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# RESEARCH PROJECT

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**Taibah University**

**Projective Modules and Their Relationship  
With Some Important Rings**

Submitted in partial fulfillment of requirements  
for the degree of

**Master of Science in Mathematics**

by

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# Abstract

This research project investigates the significant relationship between projective modules and the structural properties of important ring classes, specifically local, semilocal, semiperfect, and perfect rings. The study first lays foundational concepts from module theory. It then delves into the characteristics of these special rings and introduces the essential notion of projective covers. By developing these tools, including the concept of  $R$ -projectivity, the project establishes key characterizations. The main findings demonstrate that a ring is semiperfect if and only if every simple right module possesses an  $R$ -projective cover, and a ring is right perfect if and only if every semisimple right module has an  $R$ -projective cover. These results provide significant criteria for classifying these ring types through the lens of projective modules and their covers.

# Acknowledgements

First and foremost, I would like to express my sincere gratitude to my supervisor, **Professor Yasser AbdelWahab**, for his invaluable guidance, support, and patience throughout this research project. His expertise and insightful feedback were instrumental in shaping this work.

I would also like to extend a special thanks to **Professor Salama Daoud**, whose consistent advice and support throughout my two years of Master's studies were greatly appreciated. I am also deeply grateful to **Dr. Ahmed Noubi Sayed Elsayy** for his support in referring my research project.

I extend my thanks to the **Department of Mathematics** and the **College of Science** at **Taibah University** for providing the necessary resources and a conducive academic environment for this research.

Finally, I wish to express my deepest appreciation to my family and friends for their unwavering encouragement and understanding during my Master's studies. Their support was essential to the completion of this project.

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# Nomenclature

$R$  A ring with unity

$M_R$  A right  $R$ -module

$\bigoplus_{i \in I} M_i$  Direct sum of modules  $M_i$

$M \cong N$  Modules  $M$  and  $N$  are isomorphic

$R_R$  The ring  $R$  considered as a right module over itself

$F = \bigoplus_{i \in I} R$  A free  $R$ -module with basis indexed by  $I$

$\varphi : M \rightarrow N$  An  $R$ -module homomorphism from  $M$  to  $N$

$\ker \varphi$  The kernel of the homomorphism  $\varphi$

$\text{Im } \varphi$  The image of the homomorphism  $\varphi$

$\text{Hom}_R(M, N)$  The set of all  $R$ -module homomorphisms from  $M$  to  $N$

$\text{End}_R(M)$  The endomorphism ring of the module  $M$ ,  $\text{Hom}_R(M, M)$

$0$  The zero element of a module or ring, or the zero module

$1$  The multiplicative identity of a ring

$e \in R$  An idempotent element in the ring  $R$ ,  $e^2 = e$

$\{e_1, \dots, e_n\}$  A set of pairwise orthogonal idempotents,  $e_i e_j = 0$  for  $i \neq j$

$J(R)$  The Jacobson radical of the ring  $R$

$\text{rad } M$  The radical of the module  $M$  (intersection of all maximal submodules)

$N \hookrightarrow M$   $N$  is a submodule of  $M$

$S \ll M$   $S$  is a small (or superfluous) submodule of  $M$

$I^n$  The  $n$ -th power of an ideal  $I$

**right T-nilpotent** An ideal  $A$  where for any sequence  $a_1, a_2, \dots$  in  $A$ , there exists  $k$  such that  $a_k \cdots a_1 = 0$

$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$  is **exact**  $\text{Im } \alpha = \ker \beta$

$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  A short exact sequence

**Simple module** A nonzero module with no proper nontrivial submodules

**Semisimple module** A module that is a direct sum of simple modules (or where every submodule is a direct summand)

**Semisimple ring** A ring that is semisimple as a right module over itself

**Free module** A module with a basis

**Finitely generated module** A module generated by a finite set of elements

**Projective module** A module satisfying the lifting property (Definition 1.23)

**Local ring** A nonzero ring with a unique maximal right ideal

**Semilocal ring** A ring  $R$  such that  $R/J(R)$  is a right Artinian (or semisimple) ring

**Semiperfect ring** A semilocal ring where idempotents lift modulo  $J(R)$

**Right perfect ring** A ring  $R$  where  $R/J(R)$  is semisimple and  $J(R)$  is right T-nilpotent

**Semiprimary ring** A ring  $R$  where  $R/J(R)$  is semisimple and  $J(R)$  is nilpotent

**Semiprimitive ring** A ring  $R$  with  $J(R) = 0$

**Right Bass ring** A ring where every nonzero right  $R$ -module has a maximal submodule

**R-projective module** A module satisfying the lifting property relative to the ring  $R$  (Definition 3.1)



$P(M)$  A projective cover of the module  $M$  (a projective module  $P$  with an epimorphism  $\varphi : P \rightarrow M$  such that  $\ker \varphi \ll P$ )

**R-projective cover** An R-projective module that is a projective cover

# Introduction

Module theory provides a powerful framework for studying the structure of rings, extending concepts from linear algebra over fields to the more general setting of modules over rings. Within this rich theory, the concept of projectivity stands out as particularly fundamental and fruitful. Projective modules, initially arising as generalizations of free modules, possess unique lifting properties that make them essential tools in homological algebra and representation theory. Their close relationship with exact sequences, particularly split exact sequences, highlights their structural significance.

This research project delves into the fascinating interplay between projective modules and the properties of specific classes of rings. While free modules are always projective, the converse is not true, leading to natural questions about which rings have the property that all projective modules are free, or how projectivity characterizes the ring itself. We aim to thoroughly investigate the notion of projectivity and explore how it illuminates the structure of several important types of rings, namely local, semilocal, semiperfect, and perfect rings.

The research project is structured as follows:

- **Chapter 1** lays the necessary groundwork. We begin by reviewing essential concepts from module theory, including homomorphisms, free and finitely generated modules, radicals (Jacobson and module radicals), idempotents and their lifting properties, small (superfluous) submodules, nilpotency, and the theory of exact sequences. This foundation leads into the formal definition and exploration of projective modules, establishing their key properties and equivalences, such as their relationship to direct summands of free modules and the exactness of the Hom functor.
- **Chapter 2** shifts focus to specific classes of rings whose structure is deeply con-

nected to projectivity. We introduce local and semilocal rings, examining their defining characteristics, particularly concerning their Jacobson radicals and quotient structures (e.g.,  $R/J(R)$  being a division ring for local  $R$ , or semisimple for semilocal  $R$ ). We prove a key result (Theorem 2.1) showing that finitely generated projective modules over local rings are necessarily free. This chapter also introduces the critical concepts of semiperfect and perfect rings, along with the closely related notion of projective covers – epimorphisms from projective modules with small kernels.

- **Chapter 3** presents the main results of this project, culminating in detailed characterizations of semiperfect and perfect rings. Building upon the concepts developed earlier, we introduce the notion of  $R$ -projectivity (projectivity relative to the ring  $R$  itself) and  $R$ -projective covers. We then prove theorems (Theorem 3.1 and Theorem 3.2) establishing the equivalence between a ring being semiperfect or perfect and various conditions involving projective covers. Notably, these theorems demonstrate that a ring  $R$  is semiperfect if and only if every simple right  $R$ -module has an  $R$ -projective cover, and  $R$  is right perfect if and only if every semisimple right  $R$ -module has an  $R$ -projective cover. These characterizations offer a potentially streamlined approach by focusing on specific module classes (simple or semisimple) and the refined concept of  $R$ -projectivity.

Through this structured exploration, this research project seeks to provide a clear and comprehensive account of projective modules and demonstrate their profound connection to the structure theory of local, semilocal, semiperfect, and perfect rings.

# Chapter 1

## Projective Modules

Building on the projective module concepts from Chapter 1, this chapter explores the deep connections between projectivity and the structure of specific ring classes, including local, semilocal, semiperfect, and perfect rings. We introduce the crucial notion of projective covers and examine their properties, setting the stage for their use in characterizing these rings in the subsequent chapter. A key result regarding finitely generated projective modules over local rings is also presented.

### 1.1 Essentials of Module Theory

#### 1.1.1 Homomorphisms

**Definition 1.1.** Let  $R$  be a ring and let  $M$  and  $N$  be  $R$ -modules.

1. A map  $\varphi : M \rightarrow N$  is an  $R$ -module homomorphism if it respects the  $R$ -module structures of  $M$  and  $N$ , i.e.,
  - (a)  $\varphi(x + y) = \varphi(x) + \varphi(y)$ , for all  $x, y \in M$ , and
  - (b)  $\varphi(xr) = \varphi(x)r$ , for all  $r \in R, x \in M$ .
2. An  $R$ -module homomorphism is an isomorphism (of  $R$ -modules) if it is both a monomorphism (injective) and an epimorphism (surjective). The modules  $M$  and  $N$

are said to be isomorphic, denoted  $M \cong N$ , if there is some  $R$ -module isomorphism  $\varphi : M \rightarrow N$ .

3. If  $\varphi : M \rightarrow N$  is an  $R$ -module homomorphism, let  $\ker \varphi = \{m \in M \mid \varphi(m) = 0\}$  (the kernel of  $\varphi$ ) and let  $\varphi(M) = \{n \in N \mid n = \varphi(m) \text{ for some } m \in M\}$  (the image of  $\varphi$ , as usual).
4. Let  $M$  and  $N$  be  $R$ -modules and define  $\text{Hom}_R(M, N)$  to be the set of all  $R$ -module homomorphisms from  $M$  into  $N$ .
5. The ring  $\text{Hom}_R(M, M)$  is called the *endomorphism ring* of  $M$  and will often be denoted by  $\text{End}_R(M)$ , or just  $\text{End}(M)$ . Elements of  $\text{End}(M)$  are called *endomorphisms*.

**Example 1.1.** Let  $M$  be a module,  $N$  a submodule of  $M$ , and  $X, Y$  be modules. We now introduce several key maps related to these modules.

**1. Natural Projection:** The map  $\pi : M \rightarrow M/N$ , defined by

$$\pi(m) = m + N \quad \text{for all } m \in M,$$

is an epimorphism called the *natural projection* of  $M$  onto the quotient module  $M/N$ .

**2. Canonical Projections:** The maps  $\pi_X : X \oplus Y \rightarrow X$  and  $\pi_Y : X \oplus Y \rightarrow Y$ , defined by

$$\pi_X(x, y) = x \quad \text{and} \quad \pi_Y(x, y) = y \quad \text{for all } (x, y) \in X \oplus Y,$$

are called the *canonical projections*. These maps are epimorphisms.

**3. Natural Injection:** The inclusion map  $i : N \rightarrow M$ , defined by

$$i(n) = n \quad \text{for all } n \in N,$$

is called the *natural injection*. This map is a monomorphism.

**4. Canonical Injections:** The maps  $i_X : X \rightarrow X \oplus Y$  and  $i_Y : Y \rightarrow X \oplus Y$ , defined by

$$i_X(x) = (x, 0) \quad \text{and} \quad i_Y(y) = (0, y) \quad \text{for all } x \in X, y \in Y,$$

are called the *canonical injections*. These maps are monomorphisms.

### 1.1.2 Free & Finitely Generated Modules

**Definition 1.2.** Let  $R$  be a ring. A module  $F$  over  $R$  is called a **free module** if it satisfies one of the following equivalent conditions:

1.  $F$  has a **basis**, i.e., a linearly independent set that spans  $F$ .
2.  $F$  is isomorphic to a direct sum of copies of  $R$ , i.e.,  $F \cong \bigoplus_{i \in I} R$  for some set  $I$ .

**Definition 1.3.** Let  $M$  be a module over a ring  $R$ . A subset  $S \subseteq M$  is called a **generating set** for  $M$  if every element of  $M$  can be written as a finite linear combination of elements of  $S$  with coefficients in  $R$ . That is, for every  $m \in M$ , there exist  $s_1, s_2, \dots, s_n \in S$  and  $r_1, r_2, \dots, r_n \in R$  such that

$$m = r_1 s_1 + r_2 s_2 + \dots + r_n s_n.$$

**Remark 1.1.** Every module  $M$  over a ring  $R$  has a generating set. In particular,  $M$  itself is always a generating set.

**Definition 1.4.** Let  $R$  be a ring. A module  $M$  over  $R$  is called **finitely generated** if there exists a finite subset  $\{m_1, m_2, \dots, m_n\} \subseteq M$  such that every element of  $M$  can be written as a finite linear combination of these elements with coefficients in  $R$ .

**Proposition 1.1.** If  $M$  is an  $R$ -module and  $M$  can be generated by  $n$  elements, and  $N$  is a submodule of  $M$ , then  $M/N$  can be generated by  $n$  elements.

*Proof.* Suppose that  $M$  is a finitely generated  $R$ -module and let  $m_1, m_2, \dots, m_n$  be its

generators. So, any element of  $M$  can be expressed as

$$m = \sum_{i=1}^n m_i r_i$$

with  $r_i \in R$ . Then

$$m + N = \sum_{i=1}^n (m_i + N) r_i$$

which shows that the  $n$  elements  $m_1 + N, \dots, m_n + N$  generate  $M/N$ .  $\square$

**Proposition 1.2.** Every  $R$ -module  $M$  is an epimorphic image of a free module.

*Proof.* Let  $S = \{m_i\}_{i \in I}$  be a generating set for  $M$ . This means every element of  $M$  can be written as a finite linear combination of elements from  $S$  with coefficients in  $R$ .

Consider the free  $R$ -module  $F$  with basis  $\{f_i\}_{i \in I}$ , where each  $f_i$  corresponds to  $m_i \in S$ . Formally,  $F = \bigoplus_{i \in I} f_i R$ . This means every element of  $F$  can be uniquely written as a finite sum  $\sum_{i \in I} f_i r_i$  with  $r_i \in R$ .

Define the homomorphism  $\phi : F \rightarrow M$  by  $\phi(\sum_{i \in I} f_i r_i) = \sum_{i \in I} m_i r_i$ .

Since  $S$  generates  $M$ , for any  $m \in M$ , there exist  $r_i \in R$  such that  $m = \sum_{i \in I} m_i r_i$ . Thus,  $\phi$  maps  $\sum_{i \in I} f_i r_i$  to  $m$ , showing that  $\phi$  is surjective.  $\square$

**Corollary 1.1.** Every  $R$ -module  $M$  is isomorphic to a quotient module of a free module.

*Proof.* Let  $M$  be a right  $R$ -module. By Proposition 1.2, there exists a free module  $F$  and an epimorphism  $\varphi : F \rightarrow M$ .

Since  $\varphi$  is surjective, by the First Isomorphism Theorem for modules, we have:

$$M \cong F / \ker(\varphi).$$

This shows that  $M \cong F / \ker(\varphi)$ , proving that  $M$  is isomorphic to a quotient of the free module  $F$ .  $\square$

**Lemma 1.1** (Kasch and Wallace [14], (Theorem 2.3.11)). *Let  $M$  be a finitely generated  $R$ -module. Then every proper submodule  $N \subsetneq M$  is contained in some maximal submodule*

of  $M$ .

**Definition 1.5** (Simple Module). A right module  $M$  over a ring  $R$  is called *simple* if it is nonzero and has no nontrivial submodules, i.e., its only submodules are 0 and  $M$ .

**Definition 1.6** (Semisimple Module). A right module  $M$  over a ring  $R$  is called *semisimple* if every submodule of  $M$  is a direct summand, or equivalently, if  $M$  is isomorphic to a direct sum of simple right modules.

**Definition 1.7** (Semisimple Ring). A ring  $R$  is called *semisimple* if it is semisimple as a right module over itself, that is, if the right regular module  $R_R$  is semisimple.

**Proposition 1.3.** Let  $M$  be a semisimple module over a ring  $R$ . If  $M$  is finitely generated, then  $M$  is a finite direct sum of simple modules.

*Proof.* Assume that  $M$  is a semisimple module over  $R$ . By definition of semisimplicity,  $M$  can be expressed as a direct sum of simple submodules:

$$M = \bigoplus_{i \in I} M_i,$$

where each  $M_i$  is a simple module and  $I$  is an index set.

Suppose  $M$  is finitely generated. Then there exists a finite set of generators  $\{m_1, m_2, \dots, m_n\} \subseteq M$  such that every element of  $M$  can be written as a finite linear combination of these generators with coefficients in  $R$ .

Since each generator  $m_j$  (for  $j = 1, 2, \dots, n$ ) lies in  $M$ , and  $M$  is the direct sum of the  $M_i$ 's, each  $m_j$  can be expressed as a finite sum of elements from the  $M_i$ 's. Consequently, there exists a finite subset  $J \subseteq I$  such that:

$$\{m_1, m_2, \dots, m_n\} \subseteq \bigoplus_{i \in J} M_i.$$

Because  $M$  is generated by  $\{m_1, m_2, \dots, m_n\}$ , it follows that:

$$M \subseteq \bigoplus_{i \in J} M_i.$$



On the other hand,  $\bigoplus_{i \in J} M_i$  is a submodule of  $M$ . Therefore, we have:

$$M = \bigoplus_{i \in J} M_i.$$

Since  $J$  is finite,  $M$  is a finite direct sum of simple modules. This completes the proof.  $\square$

**Proposition 1.4.** Let  $R$  be a semisimple ring. Then,  $M_R$  is also semisimple.

*Proof.* Assume that  $R$  is semisimple, and consider the right  $R$ -module  $M$ . By Corollary 1.1, we know that

$$M \cong F/K,$$

where  $F$  is a free  $R$ -module and  $K$  is a submodule of  $F$ . By definition,  $F \cong \bigoplus_{i \in I} R$ , so  $F$  is also semisimple.

Since any quotient of a semisimple module is also semisimple, it follows that the quotient module  $M \cong F/K$  is semisimple.  $\square$

**Definition 1.8** (Socle). Let  $M$  be a right  $R$ -module. The socle of  $M$ , denoted by  $\text{Soc}(M)$ , is the sum of all simple right submodules of  $M$ . If there are no such submodules, then  $\text{Soc}(M) = 0$ .

**Remark 1.2.** 1. If  $M = R_R$ , then  $\text{Soc}(R_R)$  is the sum of all minimal right ideals of  $R$  and it is a right ideal of  $R$ .

2. If  $I$  is a minimal ideal in  $R$ , then for any  $x \in R$  either  $Ix = 0$  or  $Ix$  is a minimal right ideal, and in both cases  $Ix \subseteq \text{Soc}(R_R)$ .

3. For a semisimple module  $M$  we have  $\text{Soc}(M) = M$ .

**Proposition 1.5.** The following are equivalent for a ring  $R$ :

1. Every maximal right ideal of  $R$  is a direct summand.
2.  $R_R$  is semisimple.

*Proof.* (1)  $\Rightarrow$  (2):

Suppose that every maximal right ideal of  $R$  is a direct summand. If  $\text{Soc}(R) = R$ , we are done. Assume for the sake of contradiction that  $\text{Soc}(R) \neq R$ . Then  $\text{Soc}(R)$  is properly contained in some maximal right ideal  $A$ , whose existence is guaranteed by Lemma 1.1 since  $R_R$  is cyclic (generated by 1) and hence finitely generated. The hypothesis implies that  $A$  is a direct summand of  $R$ , so we can write  $R = A \oplus T$  for some right ideal  $T \subseteq R$ .

By the second isomorphism theorem,

$$\frac{A+T}{A} \cong \frac{T}{A \cap T},$$

and since  $R = A + T$ , we have  $R/A \cong T$ .

Now,  $A$  is maximal, so  $R/A$  is a simple right  $R$ -module, and hence  $T$  is simple. This implies that  $T \subseteq \text{Soc}(R) \subseteq A$ . Therefore,  $T \subseteq A$ , but  $R = A \oplus T$  implies  $A \cap T = 0$  and  $T \neq 0$  (since  $A$  is proper). Thus  $T \subseteq A \cap T = 0$ , which forces  $T = 0$ , a contradiction. Therefore,  $\text{Soc}(R) = R$ , and  $R$  is semisimple.

(2)  $\Rightarrow$  (1):

If  $R$  is semisimple, then every right ideal is a direct summand, and in particular, every maximal right ideal is a direct summand.  $\square$

**Proposition 1.6.** Every submodule of a semisimple module is semisimple.

*Proof.* Every submodule of a semisimple module is semisimple because semisimple modules are direct sums of simple submodules, and every submodule of a semisimple module is itself a direct summand.  $\square$

**Lemma 1.2.** Let  $K$  be a simple module, and let  $f: K \rightarrow M$  be a module homomorphism. Then exactly one of the following holds:

1. If  $\ker(f) = 0$ , then  $f$  is injective and  $K \cong \text{im}(f)$ .
2. If  $\ker(f) = K$ , then  $f$  is the zero homomorphism and  $f(K) = 0$ .

*Proof.* Since  $K$  is simple, its only submodules are 0 and  $K$ . The kernel of  $f$ , being a submodule of  $K$ , must therefore be either 0 or  $K$ .

- If  $\ker(f) = 0$ , then  $f$  is injective. By the First Isomorphism Theorem, it follows that  $K \cong \text{im}(f)$ .
- If  $\ker(f) = K$ , then  $f$  maps every element of  $K$  to zero. Hence,  $f$  is the zero homomorphism and  $f(K) = 0$ .

□

**Lemma 1.3.** *Let  $M$  be a semisimple module and  $f: M \rightarrow N$  a surjective homomorphism. Then  $N$  is semisimple.*

*Proof.* Since  $M$  is semisimple, it decomposes as a direct sum of simple modules:

$$M = \bigoplus_{i \in I} K_i, \quad \text{where each } K_i \text{ is simple.}$$

By surjectivity,  $N = f(M) = \sum_{i \in I} f(K_i)$ . By Lemma 1.2, each  $f(K_i)$  is either:

- Isomorphic to  $K_i$  (if  $\ker(f|_{K_i}) = 0$ ), or
- Zero (if  $\ker(f|_{K_i}) = K_i$ ).

Thus,  $N$  is a sum of simple modules (the nonzero  $f(K_i)$ ), and hence semisimple. □

### 1.1.3 Radicals

**Definition 1.9** ( $J(R)$  or  $\text{rad } R$ ). For a ring  $R$ , the **Jacobson radical**, denoted by  $J(R)$  or sometimes  $\text{rad } R$ , is the intersection of all maximal right ideals of  $R$ .

**Definition 1.10** ( $\text{rad } M$ ). For any right  $R$ -module  $M_R$ , we define  $\text{rad } M$  to be the intersection of all the **maximal submodules** of  $M$ . If there are no maximal submodules in  $M$ , we define  $\text{rad } M$  to be  $M$ .

**Remark 1.3.** The module radical  $\text{rad } M$  generalizes the Jacobson radical  $J(R)$  to modules. For the right regular module  $R_R$ ,  $\text{rad } R_R = J(R)$ . If  $R \neq 0$ , maximal right ideals exist, so  $J(R) \neq R$ . Similarly, for a finitely generated module  $M_R$ , maximal submodules exist, and  $\text{rad } M \neq M$ . However, for non-finitely-generated modules,  $\text{rad } M = M$  may occur, as maximal submodules need not exist.

**Proposition 1.7.** Let  $M$  be a right  $R$ -module. For every submodule  $K$  of  $M$ , the quotient module  $(K + \text{rad } M)/K$  is contained in  $\text{rad}(M/K)$  as a submodule.

*Proof.* Let  $M$  be a right  $R$ -module and let  $K$  be a submodule of  $M$ . Since  $K$  and  $\text{rad } M$  are both submodules of  $M$ , their sum  $K + \text{rad } M$  is also a submodule of  $M$ .

Consider the quotient  $(K + \text{rad } M)/K$ . This is a submodule of  $M/K$  because  $K + \text{rad } M$  contains  $K$ .

Let  $x \in K + \text{rad } M$ . Then  $x = k + m$  where  $k \in K$  and  $m \in \text{rad } M$ . In  $M/K$  we have  $x + K = m + K$ .

Since  $m \in \text{rad } M$ ,  $m$  is in every maximal submodule of  $M$ . Therefore,  $m + K$  is in every maximal submodule of  $M/K$ , which means  $m + K \in \text{rad}(M/K)$ .

Thus  $(K + \text{rad } M)/K \subseteq \text{rad}(M/K)$ .

Therefore,  $(K + \text{rad } M)/K$  is a submodule of  $M/K$  that is contained in  $\text{rad}(M/K)$ , completing the proof.  $\square$

**Proposition 1.8.** For  $y \in R$ , the following are equivalent:

1.  $y \in J(R)$
2.  $1 - yx$  is right-invertible for any  $x \in R$
3.  $My = 0$  for any simple right  $R$ -module  $M$ .

*Proof.* (1)  $\implies$  (2)

Assume  $y \in J(R)$ . We must show that for any  $x \in R$ ,  $1 - yx$  is right-invertible.

Suppose for contradiction that for some  $x \in R$ ,  $1 - yx$  is not right-invertible. Then the right ideal  $(1 - yx)R$  is proper, i.e.,  $(1 - yx)R \neq R$ .

Since  $(1 - yx)R$  is proper, it is contained in some maximal right ideal  $\mathfrak{m}$  (by Zorn's lemma). But  $1 - yx \in (1 - yx)R \subseteq \mathfrak{m}$ .

Since  $y \in J(R)$ ,  $y$  lies in every maximal right ideal, so  $y \in \mathfrak{m}$ . Thus,  $yx \in \mathfrak{m}$ .

Now,  $1 - yx \in \mathfrak{m}$  and  $yx \in \mathfrak{m}$  imply:

$$1 = (1 - yx) + yx \in \mathfrak{m}.$$

But  $1 \in \mathfrak{m}$  forces  $\mathfrak{m} = R$ , contradicting that  $\mathfrak{m}$  is maximal.

Therefore,  $1 - yx$  must be right-invertible for all  $x \in R$ .

**(2)  $\implies$  (3)**

Assume (2) holds, but (3) fails. That is, there exists a simple right  $R$ -module  $M$  and an element  $m \in M$  with  $m \neq 0$  such that  $my \neq 0$ .

Since  $M$  is simple, the submodule generated by  $my$  must be all of  $M$  (because  $M$  has no proper nonzero submodules). Thus, there exists  $x \in R$  such that:

$$m = (my)x.$$

This can be rewritten as:

$$m - m(yx) = 0 \implies m(1 - yx) = 0.$$

By hypothesis (2),  $1 - yx$  is right-invertible, so there exists  $z \in R$  such that:

$$(1 - yx)z = 1.$$

Multiplying both sides of  $m(1 - yx) = 0$  by  $z$  on the right gives:

$$m(1 - yx)z = 0 \implies m \cdot 1 = 0 \implies m = 0.$$

This contradicts our choice of  $m \neq 0$ . Therefore, (3) must hold.

$$(3) \implies (1)$$

Assume (3) holds, i.e.,  $y$  annihilates every simple right  $R$ -module.

For any maximal right ideal  $\mathfrak{m}$ , the quotient  $R/\mathfrak{m}$  is a simple right  $R$ -module. By (3),

$$y \cdot (R/\mathfrak{m}) = 0,$$

which implies  $y \in \mathfrak{m}$ .

Since this holds for all maximal right ideals  $\mathfrak{m}$ , by definition we have  $y \in J(R)$ .  $\square$

**Proposition 1.9.** Let  $R$  be a ring with Jacobson radical  $J(R)$ , and let  $M$  be a simple right  $R$ -module. Then  $M$  is annihilated by  $J(R)$ , i.e.,  $M \cdot J(R) = 0$ .

*Proof.* Let  $M$  be a simple right  $R$ -module, meaning  $M \neq \{0\}$  and its only submodules are  $\{0\}$  and  $M$ .

Fix  $x \in J(R)$  and  $m \in M$ . We show that  $m \cdot x = 0$ . Suppose, for contradiction, that  $m \cdot x \neq 0$ .

Since  $M$  is simple, the submodule generated by  $m \cdot x$  must be all of  $M$ . Thus, there exists  $r \in R$  such that

$$m \cdot x \cdot r = m.$$

Rewriting this gives

$$m - m \cdot xr = 0 \quad \text{or equivalently} \quad m \cdot (1 - xr) = 0.$$

By Proposition 1.8, since  $x \in J(R)$ , the element  $1 - xr$  is right-invertible. Hence, there exists  $z \in R$  such that

$$(1 - xr)z = 1.$$

Multiplying  $m \cdot (1 - xr) = 0$  by  $z$  on the right yields

$$m \cdot (1 - xr)z = m \cdot 1 = m = 0,$$

contradicting  $m \neq 0$ .

Therefore,  $m \cdot x = 0$  must hold for all  $x \in J(R)$  and  $m \in M$ , proving  $M \cdot J(R) = 0$ .  $\square$

**Proposition 1.10.** Let  $M$  be a right  $R$ -module. Then  $\text{rad}(M/\text{rad } M) = 0$ .

*Proof.* Let  $\mathcal{A} = \text{rad } M$ . By definition,  $\mathcal{A}$  is the intersection of all maximal submodules of  $M$ .

$$\mathcal{A} = \bigcap \{K \mid K \text{ is a maximal submodule of } M\}$$

If  $M$  has no maximal submodules, then  $\mathcal{A} = M$ , and  $M/\mathcal{A} = M/M = 0$ . The radical of the zero module is 0, so the statement holds in this case.

Assume  $M$  has maximal submodules. Let  $\mathcal{K}$  be the set of all maximal submodules of  $M$ . So,  $\mathcal{A} = \bigcap_{K \in \mathcal{K}} K$ .

Consider the quotient module  $M/\mathcal{A}$ . We want to find  $\text{rad}(M/\mathcal{A})$ . By definition, this is the intersection of all maximal submodules of  $M/\mathcal{A}$ .

Let  $L$  be a submodule of  $M/\mathcal{A}$ . By the Lattice Isomorphism Theorem for modules,  $L$  corresponds uniquely to a submodule  $K$  of  $M$  such that  $\mathcal{A} \subseteq K$ , with  $L = K/\mathcal{A}$ . Furthermore,  $L$  is a maximal submodule of  $M/\mathcal{A}$  if and only if  $K$  is a maximal submodule of  $M$ .

Since  $\mathcal{A} = \text{rad } M$  is contained in every maximal submodule  $K$  of  $M$  (i.e., for all  $K \in \mathcal{K}$ ,  $\mathcal{A} \subseteq K$ ), the set of maximal submodules of  $M/\mathcal{A}$  is precisely:

$$\{K/\mathcal{A} \mid K \in \mathcal{K}\}$$

Therefore, the radical of  $M/\mathcal{A}$  is:

$$\text{rad}(M/\mathcal{A}) = \bigcap \{K/\mathcal{A} \mid K \in \mathcal{K}\}$$

Using the property for intersections of quotient submodules, which states that  $\bigcap (K_\alpha/\mathcal{A}) = (\bigcap K_\alpha)/\mathcal{A}$ , we have:

$$\text{rad}(M/\mathcal{A}) = \left( \bigcap_{K \in \mathcal{K}} K \right) / \mathcal{A}$$

By the definition of  $\mathcal{A}$ , the term in the parenthesis is exactly  $\mathcal{A}$ :

$$\text{rad}(M/\mathcal{A}) = \mathcal{A}/\mathcal{A}$$

The quotient of a submodule by itself is the zero submodule.

$$\text{rad}(M/\mathcal{A}) = 0$$

Thus,  $\text{rad}(M/\text{rad } M) = 0$ . □

The proofs of the following lemmas can be found in [13], Lemma 3.4.11 and Lemma 3.4.12, respectively:

**Lemma 1.4** (Nakayama's Lemma). *Let  $N$  be a submodule of a finitely generated module  $M$  and  $N + MJ(R) = M$ . Then  $N = M$ .*

**Lemma 1.5** (Nakayama's Lemma Version 2). *Let  $N$  be a submodule of a finitely generated module  $M$  over a ring  $R$ . If  $N + M = M$ , then  $N = M$ .*

### 1.1.4 Idempotents

**Definition 1.11.** Let  $R$  be a ring.



1. An element  $e \in R$  is called an **idempotent** if  $e^2 = e$ .
2. Two idempotents  $e$  and  $f$  are called **orthogonal** if  $ef = fe = 0$ .
3. A set  $\{e_1, e_2, \dots, e_n\} \subseteq R$  is called **pairwise orthogonal idempotents** if  $e_i^2 = e_i$  for all  $i$  and  $e_i e_j = 0$  for all  $i \neq j$ .
4. An equality  $1 = e_1 + e_2 + \dots + e_n$ , where  $e_1, e_2, \dots, e_n$  are pairwise orthogonal **idempotents**, is called a **decomposition of the identity** of the ring  $R$ .
5. An idempotent  $e \in R$  is said to be **primitive** if  $e$  has no decomposition into a sum of nonzero orthogonal idempotents  $e = e_1 + e_2$  in  $R$  with  $e_1, e_2$  orthogonal and nonzero.

**Theorem 1.1** (Right Peirce Decomposition). *Let  $R$  be a ring and  $e \in R$  an idempotent. Then  $R = eR \oplus (1 - e)R$ , as right  $R$ -modules.*

*Proof.* For any  $r \in R$ ,  $r = (e + (1 - e))r = er + (1 - e)r$ , so  $R = eR + (1 - e)R$ . For directness, take  $x \in eR \cap (1 - e)R$ . Then  $x = er = (1 - e)s$ , so  $ex = er = e(1 - e)s = 0$ , hence  $x = 0$ .  $\square$

**Proposition 1.11.** Let  $R$  be a ring such that  $R = A \oplus B$ , where  $A$  and  $B$  are ideals of  $R$ . Then there exists an idempotent  $e \in R$  such that  $A = eR$ .

*Proof.* Since  $R = A \oplus B$ , the identity element decomposes uniquely as  $1 = e + f$  with  $e \in A$  and  $f \in B$ . Multiplying by  $e$  yields  $e = e^2 + ef$ . Because  $ef \in A \cap B = \{0\}$ , it follows that  $e$  is idempotent.

For any  $a \in A$ , we have  $a = a \cdot 1 = a(e + f) = ae + af$ . Since  $af \in A \cap B = \{0\}$ , this shows  $a = ae \in eR$ , proving  $A \subseteq eR$ . Conversely, any  $er \in eR$  (with  $r \in R$ ) decomposes as  $er = e(a + b) = ea + eb$  where  $a \in A$ ,  $b \in B$ . Since  $eb \in A \cap B = \{0\}$ , we get  $er = ea \in A$ , establishing  $eR \subseteq A$ .

Thus  $A = eR$ , where  $e$  is the constructed idempotent.  $\square$

**Definition 1.12.** An idempotent  $e \in R$  is called **local** if the ring  $eRe$  is local. A local idempotent is always a primitive idempotent.

**Definition 1.13.** Let  $I$  be an ideal in a ring  $R$  and let  $g + I$  be an idempotent element of  $R/I$ . We say that this idempotent can be *lifted (to  $e$ ) modulo  $I$*  in case there is an idempotent  $e \in R$  such that  $g + I = e + I$ . We say that *idempotents lift modulo  $I$*  in case every idempotent in  $R/I$  can be lifted to an idempotent in  $R$ .

**Remark 1.4.** Let  $R$  be a ring, and let  $I$  be an ideal of  $R$ . An idempotent  $\bar{e} \in R/I$  is said to **lift modulo  $I$**  if there exists an idempotent  $e \in R$  such that  $\pi(e) = e + I = \bar{e}$ , where  $\pi : R \rightarrow R/I$  is the natural projection map.

**Remark 1.5.** The concept of lifting idempotents modulo an ideal refers to finding an idempotent  $e$  in  $R$  such that  $e - g \in I$ . This means that  $e$  and  $g$  are congruent modulo  $I$ , and thus  $e + I = g + I$  in  $R/I$ . The idempotent  $e$  in  $R$  is a lift of the idempotent  $g + I$  in  $R/I$ .

The idempotents in a ring  $R$  correspond to idempotents in every quotient ring of  $R$ . However, idempotent cosets in a quotient ring of  $R$  do not necessarily have idempotent representatives in  $R$ .

For example,  $\mathbb{Z}$  has only two idempotents, while  $\mathbb{Z}_6$  has four. The proof of this statement will be provided in the following proposition, and a counterexample illustrating the distinction will be presented in the subsequent example.

**Proposition 1.12.** Let  $R$  be a ring and  $I \subseteq R$  an ideal. Then idempotents in  $R$  always map to idempotents in  $R/I$ .

*Proof.* Let  $e \in R$  be an idempotent, meaning that  $e = e^2$ .

Consider the coset  $e + I$  in the quotient ring  $R/I$ . We need to show that  $e + I$  is an idempotent in  $R/I$ , i.e., that  $(e + I)^2 = e + I$ .

Observe that

$$(e + I)^2 = (e + I)(e + I) = e^2 + I = e + I.$$

Thus, the coset  $e + I$  is an idempotent in  $R/I$ .

This completes the proof. □

**Example 1.2.** The converse of the previous proposition, namely that an idempotent in the quotient ring  $R/I$  can always be mapped to an idempotent in  $R$ , is not true in general. To illustrate this, consider the ring of integers  $\mathbb{Z}$  and the quotient ring  $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}_6$ .

It is well-known that  $\mathbb{Z}$  has exactly two idempotents: 0 and 1. However,  $\mathbb{Z}/6\mathbb{Z}$  has four idempotents:

$$0 + 6\mathbb{Z} \quad \text{since } (0 + 6\mathbb{Z})^2 = 0 + 6\mathbb{Z},$$

$$1 + 6\mathbb{Z} \quad \text{since } (1 + 6\mathbb{Z})^2 = 1 + 6\mathbb{Z},$$

$$3 + 6\mathbb{Z} \quad \text{since } (3 + 6\mathbb{Z})^2 = 9 + 6\mathbb{Z} = 3 + 6\mathbb{Z},$$

$$4 + 6\mathbb{Z} \quad \text{since } (4 + 6\mathbb{Z})^2 = 16 + 6\mathbb{Z} = 4 + 6\mathbb{Z}.$$

Thus, the quotient ring  $\mathbb{Z}/6\mathbb{Z}$  contains four distinct idempotents. However, only the elements  $0 + 6\mathbb{Z}$  and  $1 + 6\mathbb{Z}$  have corresponding idempotent representatives in  $\mathbb{Z}$ . Specifically, there do not exist any idempotents  $e \in \mathbb{Z}$  such that  $e + 6\mathbb{Z} = 3 + 6\mathbb{Z}$  or  $e + 6\mathbb{Z} = 4 + 6\mathbb{Z}$ .

From this example, we can conclude two important points: First, not every idempotent in  $R/I$  can be mapped to an idempotent in  $R$ . Second, not all idempotents in  $R/I$  can be lifted modulo  $I = 6\mathbb{Z}$ . Specifically, only the idempotents  $0 + 6\mathbb{Z}$  and  $1 + 6\mathbb{Z}$  have corresponding idempotents in  $\mathbb{Z}$ , while  $3 + 6\mathbb{Z}$  and  $4 + 6\mathbb{Z}$  do not. This demonstrates that idempotent lifting modulo  $I$  is not guaranteed. In fact, for idempotent lifting modulo  $I$  to be possible, every idempotent in  $R/I$  must be liftable to an idempotent in  $R$ .

**Definition 1.14 (Nil-ideal).** An element  $a$  is called **nilpotent** if there exists a positive integer  $n$  such that  $a^n = 0$ . An ideal is called a **nil-ideal** if all its elements are **nilpotent**.

**Proposition 1.13.** If  $I$  is a nil-ideal in a ring  $R$ , then an idempotent lifts modulo  $I$ .

*Proof.* Suppose  $I$  is a nil-ideal of  $R$ , and let  $g + I$  be an idempotent in  $R/I$ . Then:

$$(g + I)^2 = g^2 + I = g + I,$$

which implies  $g^2 - g \in I$ .

We will prove the statement by constructing an idempotent  $e$  from the given element  $g$ .

Set  $r = g^2 - g$  and define:

$$g_1 = g + r - 2gr.$$

First, observe that:

$$gr = g(g^2 - g) = g^3 - g^2 \quad \text{and} \quad rg = (g^2 - g)g = g^3 - g^2.$$

Thus,  $gr = rg$ .

Next, calculate  $g_1^2 - g_1$ :

$$g_1^2 - g_1 = (g + r - 2gr)^2 - (g + r - 2gr).$$

Expanding this, we obtain:

$$g^2 + r^2 + 4g^2r^2 + 2gr - 4g^2r - 4gr^2 - g - r + 2gr.$$

Simplifying further:

$$g_1^2 - g_1 = r^2 + 4r^3 - 4r^2 = r^2(4r - 3).$$

Set  $r_1 = g_1^2 - g_1 = r^2(4r - 3)$ . Since  $r = g^2 - g \in I$ , it follows that  $r^2 \in I$ , and thus  $r_1 \in I$ .

Now, define  $g_2 = g_1 + r_1 - 2g_1r_1$ . By a similar calculation, we obtain:

$$g_2^2 - g_2 = r_1^2(4r_1 - 3).$$

Set  $r_2 = g_2^2 - g_2 = r_1^2(4r_1 - 3)$ . Since  $r_1 \in I$ , we have  $r_1^2 \in I$ , and thus  $r_2 \in I$ .

Continuing this process, we define:

$$g_i = g_{i-1} + r_{i-1} - 2g_{i-1}r_{i-1},$$

and observe that:

$$r_i = g_i^2 - g_i = r_{i-1}^2(4r_{i-1} - 3).$$

Since  $I$  is a nil ideal, there exists an integer  $k > 0$  such that  $r^k = 0$ . Thus, after a finite number of steps, say  $n$ , we have:

$$r_n = g_n^2 - g_n = 0.$$

This implies  $g_n^2 = g_n$ , so  $g_n$  is idempotent.

Now, observe that:

$$g_1 - g = (g + r - 2gr) - g = r(1 - 2g) = (1 - 2g)(g^2 - g).$$

Since  $r = g^2 - g \in I$ , it follows that  $g_1 - g \in I$ . Inductively, for each  $i \in \{1, 2, \dots, n\}$ , we have:

$$g_i - g_{i-1} \in I.$$

Thus, the cumulative difference satisfies:

$$g_n - g = (g_n - g_{n-1}) + (g_{n-1} - g_{n-2}) + \dots + (g_2 - g_1) + (g_1 - g).$$

Since each term  $g_i - g_{i-1} \in I$  and  $I$  is closed under addition, we conclude that  $g_n - g \in I$ .

Finally, set  $e = g_n$ , the idempotent in  $R$  constructed through our iterative process. Since  $e - g \in I$ ,  $e$  lifts  $g$  modulo  $I$ . □

**Definition 1.15** (Right Artinian Ring). A ring  $R$  is called a right Artinian ring if it satisfies the descending chain condition on right ideals, meaning that every descending chain of right ideals in  $R$  eventually stabilizes. In other words, for any sequence of right ideals  $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ , there exists an integer  $n$  such that  $I_n = I_{n+1} = I_{n+2} = \dots$ .

**Proposition 1.14** (C. Hopkins). The Jacobson radical  $J(R)$  of a right Artinian ring  $R$  is nilpotent.

*Proof.* Let  $J(R)$  denote the Jacobson radical of the right Artinian ring  $R$ .

Consider the sequence of natural powers  $J(R)^n$  of the radical  $J(R)$ :

$$J(R) \supseteq J(R)^2 \supseteq J(R)^3 \supseteq \dots$$

Since  $R$  is Artinian, the descending chain of ideals is stationary, i.e., there exists a positive integer  $n$  such that:

$$J(R)^n = J(R)^{n+k} \quad \text{for all } k \in \mathbb{Z}^+.$$

Let  $X = J(R)^n$  be the minimal element of this chain. Clearly,  $X^2 = X$ , so  $X$  is idempotent.

Suppose, for the sake of contradiction, that  $X \neq 0$ . Let  $Y$  be a minimal right ideal of  $R$  such that  $Y \subset X$  and  $YX \neq 0$ . Then, there exists some  $y \in Y$  such that  $yX \neq 0$ .

Now, consider the expression:

$$(yX)X = yX^2 = yX \neq 0.$$

Since  $yX$  is a nonzero right ideal contained within  $Y$ , and  $Y$  is a minimal right ideal, it follows that  $yX = Y$ . Hence, we have  $yx = y$  for some  $x \in X$ .

Next, we compute:

$$y(1 - x) = y - yx = 0.$$

Since  $x \in X \subseteq J(R)$ , and by the properties of the Jacobson radical,  $1 - x$  is invertible in  $R$ , we conclude that  $1 - x \neq 0$ . Therefore,  $y = 0$ , which contradicts the assumption that  $yX \neq 0$ .

Thus, the assumption that  $X \neq 0$  must be false, implying that  $X = 0$ . Consequently, we have  $J(R)^n = X = 0$ , and therefore  $J(R)$  is nilpotent.

□

**Corollary 1.2.** *Idempotents can be lifted modulo the Jacobson radical of an Artinian ring.*

*Proof.* Since every nilpotent ideal is a nil-ideal, the proof follows from Proposition 1.14 and Proposition 1.13. □

**Remark 1.6.** In general, if we have a pair of orthogonal idempotents  $g_1 + I$  and  $g_2 + I$  in  $R/I$ , which lift to idempotents  $e_1, e_2 \in R$ , there is no guarantee that  $e_1$  and  $e_2$  will be orthogonal. However, in some special cases (e.g., nil-ideals), orthogonality of idempotents can be preserved.

### 1.1.5 Small Submodules

**Definition 1.16** (Small Submodule). A submodule  $S$  of a module  $M$  is called **small** (or **superfluous**), denoted by  $S \ll M$ , if for any submodule  $N$  of  $M$ , the equality  $S + N = M$  implies  $N = M$ .

**Example 1.3.** Consider  $\mathbb{Z}_{10}$  as a  $\mathbb{Z}$ -module. The submodules  $\langle 2 \rangle$  and  $\langle 5 \rangle$  are the only proper submodules of  $\mathbb{Z}_{10}$ , where:

$$\langle 2 \rangle = \{0, 2, 4, 6, 8\}, \quad \langle 5 \rangle = \{0, 5\}.$$

The sum of these submodules is:

$$\langle 2 \rangle + \langle 5 \rangle = \{a + b : a \in \langle 2 \rangle, b \in \langle 5 \rangle\} = \{0, 2, 4, 6, 8, 5, 7, 9, 1, 3\} = \mathbb{Z}_{10}.$$

Thus,  $\langle 2 \rangle + \langle 5 \rangle = \mathbb{Z}_{10}$ , but neither  $\langle 2 \rangle$  nor  $\langle 5 \rangle$  equals  $\mathbb{Z}_{10}$ . Therefore,  $\langle 2 \rangle$  and  $\langle 5 \rangle$  are not small submodules.

**Example 1.4.** Consider  $\mathbb{Z}_8$  as a  $\mathbb{Z}$ -module. The submodules  $\langle 2 \rangle$  and  $\langle 4 \rangle$  are the only proper submodules of  $\mathbb{Z}_8$ , where:

$$\langle 2 \rangle = \{0, 2, 4, 6\}, \quad \langle 4 \rangle = \{0, 4\}.$$

The sum of these submodules is:

$$\langle 2 \rangle + \langle 4 \rangle = \{a + b : a \in \langle 2 \rangle, b \in \langle 4 \rangle\} = \{0, 2, 4, 6\} \neq \mathbb{Z}_8.$$

Thus, the only possible way to have  $\langle 2 \rangle + N = \mathbb{Z}_8$  is if  $N = \mathbb{Z}_8$ . Similarly, the only possible way to have  $\langle 4 \rangle + N = \mathbb{Z}_8$  is if  $N = \mathbb{Z}_8$ . Hence, both  $\langle 2 \rangle$  and  $\langle 4 \rangle$  are small submodules in  $\mathbb{Z}_8$ .

**Example 1.5.** Let  $R$  be an Artinian ring with Jacobson radical  $J(R)$ , and let  $M$  be a right  $R$ -module. By Nakayama's lemma, the submodule  $MJ(R)$  is small in  $M$ , denoted  $MJ(R) \ll M$ . To see this, suppose  $MJ(R) + N = M$  for some submodule  $N$  of  $M$ . Nakayama's lemma implies that  $N = M$ , since  $MJ(R)$  is contained in the Jacobson radical and thus "small" in  $M$ . Therefore,  $MJ(R)$  is a small submodule of  $M$ .

**Example 1.6.** Let  $M$  be a finitely generated right  $R$ -module and  $A \subseteq J(R)$  a right ideal contained in the Jacobson radical of  $R$ . Then  $MA$  is a small submodule of  $M$ .

*Proof.* Suppose  $N \subseteq M$  satisfies

$$MA + N = M.$$

Passing to the quotient module  $M/N$ , we have

$$\frac{MA + N}{N} = \frac{M}{N}.$$

This simplifies to

$$\frac{MA}{N} = \frac{M}{N},$$

since  $\frac{N}{N} = 0$ .

Because  $M$  is finitely generated,  $M/N$  is also finitely generated. Since  $A \subseteq J(R)$ , by Nakayama's Lemma we conclude that

$$M/N = 0.$$

Thus  $N = M$ , proving that  $MA$  is small in  $M$  (i.e.,  $MA \ll M$ ). □



**Example 1.7.** Let  $N$  be a maximal submodule of a module  $M$ . Then  $N$  contains every small submodule  $S \ll M$ . To see this, suppose for contradiction that  $S \not\subseteq N$ . Since  $N$  is maximal, the submodule  $S + N$  must equal  $M$ . However, because  $S \ll M$ , the equality  $S + N = M$  implies  $N = M$ . This contradicts the fact that  $N$  is a proper submodule of  $M$ . Therefore,  $N$  must contain every small submodule  $S \ll M$ .

**Example 1.8.** Let  $M$  be a module and  $M'$  a submodule of  $M$ . If  $S \ll M'$ , then  $S \ll M$ .

*Proof.* Assume  $S \ll M'$  and let  $N \hookrightarrow M$  satisfy  $S + N = M$ . Since  $M' \subseteq M$ , intersecting with  $M'$  gives:

$$\begin{aligned} (S + N) \cap M' &= (S \cap M') + (N \cap M') \\ &= S + (N \cap M'), \end{aligned}$$

while:

$$M \cap M' = M'.$$

Thus:

$$S + (N \cap M') = M'.$$

Since  $S \ll M'$ , we have:

$$N \cap M' = M',$$

so  $M' \subseteq N$ .

Now  $S + N = M$  with  $S \subseteq M' \subseteq N$  implies:

$$N = M.$$

Therefore,  $S \ll M$ . □

**Example 1.9.** Let  $M$  be a module over a ring  $R$ .

1. If  $S \ll M$ , then every submodule  $T \subseteq S$  satisfies  $T \ll M$ .
2. If  $S_i \ll M$  for  $1 \leq i \leq n$ , then  $\sum_{i=1}^n S_i \ll M$ .

*Proof.* **Part 1: Submodules of small submodules are small.** Let  $S \ll M$ , and let  $T \subseteq S$ . Suppose  $T + N = M$  for some submodule  $N \subseteq M$ . Since  $T \subseteq S$ , we have:

$$S + N \supseteq T + N = M \implies S + N = M.$$

Because  $S \ll M$ , this forces  $N = M$ . Thus,  $T \ll M$ .

**Part 2: Finite sums of small submodules are small.** We proceed by induction on  $n$ .

**Base case ( $n = 2$ ):** Let  $S_1, S_2 \ll M$ . Suppose  $(S_1 + S_2) + N = M$ . Rewrite this as:

$$S_1 + (S_2 + N) = M.$$

Since  $S_1 \ll M$ , we deduce  $S_2 + N = M$ . But  $S_2 \ll M$ , so  $N = M$ . Hence,  $S_1 + S_2 \ll M$ .

**Inductive step:** Assume the claim holds for  $n = k$ . For  $n = k + 1$ , write:

$$\sum_{i=1}^{k+1} S_i = \left( \sum_{i=1}^k S_i \right) + S_{k+1}.$$

By the induction hypothesis,  $\sum_{i=1}^k S_i \ll M$ . Since  $S_{k+1} \ll M$ , the base case implies  $\sum_{i=1}^{k+1} S_i \ll M$ .  $\square$

**Example 1.10.** Suppose  $S_i \ll M_i$  (for  $1 \leq i \leq n$ ). Then

$$\bigoplus_{i=1}^n S_i \ll \bigoplus_{i=1}^n M_i.$$

*Proof.* For each  $i$ , since  $S_i \ll M_i$  and  $M_i \hookrightarrow \bigoplus_{i=1}^n M_i$ , it follows that  $S_i \ll \bigoplus_{i=1}^n M_i$  (see Example 1.8 ).

By Example 1.9 , the sum of small submodules is small. Since each  $S_i$  is small in  $\bigoplus_{i=1}^n M_i$ , their sum  $\sum_{i=1}^n S_i$  is also small in  $\bigoplus_{i=1}^n M_i$ .

Because the  $S_i$  are independent (i.e.,  $S_i \cap \sum_{j \neq i} S_j = 0$  for all  $i$ ), we have  $\sum_{i=1}^n S_i \cong$

$\bigoplus_{i=1}^n S_i$ . Thus,

$$\bigoplus_{i=1}^n S_i \ll \bigoplus_{i=1}^n M_i. \quad \square$$

**Proposition 1.15.** Homomorphic image of small submodules is small That is, let  $\phi : M \rightarrow N$  be a module homomorphism. If  $S \ll M$ , then  $\phi(S) \ll N$ .

*Proof.* Let  $\phi(S) + X = N$  for some submodule  $X \subseteq N$ . Then, for an arbitrary  $m \in M$ , we have:

$$\phi(m) = \phi(s) + x \quad \text{for some } s \in S, x \in X.$$

Equivalently,

$$\phi(m - s) = x, \quad \text{so } m - s \in \phi^{-1}(X).$$

Therefore,

$$m \in S + \phi^{-1}(X).$$

Hence,

$$M = S + \phi^{-1}(X).$$

Since  $S \ll M$ , it follows that  $M = \phi^{-1}(X)$ .

Applying  $\phi$  to both sides, Lemma 1.10 yields:

$$\phi(M) = \phi(\phi^{-1}(X)) = X \cap \text{Im } \phi.$$

Since  $X \cap \text{Im } \phi \subseteq X$ , it follows that  $\phi(M) \subseteq X$ .

Also,  $\phi(S) \subseteq \phi(M)$ , so it follows that

$$\phi(S) \subseteq X.$$

From the assumption  $\phi(S) + X = N$  and the fact that  $\phi(S) \subseteq X$ , we conclude that  $X = N$ .

Thus,  $\phi(S) \ll N$ , which is what was to be shown.  $\square$

**Proposition 1.16.** Let  $R$  be a ring, and let  $f : M \rightarrow N$  be an  $R$ -module isomorphism. If  $S$  is a small submodule of  $M$  (denoted  $S \ll M$ ), then its image  $f(S)$  is a small submodule of  $N$  (denoted  $f(S) \ll N$ ).

*Proof.* Let  $f : M \rightarrow N$  be an  $R$ -module isomorphism and  $S \ll M$ . The image  $f(S)$  is a submodule of  $N$ . To show  $f(S) \ll N$ , let  $K$  be any submodule of  $N$  such that  $f(S) + K = N$ . We must show  $K = N$ .

Since  $f$  is an isomorphism, its inverse  $f^{-1} : N \rightarrow M$  exists and is also an isomorphism. Apply  $f^{-1}$  to  $f(S) + K = N$ :

$$f^{-1}(f(S) + K) = f^{-1}(N)$$

As  $f^{-1}$  is a homomorphism:

$$f^{-1}(f(S)) + f^{-1}(K) = f^{-1}(N)$$

Since  $f^{-1}(f(S)) = S$  and  $f^{-1}(N) = M$ , this simplifies to:

$$S + f^{-1}(K) = M$$

Let  $L = f^{-1}(K)$ . Since  $K$  is a submodule of  $N$  and  $f^{-1}$  is a homomorphism,  $L$  is a submodule of  $M$ . The equation becomes  $S + L = M$ . By hypothesis,  $S \ll M$ , so  $S + L = M$  implies  $L = M$ . Thus,  $f^{-1}(K) = M$ .

Apply  $f$  to both sides of  $f^{-1}(K) = M$ :

$$f(f^{-1}(K)) = f(M)$$

Since  $f(f^{-1}(K)) = K$  and  $f(M) = N$ , we conclude  $K = N$ .

Thus,  $f(S)$  is a small submodule of  $N$ . Hence, isomorphisms preserve smallness.  $\square$

**Proposition 1.17.** Let  $M$  be an  $R$ -module. Then

1.  $\text{rad } M$  is the sum of all small submodules of  $M$ .
2.  $MJ(R) \subseteq \text{rad } M$ , where  $J(R)$  is the Jacobson radical of the ring  $R$ . The equality  $MJ(R) = \text{rad } M$  holds if  $R$  is a semi-local ring.

*Proof.* (1) Suppose  $M$  is an  $R$ -module, and let  $T = \sum \{S : S \ll M\}$  be the sum of all small submodules of  $M$ . By Example 2.5, since every small submodule  $S$  is contained in every maximal submodule and hence every small module is contained in  $\text{rad } M$ , their sum  $T$  is also contained in  $\text{rad } M$ . Thus,  $T \subseteq \text{rad } M$ .

For the reverse inclusion  $\text{rad } M \subseteq T$ , it suffices to prove that for any  $m \in \text{rad } M$ , the submodule  $mR$  is small in  $M$ . Since If  $mR \ll M$ , then  $mR$  is one of the summands in  $T$ , so  $m \in T$ . If every  $m \in \text{rad } M$  generates a small submodule  $mR$ , then  $\text{rad } M \subseteq T$ .

For any  $m \in \text{rad } M$ , we will show that  $mR \ll M$ . Assume for the sake of contradiction that  $mR$  is not small. Then there exists a submodule  $N$  of  $M$  such that  $mR + N = M$  but  $N \neq M$ .

Consider the quotient module  $M/N$ . Since  $mR + N = M$ ,  $M/N$  is a cyclic module generated by  $\langle m + N \rangle$ . Since  $M/N$  is a non-zero cyclic module and hence finitely generated, it has a maximal submodule  $N'/N$ . This means  $N'$  is a maximal submodule of  $M$  containing  $N$ . Since  $m \in \text{rad } M$ ,  $m$  must be in every maximal submodule of  $M$ . However,  $m \notin N'$  because  $N'/N$  is a maximal submodule of  $M/N$  and  $m + N$  generates  $M/N$ . So, if  $m \in N'$ , then  $N'/N = M/N$ , and  $N'$  is no longer a maximal submodule of  $M$ . This contradicts  $m \in \text{rad } M$ . Since  $N'$  is maximal and  $m \notin N'$ , but  $mR + N = M$ , we have a contradiction. Therefore,  $mR \ll M$ , and  $\text{rad } M \subseteq T$ .  $\square$

**Proposition 1.18.** Let  $M$  be an  $R$ -module. The following statements hold:

1. If  $M'$  is a submodule of  $M$ , then  $\text{rad } M' \subseteq \text{rad } M$ .
2.  $\text{rad } \left( \bigoplus_{i \in I} M_i \right) = \bigoplus_{i \in I} \text{rad } M_i$ .
3. If  $F_R$  is a free  $R$ -module, then  $\text{rad } F = F \cdot J(R)$ , where  $J(R)$  is the Jacobson radical of  $R$ .

*Proof.* 1. Let  $M'$  be a submodule of  $M$ . By Proposition 1.17,  $\text{rad } M'$  is the sum of all small submodules  $S$  of  $M'$ . Since each small submodule  $S$  of  $M'$  is also a small submodule of  $M$  (by Example 1.8), it follows that  $\text{rad } M' \subseteq \text{rad } M$ .

2. Let  $I = \{1, 2\}$  (the general case follows by induction). Set  $T = \text{rad}(M_1 \oplus M_2)$ . Since  $M_1 \subseteq M_1 \oplus M_2$  and  $M_2 \subseteq M_1 \oplus M_2$ , we have  $\text{rad } M_1 \subseteq \text{rad}(M_1 \oplus M_2)$  and  $\text{rad } M_2 \subseteq \text{rad}(M_1 \oplus M_2)$ . Therefore,  $\text{rad } M_1 \oplus \text{rad } M_2 \subseteq \text{rad}(M_1 \oplus M_2)$ .

For the reverse inclusion, let  $m = (m_1, m_2) \in \text{rad}(M_1 \oplus M_2)$ . For any maximal submodule  $N \subseteq M_1$ , the submodule  $N \oplus M_2$  is maximal in  $M_1 \oplus M_2$ . Since  $m \in \text{rad}(M_1 \oplus M_2)$ , it must lie in every maximal submodule of  $M_1 \oplus M_2$ , including  $N \oplus M_2$ . This implies that  $m_1 \in N$  for every maximal submodule  $N$  of  $M_1$ , and thus  $m_1 \in \text{rad } M_1$ . Similarly,  $m_2 \in \text{rad } M_2$ . Therefore,  $m = (m_1, m_2) \in \text{rad } M_1 \oplus \text{rad } M_2$ , and hence  $T \subseteq \text{rad } M_1 \oplus \text{rad } M_2$ . Combining both inclusions, we conclude that  $\text{rad}(M_1 \oplus M_2) = \text{rad } M_1 \oplus \text{rad } M_2$ .

3. Let  $F$  be a free  $R$ -module. Then  $F \cong \bigoplus_{i \in I} R$ . By part (2), we have  $\text{rad } F = \text{rad } \left( \bigoplus_{i \in I} R \right) = \bigoplus_{i \in I} \text{rad } R$ . Since  $F \cong \bigoplus_{i \in I} R$ , any element  $f \in F$  can be written as  $f = (r_i)_{i \in I}$ , where each  $r_i \in R$ . Multiplying  $f$  by an element  $r \in \text{rad } R = J(R)$  gives  $fr = (r_i r)_{i \in I}$ . Therefore,  $F \cdot \text{rad } R = F \cdot J(R)$  consists of all elements of the form  $(r_i r)_{i \in I}$ , which is exactly  $\bigoplus_{i \in I} \text{rad } R = \bigoplus_{i \in I} J(R)$ . Thus,  $\text{rad } F = F \cdot J(R)$ . This completes the proof.

□

**Lemma 1.6.** Let  $\phi : M \rightarrow N$  be a homomorphism. Then,

$$\phi(\text{rad } M) \hookrightarrow \text{rad } N.$$

*Proof.* Since  $\text{rad } M = \sum_{A \ll M} A$ , it follows that

$$\phi(\text{rad } M) = \phi\left(\sum_{A \ll M} A\right) = \sum_{A \ll M} \phi(A).$$

By Proposition 1.15,  $\phi(A) \ll N$  for every  $A \ll M$ .

Since each term in the sum  $\sum_{A \ll M} \phi(A)$  is a small submodule of  $N$ , their sum  $\phi(\text{rad } M)$  must be a submodule of the sum of all small submodules of  $N$ .

That is,

$$\phi(\text{rad } M) \hookrightarrow \text{rad } N. \quad \square$$

□

**Corollary 1.3.** *Let  $\varphi : M \rightarrow N$  be an epimorphism such that  $\ker \varphi \ll M$ . Then*

1.  $\varphi(\text{rad } M) = \text{rad } N$
2.  $\varphi^{-1}(\text{rad } N) = \text{rad } M$

*Proof.* We prove the two parts separately.

**Proof of (1):** We will establish the equality by showing two inclusions.

First, by Lemma 1.6, we have  $\varphi(\text{rad } M) \subseteq \text{rad } N$ .

For the reverse inclusion, let  $U$  be a small submodule of  $N$  (denoted  $U \ll N$ ), and let  $A$  be a submodule of  $M$ . Suppose that

$$A + \varphi^{-1}(U) = M$$

Since  $\varphi$  is an epimorphism, we can apply  $\varphi$  to both sides. Applying  $\varphi$  and using Lemma 1.10 (part 2, which states  $\varphi(\varphi^{-1}(U)) = U \cap \text{Im } \varphi$ ) yields:

$$\varphi(A) + (U \cap \text{Im } \varphi) = \varphi(M)$$

Since  $\varphi$  is an epimorphism,  $\text{Im } \varphi = N$  and  $\varphi(M) = N$ . Also, as  $U$  is a submodule of  $N$ , we have  $U \cap N = U$ . Thus, the equation simplifies to:

$$\varphi(A) + U = N$$

Because  $U \ll N$ , the definition of a small submodule implies that

$$\varphi(A) = N$$

Applying the preimage  $\varphi^{-1}$  to both sides gives

$$\varphi^{-1}(\varphi(A)) = \varphi^{-1}(N)$$

Since  $\varphi$  is surjective,  $\varphi^{-1}(N) = M$ . Using Lemma 1.10 (part 1, which states  $\varphi^{-1}(\varphi(A)) = A + \ker \varphi$ ), the equation becomes:

$$A + \ker \varphi = M$$

By hypothesis,  $\ker \varphi$  is small in  $M$  ( $\ker \varphi \ll M$ ). Therefore, it follows directly from the definition of a small submodule that if  $A + \ker \varphi = M$ , then  $A$  must equal  $M$ :

$$A = M$$

Thus, the initial assumption  $A + \varphi^{-1}(U) = M$  implies  $A = M$ . This demonstrates that  $\varphi^{-1}(U)$  is a small submodule of  $M$ , i.e.,  $\varphi^{-1}(U) \ll M$ .

Now, since  $\text{rad } M$  is defined as the sum of all small submodules of  $M$ , it must contain  $\varphi^{-1}(U)$ :

$$\varphi^{-1}(U) \subseteq \text{rad } M$$

Applying  $\varphi$  to both sides of this inclusion yields

$$\varphi(\varphi^{-1}(U)) \subseteq \varphi(\text{rad } M)$$

By Lemma 1.10 (part 2,  $\varphi(\varphi^{-1}(U)) = U \cap \text{Im } \varphi$ ), and noting  $\text{Im } \varphi = N$  because  $\varphi$  is an



epimorphism, we have:

$$U \cap N \subseteq \varphi(\text{rad } M)$$

Since  $U \subseteq N$ ,  $U \cap N = U$ , and the inclusion becomes:

$$U \subseteq \varphi(\text{rad } M)$$

Because  $U$  was an arbitrary small submodule of  $N$ , this inclusion holds for all  $U \ll N$ . The radical  $\text{rad } N$ , being the sum of all such small submodules of  $N$ , must therefore also be contained within  $\varphi(\text{rad } M)$ . That is,

$$\text{rad } N \subseteq \varphi(\text{rad } M)$$

Combining the two inclusions ( $\varphi(\text{rad } M) \subseteq \text{rad } N$  and  $\text{rad } N \subseteq \varphi(\text{rad } M)$ ), we conclude that

$$\text{rad } N = \varphi(\text{rad } M)$$

This completes the proof of part (1).

**Proof of (2):** From Part (1), we established that  $\varphi(\text{rad } M) = \text{rad } N$ . By hypothesis,  $\ker \varphi \ll M$ . Since  $\text{rad } M$  is the sum of all small submodules of  $M$ , the condition  $\ker \varphi \ll M$  implies, by definition of the radical, that:

$$\ker \varphi \subseteq \text{rad } M$$

This inclusion implies that

$$\text{rad } M = \text{rad } M + \ker \varphi$$

Using Lemma 1.10 (part 1, which states  $\varphi^{-1}(\varphi(X)) = X + \ker \varphi$ ) with  $X = \text{rad } M$ , we have:

$$\varphi^{-1}(\varphi(\text{rad } M)) = \text{rad } M + \ker \varphi$$

Substituting  $\text{rad } M + \ker \varphi = \text{rad } M$  (from the previous step), we get

$$\varphi^{-1}(\varphi(\text{rad } M)) = \text{rad } M$$

Finally, since  $\varphi(\text{rad } M) = \text{rad } N$  from Part (1), we can substitute  $\text{rad } N$  for  $\varphi(\text{rad } M)$  on the left side:

$$\varphi^{-1}(\text{rad } N) = \text{rad } M$$

This completes the proof of part (2).  $\square$

**Lemma 1.7.**  *$M$  is finitely generated if and only if  $\text{rad } M \ll M$  and  $M/\text{rad } M$  is finitely generated. In particular,  $\text{rad } R \ll R$ .*

*Proof.* (  $\implies$  ) Let  $M$  be a finitely generated right  $R$ -module. Let  $\text{rad } M + C = M$ . Suppose  $C \neq M$ . Since  $M$  is finitely generated,  $C$  is contained in a maximal submodule  $X \subset M$  (by Lemma 1.1). Since  $\text{rad } M$  is the intersection of all maximal submodules of  $M$ , we have  $\text{rad } M \subseteq X$ . As both  $\text{rad } M$  and  $C$  are contained in  $X$ , their sum  $\text{rad } M + C$  is also contained in  $X$ .

By assumption,  $\text{rad } M + C = M$ , so  $M \subseteq X$ , this contradicts  $X$  being a proper submodule. Thus,  $C = M$  and therefore  $\text{rad } M \ll M$ .

Furthermore, as a direct consequence of Proposition 1.1,  $M/\text{rad } M$  is finitely generated.

(  $\impliedby$  ) Assume  $\text{rad } M \ll M$  and that  $M/\text{rad } M$  is finitely generated. Let  $\{x_1 + \text{rad } M, \dots, x_n + \text{rad } M\}$  be a generating set for  $M/\text{rad } M$ . The submodule  $N = \langle x_1, \dots, x_n \rangle$  is finitely generated and satisfies  $N + \text{rad } M = M$ . Since  $\text{rad } M \ll M$ , this implies that  $N = M$ . Hence,  $M$  is finitely generated.  $\square$

**Lemma 1.8.** *Let  $R$  be a ring,  $J(R) = J$  its Jacobson radical, and  $C$  a maximal ideal. Suppose that  $R = C + B$  for some ideal  $B$ , and  $C \cap B \ll R$ . Then, we have the following isomorphism:*

$$R/J \cong C/J \oplus (B + J)/J.$$

*Proof.* We will prove the lemma by showing both inclusions.

( $\supseteq$ ) **Inclusion:** Since  $R = C + B$ , we have  $R/J = (C + B)/J$ . Clearly,  $C \subseteq C + B$ , so we obtain  $C/J \subseteq (C + B)/J = R/J$ .

Next, since we do not know if  $J \subseteq B$ , we observe that  $J \subseteq B + J$ , so we can consider the

quotient  $(B + J)/J$ . Moreover, since  $J \subseteq C$  (because  $J$  is the intersection of all maximal ideals and  $C$  is a maximal ideal), we have:

$$\frac{B + J}{J} \subseteq \frac{B + C}{J} = R/J.$$

Thus, we conclude that:

$$C/J \subseteq R/J \quad \text{and} \quad (B + J)/J \subseteq R/J.$$

Therefore, their sum is also a subset of  $R/J$ , i.e.,

$$C/J + (B + J)/J \subseteq R/J.$$

( $\subseteq$ ) **Inclusion:** For the reverse inclusion, we take any element  $x \in R/J$ . Since  $R = C + B$ , we can write  $x = c + b + J$ , where  $c \in C$  and  $b \in B$ . Therefore, we have:

$$x = c + J + b + j + J = (c + J) + (b + j + J) = (c + J) + (b + J).$$

Thus,  $x \in C/J + (B + J)/J$ , which shows that  $R/J \subseteq C/J + (B + J)/J$ .

Finally, we need to show that the intersection  $C/J \cap (B + J)/J$  is trivial.

**Intersection:** We will now show that:

$$\frac{C}{J} \cap \frac{B + J}{J} = \{0\}.$$

Take  $x \in \frac{C}{J} \cap \frac{B + J}{J}$ . Then we can write:

$$x = c + J = b + j + J \quad \text{for some } c \in C, b \in B, j \in J.$$

This implies:

$$c - b - j \in J.$$

Since  $j \in J$ , we have:

$$c - b \in J \subseteq C.$$

Thus, there exists  $y \in C$  such that:

$$c - b = y \in C.$$

Rewriting gives:

$$b = c - y \in C \cap B.$$

Because  $C \cap B \ll R$  and  $J$  is the sum of all small submodules, we have:

$$C \cap B \subseteq J.$$

Therefore,  $b \in J$ , and consequently:

$$x = b + j + J = 0 + J.$$

This shows that the only element in the intersection is zero, completing the proof. □

**Proposition 1.19.** Every nonzero semisimple module has a maximal submodule.

*Proof.* Let  $M$  be a nonzero semisimple module. By semisimplicity, every submodule of  $M$  is a direct summand. Consider any small submodule  $K \ll M$ . We can decompose  $M$  as  $M = K \oplus T$  for some submodule  $T$ . However, since  $K$  is small, the equality  $M = K + T$  forces  $T = M$ . Consequently,  $K = K \cap T = K \cap M = 0$ , proving that 0 is the only small submodule of  $M$ . By Proposition 1.17, the radical  $\text{rad } M$  (being the sum of all small submodules) must be 0. Since  $\text{rad } M \neq M$ , we conclude that  $M$  admits a maximal submodule. □

### 1.1.6 Nilpotency

**Definition 1.17 (Nilpotent).** An ideal  $\mathcal{I}$  is called **nilpotent** if there exists a natural number  $n$  such that  $\mathcal{I}^n = 0$ .

Note that  $\mathcal{I}^n = 0$  means that  $a_1 a_2 \dots a_n = 0$  for any  $n$  elements  $a_1, a_2, \dots, a_n \in \mathcal{I}$ .

**Definition 1.18 (T-nilpotent).** An ideal  $\mathcal{I}$  is called right  $T$ -nilpotent if for any sequence  $a_1, a_2, \dots, a_n$  of elements  $a_i \in \mathcal{I}$ , there exists a positive integer  $k$  such that  $a_k a_{k-1} \dots a_1 = 0$ .

**Proposition 1.20.** For any right ideal  $A \subseteq R$ , the following are equivalent:

1.  $A$  is right  $T$ -nilpotent.
2. For any right  $R$ -module  $M$ ,  $MA = M \implies M = 0$ .
3. For right  $R$ -modules  $N \subseteq M$ ,  $MA + N = M \implies N = M$ .

*Proof.* **(1)  $\implies$  (2):** Let  $A \subseteq R$  be a right  $T$ -nilpotent ideal, and let  $M$  be a right  $R$ -module such that  $MA = M$ . We aim to show that  $M = 0$ .

Since  $MA = M$ , for any  $m \in M$ , there exist elements  $m_1 \in M$  and  $a_1 \in A$  such that  $m = m_1 a_1$ . Continuing this process recursively, since  $m_1 \in M$ , there exist elements  $m_2 \in M$  and  $a_2 \in A$  such that  $m_1 = m_2 a_2$ . Substituting this into the equation for  $m$ , we get  $m = m_2 a_2 a_1$ . This process can be repeated indefinitely: for each  $m_n \in M$  and  $a_n \in A$ , there exist elements  $m_{n+1} \in M$  and  $a_{n+1} \in A$  such that  $m_n = m_{n+1} a_{n+1}$ . Thus, after  $n$  steps, we have the following equation for  $m$ :

$$m = m_n a_n a_{n-1} \dots a_1$$

Since  $A$  is right  $T$ -nilpotent, there exists some integer  $n$  such that  $a_n a_{n-1} \dots a_1 = 0$ . Substituting this into the equation for  $m$ , we get  $m = m_n \cdot 0 = 0$ .

Since  $m \in M$  was arbitrary, we conclude that  $M = 0$ .

**(2)  $\implies$  (3):** Follows trivially

**(3)  $\implies$  (1):** Assume  $A$  is not right  $T$ -nilpotent. Then, there exists an infinite sequence  $\{a_n\} \subseteq A$  such that  $a_n a_{n-1} \dots a_1 \neq 0$  for all  $n$ . Construct the right  $R$ -module  $M$  as follows:

Let  $M$  be generated by elements  $\{e_1, e_2, e_3, \dots\}$  with relations  $e_i = e_{i+1}a_i$  for all  $i$ .

This module  $M$  is non-zero because the relations do not collapse all generators to zero (since  $a_n a_{n-1} \cdots a_1 \neq 0$ ). Now, consider the submodule  $N = 0 \subseteq M$ . Observe that:

$$MA + N = MA + 0 = MA.$$

Since  $e_i = e_{i+1}a_i \in MA$  (as  $a_i \in A$ ), all generators  $e_i$  lie in  $MA$ . Hence,  $MA = M$ . By condition (3),  $MA + N = M$  implies  $N = M$ . However,  $N = 0$ , so  $M = 0$ , contradicting the construction of  $M \neq 0$ .

Thus, the assumption that  $A$  is not right  $T$ -nilpotent leads to a contradiction. Therefore,  $A$  must be right  $T$ -nilpotent. Hence, (3)  $\implies$  (1).

□

**Lemma 1.9.** *The following statements are equivalent for a right ideal  $A$  of  $R$ :*

1.  $A$  is right  $T$ -nilpotent.
2. If  $M_R \neq 0$  is any right module then  $MA \ll M$ .
3. If  $M_R \neq 0$  is any right module then  $MA \neq M$ .
4. If  $F$  is a countably generated free module then  $FA \ll F$ .

*Proof.* (1)  $\implies$  (2): Assume that  $A$  is right  $T$ -nilpotent. Let  $M \neq 0$  be a right  $R$ -module, and suppose  $MA + X = M$  for some submodule  $X \subseteq M$ . We prove that  $X = M$ .

Suppose  $X \subsetneq M$ . Choose  $m \in M/X$ , i.e.,  $m \in M$  but  $m \notin X$ . Since  $MA + X = M$ , we can write:

$$m = m_1 a_1 + x_1 \quad \text{for some } m_1 \in M, a_1 \in A, x_1 \in X.$$

As  $m \notin X$ , it follows that  $m_1a_1 \notin X$  (otherwise  $m \in X$ , which is a contradiction). Apply the same decomposition to  $m_1$ :

$$m_1 = m_2a_2 + x_2 \quad \text{for some } m_2 \in M, a_2 \in A, x_2 \in X.$$

Substitute this back into the expression for  $m$ :

$$m = (m_2a_2 + x_2)a_1 + x_1 = m_2a_2a_1 + \underbrace{x_2a_1 + x_1}_{\in X}.$$

Since  $m \notin X$ , it follows that  $m_2a_2a_1 \notin X$ . We repeat this process recursively to generate a sequence  $\{a_1, a_2, \dots\} \subseteq A$ . By  $T$ -nilpotency, there exists an integer  $n$  such that:

$$a_na_{n-1} \cdots a_1 = 0.$$

After  $n$  steps, we have:

$$m = m_na_n \cdots a_1 + \underbrace{\left( \sum_{k=1}^n x_k a_{k-1} \cdots a_1 \right)}_{\in X}.$$

Since  $a_n \cdots a_1 = 0$ , this collapses to  $m \in X$ , which contradicts  $m \notin X$ . Therefore, we must have  $X = M$ , and hence  $MA \ll M$ .

(2)  $\implies$  (3): This implication is straightforward and holds trivially.

(3)  $\implies$  (4): Let  $M_R \neq 0$  and assume that  $MA \neq M$  for any right ideal  $A$  of  $R$ . Let  $F_R$  be a countably generated free right  $R$ -module.

Suppose that  $FA + X = F$  for some submodule  $X \subseteq F$ . We aim to show that  $X = F$ .

For contradiction, assume  $X \neq F$ . In this case, the quotient module  $M = F/X$  is non-zero. The image of  $FA$  under the quotient map is  $(F/X)A$ . Since  $FA + X = F$ , every element of  $M = F/X$  can be written as  $f + X = fa + X$  for some  $f \in F, a \in A$ .

Thus, we have  $MA = (F/X)A = F/X = M$ , which means that the quotient module  $M$  is generated by the image of  $FA$  in  $F/X$ . However, this contradicts the assumption that

$MA \neq M$ . Therefore, our assumption that  $X \neq F$  must be false.

Consequently, we conclude that  $X = F$ , and hence  $FA \ll F$ .

(4)  $\implies$  (1): Assume that  $F_R$  is a countably generated free module and that  $FA \ll F$ , where  $A \subseteq R$  is any ideal.

Let  $B = \{f_1, f_2, f_3, \dots\}$  be the countably infinite basis of  $F$ . For a sequence  $a_1, a_2, \dots$  in  $A$ , define the submodule

$$G = (f_1 - f_2a_1)R + (f_2 - f_3a_2)R + \dots$$

For any basis element  $f_i$ , we can write

$$f_i = (f_i - f_{i+1}a_i) + f_{i+1}a_i,$$

where  $(f_i - f_{i+1}a_i) \in G$  and  $f_{i+1}a_i \in FA$ .

Thus,  $G + FA$  contains all the basis elements, so we have  $G + FA = F$ .

By hypothesis, since  $FA \ll F$ , it follows that  $G = F$ .

Therefore, every element of  $F$  belongs to  $G$ . Since  $G$  is generated by infinitely many elements  $(f_i - f_{i+1}a_i)$ , any specific element in  $G$  (such as  $f_1$ ) can be written as a finite  $R$ -linear combination of these generators. Hence, there exists an integer  $n$  such that

$$f_1 = (f_1 - f_2a_1)r_1 + (f_2 - f_3a_2)r_2 + \dots + (f_n - f_{n+1}a_n)r_n$$

for some  $r_1, r_2, \dots, r_n \in R$ .

Expanding the right-hand side and matching coefficients of  $f_1, f_2, \dots, f_{n+1}$ , we derive the



following system of equations:

$$\begin{aligned}
 1 &= r_1, \\
 0 &= -a_1 r_1 + r_2, \\
 0 &= -a_2 r_2 + r_3, \\
 &\vdots \\
 0 &= -a_{n-1} r_{n-1} + r_n, \\
 0 &= -a_n r_n.
 \end{aligned}$$

From this, we find  $r_1 = 1$ ,  $r_2 = a_1 r_1 = a_1$ ,  $r_3 = a_2 r_2 = a_2 a_1$ , and so on, leading to  $r_n = a_{n-1} \dots a_1$ , with the final equation  $0 = a_n r_n = a_n a_{n-1} \dots a_1$ .

To conclude  $T$ -nilpotency, we observe that  $a_n \dots a_1 = 0$ .

Since the sequence  $a_1, a_2, \dots$  was arbitrary, this shows that every sequence in  $A$  is eventually zero under multiplication. Hence,  $A$  is right  $T$ -nilpotent.

□

**Example 1.11.** Let  $M = M_R$  be a right  $R$ -module, and  $A \subseteq J(R)$  a right ideal. If  $A$  is right  $T$ -nilpotent, then  $MA \ll M$  (i.e.,  $MA$  is superfluous in  $M$ ).

*Proof.* We want to show that  $MA \ll M$ , which means that for any submodule  $N \subseteq M$ ,

$$MA + N = M \implies N = M.$$

This is exactly the statement of Proposition 1.20(3) for the module  $M$  and submodule  $N$ . Since  $A$  is right  $T$ -nilpotent by assumption, Proposition 1.20(3) guarantees that the implication holds for all right  $R$ -modules. Therefore,  $MA \ll M$  holds. □

**Proposition 1.21.** Let  $P_R$  be a projective right  $R$ -module and let  $A \subseteq J(R)$  be a right  $T$ -nilpotent ideal. Then the natural surjection  $\pi : P \rightarrow P/PA$  is a projective cover of  $P/PA$ .

*Proof.* To prove that  $\pi : P \rightarrow P/PA$  is a projective cover, we check the necessary conditions:

1. **Projectivity:** By assumption,  $P$  is a projective right  $R$ -module.
2. **Superfluous kernel:** We need to show that  $\ker \pi = PA$  is superfluous in  $P$ , that is,  $PA \ll P$ .

From Proposition 1.20(3) and Example 1.11, since  $A$  is a right  $T$ -nilpotent ideal and  $A \subseteq J(R)$ , we know that for any right  $R$ -module  $M$ ,  $MA \ll M$ . Applying this to  $M = P$ , we obtain:

$$PA = \ker \pi \ll P.$$

Since  $\pi$  is surjective by construction,  $P$  is projective, and  $\ker \pi = PA$  is superfluous, we conclude that  $\pi : P \rightarrow P/PA$  satisfies all the conditions of a projective cover.

□

### 1.1.7 Exact Sequences

**Definition 1.19.** 1. The pair of homomorphisms  $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$  is said to be **exact** (at  $Y$ ) if  $\text{Im } \alpha = \text{Ker } \beta$ .

2. A sequence  $\cdots \xrightarrow{\xi_{n-1}} X_{n-1} \xrightarrow{\xi_n} X_n \xrightarrow{\xi_{n+1}} X_{n+1} \xrightarrow{\xi_{n+2}} \cdots$  of homomorphisms is said to be an **exact sequence** if it is exact at every  $X_n$  (i.e.,  $\text{Im } \xi_i = \text{Ker } \xi_{i+1}$  for all  $i$ ).

**Remark 1.7.** Exactness means the image of one homomorphism exactly equals the kernel of the next homomorphism in the sequence.

**Proposition 1.22.** Let  $X, Y$ , and  $Z$  be  $R$ -modules over some ring  $R$ . Then:

1. The sequence  $0 \rightarrow X \xrightarrow{\psi} Y$  is exact (at  $X$ ) if and only if  $\psi$  is a **monomorphism**.
2. The sequence  $Y \xrightarrow{\varphi} Z \rightarrow 0$  is exact (at  $Z$ ) if and only if  $\varphi$  is an **epimorphism**.

- Proof.* 1. The (uniquely defined) homomorphism  $0 \longrightarrow X$  has image 0 in  $X$ . This will be the kernel of  $\psi$  if and only if  $\psi$  is a monomorphism.
2. Similarly, the kernel of the (uniquely defined) zero homomorphism  $Z \longrightarrow 0$  is all of  $Z$ , which is the image of  $\varphi$  if and only if  $\varphi$  is an epimorphism.

□

**Corollary 1.4.** *The sequence  $0 \longrightarrow X \xrightarrow{\psi} Y \xrightarrow{\varphi} Z \longrightarrow 0$  is exact if and only if  $\psi$  is a monomorphism,  $\varphi$  is an epimorphism, and  $\text{Im } \psi = \text{Ker } \varphi$ .*

*Proof.* Suppose The sequence  $0 \longrightarrow X \xrightarrow{\psi} Y \xrightarrow{\varphi} Z \longrightarrow 0$  is exact.

Since the first mapping  $0 \longrightarrow X$  has image 0, the exactness of  $0 \longrightarrow X \xrightarrow{\psi} Y$  indicates that  $\psi$  is a monomorphism.

Since the last mapping  $Z \longrightarrow 0$  is the zero homomorphism with kernel =  $Z$ , the exactness of  $Y \xrightarrow{\varphi} Z \longrightarrow 0$  indicates that  $\varphi$  is an epimorphism.

The exactness of the sequence  $0 \longrightarrow X \xrightarrow{\psi} Y \xrightarrow{\varphi} Z \longrightarrow 0$  implies that  $\text{Im } \psi = \text{Ker } \varphi$ .

Conversely, if  $\psi$  is a monomorphism,  $\varphi$  is an epimorphism, and  $\text{Im } \psi = \text{Ker } \varphi$ , then the sequence is exact by definition and Proposition 1.22.

□

**Remark 1.8.** 1. Note that the homomorphism on the left and right of

$$0 \longrightarrow X \xrightarrow{\psi} Y \xrightarrow{\varphi} Z \longrightarrow 0$$

need not be further specified since there is only one homomorphism from the 0 module to any given module and there is only one homomorphism from any given module to the 0 module.

2. The assertion that the sequence is exact (at  $X$ ) is just that  $\text{Ker } \psi = \text{Im } 0 = \{0\}$ , i.e.,  $\psi$  is a monomorphism.

3. The assertion that the sequence is exact (at  $Z$ ) is just that  $Z = \text{Ker } 0 = \text{Im } \varphi$ , i.e.,  $\varphi$  is an epimorphism.
4. The assertion that the sequence is exact (at  $Y$ ) is just that  $\text{Ker } \varphi = \text{Im } \psi$ .
5. Since  $\varphi : Y \longrightarrow Z$  is an epimorphism, the first isomorphism theorem implies  $Y/\text{Ker } \varphi \cong Z$ , and then exactness (at  $Y$ ) gives  $\text{Ker } \varphi = \text{Im } \psi$ , which then implies  $Y/\text{Im } \psi \cong Z$ .

**Definition 1.20.** Let  $D$  be a fixed  $R$ -module. For any  $R$ -module  $L$ , consider the set of all  $R$ -module homomorphisms from  $D$  to  $L$ , denoted by  $\text{Hom}_R(D, L)$ . Given an  $R$ -module homomorphism  $\varphi : L \rightarrow M$ , we define the induced homomorphism:

$$\bar{\varphi} : \text{Hom}_R(D, L) \rightarrow \text{Hom}_R(D, M)$$

by

$$\bar{\varphi}(f) = \varphi \circ f \quad \text{for all } f \in \text{Hom}_R(D, L).$$

**Proposition 1.23.** Let  $D, L, M$ , and  $N$  be  $R$ -modules. Then the following sequence

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N$$

is exact if and only if the sequence

$$0 \longrightarrow \text{Hom}_R(D, L) \xrightarrow{\bar{\psi}} \text{Hom}_R(D, M) \xrightarrow{\bar{\varphi}} \text{Hom}_R(D, N)$$

is exact.

*Proof.* Suppose  $D, L, M$ , and  $N$  are  $R$ -modules, and the sequence

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N$$

is exact.

(i) We first show that  $\bar{\psi}$  is a monomorphism.

Let  $\bar{\psi}(f_1) = \bar{\psi}(f_2)$ , where  $f_1, f_2 \in \text{Hom}_R(D, L)$ . Then

$$\psi \circ f_1 = \psi \circ f_2,$$

which implies

$$(\psi \circ f_1)(x) = (\psi \circ f_2)(x) \quad \text{for all } x \in D.$$

Since  $\psi$  is a monomorphism (by hypothesis), it follows that

$$f_1(x) = f_2(x) \quad \text{for all } x \in D.$$

Thus,  $f_1 = f_2$ , and therefore,  $\bar{\psi}$  is a monomorphism.

(ii) Next, we show that  $\text{Im } \bar{\psi} = \ker \bar{\varphi}$ .

First, we show that  $\text{Im } \bar{\psi} \subseteq \ker \bar{\varphi}$ .

Let  $g \in \text{Im } \bar{\psi}$ . Then  $g = \bar{\psi}(f)$  for some  $f \in \text{Hom}_R(D, L)$ . Therefore,

$$g = \psi \circ f \quad \text{and} \quad \varphi \circ g = \varphi \circ (\psi \circ f) = \varphi(\psi(f)).$$

Since  $\text{Im } \psi = \ker \varphi$ , we know that  $\varphi \circ \psi = 0$ , so

$$\varphi \circ g = 0 \quad \Rightarrow \quad g \in \ker \bar{\varphi}.$$

Thus,  $\text{Im } \bar{\psi} \subseteq \ker \bar{\varphi}$ .

Next, we show the reverse inclusion  $\ker \bar{\varphi} \subseteq \text{Im } \bar{\psi}$ .

Let  $g \in \ker \bar{\varphi}$ . Then  $\varphi \circ g = 0$ . Since  $\text{Im } \psi = \ker \varphi$ , we have  $g \in \text{Im } \psi$ , i.e.,  $g = \psi \circ f$  for some  $f \in \text{Hom}_R(D, L)$ . Hence,  $g \in \text{Im } \bar{\psi}$ .

Therefore,  $\ker \bar{\varphi} \subseteq \text{Im } \bar{\psi}$ , and we conclude that  $\text{Im } \bar{\psi} = \ker \bar{\varphi}$ .

Thus, the sequence

$$0 \longrightarrow \text{Hom}_R(D, L) \xrightarrow{\bar{\psi}} \text{Hom}_R(D, M) \xrightarrow{\bar{\varphi}} \text{Hom}_R(D, N)$$

is exact.

**Conversely**, suppose that the second sequence is exact for any  $D$ . We will show that the first sequence is exact as well.

Let  $D = \ker \psi$ .

We observe that the map  $\bar{\psi} : \text{Hom}_R(\ker \psi, L) \longrightarrow \text{Hom}_R(\ker \psi, M)$  is a monomorphism.

Now, let  $i : \ker \psi \longrightarrow L$  be the inclusion map.

Thus, since  $\bar{\psi}$  is a monomorphism. This means that  $\bar{\psi}(i) = 0$  which implies  $i = 0$ .

Since  $i : \ker \psi \longrightarrow L$  is the inclusion map and  $i = 0$ , it must be that  $\ker \psi = 0$ . Therefore,  $\psi$  is a monomorphism.

Next, we show that  $\text{Im } \psi = \ker \varphi$ .

Let  $D = L$ . Then

$$\psi \circ id_D = \bar{\psi}(id_D) \in \text{Im } \bar{\psi} = \ker \bar{\varphi}.$$

Thus,

$$\varphi \circ \psi = \bar{\varphi}(\psi) = 0 \quad \Rightarrow \quad \text{Im } \psi \subseteq \ker \varphi.$$

Take  $D = \ker \varphi$  and let  $\sigma : \ker \varphi \longrightarrow M$ , where  $\ker \varphi$  is a submodule of  $N$

$$\bar{\varphi}(\sigma) = \varphi \circ \sigma = 0,$$

which implies that  $\sigma \in \ker \bar{\varphi} = \text{Im } \bar{\psi}$ .

Therefore,  $\sigma = \bar{\psi}(\theta)$  for some  $\theta \in \text{Hom}_R(D, M)$ , so  $\sigma = \psi \circ \theta$ .

Thus,  $\ker \varphi = \text{Im } \sigma \subseteq \text{Im } \psi$ , and we conclude that the first sequence is exact.  $\square$

**Example 1.12.** Consider the exact sequence:

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$$

This induces the sequence:

$$0 \longrightarrow \text{Hom}_R(D, L) \xrightarrow{\bar{\psi}} \text{Hom}_R(D, M) \xrightarrow{\bar{\varphi}} \text{Hom}_R(D, N) \longrightarrow 0$$

However, this sequence is **not** necessarily exact.

**Counterexample:** To illustrate this, consider the following short exact sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

Let  $D = \mathbb{Z}/2\mathbb{Z}$  and consider the identity map  $\text{id}_{\mathbb{Z}/2\mathbb{Z}}$ . The diagram below shows that no homomorphism  $h \in \text{Hom}_R(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$  satisfies  $\pi \circ h = \text{id}_{\mathbb{Z}/2\mathbb{Z}}$ :

$$\begin{array}{ccccc} & & \mathbb{Z}/2\mathbb{Z} & & \\ & \nwarrow \bar{h} & \downarrow \text{id}_{\mathbb{Z}/2\mathbb{Z}} & & \\ \mathbb{Z} & \xleftarrow{\pi} & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \end{array}$$

where  $M = \mathbb{Z}$ ,  $N = \mathbb{Z}/2\mathbb{Z}$ , and  $D = \mathbb{Z}/2\mathbb{Z}$ .

*Proof.* Any homomorphism  $h \in \text{Hom}_R(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$  must preserve the module structure, meaning:

$$h(a + b) = h(a) + h(b) \quad \forall a, b \in \mathbb{Z}/2\mathbb{Z}$$

In  $\mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}_2 = \{0, 1\}$ , the element 1 has order 2, meaning  $1 + 1 = 0$ . Now, suppose  $h(1) = n$  for some  $n \in \mathbb{Z}$ . By the additivity property of homomorphisms, we have:

$$h(1 + 1) = h(1) + h(1) = n + n = 2n$$

Since  $1 + 1 = 0$  in  $\mathbb{Z}/2\mathbb{Z}$ , we must also have  $h(0) = 2n$ . However, the homomorphism

property implies  $h(0) = 0$ . Thus,  $2n = 0$ , which forces  $n = 0$ . Therefore,  $h(1) = 0$ .

Thus, the homomorphism  $h$  maps every element of  $\mathbb{Z}/2\mathbb{Z}$  to 0 in  $\mathbb{Z}$ , implying that  $\pi \circ h \neq \text{id}_{\mathbb{Z}/2\mathbb{Z}}$ .  $\square$

**Conclusion:** This counterexample shows that even if

$$M \xrightarrow{\varphi} N \longrightarrow 0$$

is exact, the induced sequence

$$\text{Hom}_R(D, M) \xrightarrow{\bar{\varphi}} \text{Hom}_R(D, N) \longrightarrow 0$$

is not necessarily exact.

**Definition 1.21** (Direct Summand). If  $M$  is an  $R$ -module and  $M_1 \subseteq M$  is a submodule, we say that  $M_1$  is a **direct summand** of  $M$ , or is **complemented** in  $M$ , if there is a submodule  $M_2 \subseteq M$  such that  $M \cong M_1 \oplus M_2$ .

**Definition 1.22.** Consider the sequence of  $R$ -modules and  $R$ -module homomorphisms:

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$$

This sequence is said to be a **split exact sequence** (or simply **split**) if it is exact and if  $\text{Im}\psi = \text{Ker}\varphi$  is a direct summand of  $M$ .

**Lemma 1.10.** Let  $M$  and  $L$  be  $R$ -modules,  $N$  and  $D$  submodules of  $M$  and  $L$  respectively, and  $\psi : M \longrightarrow L$  an  $R$ -module homomorphism. Then we have:

1.  $\psi^{-1}(\psi(N)) = N + \ker \psi$
2.  $\psi(\psi^{-1}(D)) = D \cap \text{Im} \psi$

*Proof.* **Part (1):** We will show two inclusions.

$$(i) \quad \psi^{-1}(\psi(N)) \subseteq N + \ker \psi$$



$$(ii) \ N + \ker \psi \subseteq \psi^{-1}(\psi(N))$$

(i) Let  $x \in \psi^{-1}(\psi(N)) = \{x \in M : \psi(x) \in \psi(N)\}$ , i.e.,  $\psi(x) \in \psi(N)$ . This means there exists  $n \in N$  such that  $\psi(x) = \psi(n)$ . It follows that  $\psi(x - n) = \psi(x) - \psi(n) = 0$ , which implies  $x - n \in \ker \psi$ . Therefore,  $x = n + k$ , where  $k = x - n \in \ker \psi$ . Hence,  $x \in N + \ker \psi$ .

(ii) Now let  $x \in N + \ker \psi$ . This means  $x = n + k$ , where  $n \in N$  and  $k \in \ker \psi$ . Applying  $\psi$  to  $x$ , we get:

$$\psi(x) = \psi(n + k) = \psi(n) + \psi(k) = \psi(n) + 0 = \psi(n).$$

Since  $\psi(n) = \psi(x) \in \psi(N)$ , by the definition of  $\psi^{-1}(\psi(N)) = \{x \in M : \psi(x) \in \psi(N)\}$ , it follows that  $x \in \psi^{-1}(\psi(N))$ . Thus, by (i) and (ii), we conclude that  $\psi^{-1}(\psi(N)) = N + \ker \psi$ .

**Part (2):** We will again show two inclusions:

$$(i) \ \psi(\psi^{-1}(D)) \subseteq D \cap \text{Im } \psi$$

$$(ii) \ D \cap \text{Im } \psi \subseteq \psi(\psi^{-1}(D))$$

(i) Let  $y \in \psi(\psi^{-1}(D)) = \{y \in L \mid y = \psi(x) \text{ for some } x \in \psi^{-1}(D)\}$ . This means there exists some  $x \in \psi^{-1}(D)$  such that  $\psi(x) = y$ . Since  $x \in \psi^{-1}(D)$ , we have,  $\psi(x) \in D$ . Therefore,  $y = \psi(x) \in D$ . Additionally, since  $y = \psi(x)$ , it follows that  $y \in \text{Im } \psi$ . Hence,  $y \in D \cap \text{Im } \psi$ .

(ii) Now let  $y \in D \cap \text{Im } \psi$ . This implies  $y \in D$  and  $y \in \text{Im } \psi$ . Since  $y \in \text{Im } \psi$ , there exists some  $x \in M$  such that  $\psi(x) = y$ . Because  $y \in D$  and  $\psi(x) = y$ , it follows that  $x \in \psi^{-1}(D)$ . Applying  $\psi$  to  $x$  gives  $\psi(x) = y$ , and therefore,  $y = \psi(x)$  is in the image of  $\psi$  applied to  $\psi^{-1}(D)$ , which is exactly  $\psi(\psi^{-1}(D))$ .

By (i) and (ii), it follows that  $\psi(\psi^{-1}(D)) = D \cap \text{Im } \psi$ . Thus, part (2) is proven.  $\square$

**Lemma 1.11.** *Let  $X, Y$ , and  $Z$  be  $R$ -modules, and consider the sequence of homomorphisms  $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ . Then:*

$$1. \ker(\beta \circ \alpha) = \alpha^{-1}(\ker \beta)$$

$$2. \operatorname{Im}(\beta \circ \alpha) = \beta(\operatorname{Im} \alpha)$$

*Proof. Part (1)*

We establish the equality by proving two inclusions.

$$(i) \operatorname{Ker}(\beta \circ \alpha) \subseteq \alpha^{-1}(\operatorname{Ker} \beta)$$

$$(ii) \alpha^{-1}(\operatorname{Ker} \beta) \subseteq \operatorname{Ker}(\beta \circ \alpha)$$

(i) Let  $x \in \operatorname{Ker}(\beta \circ \alpha)$ . Then  $(\beta \circ \alpha)(x) = 0$ , which implies  $\beta(\alpha(x)) = 0$ . Consequently,  $\alpha(x) \in \operatorname{Ker} \beta$ . Since  $\alpha^{-1}(\operatorname{Ker} \beta) = \{x \in X : \alpha(x) \in \operatorname{Ker} \beta\}$ . Therefore,  $x \in \alpha^{-1}(\operatorname{Ker} \beta)$ .

(ii) Let  $x \in \alpha^{-1}(\operatorname{Ker} \beta)$ . Then  $\alpha(x) \in \operatorname{Ker} \beta$ , which implies  $\beta(\alpha(x)) = 0$ . Hence,  $(\beta \circ \alpha)(x) = 0$ , and it follows that  $x \in \operatorname{Ker}(\beta \circ \alpha)$ .

**Part (2)**

By definition,  $\operatorname{Im}(\beta \circ \alpha) = \{z \in Z : z = (\beta \circ \alpha)(x) \text{ for some } x \in X\}$ .

Thus,  $\operatorname{Im}(\beta \circ \alpha) = (\beta \circ \alpha)(X) = \beta(\alpha(X)) = \beta(\operatorname{Im} \alpha)$ .

Therefore,  $\operatorname{Im}(\beta \circ \alpha) = \beta(\operatorname{Im} \alpha)$ .

□

**Lemma 1.12.** *Let the diagram*

$$\begin{array}{ccc} & & D \\ & \nearrow f & \downarrow \lambda \\ N & \xrightarrow{g} & M \end{array}$$

*be commutative, i.e.,  $g \circ f = \lambda$ . Then*

$$1. \operatorname{Im} f + \ker g = g^{-1}(\operatorname{Im} \lambda)$$

$$2. \operatorname{Im} f \cap \ker g = f(\ker \lambda)$$

Furthermore, the following special cases hold:

(i) If  $\lambda$  is an epimorphism, then  $\operatorname{Im} f + \ker g = g^{-1}(M) = N$ .

(ii) If  $\lambda$  is a monomorphism, then  $\operatorname{Im} f \cap \ker g = f(0) = 0$ .

(iii) If  $\lambda$  is an isomorphism, then  $\operatorname{Im} f \oplus \ker g = N$ .

*Proof.* (1) Since  $\lambda = g \circ f$ , it follows that  $\operatorname{Im} \lambda = \operatorname{Im}(g \circ f) = g(\operatorname{Im} f)$ . Therefore,  $g^{-1}(\operatorname{Im} \lambda) = g^{-1}(g(\operatorname{Im} f))$ . Because  $\operatorname{Im} f \subseteq N$  is a submodule, Lemma 1.10 implies

$$g^{-1}(g(\operatorname{Im} f)) = \operatorname{Im} f + \ker g.$$

(2) We have  $\ker \lambda = \ker(g \circ f) = f^{-1}(\ker g)$  by Lemma 1.11. Applying  $f$  yields

$$f(\ker \lambda) = f(f^{-1}(\ker g)).$$

Since  $\ker g$  is a submodule of  $N$ , Lemma 1.10 implies

$$f(f^{-1}(\ker g)) = \ker g \cap \operatorname{Im} f.$$

□

**Proposition 1.24.** The following statements are equivalent for the exact sequence

$$0 \longrightarrow X \xrightarrow{f} M \xrightarrow{g} Y \longrightarrow 0$$

(1) The sequence splits.

(2) There exists a homomorphism  $\bar{f} : M \longrightarrow X$  such that  $\bar{f} \circ f = \operatorname{id}_X$ .

(3) There exists a homomorphism  $\bar{g} : Y \longrightarrow M$  such that  $g \circ \bar{g} = \operatorname{id}_Y$ .

*Proof.* (1)  $\implies$  (2): Suppose that the sequence splits. Then  $M \cong \text{Im } f \oplus M_1$  for some submodule  $M_1 \subseteq M$ .

Let  $\pi : M \longrightarrow \text{Im } f$  be the canonical projection defined by  $\pi(f(x) + m_1) := f(x)$ , where  $f(x) \in \text{Im } f$  and  $m_1 \in M_1$ .

Define the restricted function  $f_0 : X \longrightarrow \text{Im } f$  by  $f_0(x) = f(x)$  for all  $x \in X$ . Since  $f$  is injective,  $f_0$  is also injective. Moreover, by definition,  $f_0$  is surjective onto  $\text{Im } f$ . Therefore,  $f_0$  is an isomorphism and hence invertible.

Let  $f_0^{-1} : \text{Im } f \longrightarrow X$  be the inverse of  $f_0$ . Now, consider  $\bar{f} := f_0^{-1} \circ \pi$ . Then for any  $x \in X$ , we have

$$\bar{f}(f(x)) = f_0^{-1}(\pi(f(x))) = f_0^{-1}(f(x)) = x.$$

Thus,  $\bar{f} \circ f = \text{id}_X$ .

(2)  $\implies$  (3): Suppose there exists  $\bar{f} : M \longrightarrow X$  such that  $\bar{f} \circ f = \text{id}_X$ .

Observe that for any  $m \in M$ ,

$$\begin{aligned} \bar{f}(m - f(\bar{f}(m))) &= \bar{f}(m) - \bar{f}(f(\bar{f}(m))) \\ &= \bar{f}(m) - (\bar{f} \circ f) \circ \bar{f}(m) \\ &= \bar{f}(m) - (\text{id}_X \circ \bar{f})(m) \\ &= \bar{f}(m) - \bar{f}(m) \\ &= 0. \end{aligned}$$

Hence,  $m - f(\bar{f}(m)) \in \ker \bar{f}$ .

Notice that  $m$  can be expressed as

$$m = f(\bar{f}(m)) + (m - f(\bar{f}(m))),$$

where  $f(\bar{f}(m)) \in \text{Im } f$  and  $m - f(\bar{f}(m)) \in \ker \bar{f}$ .

Since every  $m \in M$  can be written in this form, we conclude that  $M = \text{Im } f + \ker \bar{f}$ .

Now, suppose  $f(y) = m \in \text{Im } f \cap \ker \bar{f}$ . Then  $0 = \bar{f}(m) = \bar{f}(f(y)) = y$ , so  $y = 0$ , and thus  $\text{Im } f \cap \ker \bar{f} = \{0\}$ . Therefore,  $M = \text{Im } f \oplus \ker \bar{f}$ .

Next, define a homomorphism  $\bar{g} : Y \longrightarrow M$  by  $\bar{g}(y) = m - f(\bar{f}(m))$ , where  $g(m) = y$ .

Since  $g$  is surjective, there exists an  $m \in M$  for each  $y \in Y$ . However,  $m$  may not be unique because multiple elements in  $M$  can map to the same  $y$  under  $g$ . Therefore, we must verify that  $\bar{g}$  is well-defined.

Suppose  $g(m) = y = g(m')$ . We must verify that  $\bar{g}(g(m)) = \bar{g}(g(m'))$ , i.e.,

$$m - f(\bar{f}(m)) = m' - f(\bar{f}(m')).$$

First, observe that

$$0 = g(m) - g(m') = g(m - m'),$$

so  $m - m' \in \ker g = \text{Im } f$ . Then  $m - m' = f(x)$  for some  $x \in X$ .

Now, consider the difference

$$\begin{aligned} & (m - f(\bar{f}(m))) - (m' - f(\bar{f}(m'))) \\ &= (m - m') - (f(\bar{f}(m)) - f(\bar{f}(m'))) \\ &= f(x) - f(\bar{f}(m) - \bar{f}(m')). \end{aligned}$$

So the difference is in  $\text{Im } f$ . Applying  $\bar{f}$  to the difference, we get

$$\begin{aligned}
 \bar{f}((m - f(\bar{f}(m))) - (m' - f(\bar{f}(m')))) &= \bar{f}(f(x) - f(\bar{f}(m) - \bar{f}(m'))) \\
 &= x - (\bar{f}(m) - \bar{f}(m')) \\
 &= x - \bar{f}(m - m') \\
 &= x - \bar{f}(f(x)) \\
 &= x - x \\
 &= 0.
 \end{aligned}$$

So, the difference lies in  $\ker \bar{f}$  as well and hence lies in the intersection. Since  $\text{Im } f \cap \ker \bar{f} = \{0\}$ , the difference must be zero. This means

$$(m - f(\bar{f}(m))) = (m' - f(\bar{f}(m'))),$$

and hence  $\bar{g}(y)$  is the same regardless of the choice of  $m$ . Therefore,  $\bar{g}$  is well-defined.

Finally, it is clear from the construction of  $\bar{g}$  that

$$g \circ \bar{g}(y) = g(m - f(\bar{f}(m))) = g(m) - g(f(\bar{f}(m))).$$

Since the exactness of the sequence implies  $\text{Im } f = \ker g$ , we have  $g(f(\bar{f}(m))) = 0$ . Therefore,

$$g \circ \bar{g}(y) = g(m) = y,$$

and hence  $g \circ \bar{g} = \text{id}_Y$ .

This shows that (2)  $\implies$  (3).

(3)  $\implies$  (1): Suppose there exists  $\bar{g} : Y \longrightarrow M$  such that  $g \circ \bar{g} = \text{id}_Y$ .

Consider the following commutative diagram:

$$\begin{array}{ccc}
 & & Y \\
 & \nearrow \bar{g} & \downarrow \text{id}_Y \\
 M & \xrightarrow{g} & Y
 \end{array}$$

Since  $\text{id}_Y$  is an isomorphism Lemma 1.12 implies, then  $\text{Im } \bar{g} \oplus \ker g = M$  and hence  $\ker g = \text{Im } f$  is a direct summand of  $M$ . Thus, the sequence splits.  $\square$

## 1.2 Projective Modules

**Definition 1.23.** A module  $P$  is called **projective** if, for any epimorphism  $\varphi : M \rightarrow N$  and any homomorphism  $g : P \rightarrow N$ , there exists a homomorphism  $h : P \rightarrow M$  such that  $\varphi \circ h = g$ .

In other words, any diagram of the form

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow g & & \\ M & \xrightarrow{\varphi} & N & \longrightarrow & 0 \end{array}$$

where the bottom row is exact, can be completed to the following commutative diagram:

$$\begin{array}{ccccc} & & P & & \\ & \swarrow h & \downarrow g & & \\ M & \xrightarrow{\varphi} & N & \longrightarrow & 0 \end{array}$$

**Proposition 1.25.** An  $R$ -module  $P$  is projective if and only if  $\text{Hom}_R(P, -)$  is an exact functor.

i.e., For any  $R$ -modules  $L, M$ , and  $N$ ,

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$$

is a short exact sequence if and only if

$$0 \longrightarrow \text{Hom}_R(P, L) \xrightarrow{\bar{\psi}} \text{Hom}_R(P, M) \xrightarrow{\bar{\varphi}} \text{Hom}_R(P, N) \longrightarrow 0$$

is a short exact sequence.

*Proof.* Let  $P$  be a projective  $R$ -module.



Suppose that

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$$

is an arbitrary exact sequence of  $R$ -modules.

We want to show that

$$0 \longrightarrow \text{Hom}_R(P, L) \xrightarrow{\bar{\psi}} \text{Hom}_R(P, M) \xrightarrow{\bar{\varphi}} \text{Hom}_R(P, N) \longrightarrow 0$$

is exact.

By proposition 1.23, we know that

$$0 \longrightarrow \text{Hom}_R(P, L) \xrightarrow{\bar{\psi}} \text{Hom}_R(P, M) \xrightarrow{\bar{\varphi}} \text{Hom}_R(P, N)$$

is exact for any module  $P$ . So, it remains to show that  $\bar{\varphi}$  is an epimorphism.

Let  $g \in \text{Hom}_R(P, N)$ . Since  $P$  is projective, there exists  $h : P \rightarrow M$  such that  $\varphi \circ h = g$ .

Then we have the following commutative diagram:

$$\begin{array}{ccccc} & & P & & \\ & \swarrow \exists h & \downarrow g & & \\ M & \xrightarrow{\varphi} & N & \longrightarrow & 0 \end{array}$$

This implies that  $\bar{\varphi}(h) = g$ , where  $h \in \text{Hom}_R(P, M)$ .

Therefore,  $\bar{\varphi}$  is an epimorphism, and thus the second sequence is exact.

**Conversely**, consider the diagram:

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow g & & \\ M & \xrightarrow{\varphi} & N & \longrightarrow & 0 \end{array}$$

and the exact sequence:

$$0 \longrightarrow \ker(\varphi) \xrightarrow{i} M \xrightarrow{\varphi} N \longrightarrow 0$$

This sequence is exact since  $i$  is a monomorphism and  $\varphi$  is an epimorphism.

Thus,

$$0 \longrightarrow \text{Hom}_R(P, \ker(\varphi)) \xrightarrow{\bar{i}} \text{Hom}_R(P, M) \xrightarrow{\bar{\varphi}} \text{Hom}_R(P, N) \longrightarrow 0$$

is exact as well. In particular,  $\bar{\varphi}$  is an epimorphism.

Since  $g \in \text{Hom}(P, N)$ ,  $\exists h \in \text{Hom}(P, M)$  such that  $\bar{\varphi}(h) = g$ , implying  $\varphi \circ h = g$ .

Therefore,  $P$  is projective. □

**Lemma 1.13** (Hazewinkel, Gubareni, and Kirichenko [13] (Proposition 4.2.2)). *(Universal Property of Direct Sums)*

Let  $\varphi_i : X_i \rightarrow Y$  be a set of homomorphisms of  $R$ -modules, where  $i \in I$ . Then, there exists a unique homomorphism  $\xi : \bigoplus_{i \in I} X_i \rightarrow Y$  such that the following diagrams commute for all  $i \in I$ :

$$\begin{array}{ccc} X_i & \xrightarrow{\varphi_i} & Y \\ \sigma_i \downarrow & \nearrow \xi & \\ \bigoplus_{i \in I} X_i & & \end{array}$$

**Proposition 1.26.** A direct sum  $P = \bigoplus_{j \in J} P_j$  of modules is a projective module if and only if each  $P_j$  is projective.

*Proof.* Suppose  $P = \bigoplus_{j \in J} P_j$  is projective, and let  $\pi_j : P \rightarrow P_j$  be a canonical projection homomorphism from  $P$  to  $P_j$ . Thus,  $g_j \circ \pi_j$  is a homomorphism from  $P$  to  $N$ .

Since  $P$  is projective, there exists a homomorphism  $h : P \rightarrow M$  such that  $\varphi \circ h = g_j \circ \pi_j$ .

$$\begin{array}{ccccc} & & P = \bigoplus_{j \in J} P_j & & \\ & \swarrow \exists h & \downarrow \pi_j & & \\ & & P_j & & \\ & & \downarrow g_j & & \\ M & \xrightarrow{\varphi} & N & \longrightarrow & 0 \end{array}$$

Thus, if  $i_j : P_j \rightarrow P$  is the canonical injection map, define  $f_j : P_j \rightarrow M$  by  $f_j := h \circ i_j$ .

We have

$$\varphi \circ f_j = \varphi \circ (h \circ i_j) = (\varphi \circ h) \circ i_j = (g_j \circ \pi_j) \circ i_j = g_j \circ (\pi_j \circ i_j) = g_j \circ \text{id}_{P_j} = g_j.$$

$$\begin{array}{ccccc} & & P_j & & \\ & \swarrow f_j & \downarrow g_j & & \\ M & \xrightarrow{\varphi} & N & \longrightarrow & 0 \end{array}$$

Hence,  $P_j$  is projective for all  $j \in J$ .

**Conversely**, Assume that each  $P_j$  is projective and consider a diagram

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow g & & \\ M & \xrightarrow{\varphi} & N & \longrightarrow & 0 \end{array}$$

with the bottom row exact. For each  $j \in J$ , define a homomorphism  $\zeta_j : P_j \rightarrow N$  by  $\zeta_j = g \circ i_j$ .

Since  $P_j$  is projective for all  $j \in I$ , there exist homomorphisms  $h_j : P_j \rightarrow M$ , such that  $\varphi \circ h_j = g \circ i_j$ .

$$\begin{array}{ccccc} & & P_j & & \\ & \swarrow \exists h_j & \downarrow i_j & & \\ & & P = \bigoplus_{j \in J} P_j & & \\ & & \downarrow g & & \\ M & \xrightarrow{\varphi} & N & \longrightarrow & 0 \end{array}$$

Next, by the Universal Property of Direct Sums, there exists a unique homomorphism  $\xi : P \rightarrow M$  such that  $\xi \circ i_j = h_j$ .

Applying  $\varphi$  to both sides, we obtain:

$$\varphi \circ (\xi \circ i_j) = \varphi \circ h_j = g \circ i_j.$$

This gives:

$$\varphi \circ \xi = g.$$

Thus,  $P$  is projective

□

**Corollary 1.5.** *Every direct summand of a projective module is projective.*

*Proof.* Let  $P$  be a projective module, and let  $A$  be a direct summand of  $P$ . Then, we can write  $P$  as a direct sum:

$$P = A \oplus B$$

for some module  $B$ .

We want to show that  $A$  is projective. To do so, consider a homomorphism  $f : A \rightarrow N$ , where  $N$  is some module. This situation can be represented in the following diagram:

$$\begin{array}{ccccc} & & A & & \\ & & \downarrow f & & \\ M & \xrightarrow{\varphi} & N & \longrightarrow & 0 \end{array}$$

Since  $P$  is projective, there exists a homomorphism  $h : P \rightarrow M$  such that  $\varphi \circ h = f \circ \pi_A$ , where  $\pi_A : P \rightarrow A$  is the canonical projection map from  $P$  to  $A$ . The diagram becomes:

$$\begin{array}{ccccc} & & P = A \oplus B & & \\ & & \downarrow \pi_A & & \\ & \swarrow \exists h & A & & \\ M & \xrightarrow{\varphi} & N & \longrightarrow & 0 \end{array}$$

Let  $i_A : A \rightarrow P$  be the canonical injective map, where  $i_A$  is the inclusion map into the

direct sum  $P = A \oplus B$ . Since  $\varphi \circ h = f \circ \pi_A$ , we define a homomorphism  $\lambda : A \rightarrow M$  by

$$\lambda := h \circ i_A.$$

It follows that  $\lambda$  is a homomorphism, and we can verify the equality:

$$\varphi \circ \lambda = \varphi \circ (h \circ i_A) = f \circ (\pi_A \circ i_A) = f \circ id_A = f.$$

Therefore, we have shown that  $A$  is projective.  $\square$

**Proposition 1.27.** Free modules are projective

*Proof.* Let  $F$  be a free module with basis  $\{f_i \in F : i \in I\}$ . Consider the following diagram:

$$\begin{array}{ccccc} & & F & & \\ & & \downarrow \xi & & \\ M & \xrightarrow{\varphi} & N & \longrightarrow & 0 \end{array}$$

Since any module is an epimorphic image of a free module, we have  $\xi(f_i) = n_i \in N$ . Now, as  $\varphi$  is an epimorphism, there exists  $m_i \in M$  such that  $\varphi(m_i) = n_i$ .

Define  $h : F \rightarrow M$  by  $h(\sum_{i \in I} f_i a_i) = \sum_{i \in I} m_i a_i$ .

The map  $h$  is well-defined since every  $f \in F$  can be uniquely expressed in terms of the basis elements. Moreover,  $h$  is a homomorphism. Thus, we obtain the following diagram:

$$\begin{array}{ccccc} & & F & & \\ & \swarrow \exists h & \downarrow \xi & & \\ M & \xrightarrow{\varphi} & N & \longrightarrow & 0 \end{array}$$

Next, observe that:

$$\varphi \circ h \left( \sum_{i \in I} f_i a_i \right) = \varphi \left( \sum_{i \in I} m_i a_i \right) = \sum_{i \in I} \varphi(m_i) a_i = \sum_{i \in I} n_i a_i = \sum_{i \in I} \xi(f_i) a_i = \xi \left( \sum_{i \in I} f_i a_i \right) = \xi(f)$$

Therefore,  $\varphi \circ h = \xi$ , and thus  $F$  is projective.  $\square$

**Proposition 1.28.** [Test For Projectivity] Let  $P$  and  $M$  be modules over a ring  $R$ . The following are equivalent:

1.  $P$  is a projective module.
2. For every epimorphism  $\pi : M \rightarrow M/N$ , where  $N$  is a submodule of  $M$ , and for every homomorphism  $f : P \rightarrow M/N$ , there exists a homomorphism  $\lambda : P \rightarrow M$  such that  $\pi \circ \lambda = f$ .

This can be visualized by the following commutative diagram:

$$\begin{array}{ccccc} & & P & & \\ & \swarrow \exists \lambda & \downarrow f & & \\ M & \xrightarrow{\pi} & M/N & \longrightarrow & 0 \end{array}$$

*Proof.* ( $1 \Rightarrow 2$ ) Assume  $P$  is projective. By definition, for any epimorphism  $\varphi : M \rightarrow L$  and homomorphism  $h : P \rightarrow L$ , there exists a lifting  $\lambda : P \rightarrow M$  such that  $\varphi \circ \lambda = h$ . This situation can be visualized by the following diagram:

$$\begin{array}{ccc} & P & \\ & \downarrow h & \\ M & \xrightarrow{\varphi} & L \end{array}$$

Now, let  $L = M/N$ . The natural projection  $\pi : M \rightarrow M/N$  is an epimorphism. For any  $f : P \rightarrow M/N$ , the definition of projectivity guarantees the existence of  $\lambda : P \rightarrow M$  such that  $\pi \circ \lambda = f$ . The situation is represented by the following diagram:

$$\begin{array}{ccccc} & & P & & \\ & \swarrow \exists \lambda & \downarrow f & & \\ M & \xrightarrow{\pi} & M/N & \longrightarrow & 0 \end{array}$$

This shows that (2) follows directly from (1) as a special case.

( $2 \Rightarrow 1$ )

Suppose the following diagram is commutative:

$$\begin{array}{ccccc}
 & & P & & \\
 & \swarrow \exists \lambda & \downarrow f & & \\
 M & \xrightarrow{\pi} & M/N & \longrightarrow & 0
 \end{array}$$

i.e.,  $\lambda \circ \pi = f$ .

Now consider the following diagram, where the bottom row is exact:

$$\begin{array}{ccccc}
 & & P & & \\
 & & \downarrow g & & \\
 M & \xrightarrow{\varphi} & L & \longrightarrow & 0
 \end{array}$$

To show that  $P$  is projective, we need to find a map  $h$  such that  $\varphi \circ h = g$ .

Since  $\varphi$  is an epimorphism, the first isomorphism theorem implies that  $L \cong M/\ker(\varphi)$ .

Let  $\xi : L \longrightarrow M/\ker(\varphi)$  be the corresponding isomorphism. Let  $\sigma : M \longrightarrow M/\ker(\varphi)$  be the homomorphism given by  $\sigma = \xi \circ \varphi$ . Thus, the diagram becomes:

$$\begin{array}{ccccc}
 & & P & & \\
 & & \downarrow g & & \\
 M & \xrightarrow{\varphi} & L & \longrightarrow & 0 \\
 & \searrow \sigma & \downarrow \xi & & \\
 & & M/\ker(\varphi) & & 
 \end{array}$$

Next, consider the situation represented by the following diagram:

$$\begin{array}{ccccc}
 & & P & & \\
 & \swarrow \exists \bar{h} & \downarrow \xi \circ g & & \\
 M & \xrightarrow{\sigma} & M/\ker(\varphi) & \longrightarrow & 0
 \end{array}$$

By hypothesis, there exists a homomorphism  $\bar{h} : P \longrightarrow M$  such that  $\sigma \circ \bar{h} = \xi \circ g$ .

That is,

$$(\xi \circ \varphi) \circ \bar{h} = \xi \circ g$$

Applying  $\xi^{-1}$  on both sides:

$$\xi^{-1} \circ (\xi \circ \varphi) \circ \bar{h} = \xi^{-1} \circ (\xi \circ g)$$

$$\Rightarrow \varphi \circ \bar{h} = g$$

$$\begin{array}{ccccc} & & P & & \\ & \swarrow \exists \bar{h} & \downarrow g & & \\ M & \xrightarrow{\varphi} & L & \longrightarrow & 0 \end{array}$$

Hence,  $P$  is projective.

□

Now, we present an alternative proof of the theorem: *Every free module is projective.*

Suppose  $F$  is a free  $R$ -module, so  $F \cong \bigoplus_{i \in I} R$  for some index set  $I$ .

Our goal is to show that  $F$  is projective. First, we establish that  $R$ , as a module over itself, is projective. Then, applying the fact that a direct sum of projective modules is projective, together with the test for projectivity, we conclude that  $F$ , being a direct sum of projective modules, is projective.

**Proposition 1.29.** Let  $R$  be a ring. Then  $R$ , considered as a **right**  $R$ -module (denoted  $R_R$ ), is projective.

*Proof.* To show that the ring  $R$  is projective as a right  $R$ -module, we utilize Proposition 1.28. Let  $\pi : M \rightarrow M/N$  be a surjective right  $R$ -module homomorphism and let  $f : R \rightarrow M/N$  be an arbitrary right  $R$ -module homomorphism. We seek a right  $R$ -module homomorphism  $\lambda : R \rightarrow M$  such that  $\pi \circ \lambda = f$ . This requirement is illustrated by the following diagram, where the dashed arrow represents the homomorphism  $\lambda$  to be constructed:

$$\begin{array}{ccccc} & & R & & \\ & \swarrow \exists \lambda & \downarrow f & & \\ M & \xrightarrow{\pi} & M/N & \longrightarrow & 0 \end{array}$$

Consider the element  $f(1) \in M/N$ , where  $1$  is the multiplicative identity in  $R$ . As  $\pi$



is surjective, there exists an element  $m \in M$  such that  $\pi(m) = f(1)$ . We define a map  $\lambda : R \rightarrow M$  by  $\lambda(r) = m \cdot r$  for all  $r \in R$ . We verify that  $\lambda$  is a right  $R$ -module homomorphism. For any  $r_1, r_2, s \in R$ :

- Additivity:  $\lambda(r_1 + r_2) = m(r_1 + r_2) = mr_1 + mr_2 = \lambda(r_1) + \lambda(r_2)$ , using the right distributive property in  $M$ .
- Compatibility with scalar multiplication:  $\lambda(r_1 s) = m(r_1 s) = (mr_1)s = \lambda(r_1)s$ , using the associativity property in  $M$ .

Thus,  $\lambda$  is indeed a right  $R$ -module homomorphism.

Next, we check if the diagram commutes, i.e., if  $\pi \circ \lambda = f$ . For any  $r \in R$ :

$$(\pi \circ \lambda)(r) = \pi(\lambda(r)) = \pi(m \cdot r)$$

Since  $\pi$  is a right  $R$ -module homomorphism,  $\pi(m \cdot r) = \pi(m) \cdot r$ . By our choice of  $m$ ,  $\pi(m) = f(1)$ , so  $\pi(m) \cdot r = f(1) \cdot r$ . Furthermore, since  $f$  is a right  $R$ -module homomorphism,  $f(r) = f(1 \cdot r) = f(1) \cdot r$ . Combining these, we get:

$$(\pi \circ \lambda)(r) = \pi(m) \cdot r = f(1) \cdot r = f(r)$$

This equality holds for all  $r \in R$ , confirming that  $\pi \circ \lambda = f$ .

We have successfully constructed the required lifting homomorphism  $\lambda$ , making the diagram commute:

$$\begin{array}{ccccc} & & R & & \\ & \swarrow \lambda & \downarrow f & & \\ M & \xrightarrow{\pi} & M/N & \longrightarrow & 0 \end{array}$$

Therefore, by Proposition 1.28,  $R$ , considered as a right  $R$ -module, is projective.  $\square$

**Corollary 1.6.** *Every free right  $R$ -module is projective.*

*Proof.* Let  $F$  be an arbitrary free right  $R$ -module. By definition,  $F$  is isomorphic to a direct sum of copies of  $R$ , i.e.,  $F \cong \bigoplus_{i \in I} R_i$ , where each  $R_i$  is isomorphic to  $R_R$ .

Since  $R_R$  is projective, by Proposition 1.26, a direct sum of projective modules is projective. Therefore, since each summand  $R_i$  is projective, it follows that  $F$  is projective as well.

This completes the proof.  $\square$

**Proposition 1.30.** Let  $R$  be a ring. For an  $R$ -module  $P$ , the following are equivalent:

1.  $P$  is projective.
2. Every short exact sequence  $0 \rightarrow N \rightarrow M \xrightarrow{\pi} P \rightarrow 0$  splits.
3.  $P$  is a direct summand of a free  $R$ -module.

*Proof.* **(1  $\Rightarrow$  2)** Suppose  $P$  is a projective  $R$ -module and consider the diagram

$$0 \longrightarrow N \longrightarrow M \xrightarrow{\pi} P \longrightarrow 0$$

where  $\pi$  is an epimorphism. Let  $\text{id}_P : P \rightarrow P$  be the identity map on  $P$ . Since  $P$  is projective, there exists  $h : P \rightarrow M$  such that  $\pi \circ h = \text{id}_P$ .

We can then visualize the situation with the following diagram:

$$\begin{array}{ccccccc} & & & & P & & \\ & & & \swarrow \exists h & \downarrow \text{id}_P & & \\ 0 & \longrightarrow & N & \longrightarrow & M & \xrightarrow{\pi} & P \longrightarrow 0 \end{array}$$

So then the sequence splits.

**(2  $\Rightarrow$  3)** Suppose every short exact sequence splits. Since any module is an epimorphic image of a free module, there exists a free module  $F$  and an epimorphism  $\pi : F \rightarrow P$ .

So we get the short exact sequence

$$0 \longrightarrow \ker(\pi) \xrightarrow{i} F \xrightarrow{\pi} P \longrightarrow 0$$

Since the sequence splits, we conclude that  $F = \ker(\pi) \oplus P$ .

Hence, Corollary 1 implies that  $F$  is projective.

**(3  $\Rightarrow$  1)** Suppose  $P$  is a direct summand of a free module, so that

$$F = P \oplus A, \quad \text{where } F \text{ is free.}$$

Since every free module is projective, we have

$$\Rightarrow F \text{ is projective.}$$

And since a direct summand of a projective module is projective, we conclude

$$\Rightarrow P \text{ is projective.}$$

□

**Theorem 1.2.** *For a ring  $R$ , the following statements are equivalent:*

1.  $R$  is a semisimple ring.
2. Any  $R$ -module  $M$  is projective.

*Proof.* **(1)  $\Rightarrow$  (2):**

Assume  $R$  is a semisimple ring. Then the right regular module  $R_R$  decomposes into a direct sum of simple submodules, which means that  $R_R$  is a semisimple module. Any free  $R$ -module  $F$  is a direct sum of copies of  $R_R$ , and since  $R_R$  is semisimple,  $F$  is also a semisimple module.

Let  $M$  be any  $R$ -module. By Corollary 1.1,  $M$  is isomorphic to a quotient of some free  $R$ -module  $F$ , i.e.,  $M \cong F/K$ , where  $K$  is a submodule of  $F$ .

Since  $F$  is semisimple, any submodule  $K$  of  $F$  is a direct summand, so we can write

$F = K \oplus N$  for some submodule  $N$  of  $F$ . Thus, we have:

$$M \cong F/K = \frac{K \oplus N}{K} \cong N.$$

By Proposition 1.30, a module is projective if and only if it is a direct summand of a free module. Since  $N$  is a direct summand of  $F$ , it follows that  $N$  is projective, and consequently  $M$  is projective.

(2)  $\implies$  (1):

Suppose that any  $R$ -module is projective. Let  $M$  be an arbitrary  $R$ -module and let  $N \subseteq M$  be a submodule of  $M$ . By hypothesis, since every  $R$ -module is projective, the quotient module  $M/N$  is projective.

Consider the short exact sequence:

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0.$$

Since  $M/N$  is projective, this sequence splits (by Proposition 1.30), meaning that  $N$  is a direct summand of  $M$ .

Because  $N \hookrightarrow M$  was arbitrary, every submodule of  $M$  is a direct summand. By definition of semisimplicity, this implies that  $M$  is semisimple.

Since  $M$  was arbitrary, we conclude that every right  $R$ -module is semisimple, including the regular module  $R_R$ . Therefore,  $R$  is a semisimple ring.  $\square$

The following theorem demonstrates a remarkable property of projective modules: any nonzero projective module, whether finitely generated or not, always possesses a maximal submodule.

**Theorem 1.3.** *Let  $P$  be a nonzero projective right  $R$ -module. Then, the radical of  $P$ , denoted by  $\text{rad } P$ , satisfies*

$$\text{rad } P = P \cdot J(R) \subsetneq P,$$

where  $J(R)$  represents the Jacobson radical of the ring  $R$ .

*Proof.* Let  $P$  be a nonzero projective right  $R$ -module.

By Proposition 1.30, we can take a suitable projective  $R$ -module  $Q$  such that:

$$F = P \oplus Q$$

where  $F$  is a free  $R$ -module. Since  $F$  is free, we have:

$$F = \bigoplus_{i \in I} f_i R$$

where  $\{f_i\}_{i \in I}$  is a basis for  $F$ .

Since  $F = P \oplus Q$ , Proposition 1.18 (Part 2) implies:

$$\text{rad } F = \text{rad } P \oplus \text{rad } Q$$

Moreover, Part (3) of Proposition 1.18 implies:

$$\text{rad } F = FJ(R)$$

Thus, we have:

$$\text{rad } F = FJ(R) = PJ(R) \oplus QJ(R)$$

On the other hand, multiplying  $F = P \oplus Q$  by  $J(R)$  gives:

$$FJ(R) = PJ(R) \oplus QJ(R)$$

and, by transitivity, the equality:

$$\text{rad } P \oplus \text{rad } Q = PJ(R) \oplus QJ(R)$$

holds. Therefore, we conclude that:

$$\text{rad } P = PJ(R)$$

which establishes the first part of the theorem.

Next, we show that  $PJ(R)$  is a proper subset of  $P$ , i.e.,  $PJ(R) \neq P$ . Assume, for the sake of contradiction, that:

$$PJ(R) = P$$

Since  $F$  is a free right  $R$ -module with basis  $\{f_i\}_{i \in I}$ , any element  $p \in P = PJ(R) \subseteq F$  can be expressed as a finite linear combination:

$$p = \sum_{i=1}^n f_i r_i$$

where  $r_i \in R$ .

Consider the canonical projection:

$$\pi : F = P \oplus Q \rightarrow P$$

which takes a pair  $(p, q)$  and returns  $p$  for  $p \in P$  and  $q \in Q$ .

Applying  $\pi$  to a basis element  $f_i$ , we must have  $\pi(f_i) \in P$ , since  $P$  is a submodule of  $F$ . Therefore,  $\pi(f_i)$  can be expressed as a finite linear combination of the basis elements

$\{f_i\}_{i \in I}$ , but with coefficients from  $J(R)$  instead of  $R$ , because  $P = PJ(R)$ . Hence, we can write:

$$\pi(f_i) = \sum_{j=1}^m f_j a_{ij}$$

where  $a_{ij} \in J(R)$  and  $m \geq n$ . The inequality  $m \geq n$  arises because  $\pi(f_j)$  might involve more basis elements than the original  $n$  elements in  $p = \sum_{i=1}^n f_i r_i$ .

Since we are dealing with coefficients from  $J(R) \subseteq R$ , we have:

$$p = \pi(p) = \pi\left(\sum_{i=1}^n f_i r_i\right) = \sum_{i=1}^n \pi(f_i) r_i$$

Substituting  $\pi(f_i) = \sum_{j=1}^m f_j a_{ij}$ , we get:

$$p = \sum_{i=1}^n \left( \sum_{j=1}^m f_j a_{ij} \right) r_i$$

Since the sums are finite, the order of summation can be interchanged:

$$p = \sum_{j=1}^m f_j \left( \sum_{i=1}^n a_{ij} r_i \right)$$

Next, comparing the two expressions for  $p$ :

$$1. p = \sum_{j=1}^n f_j r_j \quad 2. p = \sum_{j=1}^m f_j \left( \sum_{i=1}^n a_{ij} r_i \right)$$

We observe that, for all  $j \in \{1, 2, \dots, n\}$ , the coefficients must be equal due to the basis property. Thus, we have:

$$r_j = \sum_{i=1}^n a_{ji} r_i$$

Rearranging this, we get:

$$r_j - \sum_{i=1}^n a_{ji}r_i = 0$$

Using the Kronecker delta  $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ , we can write:

$$r_j = \sum_{i=1}^n \delta_{ji}r_i$$

This works because:

- When  $i = j$ ,  $\delta_{ij} = 1$ , so  $\delta_{ij}r_i = r_j$  - When  $i \neq j$ ,  $\delta_{ij} = 0$ , so  $\delta_{ij}r_i = 0$

Thus, the sum  $\sum_{i=1}^n$  "picks out" the term  $r_j$  and ignores all other terms.

Substituting  $r_j = \sum_{i=1}^n \delta_{ij}r_i$  into  $r_j - \sum_{i=1}^n a_{ij}r_i = 0$ , we get:

$$\sum_{i=1}^n \delta_{ij}r_i - \sum_{i=1}^n a_{ij}r_i = 0$$

or equivalently:

$$\sum_{i=1}^n (\delta_{ij} - a_{ij})r_i = 0, \quad \text{for } j = 1, \dots, n$$

This system of equations can be written in matrix form as:

$$(I_n - A)r = 0$$

where:

-  $I_n$  is the  $n \times n$  identity matrix, -  $A = (a_{ij})$  is the  $n \times n$  matrix with entries  $a_{ij} \in J(R)$ ,



-  $r = (r_1, \dots, r_n)^T$  is the column vector of coefficients, -  $0$  is the zero vector.

Since  $A$  has entries in  $J(R)$ , the matrix  $I_n - A$  is invertible, as  $1 - j$  is invertible for any  $j \in J(R)$ . Therefore, the only solution to the system  $(I_n - A)r = 0$  is:

$$r = 0$$

i.e.,  $r_1 = r_2 = \dots = r_n = 0$ .

Recall that  $p = \sum_{i=1}^n f_i r_i$ . If  $r_1 = r_2 = \dots = r_n = 0$ , then:

$$p = 0$$

This contradicts the assumption that  $p$  is a nonzero element of  $P$ .

Thus, the assumption  $PJ(R) = P$  must be false, and we conclude that:

$$PJ(R) \subsetneq P$$

This completes the proof of the theorem.

□

**Definition 1.24.** Let  $R$  be a ring and  $M$  a right  $R$ -module. The **annihilator** of  $M$  in  $R$ , denoted by  $\text{Ann}_R(M)$ , is defined as:

$$\text{Ann}_R(M) = \{r \in R \mid mr = 0 \text{ for all } m \in M\}.$$

It is a (two-sided) ideal of  $R$  consisting of all elements of  $R$  that act as zero on every element of  $M$ .

**Lemma 1.14.** Every cyclic right  $R$ -module  $mR$  is isomorphic to  $R/\text{Ann}_R(M)$ .

*Proof.* Define a map  $f : R \longrightarrow mR$  by  $f(r) = mr$ . This is a well-defined  $R$ -module

homomorphism. It is surjective because  $mR$  is generated by  $m$ .

The kernel of  $f$  is:

$$\ker f = \{r \in R \mid mr = 0\} = \{r \in R \mid r \in \text{Ann}_R(M)\} = \text{Ann}_R(M).$$

By the First Isomorphism Theorem, we have:

$$R / \text{Ann}_R(M) \cong mR = M.$$

□

**Lemma 1.15.** *If  $M = mR$  is a cyclic projective right  $R$ -module, then  $M$  is isomorphic to a direct summand of  $R$ .*

*Proof.* Since  $M = mR$  is cyclic, by Lemma 1.14, we have:

$$M \cong R/I \quad \text{where } I = \text{Ann}_R(M).$$

Since  $M$  is projective, the short exact sequence:

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

splits. Hence,

$$R \cong I \oplus (R/I),$$

which implies that  $M \cong R/I$  is isomorphic to a direct summand of  $R$ . □

**Example 1.13.** The ring of integers  $\mathbb{Z}$  is a projective  $\mathbb{Z}$ -module. To see this, observe that  $\mathbb{Z}$  is a free  $\mathbb{Z}$ -module with  $\{1\}$  as a basis. Every element  $z \in \mathbb{Z}$  can be written uniquely as  $z = n \cdot 1$  for some  $n \in \mathbb{Z}$ , showing that  $\{1\}$  spans  $\mathbb{Z}$ . Additionally,  $\{1\}$  is linearly independent, as the equation  $n \cdot 1 = 0$  implies  $n = 0$ . Since  $\mathbb{Z}$  is free and every free module is projective, it follows that  $\mathbb{Z}$  is projective.

**Example 1.14.** Every free module is projective, but the converse is not true: not every projective module is free. To illustrate this, consider the ring  $R = \mathbb{Z}_6$ . The ideals  $\mathbb{Z}_3$

and  $\mathbb{Z}_2$  can be viewed as  $\mathbb{Z}_6$ -modules. By the Chinese Remainder Theorem, there is a  $\mathbb{Z}_6$ -module isomorphism:

$$\mathbb{Z}_6 \cong \mathbb{Z}_3 \oplus \mathbb{Z}_2.$$

This decomposition shows that  $\mathbb{Z}_6$  splits into a direct sum of  $\mathbb{Z}_3$  and  $\mathbb{Z}_2$ . Since  $\mathbb{Z}_6$  is a free  $\mathbb{Z}_6$ -module (with basis  $\{1\}$ ), and direct summands of free modules are projective Proposition 1.30, it follows that both  $\mathbb{Z}_3$  and  $\mathbb{Z}_2$  are projective  $\mathbb{Z}_6$ -modules. However, neither  $\mathbb{Z}_3$  nor  $\mathbb{Z}_2$  is a free  $\mathbb{Z}_6$ -module. This can be shown in two ways:

### **Solution 1: Linear Independence Argument**

A free  $\mathbb{Z}_6$ -module must have a basis consisting of linearly independent elements. However, in  $\mathbb{Z}_3$ , every element  $x$  satisfies  $3x = 0$ , and in  $\mathbb{Z}_2$ , every element  $y$  satisfies  $2y = 0$ . These relations violate linear independence, as they imply that no nonempty subset of  $\mathbb{Z}_3$  or  $\mathbb{Z}_2$  can be linearly independent over  $\mathbb{Z}_6$ . Therefore,  $\mathbb{Z}_3$  and  $\mathbb{Z}_2$  cannot have a basis, and thus they are not free  $\mathbb{Z}_6$ -modules.

### **Alternative Solution: Cardinality Argument**

Suppose, for contradiction, that  $\mathbb{Z}_2$  is a free  $\mathbb{Z}_6$ -module. If  $\mathbb{Z}_2$  were free, it would have to be isomorphic to a direct sum of copies of  $\mathbb{Z}_6$ , since  $\mathbb{Z}_6$  is the free  $\mathbb{Z}_6$ -module of rank 1. That is, there would exist an isomorphism:

$$\mathbb{Z}_2 \cong \mathbb{Z}_6^{(I)},$$

where  $\mathbb{Z}_6^{(I)}$  denotes the direct sum of  $|I|$  copies of  $\mathbb{Z}_6$ , and  $I$  is some index set. However, this is impossible because:

- The group  $\mathbb{Z}_2$  has exactly 2 elements.
- The group  $\mathbb{Z}_6$  has exactly 6 elements, and thus any direct sum of copies of  $\mathbb{Z}_6$  must have a cardinality divisible by 6 (or be infinite).

Since 2 is not divisible by 6,  $\mathbb{Z}_2$  cannot be isomorphic to  $\mathbb{Z}_6^{(I)}$  for any index set  $I$ . Therefore,  $\mathbb{Z}_2$  cannot be a free  $\mathbb{Z}_6$ -module. A similar argument applies to  $\mathbb{Z}_3$ , showing that it is also not free.

**Example 1.15.** Let  $R$  be a ring and  $e \in R$  an idempotent element ( $e^2 = e$ ). Then the right ideal  $eR$  is projective.

*Proof.* By the Right Peirce Decomposition Theorem,  $R = eR \oplus (1 - e)R$  shows  $eR$  is a direct summand of the free module  $R_R$ . Direct summands of free modules are projective.  $\square$

**Remark 1.9.** Proposition 1.30 shows that (1) direct summands of free modules are projective, but (2) they are not necessarily free. For example,  $\mathbb{Z}_3$  and  $\mathbb{Z}_2$  as  $\mathbb{Z}_6$ -modules satisfy  $\mathbb{Z}_6 \cong \mathbb{Z}_3 \oplus \mathbb{Z}_2$ , so both are projective but neither is free.

# Chapter 2

## Special Rings and Projective Covers

Building upon the foundational concepts of projective modules from Chapter 1, this chapter shifts focus to specific classes of rings whose structure is intimately related to projectivity. We introduce local and semilocal rings, exploring their defining properties, particularly concerning their Jacobson radicals and quotient structures. A key result, Theorem 2.1, demonstrates that finitely generated projective modules over local rings possess the strong property of being free. We then introduce the important concepts of semiperfect and perfect rings, laying the groundwork for their characterization in the next chapter. Central to this discussion is the notion of a projective cover, which is defined and examined, preparing for its crucial role in the main theorems of this research.

### 2.1 Local Rings & Semilocal Rings

#### 2.1.1 Local Rings

**Definition 2.1** (Local Ring). A nonzero ring  $R$  is called local if it has a unique maximal right ideal.

**Proposition 2.1** (Hazewinkel, Gubareni, and Kirichenko [13](Proposition 10.1.1)). TFAE for a ring  $R$  with Jacobson radical  $J(R)$

1.  $R$  is local.

2.  $J(R)$  is the unique maximal right ideal of  $R$ .
3. All non-invertible elements of  $R$  form a proper ideal.
4.  $J(R)$  is the set of all non-invertible elements of  $R$ .
5. The quotient  $R/J(R)$  is a division ring.

**Theorem 2.1.** *If  $R$  is a local ring, then each finitely generated projective  $R$ -module is free.*

*Proof.* Let  $R$  be a local ring with Jacobson Radical  $J(R)$ , and let  $P$  be a finitely generated projective  $R$ -module.

As  $R$  is a local ring, the quotient ring  $R/J(R)$  is a division ring. Since  $P$  is a finitely generated  $R$ -module, the quotient module  $P/PJ(R)$  is finitely generated over  $R$  by the images of the generators of  $P$ . Furthermore, because  $J(R)$  annihilates  $P/PJ(R)$ ,  $P/PJ(R)$  can be viewed as a finitely generated module over the division ring  $R/J(R)$ . Any finitely generated module over a division ring is a finite-dimensional vector space and thus is free; therefore,  $P/PJ(R)$  is a finitely generated free  $R/J(R)$ -module.

Thus,  $P/PJ(R) \cong \bigoplus_{i=1}^n (R/J(R))$ .

Let  $F$  be the free  $R$ -module  $F = \bigoplus_{i=1}^n R$ .

Moreover, we have

$$FJ(R) \cong \bigoplus_{i=1}^n RJ(R) = \underbrace{(R \oplus \cdots \oplus R)}_{n \text{ times}} J(R),$$

which gives

$$FJ(R) \cong \underbrace{J(R) \oplus \cdots \oplus J(R)}_{n \text{ times}} = \bigoplus_{i=1}^n J(R).$$

Thus, the quotient

$$F/FJ(R) \cong \left( \bigoplus_{i=1}^n R / \bigoplus_{i=1}^n J(R) \right) = \bigoplus_{i=1}^n (R/J(R)).$$

Since  $P/PJ(R) \cong F/FJ(R)$ , let

$$\psi : F/FJ(R) \longrightarrow P/PJ(R)$$

be the corresponding isomorphism of  $R/J(R)$ -modules, and let

$$\pi : F \longrightarrow F/FJ(R) \quad \text{and} \quad \sigma : P \longrightarrow P/PJ(R)$$

be the natural projections. Then,  $\alpha = \psi \circ \pi$  is a homomorphism from  $F$  to  $P/PJ(R)$ .

Since  $F$  is a free module, and thus a projective module, there exists a homomorphism

$$\varphi : F \longrightarrow P$$

such that the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & FJ(R) & \longrightarrow & F & \xrightarrow{\pi} & F/FJ(R) \longrightarrow 0 \\ & & & & \downarrow \exists \varphi & \searrow \alpha & \downarrow \psi \\ 0 & \longrightarrow & PJ(R) & \longrightarrow & P & \xrightarrow{\sigma} & P/PJ(R) \longrightarrow 0 \end{array}$$

is commutative, i.e.,  $\sigma \circ \varphi = \alpha = \psi \circ \pi$ .

We shall show that  $\varphi$  is an isomorphism.

### 1. Surjectivity of $\varphi$

For  $f \in F$ ,

$$\psi \circ \pi(f) = \psi(\pi(f)) = \psi(f + FJ(R)) \quad \text{and} \quad \sigma \circ \varphi(f) = \sigma(\varphi(f)) = \varphi(f) + PJ(R).$$

By commutativity of the diagram,

$$\psi(f + FJ(R)) = \varphi(f) + PJ(R).$$

Since  $\psi$  is surjective,

$$\text{Im}(\varphi) + PJ(R) = P.$$

Since  $P$  is finitely generated and  $PJ(R) \subset P$ , Nakayama's lemma gives

$$\text{Im}(\varphi) = P.$$

Thus,  $\varphi$  is surjective.

## 2. Injectivity (Monomorphism)

Consider the exact sequence:

$$0 \longrightarrow \text{Ker}(\varphi) \longrightarrow F \xrightarrow{\varphi} P \longrightarrow 0$$

Since  $P$  is projective, the exact sequence splits, i.e., there exists  $\varphi' : P \longrightarrow F$  such that  $\varphi \circ \varphi' = 1_P$ .

Thus, we have:

$$F = W \oplus X \cong \text{Ker}(\varphi) \oplus P.$$

It follows that:

$$FJ(R) = (W \oplus X)J(R) = WJ(R) \oplus XJ(R).$$

As  $W \cong \text{Ker}(\varphi)$  is a submodule of  $F$ , let  $f \in W$ . Then,  $\varphi(f) = 0$ , and by the natural projection  $\sigma$ , we have  $\sigma(\varphi(f)) = 0 = \psi(\pi(f))$ . Since  $\psi$  is an isomorphism, this implies  $\pi(f) = 0 = f + FJ(R)$ , or equivalently,  $f \in FJ(R)$ . Hence,  $W \subseteq FJ(R)$ .

Since  $W \cong \text{Ker}(\varphi) \subseteq FJ(R)$  and  $FJ(R) = WJ(R) \oplus XJ(R)$ , we have

$$W = W \cap FJ(R) = [W \cap WJ(R)] \oplus [W \cap XJ(R)] = WJ(R) \oplus [W \cap XJ(R)].$$

Thus,  $W$  decomposes as a direct sum. Note that  $WJ(R) \subseteq W$  ensures the validity of this decomposition.

$$W = WJ(R) \oplus [W \cap XJ(R)].$$



However, since  $W \cap XJ(R) \subseteq W \cap X = 0$ ,

$$W = WJ(R).$$

Since  $W$  is a direct summand of a finitely generated module, it follows that  $W$  is finitely generated. By Nakayama's Lemma, we conclude

$$W = 0.$$

Therefore,  $\varphi$  is a monomorphism.

Hence,  $\varphi$  is an isomorphism, and  $P$  is free. □

**Remark 2.1.** Kaplansky [17] This theorem still holds in a more general setting. Namely, I. Kaplansky proved that all projective modules over a local ring are free.

### 2.1.2 Semilocal Rings

**Definition 2.2** (Semilocal). A ring  $R$  is called **semilocal** if the quotient ring  $\bar{R} = R/J(R)$ , is a right Artinian ring, or equivalently if  $\bar{R}$  is a semisimple ring.

**Proposition 2.2.** An epimorphic image of a semilocal  $R$ -module  $M$  is semilocal.

*Proof.* Let  $N$  be an epimorphic image of a semilocal  $R$ -module  $M$ , i.e., there exists an epimorphism  $\phi : M \rightarrow N$ . Set  $K = \ker \phi$ . Then, by the First Isomorphism Theorem, we have  $M/K \cong N$ , so  $N$  is essentially a quotient module of  $M$ .

Since  $M$  is semilocal, the module  $M/\text{rad}M$  is semisimple. Observe that

$$\text{rad}M \subseteq K + \text{rad}M \subseteq M,$$

so there is a natural epimorphism

$$\phi : M/\text{rad}M \rightarrow M/(K + \text{rad}M).$$

By Lemma 1.3, an epimorphic image of a semisimple module is semisimple. Therefore,  $M/(K + \text{rad}M)$  is semisimple.

Moreover, since  $K \subseteq K + \text{rad}M$ , the Third Isomorphism Theorem implies

$$\frac{M/K}{(K + \text{rad}M)/K} \cong \frac{M}{K + \text{rad}M},$$

and hence

$$\frac{M/K}{(K + \text{rad}M)/K} \text{ is semisimple.}$$

By Proposition 1.7, we have

$$(K + \text{rad}M)/K \subseteq \text{rad}(M/K).$$

This inclusion gives rise to an epimorphism

$$\xi : \frac{M/K}{(K + \text{rad}M)/K} \rightarrow \frac{M/K}{\text{rad}(M/K)}.$$

Again, by Lemma 1.3, the module  $\frac{M/K}{\text{rad}(M/K)}$  is semisimple. Therefore,  $M/K$  is semilocal.

Since  $N \cong M/K$ , it follows that  $N$  is semilocal, as required.  $\square$

**Proposition 2.3.** The following are equivalent for a ring  $R$ :

1.  $R$  is semilocal.
2. Every right  $R$ -module with  $\text{rad} M = 0$  is semisimple.
3. Every right  $R$ -module is semilocal.

*Proof.* (1)  $\implies$  (2)

Suppose that  $R$  is semilocal, and let  $M$  be a right  $R$ -module with  $\text{rad} M = 0$ . By Proposition 1.2,  $M$  is an epimorphic image of a free module  $F$ . So there exists an

epimorphism  $\varphi : F \rightarrow M$ . Since  $F$  is free, we have

$$F/\text{rad } F \cong \bigoplus_{i \in I} R/\text{rad}(\bigoplus_{i \in I} R),$$

or equivalently, Proposition 1.18 gives us that

$$F/\text{rad } F \cong \bigoplus_{i \in I} (R/\text{rad } R).$$

$R/\text{rad } R$  is semisimple (because  $R$  is semilocal), so  $\bigoplus_{i \in I} R/\text{rad } R$  is also semisimple, thus  $F/\text{rad } F$  is semisimple. Hence  $F$  is semilocal.

Since Proposition 2.2 dictates that every epimorphic image of a semilocal module is semilocal, it follows that  $M$  is semilocal and hence  $M/\text{rad } M$  is semisimple. Since  $\text{rad } M = 0$ ,  $M$  itself is semisimple.

**(2)  $\implies$  (3)**

Assume that any right  $R$ -module  $X$  with  $\text{rad } X = 0$  is semisimple. Let  $M$  be an arbitrary right  $R$ -module. Then  $M/\text{rad } M$  satisfies  $\text{rad}(M/\text{rad } M) = 0$  (*Proposition 1.10*). By hypothesis,  $M/\text{rad } M$  is semisimple. Therefore,  $M$  is semilocal.

**(3)  $\implies$  (1)**

Assume that every right  $R$ -module is semilocal. Then, in particular,  $R_R$  is semilocal.  $\square$

## 2.2 Projective Covers

**Definition 2.3** (Projective Cover). A projective module  $P$  is called a *projective cover* of a module  $M$  and it is denoted by  $P(M)$  if there is an epimorphism  $\varphi : P \rightarrow M$  such that  $\text{Ker}(\varphi)$  is a small submodule in  $P$ .

**Example 2.1.** Let  $P_R$  be a finitely generated projective  $R$ -module, and let  $A \subseteq J(R)$  be a right ideal. Then the natural projection  $\pi : P \rightarrow P/PA$  is a projective cover of  $P/PA$ .

*Proof.* Since  $P$  is projective by assumption, we only need to verify that  $\ker \pi = PA$  is small in  $P$ . The surjectivity of  $\pi$  is immediate from its definition. By Example 1.6, since  $P$  is finitely generated and  $A \subseteq J(R)$ , the submodule  $PA$  must be small in  $P$ . Therefore,  $(P, \pi)$  is indeed a projective cover.  $\square$

**Example 2.2.** Let  $eR$  be a right ideal where  $e$  is an idempotent and  $A \subseteq J(R)$ . Consider the natural projection  $\pi : eR \rightarrow eR/eA$ .

Then  $eR/eA$  has  $eR$  as its projective cover.

*Proof.* First, note that  $eR$  is a regular right  $R$ -module. By Example 1.15, this implies that  $eR$  is a projective right  $R$ -module.

The map  $\pi : eR \rightarrow eR/eA$  is an epimorphism with kernel  $eA$ . Since  $eR$  is cyclic (and hence finitely generated), it follows from Example 2.1 that  $(\pi, eR)$  is indeed a projective cover of  $eR/eA$ .  $\square$

**Lemma 2.1.** Let  $\sigma_i : P_i \rightarrow M_i$  be  $R$ -module homomorphisms for  $1 \leq i \leq n$ . Define the direct sum map

$$\bigoplus \sigma_i : \bigoplus_{i=1}^n P_i \rightarrow \bigoplus_{i=1}^n M_i$$

by

$$\left( \bigoplus \sigma_i \right) (p_1, \dots, p_n) = (\sigma_1(p_1), \dots, \sigma_n(p_n)).$$

Then,

$$\ker \left( \bigoplus \sigma_i \right) = \bigoplus_{i=1}^n \ker(\sigma_i).$$

*Proof.* Let  $P = \bigoplus_{i=1}^n P_i$  and  $M = \bigoplus_{i=1}^n M_i$ , and let  $\sigma = \bigoplus \sigma_i: P \rightarrow M$  be defined as above.

We compute the kernel of  $\sigma$  directly:

$$\begin{aligned} \ker(\sigma) &= \{(p_1, \dots, p_n) \in P \mid \sigma(p_1, \dots, p_n) = 0\} \\ &= \left\{ (p_1, \dots, p_n) \in \bigoplus_{i=1}^n P_i \mid (\sigma_1(p_1), \dots, \sigma_n(p_n)) = (0, \dots, 0) \right\} \\ &= \left\{ (p_1, \dots, p_n) \in \bigoplus_{i=1}^n P_i \mid \sigma_i(p_i) = 0 \text{ for all } i \right\} \\ &= \left\{ (p_1, \dots, p_n) \in \bigoplus_{i=1}^n P_i \mid p_i \in \ker(\sigma_i) \text{ for all } i \right\} \\ &= \bigoplus_{i=1}^n \ker(\sigma_i). \end{aligned}$$

Hence,

$$\ker \left( \bigoplus \sigma_i \right) = \bigoplus_{i=1}^n \ker(\sigma_i).$$

□

**Lemma 2.2.** Let  $\sigma_i: P_i \rightarrow M_i$  be  $R$ -module homomorphisms for  $1 \leq i \leq n$ . Define the direct sum map

$$\bigoplus \sigma_i: \bigoplus_{i=1}^n P_i \rightarrow \bigoplus_{i=1}^n M_i$$

by

$$\left( \bigoplus \sigma_i \right) (p_1, \dots, p_n) = (\sigma_1(p_1), \dots, \sigma_n(p_n)).$$

Then, the map  $\bigoplus \sigma_i$  is surjective if and only if each  $\sigma_i: P_i \rightarrow M_i$  is surjective.

*Proof.* Let  $P = \bigoplus_{i=1}^n P_i$  and  $M = \bigoplus_{i=1}^n M_i$ , and let  $\sigma = \bigoplus \sigma_i: P \rightarrow M$  be defined as above.

We will show that  $\bigoplus \sigma_i$  is surjective if and only if each  $\sigma_i$  is surjective.

**(1)  $\Rightarrow$  Direction:** If  $\bigoplus \sigma_i$  is surjective, then each  $\sigma_i$  is surjective.

Suppose that  $\bigoplus \sigma_i: P \rightarrow M$  is surjective. This means that for every element  $(m_1, \dots, m_n) \in$

$M = \bigoplus_{i=1}^n M_i$ , there exists a tuple  $(p_1, \dots, p_n) \in P = \bigoplus_{i=1}^n P_i$  such that:

$$\bigoplus \sigma_i(p_1, \dots, p_n) = (\sigma_1(p_1), \dots, \sigma_n(p_n)) = (m_1, \dots, m_n).$$

For each  $i$ , this implies that  $\sigma_i(p_i) = m_i$ . Since this holds for any choice of  $m_i \in M_i$ , it follows that  $\sigma_i$  is surjective for each  $i$ .

**(2)  $\Leftarrow$  Direction: If each  $\sigma_i$  is surjective, then  $\bigoplus \sigma_i$  is surjective.**

Suppose that each  $\sigma_i: P_i \rightarrow M_i$  is surjective. This means that for every  $m_i \in M_i$ , there exists a  $p_i \in P_i$  such that  $\sigma_i(p_i) = m_i$ .

Now, consider an arbitrary element  $(m_1, \dots, m_n) \in \bigoplus_{i=1}^n M_i$ . Since each  $\sigma_i$  is surjective, for each  $m_i \in M_i$ , there exists a  $p_i \in P_i$  such that  $\sigma_i(p_i) = m_i$ . Thus, the tuple  $(p_1, p_2, \dots, p_n) \in \bigoplus_{i=1}^n P_i$  satisfies:

$$\bigoplus \sigma_i(p_1, p_2, \dots, p_n) = (\sigma_1(p_1), \sigma_2(p_2), \dots, \sigma_n(p_n)) = (m_1, m_2, \dots, m_n).$$

Therefore,  $\bigoplus \sigma_i$  is surjective. □

**Proposition 2.4.** Let  $\sigma_i: P_i \rightarrow M_i$  be projective covers for  $1 \leq i \leq n$ . Then  $\bigoplus_{i=1}^n \sigma_i: \bigoplus_{i=1}^n P_i \rightarrow \bigoplus_{i=1}^n M_i$  is also a projective cover.

*Proof.* Given for each  $i$ ,  $\sigma_i: P_i \rightarrow M_i$  is a projective cover, so  $\ker \sigma_i \ll P_i$ .

Since each  $P_i$  is projective, Proposition 1.18 implies that  $\bigoplus_{i=1}^n P_i$  is also projective.

Since each  $\sigma_i: P_i \rightarrow M_i$  is surjective, Lemma 2.3 implies  $\bigoplus_{i=1}^n \sigma_i: \bigoplus_{i=1}^n P_i \rightarrow \bigoplus_{i=1}^n M_i$  is also surjective.

Lemma 2.2 gives  $\ker (\bigoplus_{i=1}^n \sigma_i) = \bigoplus_{i=1}^n \ker \sigma_i$ .

By Example 1.10, since  $\ker \sigma_i \ll P_i$  for all  $i$ , we have  $\bigoplus_{i=1}^n \ker \sigma_i \ll \bigoplus_{i=1}^n P_i$ .

Thus  $(\bigoplus_{i=1}^n \sigma_i, \bigoplus_{i=1}^n P_i)$  is a projective cover of  $\bigoplus_{i=1}^n M_i$ . □

**Lemma 2.3.** *Let  $P \xrightarrow{\varphi} M \rightarrow 0$  be a projective cover of an  $R$ -module  $M$ . Then*

$$\text{Ker } \varphi \subseteq \text{rad} P.$$

*Proof.* By definition,  $\text{Ker } \varphi$  is a small submodule of  $P$ . By Proposition 2.6, the radical  $\text{rad} P$  is the sum of all small submodules of  $P$ . Since  $\text{Ker } \varphi$  is a small submodule, it follows that

$$\text{Ker } \varphi \subseteq \text{rad} P.$$

□

**Corollary 2.1.** *If  $P \xrightarrow{\varphi} U \rightarrow 0$  is a projective cover of a simple module  $U$ , then  $\text{Ker } \varphi = \text{rad} P$ . Consequently, the projective cover of a simple module has exactly one maximal submodule.*

*Proof.* Let  $P \xrightarrow{\varphi} U \rightarrow 0$  be a projective cover of the simple module  $U$ . By Lemma 2.3, we know that  $\text{Ker } \varphi \subseteq \text{rad} P$ .

To prove the reverse inclusion, we proceed as follows. Since  $\varphi : P \rightarrow U$  is an epimorphism, the First Isomorphism Theorem implies:

$$P / \text{Ker } \varphi \cong U.$$

Since  $U$  is simple, it follows that  $\text{Ker } \varphi$  is a maximal submodule of  $P$ . By definition,  $\text{rad} P$  is the intersection of all maximal submodules of  $P$ , and we deduce that:

$$\text{rad} P \subseteq \text{Ker } \varphi.$$

Thus, combining both inclusions, we conclude that:

$$\text{Ker } \varphi = \text{rad} P.$$

□

**Lemma 2.4** (Anderson and Fuller [3] (Proposition 27.4)). *Let  $R$  be a ring and let  $A$  be an ideal of  $R$  with  $A \subseteq J(R)$ . Then the following are equivalent:*

1. *Idempotents lift modulo  $A$ ;*
2. *Every direct summand of the right  $R$ -module  $R/A$  has a projective cover.*

**Lemma 2.5** (Bass Lemma). *Let  $U \rightarrow P$  with  $P$  projective  $R$ -module and let  $P/U$  have a projective cover. Then there is a decomposition  $P = P_1 \oplus P_2$  with  $P_1 \subseteq U$  and  $U \cap P_2 \ll P$ .*

*In particular, if  $A$  is a right ideal of a ring  $R$  and  $R/A$  has a projective cover, then  $R = P_1 \oplus P_2$  with  $P_1 \subseteq A$  and  $A \cap P_2 \ll P$ .*

*Proof.* Let  $f : N \rightarrow P/U$  be a projective cover of  $P/U$ , where  $U$  is a submodule of  $P$ . This means that  $N$  is projective and  $f$  is an epimorphism with a superfluous kernel. Let  $\eta : P \rightarrow P/U$  be the natural projection. Consider the following diagram:

$$\begin{array}{ccc} & & P \\ & \nearrow \exists \alpha & \downarrow \eta \\ N & \xrightarrow{f} & P/U \end{array}$$

Since  $P$  is projective, there exists a homomorphism  $\alpha : P \rightarrow N$  such that  $f \circ \alpha = \eta$ . By Lemma 1.12, and due to the commutativity of the diagram and the surjectivity of  $\eta$ , we obtain the relation

$$\text{Im} \alpha + \ker f = N.$$

Since  $\ker f \ll N$ , it follows that  $\text{Im} \alpha = N$ , so  $\alpha$  is an epimorphism. Consider the short exact sequence:

$$0 \longrightarrow \ker \alpha \xrightarrow{i} P \xrightarrow{\alpha} N \longrightarrow 0$$

From the projectivity of  $N$ , Proposition 1.30 implies that this sequence splits, i.e.,  $P = \ker \alpha \oplus T$  for some submodule  $T \subseteq P$ .

Now,  $\ker \alpha \subseteq \ker \eta$ . Indeed, since  $\eta = f \circ \alpha$ , if  $x \in \ker \alpha$ , then  $\eta(x) = f(\alpha(x)) = f(0) = 0$ , so  $x \in \ker \eta$ . Therefore,  $\ker \alpha \subseteq \ker \eta = U$ .



Next, we need to prove that  $U \cap T \ll P$ . Let  $\lambda = \alpha|_T : T \rightarrow N$  be the restriction of  $\alpha : P \rightarrow N$  to the submodule  $T \subseteq P$ . Since  $P = \ker \alpha \oplus T$ , the intersection  $T \cap \ker \alpha$  is trivial, which ensures that  $\lambda$  is injective. For surjectivity, every  $n \in N$  lifts to some  $p = k + t \in P$ , where  $k \in \ker \alpha$  and  $t \in T$ . We have

$$\alpha(p) = \alpha(k + t) = \alpha(k) + \alpha(t) = \alpha(t) = \lambda(t) = n.$$

Thus,  $\lambda$  is both injective and surjective, so it is an isomorphism. Therefore,  $T \cong N$ .

Since  $\ker f \ll N$ , Lemma 1.11 implies  $\ker(f \circ \lambda) = \lambda^{-1}(\ker f)$ . Since  $\lambda$  is an isomorphism, and Proposition 1.16 tells us that isomorphisms preserve smallness, it follows that  $\lambda^{-1}(\ker f) \ll T$ .

We also have  $\ker(f \circ \lambda) = \{x \in T : f(\alpha(x)) = 0\}$  and  $\ker(\eta|_T) = \{x \in T : \eta(x) = 0\}$ , which implies  $\ker(f \circ \lambda) = \ker(\eta|_T)$ . Since  $\eta|_T : T \rightarrow P/U$  is given by  $t \mapsto t + U$ , we have

$$\ker(\eta|_T) = \{t \in T : t \in U\} = T \cap U.$$

Thus,  $\ker(f \circ \lambda) = T \cap U$ .

To conclude, we have  $T \cap U = \lambda^{-1}(\ker f) \ll T$ , and since  $T \subseteq P$ , it follows by Example 1.8 that  $T \cap U \ll P$ . The last statement regarding rings is clear.  $\square$

## 2.3 Perfect & Semiperfect Rings

### 2.3.1 Semiperfect Rings

**Definition 2.4** (Semiperfect ring). A ring  $R$  is called semiperfect if  $R$  is semilocal, and idempotents of  $R/J(R)$  can be lifted to  $R$ .

### 2.3.2 Perfect Rings

**Definition 2.5** (Right Perfect Ring). A ring  $R$  with Jacobson radical  $J = J(R)$  is called **right perfect** if its quotient ring  $R/J$  is semisimple and the Jacobson radical  $J$  is right T-nilpotent.

### 2.3.3 Related Concepts and Properties

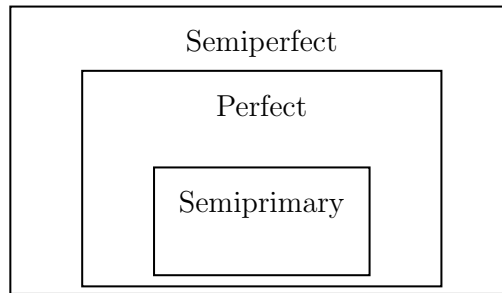


Figure 2.1: Relationship between Semiperfect, Perfect, and Semiprimary Rings.

Figure 2.1 illustrates the inclusion relationships between Semiperfect, Perfect, and Semiprimary rings. While all three classes require the quotient ring  $R/J(R)$  to be semisimple, they differ in the conditions placed upon the Jacobson radical  $J(R)$ :

- Semiperfect: Requires idempotents to lift modulo  $J(R)$ .
- Perfect: Requires  $J(R)$  to be right T-nilpotent.
- Semiprimary: Requires  $J(R)$  to be nilpotent.

The hierarchy **Semiprimary**  $\subset$  **Perfect**  $\subset$  **Semiperfect**, depicted in the figure, arises directly from these conditions:

1. Since a nilpotent  $J(R)$  is necessarily right T-nilpotent, every Semiprimary ring is also Perfect.
2. Every Perfect ring is also semiperfect. To see why every Perfect ring is semiperfect, recall that a ring is Perfect if  $J(R)$  is right T-nilpotent (and  $R/J(R)$  is semisimple).

A standard result is that any right T-nilpotent ideal is also a nil-ideal. Specifically, if  $J(R)$  is right T-nilpotent, consider any element  $a \in J(R)$  and the sequence  $a, a, a, \dots$ . By definition of right T-nilpotence, there must exist some integer  $k$  such that the product  $a \cdot a \dots a$  ( $k$  times) equals 0. This product is precisely  $a^k$ , showing that every element  $a \in J(R)$  is nilpotent. Furthermore, *Proposition 1.13* states that if  $J(R)$  is a nil-ideal, then idempotents lift modulo  $J(R)$ . Therefore, the condition that  $J(R)$  is right T-nilpotent (for Perfect rings) implies that  $J(R)$  is a nil-ideal, which in turn implies that idempotents lift modulo  $J(R)$ . Thus, every Perfect ring is semiperfect.

Therefore, Semiperfect rings form the most general class among the three shown.

**Definition 2.6** (Semiprimary ring). A ring  $R$  with Jacobson radical  $J(R)$  is said to be semiprimary if it possesses the following two conditions:

1.  $J(R)$  is nilpotent.
2.  $R/J(R)$  is semisimple.

**Definition 2.7** (Semiprimitive). A ring  $R$  is called semiprimitive if its Jacobson radical is equal to zero.

**Corollary 2.2.** *If  $R$  is a semiprimitive ring and  $P(U)$  is a projective cover of a simple  $R$ -module  $U$ , then  $P(U) \simeq U$ .*

*Proof.* Let  $P(U) \xrightarrow{\varphi} U \longrightarrow 0$  be a projective cover of a simple  $R$ -module  $U$ . Since  $R$  is semiprimitive, the Jacobson radical of  $R$ , denoted  $J(R)$ , satisfies  $J(R) = 0$ . By Theorem 1.3, the radical of  $P(U)$ , denoted  $\text{rad } P(U)$ , is given by:

$$\text{rad } P(U) = P(U) \cdot J(R).$$

Substituting  $J(R) = 0$ , we obtain:

$$\text{rad } P(U) = P(U) \cdot 0 = 0.$$

Thus,  $\text{rad } P(U) = 0$ .

From Lemma 2.3, we know that  $\text{Ker } \varphi \subseteq \text{rad } P(U)$ . Since  $\text{rad } P(U) = 0$ , it follows that:

$$\text{Ker } \varphi \subseteq 0.$$

This implies  $\text{Ker } \varphi = 0$ .

Since  $\varphi : P(U) \rightarrow U$  is surjective (by definition of a projective cover) and  $\text{Ker } \varphi = 0$ , the map  $\varphi$  is also injective. Therefore,  $\varphi$  is an isomorphism, and we conclude:

$$P(U) \cong U.$$

□

**Lemma 2.6.** *A semiprimitive ring in which every cyclic right module has a projective cover is semisimple.*

*Proof.* Let  $R$  be a semiprimitive ring, i.e.,  $J(R) = 0$  and suppose every cyclic right  $R$ -module has a projective cover.

We shall show that  $R$  is the sum of all its minimal right ideals. Let  $S = \text{Soc}(R_R)$  be the sum of all minimal right ideals of the ring  $R$ .

Assume for the sake of contradiction, that  $S \neq R$ . Then  $S$  is contained in a maximal right ideal  $I$  of the ring  $R$ . We claim that the module  $R/I = U$  is simple.

Suppose  $M$  is a submodule of  $R/I$ . Then  $M$  corresponds to a right ideal  $K$  of  $R$  such that  $I \subseteq K \subseteq R$ . Since  $I$  is maximal, we have:

- If  $K = I$ , then  $M = 0$ .
- If  $K = R$ , then  $M = R/I$ .

Hence  $R/I = U$  is a simple module, proving the claim.

Next, we claim that  $U = R/I$  is cyclic since it is generated by the coset  $1+I$ . Specifically, every element in  $U$  can be written as  $(1+I)r$  for some  $r \in R$ .

The hypothesis implies that  $U$  has a projective cover  $P(U)$ . Moreover Corollary 2.2 implies  $P(U) \cong U$ .

Consider the short exact sequence

$$0 \longrightarrow I \xrightarrow{i} R \xrightarrow{\pi} U \longrightarrow 0$$

Since  $U \cong P(U)$  is projective, this exact sequence splits by Proposition 1.30 which means  $R \cong I \oplus U$ .

Since  $R \cong I \oplus U$  we have  $I \cap U = 0$  and since  $S \subseteq I$  (by maximality of  $I$ ) we obtain  $U \not\subseteq S$ . However  $U$  is a simple module, and since  $S = \text{Soc}(R_R)$  contains all minimal right ideals (and thus all simple submodules).

This means  $U$  should be contained in  $S$ . Contradicting  $U \not\subseteq S$ . Thus our assumption that  $S \neq R$  must be incorrect and hence  $R = \text{Soc}(R_R)$  is a sum of simple modules (minimal ideals) which implies  $R$  is a semisimple ring, completing the proof.

□

# Chapter 3

## Projective Covers and Ring Characterizations

This chapter presents the culmination of our investigation with the main characterizations of semiperfect and perfect rings. We leverage the concepts of projective modules, projective covers, and projectivity developed in earlier chapters. The key theorems demonstrate fundamental equivalences based on the existence of projective covers for specific types of modules, highlighting their power in classifying ring structures.

**Definition 3.1.** A right  $R$ -module  $M$  is called  $R$ -projective (projective relative to  $R$ ) if, for any epimorphism  $g : R \rightarrow K$  and any homomorphism  $f : M \rightarrow K$ , there exists a homomorphism  $h : M \rightarrow R$  such that  $f = h \circ g$ . In other words, the following diagram commutes:

$$\begin{array}{ccccc} & & M & & \\ & \swarrow \exists h & \downarrow f & & \\ R & \xrightarrow{g} & K & \longrightarrow & 0 \end{array}$$

**Remark 3.1.** A right  $R$ -module  $M$  is said to be *projective* if it is projective relative to every  $R$ -module.

**Definition 3.2.** An  $R$ -projective cover of a right  $R$ -module  $M$  is an  $R$ -projective module  $P$  which serves as a projective cover to  $M$ . That is,

1.  $P$  is an  $R$ -projective module.
2. There exists an epimorphism  $\xi : P \rightarrow M$  such that  $\text{Ker } \xi \ll P$ .

### 3.1 Characterization of Semiperfect Rings

**Proposition 3.1** (Clark et al. [11] (4.2(3), p26)). If  $M$  is finitely generated  $R$ -projective (projective relative to  $R$ ), then  $M$  is projective.

**Corollary 3.1.** *Let  $R$  be a local ring. Then every finitely generated  $R$ -projective right  $R$ -module is free.*

*Proof.* Let  $R$  be a local ring and let  $M$  be a finitely generated  $R$ -projective right  $R$ -module. Since  $M$  is finitely generated and  $R$ -projective, it follows from Proposition 3.1 that  $M$  is a projective right  $R$ -module. Given that  $R$  is a local ring and  $M$  is a finitely generated projective  $R$ -module, Theorem 2.1 states that such a module must be free. Therefore, we conclude that  $M$  is a free  $R$ -module.  $\square$

**Theorem 3.1.** *Let  $R$  be a ring and  $J := J(R)$ , then the following statements are equivalent for a ring  $R$ :*

1.  $R$  is semiperfect.
2. Every finitely generated right  $R$ -module has a projective cover.
3. Every cyclic right  $R$ -module has a projective cover.
4. Every simple right  $R$ -module has a projective cover.
5. Every simple right  $R$ -module has an  $R$ -projective (projective relative to  $R$ ) cover.

*Proof.* (1)  $\implies$  (2)

Let  $R$  be a semiperfect ring and  $M$  a finitely generated right  $R$ -module. By Proposition 1.1, the quotient module  $M/\text{rad } M$  is also a finitely generated right  $R$ -module.

Since  $R$  is semilocal, we conclude that  $M$  is semilocal as well, by Proposition 2.3. Hence,  $M/\text{rad } M$  is semisimple.

By the semisimplicity and finite generation of  $M/\text{rad } M$ , we apply Proposition 1.3, which allows us to express  $M/\text{rad } M$  as a finite direct sum of simple modules:

$$M/\text{rad } M = \bigoplus_{i=1}^n K_i$$

where each  $K_i$  is a simple right  $R$ -module.

Let  $J(R)$  be the Jacobson radical of  $R$ . Proposition 1.9 ensures that  $J(R)$  annihilates simple modules. Thus, the action of  $R$  on each  $K_i$  factors through  $R/J(R)$ , and we can regard each  $K_i$  as a simple right  $R/J(R)$ -module. Since  $R/J(R)$  is semisimple, Theorem 1.2 guarantees that each  $K_i$  is a projective right  $R/J(R)$ -module.

Furthermore, as each  $K_i$  is simple, it must be cyclic. Since  $K_i$  is also projective, by Lemma 1.15, we conclude that  $K_i$  is isomorphic to a direct summand of  $R/J(R)$ .

Since  $\overline{R} = R/J(R)$  is semisimple, every right ideal in  $\overline{R}$  is a direct summand. By Proposition 1.11, every ideal of  $\overline{R}$  is of the form  $g_i \overline{R}$ , where  $g_i$  is an idempotent in  $\overline{R}$ . Therefore,  $K_i \cong g_i \overline{R}$ . Since idempotents lift modulo  $J(R)$ , there exists an idempotent  $e_i \in R$  such that  $\overline{e_i} = g_i$ .

Next, consider the map  $\varphi_i : e_i R \rightarrow \overline{e_i} \overline{R}$  defined by  $\varphi_i(e_i r) = \overline{e_i} \overline{r}$ . This map is an epimorphism with kernel:

$$\begin{aligned} \ker \varphi_i &= \{e_i r \in e_i R \mid \overline{e_i} \overline{r} = 0_{\overline{R}}\} \\ &= \{e_i r \in e_i R \mid e_i r + J(R) = J(R)\} \\ &= \{e_i r \in e_i R \mid e_i r \in J(R)\} \\ &= e_i R \cap J(R) \end{aligned}$$

which is the portion of  $J(R)$  lying in  $e_i R$ , specifically  $e_i J(R)$ . By the First Isomorphism Theorem, we obtain:

$$e_i R / e_i J(R) \cong \overline{e_i} \overline{R} = g_i \overline{R} \cong K_i$$



By Example 2.2,  $\psi_i : e_i R \rightarrow e_i R / e_i J(R) \cong K_i$  is a projective cover for  $e_i R / e_i J(R)$  for each  $i$ . Moreover, by Proposition 2.4, the map:

$$\psi = \bigoplus_{i=1}^n \psi_i : P = \bigoplus_{i=1}^n e_i R \rightarrow \bigoplus_{i=1}^n e_i R / e_i J(R) \cong \bigoplus_{i=1}^n K_i = M / \text{rad } M$$

is a projective cover for  $M / \text{rad } M$ .

Consider the following commutative diagram:

$$\begin{array}{ccc} & & P \\ & \swarrow \exists \alpha & \downarrow \psi \\ M & \xrightarrow{\beta} & M / \text{rad } M \end{array}$$

where  $\beta$  is the natural projection. Since  $P$  is projective by Proposition 1.26, there exists a homomorphism  $\alpha : M \rightarrow P$  such that:

$$\beta \circ \alpha = \psi$$

Since  $\psi$  is an epimorphism, it follows by Lemma 1.12 that:

$$\text{Im } \alpha + \ker \beta = M$$

and since  $\ker \beta = \text{rad } M$ , we have:

$$\text{Im } \alpha + \text{rad } M = M$$

Because  $M$  is finitely generated, by Lemma 1.7,  $\text{rad } M \ll M$ , so  $\text{Im } \alpha = M$ . Hence,  $\alpha$  is an epimorphism. Moreover, we observe that  $\psi \circ \alpha = \beta$ . Suppose  $p \in \ker \alpha$ . Then, by definition,  $\alpha(p) = 0$ , and consequently  $\psi(p) = \beta(\alpha(p)) = \beta(0) = 0$ . Thus, we have  $\ker \alpha \subseteq \ker \psi$ .

Since  $\ker \alpha \subseteq \ker \psi \ll P$ , by Example 1.9, we conclude that  $\ker \alpha \ll P$ . Therefore,  $(P, \alpha)$  is a projective cover of  $M$ .

(2)  $\implies$  (3)  $\implies$  (4)  $\implies$  (5) clear

(5)  $\implies$  (4)

Assume that every simple right  $R$ -module has an  $R$ -projective cover. Let  $f : P \longrightarrow K$  be an  $R$ -projective cover, where  $P$  is an  $R$ -projective module,  $f$  is an epimorphism, and  $\ker f \ll P$ , with  $K$  being a simple right  $R$ -module. By Corollary 2.1, it follows that  $\ker f = \text{rad } P$ .

Thus,  $\text{rad } P \ll P$ . Since  $K$  is simple, it is cyclic and therefore finitely generated. Consequently,  $P/\text{rad } P = P/\ker f \cong K$  is also finitely generated. By Lemma 1.7, we conclude that  $P$  is finitely generated as well. Furthermore, since  $P$  is a finitely generated  $R$ -projective module, Proposition 3.1 implies that  $P$  is projective.

(4)  $\implies$  (1)

Suppose that every simple right  $R$ -module has a projective cover. First, we will show that  $R$  is semilocal, meaning that  $R/J$  is semisimple, where  $J = J(R)$  is the Jacobson radical of  $R$ .

Define  $\overline{R} = R/J$ , and let  $\overline{C} = C/J$  be a maximal right ideal of  $\overline{R}$ . The quotient ring  $\overline{R}/\overline{C}$  is simple. Since  $\overline{R}/\overline{C} = \frac{R/J}{C/J}$ , where  $J \subseteq C \subseteq R$ , the Third Isomorphism Theorem implies that  $\overline{R}/\overline{C} \cong R/C$ . Therefore,  $R/C$  is a simple  $R$ -module.

By hypothesis,  $R/C$  has a projective cover. Thus, by Lemma 2.5, it follows that:

$$R = T \oplus B$$

where  $T \subseteq C$  and  $C \cap B \ll R$ .

Since  $R = T + B$  and  $T \subseteq C$ , we can express  $R$  as:

$$R = C + B.$$

Since  $C \cap B \ll R$  and Proposition 1.17 implies that  $C \cap B \subseteq J$ , Lemma 1.8 further

implies that:

$$R/J = \frac{C+B}{J} \cong \frac{C}{J} \oplus \frac{B+J}{J}.$$

Thus,  $\overline{C} = C/J$  is a direct summand of  $\overline{R}$ .

Since  $\overline{C}$  was an arbitrary maximal ideal of  $R/J$ , it follows that every maximal right ideal of  $R/J$  is a direct summand of  $R/J$ . Hence, by Proposition 1.6, we conclude that  $R/J$  is semisimple.

Next, we show that idempotents lift modulo  $J$ . Let  $A/J$  be a direct summand of  $R/J$ . Since  $R_R$  is cyclic (generated by 1) and hence finitely generated, it follows from Proposition 1.1 that the quotient  $R/J$  is also finitely generated. Moreover, the semisimplicity of  $R/J$  implies the semisimplicity of the submodule  $A/J$ . Hence, by Proposition 1.3, we have:

$$A/J \cong \bigoplus_{i=1}^n K_i,$$

where each  $K_i$  is a simple  $R$ -module.

By hypothesis, each  $K_i$  for  $i \in \{1, \dots, n\}$  has a projective cover, denoted by  $f_i : P_i \longrightarrow K_i$ . Therefore, by Proposition 2.4, the map

$$f = \bigoplus_{i=1}^n f_i : P = \bigoplus_{i=1}^n P_i \longrightarrow \bigoplus_{i=1}^n K_i \cong A/J$$

is a projective cover of  $A/J$ .

Thus, every direct summand  $A/J$  of the right  $R$ -module  $R/J$  has a projective cover. Therefore, by Lemma 2.4, we conclude that  $R$  is semiperfect.

□

## 3.2 Characterization of Perfect Rings

**Definition 3.3.** A ring  $R$  is called right Bass if every nonzero right  $R$ -module has a maximal submodule.

**Lemma 3.1.** *The following conditions are equivalent for a ring  $R$ :*

1.  $R$  is right Bass (every nonzero right  $R$ -module has a maximal submodule)
2. Every nonzero right  $R$ -module  $M$  satisfies  $\text{rad } M \neq M$
3. Every nonzero right  $R$ -module  $M$  has  $\text{rad } M \ll M$  (the radical is small)

*Proof.* We prove the implications cyclically.

(1)  $\implies$  (2): Assume  $R$  is right Bass and let  $M$  be a nonzero right  $R$ -module. Then  $M$  contains a maximal submodule  $N$ . Since  $\text{rad } M$  is the intersection of all maximal submodules, we have  $\text{rad } M \subseteq N \subsetneq M$ , hence  $\text{rad } M \neq M$ .

(2)  $\implies$  (3): Let  $M$  be nonzero with  $\text{rad } M \neq M$ . Suppose  $\text{rad } M + L = M$  for some  $L \subseteq M$ . If  $L \neq M$ , then  $M/L$  is nonzero, so by (2),  $\text{rad}(M/L) \neq M/L$ . Thus there exists maximal  $N \supseteq L$  in  $M$  satisfying:

$$\text{rad } M + L \subseteq N \subsetneq M$$

contradicting  $\text{rad } M + L = M$ . Therefore  $L = M$ , proving  $\text{rad } M \ll M$ .

(3)  $\implies$  (1): Suppose every nonzero  $M$  has  $\text{rad } M \ll M$ . If some nonzero  $M$  had no maximal submodules, we would have  $\text{rad } M = M$ . Then for  $L = 0$  we get  $M + 0 = M$  but  $0 \neq M$ , contradicting  $M \ll M$ . Thus  $M$  must have maximal submodules.  $\square$

**Lemma 3.2.** *If  $R$  is a right Bass ring and  $R/J(R)$  is semisimple, then idempotents lift modulo  $J(R)$ .*

*Proof.* We start by recalling Lemma 3.1, which asserts that a ring  $R$  is right Bass if and only if every nonzero right  $R$ -module  $M$  satisfies  $\text{rad } M \ll M$ .

Consider a countably generated, nonzero, free right  $R$ -module  $F$ . As  $F$  is nonzero and  $R$  is a right Bass ring, Lemma 3.1 ensures that  $\text{rad } F \ll F$ . Since  $F$  is free, it is projective. Applying Theorem 1.3, we obtain:

$$\text{rad } F = F \cdot J(R).$$

Thus,  $F \cdot J(R) \ll F$ , and we can invoke Lemma 1.9, which shows that  $J(R)$  is right T-nilpotent. Therefore,  $J(R)$  is a nil-ideal.

Finally, Proposition 1.13 guarantees that idempotents lift modulo a nil-ideal, concluding the proof that idempotents lift modulo  $J(R)$ .  $\square$

**Proposition 3.2** (Ketkar and Vanaja [18]). A right  $R$ -module  $M$  over a semiperfect ring  $R$  is projective if it satisfies two conditions:  $M$  is  $R$ -projective, and its radical,  $\text{rad } M$ , is small in  $M$ .

**Theorem 3.2.** *The following statements are equivalent for a ring  $R$ :*

1.  $R$  is right perfect.
2.  $R/J(R)$  is semisimple and  $R$  is right Bass.
3. Every right  $R$ -module has a projective cover.
4. Every semisimple right  $R$ -module has a projective cover.
5. Every semisimple right  $R$ -module has an  $R$ -projective cover (projective relative to  $R$ ).

*Proof.* (1)  $\implies$  (2)

Assume that  $R$  is right perfect. Then,  $R/J(R)$  is semisimple. It remains to prove that  $R$  is right Bass. Let  $M$  be a nonzero right  $R$ -module. Since  $J = J(R)$  is right T-nilpotent, it follows from Lemma 1.9 that  $MJ \neq M$ , which implies that  $M/MJ \neq 0$ . Moreover, since  $J$  annihilates  $M/MJ$ , we can treat  $M/MJ$  as a right  $R/J$ -module. Given that  $R/J$  is semisimple, it follows from Proposition 1.4 that  $M/MJ$ , as a right  $R/J$ -module, is also semisimple. Since  $M/MJ$  is a nonzero semisimple module, it follows from Proposition 1.19 that  $M/MJ$  admits a maximal submodule. Therefore,  $M$  admits a maximal submodule as well. Hence,  $R$  is right Bass.

(2)  $\implies$  (3) Assume that  $R/J(R)$  is semisimple and that  $R$  is a right Bass ring. Let  $M$  be a right  $R$ -module. Since  $R$  is semilocal, Proposition 2.3 implies that  $M$  is also semilocal.

Thus, the quotient module  $M/\text{rad } M$  is semisimple and can be expressed as a direct sum of simple modules:

$$M/\text{rad } M = \bigoplus_{i \in I} K_i,$$

where each  $K_i$  is a simple right  $R$ -module for  $i \in I$ .

By Proposition 1.9, each  $K_i$  is annihilated by  $J(R)$ , meaning that the action of  $R$  on  $K_i$  factors through  $R/J(R)$ . This allows us to view  $M/\text{rad } M$  as an  $R/J(R)$ -module.

Since  $R/J(R)$  is semisimple, we can apply Theorem 1.2, which tells us that  $M/\text{rad } M$  is a projective  $R/J(R)$ -module. Consequently, Proposition 1.26 implies that each  $K_i$  is a projective  $R/J(R)$ -module.

Moreover, as each  $K_i$  is simple (and thus cyclic), by Lemma 1.15, each  $K_i$  is isomorphic to a direct summand of  $\overline{R} = R/J(R)$ .

Proposition 1.11 further indicates that every direct summand of  $\overline{R}$  is isomorphic to  $g_i \overline{R}$ , where  $g_i$  is an idempotent in  $\overline{R}$ . Therefore, each  $K_i$  is isomorphic to  $g_i \overline{R}$ .

By Lemma 3.2, which asserts that idempotents lift modulo  $J(R)$ , there exists an idempotent  $e_i \in R$  such that  $\overline{e_i} = g_i$ . Thus, we can conclude that  $K_i \cong g_i \overline{R} = \overline{e_i} \overline{R}$ .

Next, we define the map  $\varphi_i : e_i R \rightarrow \overline{e_i} \overline{R}$  by  $\varphi_i(e_i r) = \overline{e_i r}$ . This map is an epimorphism with kernel  $e_i J(R)$ .

By the First Isomorphism Theorem, we have the following isomorphism:

$$e_i R / e_i J(R) \cong \overline{e_i} \overline{R} = g_i \overline{R} \cong K_i.$$

Now, consider the natural projection map:

$$\psi_i : e_i R \rightarrow e_i R / e_i J(R) \cong K_i.$$

According to Example 2.2, the pair  $(e_i R, \psi_i)$  forms a projective cover.

Additionally, Proposition 2.4 implies that the map

$$\psi := \bigoplus_{i \in I} \psi_i : P := \bigoplus_{i \in I} e_i R \rightarrow \bigoplus_{i \in I} (e_i R / e_i J(R))$$

is a projective cover for  $M / \text{rad } M = \bigoplus_{i \in I} K_i$ .

Since each  $e_i R$  is projective, Proposition 1.26 guarantees that  $P = \bigoplus_{i \in I} e_i R$  is also projective.

Next, we examine the following commutative diagram:

$$\begin{array}{ccc} & & P \\ & \nwarrow \exists \alpha & \downarrow \psi \\ M & \xrightarrow{\eta} & M / \text{rad } M \end{array}$$

Here,  $\eta$  is the natural projection map.

By the projectivity of  $P$ , there exists a map  $\alpha : P \rightarrow M$  such that  $\eta \circ \alpha = \psi$ .

From the commutativity of the diagram, Lemma 1.12 implies:

$$M = \ker \eta + \text{Im } \alpha.$$

Since  $\ker \eta = \text{rad } M$  and by Lemma 3.1, as  $R$  is a right Bass ring, we have  $\text{rad } M \ll M$ .

Therefore, we conclude that  $M = \text{Im } \alpha$ .

Thus,  $\alpha$  is an epimorphism. Now, we observe that  $\ker \alpha \subseteq \ker \psi$ . To demonstrate this, let  $x \in \ker \alpha$ . Then:

$$\psi(x) = \eta(\alpha(x)) = \eta(0) = 0,$$

which implies  $x \in \ker \psi$ . Since  $\ker \psi \ll P$ , it follows that  $\ker \alpha \ll P$ .

Hence, the pair  $(P, \alpha)$  is a projective cover of  $M$ . Therefore, every right  $R$ -module has a projective cover.

(3)  $\implies$  (4)

Since every right  $R$ -module has a projective cover (3), this naturally implies that every semisimple right  $R$ -module also has a projective cover (4).

(4)  $\implies$  (5)

Clear.

(5)  $\implies$  (4)

Assume that statement (5) holds, namely that every semisimple right  $R$ -module has an  $R$ -projective cover. Consider an arbitrary semisimple right  $R$ -module  $K$ . By assumption,  $K$  possesses an  $R$ -projective cover, which we denote by  $f : P \rightarrow K$ . Our goal is to demonstrate that this cover is, in fact, a projective cover, which requires showing that  $P$  is a projective module.

We first establish that  $\ker f = \text{rad } P$ . The definition of an  $R$ -projective cover implies that  $\ker f$  is a small submodule of  $P$  ( $\ker f \ll P$ ). Since  $\text{rad } P$  is the sum of all small submodules of  $P$ , the inclusion  $\ker f \subseteq \text{rad } P$  holds directly.

For the reverse inclusion, we note that  $f$  is an epimorphism and  $\ker f \ll P$ . Applying Corollary 1.3(1), we obtain:

$$f(\text{rad } P) = \text{rad } K$$

As  $K$  is given to be semisimple, Proposition 1.19 implies that its radical is zero:

$$\text{rad } K = 0$$

Consequently,  $f(\text{rad } P) = 0$ . This indicates that  $\text{rad } P$  is contained within the kernel of  $f$ :

$$\text{rad } P \subseteq \ker f$$

Combining the two inclusions yields the equality  $\text{rad } P = \ker f$ .



The initial hypothesis (5) implies, in particular, that every simple right  $R$ -module admits an  $R$ -projective cover. According to Theorem 3.1 (specifically, the implication (5)  $\implies$  (1)), this condition is sufficient to conclude that the ring  $R$  is semiperfect.

We have thus shown that  $P$  is an  $R$ -projective module over the semiperfect ring  $R$ . Furthermore, its radical,  $\text{rad } P$ , coincides with  $\ker f$ , which is small in  $P$ . Invoking Proposition 3.2, these conditions together imply that  $P$  is a projective module.

Therefore, the map  $f : P \rightarrow K$ , initially given as an  $R$ -projective cover, satisfies all the requirements of a projective cover (since  $P$  is projective,  $f$  is an epimorphism, and  $\ker f \ll P$ ). Because  $K$  was chosen as an arbitrary semisimple right  $R$ -module, we conclude that every such module possesses a projective cover, thereby establishing statement (4).

(4)  $\implies$  (1)

Assume that every semisimple right  $R$ -module possesses a projective cover. Our goal is to demonstrate that the ring  $R$  is right perfect.

From the initial assumption, it follows specifically that every simple right  $R$ -module admits a projective cover. By Theorem 3.1 (invoking the implication (4)  $\implies$  (1)), this condition suffices to establish that the ring  $R$  is semiperfect. Consequently, the quotient ring  $R/J(R)$  is semisimple. The remaining task is to prove that the Jacobson radical,  $J(R)$ , is right T-nilpotent.

According to Lemma 1.9, establishing the right T-nilpotency of  $J(R)$  is equivalent to proving that for any countably generated free right  $R$ -module  $F$ , the submodule  $FJ(R)$  is small in  $F$  (denoted  $FJ(R) \ll F$ ).

Since  $F$  is, by definition, a free module, it is inherently projective. Theorem 1.3 states that for a projective module  $F$ , its radical is given by  $\text{rad } F = FJ(R)$ . Therefore, the objective simplifies to showing that  $\text{rad } F \ll F$ .

Given that  $R$  is semiperfect (and thus semilocal), Proposition 2.3 asserts that the module  $F_R$  must also be semilocal. This implies that the quotient module  $F/\text{rad } F$  is semisimple. Based on our initial hypothesis, since  $F/\text{rad } F$  is semisimple, it must have a projective

cover.

Invoking Lemma 2.5 (Bass Lemma), because  $F$  is projective and  $F/\text{rad } F$  possesses a projective cover,  $F$  admits a decomposition of the form:

$$F = P \oplus Q$$

where  $P \subseteq \text{rad } F$ , and the intersection  $Q \cap \text{rad } F$  is small in  $Q$  ( $Q \cap \text{rad } F \ll Q$ ).

As previously established using Theorem 1.3,  $\text{rad } F = FJ(R)$ . The condition  $P \subseteq \text{rad } F$  thus yields the inclusion:

$$P \subseteq \text{rad } F = FJ(R).$$

Substituting the decomposition  $F = P \oplus Q$  into the expression for the radical gives:

$$\begin{aligned} \text{rad } F &= (P \oplus Q)J(R) \\ &= PJ(R) \oplus QJ(R). \end{aligned}$$

Hence, we have the equality:

$$\text{rad } F = FJ(R) = PJ(R) \oplus QJ(R).$$

Consider an arbitrary element  $p \in P$ . Since  $P \subseteq \text{rad } F = PJ(R) \oplus QJ(R)$ ,  $p$  can be written as:

$$p = p' + q' \quad \text{where } p' \in PJ(R) \text{ and } q' \in QJ(R).$$

The decomposition  $F = P \oplus Q$  implies  $P \cap Q = \{0\}$ . The component  $q'$  belongs to  $QJ(R)$  and thus to  $Q$ . Furthermore, since  $p \in P$  and  $p' \in PJ(R) \subseteq P$ , the difference  $q' = p - p'$  must also be an element of  $P$ . Consequently,  $q'$  lies in the intersection  $P \cap Q = \{0\}$ , forcing  $q' = 0$ . This demonstrates that  $p = p'$ , which means  $p \in PJ(R)$ .

Since the choice of  $p \in P$  was arbitrary, it follows that every element of  $P$  is contained in  $PJ(R)$ , leading to the inclusion:

$$P \subseteq PJ(R).$$

Conversely,  $PJ(R)$  is inherently a submodule of  $P$ , so  $PJ(R) \subseteq P$ . Combining these inclusions yields the equality:

$$P = PJ(R).$$

Recall that  $P$  is a direct summand of the projective module  $F$ . By Corollary 1.3, direct summands of projective modules are themselves projective, so  $P$  is projective. Applying Theorem 1.3 to the now established projective module  $P$ , we have  $\text{rad } P = PJ(R)$ . Substituting this into the equality  $P = PJ(R)$  gives:

$$P = \text{rad } P.$$

However, Theorem 1.3 also stipulates that for any nonzero projective module  $X$ , its radical  $\text{rad } X$  must be a proper submodule, i.e.,  $\text{rad } X \neq X$ . Since  $P$  is projective and satisfies  $P = \text{rad } P$ , this necessitates that  $P$  must be the zero module ( $P = 0$ ).

With  $P = 0$ , the decomposition  $F = P \oplus Q$  simplifies to  $F = Q$ . Consequently, the expression for the radical becomes:

$$\text{rad } F = FJ(R) = QJ(R).$$

The condition  $Q \cap \text{rad } F \ll Q$ , obtained from Lemma 2.5, transforms upon substituting  $Q = F$  into:

$$F \cap \text{rad } F \ll F.$$

Given that  $\text{rad } F$  is a submodule of  $F$ , the intersection  $F \cap \text{rad } F$  is simply  $\text{rad } F$ . Thus, the condition becomes:

$$\text{rad } F \ll F.$$

Finally, since  $\text{rad } F = FJ(R)$ , the condition  $\text{rad } F \ll F$  allows us to apply Lemma 1.9 (specifically, the implication (4)  $\implies$  (1)) to conclude that  $J(R)$  is right T-nilpotent.

In conclusion, we have established that  $R/J(R)$  is semisimple and  $J(R)$  is right T-nilpotent. These are the defining conditions for  $R$  to be a right perfect ring.  $\square$

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