N of M, the equality $S + N = M$ implies $N = M$.

Example 1.3. Consider $\mathbb{Z}_{10}$ as a $\mathbb{Z}$-module. The submodules $\langle 2 \rangle$ and $\langle 5 \rangle$ are the only proper submodules of $\mathbb{Z}_{10}$, where:

$$\langle 2 \rangle = \{0, 2, 4, 6, 8\}, \quad \langle 5 \rangle = \{0, 5\}.$$

The sum of these submodules is:

$$\langle 2\rangle + \langle 5\rangle = \{a+b : a \in \langle 2\rangle, b \in \langle 5\rangle\} = \{0,2,4,6,8,5,7,9,1,3\} = \mathbb{Z}_{10}.$$

Thus, $\langle 2\rangle + \langle 5\rangle = \mathbb{Z}_{10}$, but neither $\langle 2\rangle$ nor $\langle 5\rangle$ equals $\mathbb{Z}_{10}$. Therefore, $\langle 2\rangle$ and $\langle 5\rangle$ are not small submodules.

Example 1.4. Consider $\mathbb{Z}_8$ as a $\mathbb{Z}$-module. The submodules $\langle 2\rangle$ and $\langle 4\rangle$ are the only proper submodules of $\mathbb{Z}_8$, where:

$$\langle 2\rangle = \{0,2,4,6\}, \quad \langle 4\rangle = \{0,4\}.$$

The sum of these submodules is:

$$\langle 2\rangle + \langle 4\rangle = \{a+b : a \in \langle 2\rangle, b \in \langle 4\rangle\} = \{0,2,4,6\} \neq \mathbb{Z}_8.$$

Thus, the only possible way to have $\langle 2\rangle + N = \mathbb{Z}_8$ is if $N = \mathbb{Z}_8$. Similarly, the only possible way to have $\langle 4\rangle + N = \mathbb{Z}_8$ is if $N = \mathbb{Z}_8$. Hence, both $\langle 2\rangle$ and $\langle 4\rangle$ are small submodules in $\mathbb{Z}_8$.

Example 1.5. Let R be an Artinian ring with Jacobson radical $J(R)$, and let M be a right R-module. By Nakayama's lemma, the submodule $MJ(R)$ is small in M, denoted $MJ(R) \ll M$. To see this, suppose $MJ(R) + N = M$ for some submodule N of M. Nakayama's lemma implies that $N = M$, since $MJ(R)$ is contained in the Jacobson radical and thus ”small” in M. Therefore, $MJ(R)$ is a small submodule of M.

Example 1.6. Let M be a finitely generated right R-module and $A \subseteq J(R)$ a right ideal contained in the Jacobson radical of R. Then MA is a small submodule of M.

Proof. Suppose $N \subseteq M$ satisfies

$$MA + N = M.$$

Passing to the quotient module M/N, we have

$$\frac{MA+N}{N} = \frac{M}{N}.$$

This simplifies to

$$\frac{MA}{N} = \frac{M}{N},$$

since $\frac{N}{N} = 0$.

Because M is finitely generated, M/N is also finitely generated. Since $A \subseteq J(R)$, by Nakayama's Lemma we conclude that

$$M/N = 0.$$

Thus $N = M$, proving that MA is small in M (i.e., $MA \ll M$). $\square$

Example 1.7. Let N be a maximal submodule of a module M. Then N contains every small submodule $S \ll M$. To see this, suppose for contradiction that $S \nsubseteq N$. Since N is maximal, the submodule $S + N$ must equal M. However, because $S \ll M$, the equality $S + N = M$ implies $N = M$. This contradicts the fact that N is a proper submodule of M. Therefore, N must contain every small submodule $S \ll M$.

Example 1.8. Let M be a module and M' a submodule of M. If $S \ll M'$, then $S \ll M$.

Proof. Assume $S \ll M'$ and let $N \hookrightarrow M$ satisfy $S+N = M$. Since $M' \subseteq M$, intersecting with M' gives:

$$\begin{aligned}(S + N) \cap M' &= (S \cap M') + (N \cap M') \\ &= S + (N \cap M'),\end{aligned}$$

while:

$$M \cap M' = M'.$$

Thus:

$$S + (N \cap M') = M'.$$

Since $S \ll M'$, we have:

$$N \cap M' = M',$$

so $M' \subseteq N$.

Now $S+N=M$ with $S \subseteq M' \subseteq N$ implies:

$$N = M.$$

Therefore, $S \ll M$. □

Example 1.9. Let M be a module over a ring R.

1. If $S \ll M$, then every submodule $T \subseteq S$ satisfies $T \ll M$.
2. If $S_i \ll M$ for $1 \leq i \leq n$, then $\sum_{i=1}^{n} S_i \ll M$.

Proof. **Part 1: Submodules of small submodules are small.** Let $S \ll M$, and let $T \subseteq S$. Suppose $T+N=M$ for some submodule $N \subseteq M$. Since $T \subseteq S$, we have:

$$S + N \supseteq T + N = M \implies S + N = M.$$

Because $S \ll M$, this forces $N = M$. Thus, $T \ll M$.

Part 2: Finite sums of small submodules are small. We proceed by induction on n.

Base case ($n = 2$): Let $S_1, S_2 \ll M$. Suppose $(S_1 + S_2) + N = M$. Rewrite this as:

$$S_1 + (S_2 + N) = M.$$

Since $S_1 \ll M$, we deduce $S_2 + N = M$. But $S_2 \ll M$, so $N = M$. Hence, $S_1 + S_2 \ll M$.

Inductive step: Assume the claim holds for $n = k$. For $n = k+1$, write:

$$\sum_{i=1}^{k+1} S_i = \left(\sum_{i=1}^{k} S_i \right) + S_{k+1}.$$

By the induction hypothesis, $\sum_{i=1}^{k} S_i \ll M$. Since $S_{k+1} \ll M$, the base case implies $\sum_{i=1}^{k+1} S_i \ll M$. □

Example 1.10. Suppose $S_i \ll M_i$ (for $1 \le i \le n$). Then

$$\bigoplus_{i=1}^{n} S_i \ll \bigoplus_{i=1}^{n} M_i.$$

Proof. For each i, since $S_i \ll M_i$ and $M_i \hookrightarrow \bigoplus_{i=1}^n M_i$, it follows that $S_i \ll \bigoplus_{i=1}^n M_i$ (see Example 1.7).

By Example 1.8, the sum of small submodules is small. Since each S_i is small in $\bigoplus_{i=1}^n M_i$, their sum $\sum_{i=1}^n S_i$ is also small in $\bigoplus_{i=1}^n M_i$.

Because the S_i are independent (i.e., $S_i \cap \sum_{j \neq i} S_j = 0$ for all i), we have $\sum_{i=1}^n S_i \cong \bigoplus_{i=1}^n S_i$. Thus,

$$\bigoplus_{i=1}^{n} S_i \ll \bigoplus_{i=1}^{n} M_i.$$

□

Proposition 1.14. Let R be a ring, and let $f : M \to N$ be an R-module isomorphism. If S is a small submodule of M (denoted $S \ll M$), then its image $f(S)$ is a small submodule of N (denoted $f(S) \ll N$).

Proof. Let $f : M \to N$ be an R-module isomorphism and $S \ll M$. The image $f(S)$ is a submodule of N. To show $f(S) \ll N$, let K be any submodule of N such that $f(S) + K = N$. We must show $K = N$.

Since f is an isomorphism, its inverse $f^{-1} : N \to M$ exists and is also an isomorphism. Apply f^{-1} to $f(S) + K = N$:

$$f^{-1}(f(S) + K) = f^{-1}(N)$$

As f^{-1} is a homomorphism:

$$f^{-1}(f(S)) + f^{-1}(K) = f^{-1}(N)$$

Since $f^{-1}(f(S)) = S$ and $f^{-1}(N) = M$, this simplifies to:

$$S + f^{-1}(K) = M$$

Let $L = f^{-1}(K)$. Since K is a submodule of N and f^{-1} is a homomorphism, L is a submodule of M. The equation becomes $S + L = M$. By hypothesis, $S \ll M$, so $S + L = M$ implies $L = M$. Thus, $f^{-1}(K) = M$.

Apply f to both sides of $f^{-1}(K) = M$:

$$f(f^{-1}(K)) = f(M)$$

Since $f(f^{-1}(K)) = K$ and $f(M) = N$, we conclude $K = N$.

Thus, $f(S)$ is a small submodule of N. Hence, isomorphisms preserve smallness. $\square$

Proposition 1.15. Let M be an R-module. Then

1. $\operatorname{rad} M$ is the sum of all small submodules of M.
2. $MJ(R) \subseteq \operatorname{rad} M$, where $J(R)$ is the Jacobson radical of the ring R. The equality $MJ(R) = \operatorname{rad} M$ holds if R is a semi-local ring.

Proof. (1) Suppose M is an R-module, and let $T = \sum\{S : S \ll M\}$ be the sum of all small submodules of M. By Example 2.5, since every small submodule S is contained in every maximal submodule and hence every small module is contained in $\operatorname{rad} M$, their sum T is also contained in $\operatorname{rad} M$. Thus, $T \subseteq \operatorname{rad} M$.

For the reverse inclusion $\operatorname{rad} M \subseteq T$, it suffices to prove that for any $m \in \operatorname{rad} M$, the submodule mR is small in M. Since If $mR \ll M$, then mR is one of the summands in T, so $m \in T$. If every $m \in \operatorname{rad} M$ generates a small submodule mR, then $\operatorname{rad} M \subseteq T$.

For any $m \in \operatorname{rad} M$, we will show that $mR \ll M$. Assume for the sake of contradiction that mR is not small. Then there exists a submodule N of M such that $mR + N = M$ but $N \neq M$.

Consider the quotient module M/N. Since $mR + N = M$, M/N is a cyclic module generated by $\langle m + N\rangle$. Since M/N is a non-zero cyclic module and hence finitely generated, it has a maximal submodule N'/N. This means N' is a maximal submodule of M containing N. Since $m \in \operatorname{rad} M$, m must be in every maximal submodule of M. However,

$m \notin N'$ because N'/N is a maximal submodule of M/N and $m+N$ generates M/N. So, if $m \in N'$, then $N'/N = M/N$, and N' is no longer a maximal submodule of M. This contradicts $m \in \operatorname{rad} M$. Since N' is maximal and $m \notin N'$, but $mR + N = M$, we have a contradiction. Therefore, $mR \ll M$, and $\operatorname{rad} M \subseteq T$. □

Proposition 1.16. Let M be an R-module. The following statements hold:

1. If M' is a submodule of M, then $\operatorname{rad} M' \subseteq \operatorname{rad} M$.
2. $\operatorname{rad}\left(\bigoplus_{i\in I} M_i\right) = \bigoplus_{i\in I} \operatorname{rad} M_i$.
3. If F_R is a free R-module, then $\operatorname{rad} F = F \cdot J(R)$, where $J(R)$ is the Jacobson radical of R.

Proof.

1. Let M' be a submodule of M. By Proposition 1.15, $\operatorname{rad} M'$ is the sum of all small submodules S of M'. Since each small submodule S of M' is also a small submodule of M (by Example 1.8), it follows that $\operatorname{rad} M' \subseteq \operatorname{rad} M$.

2. Let $I = \{1, 2\}$ (the general case follows by induction). Set $T = \operatorname{rad}(M_1 \oplus M_2)$. Since $M_1 \subseteq M_1 \oplus M_2$ and $M_2 \subseteq M_1 \oplus M_2$, we have $\operatorname{rad} M_1 \subseteq \operatorname{rad}(M_1 \oplus M_2)$ and $\operatorname{rad} M_2 \subseteq \operatorname{rad}(M_1 \oplus M_2)$. Therefore, $\operatorname{rad} M_1 \oplus \operatorname{rad} M_2 \subseteq \operatorname{rad}(M_1 \oplus M_2)$.

   For the reverse inclusion, let $m = (m_1, m_2) \in \operatorname{rad}(M_1 \oplus M_2)$. For any maximal submodule $N \subseteq M_1$, the submodule $N \oplus M_2$ is maximal in $M_1 \oplus M_2$. Since $m \in \operatorname{rad}(M_1 \oplus M_2)$, it must lie in every maximal submodule of $M_1 \oplus M_2$, including $N \oplus M_2$. This implies that $m_1 \in N$ for every maximal submodule N of M_1, and thus $m_1 \in \operatorname{rad} M_1$. Similarly, $m_2 \in \operatorname{rad} M_2$. Therefore, $m = (m_1, m_2) \in \operatorname{rad} M_1 \oplus \operatorname{rad} M_2$, and hence $T \subseteq \operatorname{rad} M_1 \oplus \operatorname{rad} M_2$. Combining both inclusions, we conclude that $\operatorname{rad}(M_1 \oplus M_2) = \operatorname{rad} M_1 \oplus \operatorname{rad} M_2$.

3. Let F be a free R-module. Then $F \cong \bigoplus_{i\in I} R$. By part (2), we have $\operatorname{rad} F = \operatorname{rad}\left(\bigoplus_{i\in I} R\right) = \bigoplus_{i\in I} \operatorname{rad} R$. Since $F \cong \bigoplus_{i\in I} R$, any element $f \in F$ can be written as $f = (r_i)_{i\in I}$, where each $r_i \in R$. Multiplying f by an element $r \in \operatorname{rad} R = J(R)$ gives $fr = (r_i r)_{i\in I}$. Therefore, $F \cdot \operatorname{rad} R = F \cdot J(R)$ consists of all elements of the form $(r_i r)_{i\in I}$, which is exactly $\bigoplus_{i\in I} \operatorname{rad} R = \bigoplus_{i\in I} J(R)$. Thus, $\operatorname{rad} F = F \cdot J(R)$. This completes the proof.

□

Lemma 1.4. *M is finitely generated if and only if* $\operatorname{rad} M \ll M$ *and* $M/\operatorname{rad} M$ *is finitely generated. In particular,* $\operatorname{rad} R \ll R$.

Proof. ($\Longrightarrow$) Let M be a finitely generated right R-module. Let $\operatorname{rad} M + C = M$. Suppose $C \neq M$. Since M is finitely generated, C is contained in a maximal submodule $X \subset M$ (by Lemma 1.1). Since $\operatorname{rad} M$ is the intersection of all maximal submodules of M, we have $\operatorname{rad} M \subseteq X$. As both $\operatorname{rad} M$ and C are contained in X, their sum $\operatorname{rad} M + C$ is also contained in X.

By assumption, $\operatorname{rad} M + C = M$, so $M \subseteq X$, this contradicts X being a proper submodule. Thus, $C = M$ and therefore $\operatorname{rad} M \ll M$.

Furthermore, as a direct consequence of Proposition 1.1, $M/\operatorname{rad} M$ is finitely generated.

($\Longleftarrow$) Assume $\operatorname{rad} M \ll M$ and that $M/\operatorname{rad} M$ is finitely generated. Let $\{x_1 + \operatorname{rad} M, \ldots, x_n + \operatorname{rad} M\}$ be a generating set for $M/\operatorname{rad} M$. The submodule $N = \langle x_1, \ldots, x_n \rangle$ is finitely generated and satisfies $N + \operatorname{rad} M = M$. Since $\operatorname{rad} M \ll M$, this implies that $N = M$. Hence, M is finitely generated. □

Lemma 1.5. *Let R be a ring, $J(R) = J$ its Jacobson radical, and C a maximal ideal. Suppose that $R = C + B$ for some ideal B, and $C \cap B \ll R$. Then, we have the following isomorphism:*

$$R/J \cong C/J \oplus (B + J)/J.$$

Proof. We will prove the lemma by showing both inclusions.

($\supseteq$) Inclusion: Since $R = C + B$, we have $R/J = (C + B)/J$. Clearly, $C \subseteq C + B$, so we obtain $C/J \subseteq (C + B)/J = R/J$.

Next, since we do not know if $J \subseteq B$, we observe that $J \subseteq B + J$, so we can consider the quotient $(B + J)/J$. Moreover, since $J \subseteq C$ (because J is the intersection of all maximal ideals and C is a maximal ideal), we have:

$$\frac{B + J}{J} \subseteq \frac{B + C}{J} = R/J.$$

Thus, we conclude that:

$$C/J \subseteq R/J \quad \text{and} \quad (B+J)/J \subseteq R/J.$$

Therefore, their sum is also a subset of R/J, i.e.,

$$C/J + (B+J)/J \subseteq R/J.$$

($\subseteq$) Inclusion: For the reverse inclusion, we take any element $x \in R/J$. Since $R = C+B$, we can write $x = c + b + J$, where $c \in C$ and $b \in B$. Therefore, we have:

$$x = c + J + b + j + J = (c+J) + (b+j+J) = (c+J) + (b+J).$$

Thus, $x \in C/J + (B+J)/J$, which shows that $R/J \subseteq C/J + (B+J)/J$.

Finally, we need to show that the intersection $C/J \cap (B+J)/J$ is trivial.

Intersection: We will now show that:

$$\frac{C}{J} \cap \frac{B+J}{J} = \{0\}.$$

Take $x \in \frac{C}{J} \cap \frac{B+J}{J}$. Then we can write:

$$x = c + J = b + j + J \quad \text{for some } c \in C, b \in B, j \in J.$$

This implies:

$$c - b - j \in J.$$

Since $j \in J$, we have:

$$c - b \in J \subseteq C.$$

Thus, there exists $y \in C$ such that:

$$c - b = y \in C.$$

Rewriting gives:

$$b = c - y \in C \cap B.$$

Because $C \cap B \ll R$ and J is the sum of all small submodules, we have:

$$C \cap B \subseteq J.$$

Therefore, $b \in J$, and consequently:

$$x = b + j + J = 0 + J.$$

This shows that the only element in the intersection is zero, completing the proof.

□

Proposition 1.17. Every nonzero semisimple module has a maximal submodule.

Proof. Let M be a nonzero semisimple module. By semisimplicity, every submodule of M is a direct summand. Consider any small submodule $K \ll M$. We can decompose M as $M = K \oplus T$ for some submodule T. However, since K is small, the equality $M = K + T$ forces $T = M$. Consequently, $K = K \cap T = K \cap M = 0$, proving that 0 is the only small submodule of M. By Proposition 1.15, the radical $\operatorname{rad} M$ (being the sum of all small submodules) must be 0. Since $\operatorname{rad} M \neq M$, we conclude that M admits a maximal submodule. □

1.1.6 Nilpotency

Definition 1.15. An ideal A is called **right** T-nilpotent if for any sequence $a_1, a_2, \ldots, a_n$ of elements $a_i \in A$, there exists a positive integer k such that $a_k a_{k-1} \ldots a_1 = 0$.

Proposition 1.18. For any right ideal $A \subseteq R$, the following are equivalent:

1. A is right T-nilpotent.

2. For any right R-module M, $MA = M \implies M = 0$.

3. For right R-modules $N \subseteq M$, $MA + N = M \implies N = M$.

Proof. **(1) $\implies$ (2):** Let $A \subseteq R$ be a right T-nilpotent ideal, and let M be a right R-module such that $MA = M$. We aim to show that $M = 0$.

Since $MA = M$, for any $m \in M$, there exist elements $m_1 \in M$ and $a_1 \in A$ such that $m = m_1 a_1$. Continuing this process recursively, since $m_1 \in M$, there exist elements $m_2 \in M$ and $a_2 \in A$ such that $m_1 = m_2 a_2$. Substituting this into the equation for m, we get $m = m_2 a_2 a_1$. This process can be repeated indefinitely: for each $m_n \in M$ and $a_n \in A$, there exist elements $m_{n+1} \in M$ and $a_{n+1} \in A$ such that $m_n = m_{n+1} a_{n+1}$. Thus, after n steps, we have the following equation for m:

$$m = m_n a_n a_{n-1} \dots a_1$$

Since A is right T-nilpotent, there exists some integer n such that $a_n a_{n-1} \dots a_1 = 0$. Substituting this into the equation for m, we get $m = m_n \cdot 0 = 0$.

Since $m \in M$ was arbitrary, we conclude that $M = 0$.

(2) $\implies$ (3): Follows trivially

(3) $\implies$ (1): Assume A is not right T-nilpotent. Then, there exists an infinite sequence $\{a_n\} \subseteq A$ such that $a_n a_{n-1} \cdots a_1 \neq 0$ for all n. Construct the right R-module M as follows:

Let M be generated by elements $\{e_1, e_2, e_3, \dots\}$ with relations $e_i = e_{i+1} a_i$ for all i.

This module M is non-zero because the relations do not collapse all generators to zero (since $a_n a_{n-1} \cdots a_1 \neq 0$). Now, consider the submodule $N = 0 \subseteq M$. Observe that:

$$MA + N = MA + 0 = MA.$$

Since $e_i = e_{i+1}a_i \in MA$ (as $a_i \in A$), all generators e_i lie in MA. Hence, $MA = M$. By condition (3), $MA + N = M$ implies $N = M$. However, $N = 0$, so $M = 0$, contradicting the construction of $M \neq 0$.

Thus, the assumption that A is not right T-nilpotent leads to a contradiction. Therefore, A must be right T-nilpotent. Hence, (3) $\implies$ (1).

□

Lemma 1.6. *The following statements are equivalent for a right ideal A of R:*

1. *A is right T-nilpotent.*
2. *If $M_R \neq 0$ is any right module then $MA \ll M$.*
3. *If $M_R \neq 0$ is any right module then $MA \neq M$.*
4. *If F is a countably generated free module then $FA \ll F$.*

Proof. (1) $\implies$ (2): Assume that A is right T-nilpotent. Let $M \neq 0$ be a right R-module, and suppose $MA + X = M$ for some submodule $X \subseteq M$. We prove that $X = M$.

Suppose $X \subsetneq M$. Choose $m \in M/X$, i.e., $m \in M$ but $m \notin X$. Since $MA + X = M$, we can write:

$$m = m_1 a_1 + x_1 \quad \text{for some } m_1 \in M,\, a_1 \in A,\, x_1 \in X.$$

As $m \notin X$, it follows that $m_1 a_1 \notin X$ (otherwise $m \in X$, which is a contradiction). Apply the same decomposition to m_1:

$$m_1 = m_2 a_2 + x_2 \quad \text{for some } m_2 \in M,\, a_2 \in A,\, x_2 \in X.$$

Substitute this back into the expression for m:

$$m = (m_2 a_2 + x_2)a_1 + x_1 = m_2 a_2 a_1 + \underbrace{x_2 a_1 + x_1}_{\in X}.$$

Since $m \notin X$, it follows that $m_2 a_2 a_1 \notin X$. We repeat this process recursively to generate a sequence $\{a_1, a_2, \ldots\} \subseteq A$. By T-nilpotency, there exists an integer n such that:

$$a_n a_{n-1} \cdots a_1 = 0.$$

After n steps, we have:

$$m = m_n a_n \cdots a_1 + \underbrace{\left(\sum_{k=1}^{n} x_k a_{k-1} \cdots a_1 \right)}_{\in X}.$$

Since $a_n \cdots a_1 = 0$, this collapses to $m \in X$, which contradicts $m \notin X$. Therefore, we must have $X = M$, and hence $MA \ll M$.

(2) $\implies$ (3): This implication is straightforward and holds trivially.

(3) $\implies$ (4): Let $M_R \neq 0$ and assume that $MA \neq M$ for any right ideal A of R. Let F_R be a countably generated free right R-module.

Suppose that $FA + X = F$ for some submodule $X \subseteq F$. We aim to show that $X = F$.

For contradiction, assume $X \neq F$. In this case, the quotient module $M = F/X$ is non-zero. The image of FA under the quotient map is $(F/X)A$. Since $FA + X = F$, every element of $M = F/X$ can be written as $f + X = fa + X$ for some $f \in F$, $a \in A$.

Thus, we have $MA = (F/X)A = F/X = M$, which means that the quotient module M is generated by the image of FA in F/X. However, this contradicts the assumption that $MA \neq M$. Therefore, our assumption that $X \neq F$ must be false.

Consequently, we conclude that $X = F$, and hence $FA \ll F$.

(4) $\implies$ (1): Assume that F_R is a countably generated free module and that $FA \ll F$, where $A \subseteq R$ is any ideal.

Let $B = \{f_1, f_2, f_3, \ldots\}$ be the countably infinite basis of F. For a sequence $a_1, a_2, \ldots$ in A, define the submodule

$$G = (f_1 - f_2 a_1)R + (f_2 - f_3 a_2)R + \ldots$$

For any basis element f_i, we can write

$$f_i = (f_i - f_{i+1}a_i) + f_{i+1}a_i,$$

where $(f_i - f_{i+1}a_i) \in G$ and $f_{i+1}a_i \in FA$.

Thus, $G + FA$ contains all the basis elements, so we have $G + FA = F$.

By hypothesis, since $FA \ll F$, it follows that $G = F$.

Therefore, every element of F belongs to G. Since G is generated by infinitely many elements $(f_i - f_{i+1}a_i)$, any specific element in G (such as f_1) can be written as a finite R-linear combination of these generators. Hence, there exists an integer n such that

$$f_1 = (f_1 - f_2a_1)r_1 + (f_2 - f_3a_2)r_2 + \cdots + (f_n - f_{n+1}a_n)r_n$$

for some $r_1, r_2, \ldots, r_n \in R$.

Expanding the right-hand side and matching coefficients of $f_1, f_2, \ldots, f_{n+1}$, we derive the following system of equations:

$$\begin{aligned}
1 &= r_1, \\
0 &= -a_1r_1 + r_2, \\
0 &= -a_2r_2 + r_3, \\
&\vdots \\
0 &= -a_{n-1}r_{n-1} + r_n, \\
0 &= -a_nr_n.
\end{aligned}$$

From this, we find $r_1 = 1$, $r_2 = a_1r_1 = a_1$, $r_3 = a_2r_2 = a_2a_1$, and so on, leading to $r_n = a_{n-1}\ldots a_1$, with the final equation $0 = a_nr_n = a_na_{n-1}\ldots a_1$.

To conclude T-nilpotency, we observe that $a_n \ldots a_1 = 0$.

Since the sequence $a_1, a_2, \ldots$ was arbitrary, this shows that every sequence in A is eventually zero under multiplication. Hence, A is right T-nilpotent.

□

Example 1.11. Let $M = M_R$ be a right R-module, and $A \subseteq J(R)$ a right ideal. If A is right T-nilpotent, then $MA \ll M$ (i.e., MA is superfluous in M).

Proof. We want to show that $MA \ll M$, which means that for any submodule $N \subseteq M$,

$$MA + N = M \implies N = M.$$

This is exactly the statement of Proposition 1.18(3) for the module M and submodule N. Since A is right T-nilpotent by assumption, Proposition 1.18(3) guarantees that the implication holds for all right R-modules. Therefore, $MA \ll M$ holds. □

Proposition 1.19. Let P_R be a projective right R-module and let $A \subseteq J(R)$ be a right T-nilpotent ideal. Then the natural surjection $\pi : P \to P/PA$ is a projective cover of P/PA.

Proof. To prove that $\pi : P \to P/PA$ is a projective cover, we check the necessary conditions:

1. **Projectivity**: By assumption, P is a projective right R-module.

2. **Superfluous kernel**: We need to show that $\ker \pi = PA$ is superfluous in P, that is, $PA \ll P$.

   From Proposition 1.18(3) and Example 1.11, since A is a right T-nilpotent ideal and $A \subseteq J(R)$, we know that for any right R-module M, $MA \ll M$. Applying this to $M = P$, we obtain:

$$PA = \ker \pi \ll P.$$

Since π is surjective by construction, P is projective, and $\ker \pi = PA$ is superfluous, we conclude that $\pi : P \to P/PA$ satisfies all the conditions of a projective cover.

□

1.1.7 Exact Sequences

Definition 1.16. 1. The pair of homomorphisms $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ is said to be **exact** (at Y) if Im $\alpha =$ Ker β.

2. A sequence $\cdots \xrightarrow{\xi_{n-1}} X_{n-1} \xrightarrow{\xi_n} X_n \xrightarrow{\xi_{n+1}} X_{n+1} \xrightarrow{\xi_{n+2}} \cdots$ of homomorphisms is said to be an **exact sequence** if it is exact at every X_n (i.e., Im $\xi_i =$ Ker ξ_{i+1} for all i).

Remark 1.7. Exactness means the image of one homomorphism exactly equals the kernel of the next homomorphism in the sequence.

Proposition 1.20. Let X, Y, and Z be R-modules over some ring R. Then:

1. The sequence $0 \longrightarrow X \xrightarrow{\psi} Y$ is exact (at X) if and only if ψ is a **monomorphism**.
2. The sequence $Y \xrightarrow{\varphi} Z \longrightarrow 0$ is exact (at Z) if and only if φ is an **epimorphism**.

Proof. 1. The (uniquely defined) homomorphism $0 \longrightarrow X$ has image 0 in X. This will be the kernel of ψ if and only if ψ is a monomorphism.

2. Similarly, the kernel of the (uniquely defined) zero homomorphism $Z \longrightarrow 0$ is all of Z, which is the image of φ if and only if φ is an epimorphism.

□

Corollary 1.3. *The sequence* $0 \longrightarrow X \xrightarrow{\psi} Y \xrightarrow{\varphi} Z \longrightarrow 0$ *is exact if and only if* ψ *is a monomorphism,* φ *is an epimorphism, and Im* $\psi =$ *Ker* φ.

Proof. Suppose The sequence $0 \longrightarrow X \xrightarrow{\psi} Y \xrightarrow{\varphi} Z \longrightarrow 0$ is exact.

Since the first mapping $0 \longrightarrow X$ has image 0, the exactness of $0 \longrightarrow X \xrightarrow{\psi} Y$ indicates that ψ is a monomorphism.

Since the last mapping $Z \longrightarrow 0$ is the zero homomorphism with kernel $= Z$, the exactness of $Y \xrightarrow{\varphi} Z \longrightarrow 0$ indicates that φ is an epimorphism.

The exactness of the sequence $0 \longrightarrow X \xrightarrow{\psi} Y \xrightarrow{\varphi} Z \longrightarrow 0$ implies that Im $\psi =$ Ker φ.

Conversely, if ψ is a monomorphism, φ is an epimorphism, and Im $\psi =$ Ker φ, then the sequence is exact by definition and Proposition 1.20.

$\square$

Remark 1.8.

1. Note that the homomorphism on the left and right of

$$0 \longrightarrow X \xrightarrow{\psi} Y \xrightarrow{\varphi} Z \longrightarrow 0$$

   need not be further specified since there is only one homomorphism from the 0 module to any given module and there is only one homomorphism from any given module to the 0 module.

2. The assertion that the sequence is exact (at X) is just that Ker $\psi = \text{Im } 0 = \{0\}$, i.e., ψ is a monomorphism.

3. The assertion that the sequence is exact (at Z) is just that $Z = \text{Ker } 0 = \text{Im } \varphi$, i.e., φ is an epimorphism.

4. The assertion that the sequence is exact (at Y) is just that Ker $\varphi =$ Im ψ.

5. Since $\varphi : Y \longrightarrow Z$ is an epimorphism, the first isomorphism theorem implies $Y/\text{Ker } \varphi \cong Z$, and then exactness (at Y) gives Ker $\varphi =$ Im ψ, which then implies $Y/\text{Im } \psi \cong Z$.

Definition 1.17. Let D be a fixed R-module. For any R-module L, consider the set of all R-module homomorphisms from D to L, denoted by $\text{Hom}_R(D, L)$. Given an R-module homomorphism $\varphi : L \to M$, we define the induced homomorphism:

$$\bar{\varphi} : \text{Hom}_R(D, L) \to \text{Hom}_R(D, M)$$

by

$$\bar{\varphi}(f) = \varphi \circ f \quad \text{for all} \quad f \in \text{Hom}_R(D, L).$$

Proposition 1.21. Let D, L, M, and N be R-modules. Then the following sequence

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N$$

is exact if and only if the sequence

$$0 \longrightarrow \mathrm{Hom}_R(D, L) \xrightarrow{\overline{\psi}} \mathrm{Hom}_R(D, M) \xrightarrow{\overline{\varphi}} \mathrm{Hom}_R(D, N)$$

is exact.

Proof. Suppose D, L, M, and N are R-modules, and the sequence

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N$$

is exact.

(i) We first show that $\overline{\psi}$ is a monomorphism.

Let $\overline{\psi}(f_1) = \overline{\psi}(f_2)$, where $f_1, f_2 \in \mathrm{Hom}_R(D, L)$. Then

$$\psi \circ f_1 = \psi \circ f_2,$$

which implies

$$(\psi \circ f_1)(x) = (\psi \circ f_2)(x) \quad \text{for all } x \in D.$$

Since ψ is a monomorphism (by hypothesis), it follows that

$$f_1(x) = f_2(x) \quad \text{for all } x \in D.$$

Thus, $f_1 = f_2$, and therefore, $\overline{\psi}$ is a monomorphism.

(ii) Next, we show that $\mathrm{Im}\,\overline{\psi} = \ker \overline{\varphi}$.

First, we show that $\mathrm{Im}\,\overline{\psi} \subseteq \ker \overline{\varphi}$.

Let $g \in \operatorname{Im} \overline{\psi}$. Then $g = \overline{\psi}(f)$ for some $f \in \operatorname{Hom}_R(D, L)$. Therefore,

$$g = \psi \circ f \quad \text{and} \quad \varphi \circ g = \varphi \circ (\psi \circ f) = \varphi(\psi(f)).$$

Since $\operatorname{Im} \psi = \ker \varphi$, we know that $\varphi \circ \psi = 0$, so

$$\varphi \circ g = 0 \quad \Rightarrow \quad g \in \ker \overline{\varphi}.$$

Thus, $\operatorname{Im} \overline{\psi} \subseteq \ker \overline{\varphi}$.

Next, we show the reverse inclusion $\ker \overline{\varphi} \subseteq \operatorname{Im} \overline{\psi}$.

Let $g \in \ker \overline{\varphi}$. Then $\varphi \circ g = 0$. Since $\operatorname{Im} \psi = \ker \varphi$, we have $g \in \operatorname{Im} \psi$, i.e., $g = \psi \circ f$ for some $f \in \operatorname{Hom}_R(D, L)$. Hence, $g \in \operatorname{Im} \overline{\psi}$.

Therefore, $\ker \overline{\varphi} \subseteq \operatorname{Im} \overline{\psi}$, and we conclude that $\operatorname{Im} \overline{\psi} = \ker \overline{\varphi}$.

Thus, the sequence

$$0 \longrightarrow \operatorname{Hom}_R(D, L) \xrightarrow{\overline{\psi}} \operatorname{Hom}_R(D, M) \xrightarrow{\overline{\varphi}} \operatorname{Hom}_R(D, N)$$

is exact.

Conversely, suppose that the second sequence is exact for any D. We will show that the first sequence is exact as well.

Let $D = \ker \psi$.

We observe that the map $\overline{\psi} : \operatorname{Hom}_R(\ker \psi, L) \longrightarrow \operatorname{Hom}_R(\ker \psi, M)$ is a monomorphism.

Now, let $i : \ker \psi \longrightarrow L$ be the inclusion map.

Thus, since $\overline{\psi}$ is a monomorphism. This means that $\overline{\psi}(i) = 0$ which implies $i = 0$.

Since $i : \ker \psi \longrightarrow L$ is the inclusion map and $i = 0$, it must be that $\ker \psi = 0$. Therefore, ψ is a monomorphism.

Next, we show that $\operatorname{Im} \psi = \ker \varphi$.

Let $D = L$. Then

$$\psi \circ id_D = \overline{\psi}(id_D) \in \operatorname{Im} \overline{\psi} = \ker \overline{\varphi}.$$

Thus,

$$\varphi \circ \psi = \overline{\varphi}(\psi) = 0 \quad \Rightarrow \quad \operatorname{Im} \psi \subseteq \ker \varphi.$$

Take $D = \ker \varphi$ and let $\sigma : \ker \varphi \longrightarrow M$, where $\ker \varphi$ is a submodule of N

$$\overline{\varphi}(\sigma) = \varphi \circ \sigma = 0,$$

which implies that $\sigma \in \ker \overline{\varphi} = \operatorname{Im} \overline{\psi}$.

Therefore, $\sigma = \overline{\psi}(\theta)$ for some $\theta \in \operatorname{Hom}_R(D, M)$, so $\sigma = \psi \circ \theta$.

Thus, $\ker \varphi = \operatorname{Im} \sigma \subseteq \operatorname{Im} \psi$, and we conclude that the first sequence is exact. $\square$

Example 1.12. Consider the exact sequence:

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$$

This induces the sequence:

$$0 \longrightarrow \operatorname{Hom}_R(D, L) \xrightarrow{\bar{\psi}} \operatorname{Hom}_R(D, M) \xrightarrow{\bar{\varphi}} \operatorname{Hom}_R(D, N) \longrightarrow 0$$

However, this sequence is **not** necessarily exact.

Counterexample: To illustrate this, consider the following short exact sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

Let $D = \mathbb{Z}/2\mathbb{Z}$ and consider the identity map $\mathrm{id}_{\mathbb{Z}/2\mathbb{Z}}$. The diagram below shows that no homomorphism $h \in \operatorname{Hom}_R(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$ satisfies $\pi \circ h = \mathrm{id}_{\mathbb{Z}/2\mathbb{Z}}$:

$$\begin{tikzcd} & & \mathbb{Z}/2\mathbb{Z} \arrow[dll, dashed, "\nexists h"'] \arrow[d, "\mathrm{id}_{\mathbb{Z}/2\mathbb{Z}}"] & \\ \mathbb{Z} \arrow[rr, "\pi"] & & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

where $M = \mathbb{Z}$, $N = \mathbb{Z}/2\mathbb{Z}$, and $D = \mathbb{Z}/2\mathbb{Z}$.

Proof. Any homomorphism $h \in \mathrm{Hom}_R(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$ must preserve the module structure, meaning:

$$h(a+b) = h(a) + h(b) \quad \forall a, b \in \mathbb{Z}/2\mathbb{Z}$$

In $\mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}_2 = \{0,1\}$, the element 1 has order 2, meaning $1+1=0$. Now, suppose $h(1) = n$ for some $n \in \mathbb{Z}$. By the additivity property of homomorphisms, we have:

$$h(1+1) = h(1) + h(1) = n + n = 2n$$

Since $1+1=0$ in $\mathbb{Z}/2\mathbb{Z}$, we must also have $h(0) = 2n$. However, the homomorphism property implies $h(0) = 0$. Thus, $2n = 0$, which forces $n = 0$. Therefore, $h(1) = 0$.

Thus, the homomorphism h maps every element of $\mathbb{Z}/2\mathbb{Z}$ to 0 in $\mathbb{Z}$, implying that $\pi \circ h \neq \mathrm{id}_{\mathbb{Z}/2\mathbb{Z}}$. $\square$

Conclusion: This counterexample shows that even if

$$M \xrightarrow{\varphi} N \longrightarrow 0$$

is exact, the induced sequence

$$\mathrm{Hom}_R(D, M) \xrightarrow{\bar{\varphi}} \mathrm{Hom}_R(D, N) \longrightarrow 0$$

is not necessarily exact.

Definition 1.18 (Direct Summand)**.** If M is an R-module and $M_1 \subseteq M$ is a submodule, we say that M_1 is a **direct summand** of M, or is **complemented** in M, if there is a submodule $M_2 \subseteq M$ such that $M \cong M_1 \oplus M_2$.

Definition 1.19. Consider the sequence of R-modules and R-module homomorphisms:

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$$

This sequence is said to be a **split exact sequence** (or simply **split**) if it is exact and if $\mathrm{Im}\psi = \mathrm{Ker}\varphi$ is a direct summand of M.

Remark 1.9 (For the next lemma (to be removed later)). Let $\phi : M \to L$ be a module homomorphism.

1. For a submodule N of M, the set $\phi^{-1}(\phi(N))$ is the preimage of the image of N under ϕ. Explicitly, we define it as:

$$\phi^{-1}(\phi(N)) = \{m \in M \mid \phi(m) \in \phi(N)\}.$$

   This set consists of all elements $m \in M$ such that there exists some $n \in N$ with $\phi(m) = \phi(n)$.

2. For a submodule D of L, the set $\phi(\phi^{-1}(D))$ is the image of the preimage of D under ϕ. Explicitly, we define it as:

$$\phi(\phi^{-1}(D)) = \{\phi(m) \in L \mid m \in M \text{ and } \phi(m) \in D\}.$$

   This set consists of all elements in L that are the images under ϕ of elements in M whose images are in D. In other words, for each element in $\phi(\phi^{-1}(D))$, there exists some $m \in M$ such that $\phi(m) \in D$.

Lemma 1.7. *Let M and L be R-modules, N and D submodules of M and L respectively, and $\psi : M \longrightarrow L$ an R-module homomorphism. Then we have:*

1. $\psi^{-1}(\psi(N)) = N + \ker\psi$

2. $\psi(\psi^{-1}(D)) = D \cap \mathrm{Im}\,\psi$

Proof. **Part (1)**: We will show two inclusions.

(i) $\psi^{-1}(\psi(N)) \subseteq N + \ker\psi$

(ii) $N + \ker\psi \subseteq \psi^{-1}(\psi(N))$

(i) Let $x \in \psi^{-1}(\psi(N)) = \{x \in M : \psi(x) \in \psi(N)\}$, i.e., $\psi(x) \in \psi(N)$. This means there exists $n \in N$ such that $\psi(x) = \psi(n)$. It follows that $\psi(x-n) = \psi(x) - \psi(n) = 0$, which implies $x - n \in \ker\psi$. Therefore, $x = n + k$, where $k = x - n \in \ker\psi$. Hence, $x \in N + \ker\psi$.

(ii) Now let $x \in N + \ker\psi$. This means $x = n + k$, where $n \in N$ and $k \in \ker\psi$. Applying ψ to x, we get:

$$\psi(x) = \psi(n+k) = \psi(n) + \psi(k) = \psi(n) + 0 = \psi(n).$$

Since $\psi(n) = \psi(x) \in \psi(N)$, by the definition of $\psi^{-1}(\psi(N)) = \{x \in M : \psi(x) \in \psi(N)\}$, it follows that $x \in \psi^{-1}(\psi(N))$. Thus, by (i) and (ii), we conclude that $\psi^{-1}(\psi(N)) = N + \ker\psi$.

Part (2): We will again show two inclusions:

(i) $\psi(\psi^{-1}(D)) \subseteq D \cap \operatorname{Im}\psi$

(ii) $D \cap \operatorname{Im}\psi \subseteq \psi(\psi^{-1}(D))$

(i) Let $y \in \psi(\psi^{-1}(D)) = \{y \in L \mid y = \psi(x) \text{ for some } x \in \psi^{-1}(D)\}$. This means there exists some $x \in \psi^{-1}(D)$ such that $\psi(x) = y$. Since $x \in \psi^{-1}(D)$, we have, $\psi(x) \in D$. Therefore, $y = \psi(x) \in D$. Additionally, since $y = \psi(x)$, it follows that $y \in \operatorname{Im}\psi$. Hence, $y \in D \cap \operatorname{Im}\psi$.

(ii) Now let $y \in D \cap \operatorname{Im}\psi$. This implies $y \in D$ and $y \in \operatorname{Im}\psi$. Since $y \in \operatorname{Im}\psi$, there exists some $x \in M$ such that $\psi(x) = y$. Because $y \in D$ and $\psi(x) = y$, it follows that $x \in \psi^{-1}(D)$. Applying ψ to x gives $\psi(x) = y$, and therefore, $y = \psi(x)$ is in the image of ψ applied to $\psi^{-1}(D)$, which is exactly $\psi(\psi^{-1}(D))$.

By (i) and (ii), it follows that $\psi(\psi^{-1}(D)) = D \cap \operatorname{Im}\psi$. Thus, part (2) is proven. $\square$

Lemma 1.8. *Let X, Y, and Z be R-modules, and consider the sequence of homomorphisms $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$. Then:*

1. $\ker(\beta \circ \alpha) = \alpha^{-1}(\ker \beta)$
2. $\operatorname{Im}(\beta \circ \alpha) = \beta(\operatorname{Im} \alpha)$

Proof. **Part (1)**

We establish the equality by proving two inclusions.

(i) $\operatorname{Ker}(\beta \circ \alpha) \subseteq \alpha^{-1}(\operatorname{Ker}\beta)$

(ii) $\alpha^{-1}(\operatorname{Ker}\beta) \subseteq \operatorname{Ker}(\beta \circ \alpha)$

(i) Let $x \in \operatorname{Ker}(\beta \circ \alpha)$. Then $(\beta \circ \alpha)(x) = 0$, which implies $\beta(\alpha(x)) = 0$. Consequently, $\alpha(x) \in \operatorname{Ker}\beta$. Since $\alpha^{-1}(\operatorname{Ker}\beta) = \{x \in X : \alpha(x) \in \operatorname{Ker}\beta\}$. Therefore, $x \in \alpha^{-1}(\operatorname{Ker}\beta)$.

(ii) Let $x \in \alpha^{-1}(\operatorname{Ker}\beta)$. Then $\alpha(x) \in \operatorname{Ker}\beta$, which implies $\beta(\alpha(x)) = 0$. Hence, $(\beta \circ \alpha)(x) = 0$, and it follows that $x \in \operatorname{Ker}(\beta \circ \alpha)$.

Part (2)

By definition, $\operatorname{Im}(\beta \circ \alpha) = \{z \in Z : z = (\beta \circ \alpha)(x) \text{ for some } x \in X\}$.

Thus, $\operatorname{Im}(\beta \circ \alpha) = (\beta \circ \alpha)(X) = \beta(\alpha(X)) = \beta(\operatorname{Im}\alpha)$.

Therefore, $\operatorname{Im}(\beta \circ \alpha) = \beta(\operatorname{Im}\alpha)$.

□

Lemma 1.9. *Let the diagram*

$$\begin{tikzcd}
 & D \arrow[dl, "f"'] \arrow[d, "\lambda"] \\
N \arrow[r, "g"] & M
\end{tikzcd}$$

be commutative, i.e., $g \circ f = \lambda$. Then

1. $\operatorname{Im} f + \ker g = g^{-1}(\operatorname{Im} \lambda)$

2. $\operatorname{Im} f \cap \ker g = f(\ker \lambda)$

Furthermore, the following special cases hold:

(i) *If λ is an epimorphism, then* $\operatorname{Im} f + \ker g = g^{-1}(M) = N$.

(ii) *If λ is a monomorphism, then* $\operatorname{Im} f \cap \ker g = f(0) = 0$.

(iii) *If λ is an isomorphism, then* $\operatorname{Im} f \oplus \ker g = N$.

Proof. (1) Since $\lambda = g \circ f$, it follows that $\operatorname{Im} \lambda = \operatorname{Im}(g \circ f) = g(\operatorname{Im} f)$. Therefore, $g^{-1}(\operatorname{Im} \lambda) = g^{-1}(g(\operatorname{Im} f))$. Because $\operatorname{Im} f \subseteq N$ is a submodule, Lemma 1.7 implies

$$g^{-1}(g(\operatorname{Im} f)) = \operatorname{Im} f + \ker g.$$

(2) We have $\ker \lambda = \ker(g \circ f) = f^{-1}(\ker g)$ by Lemma 1.8. Applying f yields

$$f(\ker \lambda) = f(f^{-1}(\ker g)).$$

Since $\ker g$ is a submodule of N, Lemma 1.7 implies

$$f(f^{-1}(\ker g)) = \ker g \cap \operatorname{Im} f.$$

□

Proposition 1.22. The following statements are equivalent for the exact sequence

$$0 \longrightarrow X \xrightarrow{f} M \xrightarrow{g} Y \longrightarrow 0$$

(1) The sequence splits.

(2) There exists a homomorphism $\bar{f} : M \longrightarrow X$ such that $\bar{f} \circ f = \mathrm{id}_X$.

(3) There exists a homomorphism $\bar{g} : Y \longrightarrow M$ such that $g \circ \bar{g} = \mathrm{id}_Y$.

Proof. (1) $\implies$ (2): Suppose that the sequence splits. Then $M \cong \operatorname{Im} f \oplus M_1$ for some submodule $M_1 \subseteq M$.

Let $\pi : M \longrightarrow \operatorname{Im} f$ be the canonical projection defined by $\pi(f(x) + m_1) := f(x)$, where $f(x) \in \operatorname{Im} f$ and $m_1 \in M_1$.

Define the restricted function $f_0 : X \longrightarrow \operatorname{Im} f$ by $f_0(x) = f(x)$ for all $x \in X$. Since f is injective, f_0 is also injective. Moreover, by definition, f_0 is surjective onto $\operatorname{Im} f$. Therefore, f_0 is an isomorphism and hence invertible.

Let $f_0^{-1} : \operatorname{Im} f \longrightarrow X$ be the inverse of f_0. Now, consider $\bar{f} := f_0^{-1} \circ \pi$. Then for any $x \in X$, we have

$$\bar{f}(f(x)) = f_0^{-1}(\pi(f(x))) = f_0^{-1}(f(x)) = x.$$

Thus, $\bar{f} \circ f = \mathrm{id}_X$.

(2) $\implies$ (3): Suppose there exists $\bar{f} : M \longrightarrow X$ such that $\bar{f} \circ f = \mathrm{id}_X$.

Observe that for any $m \in M$,

$$\begin{aligned}
\bar{f}(m - f(\bar{f}(m))) &= \bar{f}(m) - \bar{f}(f(\bar{f}(m))) \\
&= \bar{f}(m) - (\bar{f} \circ f) \circ \bar{f}(m) \\
&= \bar{f}(m) - (\mathrm{id}_X \circ \bar{f})(m) \\
&= \bar{f}(m) - \bar{f}(m) \\
&= 0.
\end{aligned}$$

Hence, $m - f(\bar{f}(m)) \in \ker \bar{f}$.

Notice that m can be expressed as

$$m = f(\bar{f}(m)) + (m - f(\bar{f}(m))),$$

where $f(\bar{f}(m)) \in \operatorname{Im} f$ and $m - f(\bar{f}(m))) \in \ker \bar{f}$.

Since every $m \in M$ can be written in this form, we conclude that $M = \operatorname{Im} f + \ker \bar{f}$.

Now, suppose $f(y) = m \in \operatorname{Im} f \cap \ker \bar{f}$. Then $0 = \bar{f}(m) = \bar{f}(f(y)) = y$, so $y = 0$, and thus $\operatorname{Im} f \cap \ker \bar{f} = \{0\}$. Therefore, $M = \operatorname{Im} f \oplus \ker \bar{f}$.

Next, define a homomorphism $\bar{g} : Y \longrightarrow M$ by $\bar{g}(y) = m - f(\bar{f}(m))$, where $g(m) = y$.

Since g is surjective, there exists an $m \in M$ for each $y \in Y$. However, m may not be unique because multiple elements in M can map to the same y under g. Therefore, we must verify that $\bar{g}$ is well-defined.

Suppose $g(m) = y = g(m')$. We must verify that $\bar{g}(g(m)) = \bar{g}(g(m'))$, i.e.,

$$m - f(\bar{f}(m)) = m' - f(\bar{f}(m'))).$$

First, observe that

$$0 = g(m) - g(m') = g(m - m'),$$

so $m - m' \in \ker g = \operatorname{Im} f$. Then $m - m' = f(x)$ for some $x \in X$.

Now, consider the difference

$$\begin{aligned}
&(m - f(\bar{f}(m))) - (m' - f(\bar{f}(m')))) \\
&= (m - m') - (f(\bar{f}(m)) - f(\bar{f}(m')))) \\
&= f(x) - f(\bar{f}(m) - \bar{f}(m'))).
\end{aligned}$$

So the difference is in $\operatorname{Im} f$. Applying $\bar{f}$ to the difference, we get

$$\begin{aligned}
\bar{f}\left((m - f(\bar{f}(m))) - (m' - f(\bar{f}(m')))\right) &= \bar{f}\left(f(x) - f(\bar{f}(m) - \bar{f}(m'))\right) \\
&= x - \left(\bar{f}(m) - \bar{f}(m')\right) \\
&= x - \bar{f}(m - m') \\
&= x - \bar{f}(f(x)) \\
&= x - x \\
&= 0.
\end{aligned}$$

So, the difference lies in $\ker \bar{f}$ as well and hence lies in the intersection. Since $\operatorname{Im} f \cap \ker \bar{f} = \{0\}$, the difference must be zero. This means

$$(m - f(\bar{f}(m))) = (m' - f(\bar{f}(m'))),$$

and hence $\bar{g}(y)$ is the same regardless of the choice of m. Therefore, $\bar{g}$ is well-defined.

Finally, it is clear from the construction of $\bar{g}$ that

$$g \circ \bar{g}(y) = g(m - f(\bar{f}(m)))) = g(m) - g(f(\bar{f}(m)))).$$

Since the exactness of the sequence implies $\operatorname{Im} f = \ker g$, we have $g(f(\bar{f}(m)))) = 0$. Therefore,

$$g \circ \bar{g}(y) = g(m) = y,$$

and hence $g \circ \bar{g} = \mathrm{id}_Y$.

This shows that (2) $\implies$ (3).

(3) $\implies$ (1): Suppose there exists $\bar{g} : Y \longrightarrow M$ such that $g \circ \bar{g} = \mathrm{id}_Y$.

Consider the following commutative diagram:

$$\begin{tikzcd}
 & Y \arrow[dl, "\bar{g}"'] \arrow[d, "\mathrm{id}_Y"] \\
M \arrow[r, "g"] & Y
\end{tikzcd}$$

Since id_Y is an isomorphism Lemma 1.9 implies, then $\operatorname{Im} \bar{g} \oplus \ker g = M$ and hence $\ker g = \operatorname{Im} f$ is a direct summand of M. Thus, the sequence splits. $\square$

1.2 Projective Modules

Definition 1.20. A module P is called **projective** if, for any epimorphism $\varphi : M \to N$ and any homomorphism $g : P \to N$, there exists a homomorphism $h : P \to M$ such that $\varphi \circ h = g$.

In other words, any diagram of the form

$$\begin{tikzcd} & P \arrow[d, "g"] & \\ M \arrow[r, "\varphi"] & N \arrow[r] & 0 \end{tikzcd}$$

where the bottom row is exact, can be completed to the following commutative diagram:

$$\begin{tikzcd} & P \arrow[d, "g"] \arrow[dl, dashed, "h"'] & \\ M \arrow[r, "\varphi"] & N \arrow[r] & 0 \end{tikzcd}$$

Proposition 1.23. Let D, L, M, and N be R-modules. Then the following sequence

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$$

is exact if and only if the sequence

$$0 \longrightarrow \mathrm{Hom}_R(D, L) \xrightarrow{\overline{\psi}} \mathrm{Hom}_R(D, M) \xrightarrow{\overline{\varphi}} \mathrm{Hom}_R(D, N)$$

is exact.

Proof. Suppose D, L, M, and N are R-modules, and the sequence

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$$

is exact.

(i) We first show that $\overline{\psi}$ is a monomorphism.

Let $\overline{\psi}(f_1) = \overline{\psi}(f_2)$, where $f_1, f_2 \in \mathrm{Hom}_R(D, L)$. Then

$$\psi \circ f_1 = \psi \circ f_2,$$

which implies

$$(\psi \circ f_1)(x) = (\psi \circ f_2)(x) \quad \text{for all } x \in D.$$

Since ψ is a monomorphism (by hypothesis), it follows that

$$f_1(x) = f_2(x) \quad \text{for all } x \in D.$$

Thus, $f_1 = f_2$, and therefore, $\overline{\psi}$ is a monomorphism.

(ii) Next, we show that $\mathrm{Im}(\overline{\psi}) = \mathrm{Ker}(\overline{\varphi})$.

First, we show that $\mathrm{Im}(\overline{\psi}) \subseteq \mathrm{Ker}(\overline{\varphi})$.

Let $g \in \mathrm{Im}(\overline{\psi})$. Then $g = \overline{\psi}(f)$ for some $f \in \mathrm{Hom}_R(D, L)$. Therefore,

$$g = \psi \circ f \quad \text{and} \quad \varphi \circ g = \varphi \circ (\psi \circ f) = (\varphi \circ \psi) \circ f.$$

Since $\mathrm{Im}(\psi) = \mathrm{Ker}(\varphi)$, we know that $\varphi \circ \psi = 0$, so

$$\varphi \circ g = 0 \quad \Rightarrow \quad g \in \mathrm{Ker}(\overline{\varphi}).$$

Thus, $\mathrm{Im}(\overline{\psi}) \subseteq \mathrm{Ker}(\overline{\varphi})$.

Next, we show the reverse inclusion $\mathrm{Ker}(\overline{\varphi}) \subseteq \mathrm{Im}(\overline{\psi})$.

Let $g \in \mathrm{Ker}(\overline{\varphi})$. Then $\varphi \circ g = 0$. Since $\mathrm{Im}(\psi) = \mathrm{Ker}(\varphi)$, we have $g \in \mathrm{Im}(\psi)$, i.e., $g = \psi \circ f$ for some $f \in \mathrm{Hom}_R(D, L)$. Hence, $g \in \mathrm{Im}(\overline{\psi})$.

Therefore, $\mathrm{Ker}(\overline{\varphi}) \subseteq \mathrm{Im}(\overline{\psi})$, and we conclude that $\mathrm{Im}(\overline{\psi}) = \mathrm{Ker}(\overline{\varphi})$.

Thus, the sequence

$$0 \longrightarrow \mathrm{Hom}_R(D, L) \xrightarrow{\overline{\psi}} \mathrm{Hom}_R(D, M) \xrightarrow{\overline{\varphi}} \mathrm{Hom}_R(D, N)$$

is exact.

Conversely, suppose that the second sequence is exact for any D. We will show that the first sequence is exact as well.

Let $D = \mathrm{Ker}(\psi)$.

We observe that the map $\mathrm{Hom}_R(\mathrm{Ker}(\psi), L) \longrightarrow \mathrm{Hom}_R(\mathrm{Ker}(\psi), M)$ is a monomorphism.

Now, let $i : \mathrm{Ker}(\psi) \longrightarrow L$ be the inclusion map.

Thus, we have

$$\psi \circ i = 0 \quad \Rightarrow \quad \overline{\psi}(i) = 0 \quad \Rightarrow \quad i = 0.$$

Since $\overline{\psi}$ is a monomorphism, it follows that $\mathrm{Ker}(\psi) = 0$.

Therefore, ψ is a monomorphisim

Next, we show that $\mathrm{Im}(\psi) = \mathrm{Ker}(\varphi)$.

Let $D = L$. Then

$$\psi \circ 1_D = \overline{\psi}(1_D) \in \mathrm{Im}(\overline{\psi}) = \mathrm{Ker}(\overline{\varphi}).$$

Thus,

$$\varphi \circ \psi = \overline{\varphi}(\psi) = 0 \quad \Rightarrow \quad \mathrm{Im}(\psi) \subseteq \mathrm{Ker}(\varphi).$$

Take $D = \mathrm{Ker}(\varphi)$ and let $\sigma : \mathrm{Ker}(\varphi) \longrightarrow M$, where $\mathrm{Ker}(\varphi)$ is a submodule of N

$$\overline{\varphi}(\sigma) = \varphi \circ \sigma = 0,$$

which implies that $\sigma \in \mathrm{Ker}(\overline{\varphi}) = \mathrm{Im}(\overline{\psi})$.

Therefore, $\sigma = \overline{\psi}(\theta)$ for some $\theta \in \mathrm{Hom}_R(D, M)$, so $\sigma = \psi \circ \theta$.

Thus, $\mathrm{Ker}(\varphi) = \mathrm{im}(\sigma) \subseteq \mathrm{Im}(\psi)$, and we conclude that the first sequence is exact. $\square$

Proposition 1.24. An R-module P is projective if and only if $\mathrm{Hom}_R(P, -)$ is an exact functor.

i.e., For any R-modules L, M, and N,

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$$

is a short exact sequence if and only if

$$0 \longrightarrow \mathrm{Hom}_R(P, L) \xrightarrow{\overline{\psi}} \mathrm{Hom}_R(P, M) \xrightarrow{\overline{\varphi}} \mathrm{Hom}_R(P, N) \longrightarrow 0$$

is a short exact sequence.

Proof. Let P be a projective R-module.

Suppose that

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$$

is an arbitrary exact sequence of R-modules.

We want to show that

$$0 \longrightarrow \mathrm{Hom}_R(P, L) \xrightarrow{\overline{\psi}} \mathrm{Hom}_R(P, M) \xrightarrow{\overline{\varphi}} \mathrm{Hom}_R(P, N) \longrightarrow 0$$

is exact.

By proposition 1.23, we know that

$$0 \longrightarrow \mathrm{Hom}_R(P, L) \xrightarrow{\overline{\psi}} \mathrm{Hom}_R(P, M) \xrightarrow{\overline{\varphi}} \mathrm{Hom}_R(P, N)$$

is exact for any module P. So, it remains to show that $\overline{\varphi}$ is an epimorphism.

Let $g \in \mathrm{Hom}_R(P, N)$. Since P is projective, there exists $h : P \to M$ such that $\varphi \circ h = g$.

Then we have the following commutative diagram:

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists h"'] \arrow[d, "g"] & \\ M \arrow[r, "\varphi"] & N \arrow[r] & 0 \end{tikzcd}$$

This implies that $\overline{\varphi}(h) = g$, where $h \in \text{Hom}_R(P, M)$.

Therefore, $\overline{\varphi}$ is an epimorphism, and thus the second sequence is exact.

Conversely, consider the diagram:

$$\begin{tikzcd} & P \arrow[d, "g"] & \\ M \arrow[r, "\varphi"] & N \arrow[r] & 0 \end{tikzcd}$$

and the exact sequence:

$$0 \longrightarrow \ker(\varphi) \xrightarrow{i} M \xrightarrow{\varphi} N \longrightarrow 0$$

This sequence is exact since i is a monomorphism and φ is an epimorphism.

Thus,

$$0 \longrightarrow \text{Hom}_R(P, \ker(\varphi)) \xrightarrow{\bar{i}} \text{Hom}_R(P, M) \xrightarrow{\overline{\varphi}} \text{Hom}_R(P, N) \longrightarrow 0$$

is exact as well. In particular, $\overline{\varphi}$ is an epimorphism.

Since $g \in \text{Hom}(P, N)$, $\exists h \in \text{Hom}(P, M)$ such that $\overline{\varphi}(h) = g$, implying $\varphi \circ h = g$.

Therefore, P is projective. □

Lemma 1.10. ***(Universal Property of Direct Sums)***
Let $\varphi_i : X_i \to Y$ be a set of homomorphisms of R-modules, where $i \in I$. Then, there exists a unique homomorphism $\xi : \bigoplus_{i\in I} X_i \to Y$ such that the following diagrams commute for all $i \in I$:

$$\begin{tikzcd}
X_i \arrow[r, "\varphi_i"] \arrow[d, "\sigma_i"'] & Y \\
\bigoplus_{i\in I} X_i \arrow[ur, "\xi"'] &
\end{tikzcd}$$

Proposition 1.25. A direct sum $P = \bigoplus_{j\in J} P_j$ of modules is a projective module if and only if each P_j is projective.

Proof. Suppose $P = \bigoplus_{j\in J} P_j$ is projective, and let $\pi_j : P \longrightarrow P_j$ be a canonical projection homomorphism from P to P_j. Thus, $g_j \circ \pi_j$ is a homomorphism from P to N.

Since P is projective, there exists a homomorphism $h : P \longrightarrow M$ such that $\varphi \circ h = g_j \circ \pi_j$.

$$\begin{tikzcd}
& P = \bigoplus_{j\in J} P_j \arrow[d, "\pi_j"] \arrow[ddl, dashed, "\exists h"'] & \\
& P_j \arrow[d, "g_j"] & \\
M \arrow[r, "\varphi"] & N \arrow[r] & 0
\end{tikzcd}$$

Thus, if $i_j : P_j \longrightarrow P$ is the canonical injection map, define $f_j : P_j \longrightarrow M$ by $f_j := h \circ i_j$.

We have

$$\varphi \circ f_j = \varphi \circ (h \circ i_j) = (\varphi \circ h) \circ i_j = (g_j \circ \pi_j) \circ i_j = g_j \circ (\pi_j \circ i_j) = g_j \circ \mathrm{id}_{P_j} = g_j.$$

$$\begin{tikzcd}
& P_j \arrow[d, "g_j"] \arrow[dl, dashed, "f_j"'] & \\
M \arrow[r, "\varphi"] & N \arrow[r] & 0
\end{tikzcd}$$

Hence, P_j is projective for all $j \in J$.

Conversely, Assume that each P_j is projective and consider a diagram

$$\begin{tikzcd}
& P \arrow[d, "g"] & \\
M \arrow[r, "\varphi"] & N \arrow[r] & 0
\end{tikzcd}$$

with the bottom row exact. For each $j \in J$, define a homomorphism $\zeta_j : P_j \to N$ by $\zeta_j = g \circ i_j$.

Since P_j is projective for all $j \in I$, there exist homomorphisms $h_j : P_j \to M$, such that $\varphi \circ h_j = g \circ i_j$.

$$\begin{tikzcd}
 & P_j \arrow[d, "i_j"] \arrow[dl, dashed, "\exists h_j"'] & \\
 & P = \bigoplus_{j\in J} P_j \arrow[d, "g"] & \\
M \arrow[r, "\varphi"] & N \arrow[r] & 0
\end{tikzcd}$$

Next, by the Universal Property of Direct Sums, there exists a unique homomorphism $\xi : P \to M$ such that $\xi \circ i_j = h_j$.

Applying φ to both sides, we obtain:

$$\varphi \circ (\xi \circ i_j) = \varphi \circ h_j = g \circ i_j.$$

This gives:

$$\varphi \circ \xi = g.$$

Thus, P is projective

□

Corollary 1.4. *Every direct summand of a projective module is projective.*

Proof. Let P be a projective module, and let A be a direct summand of P. Then, we can write P as a direct sum:

$$P = A \oplus B$$

for some module B.

We want to show that A is projective. To do so, consider a homomorphism $f : A \to N$, where N is some module. This situation can be represented in the following diagram:

$$\begin{tikzcd} & A \arrow[d, "f"] & \\ M \arrow[r, "\varphi"] & N \arrow[r] & 0 \end{tikzcd}$$

Since P is projective, there exists a homomorphism $h : P \to M$ such that $\varphi \circ h = f \circ \pi_A$, where $\pi_A : P \to A$ is the canonical projection map from P to A. The diagram becomes:

$$\begin{tikzcd} & P = A \oplus B \arrow[d, "\pi_A"] \arrow[ddl, dashed, "\exists h"'] & \\ & A \arrow[d, "f"] & \\ M \arrow[r, "\varphi"] & N \arrow[r] & 0 \end{tikzcd}$$

Let $i_A : A \to P$ be the canonical injective map, where i_A is the inclusion map into the direct sum $P = A \oplus B$. Since $\varphi \circ h = f \circ \pi_A$, we define a homomorphism $\lambda : A \to N$ by

$$\lambda := h \circ i_A.$$

It follows that λ is a homomorphism, and we can verify the equality:

$$\varphi \circ \lambda = \varphi \circ (h \circ i_A) = f \circ (\pi_A \circ i_A) = f \circ id_A = f.$$

Therefore, we have shown that A is projective. □

Proposition 1.26. Free modules are projective

Proof. Let F be a free module with basis $\{f_i \in F : i \in I\}$. Consider the following diagram:

$$\begin{tikzcd} & F \arrow[d, "\xi"] & \\ M \arrow[r, "\varphi"] & N \arrow[r] & 0 \end{tikzcd}$$

Since any module is an epimorphic image of a free module, we have $\xi(f_i) = n_i \in N$. Now, as φ is an epimorphism, there exists $m_i \in M$ such that $\varphi(m_i) = n_i$.

Define $h : F \to M$ by $h\left(\sum_{i\in I} f_i a_i\right) = \sum_{i\in I} m_i a_i$.

The map h is well-defined since every $f \in F$ can be uniquely expressed in terms of the basis elements. Moreover, h is a homomorphism. Thus, we obtain the following diagram:

$$\begin{tikzcd} & F \arrow[dl, dashed, "\exists h"'] \arrow[d, "\xi"] & \\ M \arrow[r, "\varphi"] & N \arrow[r] & 0 \end{tikzcd}$$

Next, observe that:

$$\varphi\circ h\left(\sum_{i\in I} f_i a_i\right) = \varphi\left(\sum_{i\in I} m_i a_i\right) = \sum_{i\in I} \varphi(m_i) a_i = \sum_{i\in I} n_i a_i = \sum_{i\in I} \xi(f_i) a_i = \xi\left(\sum_{i\in I} f_i a_i\right) = \xi(f)$$

Therefore, $\varphi \circ h = \xi$, and thus F is projective. $\square$

Proposition 1.27 (Modified Test For Projectivity)**.** Let P and M be modules over a ring R. The following are equivalent:

1. P is a projective module.
2. For every epimorphism $\pi : M \to M/N$, where N is a submodule of M, and for every homomorphism $f : P \to M/N$, there exists a homomorphism $\lambda : P \to M$ such that $\pi \circ \lambda = f$.

   This can be visualized by the following commutative diagram:

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists \lambda"'] \arrow[d, "f"] & \\ M \arrow[r, "\pi"] & M/N \arrow[r] & 0 \end{tikzcd}$$

Proof. $(1 \Rightarrow 2)$ Assume P is projective. By definition, for any epimorphism $\varphi : M \to L$ and homomorphism $h : P \to L$, there exists a lifting $\lambda : P \to M$ such that $\varphi \circ \lambda = h$. This situation can be visualized by the following diagram:

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists\lambda"'] \arrow[d, "h"] \\ M \arrow[r, "\varphi"] & L \end{tikzcd}$$

Now, let $L = M/N$. The natural projection $\pi : M \to M/N$ is an epimorphism. For any $f : P \to M/N$, the definition of projectivity guarantees the existence of $\lambda : P \to M$ such that $\pi \circ \lambda = f$. The situation is represented by the following diagram:

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists\lambda"'] \arrow[d, "f"] & \\ M \arrow[r, "\pi"] & M/N \arrow[r] & 0 \end{tikzcd}$$

This shows that (2) follows directly from (1) as a special case.

$(2 \Rightarrow 1)$

Suppose the following diagram is commutative:

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists\lambda"'] \arrow[d, "f"] & \\ M \arrow[r, "\pi"] & M/N \arrow[r] & 0 \end{tikzcd}$$

i.e., $\lambda \circ \pi = f$.

Now consider the following diagram, where the bottom row is exact:

$$\begin{tikzcd} & P \arrow[d, "g"] & \\ M \arrow[r, "\varphi"] & L \arrow[r] & 0 \end{tikzcd}$$

To show that P is projective, we need to find a map h such that $\varphi \circ h = g$.

Since φ is an epimorphism, the first isomorphism theorem implies that $L \cong M/\ker(\varphi)$.

Let $\xi : L \longrightarrow M/\ker(\varphi)$ be the corresponding isomorphism. Let $\sigma : M \longrightarrow M/\ker(\varphi)$

be the homomorphism given by $\sigma = \xi \circ \varphi$. Thus, the diagram becomes:

$$\begin{tikzcd} & P \arrow[d, "g"] & \\ M \arrow[r, "\varphi"] \arrow[dr, "\sigma"] & L \arrow[r] \arrow[d, "\xi"] & 0 \\ & M/\ker(\varphi) & \end{tikzcd}$$

Next, consider the situation represented by the following diagram:

$$\begin{tikzcd} & P \arrow[d, "\xi\circ g"] \arrow[dl, dashed, "\exists\bar{h}"'] & \\ M \arrow[r, "\sigma"] & M/\ker(\varphi) \arrow[r] & 0 \end{tikzcd}$$

By hypothesis, there exists a homomorphism $\bar{h} : P \longrightarrow M$ such that $\sigma \circ \bar{h} = \xi \circ g$.

That is,

$$(\xi \circ \varphi) \circ \bar{h} = \xi \circ g$$

Applying ξ^{-1} on both sides:

$$\xi^{-1} \circ (\xi \circ \varphi) \circ \bar{h} = \xi^{-1} \circ (\xi \circ g)$$

$$\Rightarrow \varphi \circ \bar{h} = g$$

$$\begin{tikzcd} & P \arrow[d, "g"] \arrow[dl, dashed, "\exists\bar{h}"'] & \\ M \arrow[r, "\varphi"] & L \arrow[r] & 0 \end{tikzcd}$$

Hence, P is projective.

$\square$

Now, we present an alternative proof of the theorem: *Every free module is projective.* Suppose F is a free R-module, so $F \cong \bigoplus_{i\in I} R$ for some index set I.

Our goal is to show that F is projective. First, we establish that R, as a module over itself, is projective. Then, applying the fact that a direct sum of projective modules is projective, together with the modified test for projectivity, we conclude that F, being a direct sum of projective modules, is projective.

Proposition 1.28. Every free R-module is projective.

Proof. Suppose F is a free R-module. Then $F \cong \bigoplus_{i\in I} R$ for some index set I. Let R be a module over itself and consider the following diagram with exact row:

$$\begin{tikzcd} & R \arrow[d, "f"] & \\ M \arrow[r, "\pi"] & M/N \arrow[r] & 0 \end{tikzcd}$$

Define $\lambda : R \longrightarrow M$ by $\lambda(1) = m$. Then $\lambda(r) = \lambda(1 \cdot r) = \lambda(1) \cdot r = m \cdot r$.

We show λ is well-defined. Suppose $r_1 = r_2$, then $mr_1 = mr_2$, so $\lambda(r_1) = \lambda(r_2)$. Hence λ is well-defined.

Next, We show that λ is a homomorphism:

(i)

$$\lambda(r_1 + r_2) = m(r_1 + r_2) = mr_1 + mr_2 = \lambda(r_1) + \lambda(r_2).$$

(ii)

$$\lambda(r_1 r) = m(r_1 r) = (mr_1)r = \lambda(r_1) \cdot r.$$

Hence, λ is a homomorphism.

Now, we show that $\pi \circ \lambda = f$:

$$\pi \circ \lambda(r) = \pi(mr) = mr + N = (m + N)r = f(1)r.$$

since

$$f : R \longrightarrow M/N, \quad f(r) = (m + N)r.$$

Hence, $\pi \circ \lambda = f$.

$$\begin{tikzcd}
 & P \arrow[dl, dashed, "\lambda"'] \arrow[d, "f"] & \\
M \arrow[r, "\pi"] & M/N \arrow[r] & 0
\end{tikzcd}$$

Thus, $\pi \circ \lambda = f$.

$\implies$ R is projective, and hence $F = R \oplus R \oplus \dots$ where each R is projective, so F is projective. □

□

Proposition 1.29. Let R be a ring. For an R-module P, the following are equivalent:

1. P is projective.
2. Every short exact sequence $0 \to N \to M \xrightarrow{\pi} P \to 0$ splits.
3. P is a direct summand of a free R-module.

Proof. **(1 ⇒ 2)** Suppose P is a projective R-module and consider the diagram

$$0 \longrightarrow N \longrightarrow M \xrightarrow{\pi} P \longrightarrow 0$$

where π is an epimorphism. Let $\mathrm{id}_P : P \to P$ be the identity map on P. Since P is projective, there exists $h : P \to M$ such that $\pi \circ h = \mathrm{id}_P$.

We can then visualize the situation with the following diagram:

$$\begin{tikzcd}
 & & & P \arrow[dl, dashed, "\exists h"'] \arrow[d, "id_P"] & \\
0 \arrow[r] & N \arrow[r] & M \arrow[r, "\pi"] & P \arrow[r] & 0
\end{tikzcd}$$

So then the sequence splits.

(2 ⇒ 3) Suppose every short exact sequence splits. Since any module is an epimorphic image of a free module, there exists a free module F and an epimorphism $\pi : F \to P$.

So we get the short exact sequence

$$0 \longrightarrow \ker(\pi) \xrightarrow{i} F \xrightarrow{\pi} P \longrightarrow 0$$

Since the sequence splits, we conclude that $F = \ker(\pi) \oplus P$.

Hence, Corollary 1 implies that F is projective.

(3 ⇒ 1) Suppose P is a direct summand of a free module, so that

$$F = P \oplus A, \quad \text{where } F \text{ is free.}$$

Since every free module is projective, we have

$$\Rightarrow F \text{ is projective.}$$

And since a direct summand of a projective module is projective, we conclude

$$\Rightarrow P \text{ is projective.}$$

□

Theorem 1.2. *For a ring R, the following statements are equivalent:*

1. *R is a semisimple ring.*
2. *Any R-module M is projective.*

Proof. **(1)** $\implies$ **(2)**:

Assume R is a semisimple ring. Then the right regular module R_R decomposes into a direct sum of simple submodules, which means that R_R is a semisimple module. Any free R-module F is a direct sum of copies of R_R, and since R_R is semisimple, F is also a semisimple module.

Let M be any R-module. By Corollary 1.1, M is isomorphic to a quotient of some free R-module F, i.e., $M \cong F/K$, where K is a submodule of F.

Since F is semisimple, any submodule K of F is a direct summand, so we can write $F = K \oplus N$ for some submodule N of F. Thus, we have:

$$M \cong F/K = \frac{\cancel{K} \oplus N}{\cancel{K}} \cong N.$$

By Proposition 1.29, a module is projective if and only if it is a direct summand of a free module. Since N is a direct summand of F, it follows that N is projective, and consequently M is projective.

(2) $\implies$ (1):

Suppose that any R-module is projective. Let M be an arbitrary R-module and let $N \subseteq M$ be a submodule of M. By hypothesis, since every R-module is projective, the quotient module M/N is projective.

Consider the short exact sequence:

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0.$$

Since M/N is projective, this sequence splits (by Proposition 1.29), meaning that N is a direct summand of M.

Because $N \hookrightarrow M$ was arbitrary, every submodule of M is a direct summand. By definition of semisimplicity, this implies that M is semisimple.

Since M was arbitrary, we conclude that every right R-module is semisimple, including the regular module R_R. Therefore, R is a semisimple ring. $\square$

The following theorem demonstrates a remarkable property of projective modules: any nonzero projective module, whether finitely generated or not, always possesses a maximal submodule.

Theorem 1.3. *Let P be a nonzero projective right R-module. Then, the radical of P,* *denoted by* $rad\, P$*, satisfies*

$$rad\, P = P \cdot J(R) \subsetneq P,$$

where $J(R)$ *represents the Jacobson radical of the ring* R*.*

Definition 1.21. Let R be a ring and M a right R-module. The **annihilator** of M in R, denoted by $\operatorname{Ann}_R(M)$, is defined as:

$$\operatorname{Ann}_R(M) = \{r \in R \mid mr = 0 \text{ for all } m \in M\}.$$

It is a (two-sided) ideal of R consisting of all elements of R that act as zero on every element of M.

Lemma 1.11. *Every cyclic right* R*-module* mR *is isomorphic to* $R/\operatorname{Ann}_R(M)$*.*

Proof. Define a map $f : R \longrightarrow mR$ by $f(r) = mr$. This is a well-defined R-module homomorphism. It is surjective because mR is generated by m.

The kernel of f is:

$$\ker f = \{r \in R \mid mr = 0\} = \{r \in R \mid r \in \operatorname{Ann}_R(M)\} = \operatorname{Ann}_R(M).$$

By the First Isomorphism Theorem, we have:

$$R/\operatorname{Ann}_R(M) \cong mR = M.$$

□

Lemma 1.12. *If* $M = mR$ *is a cyclic projective right* R*-module, then* M *is isomorphic to a direct summand of* R*.*

Proof. Since $M = mR$ is cyclic, by Lemma 1.11, we have:

$$M \cong R/I \quad \text{where } I = \operatorname{Ann}_R(M).$$

Since M is projective, the short exact sequence:

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

splits. Hence,

$$R \cong I \oplus (R/I),$$

which implies that $M \cong R/I$ is isomorphic to a direct summand of R. $\square$

Example 1.13. The ring of integers $\mathbb{Z}$ is a projective $\mathbb{Z}$-module. To see this, observe that $\mathbb{Z}$ is a free $\mathbb{Z}$-module with $\{1\}$ as a basis. Every element $z \in \mathbb{Z}$ can be written uniquely as $z = n \cdot 1$ for some $n \in \mathbb{Z}$, showing that $\{1\}$ spans $\mathbb{Z}$. Additionally, $\{1\}$ is linearly independent, as the equation $n \cdot 1 = 0$ implies $n = 0$. Since $\mathbb{Z}$ is free and every free module is projective, it follows that $\mathbb{Z}$ is projective.

Example 1.14. Every free module is projective, but the converse is not true: not every projective module is free. To illustrate this, consider the ring $R = \mathbb{Z}_6$. The ideals $\mathbb{Z}_3$ and $\mathbb{Z}_2$ can be viewed as $\mathbb{Z}_6$-modules. By the Chinese Remainder Theorem, there is a $\mathbb{Z}_6$-module isomorphism:

$$\mathbb{Z}_6 \cong \mathbb{Z}_3 \oplus \mathbb{Z}_2.$$

This decomposition shows that $\mathbb{Z}_6$ splits into a direct sum of $\mathbb{Z}_3$ and $\mathbb{Z}_2$. Since $\mathbb{Z}_6$ is a free $\mathbb{Z}_6$-module (with basis $\{1\}$), and direct summands of free modules are projective Proposition 1.29, it follows that both $\mathbb{Z}_3$ and $\mathbb{Z}_2$ are projective $\mathbb{Z}_6$-modules. However, neither $\mathbb{Z}_3$ nor $\mathbb{Z}_2$ is a free $\mathbb{Z}_6$-module. This can be shown in two ways:

Solution 1: Linear Independence Argument

A free $\mathbb{Z}_6$-module must have a basis consisting of linearly independent elements. However, in $\mathbb{Z}_3$, every element x satisfies $3x = 0$, and in $\mathbb{Z}_2$, every element y satisfies $2y = 0$. These relations violate linear independence, as they imply that no nonempty subset of $\mathbb{Z}_3$ or $\mathbb{Z}_2$ can be linearly independent over $\mathbb{Z}_6$. Therefore, $\mathbb{Z}_3$ and $\mathbb{Z}_2$ cannot have a basis, and thus they are not free $\mathbb{Z}_6$-modules.

Alternative Solution: Cardinality Argument

Suppose, for contradiction, that $\mathbb{Z}_2$ is a free $\mathbb{Z}_6$-module. If $\mathbb{Z}_2$ were free, it would have to be isomorphic to a direct sum of copies of $\mathbb{Z}_6$, since $\mathbb{Z}_6$ is the free $\mathbb{Z}_6$-module of rank 1.

That is, there would exist an isomorphism:

$$\mathbb{Z}_2 \cong \mathbb{Z}_6^{(I)},$$

where $\mathbb{Z}_6^{(I)}$ denotes the direct sum of $|I|$ copies of $\mathbb{Z}_6$, and I is some index set. However, this is impossible because:

- The group $\mathbb{Z}_2$ has exactly 2 elements.
- The group $\mathbb{Z}_6$ has exactly 6 elements, and thus any direct sum of copies of $\mathbb{Z}_6$ must have a cardinality divisible by 6 (or be infinite).

Since 2 is not divisible by 6, $\mathbb{Z}_2$ cannot be isomorphic to $\mathbb{Z}_6^{(I)}$ for any index set I. Therefore, $\mathbb{Z}_2$ cannot be a free $\mathbb{Z}_6$-module. A similar argument applies to $\mathbb{Z}_3$, showing that it is also not free.

Example 1.15. Let R be a ring and $e \in R$ an idempotent element ($e^2 = e$). Then the right ideal eR is projective.

Proof. By the Right Peirce Decomposition Theorem, $R = eR \oplus (1-e)R$ shows eR is a direct summand of the free module R_R. Direct summands of free modules are projective. $\square$

Remark 1.10. Proposition 1.29 shows that (1) direct summands of free modules are projective, but (2) they are not necessarily free. For example, $\mathbb{Z}_3$ and $\mathbb{Z}_2$ as $\mathbb{Z}_6$-modules satisfy $\mathbb{Z}_6 \cong \mathbb{Z}_3 \oplus \mathbb{Z}_2$, so both are projective but neither is free.

Chapter 2

Special Rings and Projective Covers

Building upon the foundational concepts of projective modules from Chapter 1, this chapter shifts focus to specific classes of rings whose structure is intimately related to projectivity. We introduce local and semilocal rings, exploring their defining properties, particularly concerning their Jacobson radicals and quotient structures. A key result, Theorem 2.1, demonstrates that finitely generated projective modules over local rings possess the strong property of being free. We then introduce the important concepts of semiperfect and perfect rings, laying the groundwork for their characterization in the next chapter. Central to this discussion is the notion of a projective cover, which is defined and examined, preparing for its crucial role in the main theorems of this research.

2.1 Local Rings and Semilocal Rings

2.1.1 Local Rings

Definition 2.1 (Local Ring)**.** A non-zero ring R is called local if it has a unique maximal right ideal.

Proposition 2.1. TFAE for a ring R with Jacobson radical J(R)

1. R is local.

2. J(R) is the unique maximal right ideal of R.

3. All non-invertible elements of R form a proper ideal.

4. J(R) is the set of all non-invertible elements of R.

5. The quotient R/J(R) is a division ring.

Proof. □

Lemma 2.1 (Nakayama's Lemma)**.** *Let N be a submodule of a finitely generated module M and $N + MJ(R) = M$. Then $N = M$.*

Proof. □

Theorem 2.1. *If R is a local ring, then each finitely generated projective R-module is free.*

Proof. Let R be a local ring with Jacobson Radical $J(R)$, and let P be a finitely generated projective R-module.

As R is a local ring, the quotient ring $R/J(R)$ is a division ring. Since P is a finitely generated R-module, the quotient module $P/PJ(R)$ is finitely generated over R by the images of the generators of P. Furthermore, because $J(R)$ annihilates $P/PJ(R)$, $P/PJ(R)$ can be viewed as a finitely generated module over the division ring $R/J(R)$. Any finitely generated module over a division ring is a finite-dimensional vector space and thus is free; therefore, $P/PJ(R)$ is a finitely generated free $R/J(R)$-module.

Thus, $P/PJ(R) \cong \bigoplus_{i=1}^{n}(R/J(R))$.

Let F be the free R-module $F = \bigoplus_{i=1}^{n} R$.

Moreover, we have

$$FJ(R) \cong \bigoplus_{i=1}^{n} RJ(R) = \underbrace{(R \oplus \cdots \oplus R)}_{n \text{ times}} J(R),$$

which gives

$$FJ(R) \cong \underbrace{J(R) \oplus \cdots \oplus J(R)}_{n \text{ times}} = \bigoplus_{i=1}^{n} J(R).$$

Thus, the quotient

$$F/FJ(R) \cong (\bigoplus_{i=1}^{n} R/\bigoplus_{i=1}^{n} J(R)) = \bigoplus_{i=1}^{n}(R/J(R)).$$

Since $P/PJ(R) \cong F/FJ(R)$, let

$$\psi : F/FJ(R) \longrightarrow P/PJ(R)$$

be the corresponding isomorphism of $R/J(R)$-modules, and let

$$\pi : F \longrightarrow F/FJ(R) \quad \text{and} \quad \sigma : P \longrightarrow P/PJ(R)$$

be the natural projections. Then, $\alpha = \psi \circ \pi$ is a homomorphism from F to $P/PJ(R)$.

Since F is a free module, and thus a projective module, there exists a homomorphism

$$\varphi : F \longrightarrow P$$

such that the following diagram

$$\begin{tikzcd}
0 \arrow[r] & FJ(R) \arrow[r] & F \arrow[r, "\pi"] \arrow[d, "\exists\varphi"'] \arrow[dr, "\alpha"] & F/FJ(R) \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & PJ(R) \arrow[r] & P \arrow[r, "\sigma"] & P/PJ(R) \arrow[r] & 0
\end{tikzcd}$$

is commutative, i.e., $\sigma \circ \varphi = \alpha = \psi \circ \pi$.

We shall show that φ is an isomorphism.

1. Surjectivity of φ

For $f \in F$,

$$\psi \circ \pi(f) = \psi(\pi(f)) = \psi(f + FJ(R)) \quad \text{and} \quad \sigma \circ \varphi(f) = \sigma(\varphi(f)) = \varphi(f) + PJ(R).$$

By commutativity of the diagram,

$$\psi(f + FJ(R)) = \varphi(f) + PJ(R).$$

Since ψ is surjective,

$$\mathrm{Im}(\varphi) + PJ(R) = P.$$

Since P is finitely generated and $PJ(R) \subset P$, Nakayama's lemma gives

$$\mathrm{Im}(\varphi) = P.$$

Thus, φ is surjective.

2. Injectivity (Monomorphism)

Consider the exact sequence:

$$0 \longrightarrow \mathrm{Ker}(\varphi) \longrightarrow F \xrightarrow{\varphi} P \longrightarrow 0$$

Since P is projective, the exact sequence splits, i.e., there exists $\varphi' : P \longrightarrow F$ such that $\varphi \circ \varphi' = 1_P$.

Thus, we have:

$$F = W \oplus X \cong \mathrm{Ker}(\varphi) \oplus P.$$

It follows that:

$$FJ(R) = (W \oplus X)J(R) = WJ(R) \oplus XJ(R).$$

As $W \cong \mathrm{Ker}(\varphi)$ is a submodule of F, let $f \in W$. Then, $\varphi(f) = 0$, and by the natural projection σ, we have $\sigma(\varphi(f)) = 0 = \psi(\pi(f))$. Since ψ is an isomorphism, this implies $\pi(f) = 0 = f + FJ(R)$, or equivalently, $f \in FJ(R)$. Hence, $W \subseteq FJ(R)$.

Since $W \cong \mathrm{Ker}(\varphi) \subseteq FJ(R)$ and $FJ(R) = WJ(R) \oplus XJ(R)$, we have

$$W = W \cap FJ(R) = [W \cap WJ(R)] \oplus [W \cap XJ(R)] = WJ(R) \oplus [W \cap XJ(R)].$$

Thus, W decomposes as a direct sum. Note that $WJ(R) \subseteq W$ ensures the validity of this decomposition.

$$W = WJ(R) \oplus [W \cap XJ(R)].$$

However, since $W \cap XJ(R) \subseteq W \cap X = 0$,

$$W = WJ(R).$$

Since W is a direct summand of a finitely generated module, it follows that W is finitely generated. By Nakayama's Lemma, we conclude

$$W = 0.$$

Therefore, φ is a monomorphism.

Hence, φ is an isomorphism, and P is free. □

Remark 2.1. This theorem still holds in a more general setting. Namely, I. Kaplansky proved that all projective modules over a local ring are free.

Definition 2.2 (Right Artinian Ring)**.** A ring R is called a right Artinian ring if it satisfies the descending chain condition on right ideals, meaning that every descending chain of right ideals in R eventually stabilizes. In other words, for any sequence of right ideals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$, there exists an integer n such that $I_n = I_{n+1} = I_{n+2} = \dots$.

2.1.2 Semilocal Rings

Definition 2.3 (Semilocal)**.** A ring R is called **semilocal** if the quotient ring $\bar{R} = R/J(R)$, is a right Artinian ring, or equivalently if $\bar{R}$ is a semisimple ring.

Proposition 2.2. For a ring R, consider the following two conditions:

1. R is semilocal.

2. R has finitely many maximal left ideals.

We have, in general, (2) $\Rightarrow$ (1). The converse holds if $R/\operatorname{rad} R$ is commutative.

Proof. □

Proposition 2.3. An epimorphic image of a semilocal R-module M is semilocal.

Proof. Let N be an epimorphic image of a semilocal R-module M, i.e., there exists an epimorphism $\phi : M \to N$. Set $K = \ker \phi$. Then, by the First Isomorphism Theorem, we have $M/K \cong N$, so N is essentially a quotient module of M.

Since M is semilocal, the module $M/\mathrm{rad}M$ is semisimple. Observe that

$$\mathrm{rad}M \subseteq K + \mathrm{rad}M \subseteq M,$$

so there is a natural epimorphism

$$\phi : M/\mathrm{rad}M \to M/(K + \mathrm{rad}M).$$

By Lemma 1.3, an epimorphic image of a semisimple module is semisimple. Therefore, $M/(K + \mathrm{rad}M)$ is semisimple.

Moreover, since $K \subseteq K + \mathrm{rad}M$, the Third Isomorphism Theorem implies

$$\frac{M/K}{(K + \mathrm{rad}M)/K} \cong \frac{M}{K + \mathrm{rad}M},$$

and hence

$$\frac{M/K}{(K + \mathrm{rad}M)/K} \text{ is semisimple.}$$

By Proposition 1.7, we have

$$(K + \mathrm{rad}M)/K \subseteq \mathrm{rad}(M/K).$$

This inclusion gives rise to an epimorphism

$$\xi : \frac{M/K}{(K + \mathrm{rad} M)/K} \to \frac{M/K}{\mathrm{rad}(M/K)}.$$

Again, by Lemma 1.3, the module $\frac{M/K}{\mathrm{rad}(M/K)}$ is semisimple. Therefore, M/K is semilocal.

Since $N \cong M/K$, it follows that N is semilocal, as required. □

Proposition 2.4. The following are equivalent for a ring R:

1. R is semilocal.
2. Every right R-module with $\operatorname{rad} M = 0$ is semilocal.
3. Every right R-module is semilocal.

Proof. **(1) $\implies$ (2)**

Suppose that R is semilocal, and let M be a right R-module with $\operatorname{rad} M = 0$. By Proposition 1.2, M is an epimorphic image of a free module F. So there exists an epimorphism $\varphi : F \to M$. Since F is free, we have

$$F/\operatorname{rad} F \cong \bigoplus_{i \in I} R,$$

or equivalently, Proposition 1.16 gives us that

$$F/\operatorname{rad} F \cong \bigoplus_{i \in I} R/\operatorname{rad} R.$$

$R/\operatorname{rad} R$ is semisimple (because R is semilocal), so $\bigoplus_{i \in I} R/\operatorname{rad} R$ is also semisimple, thus $F/\operatorname{rad} F$ is semisimple. Hence F is semilocal.

Since Proposition 2.3 dictates that every epimorphic image of a semilocal module is semilocal, it follows that M is semilocal.

(2) $\implies$ (3)

Assume that any right R-module X with $\operatorname{rad} X = 0$ is semilocal. Let M be an arbitrary right R-module. Then $M/\operatorname{rad} M$ satisfies $\operatorname{rad}(M/\operatorname{rad} M) = 0$. By hypothesis, $M/\operatorname{rad} M$ is semilocal, thus

$$\frac{M/\operatorname{rad} M}{\operatorname{rad}(M/\operatorname{rad} M)} = \frac{M/\operatorname{rad} M}{0} \cong M/\operatorname{rad} M$$

is semisimple. Therefore, M is semilocal.

(3) $\implies$ (1)

Assume that every right R-module is semilocal. Then, in particular, R_R is semilocal. $\square$

2.2 Perfect & Semiperfect Rings

2.2.1 Semiperfect Rings

Definition 2.4 (Semiperfect ring)**.** A ring R is called semiperfect if R is semilocal, and idempotents of $R/J(R)$ can be lifted to R.

2.2.2 Projective Covers

Definition 2.5 (Projective Cover)**.** A projective module P is called a *projective cover* of a module M and it is denoted by $P(M)$ if there is an epimorphism $\varphi : P \to M$ such that $\mathrm{Ker}(\varphi)$ is a small submodule in P.

Example 2.1. Let P_R be a finitely generated projective R-module, and let $A \subseteq J(R)$ be a right ideal. Then the natural projection $\pi \colon P \to P/PA$ is a projective cover of P/PA.

Proof. Since P is projective by assumption, we only need to verify that $\ker \pi = PA$ is small in P. The surjectivity of π is immediate from its definition. By Example 1.6, since P is finitely generated and $A \subseteq J(R)$, the submodule PA must be small in P. Therefore, (P, π) is indeed a projective cover. $\square$

Example 2.2. Let eR be a right ideal where e is an idempotent and $A \subseteq J(R)$. Consider the natural projection $\pi : eR \longrightarrow eR/eA$.

Then eR/eA has eR as its projective cover.

Proof. First, note that eR is a regular right R-module. By Example 1.15, this implies that eR is a projective right R-module.

The map $\pi : eR \to eR/eA$ is an epimorphism with kernel eA. Since eR is cyclic (and hence finitely generated), it follows from Example 2.1 that (π, eR) is indeed a projective cover of eR/eA. $\square$

Lemma 2.2. *Let $\sigma_i \colon P_i \to M_i$ be R-module homomorphisms for $1 \leq i \leq n$. Define the direct sum map*

$$\bigoplus \sigma_i \colon \bigoplus_{i=1}^{n} P_i \to \bigoplus_{i=1}^{n} M_i$$

by

$$\left(\bigoplus \sigma_i\right)(p_1, \ldots, p_n) = (\sigma_1(p_1), \ldots, \sigma_n(p_n)).$$

Then,

$$\ker\left(\bigoplus \sigma_i\right) = \bigoplus_{i=1}^{n} \ker(\sigma_i).$$

Proof. Let $P = \bigoplus_{i=1}^n P_i$ and $M = \bigoplus_{i=1}^n M_i$, and let $\sigma = \bigoplus \sigma_i \colon P \to M$ be defined as above.

We compute the kernel of σ directly:

$$\begin{aligned}
\ker(\sigma) &= \{(p_1, \ldots, p_n) \in P \mid \sigma(p_1, \ldots, p_n) = 0\} \\
&= \left\{(p_1, \ldots, p_n) \in \bigoplus_{i=1}^n P_i \mid (\sigma_1(p_1), \ldots, \sigma_n(p_n)) = (0, \ldots, 0)\right\} \\
&= \left\{(p_1, \ldots, p_n) \in \bigoplus_{i=1}^n P_i \mid \sigma_i(p_i) = 0 \text{ for all } i\right\} \\
&= \left\{(p_1, \ldots, p_n) \in \bigoplus_{i=1}^n P_i \mid p_i \in \ker(\sigma_i) \text{ for all } i\right\} \\
&= \bigoplus_{i=1}^n \ker(\sigma_i).
\end{aligned}$$

Hence,

$$\ker\left(\bigoplus \sigma_i\right) = \bigoplus_{i=1}^n \ker(\sigma_i).$$

$\square$

Lemma 2.3. *Let $\sigma_i \colon P_i \to M_i$ be R-module homomorphisms for $1 \leq i \leq n$. Define the direct sum map*

$$\bigoplus \sigma_i \colon \bigoplus_{i=1}^n P_i \to \bigoplus_{i=1}^n M_i$$

by

$$\left(\bigoplus \sigma_i\right)(p_1, \ldots, p_n) = (\sigma_1(p_1), \ldots, \sigma_n(p_n)).$$

Then, the map $\bigoplus \sigma_i$ is surjective if and only if each $\sigma_i \colon P_i \to M_i$ is surjective.

Proof. Let $P = \bigoplus_{i=1}^n P_i$ and $M = \bigoplus_{i=1}^n M_i$, and let $\sigma = \bigoplus \sigma_i \colon P \to M$ be defined as above.

We will show that $\bigoplus \sigma_i$ is surjective if and only if each σ_i is surjective.

(1) $\Rightarrow$ Direction: If $\bigoplus \sigma_i$ is surjective, then each σ_i is surjective.

Suppose that $\bigoplus \sigma_i \colon P \to M$ is surjective. This means that for every element $(m_1, \ldots, m_n) \in$

$M = \bigoplus_{i=1}^n M_i$, there exists a tuple $(p_1, \ldots, p_n) \in P = \bigoplus_{i=1}^n P_i$ such that:

$$\bigoplus \sigma_i(p_1, \ldots, p_n) = (\sigma_1(p_1), \ldots, \sigma_n(p_n)) = (m_1, \ldots, m_n).$$

For each i, this implies that $\sigma_i(p_i) = m_i$. Since this holds for any choice of $m_i \in M_i$, it follows that σ_i is surjective for each i.

(2) $\Leftarrow$ Direction: If each σ_i is surjective, then $\bigoplus \sigma_i$ is surjective.

Suppose that each $\sigma_i \colon P_i \to M_i$ is surjective. This means that for every $m_i \in M_i$, there exists a $p_i \in P_i$ such that $\sigma_i(p_i) = m_i$.

Now, consider an arbitrary element $(m_1, \ldots, m_n) \in \bigoplus_{i=1}^n M_i$. Since each σ_i is surjective, for each $m_i \in M_i$, there exists a $p_i \in P_i$ such that $\sigma_i(p_i) = m_i$. Thus, the tuple $(p_1, p_2, \ldots, p_n) \in \bigoplus_{i=1}^n P_i$ satisfies:

$$\bigoplus \sigma_i(p_1, p_2, \ldots, p_n) = (\sigma_1(p_1), \sigma_2(p_2), \ldots, \sigma_n(p_n)) = (m_1, m_2, \ldots, m_n).$$

Therefore, $\bigoplus \sigma_i$ is surjective. $\square$

Proposition 2.5. Let $\sigma_i : P_i \to M_i$ be projective covers for $1 \leq i \leq n$. Then $\bigoplus_{i=1}^n \sigma_i : \bigoplus_{i=1}^n P_i \to \bigoplus_{i=1}^n M_i$ is also a projective cover.

Proof. Given for each i, $\sigma_i : P_i \to M_i$ is a projective cover, so $\ker \sigma_i \ll P_i$.

Since each P_i is projective, Proposition 1.18 implies that $\bigoplus_{i=1}^n P_i$ is also projective.

Since each $\sigma_i : P_i \to M_i$ is surjective, Lemma 2.3 implies $\bigoplus_{i=1}^n \sigma_i : \bigoplus_{i=1}^n P_i \to \bigoplus_{i=1}^n M_i$ is also surjective.

Lemma 2.2 gives $\ker\left(\bigoplus_{i=1}^n \sigma_i\right) = \bigoplus_{i=1}^n \ker \sigma_i$.

By Example 1.10, since $\ker \sigma_i \ll P_i$ for all i, we have $\bigoplus_{i=1}^n \ker \sigma_i \ll \bigoplus_{i=1}^n P_i$.

Thus $(\bigoplus_{i=1}^n \sigma_i, \bigoplus_{i=1}^n P_i)$ is a projective cover of $\bigoplus_{i=1}^n M_i$. $\square$

Lemma 2.4. *Let $P \xrightarrow{\varphi} M \to 0$ be a projective cover of an R-module M. Then*

$$\operatorname{Ker}\varphi \subseteq \mathrm{radP}\,.$$

Proof. By definition, $\operatorname{Ker}\varphi$ is a small submodule of P. By Proposition 2.6, the radical radP is the sum of all small submodules of P. Since $\operatorname{Ker}\varphi$ is a small submodule, it follows that

$$\operatorname{Ker}\varphi \subseteq \mathrm{radP}\,.$$

□

Corollary 2.1. *If $P \xrightarrow{\varphi} U \longrightarrow 0$ is a projective cover of a simple module U, then $\operatorname{Ker}\varphi = \operatorname{rad} P$. Consequently, the projective cover of a simple module has exactly one maximal submodule.*

Proof. Let $P \xrightarrow{\varphi} U \longrightarrow 0$ be a projective cover of the simple module U. By Lemma 2.4, we know that $\operatorname{Ker}\varphi \subseteq \operatorname{rad} P$.

To prove the reverse inclusion, we proceed as follows. Since $\varphi : P \to U$ is an epimorphism, the First Isomorphism Theorem implies:

$$P/\operatorname{Ker}\varphi \cong U.$$

Since U is simple, it follows that $\operatorname{Ker}\varphi$ is a maximal submodule of P. By definition, $\operatorname{rad} P$ is the intersection of all maximal submodules of P, and we deduce that:

$$\operatorname{rad} P \subseteq \operatorname{Ker}\varphi.$$

Thus, combining both inclusions, we conclude that:

$$\operatorname{Ker}\varphi = \operatorname{rad} P.$$

□

Remark 2.2. [Lemma 27.3, Anderson and Fuller] A cyclic module M_R has a projective cover if and only if $M \cong eR/eI$ for some idempotent $e \in R$ and some right ideal $I \subseteq J(R)$. For e and I satisfying this condition, the natural map

$$eR \to eR/eI \to 0$$

is a projective cover.

Lemma 2.5. *Let R be a ring and let A be an ideal of R with $A \subseteq J(R)$. Then the following are equivalent:*

1. *Idempotents lift modulo A;*

2. *Every direct summand of the right R-module R/A has a projective cover.*

Proof. (1) $\implies$ (2) Assume idempotents lift modulo A, where $A \subseteq J(R)$. Let $R/A = X \oplus Y$ be a direct sum decomposition. By Proposition 1.10, there exists an idempotent $\bar{e} \in R/A$ such that $X = \bar{e}(R/A)$. By hypothesis, $\bar{e}$ lifts to an idempotent $e \in R$ satisfying $e + A = \bar{e}$.

Thus we have:

$$X = \bar{e}(R/A) = (e + A)(R/A) = \frac{eR + A}{A}.$$

Applying the Second Isomorphism Theorem yields:

$$\frac{eR + A}{A} \cong \frac{eR}{A \cap eR}.$$

Since A is an ideal, $A \cap eR = eA$, and consequently:

$$X \cong \frac{eR}{eA}.$$

We now verify that the natural projection $\pi : eR \to eR/eA$ is a projective cover:

1. **Projectivity**: By Example 1.15, eR is projective.

2. **Epimorphism**: The map π is clearly surjective.

3. **Small kernel**: Note that $\ker(\pi) = eA$. Since $A \subseteq J(R)$, we have $eA \subseteq eJ(R)$. By Theorem 1.3, $eJ(R) \ll eR$ (as eR is projective). Example 1.9 implies that submodules of small submodules are small, hence $eA \ll eR$.

By the hypothesis of Remark 2.2, *every* direct summand of R/A admits a projective cover.

□

Definition 2.6 (Semiprimary ring)**.** A ring R with Jacobson radical $J(R)$ is said to be semiprimary if it possesses the following two conditions:

1. $J(R)$ is nilpotent.

2. $R/J(R)$ is semisimple.

Definition 2.7 (Semiprimitive)**.** A ring R is called semiprimitive if its Jacobson radical is equal to zero.

Corollary 2.2. *If R is a semiprimitive ring and $P(U)$ is a projective cover of a simple R-module U, then $P(U) \simeq U$.*

Proof. Let $P(U) \xrightarrow{\varphi} U \longrightarrow 0$ be a projective cover of a simple R-module U. Since R is semiprimitive, the Jacobson radical of R, denoted $J(R)$, satisfies $J(R) = 0$. By Theorem 1.3, the radical of $P(U)$, denoted $\operatorname{rad} P(U)$, is given by:

$$\operatorname{rad} P(U) = P(U) \cdot J(R).$$

Substituting $J(R) = 0$, we obtain:

$$\operatorname{rad} P(U) = P(U) \cdot 0 = 0.$$

Thus, $\operatorname{rad} P(U) = 0$.

From Lemma 2.4, we know that $\operatorname{Ker}\varphi \subseteq \operatorname{rad} P(U)$. Since $\operatorname{rad} P(U) = 0$, it follows that:

$$\operatorname{Ker}\varphi \subseteq 0.$$

This implies $\operatorname{Ker}\varphi = 0$.

Since $\varphi : P(U) \to U$ is surjective (by definition of a projective cover) and $\operatorname{Ker}\varphi = 0$, the map φ is also injective. Therefore, φ is an isomorphism, and we conclude:

$$P(U) \cong U.$$

□

Lemma 2.6. *A semiprimitive ring in which every cyclic right module has a projective cover is semisimple.*

Proof. Let R be a semiprimitive ring, i.e., $J(R) = 0$ and suppose every cyclic right R-module has a projective cover.

We shall show that R is the sum of all its minimal right ideals. Let $S = \operatorname{Soc}(R_R)$ be the sum of all minimal right ideals of the ring R.

Assume for the sake of contradiction, that $S \neq R$. Then S is contained in a maximal right ideal I of the ring R. We claim that the module $R/I = U$ is simple.

Suppose M is a submodule of R/I. Then M corresponds to a right ideal K of R such that $I \subseteq K \subseteq R$. Since I is maximal, we have:

- If $K = I$, then $M = 0$.
- If $K = R$, then $M = R/I$.

Hence $R/I = U$ is a simple module, proving the claim.

Next, we claim that $U = R/I$ is cyclic since it is generated by the coset $1 + I$. Specifically, every element in U can be written as $(1 + I)r$ for some $r \in R$.

The hypothesis implies that U has a projective cover $P(U)$. Moreover Corollary 2.2 implies $P(U) \cong U$.

Consider the short exact sequence

$$0 \longrightarrow I \xrightarrow{i} R \xrightarrow{\pi} U \longrightarrow 0$$

Since $U \cong P(U)$ is projective, this exact sequence splits by Proposition 1.29 which means $R \cong I \oplus U$.

Since $R \cong I \oplus U$ we have $I \cap U = 0$ and since $S \subseteq I$ (by maximality of I) we obtain $U \nsubseteq S$. However U is a simple module, and since $S = \mathrm{Soc}(R_R)$ contains all minimal right ideals (and thus all simple submodules).

This means U should be contained in S. Contradicting $U \nsubseteq S$. Thus our assumption that $S \neq R$ must be incorrect and hence $R = \mathrm{Soc}(R_R)$ is a sum of simple modules (minimal ideals) which implies R is a semisimple ring, completing the proof.

□

Lemma 2.7 (Bass Lemma)**.** *Let $U \to P$ with P projective R-module and let P/U have a projective cover. Then there is a decomposition $P = P_1 \oplus P_2$ with $P_1 \subseteq U$ and $U \cap P_2 << P$.*

In particular, if A is a right ideal of a ring R and R/A has a projective cover, then $R = P_1 \oplus P_2$ with $P_1 \subseteq A$ and $A \cap P_2 << R$.

Proof. Let $f : N \to P/U$ be a projective cover of P/U, where U is a submodule of P. This means that N is projective and f is an epimorphism with a superfluous kernel. Let $\eta : P \to P/U$ be the natural projection. Consider the following diagram:

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists\alpha"'] \arrow[d, "\eta"] \\ N \arrow[r, "f"] & P/U \end{tikzcd}$$

Since P is projective, there exists a homomorphism $\alpha : P \to N$ such that $f \circ \alpha = \eta$. By

Lemma 1.9, and due to the commutativity of the diagram and the surjectivity of η, we obtain the relation

$$\mathrm{Im}\alpha + \ker f = N.$$

Since $\ker f \ll N$, it follows that $\mathrm{Im}\alpha = N$, so α is an epimorphism. Consider the short exact sequence:

$$0 \longrightarrow \ker \alpha \xrightarrow{i} P \xrightarrow{\alpha} N \longrightarrow 0$$

From the projectivity of N, Proposition 1.29 implies that this sequence splits, i.e., $P = \ker \alpha \oplus T$ for some submodule $T \subseteq P$.

Now, $\ker \alpha \subseteq \ker \eta$. Indeed, since $\eta = f \circ \alpha$, if $x \in \ker \alpha$, then $\eta(x) = f(\alpha(x)) = f(0) = 0$, so $x \in \ker \eta$. Therefore, $\ker \alpha \subseteq \ker \eta = U$.

Next, we need to prove that $U \cap T \ll P$. Let $\lambda = \alpha|_T : T \to N$ be the restriction of $\alpha : P \to N$ to the submodule $T \subseteq P$. Since $P = \ker \alpha \oplus T$, the intersection $T \cap \ker \alpha$ is trivial, which ensures that λ is injective. For surjectivity, every $n \in N$ lifts to some $p = k + t \in P$, where $k \in \ker \alpha$ and $t \in T$. We have

$$\alpha(p) = \alpha(k + t) = \alpha(k) + \alpha(t) = \alpha(t) = \lambda(t) = n.$$

Thus, λ is both injective and surjective, so it is an isomorphism. Therefore, $T \cong N$.

Since $\ker f \ll N$, Lemma 1.8 implies $\ker(f \circ \lambda) = \lambda^{-1}(\ker f)$. Since λ is an isomorphism, and Proposition 1.14 tells us that isomorphisms preserve smallness, it follows that $\lambda^{-1}(\ker f) \subseteq T$.

We also have $\ker(f \circ \lambda) = \{x \in T : f(\alpha(x)) = 0\}$ and $\ker(\eta|_T) = \{x \in T : \eta(x) = 0\}$, which implies $\ker(f \circ \lambda) = \ker(\eta|_T)$. Since $\eta|_T : T \to P/U$ is given by $t \mapsto t + U$, we have

$$\ker(\eta|_T) = \{t \in T : t \in U\} = T \cap U.$$

Thus, $\ker(f \circ \lambda) = T \cap U$.

To conclude, we have $T \cap U = \lambda^{-1}(\ker f) \ll T$, and since $T \subseteq P$, it follows by Example

1.8 that $T \cap U \ll P$. The last statement regarding rings is clear. $\square$

Chapter 3

Ring Characterizations via Projective Covers

This chapter presents the culmination of our investigation, focusing on the characterization of semiperfect and perfect rings through the lens of projective modules and covers. Building upon the definitions and preliminary results established in the previous chapters, we utilize the concept of R-projectivity (projectivity relative to R) and R-projective covers. The core of this chapter lies in proving two major theorems: Theorem 3.1 establishes the equivalence between a ring being semiperfect and the condition that every simple right R-module admits an R-projective cover. Subsequently, Theorem 3.2 provides a parallel characterization for right perfect rings, linking them to the existence of R-projective covers for every semisimple right R-module. These results highlight the power of projective concepts in classifying ring structures.

Definition 3.1. A right R-module M is called *R-projective* (projective relative to R) if, for any epimorphism $g : R \to K$ and any homomorphism $f : M \to K$, there exists a homomorphism $h : M \to R$ such that $f = h \circ g$. In other words, the following diagram commutes:

$$\begin{tikzcd} & M \arrow[dl, dashed, "\exists h"'] \arrow[d, "f"] & \\ R \arrow[r, "g"] & K \arrow[r] & 0 \end{tikzcd}$$

Remark 3.1. A right R-module M is said to be *projective* if it is projective relative to every R-module.

Definition 3.2. An *R-projective cover* of a right R-module M is an R-projective module P which serves as a projective cover to M. That is,

1. P is an R-projective module.
2. There exists an epimorphism $\xi : P \to M$ such that $\operatorname{Ker} \xi \ll P$.

3.1 Semiperfect Rings Result

Proposition 3.1. If M is finitely generated R-projective (projective relative to R), then M is projective.

Corollary 3.1. *Let R be a local ring. Then every finitely generated R-projective right R-module is free.*

Proof. Let R be a local ring and let M be a finitely generated R-projective right R-module. Since M is finitely generated and R-projective, it follows from Proposition 3.1 that M is a projective right R-module. Given that R is a local ring and M is a finitely generated projective R-module, Theorem 2.1 states that such a module must be free. Therefore, we conclude that M is a free R-module. $\square$

Theorem 3.1. *Let R be a ring and $J := \mathrm{J}(R)$, then the following statements are equivalent for a ring R:*

1. *R is semiperfect.*
2. *Every finitely generated right R-module has a projective cover.*

3. *Every cyclic right R-module has a projective cover.*

4. *Every simple right R-module has a projective cover.*

5. *Every simple right R-module has an R-projective (projective relative to R) cover.*

Proof. (1) $\implies$ (2)

Let R be a semiperfect ring and M a finitely generated right R-module. By Proposition 1.1, the quotient module $M/\operatorname{rad} M$ is also a finitely generated right R-module. Since R is semilocal, we conclude that M is semilocal as well, by Proposition 2.4. Hence, $M/\operatorname{rad} M$ is semisimple.

By the semisimplicity and finite generation of $M/\operatorname{rad} M$, we apply Proposition 1.3, which allows us to express $M/\operatorname{rad} M$ as a finite direct sum of simple modules:

$$M/\operatorname{rad} M = \bigoplus_{i=1}^{n} K_i$$

where each K_i is a simple right R-module.

Let $J(R)$ be the Jacobson radical of R. Proposition 1.9 ensures that $J(R)$ annihilates simple modules. Thus, the action of R on each K_i factors through $R/J(R)$, and we can regard each K_i as a simple right $R/J(R)$-module. Since $R/J(R)$ is semisimple, Theorem 1.2 guarantees that each K_i is a projective right $R/J(R)$-module.

Furthermore, as each K_i is simple, it must be cyclic. Since K_i is also projective, by Lemma 1.12, we conclude that K_i is isomorphic to a direct summand of $R/J(R)$.

Since $\overline{R} = R/J(R)$ is semisimple, every right ideal in $\overline{R}$ is a direct summand. By Proposition 1.10, every ideal of $\overline{R}$ is of the form $g_i\overline{R}$, where g_i is an idempotent in $\overline{R}$. Therefore, $K_i \cong g_i\overline{R}$. Since idempotents lift modulo $J(R)$, there exists an idempotent $e_i \in R$ such that $\overline{e_i} = g_i$.

Next, consider the map $\varphi_i : e_iR \to \overline{e_i}\overline{R}$ defined by $\varphi_i(e_ir) = \overline{e_ir}$. This map is an epimorphism with kernel:

$$\begin{aligned}\ker\varphi_i &= \{e_ir \in e_iR \mid \overline{e_ir} = 0_{\overline{R}}\}\\ &= \{e_ir \in e_iR \mid e_ir + J(R) = J(R)\}\\ &= \{e_ir \in e_iR \mid e_ir \in J(R)\}\\ &= e_iR \cap J(R)\end{aligned}$$

which is the portion of $J(R)$ lying in e_iR, specifically $e_iJ(R)$. By the First Isomorphism Theorem, we obtain:

$$e_iR/e_iJ(R) \cong \overline{e_i}\overline{R} = g_i\overline{R} \cong K_i$$

By Example 2.2, $\psi_i : e_iR \to e_iR/e_iJ(R) \cong K_i$ is a projective cover for $e_iR/e_iJ(R)$ for each i. Moreover, by Proposition 2.5, the map:

$$\psi = \bigoplus_{i=1}^{n} \psi_i : P = \bigoplus_{i=1}^{n} e_iR \to \bigoplus_{i=1}^{n} e_iR/e_iJ(R) \cong \bigoplus_{i=1}^{n} K_i = M/\operatorname{rad} M$$

is a projective cover for $M/\operatorname{rad} M$.

Consider the following commutative diagram:

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists\alpha"'] \arrow[d, "\psi"] \\ M \arrow[r, "\beta"] & M/\operatorname{rad} M \end{tikzcd}$$

where β is the natural projection. Since P is projective by Proposition 1.25, there exists a homomorphism $\alpha : M \to P$ such that:

$$\psi \circ \alpha = \beta$$

Since ψ is an epimorphism, it follows by Lemma 1.9 that:

$$\mathrm{Im}\alpha + \ker\beta = P$$

and since $\ker\beta = \operatorname{rad} M$, we have:

$$\mathrm{Im}\alpha + \operatorname{rad} M = M$$

Because M is finitely generated, by Lemma 1.4, $\operatorname{rad} M \ll M$, so $\mathrm{Im}\alpha = M$. Hence, α is an epimorphism. Moreover, we observe that $\psi \circ \alpha = \beta$. Suppose $p \in \ker\alpha$. Then, by definition, $\alpha(p) = 0$, and consequently $\psi(p) = \beta(\alpha(p)) = \beta(0) = 0$. Thus, we have $\ker\alpha \subseteq \ker\psi$.

Since $\ker\alpha \subseteq \ker\psi \ll P$, by Example 1.9, we conclude that $\ker\alpha \ll P$. Therefore, (P, α) is a projective cover of M.

(2) $\implies$ (3) $\implies$ (4) $\implies$ (5) clear

(5) $\implies$ (4)

Assume that every simple right R-module has an R-projective cover. Let $f : P \longrightarrow K$ be an R-projective cover, where P is an R-projective module, f is an epimorphism, and $\ker f \ll P$, with K being a simple right R-module. By Corollary 2.1, it follows that $\ker f = \operatorname{rad} P$.

Thus, $\operatorname{rad} P \ll P$. Since K is simple, it is cyclic and therefore finitely generated. Consequently, $P/\operatorname{rad} P = P/\ker f \cong K$ is also finitely generated. By Lemma 1.4, we conclude that P is finitely generated as well. Furthermore, since P is a finitely generated R-projective module, Proposition 3.1 implies that P is projective.

(4) $\implies$ (1)

Suppose that every simple right R-module has a projective cover. First, we will show that R is semilocal, meaning that R/J is semisimple, where $J = J(R)$ is the Jacobson radical of R.

Define $\overline{R} = R/J$, and let $\overline{C} = C/J$ be a maximal right ideal of $\overline{R}$. The quotient ring $\overline{R}/\overline{C}$ is simple. Since $\overline{R}/\overline{C} = \frac{R/J}{C/J}$, where $J \subseteq C \subseteq R$, the Third Isomorphism Theorem implies that $\overline{R}/\overline{C} \cong R/C$. Therefore, R/C is a simple R-module.

By hypothesis, R/C has a projective cover. Thus, by Lemma 2.7, it follows that:

$$R = T \oplus B$$

where $T \subseteq C$ and $C \cap B \ll R$.

Since $R = T + B$ and $T \subseteq C$, we can express R as:

$$R = C + B.$$

Since $C \cap B \ll R$ and Proposition 1.15 implies that $C \cap B \subseteq J$, Lemma 1.5 further implies that:

$$R/J = \frac{C+B}{J} \cong \frac{C}{J} \oplus \frac{B+J}{J}.$$

Thus, $\overline{C} = C/J$ is a direct summand of $\overline{R}$.

Since $\overline{C}$ was an arbitrary maximal ideal of R/J, it follows that every maximal right ideal of R/J is a direct summand of R/J. Hence, by Proposition 1.6, we conclude that R/J is semisimple.

Next, we show that idempotents lift modulo J. Let A/J be a direct summand of R/J. Since R_R is cyclic (generated by 1) and hence finitely generated, it follows from Proposition 1.1 that the quotient R/J is also finitely generated. Moreover, the semisimplicity of R/J implies the semisimplicity of the submodule A/J. Hence, by Proposition 1.3, we have:

$$A/J \cong \bigoplus_{i=1}^{n} K_i,$$

where each K_i is a simple R-module.

By hypothesis, each K_i for $i \in \{1, ..., n\}$ has a projective cover, denoted by $f_i : P_i \longrightarrow K_i$. Therefore, by Proposition 2.5, the map

$$f = \bigoplus_{i=1}^{n} f_i : P = \bigoplus_{i=1}^{n} P_i \longrightarrow \bigoplus_{i=1}^{n} K_i \cong A/J$$

is a projective cover of A/J.

Thus, every direct summand A/J of the right R-module R/J has a projective cover. Therefore, by Lemma 2.5, we conclude that R is semiperfect.

□

3.2 Ring Characterizations via Projective Covers

Definition 3.3. A ring R is called right Bass if every nonzero right R-module has a maximal submodule.

Lemma 3.1. *The following conditions are equivalent for a ring R:*

1. *R is* right Bass *(every nonzero right R-module has a maximal submodule)*
2. *Every nonzero right R-module M satisfies* $\operatorname{rad} M \neq M$
3. *Every nonzero right R-module M has* $\operatorname{rad} M \ll M$ *(the radical is small)*

Proof. We prove the implications cyclically.

(1) $\Longrightarrow$ (2): Assume R is right Bass and let M be a nonzero right R-module. Then M contains a maximal submodule N. Since $\operatorname{rad} M$ is the intersection of all maximal submodules, we have $\operatorname{rad} M \subseteq N \subsetneq M$, hence $\operatorname{rad} M \neq M$.

(2) $\Longrightarrow$ (3): Let M be nonzero with $\operatorname{rad} M \neq M$. Suppose $\operatorname{rad} M + L = M$ for some $L \subseteq M$. If $L \neq M$, then M/L is nonzero, so by (2), $\operatorname{rad}(M/L) \neq M/L$. Thus there exists maximal $N \supseteq L$ in M satisfying:

$$\operatorname{rad} M + L \subseteq N \subsetneq M$$

contradicting $\operatorname{rad} M + L = M$. Therefore $L = M$, proving $\operatorname{rad} M \ll M$.

(3) $\Longrightarrow$ (1): Suppose every nonzero M has $\operatorname{rad} M \ll M$. If some nonzero M had no maximal submodules, we would have $\operatorname{rad} M = M$. Then for $L = 0$ we get $M + 0 = M$ but $0 \neq M$, contradicting $M \ll M$. Thus M must have maximal submodules. □

Theorem 3.2. *The following statements are equivalent for a ring R:*

1. *R is right perfect.*

2. *$R/\operatorname{J}(R)$ is semisimple and R is right Bass.*

3. *Every right R-module has a projective cover.*

4. *Every semisimple right R-module has a projective cover.*

5. *Every semisimple right R-module has an R-projective cover (projective relative to R).*

Proof. (1) $\Longrightarrow$ (2)

Assume that R is right perfect. Then, $R/J(R)$ is semisimple. It remains to prove that R is right Bass. Let M be a nonzero right R-module. Since $J = J(R)$ is right T-nilpotent, it follows from Lemma 1.6 that $MJ \neq M$, which implies that $M/MJ \neq 0$. Moreover, since J annihilates M/MJ, we can treat M/MJ as a right R/J-module. Given that R/J is semisimple, it follows from Proposition 1.4 that M/MJ, as a right R/J-module, is also semisimple. Since M/MJ is a nonzero semisimple module, it follows from Proposition 1.17 that M/MJ admits a maximal submodule. Therefore, M admits a maximal submodule as well. Hence, R is right Bass.

$\square$

Appendix A

Appendix

Acknowledgements

Some text ...