Selected Functions

1) $f(x) = 1 + e^{-\cos(x-1)}$,

2) $f(x) = x \sin \pi x - (x-2) \ln x$,

3) $f(x) = \sin^5(x) + \ln\left(1 + x^2\right) - \dfrac{\cos^3(x)}{1 + x^4}$

Part A: Numerical Differentiation

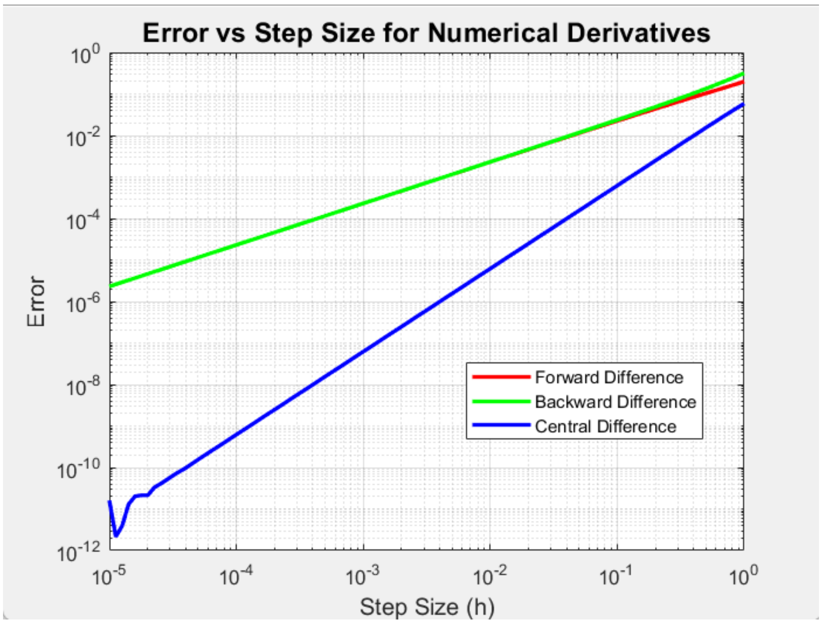
Please refer to page 4, for the detailed table records of the calculations.

Summary on the Results & Convergence

While comparing the 3 different approximation functions, it has become clearly evident that the Central Approximation is much more accurate compared to the Forward & Backward Approximations.

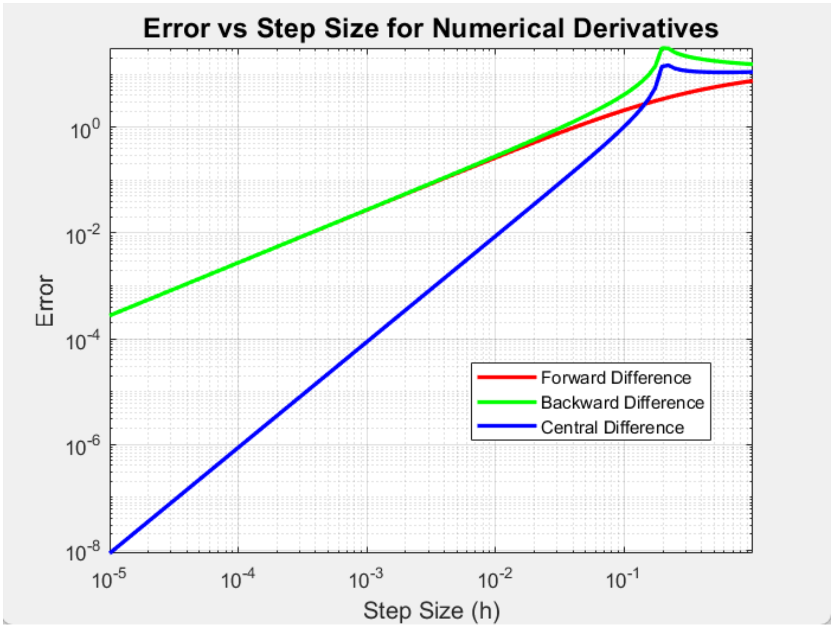
Also reducing the value of the step size of h, for example from 0.1 to 0.001 makes the calculations more accurate for all 3 Approximation functions.

When comparing to the analytical solutions. All the approximations were close to the analytical solution as long as a smaller step size was taken. Once the step size became larger, forward and backward Approximations became less accurate compared to Central.

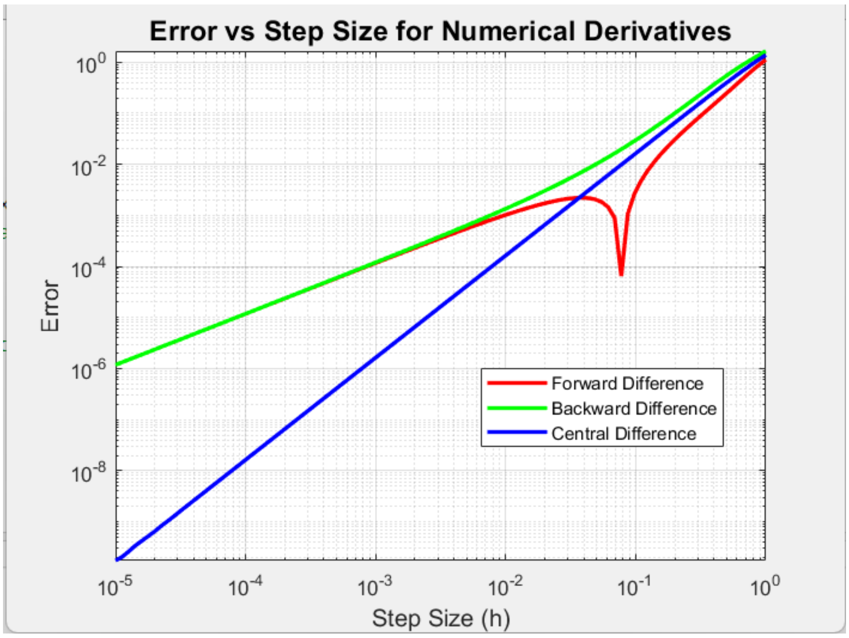


Note: We have used the first x points for each function and varied the h values to plot against the errors.

$$f(x) = 1 + e^{-\cos(x-1)}$$



$$f(x) = x\sin\pi - (x-2)\ln x$$



$$f(x) =$$

$$\sin^5(x) + \ln(1 + x^2) - \frac{\cos^3(x)}{1 + x^4}$$

Part A: Numerical Differentiation

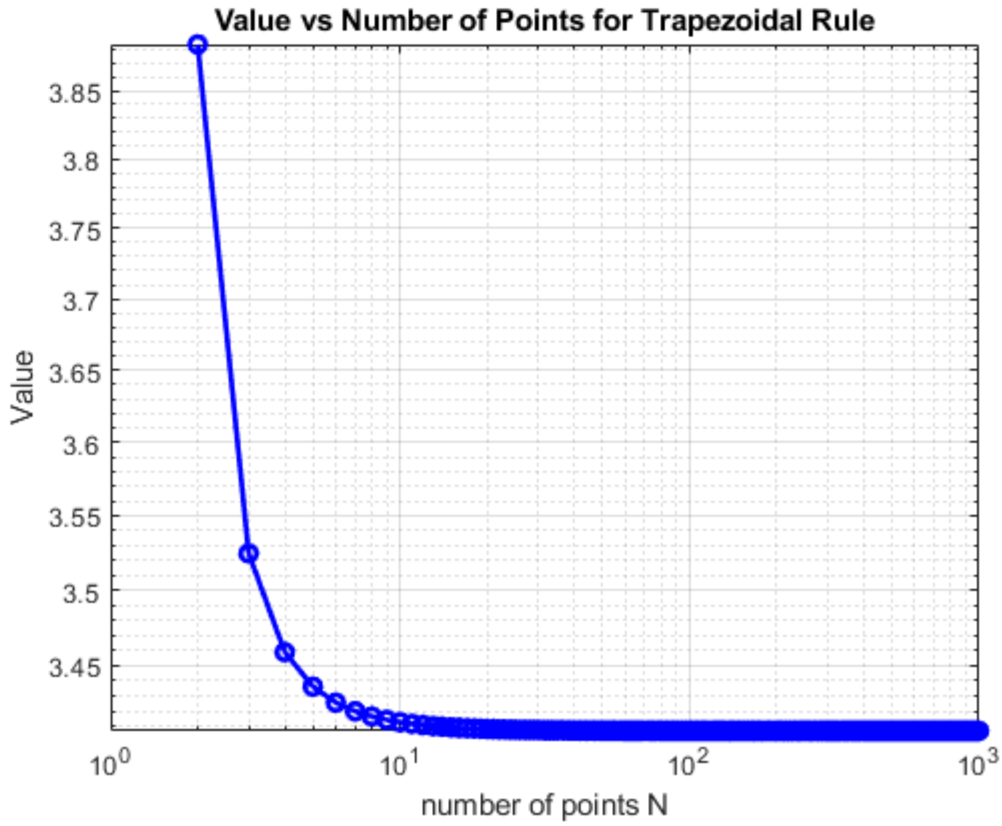
Functions	Point: x	Step:h	Forward	Backward	Central	Analytical	Error
$f(x) = 1 + e^{-\cos(x-1)}$	0.5	0.01	-0.197043	-0.201647	-0.199345	-0.199339	Forward: 0.002296 Backward: 0.002308 Central: 0.000006
	1.2	0.1	0.093966	0.055642	0.074804	0.074558	Forward: 0.019408 Backward: 0.018916 Central: 0.000246
$f(x) = xsin\pi - (x-2)lnx$	0.2	0.01	10.342877	10.893524	10.618201	10.609438	Forward: 0.266561 Backward: 0.284086 Central: 0.008763
	3.2	0.2	-1.587523	-1.485843	-1.536683	-1.538151	Forward: 0.049372 Backward: 0.052307 Central: 0.001468
$f(x) = sin^5(x) + ln(1+x^2) - \frac{cos^3(x)}{1+x^4}$	0.8	0.001	2.987851	2.987615	2.987733	2.987734	Forward: 0.000116 Backward: 0.000120 Central: 0.000002
	1.5	0.1	1.014995	1.526239	1.270617	1.275832	Forward: 0.260837 Backward: 0.250406 Central: 0.005215

Part B: Numerical Integration

Method 1 : Trapezoidal

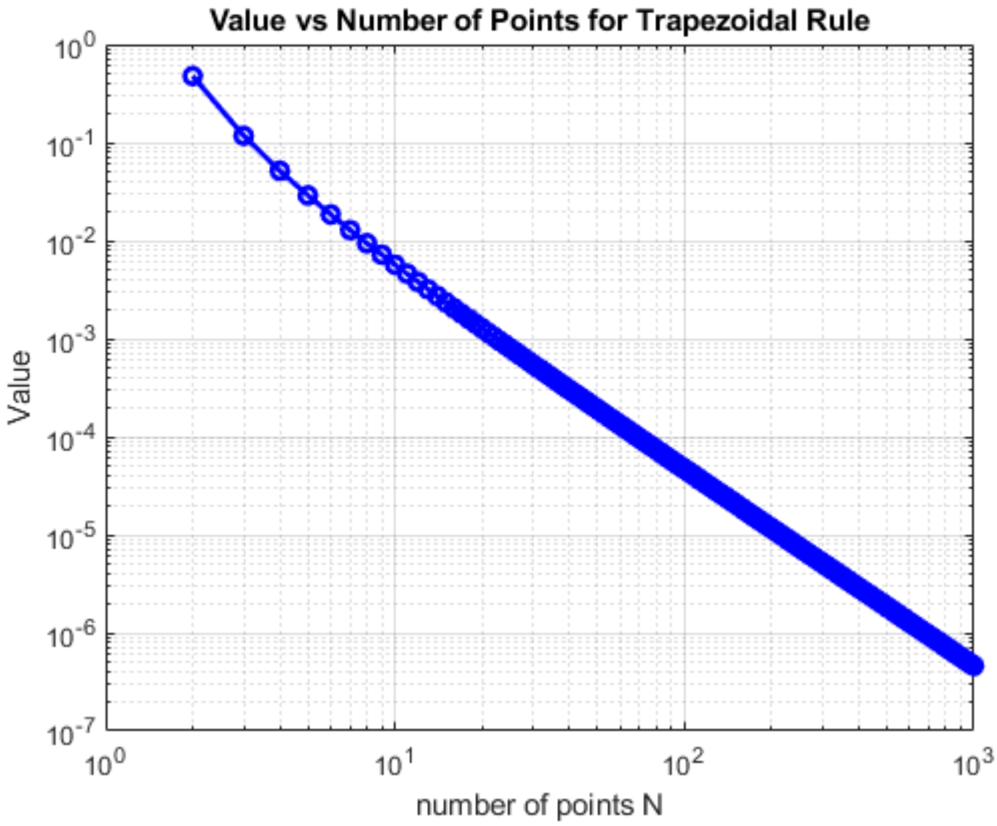
$$f(x) = 1 + e^{-\cos(x-1)}$$

Note: We have chosen the sample points to be from 2 to 1000 points.



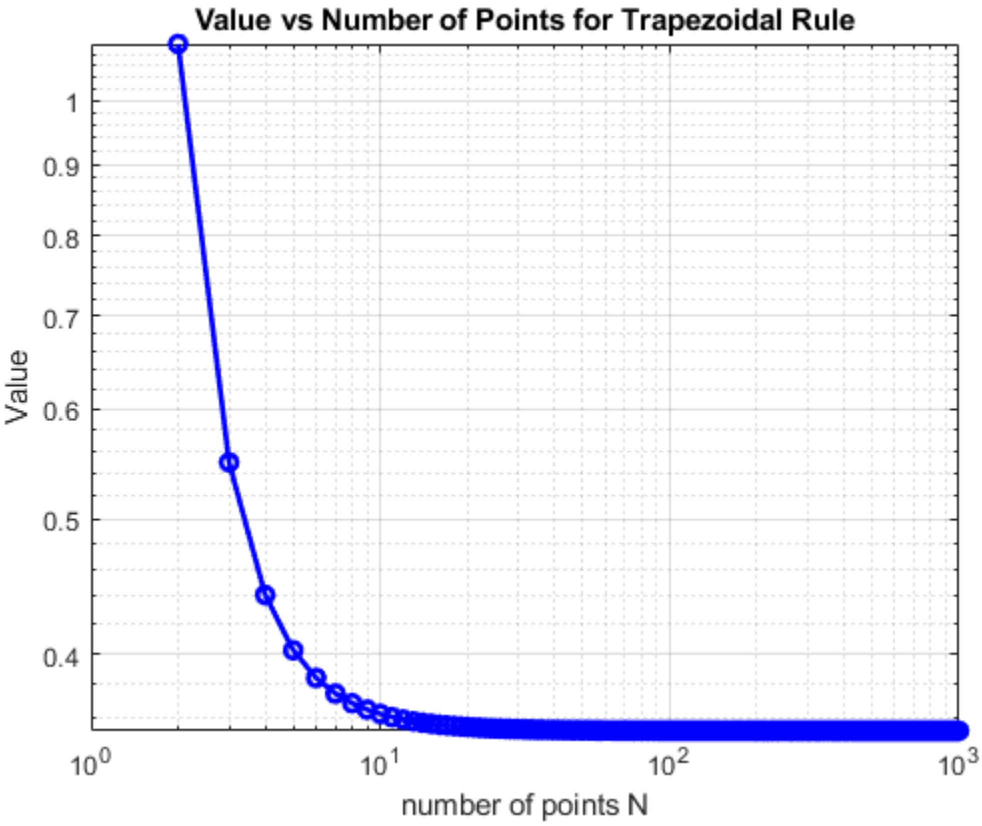
n	Approximation
2	3.8839879138
10	3.4128200531
20	3.4084134714
50	3.4073315969
100	3.4071870844
200	3.4071518018

Analytical vs numerical errors comparison using first function



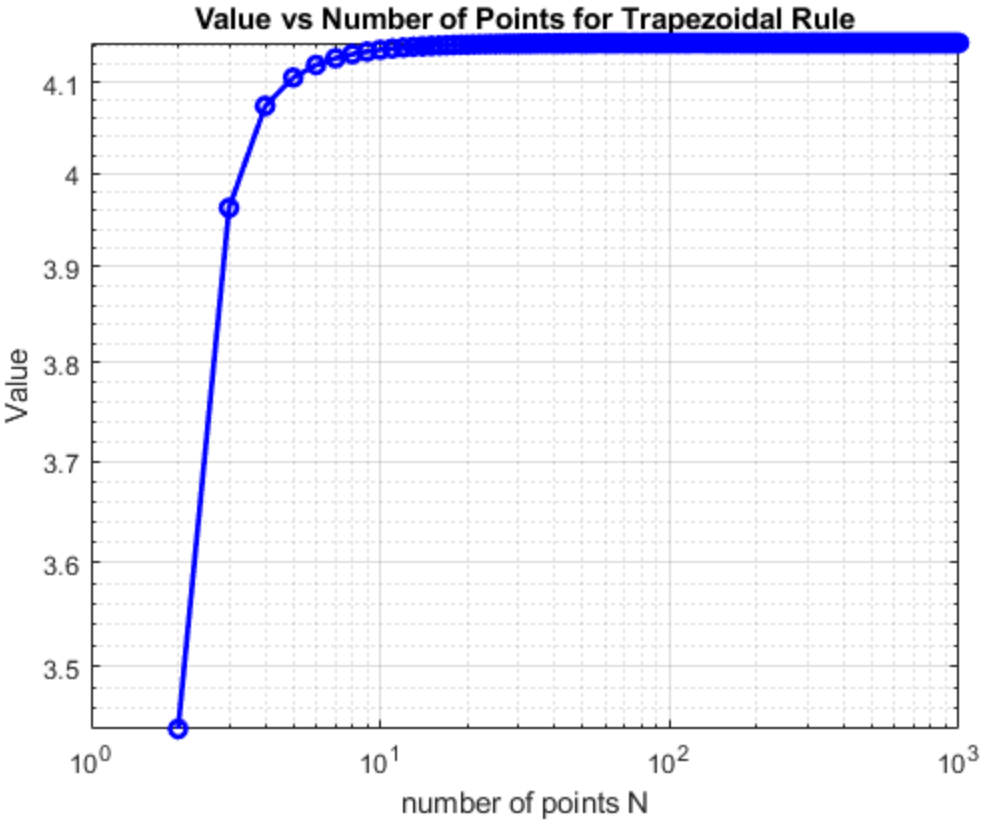
The error converges to zero, getting smaller and smaller, but improvements returns diminishes the more points you use.

$f(x) = x sin\pi - (x-2)lnx$



n	Approximation
2	-1.0986122887
10	-0.3620737235
20	-0.3543263248
50	-0.3524191783
100	-0.3521642767
200	-0.3521020373

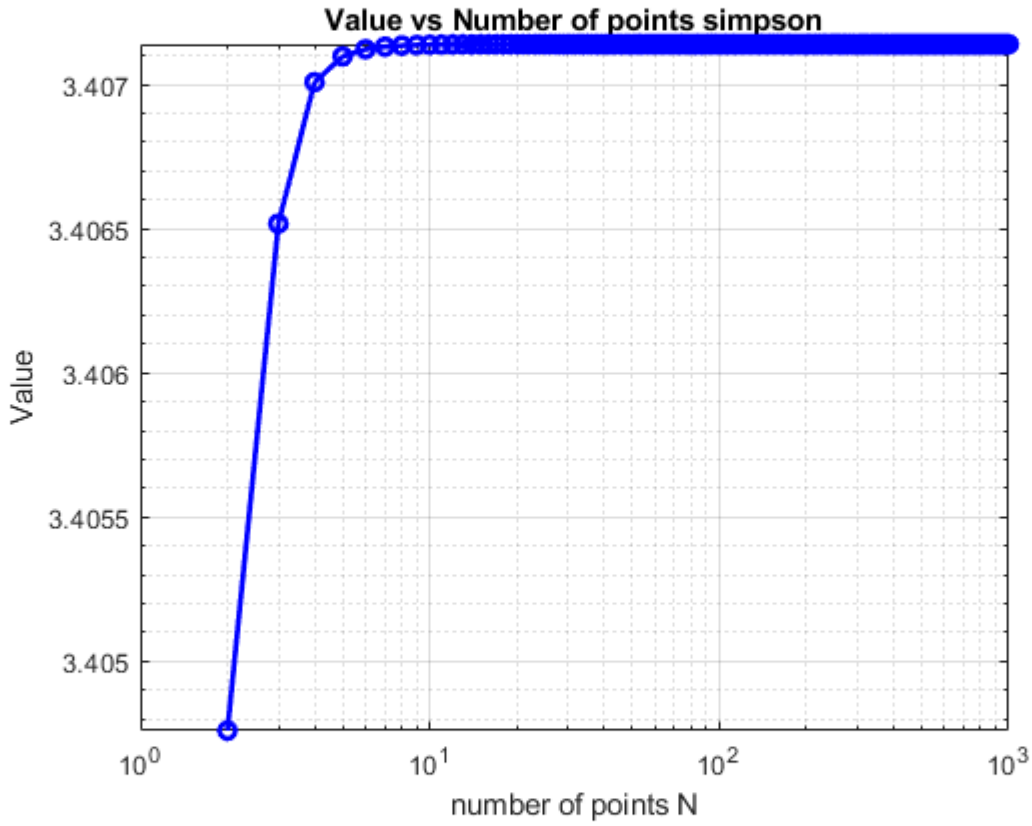
$$f(x) = sin^5(x) + ln(1 + x^2) - \frac{cos^3(x)}{1 + x^4}$$



n	Approximation
2	3.4412322848
10	4.1362452405
20	4.1421127498
50	4.1435934583
100	4.1438046118
200	4.1438617512

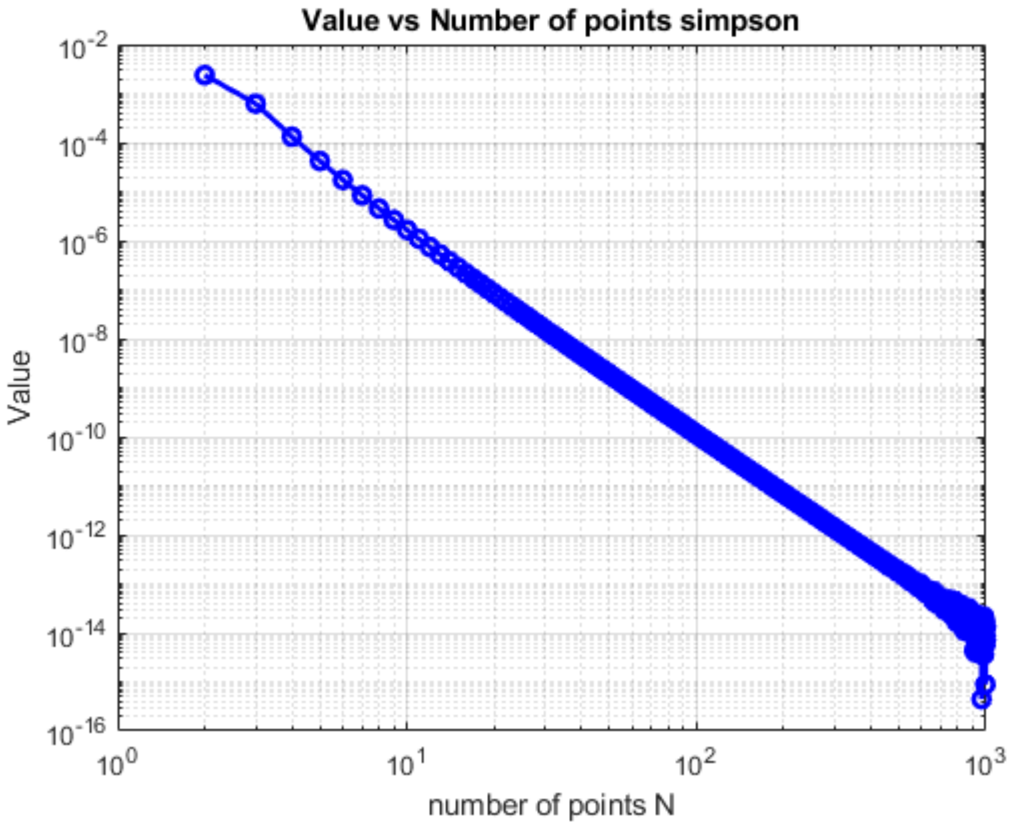
Method 2 : Simpson's

$$f(x) = 1 + e^{-\cos(x-1)}$$



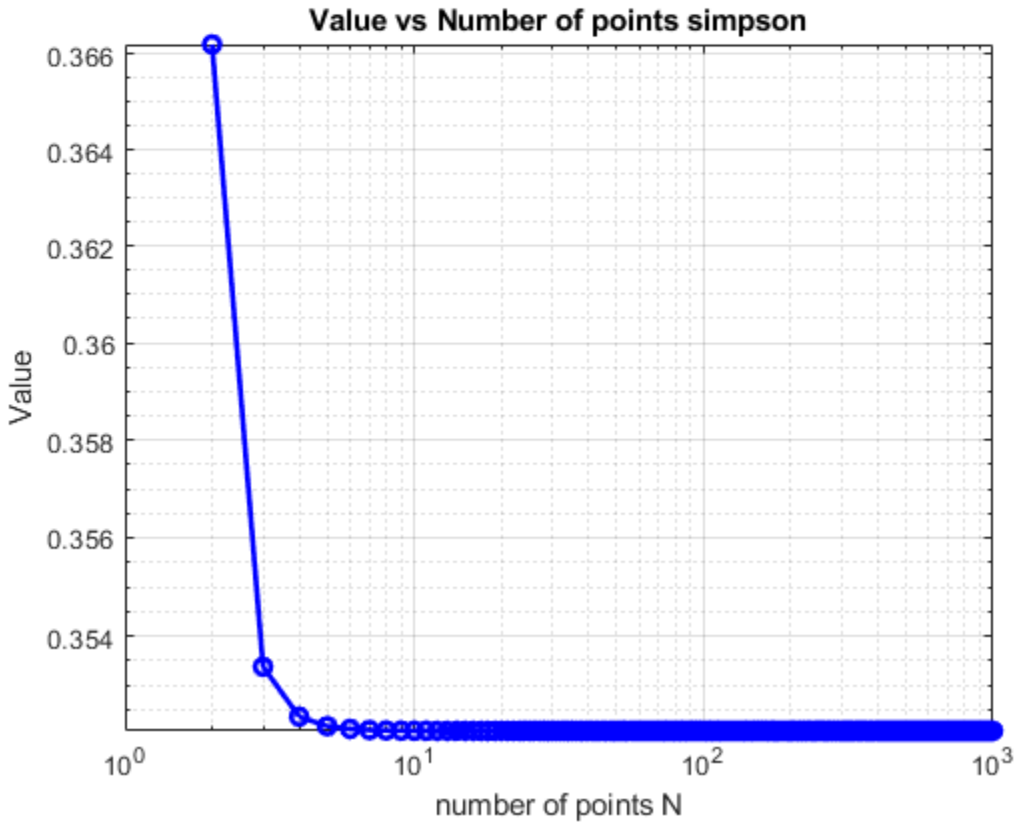
n	Approximation
2	3.4047587857
10	3.4071385405
20	3.4071401142
50	3.4071401959
100	3.4071401976
200	3.4071401977

Analytical vs numerical errors comparison using first function



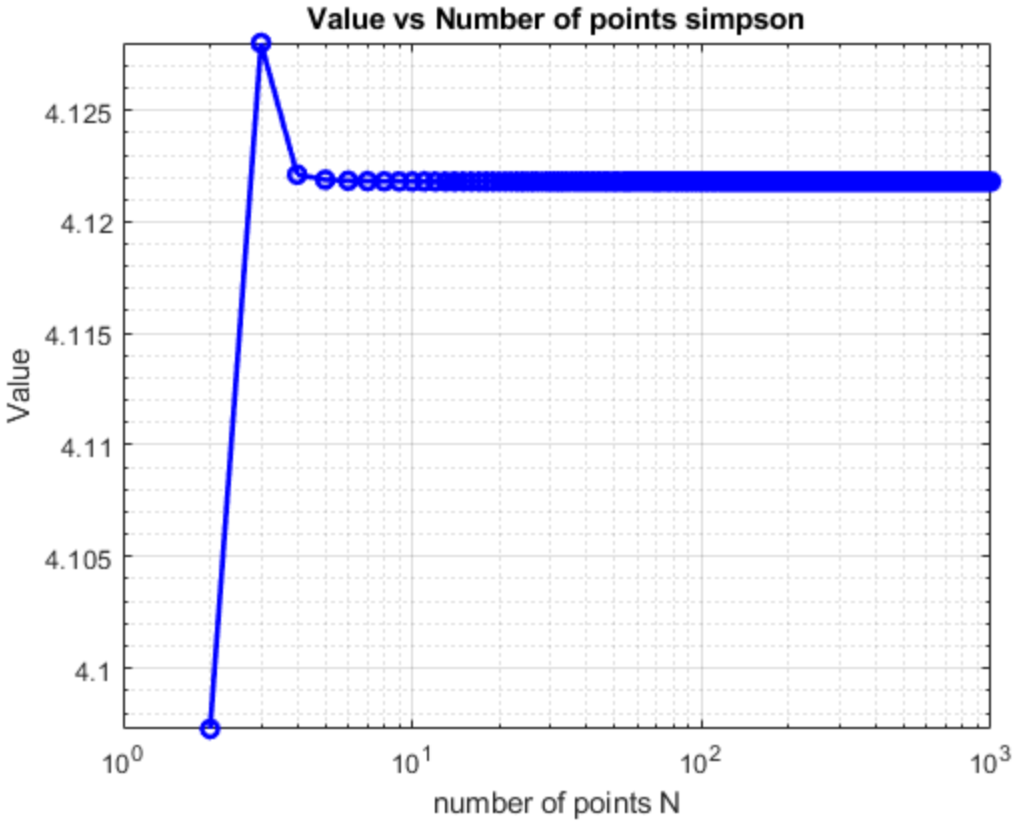
The error converges to zero, faster than trapezoidal method

$$f(x) = xsin\pi - (x-2)lnx$$



n	Approximation
2	-0.3662040962
10	-0.3520855167
20	-0.3520817683
50	-0.3520815716
100	-0.3520815673
200	-0.352081567

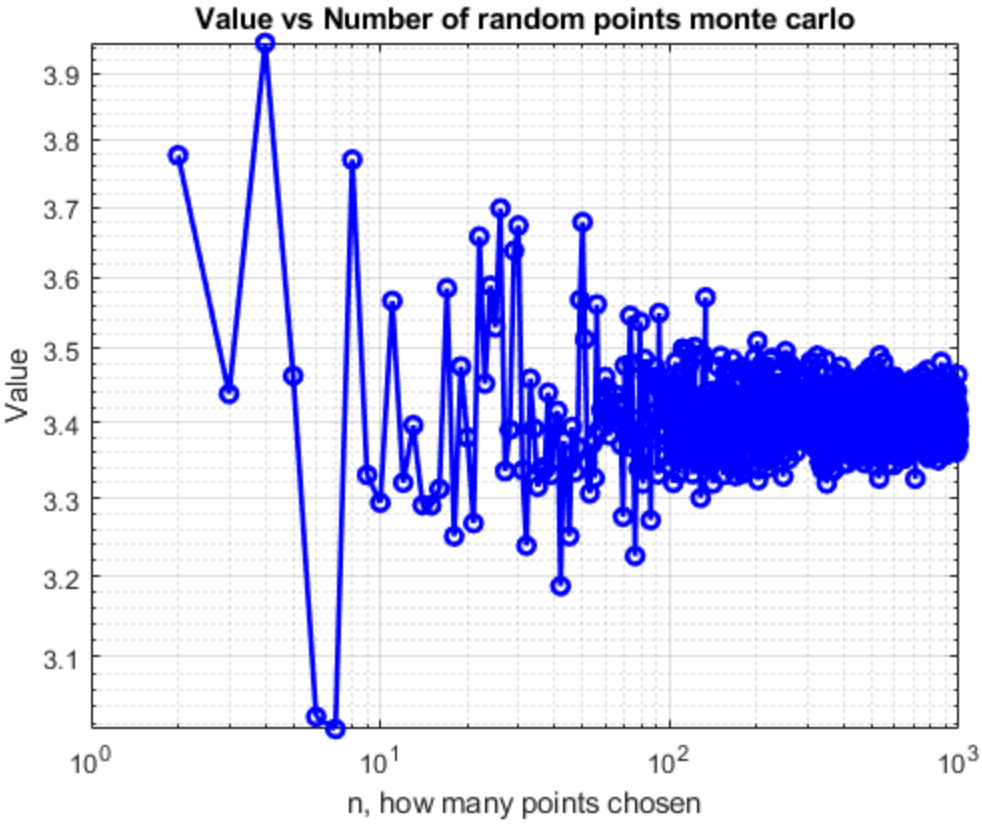
$$f(x) = sin^5(x) + ln(1 + x^2) - \frac{cos^3(x)}{1 + x^4}$$



n	Approximation
2	4.0972865215
10	4.1218121827
20	4.1218084038
50	4.1218081984
100	4.1218081939
200	4.1218081936

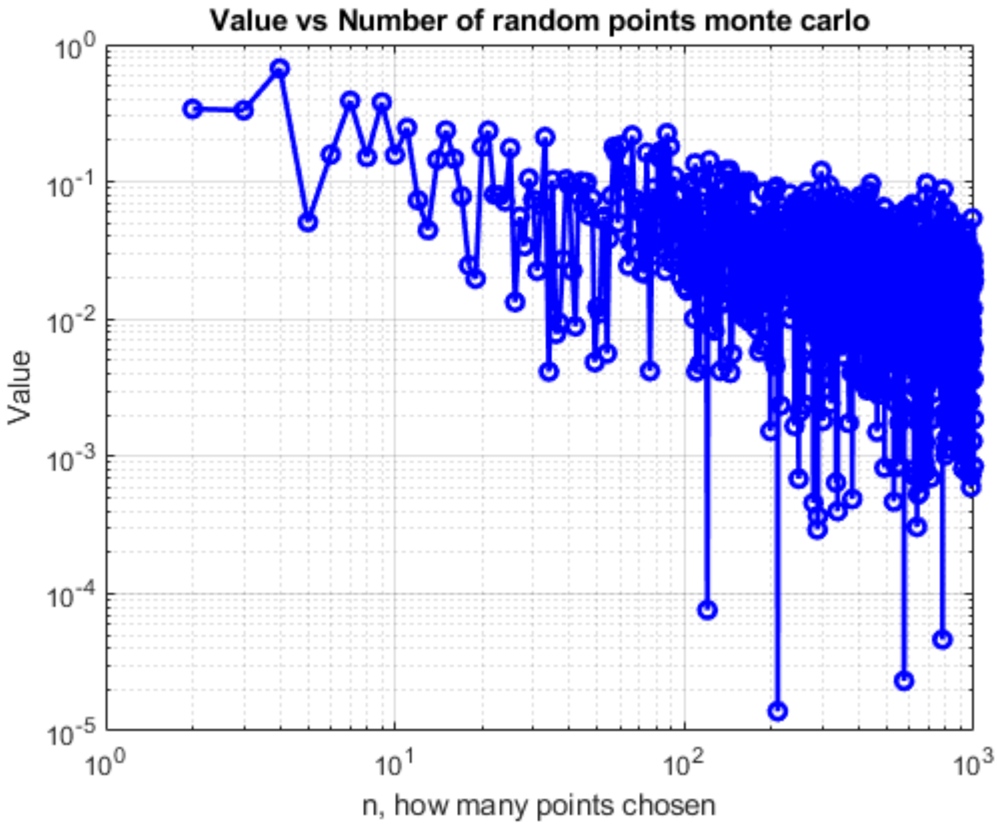
Method 2 : Monte Carlo

$$f(x) = 1 + e^{-\cos(x-1)}$$



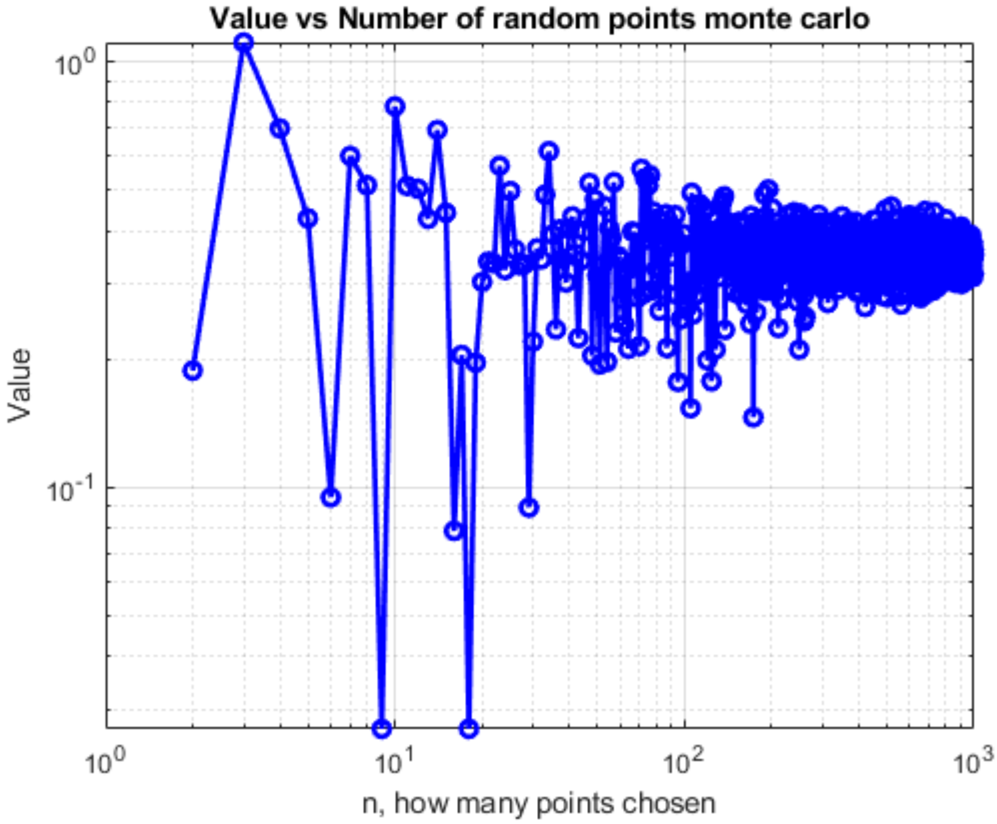
n	Approximation
2	3.5918422185
10	3.8693499186
20	3.120085636
50	3.5870484207
100	3.4650888245
200	3.480113497

Analytical vs numerical errors comparison using first function



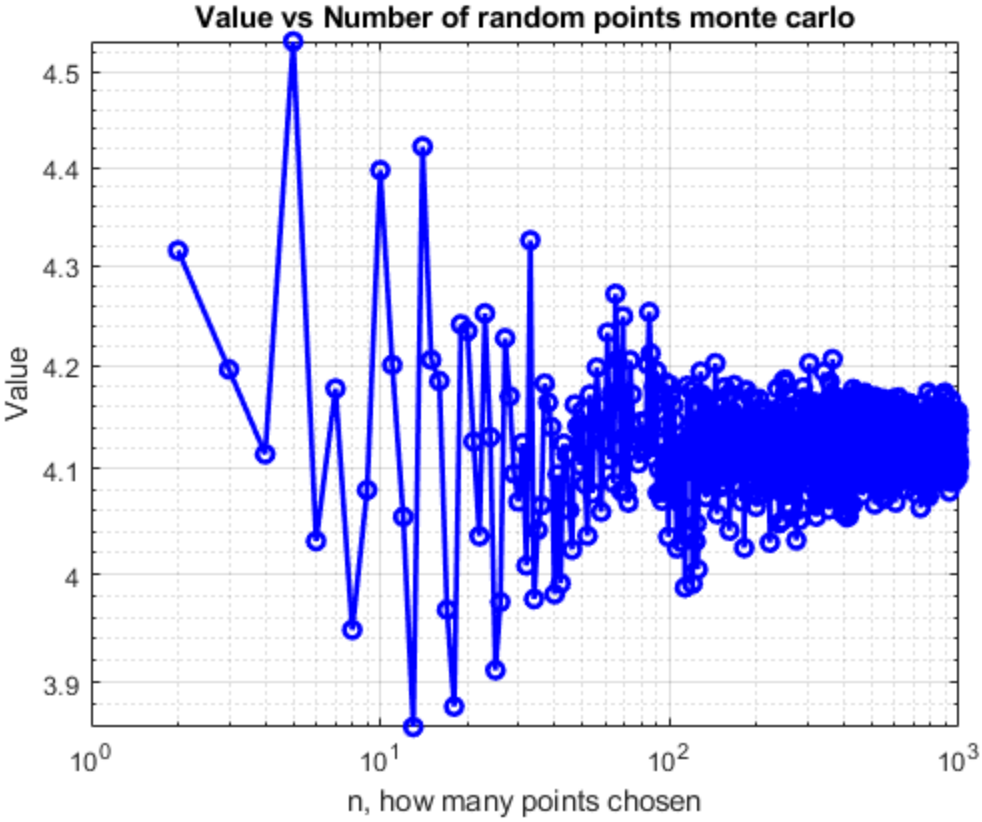
The error converges to zero on average, although it isn't always the case that more points is better than less points. It converges on a much slower pace than other methods

$$f(x) = x\sin\pi - (x-2)\ln x$$



n	Approximation
2	0.2752218822
10	-0.447854068
20	-0.2850837001
50	-0.469808259
100	-0.3518853817
200	-0.3199808638

$$f(x) = sin^5(x) + ln(1 + x^2) - \frac{cos^3(x)}{1 + x^4}$$



n	Approximation
2	3.3153938629
10	4.3889254981
20	4.2218637255
50	4.1370338157
100	4.0606286883
200	4.1669311607

Summary on the Results & Integration Convergence

In conclusion, they all work on non integrable functions, the performance of numerical integration methods varies depending on the complexity of the function and the number of points used. The Trapezoidal rule is best suited for simple, linear graphs due to the straightforward averaging approach, providing accurate results with minimal computational effort. However, as functions become more complex, its accuracy diminishes due to its inability to capture curvatures.

Simpson's performs well for medium complexity graphs, where its quadratic approximation provides better accuracy. By assigning more weight to points along curves, Simpson’s method captures more finer details of the function, capturing curvatures better, making it more reliable for functions that are not sharp but smooth. Monte Carlo integration, while less precise for one-dimensional problems, becomes a viable option for functions with higher complexity or irregular behaviour, where spikes and erratic behaviour is the norm. It is especially useful for integrals that are difficult to compute as a closed form.

A key observation is that the accuracy of all methods improves with an increasing number of points. Trapezoidal and Simpson's rules show consistent improvements in errors as the interval is divided into more segments. For Monte Carlo, while its errors reduce on average with more points, it’s still variable in nature, making it less reliable compared to the deterministic methods.

1. **Trapezoidal Rule:** Works best on simple linear graphs, but its accuracy decreases the more complex it gets.
2. **Simpson’s Rule:** The best choice for medium complexity graphs due to its ability to handle curves more accurately.
3. **Monte Carlo Integration:** Suitable for highly complex or irregular functions, particularly in higher-dimensional cases where it spikes suddenly, but requires significantly more points to achieve comparable accuracy.