Induction Examples

January 30, 2019

1 Induction

A proof by induction is just like an ordinary proof in which every step must be justified. However it employs a neat trick which allows you to prove a statement about an arbitrary number n by first proving it is true when n is 1 and then assuming it is true for n=k and showing it is true for n=k+1. The idea is that if you want to show that someone can climb to the nth floor of a fire escape, you need only show that you can climb the ladder up to the fire escape ($n = 1$) and then show that you know how to climb the stairs from any level of the fire escape ($n = n_0$) to the next level (i.e. $n = n_0 + 1$).

2 Induction Examples

$7n - 2$ is $\mathcal{O}(n)$

Need $c > 0$ and $n_0 \geq 1$ such that $7n - 2 \leq cn$ for $n \geq n_0$

This is true for $c = 7$ and $n_0 = 1$

3 Let's look at n_0 first

For starters, we cannot have negative running time so n_0 needs to be ≥ 0.

For n=0,1,2 when does the equality hold true that $7n - 2 \leq cn$ for $n \geq n_0$?

Let's restate this as:

What is the minimum value of n where $7n - 2 \leq cn$?

Let's try $n_0 = 0$

$f(n_0) = 7 * (0) - 2 = -2$

Negative, so not this.

Let's try $n_0 = 1$

$f(n_0) = 7 * (1) - 2 = 5$

This is true so, the minimum n_0 is 1.

4 Now let's look at c

For what value of c is $7n - 2 \leq cn$?

I start with the left side of the equation I want to show and proceed using the induction hypothesis and algebra to reach the right side of the equation.

Remember that n can grow from $n_0 = 1$.

For $n = 1$: $7(1) - 2 \leq c(1)$,
$5/2 = 2.5$.
$c \geq 5$
For $n = 2$: $7(2) - 2 \leq c(2)$, $12/2 = 6$. $c \geq 6$
For $n = 3$: $7(3) - 2 \leq c(3)$, $19/3 = 9.5$. $c \geq 6.3$
For $n = 4$: $7(4) - 2 \leq c(4)$, $26/4 = 13$. $c \geq 6.5$
For $n = 7$: $7(7) - 2 \leq c(7)$, $47/7 = 6.7$. $c \geq 6.7$
For $n = 100$: $7(100) - 2 \leq c(100)$, $698/100 = 6.98$. $c \geq 6.98$
For $n = 1000$: $7(1000) - 2 \leq c(100 =)$, $6998/1000 = 6.98$. $c \geq 6.998$
Let's try plotting the function $f(n) = 7n - 2$ and boundary condition $7n$

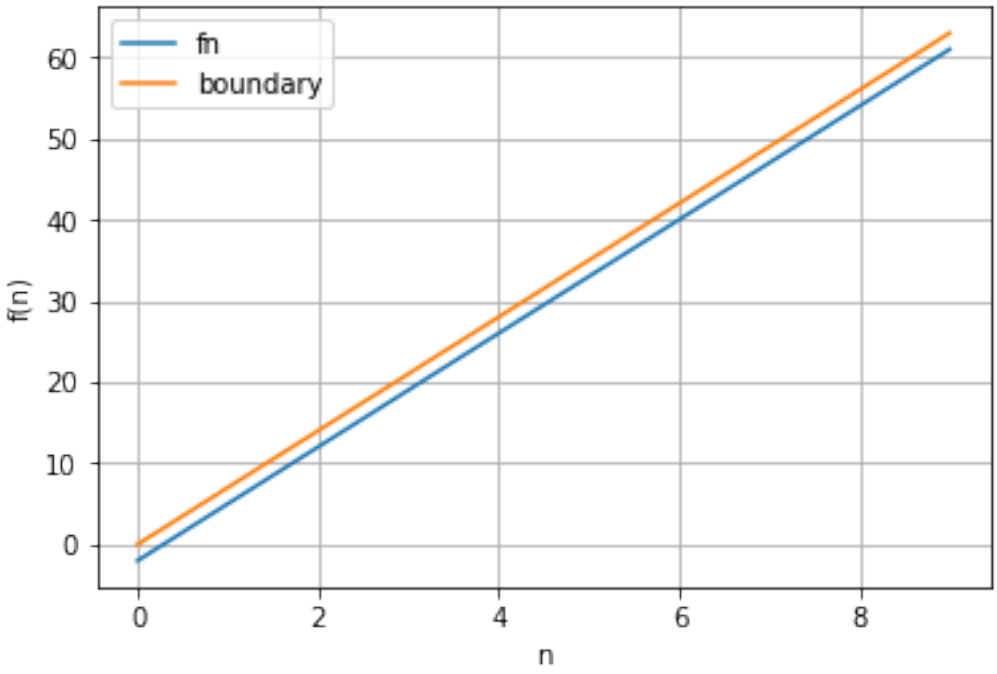
In [74]:
```
import matplotlib.pyplot as plt
import numpy as np
n=np.arange(0,10)

fn= 7*n - 2

c=7

boundary = c*n

plt.plot(n,fn)
plt.plot(n,boundary)
plt.grid()
plt.xlabel('n')
plt.ylabel('f(n)')
plt.legend(['fn','boundary'])
plt.show()
```



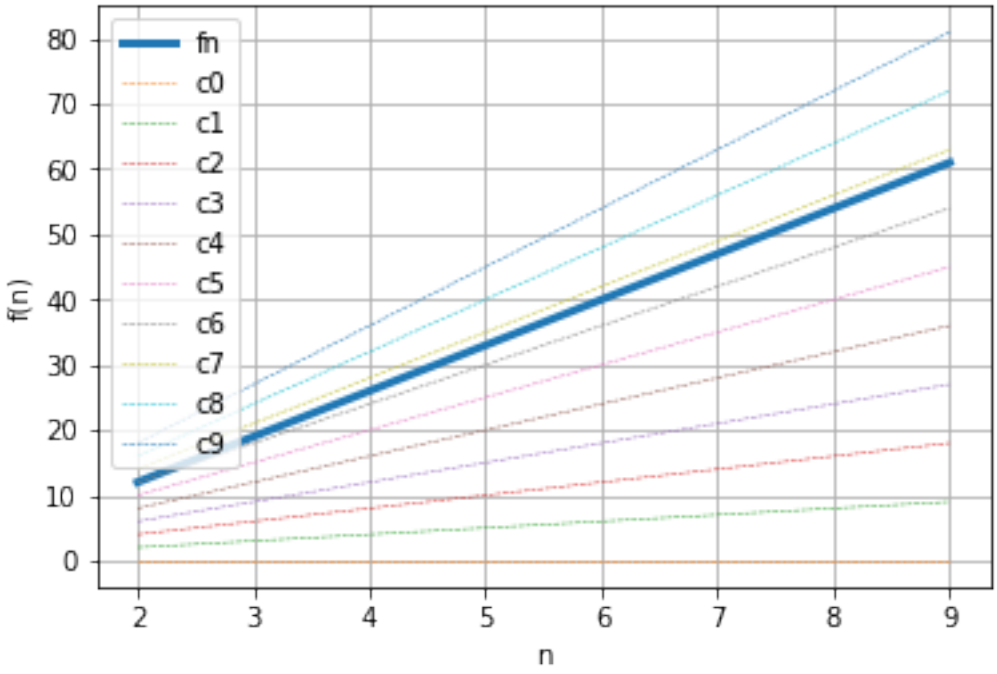
We can seen the orange boundary for $c = 7$ and $n_0 = 1$ is always above $f(n)$.

```
In [81]: fn= 7*n - 2
         plt.plot(n,fn,linewidth=3, label="f(n)")

         leg=['fn']

         for c in np.arange(10):
             boundary = c*n
             leg.append("c"+str(c))
             plt.plot(n,boundary,'--', linewidth=.5, label="c"+str(c))

         plt.grid()
         plt.xlabel('n')
         plt.ylabel('f(n)')
         plt.legend(leg)
         plt.show()
```



Here we have plotted $f(n)$ and all of the boundaries for $c = 0$ to 9 and $n = 0$ to 9. Our chosen boundary is the closest of the boundaries above the function.

5 Example 2

Our second example is $(O)n^3$.

$f(n) = 3n^3 + 20n^2 + 5$
$c = 4$ and $n_0 = 21$
Pick c greater than the coeffifient of n^3, so 4.
Pick n_0 greater than the coefficient of n^2.
Why, because $21^3 = 21(21^2)$ which is greater than $20(21^2)$.
That way the right hand size of the statement is greater than the left.

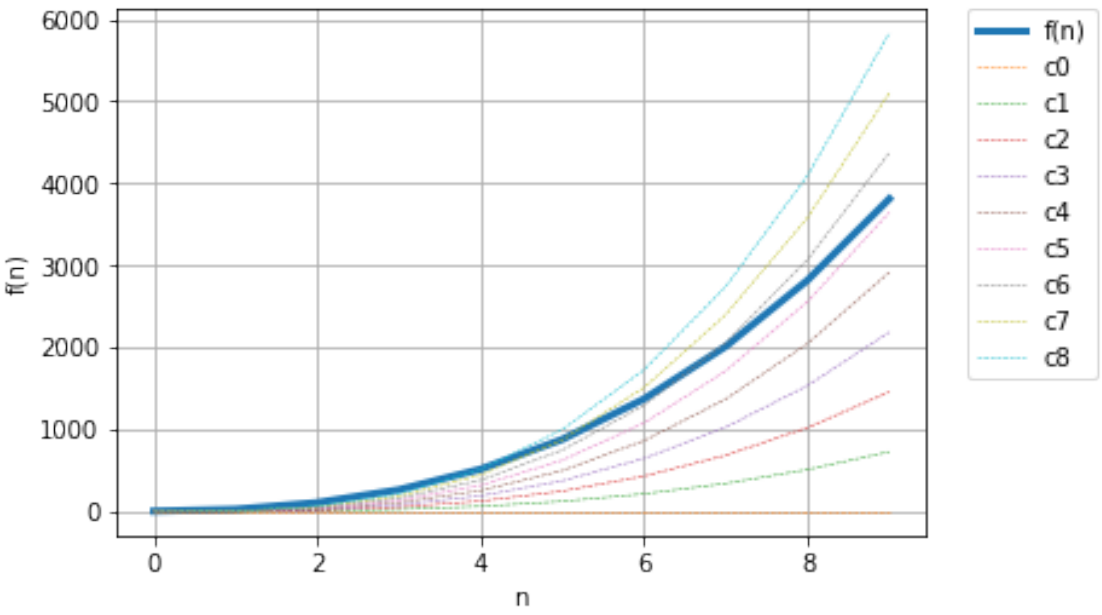
In [76]:
```
fn=3*n**3 + 20*n**2 + 5

plt.plot(n,fn,linewidth=3, label="f(n)")

for c in np.arange(9):
    boundary = c*n**3

    plt.plot(n,boundary,'--', linewidth=.5, label="c"+str(c))

plt.grid()
plt.xlabel('n')
plt.ylabel('f(n)')
plt.legend(bbox_to_anchor=(0., 1.02, 1., .102), loc=3,
           ncol=2, mode="expand", borderaxespad=0.)
plt.legend(bbox_to_anchor=(1.05, 1), loc=2, borderaxespad=0.)
plt.show()
```



6 Example 3

Our third example is $(O)log(n)$.

$f(n) = 3log(n) + 5$
$c = 8$ and $n_0 = 2$

```
In [77]: n=np.arange(2,10)

         fn=3*np.log2(n) +5

         plt.plot(n,fn,linewidth=3, label="f(n)")

         for c in np.arange(5,15):
             boundary = c*np.log2(n)

             plt.plot(n,boundary,'--', linewidth=.5, label="c"+str(c))

         plt.grid()
         plt.xlabel('n')
         plt.ylabel('f(n)')
         plt.legend(bbox_to_anchor=(0., 1.02, 1., .102), loc=3,
                    ncol=2, mode="expand", borderaxespad=0.)
         plt.legend(bbox_to_anchor=(1.05, 1), loc=2, borderaxespad=0.)
         plt.show()
```