MLP Exercise

COMP47590

Question

Using the network setup in Figure 1 perform the following tasks:

1. Perform a forward propagation pass through the network
2. Calculate the loss associated with the training instance
3. Perform a backward propagation pass through the network
4. Perform a weight update within the network using a learning rate, α, of 0.001

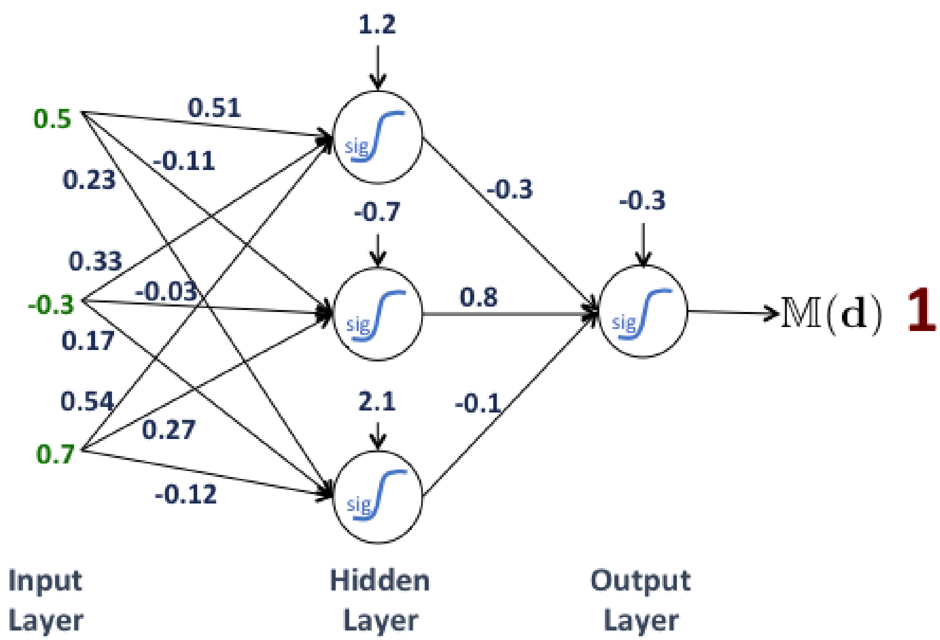

Figure 1: An MLP network and association inputs, weights, biases, and target

Solution

Setup Input, Weight and Bias Matrices

The network inputs:

$$\mathbf{d} = \begin{bmatrix} 0.5 \\ -0.3 \\ 0.7 \end{bmatrix} \tag{1}$$

Weights and biases for **Layer 1**:

$$\mathbf{W}^{[1]} = \begin{bmatrix} 0.51 & 0.33 & 0.54 \\ -0.11 & -0.03 & 0.27 \\ 0.23 & 0.17 & -0.12 \end{bmatrix} \tag{2}$$

$$\mathbf{b}^{[1]} = \begin{bmatrix} 1.2 \\ -0.7 \\ 2.1 \end{bmatrix} \tag{3}$$

Weights and biases for **Layer 2**:

$$\mathbf{W}^{[2]} = \begin{bmatrix} -0.3 & 0.8 & -0.1 \end{bmatrix} \tag{4}$$

$$\mathbf{b}^{[2]} = \begin{bmatrix} -0.3 \end{bmatrix} \tag{5}$$

Forward Propagate

To perform a forward propagation for the first layer in the network, first calculate $\mathbf{z}^{[1]}$:

$$\begin{aligned} \mathbf{z}^{[1]} &= \mathbf{W}^{[1]}\,\mathbf{d} + \mathbf{b}^{[1]} & (6) \\ &= \begin{bmatrix} 0.51 & 0.33 & 0.54 \\ -0.11 & -0.03 & 0.27 \\ 0.23 & 0.17 & -0.12 \end{bmatrix} \begin{bmatrix} 0.5 \\ -0.3 \\ 0.7 \end{bmatrix} + \begin{bmatrix} 1.2 \\ -0.7 \\ 2.1 \end{bmatrix} & (7) \\ &= \begin{bmatrix} 1.734 \\ -0.557 \\ 2.08 \end{bmatrix} & (8) \end{aligned}$$

then apply the activation function, in this case a **sigmoid function**, to calculate the activation of the nodes at **Layer 1**:

$$\begin{aligned}
\mathbf{a}^{[1]} &= \text{sigmoid}(\mathbf{z}^{[1]}) && (9)\\
&= \text{sigmoid}\left(\begin{bmatrix} 1.734 \\ -0.557 \\ 2.08 \end{bmatrix}\right) && (10)\\
&= \begin{bmatrix} 0.84992335 \\ 0.36424189 \\ 0.88894403 \end{bmatrix} && (11)
\end{aligned}$$

To perform a forward propagation for the second layer in the network, first calculate $\mathbf{z}^{[2]}$:

$$\begin{aligned}
\mathbf{z}^{[2]} &= \mathbf{W}^{[2]}\,\mathbf{a}^{[1]} + \mathbf{b}^{[2]} && (12)\\
&= \begin{bmatrix} -0.3 & 0.8 & -0.1 \end{bmatrix} \begin{bmatrix} 0.84992335 \\ 0.36424189 \\ 0.88894403 \end{bmatrix} + \begin{bmatrix} -0.3 \end{bmatrix} && (13)\\
&= \begin{bmatrix} -0.3524779 \end{bmatrix} && (14)
\end{aligned}$$

then apply the activation function, in this case a **sigmoid function**, to calculate the activation of the output nodes of the network:

$$\begin{aligned}
\mathbf{a}^{[2]} &= \text{sigmoid}(\mathbf{z}^{[2]}) && (15)\\
&= \text{sigmoid}\left(\begin{bmatrix} -0.3524779 \end{bmatrix}\right) && (16)\\
&= \begin{bmatrix} 0.41278167 \end{bmatrix} && (17)
\end{aligned}$$

Calculate Loss

$$\begin{aligned}
\text{loss} &= -\left(t \times log\left(\mathbf{a}^{[2]}\right) + (1-t) \times log\left(1 - \mathbf{a}^{[2]}\right)\right) && (18)\\
&= -\left(1 \times log\left(\begin{bmatrix} 0.41278167 \end{bmatrix}\right) + (1-1) \times log\left(1 - \begin{bmatrix} 0.41278167 \end{bmatrix}\right)\right) && (19)\\
&= 0.8848364792770356 && (20)
\end{aligned}$$

Backward Propagate

To perform a backward propagation at the second layer calculate, $d\mathbf{z}^{[2]}$:

$$\begin{aligned}
d\mathbf{z}^{[2]} &= \mathbf{a}^{[2]} - t && (21)\\
&= \begin{bmatrix} 0.41278167 \end{bmatrix} - \begin{bmatrix} 1 \end{bmatrix} && (22)\\
&= \begin{bmatrix} -0.58721833 \end{bmatrix} && (23)
\end{aligned}$$

and record $d\mathbf{W}^{[2]}$ and $d\mathbf{b}^{[2]}$:

$$\begin{aligned}
d\mathbf{W}^{[2]} &= d\mathbf{z}^{[2]}\,{\mathbf{a}^{[1]}}^{T} && (24)\\
&= \begin{bmatrix}-0.58721833\end{bmatrix}\begin{bmatrix}0.84992335 & 0.36424189 & 0.88894403\end{bmatrix} && (25)\\
&= \begin{bmatrix}-0.49909057 & -0.21388951 & -0.52200423\end{bmatrix} && (26)\\
& && (27)\\
d\mathbf{b}^{[2]} &= d\mathbf{z}^{[2]} && (28)\\
&= \begin{bmatrix}-0.58721833\end{bmatrix} && (29)
\end{aligned}$$

To perform a backward propagation at the first layer calculate, $d\mathbf{z}^{[1]}$:

$$\begin{aligned}
d\mathbf{z}^{[1]} &= {\mathbf{W}^{[2]}}^{T}\,d\mathbf{z}^{[2]} * g^{[1]'}\left(\mathbf{z}^{[1]}\right) && (30)\\
&= \begin{bmatrix}-0.3\\0.8\\-0.1\end{bmatrix}\begin{bmatrix}-0.58721833\end{bmatrix} * sigmoid'\left(\begin{bmatrix}1.734\\-0.557\\2.08\end{bmatrix}\right) && (31)\\
&= \begin{bmatrix}0.1761655\\-0.46977467\\0.05872183\end{bmatrix} * \begin{bmatrix}0.12755365\\0.23156973\\0.09872254\end{bmatrix} && (32)\\
&= \begin{bmatrix}0.02247055\\-0.10878559\\0.00579717\end{bmatrix} && (33)
\end{aligned}$$

and record $d\mathbf{W}^{[1]}$ and $d\mathbf{b}^{[1]}$:

$$\begin{aligned}
d\mathbf{W}^{[1]} &= d\mathbf{z}^{[1]}\mathbf{d}^{T} && (34)\\
&= \begin{bmatrix}0.02247055\\-0.10878559\\0.00579717\end{bmatrix}\begin{bmatrix}0.5 & -0.3 & 0.7\end{bmatrix} && (35)\\
&= \begin{bmatrix}0.01123528 & -0.00674117 & 0.01572939\\-0.0543928 & 0.03263568 & -0.07614992\\0.00289858 & -0.00173915 & 0.00405802\end{bmatrix} && (36)\\
& && (37)\\
d\mathbf{b}^{[1]} &= d\mathbf{z}^{[1]} && (38)\\
&= \begin{bmatrix}0.02247055\\-0.10878559\\0.00579717\end{bmatrix} && (39)
\end{aligned}$$

Update Weights

Update the weight and bias terms in the first layer:

$$\mathbf{W}^{[1]} = \mathbf{W}^{[1]} - \alpha\, d\mathbf{W}^{[1]} \tag{40}$$

$$= \begin{bmatrix} 0.51 & 0.33 & 0.54 \\ -0.11 & -0.03 & 0.27 \\ 0.23 & 0.17 & -0.12 \end{bmatrix} - 0.001 \times \begin{bmatrix} 0.01123528 & -0.00674117 & 0.01572939 \\ -0.0543928 & 0.03263568 & -0.07614992 \\ 0.00289858 & -0.00173915 & 0.00405802 \end{bmatrix} \tag{41}$$

$$= \begin{bmatrix} 0.50987641 & 0.33007415 & 0.53982698 \\ -0.10940168 & -0.03035899 & 0.27083765 \\ 0.22996812 & 0.17001913 & -0.12004464 \end{bmatrix} \tag{42}$$

$$\mathbf{b}^{[1]} = \mathbf{b}^{[1]} - \alpha\, d\mathbf{b}^{[1]} \tag{43}$$

$$= \begin{bmatrix} 1.2 \\ -0.7 \\ 2.1 \end{bmatrix} - 0.001 \times \begin{bmatrix} 0.02247055 \\ -0.10878559 \\ 0.00579717 \end{bmatrix} \tag{44}$$

$$= \begin{bmatrix} 1.19975282 \\ -0.69880336 \\ 2.09993623 \end{bmatrix} \tag{45}$$

Update the weight and bias terms in the second layer:

$$\mathbf{W}^{[2]} = \mathbf{W}^{[2]} - \alpha\, d\mathbf{W}^{[2]} \tag{46}$$

$$= \begin{bmatrix} -0.3 & 0.8 & -0.1 \end{bmatrix} - 0.001 \times \begin{bmatrix} -0.49909057 & -0.21388951 & -0.52200423 \end{bmatrix} \tag{47}$$

$$= \begin{bmatrix} -0.29451 & 0.80235278 & -0.09425795 \end{bmatrix} \tag{48}$$

$$\mathbf{b}^{[2]} = \mathbf{b}^{[2]} - \alpha\, d\mathbf{b}^{[2]} \tag{49}$$

$$= \begin{bmatrix} -0.3 \end{bmatrix} - 0.001 \times \begin{bmatrix} -0.58721833 \end{bmatrix} \tag{50}$$

$$= \begin{bmatrix} -0.2935406 \end{bmatrix} \tag{51}$$

Check Impact of Updating Weights

Performing a forward pass with these new weight values gives:

$$\mathbf{a}^{[2]} = \begin{bmatrix} 0.4166458 \end{bmatrix} \tag{52}$$

from which we can calculate a new loss:

$$
\begin{aligned}
\text{loss} &= -\left(t \times log\left(\mathbf{a}^{[2]}\right) + (1-t) \times log\left(1 - \mathbf{a}^{[2]}\right)\right) && (53)\\
&= -\left(1 \times log\left(\begin{bmatrix}0.4166458\end{bmatrix}\right) + (1-1) \times log\left(1 - \begin{bmatrix}0.4166458\end{bmatrix}\right)\right) && (54)\\
&= 0.8755188083022127 && (55)
\end{aligned}
$$

which is ever so slightly less than the previous value of 0.8848364792770356 so our back-propagation and weight update step worked!