Chapter 26 : Another appeal to monotonicity.

In which we once again use that nice property.

We are given f[0..M-1, 0..N-1] of int. We are told that f is ascending in both arguments. We are asked to construct a program to compute, for some X : int,

$$r = \langle + i,j : 0 \le i < M \wedge 0 \le j < N : g.(f.i.j)\rangle$$

Where

$$g.\alpha \quad = \quad 1 \quad \Leftarrow \quad \alpha \le X$$
$$g.\alpha \quad = \quad 0 \quad \Leftarrow \quad \alpha > X$$

We begin by modelling our domain.

* (0) $C.m.n \quad = \quad \langle + i,j : m \le i < M \wedge 0 \le j < n : g.(f.i.j)\rangle$

By falsifying the ranges we get

\- (1) $C.M.n \quad = \quad 0 \qquad , 0 \le n \le N$

\- (2) $C.m.0 \quad = \quad 0 \qquad , 0 \le m \le M$

We observe

$C.m.n$

= {(0)}

$\langle + i,j : m \le i < M \wedge 0 \le j < n : g.(f.i.j)\rangle$

= { Split off i = m term }

$\langle + i,j : m+1 \le i < M \wedge 0 \le j < n : g.(f.i.j)\rangle + \langle +,j : 0 \le j < n : g.(f.m.j)\rangle$

= { (0), (5) }

$C.(m+1).n + D.n$

\- (3) $C.m.n \quad = \quad C.(m+1).n + D.n \qquad , 0 \le m < M$

Similarly, we observe

$C.m.n$

= {(0)}

$\langle + i,j : m \le i < M \wedge 0 \le j < n : g.(f.i.j)\rangle$

= { Split off j = n-1 term }

$\langle + i,j : m \le i < M \wedge 0 \le j < n-1 : g.(f.i.j)\rangle + \langle + i : m \le i < M : g.(f.i.(n-1))\rangle$

= { (0), (6) }

$C.m.(n-1) + E.m$

- (4) $C.m.n = C.m.(n-1) + E.m \quad , 0 < n \leq N$

* (5) $D.n = \langle + j : 0 \leq j < n : g.(f.m.j) \rangle$

* (6) $E.m = \langle + i : m \leq i < M : g.(f.i.(n-1)) \rangle$

We now turn our attention to investigating D and E

- (7) $D.n = ? \quad \Leftarrow X < f.m.(n-1)$

- (8) $D.n = n \quad \Leftarrow f.m.(n-1) \leq X$

- (9) $E.m = 0 \quad \Leftarrow X < f.m.(n-1)$

- (10) $E.m = ? \quad \Leftarrow f.m.(n-1) \leq X$ [1]

We have now completed our model and so we turn to constructing the program.

Rewrite Postcondition.

Post : $r = C.0.N$

Invariants

P0 : $r + C.m.n = C.0.N$
P1 : $0 \leq m \leq M \wedge 0 \leq n \leq N$

Upon termination

$P0 \wedge P1 \wedge (m=M \vee n=0) \Rightarrow Post$

Establish Invariants.

$r, m, n := 0, 0, N$

Guard.

$m \neq M \wedge n \neq 0$

[1] In this case we know the answer is at least 1 but we cannot determine anything else.

Loop Body.

$$
\begin{array}{ll}
 & P0 \\
= & \quad \{\text{definition of } P0\} \\
 & r + C.m.n = C.0.N \\
= & \quad \{(3)\} \\
 & r + C.(m+1).n + D.n = C.0.N \\
= & \quad \{\text{case analysis } f.m.(n-1) \le X \ (8)\} \\
 & r + C.(m+1).n + n = C.0.N \\
= & \quad \{WP\} \\
 & (r, m := r+n, m+1).P0
\end{array}
$$

Giving us

If $f.m.n \le X \rightarrow r, m, n := r+n, m+1$

There is no point in appealing to (7) so we ignore that case.

We also observe

$$
\begin{array}{ll}
 & P0 \\
= & \quad \{\text{definition of } P0\} \\
 & r + C.m.n = C.0.N \\
= & \quad \{(4)\} \\
 & r + C.m.(n-1) + E.m = C.0.N \\
= & \quad \{\text{case analysis } X < f.m.(n-1) \ (9)\} \\
 & r + C.m.(n-1) + 0 = C.0.N \\
= & \quad \{WP\} \\
 & (r, n := r + 0, n-1).P0
\end{array}
$$

Giving us

If $X < f.m.n \rightarrow r, m, n := r + 0, n-1$

Finished Program.

$r, n, m := 0, 0, N$ $\{P0 \wedge P1\}$
; do $m \neq M \wedge n \neq 0 \rightarrow$ $\{P0 \wedge P1 \wedge m \neq M \wedge n \neq 0\}$

If $f.m.(n-1) \leq X \rightarrow r, m := r+n, m+1$
[] $X < f.m.(n-1) \rightarrow r, n := r+0, n-1$
Fi

$\{P0 \wedge P1\}$
od
$\{P0 \wedge P1 \wedge (m=M \vee n=0)\}$

This has temporal complexity $O(M+N)$.