

## Chapter 13 : Fibonacci.

*In which we look at a different shape of Invariant and look at a different way of motivating decreasing the variant  $vf$ .*

Sometimes, when we are presented with a problem we are given the domain model to start so we do not have to go and develop one ourselves. Here is such an example. Suppose we are given the fibonacci function defined as follows.

$$* (0) \text{ fib}.0 = 0$$

$$* (1) \text{ fib}.1 = 1$$

$$* (2) \text{ fib}.(n+2) = \text{fib}.n + \text{fib}.(n+1) , 0 \leq n$$

We are given a natural number  $N$  and asked to write a program to computer  $\text{fib}.N$

*Choosing invariants.*

We propose as invariants

$$\begin{aligned} P0 : \alpha * \text{fib}.n + \beta * \text{fib}.(n+1) &= \text{fib}.N \\ P1 : 0 \leq n \leq N \end{aligned}$$

*Establishing the Invariants.*

The following assignment establishes the invariants

$$n, \alpha, \beta := N, 1, 0$$

*Termination.*

We note the following.

$$\begin{aligned} & P0 \wedge P1 \wedge n=N \\ = & \quad \{\text{definitions of } P0, P1\} \\ & \alpha * \text{fib}.n + \beta * \text{fib}.(n+1) = \text{fib}.N \wedge 0 \leq n \leq N \wedge n=0 \\ = & \quad \{\text{algebra}\} \\ & \alpha * \text{fib}.0 + \beta * \text{fib}.1 = \text{fib}.N \wedge \text{true} \\ = & \quad \{\text{predicate calculus}\} \\ & \alpha * \text{fib}.0 + \beta * \text{fib}.1 = \text{fib}.N \\ = & \quad \{\text{defn. fib}\} \\ & \alpha * 0 + \beta * 1 = \text{fib}.N \end{aligned}$$

$$= \begin{array}{l} \{\text{arithmetic}\} \\ \beta = \text{fib.N} \end{array}$$

*Calculate the loop body.*

$$\begin{array}{l} \alpha * \text{fib.n} + \beta * \text{fib.(n+1)} = \text{fib.N} \\ = \begin{array}{l} \{\text{P1} \wedge n \neq 0, (2)\} \\ \alpha * \text{fib.n} + \beta * (\text{fib.(n-1)} + \text{fib.n}) = \text{fib.N} \\ = \begin{array}{l} \{*/+\} \\ \alpha * \text{fib.n} + \beta * \text{fib.(n-1)} + \beta * \text{fib.n} = \text{fib.N} \\ = \begin{array}{l} \{\text{algebra, grouping terms}\} \\ \beta * \text{fib.(n-1)} + (\alpha + \beta) * \text{fib.n} = \text{fib.N} \\ = \begin{array}{l} \{\text{WP calculus}\} \\ (n, \alpha, \beta := n-1, \beta, \alpha + \beta).P0 \end{array} \end{array} \end{array} \end{array}$$

*Finished program.*

And so the entire program is as follows.

$$\begin{array}{l} n, \alpha, \beta := N, 1, 0 \quad \{\text{P0} \wedge \text{P1}\} \\ ;\text{do } n \neq 0 \rightarrow \quad \{\text{P0} \wedge \text{P1} \wedge n \neq 0\} \\ \quad \quad \quad n, \alpha, \beta := n-1, \beta, \alpha + \beta \\ \quad \quad \quad \{\text{P0} \wedge \text{P1}\} \\ \text{od} \\ \{\beta = \text{fib.N}\} \end{array}$$