Chapter 42 : The longest minimal-prefix segment.

In which we learn some more about manipulating expressions.

Given f[0..N) of int, $\{0 \le N\}$. We are asked to determine the length of the longest minimal-prefix segment in f. More formally, we are asked to construct a program which establishes

$$\text{Post} : r = \langle \uparrow i,j : 0 \le i \le j \le N \wedge MP.i.j : j - i \rangle$$

We define a minimal-prefix segment as follows.

* (0) $MP.i.j \equiv \langle \forall\, k : i \le k < j : f.i \le f.k \rangle$ $\quad ,0 \le i \le j \le N$

From this, by appealing to the empty range, the reflexivity of $\le$ and the associativity of $\wedge$, the following theorems emerge.

\- (1) $MP.i.i$ $\quad ,0 \le i \le N$

\- (2) $MP.i.(i+1)$ $\quad ,0 \le i < N$

\- (3) $MP.i.(j+1) \equiv MP.i.j \wedge f.i \le f.j$ $\quad ,0 \le i \le j < N$

We now name the quantified expression in the postcondition and see what theorems suggest themselves.

* (4) $C.n = \langle \uparrow i,j : 0 \le i \le j \le n \wedge MP.i.j : j - i \rangle$ $\quad ,0 \le n \le N$

Appealing to the 1-point rule and (2) above we get

\- (5) $C.0 = 0$

Now we consider C.(n+1)

$$
\begin{array}{ll}
 & C.(n+1) \\
= & \quad \{(4)\} \\
 & \langle \uparrow i,j : 0 \le i \le j \le n+1 \wedge MP.i.j : j - i \rangle \\
= & \quad \{\text{range is not empty, } \uparrow \text{ associative, split off } j=n+1 \text{ term}\} \\
 & \langle \uparrow i,j : 0 \le i \le j \le n \wedge MP.i.j : j - i \rangle \uparrow \langle \uparrow i : 0 \le i \le n+1 \wedge MP.i.(n+1) : n+1 - i \rangle \\
= & \quad \{(4)\} \\
 & C.n \uparrow \langle \uparrow i : 0 \le i \le n+1 \wedge MP.i.(n+1) : n+1 - i \rangle \\
= & \quad \{\text{name and conquer, (7)}\} \\
 & C.n \uparrow D.(n+1)
\end{array}
$$

\- (6) $C.(n+1) = C.n \uparrow D.(n+1)$ $\quad ,0 \le n < N$

* (7) $D.n = \langle \uparrow i : 0 \leq i \leq n \wedge MP.i.n : n - i \rangle$ $\qquad ,0 \leq n \leq N$

Appealing to the 1-point rule and (2) above we get

- (8) $D.0 = 0$

We now explore D.(n+1)

$$
\begin{array}{ll}
 & D.(n+1) \\
= & \quad \{(7)\} \\
 & \langle \uparrow i : 0 \leq i \leq n+1 \wedge MP.i.(n+1) : n+1 - i \rangle \\
= & \quad \{\text{range not empty, split off } i = n+1 \text{ term}\} \\
 & \langle \uparrow i : 0 \leq i \leq n \wedge MP.i.(n+1) : n+1 - i \rangle \uparrow (n+1)-(n+1) \\
= & \quad \{\text{arithmetic}\} \\
 & \langle \uparrow i : 0 \leq i \leq n \wedge MP.i.(n+1) : n+1 - i \rangle \uparrow 0 \\
= & \quad \{\text{range not empty } +/\uparrow\} \\
 & (1 + \langle \uparrow i : 0 \leq i \leq n \wedge MP.i.(n+1) : n - i \rangle) \uparrow 0 \\
= & \quad \{(3)\} \\
 & (1 + \langle \uparrow i : 0 \leq i \leq n \wedge MP.i.n \wedge f.i \leq f.n : n - i \rangle) \uparrow 0
\end{array}
$$

Now we pause. Looking at the shape of the quantified expression we should probably try a case analysis. In the case where $f.i \leq f.n$ is true then that term would disappear and we would be left with D.n. However, we cannot refer to $f.i \leq f.n$ as i is a bound variable.

However, we do know that the i we are interested in is the value in the range $0 \leq i \leq n$ having the property MP.i.n which maximises the expression $n - i$. In other words the minimum value of i with these properties.

Let us return to the definition of D.n

$$
\begin{array}{ll}
 & D.n = \langle \uparrow i : 0 \leq i \leq n \wedge MP.i.n : n - i \rangle \\
= & \quad \{+/\uparrow \text{ for non-empty ranges, } 0 \leq n\ \} \\
 & D.n = n + \langle \uparrow i : 0 \leq i \leq n \wedge MP.i.n : - i \rangle \\
= & \quad \{*/\downarrow \text{ for non-empty ranges, } 0 \leq n\ \} \\
 & D.n = n - \langle \downarrow i : 0 \leq i \leq n \wedge MP.i.n : i \rangle \\
= & \quad \{\text{algebra}\} \\
 & n - D.n = \langle \downarrow i : 0 \leq i \leq n \wedge MP.i.n : i \rangle
\end{array}
$$

This gives us the required value for i. We will add this nice theorem to our model

- (9) $n - D.n = \langle \downarrow i : 0 \leq i \leq n \wedge MP.i.n : i \rangle$ $\qquad , 0 \leq n \leq N$

We can now return to considering D.(n+1)

$$
\begin{array}{ll}
 & D.(n+1) \\
= & \quad \{(7)\} \\
 & \langle \uparrow i : 0 \le i \le n+1 \wedge MP.i.(n+1) : n+1 - i \rangle \\
= & \quad \{\text{range not empty, split off } i = n+1 \text{ term}\} \\
 & \langle \uparrow i : 0 \le i \le n \wedge MP.i.(n+1) : n+1 - i \rangle \uparrow (n+1)-(n+1) \\
= & \quad \{\text{arithmetic}\} \\
 & \langle \uparrow i : 0 \le i \le n \wedge MP.i.(n+1) : n+1 - i \rangle \uparrow 0 \\
= & \quad \{\text{range not empty } +/\uparrow\} \\
 & (1 + \langle \uparrow i : 0 \le i \le n \wedge MP.i.(n+1) : n - i \rangle) \uparrow 0 \\
= & \quad \{(3)\} \\
 & (1 + \langle \uparrow i : 0 \le i \le n \wedge MP.i.n \wedge f.i \le f.n : n - i \rangle) \uparrow 0 \\
= & \quad \{\text{case analysis } f.(n - D.n) \le f.n, \text{Id}\wedge\} \\
 & (1 + \langle \uparrow i : 0 \le i \le n \wedge MP.i.n : n - i \rangle) \uparrow 0 \\
= & \quad \{(7)\} \\
 & (1 + D.n) \uparrow 0
\end{array}
$$

- (10) $D.(n+1) = (1 + D.n) \uparrow 0 \quad \Leftarrow \quad f.(n - D.n) \le f.n \quad , 0 \le n < N$

And we consider the other case.

$$
\begin{array}{ll}
 & D.(n+1) \\
= & \quad \{(7)\} \\
 & \langle \uparrow i : 0 \le i \le n+1 \wedge MP.i.(n+1) : n+1 - i \rangle \\
= & \quad \{\text{range not empty, split off } i = n+1 \text{ term}\} \\
 & \langle \uparrow i : 0 \le i \le n \wedge MP.i.(n+1) : n+1 - i \rangle \uparrow (n+1)-(n+1) \\
= & \quad \{\text{arithmetic}\} \\
 & \langle \uparrow i : 0 \le i \le n \wedge MP.i.(n+1) : n+1 - i \rangle \uparrow 0 \\
= & \quad \{\text{range not empty } +/\uparrow\} \\
 & (1 + \langle \uparrow i : 0 \le i \le n \wedge MP.i.(n+1) : n - i \rangle) \uparrow 0 \\
= & \quad \{(3)\} \\
 & (1 + \langle \uparrow i : 0 \le i \le n \wedge MP.i.n \wedge f.i \le f.n : n - i \rangle) \uparrow 0 \\
= & \quad \{\text{case analysis } f.(n - D.n) > f.n, i = n\} \\
 & (1 + \langle \uparrow i : 0 \le i \le n \wedge MP.i.n \wedge f.n \le f.n : n - n \rangle) \uparrow 0 \\
= & \quad \{\text{ bound variable does not appear in the term}\} \\
 & (1 + 0) \uparrow 0 \\
= & \quad \{\text{arithmetic}\} \\
 & 1
\end{array}
$$

- (11) $D.(n+1) = 1 \quad \Leftarrow \quad f.(n - D.n) > f.n \quad , 0 \le n < N$

Now we begin the programming task.

Post : $r = C.N$

We strengthen this to get

Post ‘: $r = C.n \wedge n = N$

Invariants.

P0 : $r = C.n \wedge d = D.n$
P1 : $0 \leq n \leq N$

Guard.

$n \neq N$

Establish invariants.

$n, r, d := 0, 0, 0$

Loop body.

$(n, r, d := n{+}1, E, E').P0$
= {test sub.}
$E = C.(n{+}1) \wedge E' = D.(n{+}1)$
= {(6)}
$E = C.n \uparrow D.(n{+}1) \wedge E' = D.(n{+}1)$
= {case analysis, $f.(n{-}D.n) \leq f.n$, (10)}
$E = C.n \uparrow (1 + D.n) \uparrow 0 \wedge E' = (1 + D.n) \uparrow 0$
= {P0}
$E = r \uparrow (1 + d) \uparrow 0 \wedge E' = (1 + d) \uparrow 0$

Which gives us

if $f.(n{-}D.n) \leq f.n \rightarrow n, r, d := n{+}1, r \uparrow (1 + d) \uparrow 0, (1 + d) \uparrow 0$

We calculate the other case

$(n, r, d := n{+}1, E, E').P0$
= {test sub.}
$E = C.(n{+}1) \wedge E' = D.(n{+}1)$
= {(6)}
$E = C.n \uparrow D.(n{+}1) \wedge E' = D.(n{+}1)$
= {case analysis, $f.(n{-}D.n) > f.n$, (11)}
$E = C.n \uparrow 0 \wedge E' = 1$
= {P0}
$E = r \uparrow 0 \wedge E' = 1$

Giving us

```
if f.(n-D.n) > f.n → n, r, d := n+1, r ↑ 0, 1
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Finished Algorithm.

```
n, r, d :+ 0, 0, 0
;do n ≠ N →
    if f.(n-D.n) ≤ f.n → n, r, d := n+1, r ↑ (1 + d) ↑ 0, (1 + d) ↑ 0
    [] f.(n-D.n) > f.n → n, r, d := n+1, r ↑ 0, 1
    fi
od
```