Exercise Solutions for Ex 02

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Solution

Ex 03.1

Sensorpositions in world frame is given as:

$$p_s = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_s \\ y_s \end{bmatrix}$$

sensor velocity $(\dot{x}_s, \dot{y}_s)$ is:

$$\dot{p}_s = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + \omega \begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix} \begin{bmatrix} x_s \\ y_s \end{bmatrix}$$

where:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} v\cos\theta \\ v\sin\theta \end{bmatrix}$$

leads to:

$$\dot{x}_s = v\cos\theta - \omega(x_s \sin\theta + y_s \cos\theta)$$

$$\dot{y}_s = v\sin\theta + \omega(x_s \cos\theta - y_s \sin\theta)$$

when $x_s = 0$:

$$\dot{x}_s = v\cos\theta - \omega y_s \cos\theta$$

$$\dot{y}_s = v\sin\theta - \omega y_s \sin\theta$$

This simplification occurs because x_s no longer contributes to the angular velocity term. Physically, this means that if the sensor is positioned directly along the y-axis of the robot's frame (i.e., no offset along the x-axis), the effect of the robot's rotation (ω) on the sensor's velocity will only scale with y_s and will not depend on any x-offset. In essence, the sensor's velocity components are more straightforward when $x_s = 0$, as the influence of rotation is isolated along one direction.

03.2

The tricycle robot with three fixed wheels and one caster wheel is over-constrained. The fixed wheels restrict lateral movement, making the robot unable to move freely without slipping, resulting in a **degenerate kinematic configuration**.

Minimal Fix

Reducing the number of fixed wheels from three to two removes the over-constraint. The robot would then have:

- **Two fixed wheels** providing necessary constraints.
- **One caster wheel** for support, with no added constraints.

Classification

With this adjustment, the robot's type becomes:

- $(\delta_m, \delta_s) = (2, 0)$: Two degrees of mobility and zero steerable wheels.

03.4

See matlab code.

03.5

Given the bicycle-like kinematic model:

$$\dot{q} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ \tan\psi / l & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}$$

a) Analysis When $\psi \to \frac{\pi}{2}$

When $\psi \to \frac{\pi}{2}$, the term $\tan\psi$ tends to infinity. As a result, the angular rate $\dot{\theta} = \frac{\tan\psi}{l} v$ becomes unbounded. This implies that the steering angle causes extreme turning behavior, and the model becomes singular, losing physical interpretability.

b) Input Transformation

The given input transformation is:

$$v_f = \frac{v}{\cos\psi}$$

From this, we can express v as:

$$v = v_f \cos\psi$$

Substituting this into the original model:

$$\dot{q} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ \tan\psi/l & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_f \cos\psi \\ \omega \end{bmatrix}$$

Simplifying each component:

$$\dot{x} = \cos\theta \cdot v_f \cos\psi = v_f \cos\theta \cos\psi$$

$$\dot{y} = \sin\theta \cdot v_f \cos\psi = v_f \sin\theta \cos\psi$$

$$\dot{\theta} = \frac{\tan\psi}{l} \cdot v_f \cos\psi = \frac{\sin\psi}{l} \cdot v_f$$

$$\dot{\psi} = \omega$$

c) Verification that v_f is the Speed of the Front Wheel

The front wheel's position C is given by:

$$C = \begin{pmatrix} x + l\cos\theta \\ y + l\sin\theta \end{pmatrix}$$

The velocity of point C is:

$$\text{Velocity of } C = \frac{d}{dt}\begin{pmatrix} x + l\cos\theta \\ y + l\sin\theta \end{pmatrix} = \begin{pmatrix} \dot{x} - l\sin\theta\dot{\theta} \\ \dot{y} + l\cos\theta\dot{\theta} \end{pmatrix}$$

Substitute the expressions:

$$\dot{x} = v_f \cos\theta\cos\psi, \quad \dot{y} = v_f \sin\theta\cos\psi, \quad \dot{\theta} = \frac{\sin\psi}{l} v_f$$

$$\text{Velocity of } C = \begin{pmatrix} v_f \cos\theta\cos\psi - l\sin\theta\left(\frac{\sin\psi}{l} v_f\right) \\ v_f \sin\theta\cos\psi + l\cos\theta\left(\frac{\sin\psi}{l} v_f\right) \end{pmatrix}$$

Simplifying:

$$= \begin{pmatrix} v_f \cos\theta\cos\psi - v_f \sin\theta\sin\psi \\ v_f \sin\theta\cos\psi + v_f \cos\theta\sin\psi \end{pmatrix}$$

$$= v_f \begin{pmatrix} \cos\theta\cos\psi - \sin\theta\sin\psi \\ \sin\theta\cos\psi + \cos\theta\sin\psi \end{pmatrix}$$

Using trigonometric identities:

$$\cos(\theta - \psi) = \cos\theta\cos\psi - \sin\theta\sin\psi$$

$$\sin(\theta - \psi) = \sin\theta\cos\psi + \cos\theta\sin\psi$$

Therefore:

$$\text{Velocity of } C = v_f \begin{pmatrix} \cos(\theta - \psi) \\ \sin(\theta - \psi) \end{pmatrix}$$

Magnitude of the Velocity

The magnitude of the velocity vector is:

$$\text{Magnitude} = \sqrt{(v_f\cos(\theta - \psi))^2 + (v_f\sin(\theta - \psi))^2}$$

$$= v_f\sqrt{\cos^2(\theta - \psi) + \sin^2(\theta - \psi)} = v_f \times 1 = v_f$$

03.6

a

We define the state vector $\mathbf{x}$ and the input vector $\mathbf{u}$ as:

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u_R \\ u_L \end{bmatrix}$$

The state-space model is given by:

$$\dot{\mathbf{x}} = \mathbf{A}(\theta)\mathbf{u}$$

where:

$$\mathbf{A}(\theta) = \begin{bmatrix} \frac{r_R}{2}\cos\theta & \frac{kr_R}{2}\cos\theta \\ \frac{r_R}{2}\sin\theta & \frac{kr_R}{2}\sin\theta \\ \frac{r_R}{b} & -\frac{kr_R}{b} \end{bmatrix}$$

This model describes the evolution of the robot's position (x, y) and orientation θ over time, given the angular velocities u_R and u_L of the right and left wheels.

b

The state-space model for a differential drive robot (DDR) with uneven wheel radii r_R and $r_L = k \cdot r_R$ is:

$$\dot{\mathbf{x}} = \mathbf{A}(\theta)\mathbf{u}$$

where:

$$\mathbf{u} = \begin{bmatrix} u_R \\ u_L \end{bmatrix}, \quad \mathbf{A}(\theta) = \begin{bmatrix} \frac{r_R}{2}\cos\theta & \frac{kr_R}{2}\cos\theta \\ \frac{r_R}{2}\sin\theta & \frac{kr_R}{2}\sin\theta \\ \frac{r_R}{b} & -\frac{kr_R}{b} \end{bmatrix}$$

With $u_R = u_L = \bar{u}$:

$$\dot{x} = \frac{r_R(1+k)}{2}\cos\theta\bar{u}$$

$$\dot{y} = \frac{r_R(1+k)}{2}\sin\theta\bar{u}$$

$$\dot{\theta} = \frac{r_R(1-k)}{b}\bar{u}$$

2. Model with Even Wheel Radii: For a DDR with both wheels having the same radius r_R:

$$\dot{\mathbf{x}} = \begin{bmatrix} r_R\cos\theta\bar{u} \\ r_R\sin\theta\bar{u} \\ 0 \end{bmatrix}$$

Comparison of Velocities

- **Uneven Wheel Radii:**

$$\dot{x} = \frac{r_R(1+k)}{2}\cos\theta\bar{u}, \quad \dot{y} = \frac{r_R(1+k)}{2}\sin\theta\bar{u}, \quad \dot{\theta} = \frac{r_R(1-k)}{b}\bar{u}$$

- **Even Wheel Radii:**

$$\dot{x} = r_R\cos\theta\bar{u}, \quad \dot{y} = r_R\sin\theta\bar{u}, \quad \dot{\theta} = 0$$

For **even wheel radii**, the robot moves in a straight line with no rotation when $u_R = u_L$. For **uneven wheel radii**, the robot moves forward and rotates, with $\dot{\theta}$ depending on the radii difference.

c

We aim to find the input transformation matrix $\mathbf{T}(k)$ such that the kinematic model with unequal wheel radii behaves like a model with equal wheel radii when given the same inputs.

Given Models

1. **Model for Unequal Wheel Radii:**

$$\dot{x} = \frac{r_R(u_R + ku_L)}{2}\cos\theta$$

$$\dot{y} = \frac{r_R(u_R + ku_L)}{2}\sin\theta$$

$$\dot{\theta} = \frac{r_R(u_R - ku_L)}{b}$$

2. **Model for Equal Wheel Radii:**

$$\dot{x} = r_R\cos\theta\bar{u}$$

$$\dot{y} = r_R\sin\theta\bar{u}$$

$$\dot{\theta} = 0$$

where $\bar{u} = \bar{u}_R = \bar{u}_L$ is the common input for the equal radii case.

Input Transformation

We want to find $\mathbf{T}(k)$ such that:

$$\mathbf{u} = \mathbf{T}(k)\mathbf{u}_{\text{eq}}$$

where:

$$\mathbf{u} = \begin{bmatrix} u_R \\ u_L \end{bmatrix}, \quad \mathbf{u}_{\text{eq}} = \begin{bmatrix} \bar{u}_R \\ \bar{u}_L \end{bmatrix} = \begin{bmatrix} \bar{u} \\ \bar{u} \end{bmatrix}$$

$$\frac{r_R(u_R + ku_L)}{2} = r_R\bar{u}$$

Simplifying:

$$u_R + ku_L = 2\bar{u}$$

For angular velocity

$$\frac{r_R(u_R - ku_L)}{b} = 0$$

Simplifying:

$$u_R - ku_L = 0$$

Solving for u_R and u_L

Adding and subtracting the equations:

- Adding: $u_R + ku_L + u_R - ku_L = 2\bar{u}$ gives:

$$2u_R = 2\bar{u} \implies u_R = \bar{u}$$

- Subtracting: $u_R + ku_L - (u_R - ku_L) = 2ku_L$ gives:

$$2ku_L = 2k\bar{u} \implies u_L = \bar{u}$$

Input Transformation Matrix

We can write this in matrix form:

$$\begin{bmatrix} u_R \\ u_L \end{bmatrix} = \mathbf{T}(k) \begin{bmatrix} \bar{u}_R \\ \bar{u}_L \end{bmatrix}$$

where:

$$\mathbf{T}(k) = \begin{bmatrix} \frac{1}{1+k} & 0 \\ 0 & \frac{1}{1-k} \end{bmatrix}$$