SSY281 - Assignment 1

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1 Question 1

1.1 Part A

The linear model approximating sin(x) as x is expressed as follows:

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ b & 0 & 0 & a \\ 0 & 0 & 0 & 1 \\ q & 0 & 0 & p \end{bmatrix}}_{Ac} x(t) + \underbrace{\begin{bmatrix} 0 \\ c \\ 0 \\ r \end{bmatrix}}_{Bc} u(t)$$

$$y(t) = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}}_{Cc} x(t)$$

Deriving the discretized model using Lemma 2.1 gives:

$$x(k+1) = \underbrace{\begin{bmatrix} 1.4421 & 0.1143 & 0.0000 & -0.0045 \\ 9.4370 & 1.4421 & 0.0000 & -0.0908 \\ 0.0237 & 0.0008 & 1.0000 & 0.0833 \\ 0.4814 & 0.0237 & 0.0000 & 0.6845 \end{bmatrix}}_{Ad} x(k) + \underbrace{\begin{bmatrix} 0.0677 \\ 1.3715 \\ 0.2530 \\ 4.7660 \end{bmatrix}}_{Bd} u(k)$$

$$y(k) = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}}_{Cd} x(k)$$

1.2 Part B

By introducing a computational delay of $\tau = 0.8h$, our system undergoes a transformation. This revised system necessitates the inclusion of the previous input $u(k-1)$ in the state vector. The adjusted state-space is represented by equation (3).

$$\xi(k+1) = \begin{bmatrix} A_d & B_1 \\ 0 & 0 \end{bmatrix} \xi(k) + \begin{bmatrix} B_2 \\ I \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} C_d & 0 \end{bmatrix} \cdot \xi(k)$$

Using Lemma 2.1 can B1 and B2 be calculated and numerical values can be obtained:

$$A_a = \begin{bmatrix} A_d & B_1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1.4421 & 0.1143 & 0.0000 & -0.0045 & 0.0649 \\ 9.4370 & 1.4421 & 0.0000 & -0.0908 & 1.0958 \\ 0.0237 & 0.0008 & 1.0000 & 0.0833 & 0.2419 \\ 0.4814 & 0.0237 & 0.0000 & 0.6845 & 3.6657 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix}$$

$$B_a = \begin{bmatrix} B_2 \\ I \end{bmatrix} = \begin{bmatrix} 0.0028 \\ 0.2757 \\ 0.0111 \\ 1.1002 \\ 1.0000 \end{bmatrix}$$

$$C_a = \begin{bmatrix} C_d & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

When I compare the matrices A_d and A_a, I see that they have the same eigenvalues. However, A_a has an extra pole at the origin. Both systems are unstable because they have a pole outside the unit circle. The poles for each system are:

$$\text{eig}(A_d) = \{1.0000, 2.4781, 0.4008, 0.6899\}$$
$$\text{eig}(A_a) = \{1.0000, 2.4781, 0.4008, 0.6899, 0.0000\}$$

s the computational delay τ approaches zero, the augmented system becomes identical to the discrete system. Essentially, when τ tends to zero, B_1 becomes zero, and B_2 becomes B_d.

2 Question 2

2.1 Part A

Through the provided code,function "MinNQ2a", it becomes possible to identify the smallest N necessary for stabilizing the system using dynamic programming. The system's stability is guaranteed by the positioning of all poles within the unit circle. The results:
N = 33,
$K = \begin{bmatrix} -46.1565 & -5.2278 & 0.01554 & 0.4031 \end{bmatrix}$
$\text{eig}(A + BK) = \{0.5584, 0.9235, 0.9523, 0.9995\}$

2.2 Part B

By utilizing the idare function, the stationary solution $P_{N=\infty}$for the Riccati equation was found. The numerical values are provided:

$$P_N = \infty = \begin{bmatrix} 4.8730e+04 & 5.3230e+03 & -1.0378e+03 & -1.1542e+03 \\ 5.3230e+03 & 5.8747e+02 & -1.1437e+02 & -1.2712e+02 \\ -1.0378e+03 & -1.1437e+02 & 1.2499e+02 & 2.6626e+01 \\ -1.1542e+03 & -1.2712e+02 & 2.6626e+01 & 2.9814e+01 \end{bmatrix}$$

Attempting to find the same stationary solution with DP and setting $(P(k+1)P(k))0.1$ as the terminal criterion, the following code was used:

This function increments the control horizon N until the terminal criterion is met. It was determined that the stationary solution was achieved when $N = 427$. The solution $P_{N=427}$, as shown in expression below:

$$P_{N=427} = \begin{bmatrix} 4.8726e+04 & 5.3225e+03 & -1.0373e+03 & -1.1541e+03 \\ 5.3225e+03 & 5.8741e+02 & -1.1431e+02 & -1.2711e+02 \\ -1.0373e+03 & -1.1431e+02 & 1.2493e+02 & 2.6613e+01 \\ -1.1541e+03 & -1.2711e+02 & 2.6613e+01 & 2.9811e+01 \end{bmatrix}$$

2.3 Part C

To get the N that makes the system stable can we use the following code where $P_f = P_\infty$ where N increases untill the system becames stable. To achieve a stable closed loop, we've set the control horizon to $N = 1$. This decision is rooted in aligning Pf (estimation error covariance) with the solution that satisfies the Riccati equation (p_∞).

```
clc;
A = [1.0041 0.0100 0 0; 0.8281 1.0041 0 -0.0093; 0.0002 0.0000 1 0.0098; 0.0491 0.0002
     0 0.9629];
B  = [0.0007; 0.1398; 0.0028; 0.5605];
Q = eye(4);
Pf = P_inf_DP_2b;
R = 1;

[~,~,K_2C,N_2C] = MinNQ2a(A,B,Q,R,Pf)
```

Listing 1: Equations

3 Question 3

This task was handled much like the last one. With the same settings as before, stability was reached when N was 33. The function gave us these values:

$$\begin{aligned} N &= 33, \\ K &= \begin{bmatrix} -46.1565 & -5.2278 & 0.01554 & 0.4031 \end{bmatrix}, \\ \mathrm{eig}(A + BK) &= [0.5584, 0.9235, 0.9523, 0.9995]. \end{aligned}$$

These values matched those from earlier. Both the dynamic programming (DP) and batch methods gave us the same control horizon N, feedback gain K, and the same eigenvalues for $(A + BK)$.

4 Question 4

The LQ function ti minimize is:

$$V_N(x(0), u(0:N-1)) = x(N)^T P_f x(N) + \sum_{k=0}^{N-1} x(k)^T Q x(k) + u(k)^T R u(k)$$

Dynamic programming approach was chosen with initial condition $x(0) = [\frac{\pi}{38} \; 0 \; 0 \; 0]^T$.

In figure 1 shows how different controllers behave with varying planning horizons and R values. Although the paths they take differ, the final outcomes are almost the same.

The R value significantly influences controller behavior. A smaller R leads to more actuator activity. Conversely, larger R values result in less aggressive control. This effect is observable in the input plots of figure 1.

The choice of control horizon N affects state trajectories. Controllers with longer horizons, like N = 80, converge faster towards the desired state compared to those with shorter horizons like N = 40. Extremely short horizons, such as N < 33, may lead to unstable feedback.

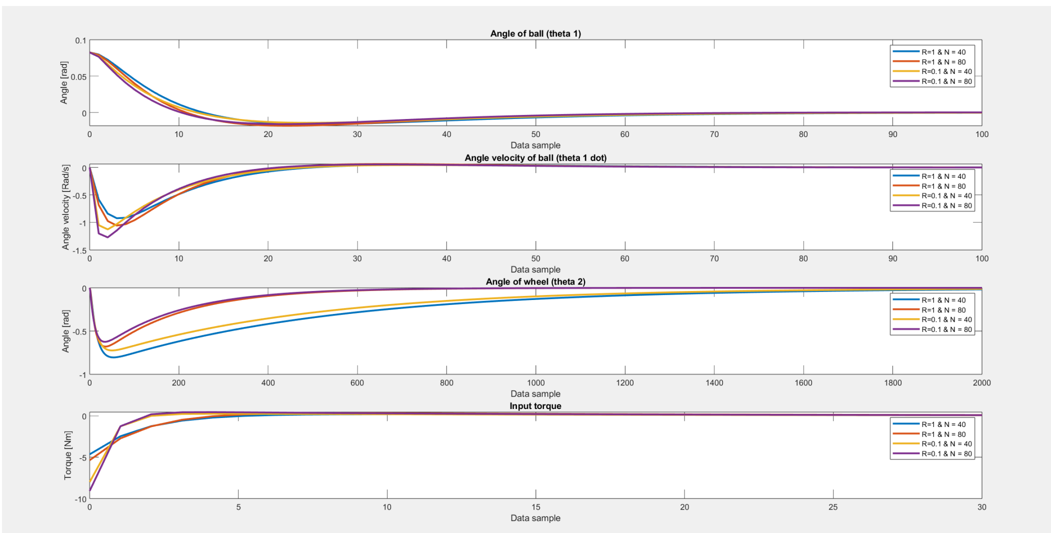

Figure 1: Different unconstrained RHCs

5 Question 5

Figure 2 illustrates the behavior of diverse restricted regulators with varying horizon lengths and R values. The limitations are: $|x_2| \leq 1$ and $|u| \leq 8$.

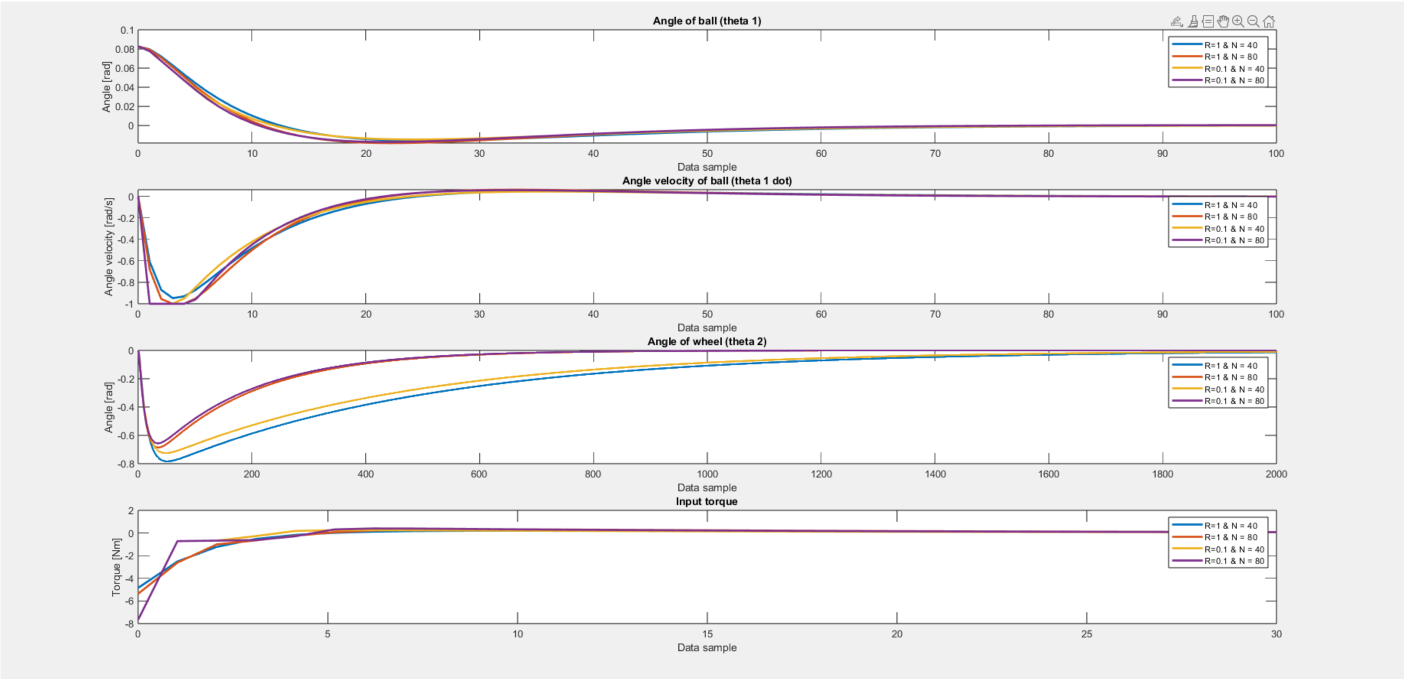

Figure 2: Various constrained RHC:s

Significant distinctions emerge between the constrained and unconstrained controllers. The primary contrast arises in the case of the purple plot (R = 0.1 & N = 80), where the angular velocity of the ball is restricted, necessitating constraint on the input as well, which is enforced.