Analysis of Algorithms

Lecture No. 6

Complexity and asymptotic Notations

Average case analysis

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Recap

- What is complexity
- Asymptotic analysis
- Types of Analysis
- Big O Notation (Worst Case Analysis)
- Big Omega (Best Case analysis)
- Examples

Today's Lecture

- Theta Notation (Average Case Analysis)
- Little o
- Little Omega
- Comparison

Asymptotic Notation

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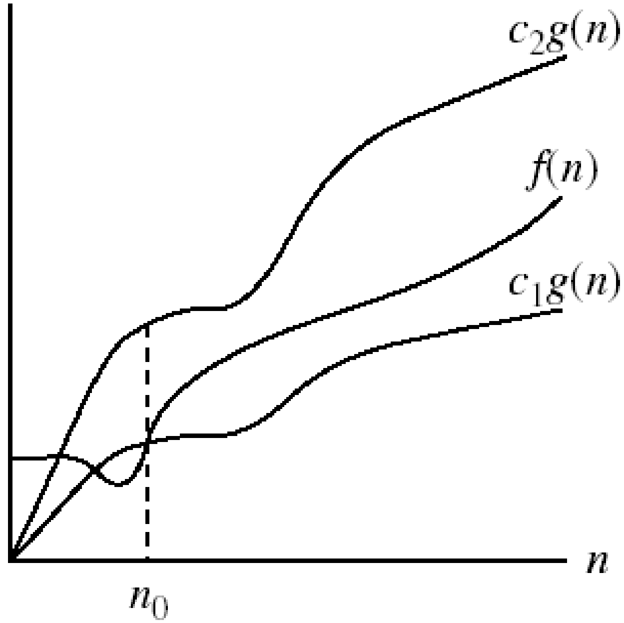
- **Theta Notation** *Θ*
- Theta Notation For non-negative functions, *f(n)* and *g(n)*, *f(n)* is theta of *g(n)* if and only if *f(n) = O(g(n))* and *f(n) = Ω(g(n))*. This is denoted as "*f(n) = Θ(g(n))*".

  This is basically saying that the function, *f(n)* is bounded both from the top and bottom by the same function, *g(n)*.

Asymptotic notations (cont.)

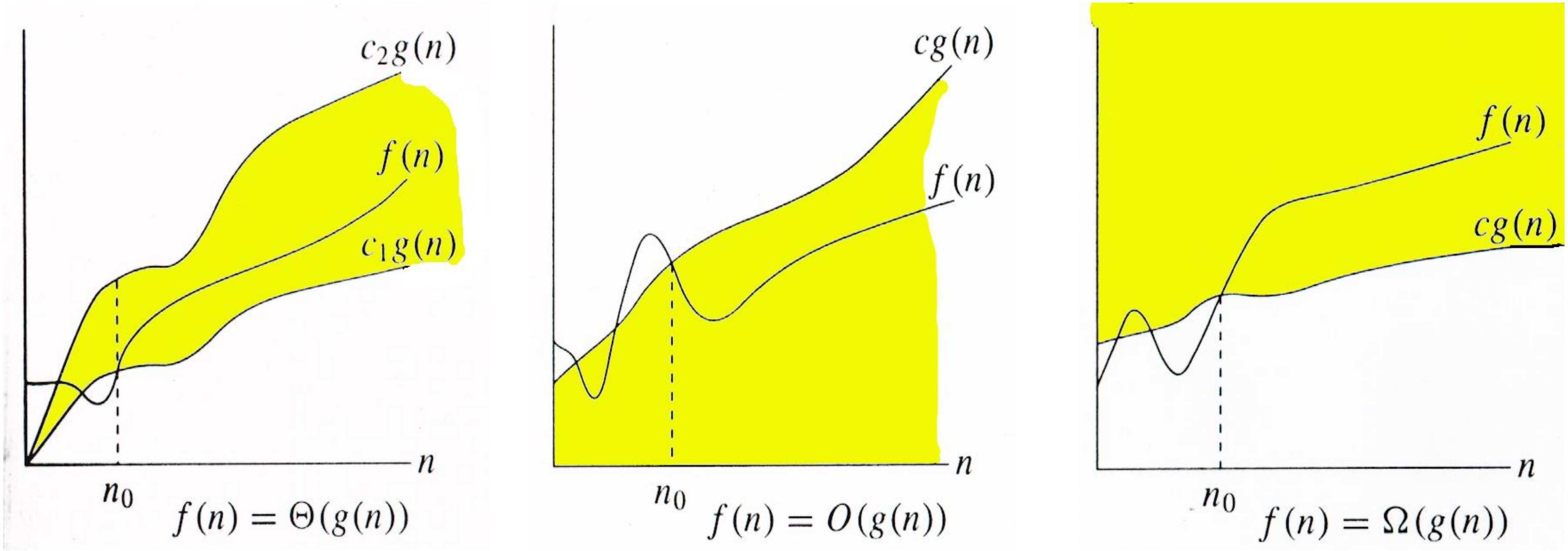
$$\Theta(g(n)) = \{f(n) : \text{ there exist positive constants } c_1, c_2, \text{ and } n_0 \text{ such that } 0 \le c_1 g(n) \le f(n) \le c_2 g(n) \text{ for all } n \ge n_0\} .$$



$\Theta(g(n))$ is the set of functions with the same order of growth as $g(n)$

$g(n)$ is an ***asymptotically tight bound*** for $f(n)$.

Relations Between Θ, *O*, Ω



Relations Between Θ, O, Ω

Theorem : For any two functions $g(n)$ and $f(n)$,

$f(n) = \Theta(g(n))$ iff

$f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

- I.e., $\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$
- In practice, asymptotically tight bounds are obtained from asymptotic upper and lower bounds.

Asymptotic Notation

Example 1: Prove that $3n^2+5n+1 \in \Theta(n^2)$

Proof

Step 1: Define f(n) and g(n)

Let:

- $f(n) = 3n^2 + 5n + 1$
- $g(n) = n^2$

We want to prove that:

Step 2: Use Θ-notation definition

A function $f(n) \in \Theta(n^2)$ if there exist constants $c_1, c_2 > 0$ and n_0 such that:

$$c_1 n^2 \le 3n^2 + 5n + 1 \le c_2 n^2 \quad \text{for all } n \ge n_0$$

For Upper bound

$$3n^2 + 5n + 1 \le c_2 n^2 \Rightarrow \frac{3n^2 + 5n + 1}{n^2} \le c_2 \Rightarrow 3 + \frac{5}{n} + \frac{1}{n^2} \le c_2$$

As $n \to \infty$, $\frac{5}{n} \to 0$, $\frac{1}{n^2} \to 0$

So, pick $n_0 = 1$, and for all $n \ge 1$:

$$3 + \frac{5}{n} + \frac{1}{n^2} \le 3 + 5 + 1 = 9 \Rightarrow f(n) \le 9n^2$$

So, we can choose $c_2 = 9$

✅ Upper Bound Proven:

$$f(n) \le 9n^2$$

Asymptotic Notation

Example 1: Prove that $3n^2+5n+1 \in \Theta(n^2)$

We want:

$$3n^2+5n+1 \geq c_1 n^2 \Rightarrow \frac{3n^2+5n+1}{n^2} \geq c_1 \Rightarrow 3+\frac{5}{n}+\frac{1}{n^2} \geq c_1$$

As $n \to \infty$, $\frac{5}{n}+\frac{1}{n^2} \to 0$

So for large n, $f(n)/n^2$ approaches 3 from above. Let's pick $n_0 = 1$, then:

$$3+\frac{5}{n}+\frac{1}{n^2} \geq 3 \Rightarrow f(n) \geq 3n^2$$

So we can take $c_1 = 3$

✅ Lower Bound Proven:

$$f(n) \geq 3n^2$$

$$c_1 n^2 \leq f(n) \leq c_2 n^2 \quad \text{for all } n \geq n_0$$

🔷 **Therefore:**

$$3n^2+5n+1 \in \Theta(n^2)$$

Asymptotic Notation

Example 2: Prove that $f(n) = 2n^2 + 3n + 6 \notin \Theta(n^3)$

Proof: Let $f(n) = 2.n^2 + 3.n + 6$, and $g(n) = n^3$

we have to show that $f(n) \notin \Theta(g(n))$

On contrary assume that $f(n) \in \Theta(g(n))$ i.e.

there exist some positive constants c_1, c_2 and n_0 such that: $c_1.g(n) \leq f(n) \leq c_2.g(n)$

$c_1.g(n) \leq f(n) \leq c_2.g(n) \Rightarrow c_1.n^3 \leq 2.n^2 + 3.n + 6 \leq c_2.\ n^3 \Rightarrow$

$c_1.n \leq 2 + 3/n + 6/n^2 \leq c_2.\ n \Rightarrow$

$c_1.n \leq 2 \leq c_2.\ n$, for large $n \Rightarrow$

which is not possible

Hence $f(n) \notin \Theta(g(n)) \Rightarrow 2.n^2 + 3.n + 6 \notin \Theta(n^3)$

Asymptotic notations (cont.)

- **Little-O Notation**

Definition: Little-o notation describes an *upper bound* that a function **strictly** falls short of — meaning the function grows *slower* than another function as $n \to \infty$.

Formally:

We write

> **f(n) ∈ o(g(n))**
> **if for every constant c > 0, there exists an n_0 such that:**
> **$f(n) < c \cdot g(n)$ for all $n \geq n_0$**

This means that **f(n)** grows *strictly slower* than **g(n)**.

- **Example 1:**
  Let **f(n) = n**, and **g(n) = n^2**
  We say **n ∈ o(n^2)** because no matter how small the constant **c**, eventually **$n < c \cdot n^2$** for sufficiently large **n**.

Asymptotic notations (cont.)

- **Little-O Notation**
- **Example 2:** $f(n) = 2n^2, g(n) = n^2$:
- We say f(n) ∈ o(g(n)) if and only if for every constant $c > 0$, there exists an n_0 such that:$f(n) < c \cdot g(n)$ for all $n \geq n_0$
- $2n^2 < c \cdot n^2$
- Divide both sides by n^2 (for $n > 0$):
- $2 < c$

This inequality **2 < c** must be **true for all c > 0**, but it's **not**:

- For **c = 1**, we get **2 > c**, so **$2n^2 < c \cdot n^2$** does **not hold**
- For **c = 1.5**, same — inequality fails.

So **$2n^2 < c \cdot n^2$** is **not true for all positive c** — it's only true when **c > 2**

– But **little-o** requires it to hold for ***every*** $c > 0$, no matter how small.

Asymptotic notations for little o

Example 3: Prove that $2n^2 \in o(n^3)$

Proof:

Assume that $f(n) = 2n^2$, and $g(n) = n^3$

$f(n) \in o(g(n))$?

Now we have to find the existence n_0 for any c

$f(n) < c.g(n)$ this is true

⍰ $2n^2 < c.n^3$ ⍰ $2 < c.n$

This is true for any c, because for any arbitrary c we can choose n_0 such that the above inequality holds.

Hence $f(n) \in o(g(n))$

Asymptotic notations for Little o

Example 4: Prove that $n^2 \notin o(n^2)$

Proof:

Assume that $f(n) = n^2$, and $g(n) = n^2$

Now we have to show that $f(n) \notin o(g(n))$

Since

$f(n) < c.g(n)$ ⍰ $n^2 < c.n^2$ ⍰ $1 \leq c$,

In our definition of small o, it was required to prove for any c but here there is a constraint over c . Hence, $n^2 \notin o(n^2)$, where $c = 1$ and $n_0 = 1$

Asymptotic notations (cont.)

- **Little Omega Notation**
- f(n)∈ω(g(n)) if:∀c>0,∃n0 such that $f(n)>c \cdot g(n)$for all n≥n0∀

> For every constant $c > 0$, there exists a constant n_0 such that for all $n \geq n_0$,

$$f(n) > c \cdot g(n)$$

- So, no matter how large the constant c, $f(n)$ eventually outgrows c·g(n).

Asymptotic notations for Little omega

Prove that $5.n^2 \in \omega(n)$

Proof:

Assume that $f(n) = 5.n^2$, and $g(n) = n$

$f(n) \in \Omega(g(n))$?

We have to prove that for any c there exists n_0 s.t.,

$c.g(n) < f(n) \quad \forall\, n \geq n_0$

$c.n < 5.n^2 \Rightarrow c < 5.n$

And hence $f(n) \in \omega(g(n))$,

Asymptotic notations Comparison

Notation	Symbol	Meaning	Definition (Inequality Form)	Growth Relation	Bound Type
Big-O	$O(g(n))$	Upper Bound	$\exists\, c > 0,\ \exists\, n_0 :$ $f(n) \leq c \cdot g(n)\ \forall n \geq n_0$	$f(n)$ grows **at most** like $g(n)$	Loose or Tight Upper Bound
Little-o	$o(g(n))$	Strict Upper Bound	$\forall\, c > 0,\ \exists\, n_0 :$ $f(n) < c \cdot g(n)\ \forall n \geq n_0$	$f(n)$ grows **strictly slower** than $g(n)$	Strict (Loose) Upper Bound
Big-Omega	$\Omega(g(n))$	Lower Bound	$\exists\, c > 0,\ \exists\, n_0 :$ $f(n) \geq c \cdot g(n)\ \forall n \geq n_0$	$f(n)$ grows **at least** like $g(n)$	Loose or Tight Lower Bound
Little-omega	$\omega(g(n))$	Strict Lower Bound	$\forall\, c > 0,\ \exists\, n_0 :$ $f(n) > c \cdot g(n)\ \forall n \geq n_0$	$f(n)$ grows **strictly faster** than $g(n)$	Strict (Loose) Lower Bound

Summery

- Asymptotic Notation