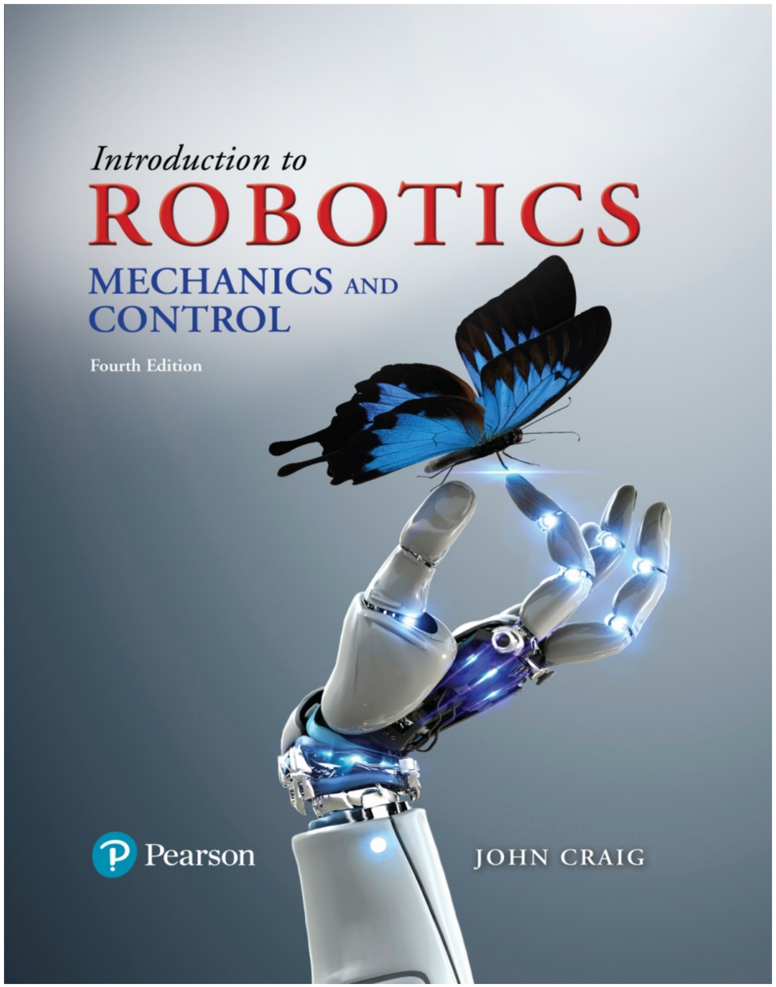
Introduction to Robotics

Mechanics and Control

4th Edition

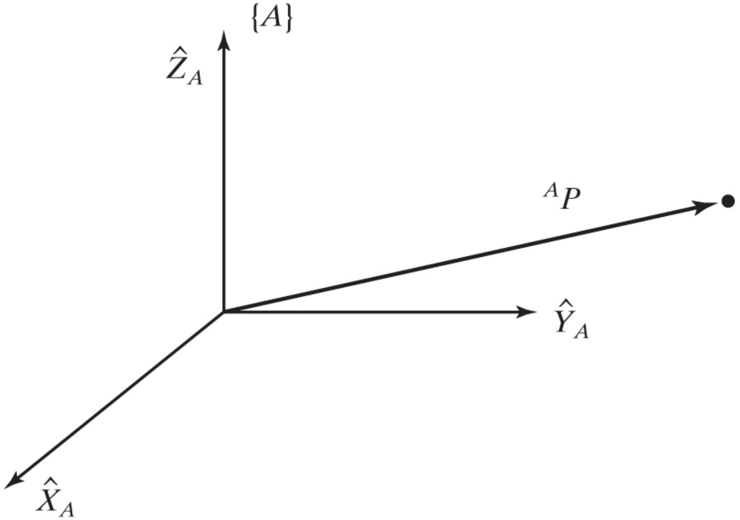


Spatial Descriptions and Transformations

A description is used to specify attributes of various objects with which a manipulation system deals. These objects are parts, tools, and the manipulator itself. In this section, we discuss the **description of positions, of orientations,** and of an entity that contains both of these descriptions: the **frame**.

Description of a position

Once a coordinate system is established, we can locate any point in the universe with a 3 x 1 position vector. Because we will often define many coordinate systems in addition to the universe coordinate system, vectors must be tagged with information identifying which coordinate system they are defined within. In this course, vectors are written with a leading superscript indicating the coordinate system to which they are referenced (unless it is clear from context)—for example, ${}^{A}P$. This means that the components of ${}^{A}P$ have numerical values that indicate distances along the axes of {A}. Each of these distances along an axis can be thought of as the result of projecting the vector onto the corresponding axis.



Vector relative to frame (example).

The figure pictorially represents a coordinate system, {A}, with three mutually orthogonal unit vectors with solid heads. A point ${}^{A}P$ is represented as a vector and can equivalently be thought of as a position in space, or simply as an ordered set of three numbers. Individual elements of a vector are given the subscripts x, y, and z:

$$ {}^{A}P = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}. $$

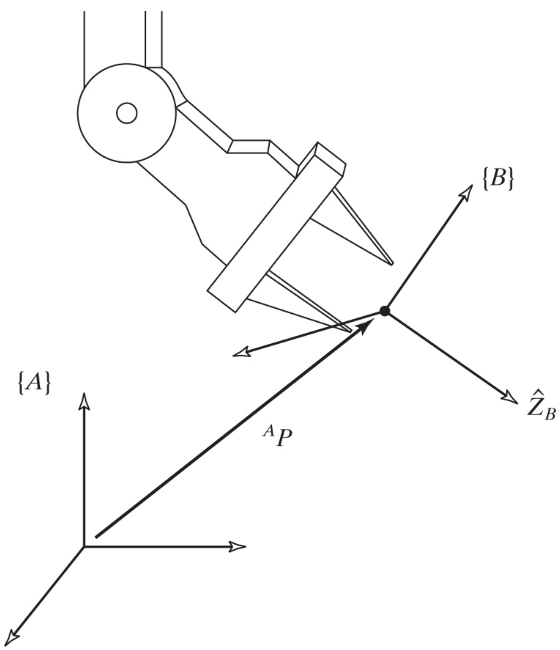
In summary, we will describe the position of a point in space with a position vector.

Other 3-tuple descriptions of the position of points, such as spherical or cylindrical coordinate representations exist.

Description of an orientation

Often, we will find it necessary not only to represent a point in space but also to describe the orientation of a body in space. For example, if vector ^{A}P locates the point directly between the fingertips of a manipulator's hand, the complete location of the hand is still not specified until its orientation is also given. Assuming that the manipulator has a sufficient number of joints, the hand could be oriented arbitrarily while keeping the point between the fingertips at the same position in space. In order to describe the orientation of a body, we will attach a coordinate system to the body and then give a description of this coordinate system relative to the reference system. Coordinate system {B} has been attached to the body in a known way. A description of {B} relative to {A} now suffices to give the orientation of the body.

Thus, positions of points are described with vectors and orientations of bodies are described with an attached coordinate system. One way to describe the body attached coordinate system, {B}, is to write the unit vectors of its three principal axes in terms of the coordinate system {A}.



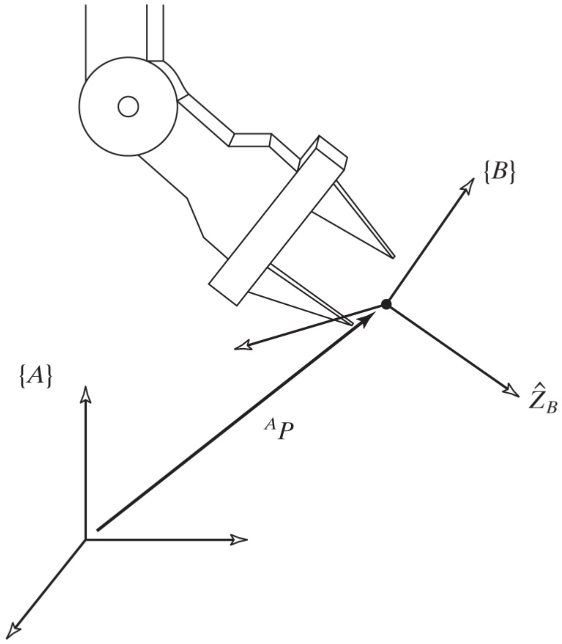
We denote the unit vectors giving the principal directions of coordinate system $\{B\}$ as $\hat{X}_B$, $\hat{Y}_B$, and $\hat{Z}_B$. When written in terms of coordinate system $\{A\}$, they are called $^{A}\hat{X}_B$, $^{A}\hat{Y}_B$, and $^{A}\hat{Z}_B$. It will be convenient if we stack these three unit vectors together as the columns of a 3×3 matrix, in the order $^{A}\hat{X}_B$, $^{A}\hat{Y}_B$, $^{A}\hat{Z}_B$. We will call this matrix a **rotation matrix**, and, because this particular rotation matrix describes $\{B\}$ relative to $\{A\}$, we name it with the notation $^{A}_{B}R$ (the choice of leading sub- and superscripts in the definition of rotation matrices will become clear in following sections):

$$^{A}_{B}R = \left[\, ^{A}\hat{X}_B \quad ^{A}\hat{Y}_B \quad ^{A}\hat{Z}_B \,\right] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}.$$

Locating an object in position and orientation.

Description of a frame

The information needed to completely specify the whereabouts of the manipulator hand is a position and an orientation. The point on the body whose position we describe could be chosen arbitrarily, however. For convenience, the point whose position we will describe is chosen as the origin of the body-attached frame. **The situation of a position and an orientation pair arises so often in robotics that we define an entity called a frame, which is a set of four vectors giving position and orientation information.** For example one vector locates the fingertip position and three more describe its orientation. Equivalently, the description of a frame can be thought of as a position vector and a rotation matrix. Note that a frame is a coordinate system where, in addition to the orientation, we give a position vector, which locates its origin relative to some other embedding frame. For example, frame {B} is described by ${}^{A}_{B}R$ and ${}^{A}P_{BORG}$, where ${}^{A}P_{BORG}$ is the vector that locates the origin of the frame {B}:



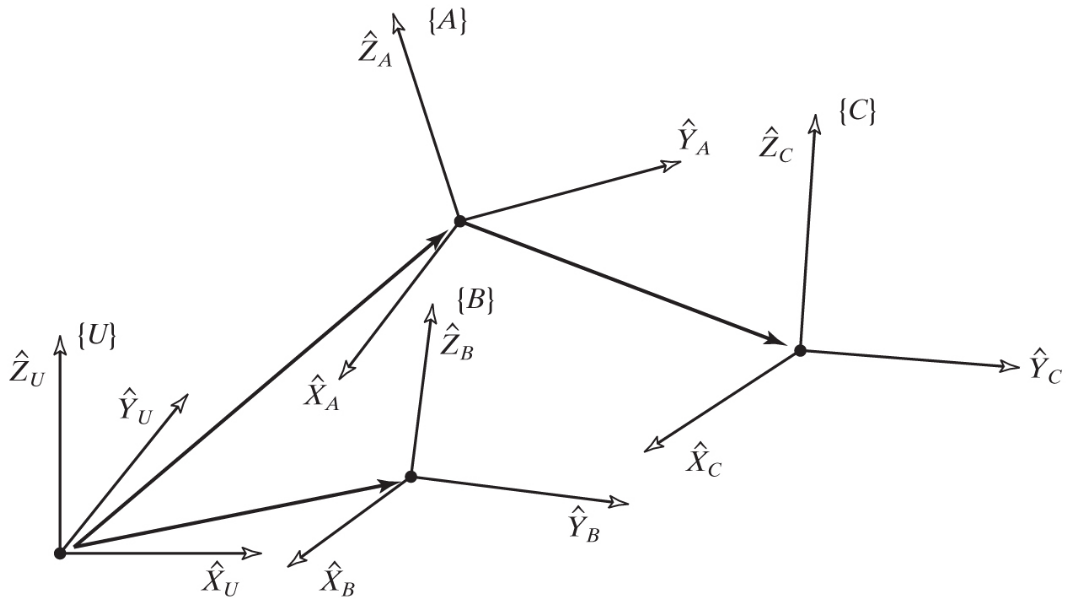
$$\{B\} = \{{}^{A}_{B}R, {}^{A}P_{BORG}\}.$$

Locating an object in position and orientation.

Several frames.

In the figure below there are three frames that are shown along with the universe coordinate system. Frames {A} and {B} are known relative to the universe coordinate system, and frame {C} is known relative to frame {A}. We introduce a graphical representation of frames, which is convenient in visualizing frames. A frame is depicted by three arrows representing unit vectors defining the principal axes of the frame. An arrow representing a vector is drawn from one origin to another. This vector represents the position of the origin at the head of the arrow in terms of the frame at the tail of the arrow. The direction of this locating arrow tells us, for example, that {C} is known relative to {A} and not vice versa.

In summary, a frame can be used as a description of one coordinate system relative to another. A frame encompasses two ideas by representing both position and orientation and so may be thought of as a generalization of those two ideas. Positions could be represented by a frame whose rotation-matrix part is the identity matrix and whose position-vector part locates the point being described. Likewise, an orientation could be represented by a frame whose position-vector part was the zero vector.



Mappings involving translated frames

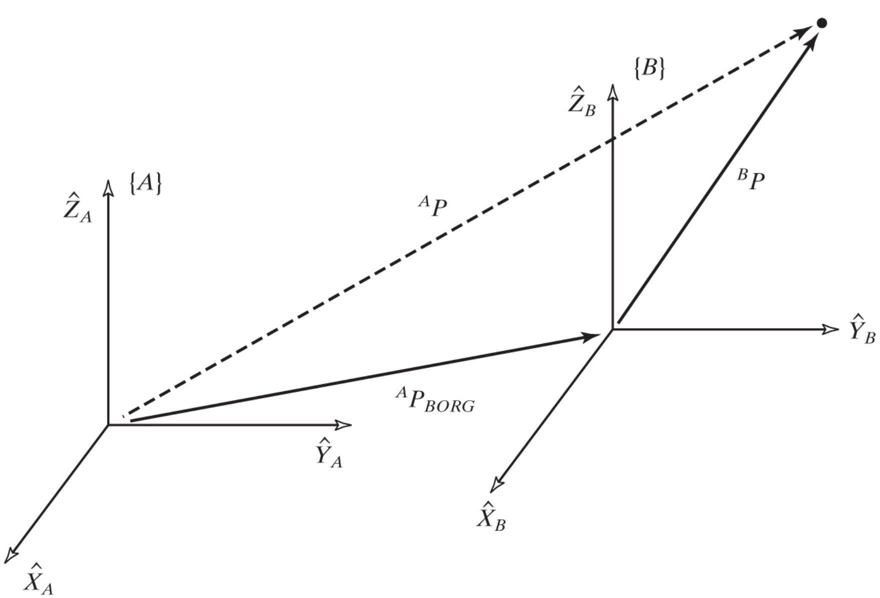
We have a position defined by the vector ${}^{B}P$. We wish to express this point in space in terms of frame {A}, when {A} has the same orientation as {B}. In this case, {B} differs from {A} only by a translation, which is given by ${}^{A}P_{BORG}$, a vector that locates the origin of {B} relative to {A}. Because both vectors are defined relative to frames of the same orientation, we calculate the description of point P relative to {A}, ${}^{A}P$, by vector addition:

$$ {}^{A}P = {}^{B}P + {}^{A}P_{BORG}. $$

Note that only in the special case of equivalent orientations may we add vectors that are defined in terms of different frames.

In this simple example, we have illustrated **mapping** a vector from one frame to another. This idea of mapping, or changing the description from one frame to another, is an extremely important concept. The quantity itself (here, a point in space) is not changed; only its description is changed This is illustrated on the left, where the point described by ${}^{B}P$ is not translated, but remains the same, and instead we have computed a new description of the same point, but now with respect to system {A}.

We say that the vector ${}^{A}P_{BORG}$ defines this mapping because all the information needed to perform the change in description is contained in ${}^{A}P_{BORG}$ (along with the knowledge that the frames had equivalent orientation).



Translational mapping.

Mappings involving rotated frames

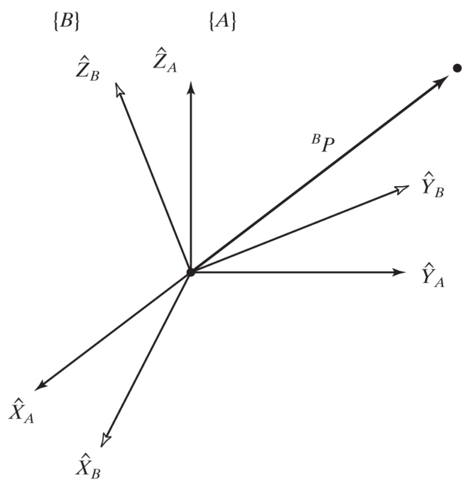
We introduced the notion of describing an orientation by three unit vectors denoting the principal axes of a body-attached coordinate system. For convenience, we stack these three unit vectors together as the columns of a 3 x 3 matrix. We will call this matrix a **rotation matrix**, and, if this particular rotation matrix describes {B} relative to {A}, we name it with the notation ${}^{A}_{B}R$.

Note that, by our definition, the columns of a rotation matrix all have unit magnitude, and, further, that these unit vectors are orthogonal. As we saw earlier, a consequence of this is that

$$ {}^{A}_{B}R = {}^{B}_{A}R^{-1} = {}^{B}_{A}R^{T}. $$

Therefore, because the columns of are the unit vectors of {B} written in {A}, the rows of are the unit vectors of {A} written in {B}. So a rotation matrix can be interpreted as a set of three column vectors or as a set of three row vectors, as follows:

$$ {}^{A}_{B}R = \left[\, {}^{A}\hat{X}_{B} \;\; {}^{A}\hat{Y}_{B} \;\; {}^{A}\hat{Z}_{B} \,\right] = \begin{bmatrix} {}^{B}\hat{X}_{A}^{T} \\ {}^{B}\hat{Y}_{A}^{T} \\ {}^{B}\hat{Z}_{A}^{T} \end{bmatrix}. $$



The situation will arise often where we know the definition of a vector with respect to some frame, {B}, and we would like to know its definition with respect to another frame, (A}, where the origins of the two frames are coincident.

Rotating the description of a vector.

$$
{}^A_BR = \begin{bmatrix} {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \end{bmatrix} = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix}. \tag{2.3}
$$

For brevity, we have omitted the leading superscripts in the rightmost matrix of (2.3). In fact, the choice of frame in which to describe the unit vectors is arbitrary as long as it is the same for each pair being dotted. The dot product of two unit vectors yields the cosine of the angle between them, so it is clear why the components of rotation matrices are often referred to as **direction cosines**.

Further inspection of (2.3) shows that the rows of the matrix are the unit vectors of $\{A\}$ expressed in $\{B\}$; that is,

$$
{}^A_BR = \begin{bmatrix} {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \end{bmatrix} = \begin{bmatrix} {}^B\hat{X}_A^T \\ {}^B\hat{Y}_A^T \\ {}^B\hat{Z}_A^T \end{bmatrix}. \tag{2.4}
$$

Hence, ${}^B_AR$, the description of frame $\{A\}$ relative to $\{B\}$, is given by the transpose of (2.3); that is,

$$
{}^B_AR = {}^A_BR^T. \tag{2.5}
$$

This suggests that the inverse of a rotation matrix is equal to its transpose, a fact that can be easily verified as

$$
{}^A_BR^T \, {}^A_BR = \begin{bmatrix} {}^A\hat{X}_B^T \\ {}^A\hat{Y}_B^T \\ {}^A\hat{Z}_B^T \end{bmatrix} \begin{bmatrix} {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \end{bmatrix} = I_3, \tag{2.6}
$$

where I_3 is the 3×3 identity matrix. Hence,

$$
{}^A_BR = {}^B_AR^{-1} = {}^B_AR^T. \tag{2.7}
$$

Indeed, from linear algebra [1], we know that the inverse of a matrix with orthonormal columns is equal to its transpose. We have just shown this geometrically.

This computation is possible when a description of the orientation of $\{B\}$ is known relative to $\{A\}$. This orientation is given by the rotation matrix ${}^A_BR$, whose columns are the unit vectors of $\{B\}$ written in $\{A\}$.

In order to calculate ${}^AP$, we note that the components of any vector are simply the projections of that vector onto the unit directions of its frame. The projection is calculated as the vector dot product. Thus, we see that the components of ${}^AP$ may be calculated as

$$
\begin{aligned}
{}^Apx &= {}^B\hat{X}_A \cdot {}^BP, \\
{}^Ap_y &= {}^B\hat{Y}_A \cdot {}^BP, \\
{}^Ap_z &= {}^B\hat{Z}_A \cdot {}^BP.
\end{aligned}
\tag{2.12}
$$

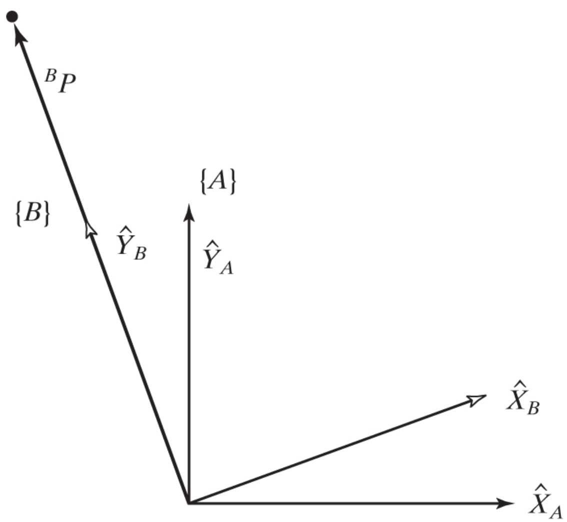
In order to express (2.13) in terms of a rotation matrix multiplication, we note from (2.11) that the *rows* of ${}^A_BR$ are ${}^B\hat{X}_A$, ${}^B\hat{Y}_A$, and ${}^B\hat{Z}_A$. So (2.13) may be written compactly, by using a rotation matrix, as

$$
{}^AP = {}^A_BR\,{}^BP. \tag{2.13}
$$

Equation 2.13 implements a mapping—that is, it changes the description of a vector—from ${}^BP$, which describes a point in space relative to $\{B\}$, into ${}^AP$, which is a description of the same point, but expressed relative to $\{A\}$.

We now see that our notation is of great help in keeping track of mappings and frames of reference. A helpful way of viewing the notation we have introduced is to imagine that leading subscripts cancel the leading superscripts of the following entity, for example the Bs in (2.13).

Example:
$\{B\}$ rotated 30 degrees about $\hat{Z}$.



Writing the unit vectors of $\{B\}$ in terms of $\{A\}$ and stacking them as the columns of the rotation matrix, we obtain

$$ {}^{A}_{B}R = \begin{bmatrix} 0.866 & -0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix}. \tag{2.14} $$

Given

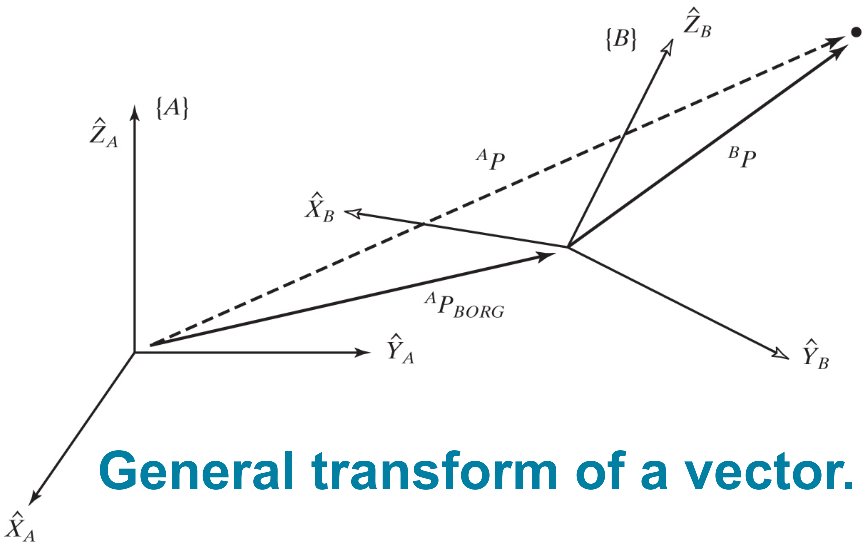
$$ {}^{B}P = \begin{bmatrix} 0.0 \\ 2.0 \\ 0.0 \end{bmatrix}, \tag{2.15} $$

we calculate ^{A}P as

$$ {}^{A}P = {}^{A}_{B}R\, {}^{B}P = \begin{bmatrix} -1.000 \\ 1.732 \\ 0.000 \end{bmatrix}. \tag{2.16} $$

Here, ${}^{A}_{B}R$ acts as a mapping that is used to describe ^{B}P relative to frame $\{A\}$, ^{A}P. As was introduced in the case of translations, it is important to remember that, viewed as a mapping, the original vector P is not changed in space. Rather, we compute a new description of the vector relative to another frame.

Mappings involving general frames



General transform of a vector.

Very often, we know the description of a vector with respect to some frame {B}, and we would like to know its description with respect to another frame, {A}. We now consider the general case of mapping. Here, the origin of frame {B} is not coincident with that of frame {A} but has a general vector offset. The vector that locates {B}'s origin is called $^AP_{BORG}$. Also {B} is rotated with respect to {A}, as described by ^A_BR. Given ^BP we wish to compute ^AP.

The description of frame {B} relative to (A} is ^A_BT.

We can first change ${}^{B}P$ to its description relative to an intermediate frame that has the same orientation as $\{A\}$, but whose origin is coincident with the origin of $\{B\}$. This is done by premultiplying by ${}^{A}_{B}R$ as in the last section. We then account for the translation between origins by simple vector addition, as before, and obtain

$$ {}^{A}P = {}^{A}_{B}R\, {}^{B}P + {}^{A}P_{BORG}. \tag{2.17} $$

Equation 2.17 describes a general transformation mapping of a vector from its description in one frame to a description in a second frame. Note the following interpretation of our notation as exemplified in (2.17): the B's cancel, leaving all quantities as vectors written in terms of A, which may then be added.

The form of (2.17) is not as appealing as the conceptual form

$$ {}^{A}P = {}^{A}_{B}T\, {}^{B}P. \tag{2.18} $$

That is, we would like to think of a mapping from one frame to another as an operator in matrix form. This aids in writing compact equations and is conceptually clearer than (2.17). In order that we may write the mathematics given in (2.17) in the matrix operator form suggested by (2.18), we define a 4×4 matrix operator and use 4×1 position vectors, so that (2.18) has the structure

$$ \begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = \left[\begin{array}{ccc|c} & {}^{A}_{B}R & & {}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array} \right] \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix}. \tag{2.19} $$

In other words,

1. a "1" is added as the last element of the 4×1 vectors;
2. a row "[0 0 0 1]" is added as the last row of the 4×4 matrix.

We adopt the convention that a position vector is 3×1 or 4×1, depending on whether it appears multiplied by a 3×3 matrix or by a 4×4 matrix. It is readily seen that (2.19) implements

$$ \begin{aligned} {}^{A}P &= {}^{A}_{B}R\, {}^{B}P + {}^{A}P_{BORG} \\ 1 &= 1. \end{aligned} \tag{2.20} $$

The 4×4 matrix in (2.19) is called a **homogeneous transform**. For our purposes, it can be regarded purely as a construction used to cast the rotation and translation of the general transform into a single matrix form. In other fields of study, it can be used to compute perspective and scaling operations (when the last row is other than "[0 0 0 1]" or the rotation matrix is not orthonormal). The interested reader should see [2].

Often, we will write an equation like (2.18) without any notation indicating that it is a homogeneous representation, because it is obvious from context. Note that, although homogeneous transforms are useful in writing compact equations, a computer program to transform vectors would generally not use them, because of time wasted multiplying ones and zeros. Thus, this representation is mainly for our convenience when thinking and writing equations down on paper.

Example: Frame {*B*} rotated and translated.

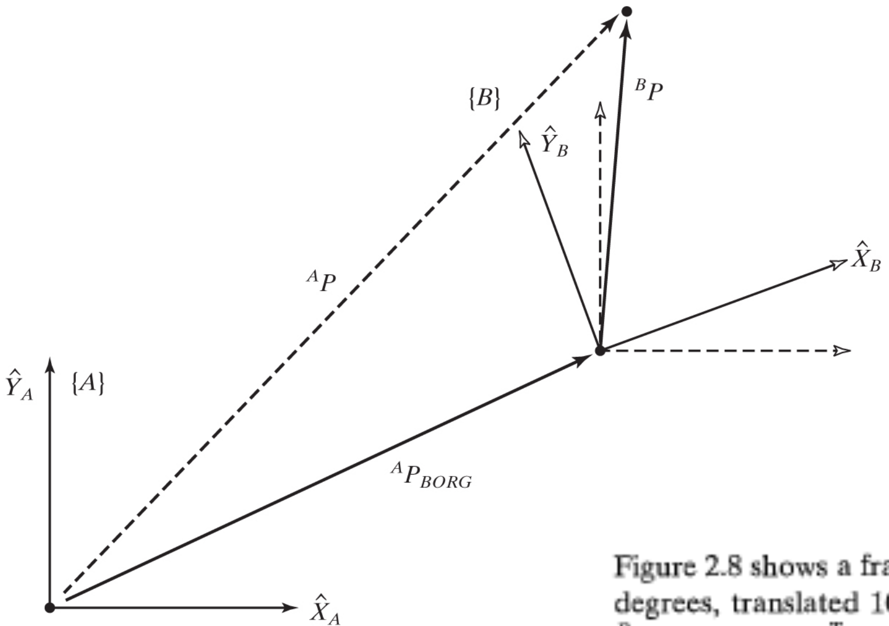


Figure 2.8 shows a frame $\{B\}$, which is rotated relative to frame $\{A\}$ about $\hat{Z}$ by 30 degrees, translated 10 units in $\hat{X}_A$, and translated 5 units in $\hat{Y}_A$. Find ^AP, where $^BP = [3.0\,7.0\,0.0]^T$.

The definition of frame $\{B\}$ is

$$^A_BT = \begin{bmatrix} 0.866 & -0.500 & 0.000 & 10.0 \\ 0.500 & 0.866 & 0.000 & 5.0 \\ 0.000 & 0.000 & 1.000 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.21}$$

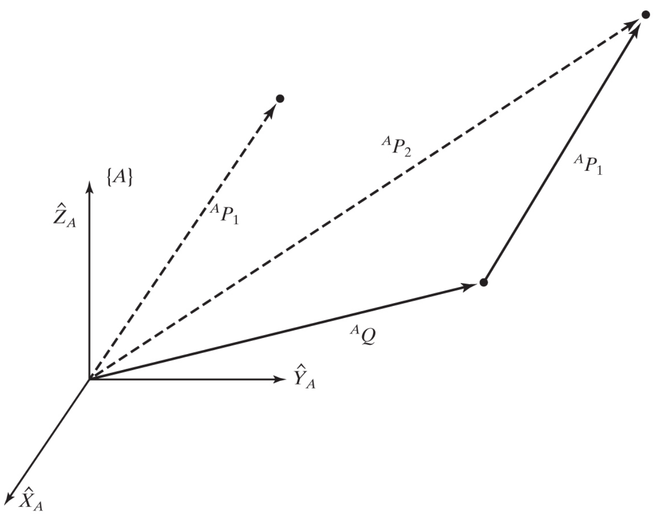
Given

$$^BP = \begin{bmatrix} 3.0 \\ 7.0 \\ 0.0 \end{bmatrix}, \tag{2.22}$$

we use the definition of $\{B\}$ just given as a transformation:

$$^AP = {}^A_BT\, {}^BP = \begin{bmatrix} 9.098 \\ 12.562 \\ 0.000 \end{bmatrix}. \tag{2.23}$$

Translation operator.



A translation moves a point in space a finite distance along a given vector direction. With this interpretation of actually translating the point in space, only one coordinate system need be involved. It turns out that translating the point in space is accomplished with the same mathematics as mapping the point to a second frame. Almost always, it is very important to understand which interpretation of the mathematics is being used. The distinction is as simple as this: When a vector is moved "forward" relative to a frame, we may consider either that the vector moved "forward" or that the frame moved "backward." The mathematics involved in the two cases is identical; only our view of the situation is different. Figure 2.9 indicates pictorially how a vector $^AP_1$ is translated by a vector ^AQ. Here, the vector ^AQ gives the information needed to perform the translation.

The result of the operation is a new vector $^AP_2$, calculated as

$$^AP_2 = {}^AP_1 + {}^AQ. \tag{2.24}$$

To write this translation operation as a matrix operator, we use the notation

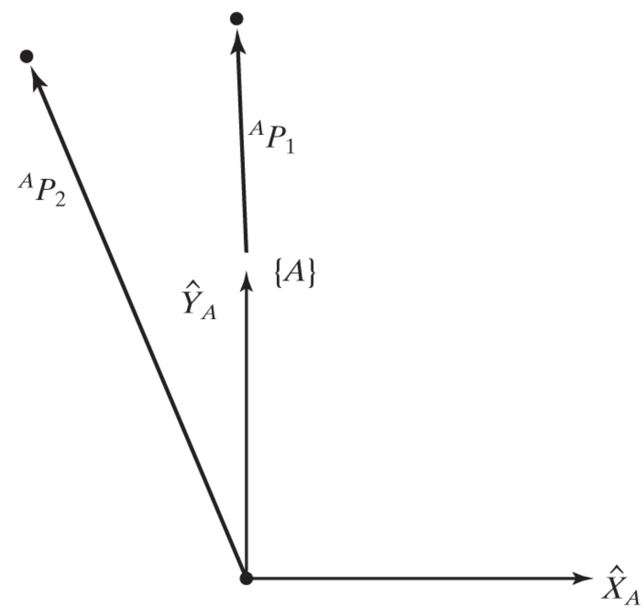
$$^AP_2 = D_Q(q)\, {}^AP_1, \tag{2.25}$$

where q is the signed magnitude of the translation along the vector direction $\hat{Q}$. The D_Q operator may be thought of as a homogeneous transform of a special simple form:

$$D_Q(q) = \begin{bmatrix} 1 & 0 & 0 & q_x \\ 0 & 1 & 0 & q_y \\ 0 & 0 & 1 & q_z \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.26}$$

where q_x, q_y, and q_z are the components of the translation vector Q and $q = \sqrt{q_x^2 + q_y^2 + q_z^2}$. Equations (2.9) and (2.24) implement the same mathematics. Note that, if we had defined $^BP_{AORG}$ (instead of $^AP_{BORG}$) in Fig. 2.4 and had used it in (2.9), then we would have seen a sign change between (2.9) and (2.24). This sign change would indicate the difference between moving the vector "forward" and moving the coordinate system "backward." By defining the location of $\{B\}$ relative to $\{A\}$ (with $^AP_{BORG}$), we cause the mathematics of the two interpretations to be the same. Now that the "D_Q" notation has been introduced, we may also use it to describe frames and as a mapping.

Figure 2.10
The vector $^{A}P_1$ rotated 30 degrees about $\hat{Z}$.



Another interpretation of a rotation matrix is as a *rotational operator* that operates on a vector $^{A}P_1$ and changes that vector to a new vector, $^{A}P_2$, by means of a rotation, R. Usually, when a rotation matrix is shown as an operator, no sub- or superscripts appear, because it is not viewed as relating two frames. That is, we may write

$$^{A}P_2 = R\,^{A}P_1. \tag{2.27}$$

Again, as in the case of translations, the mathematics described in (2.13) and in (2.27) is the same; only our interpretation is different. This fact also allows us to see *how to obtain* rotational matrices that are to be used as operators:

The rotation matrix that rotates vectors through some rotation, R, is the same as the rotation matrix that describes a frame rotated by R relative to the reference frame.

Although a rotation matrix is easily viewed as an operator, we will also define another notation for a rotational operator that clearly indicates which axis is being rotated about:

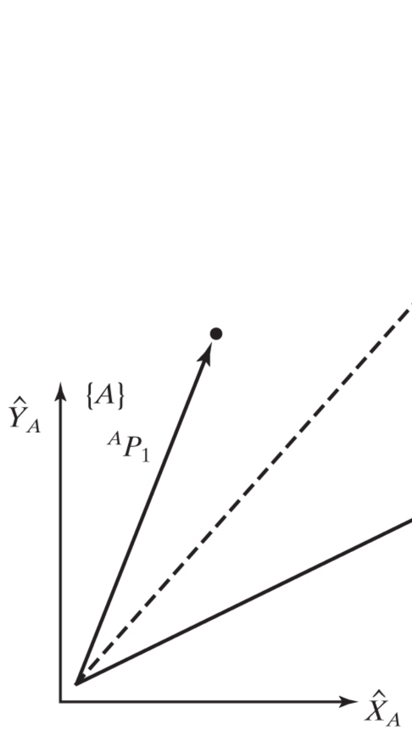
$$^{A}P_2 = R_K(\theta)\,^{A}P_1. \tag{2.28}$$

In this notation, "$R_K(\theta)$" is a rotational operator that performs a rotation about the axis direction $\hat{K}$ by θ degrees. This operator can be written as a homogeneous transform whose position-vector part is zero. For example, substitution into (2.11) yields the operator that rotates about the $\hat{Z}$ axis by θ as

$$R_z(\Theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.29}$$

Of course, to rotate a position vector, we could just as well use the 3×3 rotation-matrix part of the homogeneous transform. The "R_K" notation, therefore, may be considered to represent a 3×3 or a 4×4 matrix. Later in this chapter, we will see how to write the rotation matrix for a rotation about a general axis $\hat{K}$.

Figure 2.11
The vector $^AP_1$ rotated and translated to form $^AP_2$.



The operator T, which performs the translation and rotation, is

$$T = \begin{bmatrix} 0.866 & -0.500 & 0.000 & 10.0 \\ 0.500 & 0.866 & 0.000 & 5.0 \\ 0.000 & 0.000 & 1.000 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Given

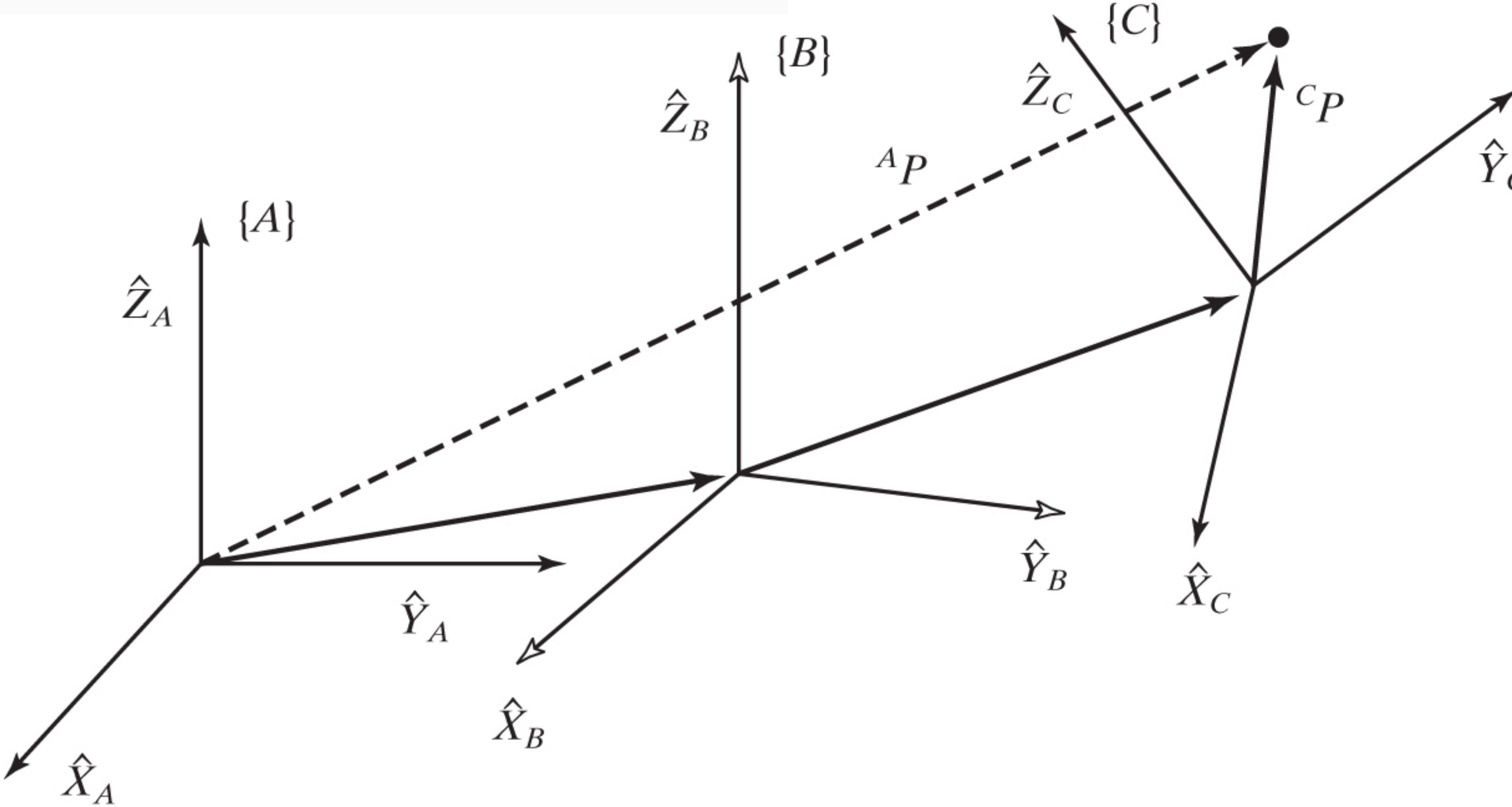
$$^AP_1 = \begin{bmatrix} 3.0 \\ 7.0 \\ 0.0 \end{bmatrix},$$

we use T as an operator:

$$^AP_2 = T\,^AP_1 = \begin{bmatrix} 9.098 \\ 12.562 \\ 0.000 \end{bmatrix}.$$

Figure 2.12
Compound frames: each is known relative to the previous one.

, we have ${}^{C}P$ and wish to find ${}^{A}P$.



Frame $\{C\}$ is known relative to frame $\{B\}$, and frame $\{B\}$ is known relative to frame $\{A\}$. We can transform ${}^{C}P$ into ${}^{B}P$ as

$$ {}^{B}P = {}^{B}_{C}T \, {}^{C}P; \tag{2.37} $$

then we can transform ${}^{B}P$ into ${}^{A}P$ as

$$ {}^{A}P = {}^{A}_{B}T \, {}^{B}P. \tag{2.38} $$

Combining (2.37) and (2.38), we get the (not unexpected) result

$$ {}^{A}P = {}^{A}_{B}T \, {}^{B}_{C}T \, {}^{C}P, \tag{2.39} $$

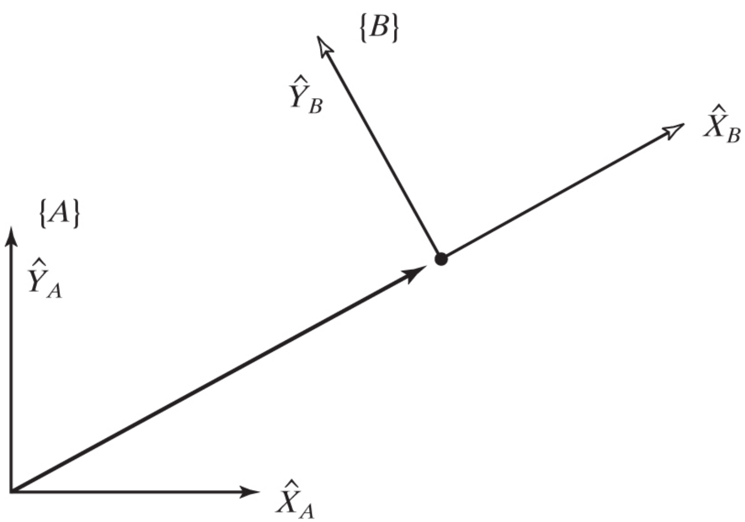
from which we could define

$$ {}^{A}_{C}T = {}^{A}_{B}T \, {}^{B}_{C}T. \tag{2.40} $$

Again, note that familiarity with the sub- and superscript notation makes these manipulations simple. In terms of the known descriptions of $\{B\}$ and $\{C\}$, we can give the expression for ${}^{A}_{C}T$ as

$$ {}^{A}_{C}T = \left[\begin{array}{ccc|c} & {}^{A}_{B}R \, {}^{B}_{C}R & & {}^{A}_{B}R \, {}^{B}P_{CORG} + {}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array} \right] \tag{2.41} $$

Inverting a transform $\{B\}$ relative to $\{A\}$.



To find ${}^B_AT$, we must compute ${}^B_AR$ and ${}^BP_{AORG}$ from ${}^A_BR$ and ${}^AP_{BORG}$. First, recall from our discussion of rotation matrices that

$$ {}^B_AR = {}^A_BR^T. \tag{2.42}$$

Next, we change the description of ${}^AP_{BORG}$ into $\{B\}$ by using (2.13):

$$ {}^B({}^AP_{BORG}) = {}^B_AR\,{}^AP_{BORG} + {}^BP_{AORG}. \tag{2.43}$$

The left-hand side of (2.43) must be zero, so we have

$$ {}^BP_{AORG} = -{}^B_AR\,{}^AP_{BORG} = -{}^A_BR^T\,{}^AP_{BORG}. \tag{2.44}$$

Using (2.42) and (2.44), we can write the form of ${}^B_AT$ as

$$ {}^B_AT = \left[\begin{array}{ccc|c} & {}^A_BR^T & & -{}^A_BR^T\,{}^AP_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]. \tag{2.45}$$

Note that, with our notation,

$$ {}^B_AT = {}^A_BT^{-1}. $$

Equation (2.45) is a general and extremely useful way of computing the inverse of a homogeneous transform.

Figure 2.14
Set of transforms forming a loop.

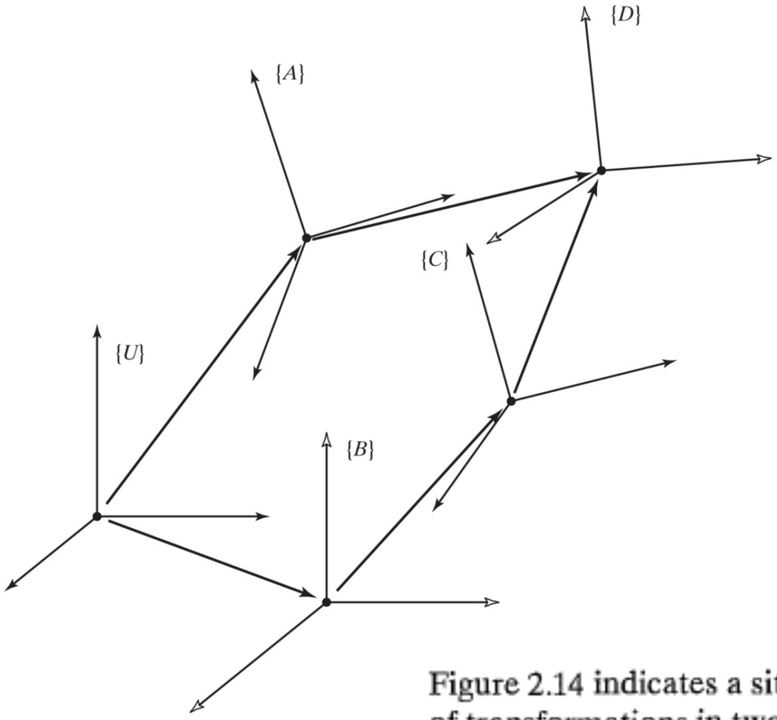


Figure 2.14 indicates a situation in which a frame $\{D\}$ can be expressed as products of transformations in two different ways. First,

$$ {}^{U}_{D}T = {}^{U}_{A}T \, {}^{A}_{D}T; \tag{2.48} $$

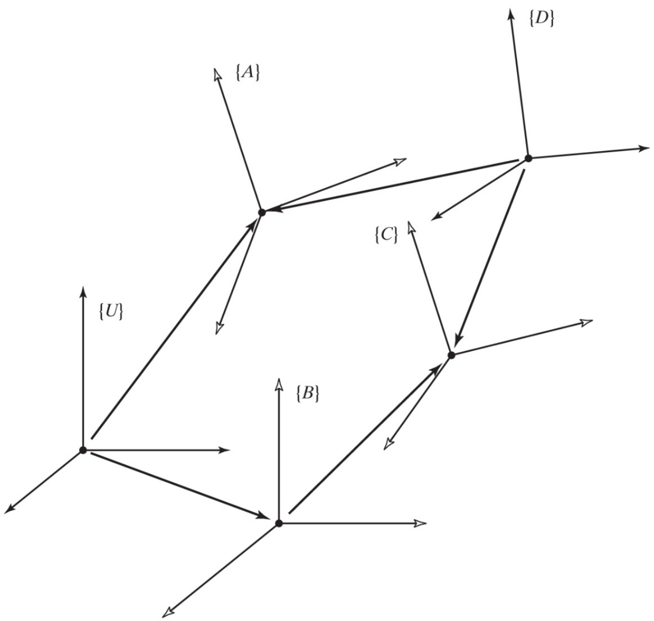
second;

$$ {}^{U}_{D}T = {}^{U}_{B}T \, {}^{B}_{C}T \, {}^{C}_{D}T. \tag{2.49} $$

We can set these two descriptions of ${}^{U}_{D}T$ equal to construct a **transform equation**:

$$ {}^{U}_{A}T \, {}^{A}_{D}T = {}^{U}_{B}T \, {}^{B}_{C}T \, {}^{C}_{D}T. \tag{2.50} $$

Figure 2.15
Example of a transform equation.



Transform equations can be used to solve for transforms in the case of n unknown transforms and n transform equations. Consider (2.50) in the case that all transforms are known except ${}^{B}_{C}T$. Here, we have one transform equation and one unknown transform; hence, we easily find its solution to be

$$ {}^{B}_{C}T = {}^{U}_{B}T^{-1}\,{}^{U}_{A}T\,{}^{A}_{D}T\,{}^{C}_{D}T^{-1}. \tag{2.51} $$

Figure 2.15 indicates a similar situation.

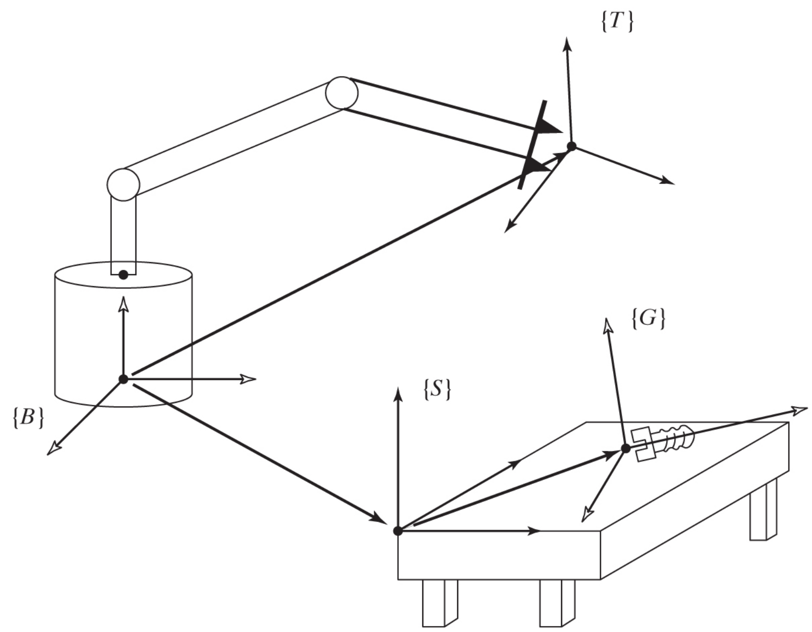
Note that, in all figures, we have introduced a *graphical* representation of frames as an arrow pointing from one origin to another origin. The arrow's direction indicates which way the frames are defined: In Fig. 2.14, frame $\{D\}$ is defined relative to $\{A\}$; in Fig. 2.15, frame $\{A\}$ is defined relative to $\{D\}$. In order to compound frames when the arrows line up, we simply compute the product of the transforms. If an arrow points the opposite way in a chain of transforms, we simply compute its inverse first. In Fig. 2.15, two possible descriptions of $\{C\}$ are

$$ {}^{U}_{C}T = {}^{U}_{A}T\,{}^{D}_{A}T^{-1}\,{}^{D}_{C}T \tag{2.52} $$

and

$$ {}^{U}_{C}T = {}^{U}_{B}T\,{}^{B}_{C}T. \tag{2.53} $$

Figure 2.16
Manipulator reaching for a bolt.



Assume that we know the transform ${}^{B}_{T}T$ in Fig. 2.16, which describes the frame at the manipulator's fingertips $\{T\}$ relative to the base of the manipulator, $\{B\}$, that we know where the tabletop is located in space relative to the manipulator's base (because we have a description of the frame $\{S\}$ that is attached to the table as shown, ${}^{B}_{S}T$), and that we know the location of the frame attached to the bolt lying on the table relative to the table frame—that is, ${}^{S}_{G}T$. Calculate the position and orientation of the bolt relative to the manipulator's hand, ${}^{T}_{G}T$.

Guided by our notation (and, it is hoped, our understanding), we compute the bolt frame relative to the hand frame as

$$ {}^{T}_{G}T = {}^{B}_{T}T^{-1}\, {}^{B}_{S}T\, {}^{S}_{G}T. \tag{2.55} $$

Figure 2.17
X–Y–Z fixed angles. Rotations are performed in the order $R_X(\gamma)$, $R_Y(\beta)$, $R_Z(\alpha)$.

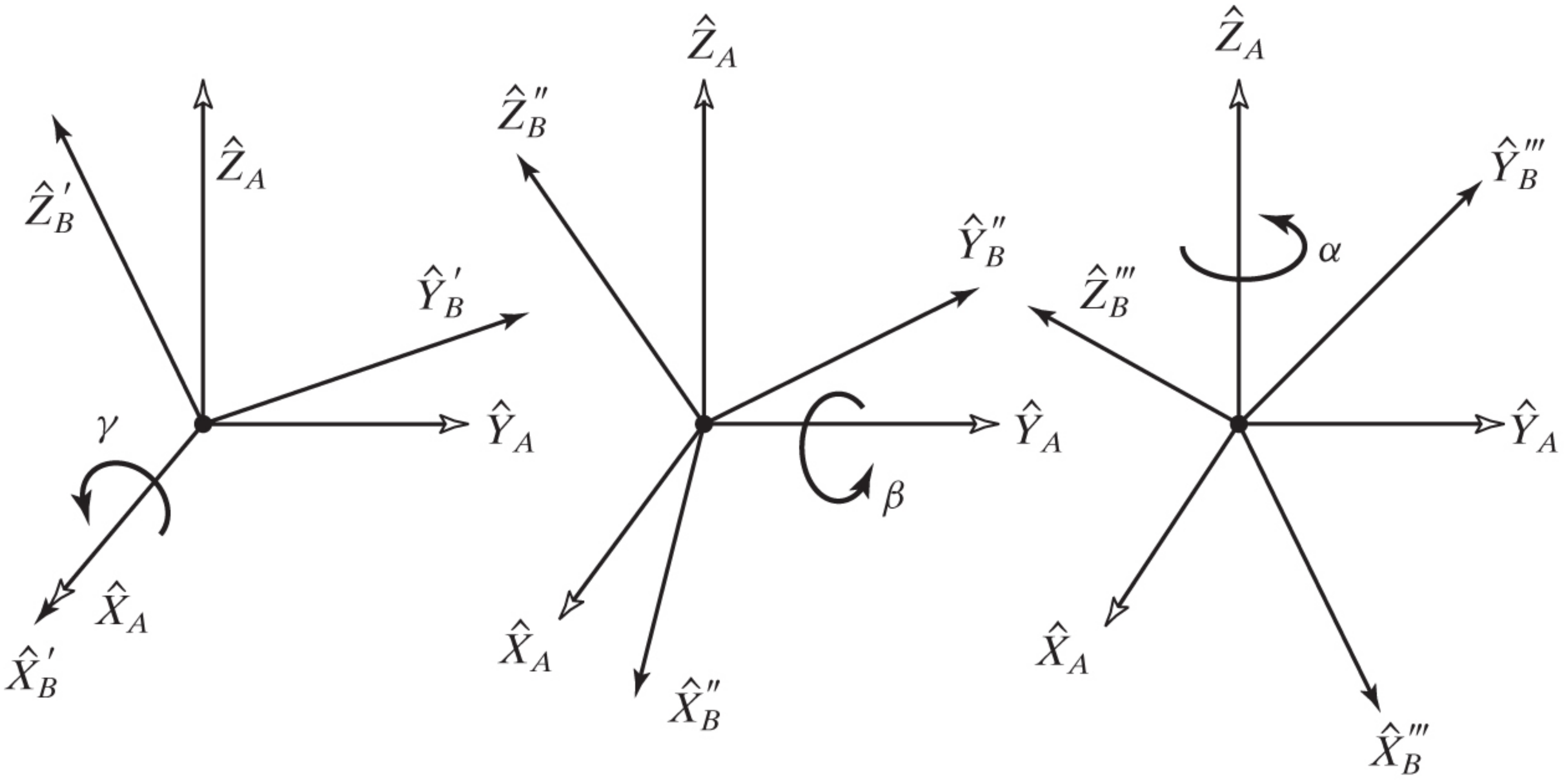


Figure 2.18
Z–Y–X Euler angles.

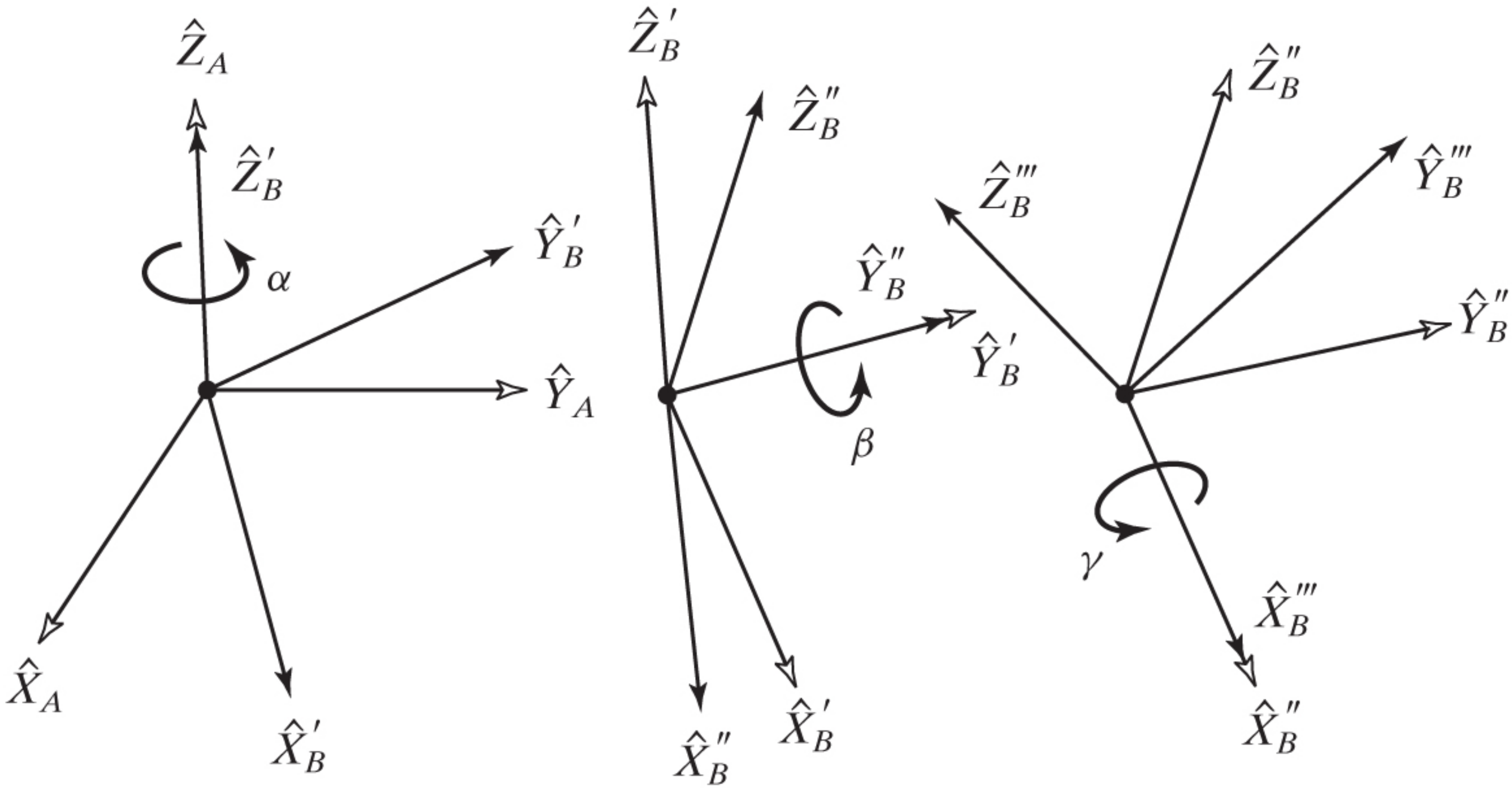


Figure 2.19
Equivalent angle–axis representation.

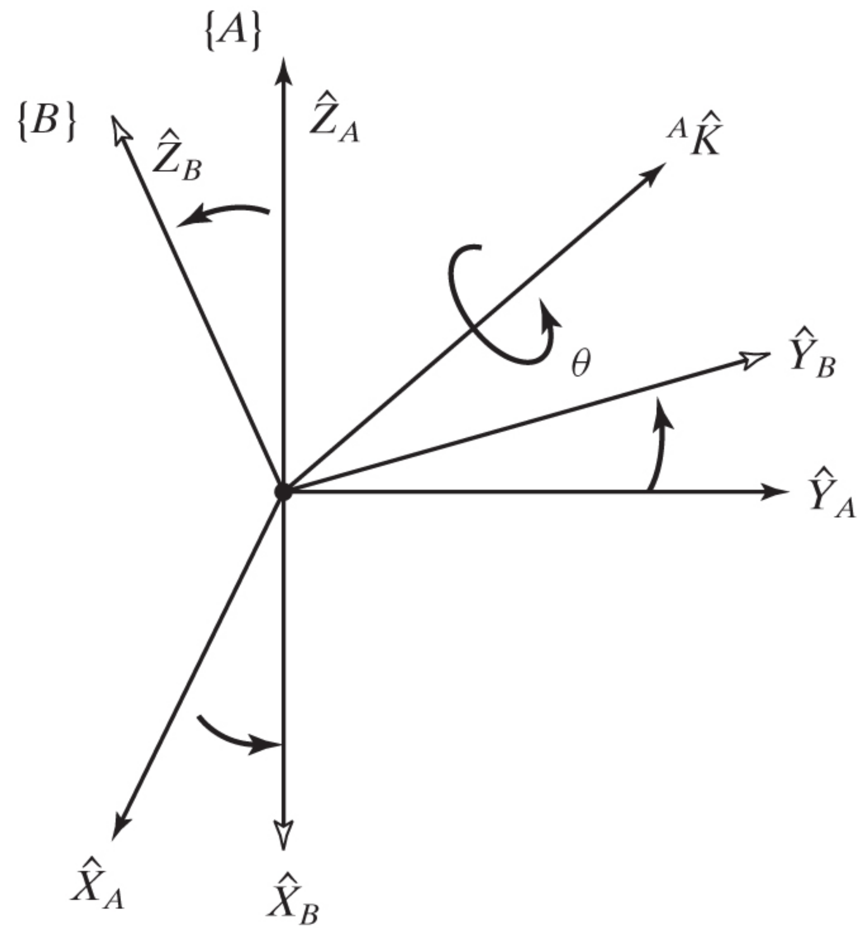


Figure 2.20
Rotation about an axis that does not pass through the origin of {*A*}. Initially, {*B*} was coincident with {*A*}.

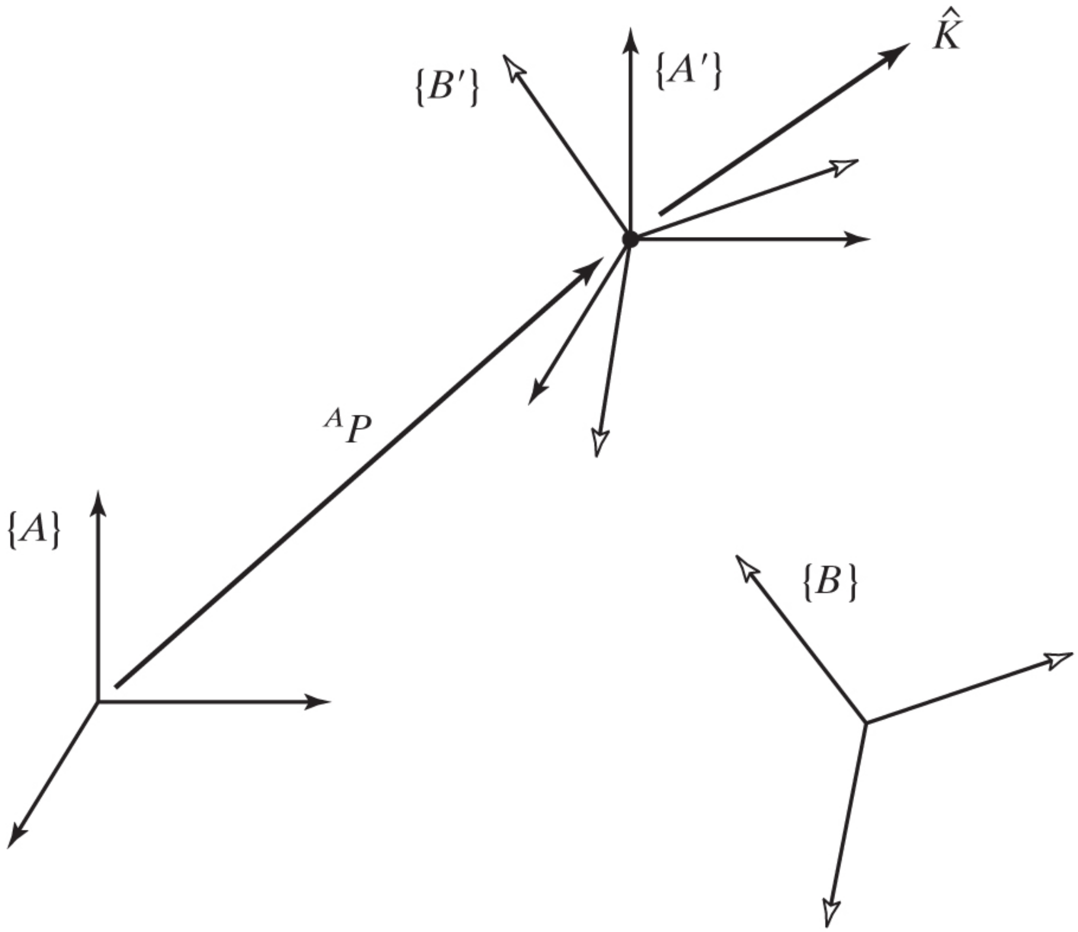


Figure 2.21
Equal velocity vectors.

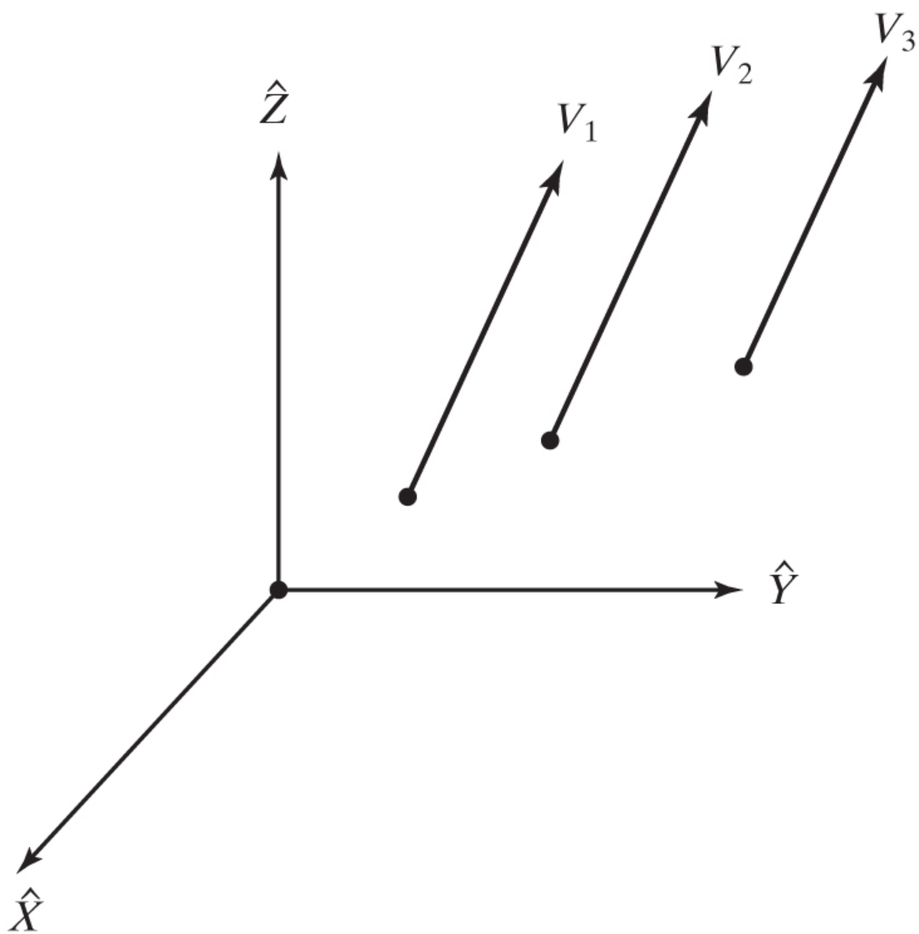


Figure 2.22
Transforming velocities.

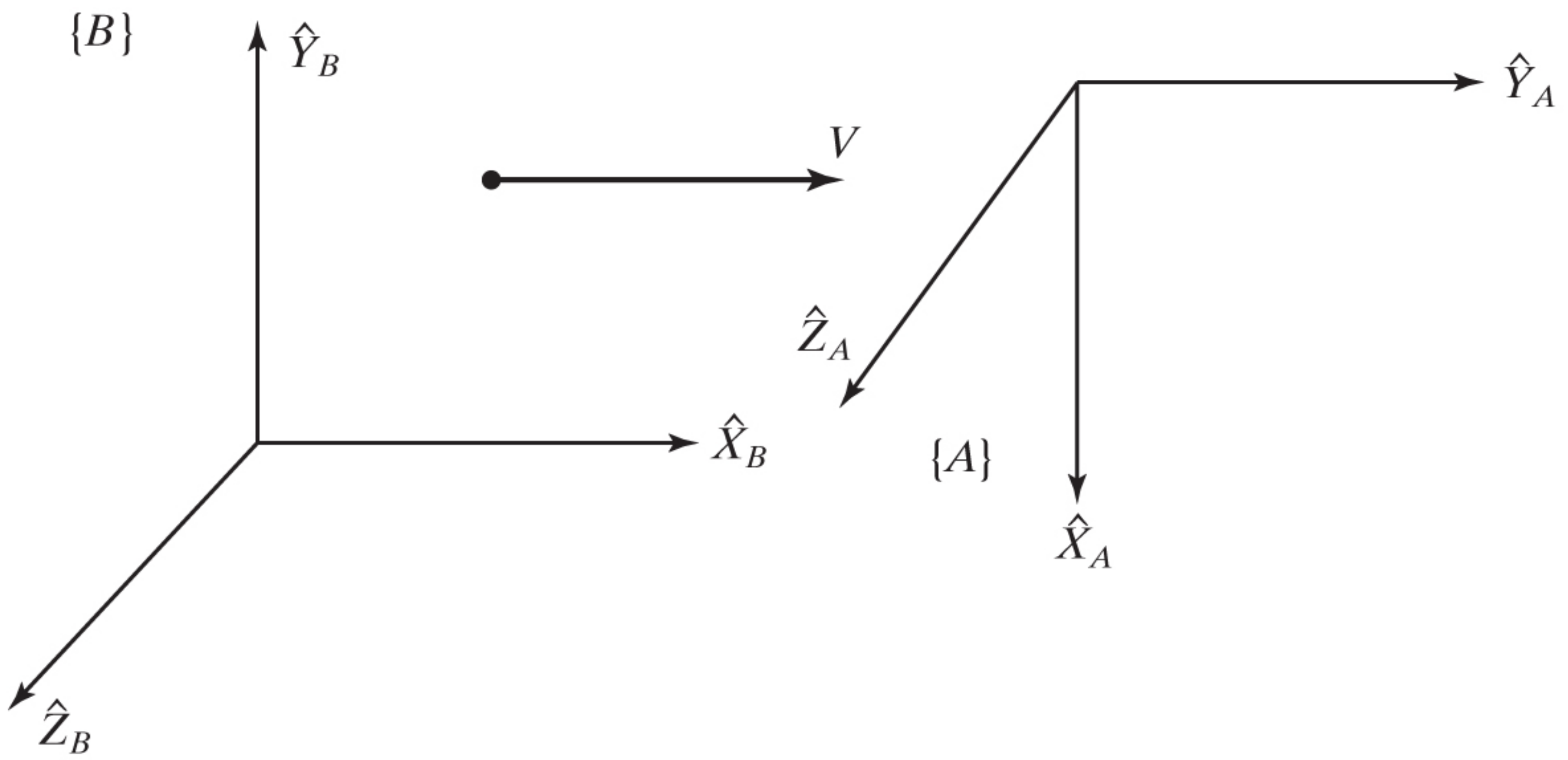


Figure 2.23
Cylindrical coordinates.

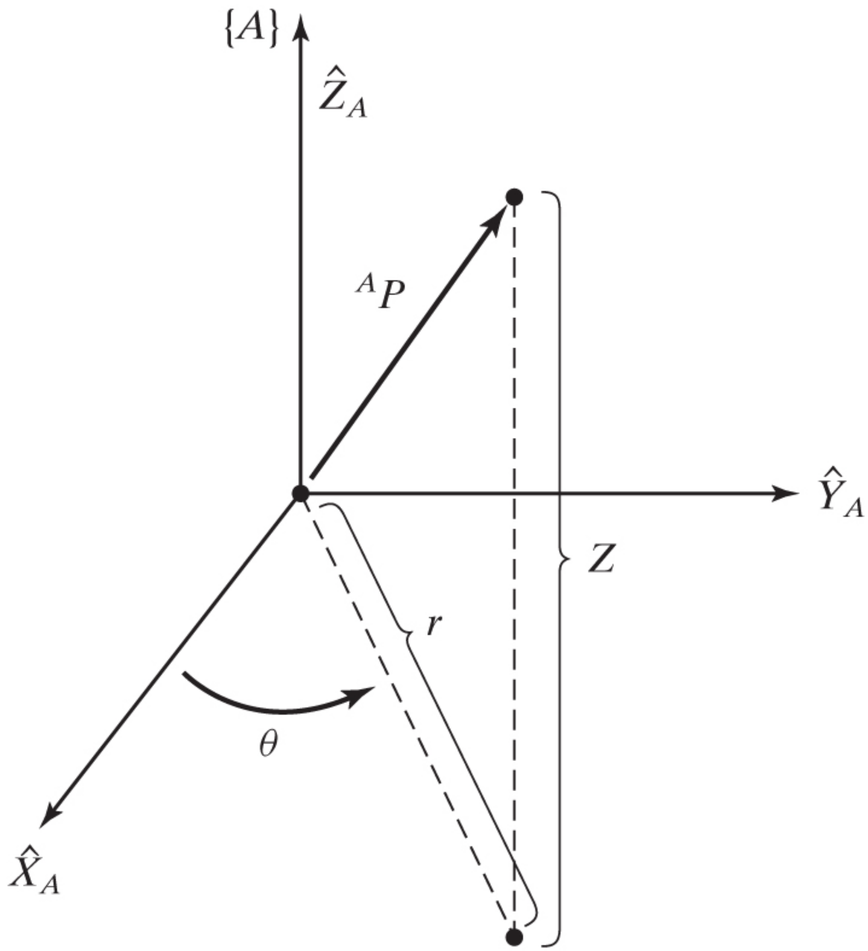


Figure 2.24
Spherical coordinates.

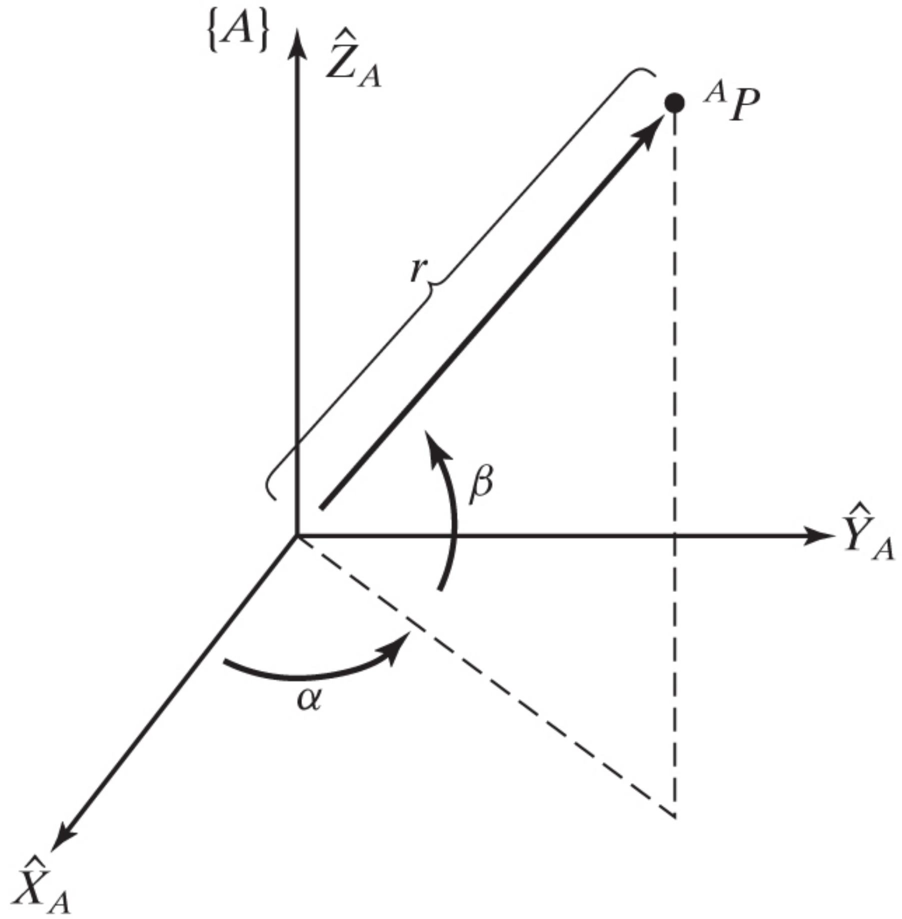


Figure 2.25
Frames at the corners of a wedge.

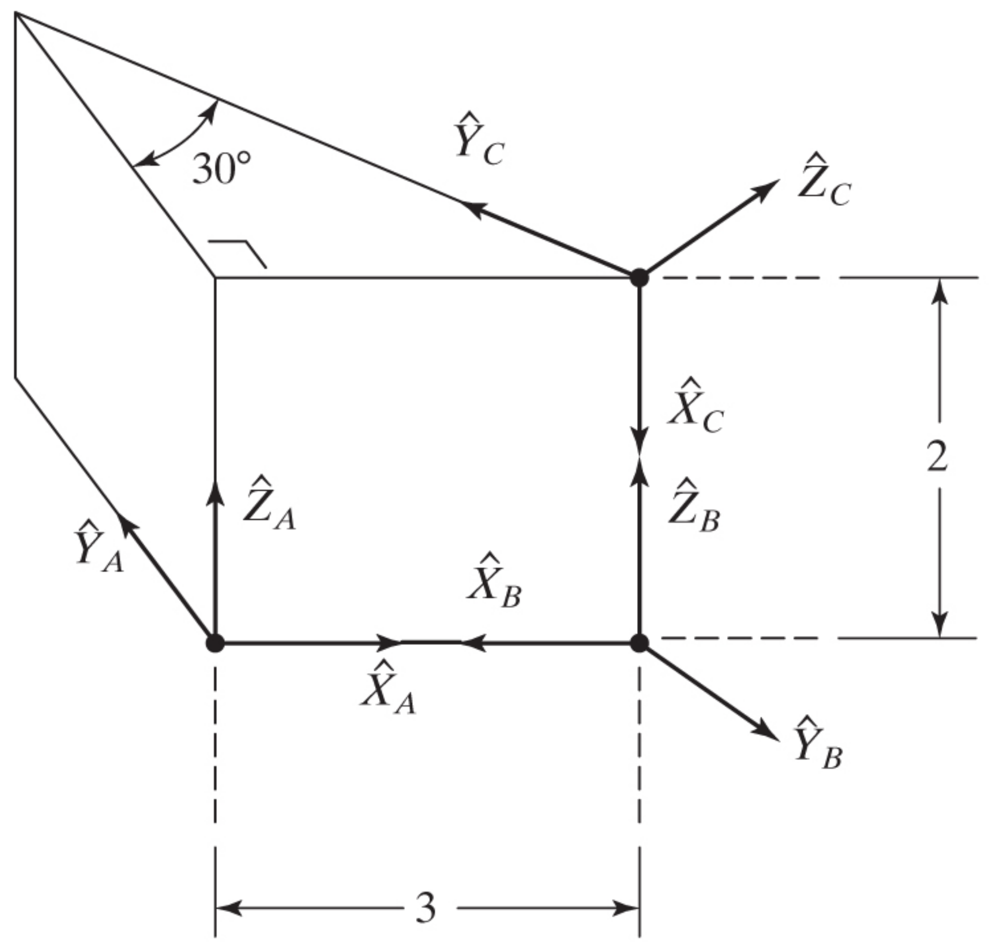


Figure 2.26
Frames at the corners of a wedge.