Research statement

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1 Introduction

My broad field of study is stochastic analysis, with a particular focus on singular SPDEs, constructive quantum field theories (QFTs), and rough analysis. I am also interested in areas such as stochastic control and regularization by noise, which I hope to explore further if the opportunity arises. My current research focuses on quantum gauge theory, in particular two and three dimensional Yang-Mills theories. The Yang-Mills theories, or broadly, QFTs are fundamental to our understanding of the physical universe. They form the basis of the Standard Model of particle physics. Quantum gauge theories, a class of QFTs, have an important feature, namely gauge symmetries which somehow describe how particles interact or change under coordinate transformation.

On the mathematical side, quantum gauge theories have inspired the development of numerous sophisticated mathematical frameworks and techniques. In particular, the problem that my research is based on the study, and possibly constructing, the Yang Mills measure. The setting starts by fixing a principal G-bundle with a base smooth manifold M and a compact structure Lie group G. On this principal bundle, which we simply assume to be trivial, one can obtain connection 1-forms $A \in \Omega^1(M, \mathfrak{g})$ (where $\mathfrak{g}$ is the Lie algebra of G). The Yang-Mills measure in this setting is formally given by the ill-defined expresssion

$$\mu_{\mathrm{YM}}(\mathrm{d}A) = \frac{1}{Z} \exp\left(-\int_M |F^A(x)|_{\mathfrak{g}}^2 \,\mathrm{d}x\right) \mathrm{d}A.$$

Making sense of this expression is a central problem in (probabilistic) QFTs.

There are many works in defining and studying the measure, most interesting works are limited to two—and occasionally three—dimension. The physically relevant case, however, lies in four dimensions, which presents significant challenges and is the basis for the Millennium Prize Problem known as the Yang-Mills existence and mass gap problem. As the list of references is long, I only mention the ones that are relevant for my research. For example, the works [**?**] and [**?**] study the measure in the case $M = \mathbb{R}^2$ and exploit the topological property of $\mathbb{R}^2$ to relate the measure to some Gaussian process. The parallel transport of the connection form sampled from the measure, also called the holonomy, solve stochastic differential equations. That yields the study of holonomies as Lie group valued Brownian motions. The relation between axial gauge and the so-called lasso fields from [?] is crucial in obtaining a gauge fixing procedure for the 2D Yang-Mills measure on the unit square (see Section 2.1). Later the measure is constructed via various techniques, similar to the one just mentioned in [?], and later with a graph approximation in [?].

Most of these works consider the holonomy and the connection form in the so-called axial gauge (e.g. in the cases of $\mathbb{R}^2$ or $[0,1]^2$). The regularity for the connection form in this case, at least if one could obtain one from the holonomy, has a Holder-Besov regularity of $C^{-1/2^-}$. By a suitable gauge fixing procedure in [?] it is shown that one can actually obtain a connection form in C^{0^-}. This was a result, or rather a corollary from a rough Uhlenbeck compactness on $\mathbb{T}^2$ proved in [?]. The result therein uses lattice approximation of the measure. One of the extensions that I worked on during my PhD is to reprove the result without using lattices, but instead rely on singular SPDE techniques (see Section 2.1). Although my approach to using singular SPDEs differs slightly, significant progress in studying the Yang-Mills measure via singular SPDEs—particularly through the Langevin dynamic—has also been made in the incomplete list of works such as [?, ?, ?, ?, ?]. Techniques from the theory of regularity structures [?] and paracontrolled calculus [?], as utilized in the aforementioned works, are also central to my research.

Another direction that one could study the measure, is through enhancing the understanding the underlying geometry. It is known in classical gauge theory that a connection 1-form A can be equivalently be understood as a covariant derivative d_A. That is the second focus for my PhD thesis. We try to generalise well-known ideas from the smooth setting to cases where the connection form is distributional, particularly $A \in C^{0^-}$ in 2D and extend the results to $A \in C^{-1/2^-}$ in 3D. More precisely, we study the operator $\mathrm{d}_A \oplus \mathrm{d}_A^*$ and its properties as an unbounded operator on a suitable Banach space (see Section 2.2).

I am planning to work on problems related to or arising from my PhD research, for example generalizing these results to different geometries or different dimensions to deepen our understanding of Yang-Mills theories (see Section 3 for more details). At the same time, I am open to exploring other directions within stochastic analysis that may diverge from or build upon my current work.

2 PhD Research Overview

2.1 Rough Uhlenbeck compactness on the unit square

This is a joint work in progress with Ilya Chevyrev (UoE) and Tom Klose (UoO).

In this work we generalise the works Chevyrev 2019 using continuum PDE techniques inspired by the original paper [?]. First we solve for Coulomb gauge $\mathrm{d}^*A = 0$ on smaller squares, and then patch the solutions in each small square together to obtain a globally defined connection form. The latter is very similar to what was also done in [?]. Our techniques differs in the sense that we solve for the Coulomb gauge directly, instead of using implicit existence results.

Classically, the result is essentially under smallness assumptions, one can find a gauge transformation such that

$$\|A^g\|_{W^{1,p}} \lesssim \|F^A\|_{L^p}.$$

In the setting of 2D Yang-Mills theory, the curvature has the same regularity as white noise which makes the L^p-norm not applicable. Furthermore, generally Besov norms of the curvature is not gauge invariant, whereas L^p-norms are. Instead, we consider so-called Lasso fields, which are somehow related to the curvature. These are gauge invariant up to a trivial action G.

The distribution of the lasso fields L^A for the 2D YM measure on the square is explicit, namely same as white noise. The lasso fields relate to axial gauge representations $\bar{A}$, in the sense that

$$\bar{A} = \int L^A.$$

In our work we consider norms for the axial gauge representative.

We first develop the theory of rough additive functions. This generalises the notion of additive functions from [?] (later generalised in [?]). The definition of additive functions to rough additive functions, is how Young integration is to rough integration. There is a natural notion for gauge transformations for such rough additive functions inspired by controlled rough paths. Then our Uhlenbeck compactness boils down whether one can transform a rough additive function in the axial gauge to the Coulomb gauge.

The latter yields a singular SPDE. We wish to solve for g such that $\mathrm{d}^* A^g = 0$. Recall that $A^g = \mathrm{Ad}_g A - \mathrm{d}g g^{-1} = \mathrm{Ad}_g A + 0^g$. We set up a system of SPDEs for essentially Ad_g and 0^g. As the bound, that we want to prove, does not care about the way we solve for these equations, or whether we solve any equation, this causes a freedom. We use the equations carefully as well with boundary conditions. We use the theory of regularity structures to solve it. However, there is so many adaptations that we have to do from the classical black box theory.

Firstly, our equation is elliptic. While this per se not that problematic, we have boundary conditions, which cause technical issues which breaks down the theory of regularity structures. This leads us to come up with an smartly chosen auxiliary modelled distribution. Secondly, we construct the model from a rough additive function (as well as the auxiliary modelled distribution). This requires a heavy machinary as it is a highly non-trivial task. We introduce, suitable sectors for regularity structure and enhanced Picard iteration which preserves these sectors. This is a sophisticated step which uses a "cosmetic" Da Prato-Debussche trick together with shifting of indices. The construction of model goes via integration identities, derivative identities and symmetry.

Afterwards, we patch the solutions in small squares together to obtain a global solution. This yields a pathwise rough Uhlenbeck compactness theorem which is neater as well as more natural. Without the additional work, or if we were to lean towwards using the black box theory, we would have a more probabilistic statement. Instead, currently, the probabilistic argument is only used in constructing a suitable version of rough additive functions for the Yang-Mills measure.

2.2 State space for 2D Yang-Mills via Dirac operators

This is joint work with Ilya Chevyrev (UoE) and Massimiliano Gubinelli (UoO).

Let $G \subset U(N)$ be a compact Lie group. Let us consider a trivial principal G-bundle with base manifold $\mathbb{T}^2$. We can consider an associated vector bundle via a representation $G \to \mathrm{GL}(\mathbb{C}^N)$. It is known that connection forms A can be viewed as a covariant derivative on a suitable associated vector bundle. In this project, we study properties of the space of connection forms via using covariant derivative d_A, or rather more precisely, the operator $\mathrm{D}_A := \mathrm{d}_A \oplus \mathrm{d}_A^*$ acting on the exterior algebra $\Omega = \bigoplus_{k=0}^{2} \Omega^k(\mathbb{T}^2, \mathbb{C}^N)$.

For a Banach space X of functions we denote by ΩX the space of differential forms constitued by functions in X. We view the operator D_A as an unbounded operator on ΩL^2. We define a suitable domain for D_A to make it a closed, self-adjoint operator with compact resolvent. We can show that the spectrum consists of discrete set eigenvalues with finite multiplicities. We can use suitable resolvent estimates and expansions to show that the orbit space $\Omega C^{\alpha-1}/\mathfrak{G}^{\alpha}$ is Polish.

Furthermore, we can establish a natural gauge invariant observables via spectral properties in the case of $G = U(N)$ and recover gauge transformations via these observables in this case.

3 Future Research Directions

In the future, I am planning to extend the results of my PhD thesis to more sophisticated settings. Broadly speaking, I am always open to explore directions in rough analysis that are not necessarily mentioned in this list.

3.1 Rough Uhlenbeck compactness on closed surfaces

This project builds on the rough Uhlenbeck compactness in my PhD thesis, extending them to a generic closed smooth surface M as the base manifold of the trivial principal G-bundle. I have been developing this in parallel, with the possibility of including preliminary findings in my thesis.

As in the classical setting, the main work involves proving the existence of the Coulomb gauge on a small Euclidean ball; in our setting, however, it suffices to consider a small Euclidean square. We need to generalise the metric of connection forms we have to use. The idea is to consider a suitably graph of the manifold and consider the supremum of the metric on each face (as the faces are homeomorphic to squares). We have to make sure that each face of the graph is small enough (in fact smaller than the injectivity radius). On each face, we use the techniques developed in the PhD thesis. We consider a further a subgraph on each face. Each subfaces is then mapped to a Euclidean square on which we have the smallness assumptions to find a Coulomb gauge as done in the rough Uhlenbeck compactness on the unit square.

The main difficulty in applying this to Yang-Mills theory lies in the fact that, unlike the square case, the lasso field—or effectively, the curvature—does not behave as white noise.

Instead, we work with a conditional measure on the initial graph of the manifold. On each face, we introduce a lasso field and derive bounds based on the conditional measure, which, while not strictly Gaussian, possesses comparable Gaussian bounds.

3.2 State space for 3D Yang-Mills via Dirac operators

Even though, the (non-Abelian) 3D Yang-Mills measure is not constructed yet, it is very relevant to study a potential space where the measure is supported on. In fact, to construct the measure via the stochastic quantisation, it is crucial to have a state space, e.g. Polish, to hope that one has nice (Markovian) properties and possibly invariant measure for which it converges to. One needs to somehow obtain a gauge group $\mathfrak{G}$ that acts on the possible state space $\mathcal{S}$ that the dynamic takes values in. This gauge group is likely be depending on the element $s \in \mathcal{S}$, which gives it likely a fibre bundle structure. There is a candidate space in [?] which is lacking a gauge group $\mathfrak{G}$. There is an equivalence relation, resembling classical gauge equivalence, defined by means of the Yang-Mills heat flow. This approach could poissbly be developed further by incorporating on one hand rough line integration, i.e. rough additive functions, from rough Uhlenbeck compactness and on the other hand the covariant derivative from Section 2.2. Even though, incorporating rough additive functions is something my advisor Ilya and I have tried during my PhD, we realised there is "more structure" that we somehow were missing.

Using the Dirac operator approach, we hope to have an alternative approach. As we have seen in Section 2.2, we were able to prove Polishness of the quotient space by merely using resolvent analysis. While such approach does not naturally carry over, I hope it is still worth exploring its boundaries. In fact, one difficulty is that the connection form in 3D is expected to be of regularity $C^{-1/2^-}$ which makes the definition of D_A really a singular operator in the sense that the resolvent equation is a singular (S)PDE. Luckily, this by itself will not cause issues as we know how to solve it. The issue is that the gauge transformation also require additional structure. This structure is not clear yet. On the other hand the gauge transformation on covariant derivative is $g\mathrm{D}_A g^{-1}$ which as an operator makes sense without too much regularity on g. For example, on L^2, we can simply take $g : \mathbb{T}^2 \to G$ measurable. We do not exclude that for the gauge transformation one might need the theory of rough additive functions. It is likely that combining these two approaches will maximise the gain.

3.3 Covariant GFF and constructing conditional Higgs measure

With Dr. Ajay Chandra, we are planning to study conditional Higgs measure, by first studying a covariant GFF. That is, for a given (rough) A, we would like to study the Gibbs-type measure

$$\mu_A(\mathrm{d}\Phi) = \frac{1}{Z_A} \exp\left(- \int_{\mathbb{T}^2} |\mathrm{d}_A \Phi(x)|^2 + c_A |\Phi(x)|^2 \mathrm{d}x \right) \mathrm{d}\Phi.$$

Of course, the way it is written, this measure is ill-defined. This measure is representing a Gaussian free field with covariance kernel given by the Green function for $(\mathrm{d}_A^* \mathrm{d}_A + c_A)$. There are several ideas that one could try: study the measure via the covariant Laplacian, Dirichlet form or through the Feynman-Kac formulation of the covariance kernel. In [?] the authors study the measure when A is smooth, so the objective of this work is to extend their work to a A of regularity C^{0^-}.

We first try to construct this measure in the Abelian case and later extend it to non-Abelian setting. The way to rigorously construct this measure is by first taking a mollified $A^\varepsilon \to A$ and consider the measure μ_{A^ε}. In the application we have in mind A is essentially Gaussian free field. In this specific setting, we expect to need to take $c_{A^\varepsilon} \to -\infty$ as a renomralisation (indeed, this can be seen by writing the covariant Laplacian in coordinates).

Studying of the covariant Laplacian requires to set up a suitable domain and studying the spectral properties. This is very similar to what I have worked on in Section 2.2 (but the operators considered are similar but also different). It usually common to alternatively study the Dirichlet form corresponding to the Laplacian. In our setting, it would be

$$\mathcal{E}_A(\varphi, \psi) = (\mathrm{d}_A \varphi, \mathrm{d}_A \psi) + c_A(\varphi, \psi).$$

The issue that could arise is that this operator is non-degenerate as we are going to take $c_A < 0$.

The last approach relies on the Feynman-Kac representation of the Green function (e.g. see [?]). Our objective is then to study the problem via stochastic differential equations and stochastic integrals of the form $\int A(X_t) \cdot \mathrm{d}X_t$ for the connection form $A \in C^{0^-}$, say $\operatorname{div} A = 0$, ensuring the integral coincides with the Stratonovich integral. Although this integral is not well-defined through Itô calculus, assuming A exhibits properties similar to a Gaussian free field (for instance, integrability along lines), we can explore the possibility of combining rough paths and stochastic sewing techniques to define this integral or at least some of its expected values. Of course, we always start by considering mollified A^ε, and at least for the expected values, there is hope as there is c_{A^ε} which can compensate divergences to keep everything stable in a suitable topology for the connection form.

3.4 Alternative proof for rough Uhlenbeck compactness in 2D

The motivation for developing an alternative proof of rough Uhlenbeck compactness is not merely to replicate the result through numerous methods. Instead, this enhances our understanding of singular SPDE techniques, rough differential geometry, and gauge-invariant quantities. For example, one could try to prove using techniques relying on implicit function theorem as classically done in [?] which is not currently available for singular SPDEs. In fact, in the project Section 2.1 from my PhD thesis, we developed several interesting techniques to reach this result, and I believe that redoing the proof could reveal even more valuable insights.

The gauge-invariant observable that we use is unfortunately not easily applicable for the Langevin dynamic. Changing the observable to a different quantity and reporving the same result could be more natural for the Langevin dynamic. One example could be the L^p-norm of the curvature smoothed by the Yang-Mills heat flow with some blow-up controlled suitably in the time parameter. Another one could be spectral quantities that can be obtained from Dirac operator D_A, for example trace of the heat kernel.

Further benefits for tweaking the result to accommodate the 2D Langeivn dynamic is that it could initiate proving pathwise global existence of the orbits under the dynamic as well as global existence for the non-Abelian Yang-Mills-Higgs. Furthermore, it could enhance our understanding of the 3D Yang-Mills measure. For instance, the result that we currently have in Section 2.1 using the axial gauge relation to lasso field which is very special property of 2 dimensions (in fact, we know even on $\mathbb{T}^2$, we do not have such relation).

3.5 Extending regularity structures for boundary singularities

In this project I am planning to extend some definitions in regularity structures [**?**] to allow for singular objects arising because of boundary considerations. The main example that I have in mind is parabolic SPDEs where one cannot classically close the fixed point equation in the space of modelled distributions. This has occurred in [**?**] for example where the treatment was an additional Da-Prato Debussche step. Generally, if one has situations where one has a large amount of elements that require to be dealt with manually, say if one would consider a Langevin dynamic in a similar framework as [**?**], then one needs a unifying technique to deal with such problems.

Furthermore, inspiration arising from equations with spatial boundary, where the boundary values has some noise which has some correlations with interior noise. This has occured in the project of Section 2.1, where we had a particular technique to deal with. Later this could require a special version of [**?**] to deal with these stochastic objects which could be a natural follow-up project.

3.6 Exploring rough Calderon problems

This is a more exploratory project. Calderon type problems, or inverse problem already arises in the study of the operator $\mathrm{D}_A = \mathrm{d}_A \oplus \mathrm{d}_A^*$ from Section 2.2. One could try to obtain information of A by studying D_A, as well as stability results, i.e. some kind of continuity of the map $\mathrm{D}_A \mapsto A$. There is many results for operators of the form $\Delta + \zeta$ as well as recently for covariant Laplacian $\mathrm{d}_A^*\mathrm{d}_A$ and characterising up to gauge transformation [**?**]. These questions are not precisely the same as the one I have mentioned, but it regardless poses the question: what happens when ζ is a 2D white noise, or A is Gaussian free field on $\mathbb{T}^2$? Of course, there are issues: firstly the corresponding equations require singular SPDE technique, secondly boundary conditions, specifically, the Dirichlet-to-Neumann map, might require special effort to be understood.

References

[FHL21] Peter K Friz, Antoine Hocquet, and Khoa Lê. Rough stochastic differential equations. *arXiv preprint arXiv:2106.10340*, 2021.