Exercise Solutions for Ex 02

Abdulrahman Hameshli

Matrikelnummer: 3768284

Solution

1 Ex 02.1

Part (a)

Given probabilities:

$$p(r) = 0.2, \quad p(b) = 0.2, \quad p(g) = 0.6$$

Probabilities of selecting an apple from each box:

$$P(\text{apple}|r) = \frac{3}{10}, \quad P(\text{apple}|b) = \frac{1}{2}, \quad P(\text{apple}|g) = \frac{3}{10}$$

Using the law of total probability:

$$P(\text{apple}) = 0.2 \cdot \frac{3}{10} + 0.2 \cdot \frac{1}{2} + 0.6 \cdot \frac{3}{10}$$

Calculating each term:

$$P(\text{apple}) = 0.06 + 0.1 + 0.18 = 0.34$$

Thus, the probability of selecting an apple is **0.34** or 34%.

Part (b):

Probabilities of selecting an orange from each box:

$$P(\text{orange}|r) = 0.4, \quad P(\text{orange}|b) = 0.5, \quad P(\text{orange}|g) = 0.3$$

Total probability of selecting an orange:

$$P(\text{orange}) = 0.2 \cdot 0.4 + 0.2 \cdot 0.5 + 0.6 \cdot 0.3 = 0.36$$

Using Bayes' theorem:

$$P(g|\text{orange}) = \frac{0.6 \cdot 0.3}{0.36} = 0.5$$

Thus, the probability that the selected orange came from the green box is **0.5** or 50%.

2 Ex 02.2

We want to calculate $p(x_i|z)$ for each position $X = (x_1, x_2, x_3)$ using Bayes' theorem:

$$p(x_i|z) = \frac{p(z|x_i) \cdot px_i)}{p(z)}$$

- Conditional probabilities:

$$p(z|x_1) = 0.8, \quad p(z|x_2) = 0.4, \quad p(z|x_3) = 0.1$$

- Prior probabilities:

$$p(x_1) = p(x_2) = p(x_3) = \frac{1}{3}$$

$$p(z) = \left(0.8 \cdot \frac{1}{3}\right) + \left(0.4 \cdot \frac{1}{3}\right) + \left(0.1 \cdot \frac{1}{3}\right) = \frac{1.3}{3} \approx 0.433$$

- For x_1:

$$p(x_1|z) = \frac{0.8 \cdot \frac{1}{3}}{0.433} \approx 0.615$$

- For x_2:

$$p(x_2|z) = \frac{0.4 \cdot \frac{1}{3}}{0.433} \approx 0.308$$

- For x_3:

$$p(x_3|z) = \frac{0.1 \cdot \frac{1}{3}}{0.433} \approx 0.077$$

$$p(x_1|z) \approx 0.615, \quad p(x_2|z) \approx 0.308, \quad p(x_3|z) \approx 0.077$$

3 02.3

a

$$\begin{aligned}
E[x+z] &= \int_{-\infty}^{\infty} (x+z)\, f(x,z)\, dx\, dz \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x+z) f_x(x) f_z(z)\, dx\, dz \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_x(x) f_z(z)\, dx\, dz + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z f_x(x) f_z(z)\, dx\, dz \\
&= \int_{-\infty}^{\infty} x f_x(x)\, dx \int_{-\infty}^{\infty} f_z(z)\, dz + \int_{-\infty}^{\infty} z f_z(z)\, dz \int_{-\infty}^{\infty} f_x(x)\, dx \\
&= E[x] + E[z]
\end{aligned}$$

$$
\begin{aligned}
\mathrm{Var}[x+z] &= E[(x+z)^2] - (E[x+z])^2 \\
&= E[x^2 + 2xz + z^2] - (E[x] + E[z])^2 \\
&= E[x^2] + 2E[xz] + E[z^2] - (E[x]^2 + 2E[x]E[z] + E[z]^2) \\
&= (E[x^2] - E[x]^2) + (E[z^2] - E[z]^2) \\
&= \mathrm{Var}[x] + \mathrm{Var}[z]
\end{aligned}
$$

3.1 b

$$
\begin{aligned}
\mathrm{Cov}[x,y] &= E[(x-\mu_x)(y-\mu_y)] \\
&= E[xy - x\mu_y - y\mu_x + \mu_x\mu_y] \\
&= E[xy] - E[x]\cdot\mu_y - E[y]\cdot\mu_x + \mu_x\cdot\mu_y \\
&= E[xy] - E[x]\cdot E[y] - E[y]\cdot E[x] + E[x]\cdot E[y] \\
&= E[xy] - E[x]\cdot E[y] \\
&= 0 \quad (\text{if } x \text{ and } y \text{ are independent, so } E[xy] = E[x]\cdot E[y])
\end{aligned}
$$

3.2 c

The matrix must be symmetric and positive semi-definite:

Symmetry: The matrix is symmetric because it is equal to its transpose, which means that its entries are mirrored across the main diagonal.

Positive Semi-Definiteness: We calculate the eigenvalues to determine if the matrix is positive semi-definite.

```
>> eig([9 4 ;4 1])

ans =

   -0.6569
   10.6569
```

Figure 1: Eigenvalues of the matrix

Since one of the eigenvalues is negative, the matrix is not positive semi-definite.

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For answers see file ex2.m

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1

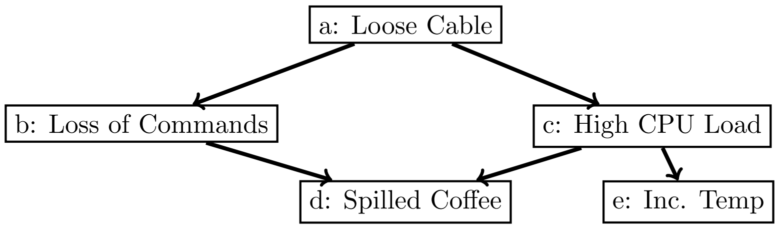

Figure 2: Probabilistic Graphical Model of Malfunctioning Coffee-Serving Robot

2

Graph A:

$$p(x) = p(x_1)p(x_2)p(x_3)p(x_4)p(x_5|x_1,x_2,x_3,x_4)p(x_6|x_2,x_3)p(x_7|x_1,x_3,x_4)p(x_8|x_5,x_6)p(x_9|x_5,x_6,x_7) \tag{1}$$

Grapg B:

$$p(x) = p(x_1)p(x_2)p(x_3)p(x_4|x_1,x_2,x_3)p(x_5|x_2)p(x_6|x_4,x_5)p(x_7|x_5)p(x_8|x_4,x_5)p(x_9|x_6,x_8)p(x_{10}|x_7,x_8) \tag{2}$$