Chapter 27 : Another appeal to monotonicity.

In which we once again use that nice property.

We are given f[0..M), g[0..N) of int. We are told that f is ascending and g is descending. We are asked to construct a program to compute the number of pairs f.i and g.j whose sum exceeds 37.

$$r = \langle + \, i,j : 0 \le i < M \wedge 0 \le j < N : h.(f.i).(g.j) \rangle$$

where

* (0) h.x.y $=$ 1 $\Leftarrow$ $x + y > 37$
* (1) h.x.y $=$ 0 $\Leftarrow$ $x + y \le 37$

We begin by modelling our domain.

* (2) C.m.n $=$ $\langle + \, i,j : m \le i < M \wedge n \le j < N : h.(f.i).(g.j) \rangle$

Emptying the components of the range in turn lead us to the following

- (3) C.M.n $=$ 0 , $0 \le n \le N$

- (4) C.m.N $=$ 0 , $0 \le m \le M$

We observe,

C.m.n
= {(2)}
$\langle + \, i,j : m \le i < M \wedge n \le j < N : h.(f.i).(g.j) \rangle$
= { Split off i = m term }
$\langle + \, i,j : m+1 \le i < M \wedge n \le j < N : h.(f.i).(g.j) \rangle + \langle + \, j : n \le j < N : h.(f.m).(g.j) \rangle$
= {(2), (7)}
C.(m+1).n + D.n

- (5) C.m.n $=$ C.(m+1).n + D.n , $0 \le m < M$

We observe,

C.m.n
= {(2)}
$\langle + \, i,j : m \le i < M \wedge n \le j < N : h.(f.i).(g.j) \rangle$
= { Split off j = n term }
$\langle + \, i,j : m \le i < M \wedge n+1 \le j < N : h.(f.i).(g.j) \rangle + \langle + \, i : m \le i < M : h.(f.i).(g.n) \rangle$
= {(2), (8)}

$C.m.(n+1) + E.m$

- (6) $C.m.n = C.m.(n+1) + E.m \quad , 0 < n \leq N$

* (7) $D.n = \langle + j : n \leq j < N : h.(f.m).(g.j) \rangle$

-*(8) $E.m = \langle + i : m \leq i < M : h.(f.i).(g.n) \rangle$

We now turn our attention to investigating D and E

- (9) $D.n = ? \quad \Leftarrow f.m + g.n > 37$

- (10) $D.n = 0 \quad \Leftarrow f.m + g.n \leq 37$

- (11) $E.m = M\text{-}m \quad \Leftarrow f.m + g.n > 37$

- (12) $E.m = ? \quad \Leftarrow f.m + g.n \leq 37$

Our postcondition can now be written as

Post : $r = C.0.0$

Invariants.
As invariants we choose

P0 : $r + C.m.n = C.0.0$
P1 : $0 \leq m \leq M \wedge 0 \leq n \leq N$

Establish Invariants.

$R, m, n := 0, 0, 0$

Upon termination

$P0 \wedge P1 \wedge (m=M \vee n=N) \Rightarrow Post$

Guard.

$m \neq M \wedge n \neq N$

Loop body

$$
\begin{array}{ll}
 & P0 \\
= & \qquad \{\text{definition of } P0\} \\
 & r + C.m.n = C.0.0 \\
= & \qquad \{(5)\} \\
 & r + C.(m+1).n + D.n = C.0.0 \\
= & \qquad \{\text{case analysis } f.m + g.n \le 37 \ (10)\} \\
 & r + C.(m+1).n + 0 = C.0.0 \\
= & \qquad \{WP\} \\
 & (r, m := r+0, m+1).P0
\end{array}
$$

Giving us

if $f.m + g.n \le 37 \rightarrow r, m := r+0 , m+1$

We also observe

$$
\begin{array}{ll}
 & P0 \\
= & \qquad \{\text{definition of } P0\} \\
 & r + C.m.n = C.0.0 \\
= & \qquad \{(11)\} \\
 & r + C.m.(n+1) + E.m = C.0.0 \\
= & \qquad \{\text{case analysis } f.m + g.n > 37 \ (1)\} \\
 & r + C.m.(n+1) + M\text{-}m = C.0.0 \\
= & \qquad \{WP\} \\
 & (r, n := r +(M\text{-}m), n+1).P0
\end{array}
$$

Giving us

if $f.m + g.n > 37 \rightarrow r, n := r +(M\text{-}m), n+1$

There is no point in appealing to (12) so we ignore that case

Finished Algorithm.

$r, n, m := 0, 0, 0 \ \{P0 \wedge P1\}$

; do $m \neq M \wedge n \neq N \rightarrow \quad \{P0 \wedge P1 \wedge m \neq M \wedge n \neq N\}$

if $f.m + g.n \le 37 \rightarrow r, m := r+0 , m+1$

[]$f.m + g.n > 37 \rightarrow r, n := r +(M\text{-}m), n+1$

fi

$\{P0 \wedge P1\}$

od

$\{P0 \wedge P1 \wedge (m=M \vee n=N)\}$