Chapter 30 : How many segments sum to X

Given f[0..N) of N^+ and X of N^+ we are asked to develop an algorithm to determine the number of segments in f which sum to X.

We define a segment sum as follows

* (0) $SS.i.j = \langle + k : i \le k < j : f.k \rangle$, $0 \le i \le j \le N$

We can phrase our postcondition as follows

$$r = \langle + i,j \ : 0 \le i \le N \wedge \ 0 \le j \le N : g.(SS.i.j).X \rangle$$

Where

*(0) $g.x.y = 1 \ \Leftarrow \quad x = y$
*(1) $g.x.y = 0 \ \Leftarrow \quad x \neq y$

We now develop a small SS-theory.

(2) $SS.i.i = 0$, $0 \le i \le N$

(3) $SS.i.(j+1) \ = \quad SS.i.j + f.j$, $0 \le i \le j < N$

Because of the type of values in f we can also state

(4) $SS.i.j < SS.i.(j+1)$, $0 \le i \le j < N$

(5) $SS.(i+1).j < SS.i.j$, $0 \le i < j \le N$

We now name and conquer the quantified expression in the postcondition

* (6) $C.m.n \quad = \quad \langle + i,j \ : m \le i \le N \ \wedge n \le j \le N : g.(SS.i.j).X \rangle$

We note that

(7) $C.N.n \quad = \quad 0$, $0 \le n \le N$

We note also that C.m.N doesn’t yield a nice result so we don’t use it (yet).

We observe

$$
\begin{array}{ll}
 & C.m.n \\
= & \quad \{(6)\} \\
 & \langle + i,j : m \le i \le N \ \wedge n \le j \le N : g.(SS.i.j).X \rangle \\
= & \quad \{ m < N \text{ split off } i{=}m \text{ term}\} \\
 & \langle + i,j : m{+}1 \le i \le N \wedge n \le j \le N : g.(SS.i.j).X \rangle + \langle + j : n \le j \le N : g.(SS.m.j).X \rangle \\
= & \quad \{(6)\ (10)\} \\
 & C.(m{+}1).n + D.n
\end{array}
$$

(8) C.m.n $\quad = \quad C.(m+1).n + D.n \qquad , 0 \le m < n \le N$

We observe

$$
\begin{array}{ll}
 & C.m.n \\
= & \quad \{(6)\} \\
 & \langle + i,j : m \le i \le N \ \wedge n \le j \le N : g.(SS.i.j).X \rangle \\
= & \quad \{ n < N \text{ split off } j{=}n \text{ term}\} \\
 & \langle + i,j : m \le i \le N \wedge n{+}1 \le j \le N : g.(SS.i.j).X \rangle + \langle + i : m \le i \le N : g.(SS.i.n).X \rangle \\
= & \quad \{(6)\ (10)\} \\
 & C.m.(n{+}1) + E.m
\end{array}
$$

(9) C.m.n $\quad = \quad C.m.(n+1) + E.m \qquad , 0 \le m \le n < N$

Where

* (10) D.n $\quad = \quad \langle + j \ : n \le j \le N : g.(SS.m.j).X \rangle$

*(11) E.m $\quad = \quad \langle + i \ : m \le i \le N : g.(SS.i.n).X \rangle$

As SS is monotonic in its 2nd argument and anti-monotonic in its 1st we can deduce the following

(12) D.n $\quad = \quad ? \qquad \Leftarrow \qquad SS.m.n < X$

(13) D.n $\quad = \quad 1 \qquad \Leftarrow \qquad SS.m.n = X$

(14) D.n $\quad = \quad 0 \qquad \Leftarrow \qquad SS.m.n > X$

(15) $E.m = 0 \Leftarrow SS.m.n < X$

(16) $E.m = 1 \Leftarrow SS.m.n = X$

(17) $E.m = ? \Leftarrow SS.m.n > X$

Our postcondition is.

Post : $r = C.0.0$

Invariants.

P0 : $r + C.m.n = C.0.0$
P1: $0 \le m \le N \ \wedge \ 0 \le n \le N$

Establish Invariants.

$r, m, n := 0,0,0$

Termination.

From P0 it is clear that $C.m.n = 0 \Rightarrow r = C.0.0$

From law (6) above we know that $C.N.n = 0$ thus

$m = N \Rightarrow r = C.0.0$

Are there any other circumstances under which $C.m.n = 0$?

My intuition is guiding me here.

Let us consider C.m.N

$$
\begin{array}{ll}
 & C.m.N \\
= & \quad \{\text{definition of } C\} \\
 & \langle + \ i,j \ : m \le i \le N \ : g.(SS.i.N).X \rangle \\
= & \quad \{\text{assume } SS.m.N < X \text{ , SS anti-monotonic in 1st argument}\} \\
 & 0
\end{array}
$$

Thus

$m = N \vee (n = N \wedge SS.m.n < X) \ \Rightarrow \ C.m.n = 0$

Guard.

$m \neq N \wedge (n \neq N \vee SS.m.n \geq X)$

Loop body.

We observe,

$$
\begin{array}{ll}
 & r + C.m.n = C.0.0 \\
= & \qquad \{ (8) \} \\
 & r + C.(m+1).n + D.n = C.0.0 \\
= & \qquad \{ \text{case } SS.m.n > X \ (14) \} \\
 & r + C.(m+1).n + 0 = C.0.0 \\
 & \qquad \{ WP \} \\
 & (r, m := r+0, m+1).P0
\end{array}
$$

We further observe,

$$
\begin{array}{ll}
 & r + C.m.n = C.0.0 \\
= & \qquad \{ (8) \} \\
 & r + C.(m+1).n + D.n = C.0.0 \\
= & \qquad \{ \text{case } SS.m.n = X \ (13) \} \\
 & r + C.(m+1).n + 1 = C.0.0 \\
 & \qquad \{ WP \} \\
 & (r, m := r+1, m+1).P0
\end{array}
$$

We further observe,

$$
\begin{array}{ll}
 & r + C.m.n = C.0.0 \\
= & \qquad \{ (9) \} \\
 & r + C.m.(n+1) + E.m = C.0.0 \\
= & \qquad \{ \text{case } SS.m.n = X \ (16) \} \\
 & r + C.m.(n+1) + 1 = C.0.0 \\
 & \qquad \{ WP \} \\
 & (r, n := r+1, n+1).P0
\end{array}
$$

We further observe,

$$
\begin{array}{ll}
 & r + C.m.n = C.0.0 \\
= & \qquad \{ (9) \} \\
 & r + C.m.(n+1) + E.m = C.0.0 \\
= & \qquad \{ \text{case } SS.m.n < X \ (15) \}
\end{array}
$$

$r + C.m.(n+1) + 0 = C.0.0$

$\{ WP \}$

$(r, n := r+0, n+1).P0$

Putting this all together we get

$r.m.n := 0,0,0$

$do\ m \neq N \wedge (n \neq N \vee SS.m.n \geq X) \rightarrow$

$$
\begin{array}{lll}
\text{if } SS.m.n < X & \rightarrow & r, n := r + 0, n + 1 \\
[]\ SS.m.n = X & \rightarrow & r, m := r+1, m+1 \\
[]\ SS.m.n = X & \rightarrow & r, n := r+1, n+1 \\
[]\ SS.m.n > X & \rightarrow & r, m := r + 0, m + 1 \\
\text{fi} & &
\end{array}
$$

od

$\{r = C.0.0\}$

As we would prefer not to have to evaluate SS.m.n each time we can always strengthen our invariants with

P2 : $y = SS.m.n$

We leave this as an exercise.