Chapter 25 : Slope search.

In which we make use of a nice mathematical property called monotonicity.

Given f[0..M,0..N] of int where $\{0 \le M \wedge 0 \le N\}$. We are told that f is ascending in both of its arguments.

The problem specification is as follows

$$\{\langle \exists\ i,j : 0 \le i \le M \wedge 0 \le j \le N : f.i.j = X \rangle\}$$

$$S$$

$$\{\ 0 \le a \le M \wedge 0 \le b \le N \wedge f.a.b = X\}$$

Domain modelling.

We make a model of our problem domain.

* (0) C.m.n = $\langle \exists\ i,j : m \le i \le M \wedge 0 \le j \le n : f.i.j = X \rangle$

From this we can derive the following theorems

- (1) C.0.N ≡ true[1]

We observe

$$
\begin{array}{ll}
 & C.m.n \\
= & \\
 & \langle \exists\ i,j : m \le i \le M \wedge 0 \le j \le n : f.i.j = X \rangle \\
= & \qquad \{\text{Split off } i = m \text{ term}\} \\
 & \langle \exists\ i,j : m+1 \le i \le M \wedge 0 \le j \le n : f.i.j = X \rangle \vee \langle \exists\ j : 0 \le j \le n : f.m.j = X \rangle \\
= & \qquad \{\ (0), (3)\ \} \\
 & C.(m+1).n \vee D.n
\end{array}
$$

So we have

- (2) C.m.n ≡ C.(m+1).n ∨ D.n ,$0 \le m < M$

* (3) D.n ≡ $\langle \exists\ j : 0 \le j \le n : f.m.j = X \rangle$

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[1] This is simply what is given in the Precondition.

Similarly, we observe

$$
\begin{array}{ll}
 & C.m.n \\
= & \\
 & \langle \exists\, i,j : m \le i \le M \wedge 0 \le j \le n : f.i.j = X \rangle \\
= & \qquad \{\text{Split off } j = n \text{ term}\} \\
 & \langle \exists\, i,j : m \le i \le M \wedge 0 \le j \le n\text{-}1 : f.i.j = X \rangle \vee \langle \exists\, i : m \le i \le M : f.i.n = X \rangle \\
= & \qquad \{ (0), (5) \} \\
 & C.m.(n\text{-}1) \vee E.m
\end{array}
$$

- (4) $C.m.n \quad \equiv \quad C.m.(n\text{-}1) \vee E.m \qquad ,0 < n \le M$

* (5) $E.m \quad \equiv \quad \langle \exists\, i : m \le i \le M : f.i.n = X \rangle$

Now let us consider D.n and E.m in turn.

- (6) $D.n \quad \equiv \quad \text{false} \quad \Leftarrow \quad f.m.n < X$

- (7) $D.n \quad \equiv \quad \text{true} \quad \Leftarrow \quad f.m.n = X$

- (8) $D.n \quad \equiv \quad ? \quad \Leftarrow \quad f.m.n > X$

- (9) $E.m \quad \equiv \quad ? \quad \Leftarrow \quad f.m.n < X$

- (10) $E.m \quad \equiv \quad \text{true} \quad \Leftarrow \quad f.m.n = X$

- (11) $E.m \quad \equiv \quad \text{false} \quad \Leftarrow \quad f.m.n > X$

Choose invariants.
For our invariants we choose the following

$P0 : C.a.b$
$P1 : 0 \le a \le M \wedge 0 \le b \le N$

Guard.
We choose as our guard

$f.a.b \neq X$

Termination.
Upon termination of the loop we note

$$P0 \wedge P1 \wedge f.a.b = X \Rightarrow Post$$

Loop Body.
We now calculate the loop body.

$$
\begin{array}{ll}
 & P0 \\
= & \quad \{\text{definition of } P0\} \\
 & C.a.b \\
= & \quad \{ (2)\} \\
 & C.(a+1).b \vee D.b \\
= & \quad \{\text{case analysis } f.a.b < X \ (6)\} \\
 & C.(a+1).b \vee false \\
= & \quad \{ ID \vee \} \\
 & C.(a+1).b \\
= & \quad \{WP\ ) \\
 & (a := a+1).P0
\end{array}
$$

Which gives us the program fragment

$$\text{if } f.a.b < X \rightarrow a := a+1$$

Because of the guard we do not consider the case f.a.b = X.

Now we seek to exploit E. We observe

$$
\begin{array}{ll}
 & P0 \\
= & \quad \{\text{definition of } P0\} \\
 & C.a.b \\
= & \quad \{ (4) \} \\
 & C.a.(b-1) \vee E.a \\
= & \quad \{\text{case analysis } f.a.b > X \ (11)\} \\
 & C.a.(b-1) \vee false \\
= & \quad \{ ID \vee \} \\
 & C.a.(b-1) \\
= & \quad \{WP\} \\
 & (b := b-1).P0
\end{array}
$$

This gives us the program fragment

$$\text{if } f.a.b > X \rightarrow b := b-1$$

Because of the guard we do not need to consider the case f.a.b = X.

Finished Program.

Putting this together we arrive at our finished program

```
a, b := 0, N {P0 ∧ P1}
;do f.a.b ≠ X →          {P0 ∧ P1 ∧ f.a.b ≠ X}

                          if f.a.b < X → a := a+1
                          [] f.a.b > X → b := b-1
                          fi

                          {P0 ∧ P1}
od
{Post}
```

This has temporal complexity O(M+N).