Chapter 10: The Linear Search Theorem.

In which we introduce the simplest searching algorithm.

Suppose we are given an array f[0..N) of integer, where $\{1 \le N\}$, which is guaranteed to contain at least one occurrence of the value X, and we are asked to find the location of the leftmost X in f, i.e. the smallest index n where f.n = X.

We begin as usual with a problem specification.

$$\{\langle \exists j : 0 \le j < N : f.j = X \rangle\}$$

$$S$$

$$\{\langle \forall j : 0 \le j < n : f.j \ne X\rangle \wedge f.n = X\}$$

As usual, we begin by developing a model of the problem domain.

* (0) $C.n \equiv \langle \forall j : 0 \le j < n : f.j \ne X\rangle$, $0 \le n \le N$

Appealing to the empty range and associativity we get the following theorems

Consider.

$$C.0$$
$$= \quad \{(0) \text{ in model }\}$$
$$\langle \forall j : 0 \le j < 0: f.j \ne X\rangle$$
$$= \quad \{ \text{ empty range } \}$$
$$true$$

\- (1) $C.0 \equiv true$

Consider

$$C.(n+1)$$
$$= \quad \{(0) \text{ in model }\}$$
$$\langle \forall j : 0 \le j < n+1: f.j \ne X\rangle$$
$$= \quad \{ \text{ split off } j = n \text{ term}\}$$
$$\langle \forall j : 0 \le j < n : f.j \ne X\rangle \wedge f.n \ne X$$
$$= \quad \{(0) \text{ in model}\}$$
$$C.n \wedge f.n \ne X$$

\- (2) $C.(n+1) \equiv C.n \wedge f.n \ne X$, $0 \le n < N$

Rewrite postcondition using the model.

$$\text{Post} : C.n \wedge f.n = X$$

Choose Invariants.

We choose as our invariants

$$P0: C.n$$
$$P1: 0 \leq n \leq N$$

Establish Invariants.

Theorem (1) in our model shows us that we can establish P0 by the assignment

$$n := 0$$

This also establishes P1.

Termination.

We note that

$$P0 \wedge P1 \wedge f.n = X \Rightarrow \text{Post}$$

Guard

We choose our loop guard to be

$$B : f.n \neq X$$

Variant.

As our variant function we choose

$$K - n$$

Where $0 \leq K < N$ and K is the (as yet unknown) index of the leftmost occurrence of X.

Calculate Loop body.

Decreasing the variant by the assignment n := n+1 is a standard step and maintains P1. Let us see what effect it has on P0

$$
\begin{array}{ll}
 & (n := n+1).\ P0 \\
= & \qquad \{\text{textual substitution}\} \\
 & C.(n+1) \\
= & \qquad \{(2)\ \text{above}\} \\
 & C.n \wedge f.n \neq X \\
= & \qquad \{\ f.n \neq X\ \text{at start of loop body}\} \\
 & C.n \\
= & \qquad \{\ P0\ \text{is invariant}\ \} \\
 & \text{true}
\end{array}
$$

Finished program.

So our finished program is

```
  n := 0
; do f.n ≠ X →

            n := n+1

  od
{C.n ∧ f.n = X}
```

Generic Solution.

This problem is just one instance of a set of problems called the Linear Searches. We will now describe this family of problems and construct the generic solution.

Suppose we are given a finite, ordered domain, f[α..β) and a predicate Q defined on the elements of f. We are also given that Q holds true at at least one point in the domain. We are asked to find the lowest point at which this is the case. We begin with our specification.

$$\{\langle \exists j : \alpha \leq j < \beta : \neg Q.(f.j) \rangle\}$$

$$S$$

$$\{\langle \forall j : \alpha \leq j < n : \neg Q.(f.j) \rangle \wedge Q.(f.n)\}$$

As usual we develop our model

* (3) $C.n \quad \equiv \quad \langle \forall j : \alpha \leq j < n : \neg Q.(f.j)\rangle \qquad , \alpha \leq n \leq \beta$

Appealing to the empty range law and associativity we get the following theorems

Consider.

$$\begin{array}{ll} & C.\alpha \\ = & \quad \{(0) \text{ in model }\} \\ & \langle \forall j : \alpha \leq j < \alpha : \neg Q.(f.j)\rangle \\ = & \quad \{ \text{ empty range } \} \\ & \text{true} \end{array}$$

\- (1) $C.\alpha \quad \equiv \quad \text{true}$

Consider

$$\begin{array}{ll} & C.(n+1) \\ = & \quad \{(0) \text{ in model }\} \\ & \langle \forall j : \alpha \leq j < n+1 : \neg Q.(f.j)\rangle \\ = & \quad \{ \text{ split off } j = n \text{ term}\} \\ & \langle \forall j : \alpha \leq j < n : \neg Q.(f.j)\rangle \wedge \neg Q.(f.n) \\ = & \quad \{(0) \text{ in model}\} \\ & C.n \wedge \neg Q.(f.n) \end{array}$$

\- (2) $C.(n+1) \quad \equiv \quad C.n \wedge \neg Q.(f.n) \qquad , \alpha \leq n < \beta$

We can now rewrite our postcondition as

$$\text{Post} : C.n \wedge Q.(f.n)$$

Choose Invariants.

We choose as our invariants

$$P0: C.n$$
$$P1: \alpha \leq n \leq \beta$$

Termination.

We note that

$$P0 \wedge P1 \wedge Q.(f.n) \Rightarrow \text{Post}$$

Establish Invariants.

Our model (4) shows us that we can establish P0 by the assignment

$$n := \alpha$$

This also establishes P1.

Guard.

We choose our loop guard to be

$$B : \neg Q.(f.n)$$

Variant.

As our variant function we choose

$$K - n$$

Where $0 \leq K < N$ and K is the (as yet unknown) index of the leftmost point where Q holds.

Calculate Loop body.

Decreasing the variant by the assignment n := n+1 is a standard step and maintains P1. Let us se what effect it has on P0

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\begin{array}{ll}
 & (n := n+1).\ P0 \\
= & \qquad \{\text{textual substitution}\} \\
 & C.(n+1) \\
= & \qquad \{(2)\ \text{above}\} \\
 & C.n \wedge \neg Q.(f.n) \\
= & \qquad \{\neg Q.(f.n)\ \text{at start of loop body}\} \\
 & C.n \\
= & \qquad \{\ P0\ \} \\
 & \text{true}
\end{array}
$$

Finished Program.

So our finished program is as follows

```
n := α
; do ¬Q.(f.n) →

            n := n+1

  od

{C.n ∧ Q.(f.n)}
```

This is called the Linear Search Theorem.

Exercises.

Find the largest natural number i, where $i*i \leq 1023$.

Given f[0..1000) of integer which is guaranteed to contain the value X more than once, construct programs to

- find the rightmost index i where f.i = X
- find the 2nd leftmost index i where f.i = X

Fred's appointment book is represented by a boolean array f[0..N) and Mary's appointment book is represented by a boolean array m[0..N). Fred is available to meet on day i if f.i = true and is busy if f.i = false. Similarly with Mary. Given that there exists a day in the future when they both are free to move, find out when the earliest day is that they can meet.