Numerical Computing

Date: Makam

⇒ Week 1: Lecture 1

Error

round off   5.43512 ≈ 5.44

truncate   5.43512 ≈ 5.43   (simply cut off)

1. Round off error

term >5 , add 1 to previous term

term < 5 , previous term remains same

2. Truncation error

approximation of an infinite mathematical procedure.

3. Absolute error

Absolute = |exact - approximate|
E_a

4. Relative error

$$E_R = \left| \frac{E_A}{exact} \right|$$

5. Percentage error

$E_R \times 100$

Date:

⇒ Week 1 : Lecture 2

→ Interpollation

Interpollation

equally spaced data | unequally spaced data

- Newton forward difference
- Newton backward difference

→ Newton forward difference formula

Let $x_0, x_1, \dots x_n$

$y_0, y_1, \dots y_m$ $\quad y = f(x)$

$y_0 \quad y_1 \quad y_2$

$x_0 \quad x_1 = x_0 + h \quad x_2 = x_0 + 2h \quad x = x_0 + ph$

$x_p = x_0 + ph$

$p = \dfrac{x - x_0}{h}$ $\quad$ here $x = x_p$

$E^p f(x) = f(x + ph)$

$E^p f(x_0) = f(x_0 + ph)$ $\quad \therefore x_p = x = x_0 + ph$

$E^p y_0 = f(x)$

$f(x) = E^p y_0$

$y = (1 + \Delta)^p y_0$ $\quad (1+x)^m = 1 + mx + \dfrac{n(n-1)x^2}{2!} \dots$

$E^p f(x) = f(x + ph)$

$E = 1 + \Delta$

$$y = \left[1 + P\Delta + \frac{P(P-1)}{2!}\Delta^2 + \frac{P(P-1)(P-2)}{3!}\Delta^3 + \dots\right] y_0$$

$$y = \left[y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \dots\right]$$

Date:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0	y_0			
x_1	y_1	$\Delta y_0 = y_1 - y_0$	$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$	
x_2	y_2	$\Delta y_1 = y_2 - y_1$	$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$	$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$
x_3	y_3	$\Delta y_2 = y_3 - y_2$		

- how much formula evaluated depends on the number of datapoints.
- 2 datapoints → linear
  3 datapoints → quadratic
  polynomial is 1 less order of # of datapoints

- Question

The following are data from the steam table

Temperature	140	150	160	170	180
Pressure	3.685	4.854	6.302	8.076	10.225

Using Newtons formula find pressure of the steam for a temperature of x°c

Step 1: Calculate h

$h = 10$

Step 2: Calculate p $\quad p = \frac{x - x_0}{h}$

$p = \frac{x - 140}{10}$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
140	3.685				
		1.169			
150	4.854		0.279		
		1.448		0.047	
160	6.302		0.326		0.002
		1.774		0.049	
170	8.076		0.375		
		2.149			
180	10.225				

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$$y = 3.685 + P(1.169) + \frac{P(P-1)\,0.279}{2} + \frac{P(P-1)(P-2)\,0.047}{6} + \frac{P(P-1)(P-2)(P-3)\,0.002}{24}$$

$$= 3.685 + 1.169P + 0.1395\,P(P-1) + 0.00783\,P(P-1)(P-2) + 0.000083\,P(P-1)(P-2)(P-3$$

⇒ Week 2 : Lecture 1

→ Newton Backward difference formula

x	y	∇y
x_0	y_0	
		$\nabla y_1 = y_1 - y_0$
x_1	y_1	
		$\nabla y_2 = y_2 - y_1$
x_2	y_2	
		$\nabla y_3 = y_3 - y_2$
x_3	y_3	

In backward, $\Delta y_i = y_i - y_{i-1}$ where $i = n$ in start (start from end/back)

$$p = \frac{x - x_m}{h}$$

Formula: $y = y_n + p\nabla y + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \dots$

Date:

Question: Apply Newton ~~Gauss's~~ forward formula to find value of $f(x)$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
2.5	24.145					
		$\Delta y_0 = -2.102$				
3.0	22.043		0.284	-0.47		
3.5	20.225	$\Delta y_1 = -1.81$	0.237		0.009	
4.0	18.644	$\Delta y_2 = -1.58$	0.199	-0.038	0.006	-0.003
4.5	17.262	$\Delta y_3 = -1.38$	0.167	-0.032		
5.0	16.047	$\Delta y_4 = -1.215$				

1. Equally spaced data so Newton's forward/backward used
2. $p = \dfrac{x - x_0}{h}$

$$= \frac{x - 2.5}{0.5}$$

$$= 2x - 5$$

3. Now by using NDDF (finding polynomial)

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$+ \frac{p(p-1)(p-2)(p-3)(p-4)}{5!}\Delta^5 y_0$$

$$= 24.145 + (2x-5)(-2.102) + \frac{(2x-5)(2x-5-1)(0.284)}{2} +$$

$$\frac{(2x-5)(2x-5-1)(2x-5-2)(-0.47)}{3!} + \frac{(2x-5)(2x-5-1)(2x-5-2)(2x-5-3)0.009}{4!}$$

$$+ \frac{(2x-5)(2x-5-1)(2x-5-2)(2x-5-3)(2x-5-4)(-0.003)}{5!}$$

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If $x = 3.2$ given in the start

$p = \frac{x - x_0}{h}$ (find value of p)

$= \frac{3.2 - 2.5}{0.5}$

$= 1.4$

Now by using NDDF formula, substitute value of p and find specific y.

- if we have to find functional value in the starting region of the values we use Newton's forward, and if in ending region, we use Newton's backward. (answer same in both cases)

→ Guass formulas

- Guass forward and backward formula is used for finding values in the central region.
- In central region, the formula we chose depends on the value of p
  - Guass forward : $p: 0 - 1$
  - Guass backward $p: -1 - 0$

- $p = \frac{x - x_0}{h}$ for both forward and backward

→ Week 2 : Lecture 2

→ Guass's forward

x		y		Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_{-2}	x_0	y_0	y_{-2}	$\Delta y_{-2} =$	$\Delta^2 y_{-2}$		
x_{-1}	x_1	y_1	y_{-1}	Δy_{-1} :	$\boxed{\Delta^2 y_{-1}}$	$\Delta^3 y_{-2}$	$\boxed{\Delta^4 y_{-2}}$
x_0	x_2	y_2	$\boxed{y_0}$	$\boxed{\Delta y_0}$	$\Delta^2 y_0$	$\boxed{\Delta^3 y_{-1}}$	
x_1	x_3	y_3	y_1	Δy_1			
x_2	x_4	y_4	y_2				

central line

formula contains odd diff below the central line + even on the central ~~line~~

Formula

$$y_p = y_0 + p\Delta y_0 + \left[\frac{p(p-1)}{1\cdot 2}\right]\Delta^2 y_{-1} + \left[\frac{(p+1)p(p-1)}{1\cdot 2\cdot 3}\right]\Delta^3 y_{-1} +$$

$$\left[\frac{(p+1)p(p-1)(p-2)}{1\cdot 2\cdot 3\cdot 4}\right]\Delta^4 y_{-2} + \cdots$$

here $0 < p < 1$

→ Guass's backward formula

x		y		∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	
x_{-2}	x_0	y_0	y_{-2}					
x_{-1}	x_1	y_1	y_{-1}	∇y_{-2}	$\nabla^2 y_{-2}$			
x_0	x_2	y_2	$\boxed{y_0}$	$\boxed{\nabla y_{-1}}$	$\boxed{\nabla^2 y_{-1}}$	$\boxed{\nabla^3 y_{-2}}$	$\boxed{\nabla^4 y_{-2}}$	- - - central line
x_1	x_3	y_3	y_1	∇y_0	$\nabla^2 y_0$	$\nabla^3 y_{-1}$		
x_2	x_4	y_4	y_2	∇y_1				

This formula contains odd differences above the central line and even differences on the line

Formula:

$$y_p = y_0 + (p)\Delta y_{-1} + \left[\frac{(p+1)(p)}{1\cdot 2}\right]\Delta^2 y_{-1} + \left[\frac{(p+1)(p)(p-1)}{1\cdot 2\cdot 3}\right]\Delta^3 y_{-2} +$$

$$\left[\frac{(p+2)(p+1)p(p-1)}{1\cdot 2\cdot 3\cdot 4}\right]\Delta^4 y_{-2} + \cdots$$

- In Guass's forward formula we chose the upward value as x_0
- In Guass's backward formula we chose the ~~back~~ down ward value as x_0

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→ Stirling's formula

Average of two Guass's formula gives the Stirling's formula

Formula:

$$y_p = y_0 + p\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \left[\frac{p^2}{2!}\right](\Delta^2 y_{-1}) + \left[\frac{p(p^2-1^2)}{3!}\right]\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right]$$

$$+ \left[\frac{p^2(p^2-1^2)}{4!}\right](\Delta^4 y_{-2}) + \dots$$

It is used to interpolate the values of the function from the value $p\left(\frac{-1}{2} \le p \le \frac{1}{2}\right)$.

Good estimates if $\frac{-1}{4} \le p \le \frac{1}{4}$

• Note:

1. Formula involves means of odd differences just above and below the central line and even differences on this line

$$\begin{array}{cccccccccc} & \Delta y_{-1} & & \Delta^3 y_{-2} & & \Delta^5 y_{-3} & & & \\ \dots y_0 & & \Delta^2 y_{-1} & & \Delta^4 y_{-2} & & \Delta^6 y_{-3} & & \text{central line} \\ & \Delta y_0 & & \Delta^3 y_{-1} & & \Delta^5 y_{-2} & & & \\ & \downarrow & & \downarrow & & \downarrow & & & \\ & \text{Mean} & & \text{Mean} & & \text{Mean} & & & \end{array}$$

2. p must lie between $\frac{-1}{2}$ to $\frac{1}{2}$ while using this formula

Date:

→ Week 3 : Lecture 1

Guass's Forward : $0 < p < 1$
Guass's Backward $-1 < p < 0$
Stirling formula : $-0.5 < p < 0.5$
best estimation $-0.25 < p < 0.25$
Bessel's formula : $0.25 < p < 0.75$

→ Bessel's Formula

Formula:

$$y_p = \frac{y_0 + y_1}{2} + \left(p - \frac{1}{2}\right)(\Delta y_0) + \left[\frac{p(p-1)}{2!}\right]\left[\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right]$$

$$+ \left[\frac{(p-\frac{1}{2})p(p-1)}{3!}\right]\Delta^3 y_{-1} +$$

$$\left[\frac{(p+1)p(p-1)(p-2)}{4!}\right]\left[\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right] + \ldots$$

- $\frac{1}{4} \leq p \leq \frac{3}{4}$

- formula invowes odd differences below the central line and means of the even differences on and below the line

$$\left.\begin{matrix} y_0 \\ y_1 \end{matrix}\right\} \Delta y_0 \quad \left.\begin{matrix} \Delta^2 y_{-1} \\ \Delta^2 y_0 \end{matrix}\right\} \Delta^3 y_{-1} \quad \left.\begin{matrix} \Delta^4 y_{-2} \\ \Delta^4 y_{-1} \end{matrix}\right\}$$

↑ Mean ↑ Mean ↑ Mean

- coefficients of all odd differences are 0 when $p = \frac{1}{2}$. Thus special case is known as the formula for interpolating to halves.

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Date:

→ Newton Devided Difference formula for unequal intervals

devided differences ($\Delta f(x_0)$, (x_0, x_1), $f(x_0, x_1)$) representation

1st order differences

$$\Delta y_0 = \frac{y_1 - y_0}{x_1 - x_0} \quad (x_0, x_1)$$

$$\Delta y_1 = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\Delta y_2 = \frac{y_3 - y_2}{x_3 - x_2}$$

$$\Delta y_3 = \frac{y_4 - y_3}{x_4 - x_3}$$

2nd OD

$$\Delta^2 y_0 = \frac{\Delta y_1 - \Delta y_0}{x_2 - x_0} \quad (x_0, x_1, x_2)$$

$$\Delta^2 y_1 = \frac{\Delta y_2 - \Delta y_1}{x_3 - x_1}$$

$$\Delta^2 y_2 = \frac{\Delta y_3 - \Delta y_2}{x_4 - x_2}$$

3rd OD (x_0, x_1, x_2, x_3)

$$\Delta^3 y_0 = \frac{\Delta^2 y_1 - \Delta^2 y_0}{x_3 - x_0}$$

$$\Delta^3 y_1 = \frac{\Delta^2 y_2 - \Delta^2 y_1}{x_4 - x_1}$$

4th OD

$$\Delta^4 y_0 = \frac{\Delta^3 y_1 - \Delta^3 y_0}{x_4 - x_0} \quad (x_0, x_1, x_2, x_3, x_4)$$

→ Question :

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	5	150				
			$\Delta y_0 = 121$			
x_1	7	392		$\Delta^2 y_0 = 24$		
			$\Delta y_1 = 265$		$\Delta^3 y_0 = 1$	
x_2	11	1452		$\Delta^2 y_1 = 32$		$\Delta^4 y_0 = 0$
			$\Delta y_2 = 457$		$\Delta^3 y_1 = 1$	
x_3	13	2366		$\Delta^2 y_2 = \cancel{48}\ 42$		
			$\Delta y_3 = 709$			
x_4	17	5202				

Formula:

$$y = y_0 + (x - x_0)\Delta y_0 + (x - x_0)(x - x_1)\Delta^2 y_0 + (x - x_0)(x - x_1)(x - x_2)\Delta^3 y_0 + \dots + (x - x_0)(x - x_1)\dots(x - x_{n-1})\Delta^n y_0$$

$$y = 150 + (x-5)(121) + (x-5)(x-7)(24) + (x-5)(x-7)(x-11)(1)$$

$$= 150 + 121x - 605 + (x^2 - 12x + 35)(24) + (x^2 - 12x + 35)(x - 11)$$

$$= 150 + 121x - 605 + 24x^2 - 288x + 840 + x^3 - 11x^2 - 12x^2 + 132x + 35x - 35$$

$$= x^3 + x^2 + 350$$

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Date:

→ Week 3 : Lecture 2

→ Lagrange's Formula

Formula: $y = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} y_1$

$+ \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} y_3 + \dots$

- use when datapoints are less (5-6)

→ Lagrange's Inverse Formula

Formula: $x = \frac{(y-y_1)(y-y_2)(y-y_3)(y-y_4)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)(y_0-y_4)} x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)(y-y_4)}{(x_0-x_0)(x-x_2)(x-x_3)(x-x_4)} + \dots$

→ Numerical Differentiation

Formula:

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots$$

($p^2 - p$ written above the second coefficient, $p^3 - 3p^2 + 2p$ above the third)

where $p = \frac{x-x_0}{h}$

for more accuracy we take derivative to find values rather than derivative of polynomial (in equal intervals)
for unequal intervals,

$$y' = \frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx}$$

Guass ↑↓ Bessel derivative

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$$\frac{dy}{dp} = 0 + \Delta y_0 (1) + \frac{\Delta^2 y_0}{2!}(2P-1) + \frac{\Delta^3 y_0}{3!}(3P^2 - 6P + 2) + \cdots$$

$$= \Delta y_0 + \frac{(2P-1)}{2!}\Delta^2 y_0 + \frac{(3P^2 - 6P + 2)}{3!}\Delta^3 y_0 + \cdots$$

$$\frac{dp}{dx} = \frac{1}{h}$$

$$\frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx}$$

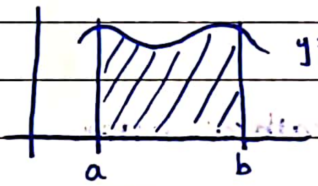
$$= \frac{1}{h}\left[\Delta y_0 + \frac{(2P-1)}{2!}\Delta^2 y_0 + \frac{(3P^2-6P+2)}{3!}\Delta^3 y_0 + \cdots\right]$$

→ Week 4 : Lecture 1

Numerical Integration

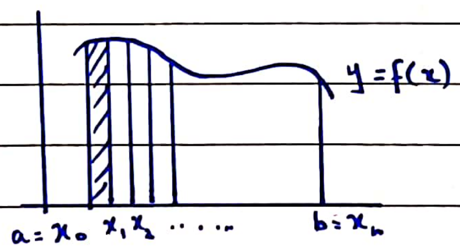
$\int \frac{x^2 e^{x/2}}{x+1} dx$ can't be integrated by parts

$$f(x) = \frac{x^2 e^{x/2}}{x+1}$$



Trapezoidal Rule

1. Devide the interval [a, b] into n equal sub-intervals



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→ Limits x_0, x_1 ↗ Newton forward formula

$$\int_{x_0}^{x_1} f(x)\,dx \approx \int_{x_0}^{x_1} p(x)\,dx = \int_{x_0}^{x_1} (y_0 + p\Delta y_0)\,dx$$

substitution

$p = \frac{x - x_0}{h}$

After substitution

$$= h\int_0^1 (y_0 + p\Delta y_0)\,dp$$

$x = x_0 + ph$

$\frac{dx}{dp} = h$

$$= h\left[y_0 p + \frac{p^2}{2!}\Delta y_0\right]_0^1$$

$dx = h\,dp$

$$= h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right]$$

when $x \to x_0, p \to 0$

$x \to x_1, p \to 1$

$$= h\left[y_0 + \frac{y_1}{2} - \frac{y_0}{2}\right]$$

$x_1 = x_0 + ph$

$x_0 + h = x_0 + ph$

$p = 1$

$$= h\left[\frac{y_0}{2} + \frac{y_1}{2}\right]$$

$$= \frac{h}{2}(y_0 + y_1)$$

→ Limits x_1, x_2

$$\int_{x_1}^{x_2} f(x)\,dx \approx \int_{x_1}^{x_2} (y_1 + p\Delta y_1)\,dx$$

limits

$p = \frac{x - x_1}{h}$

After substitution

$$= h\int_0^1 (y_1 + p\Delta y_1)\,dp$$

$x = x_1 + ph$

$\vdots$

$dx = h\,dp$

$$= \frac{h}{2}(y_1 + y_2)$$

$x \to x_1, p \to 0$

$x \to x_2, p \to 1$

$$\frac{h}{2}(y_1 + y_2)$$

$\vdots$

$$\frac{h}{2}(y_{n-1} + y_n)$$

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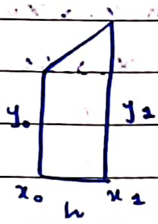
Formula

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \dots + \int_{x_{n-1}}^{x_n} f(x)\,dx$$

$$= \frac{h}{2}(y_0+y_1) + \frac{h}{2}(y_1+y_2) + \frac{h}{2}(y_2+y_3) + \dots \frac{h}{2}(y_{n-1}+y_n)$$

$$= \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+\dots+y_{n-1})\right]$$



Area $\square = hy$

Area $\Delta = \frac{1}{2}h(y_1-y_0)$

Area $= hy_0 + \frac{h}{2}(y_1-y_0)$

$= \frac{h}{2}(y_1-y_0) + hy_0$

Question: Calculate $\int_0^1 \frac{1}{1+x^2}\,dx$, using trapezoidal rule

consider $n=4$

$a=0,\ b=1$

$h = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4}$

$a = x_0 = 0$

$x_1 = x_0 + h = 0 + \frac{1}{4} = 1/4$

$x_2 = x_1 + h = \frac{1}{4} + \frac{1}{4} = 2/4$

$x_3 = x_2 + h = \frac{2}{4} + \frac{1}{4} = 3/4$

$x_4 = x_3 + h = \frac{3}{4} + \frac{1}{4} = 1$

$\int_a^b f(x)\,dx$

$y = \frac{1}{1+x^2}$

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$y_0 = \frac{1}{1+x_0^2} = \frac{1}{1+0} = 1$ $\quad$ $y_3 = \frac{1}{1+x_3^2} = \frac{1}{1+(\frac{3}{4})^2} = \frac{16}{25}$

$y_1 = \frac{1}{1+x_1^2} = \frac{1}{1+(\frac{1}{4})^2} = \frac{16}{17}$ $\quad$ $y_4 = \frac{1}{1+x_4^2} = \frac{1}{1+1^2} = \frac{1}{2}$

$y_2 = \frac{1}{1+x_2^2} = \frac{1}{1+(\frac{1}{2})^2} = \frac{4}{5}$

In formula,

$$= \frac{h}{2}\left[(y_0+y_4)+2(y_1+y_2+y_3)\right]$$

$$= \frac{1}{8}\left[\left(1+\frac{1}{2}\right)+2\left(\frac{16}{17}+\frac{4}{5}+\frac{16}{25}\right)\right]$$

$=$

⇒ Week 5: Lecture 1

Numerical Integration

Simpson's Method

$$\int_{x_0}^{x_2} f(x)\,dx = \int_{x_0}^{x_2}\left(y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0\right)dx \longrightarrow 1.$$

$p = \frac{x-x_0}{h}$

$x = x_0 + ph$

$\frac{dx}{dp} = h$

$dx = h\,dp$

when $x \to x_0$, $p \to 0$

$x \to x_2$, $p \to 2$

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1 $\Rightarrow \int_{x_0}^{x_2} f(x)\,dx = h\int_0^2 \left(y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0\right) dp$

$= h\left[y_0 p + \frac{p^2}{2!}\Delta y_0 + \left(\frac{p^3}{3} - \frac{p^2}{2}\right)\frac{\Delta^2 y_0}{2}\right]_0^2$

$= h\left[2y_0 + 2\Delta y_0 + \left(\frac{8}{3} - 2\right)\frac{\Delta^2 y_0}{2}\right]$

$= h\left[2y_0 + 2(y_1 - y_0) + \frac{\cancel{2}}{3\cdot\cancel{2}}(y_2 - 2y_1 + y_0)\right]$

$= h\left[\cancel{2y_0} + 2y_1 - \cancel{2y_0} + \frac{y_2}{3} - \frac{2y_1}{3} + \frac{y_0}{3}\right]$

$= h\left[\frac{y_0}{3} + \frac{4y_1}{3} + \frac{y_2}{3}\right]$

$= \frac{h}{3}(y_0 + 4y_1 + y_2)$

$\Delta^2 y_0 = \Delta(\Delta y_0)$
$= \Delta(y_1 - y_0)$
$= \Delta y_1 - \Delta y_0$
$= y_2 - y_1 - y_1 + y_0$
$= y_2 - 2y_1 + y_0$

$\frac{h}{3}(y_0 + 4y_1 + y_2)$

$\frac{h}{3}(y_2 + 4y_3 + y_4)$

$\vdots$

Formula: $\frac{h}{3}\left[(y_0 + y_{2N}) + 4(y_1 + y_3 + y_5 + \cdots y_{2N-1}) + 2(y_2 + y_4 + \cdots y_{2N-2})\right]$

$y = f(x) = e^{-x}$

Question $\int_0^1 e^{-x}\,dx$ $\quad h = 0.5,\ h = 0.25,\ h = 0.125$

$2N = \frac{b-a}{h}$

$x_0 = 0, 0.5, 1$

$2N = \frac{1-0}{0.5}$

$2N = 2$

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→ Errors

Trapezoidal

η = eata

$$E = -\frac{(b-a)}{12} f''(\eta) h^2 \qquad a < \eta < b$$

h power tells accuracy

h↓ e↓

Ch^2

$C\left(\frac{h}{2}\right)^2 = \frac{Ch^2}{4}$

Simpson

$$E = \frac{-(b-a)}{180} f''(\eta) h^4 \qquad a < \eta < b$$

→ for previous question

Trapezoidal: $E = -\frac{(1-0)}{12} f''(\eta)$

$f(x) = e^{-x}$

$f'(x) = -e^{-x}$

$f''(x) = e^{-x}$

$f''(\eta) = e^{-\eta}$

$$|E| = \frac{1}{12} e^{-\eta} \qquad 0 < \eta < 1$$

$$-\frac{e^{-\eta}}{12} < |E| < \frac{e^{-\eta}}{12}$$

$$-\frac{e^{-1}}{12} < |E| < \frac{1}{12}$$

lower bound ↙ ↘ upper bound

Date:

→ Week 5: Lecture 2

Romberg Integration

Step size h	$O(h^2)$	$O(h^4)$ $m=1$	$O(h^6)$ $m=2$
$h=1$	$I_h^{(0)} = 0.683939721$	$I_h^{(1)} = 0.63233680$	$I_h^2 = 0.63212087$
$\frac{h}{2} = 0.5$	$I_{h/2}^{(0)} = 0.645235190$	$I_{h/2}^1 = 0.63134175$	$I^2_{h/2} =$
$\frac{h}{4} = 0.25$	$I_{h/4}^{(0)} = 0.635409429$	$I_{h/4}^2 = 0.63212414$	
$\frac{h}{8} = 0.125$	$I_{h/8}^{(0)} = 0.632943418$		

Stop here as required accuracy achieved (no need to even complete column)

Romberg Formula based on Trapezoidal Rule

$$I_T^m(h) = \frac{4^m I_T^{m-1}(h/2) - I_T^{m-1}(h)}{4^m - 1} \qquad m = 1, 2, \ldots$$

where

$$I_T^{(0)}\left(\frac{h}{2}\right) = I_{h/2}$$

$$I_T(h) = I_h$$

$\int^1 e^{-x}\,dx = 0.6321205588$ ∴ continue solving the table until required accuracy is achieved (6dp)

$h = 1, 0.5, 0.25, 0.125$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

$I_h \quad I_{h/2} \quad I_{h/4} \quad I_{h/8}$

Calculation

$m = 1$

$$I_T^1(h) = \frac{4^1 I_T^0(h/2) - I_T^0(h)}{4-1} =$$

$$I_T^1\left(\frac{h}{2}\right) = \frac{4 I_T^0(\frac{h}{4}) - I_T^0(h/2)}{4-1} =$$

Date:

$m = 2$

$$I_h^2 = \frac{4^2 I_{h/2}^1 - I_h^1}{15} =$$

In the above table (last page)

Extrapolations : 4

Functional evaluations : 8

Romberg formula based on Simpson's Rule

$$I_s^m(h) = \frac{4^{m+1} I_s^{m-1}(h/2) - I_s^{m-1}(h)}{4^{m+1} - 1}$$

where $I_s^0\left(\frac{h}{2}\right) = I_{h/2}$

$I_s^0(h) = I_h$

⇒ Week 6 : Lecture 2

Finding Root of the equation

1. Bisection Method

- Root of the equation : $f(x) = 0$
- Intermediate Value Theorem

Statement :

If f is a function which is continuous at every point of the interval $[a,b]$ and $f(a) < 0$, $f(b) > 0$

then $\exists$ (there exists) a pt 'c' such that

$f(c) = 0$

Date:

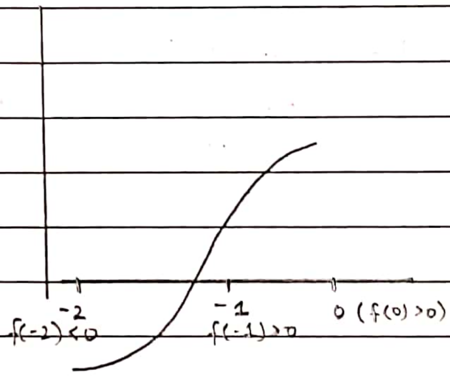
Question

$f(x) = x^3 - 2x + 1$

Put values of x in $f(x)$ and find one x where $f(x) > 0$ and one x where $f(x) < 0$

> $x = -2 \rightarrow f(x) = -3$

> $x = 0 \rightarrow f(x) = 1$



- $[-2, 0]$ with $a = -2$, $b = 0$

$c = \frac{a+b}{2} = \frac{-2+0}{2} = -1$

$f(c) = f(-1) = 2 > 0$ hence we will consider interval $[-2, -1]$

- $[-2, -1]$

$c = \frac{-2-1}{2} = -1.5$

$f(-1.5) = 0.625 > 0$

$\therefore$ new interval $[-2, -1.5]$

* smaller the initial interval less the # of iterations required.

- $[-2, -1.5]$

$c = \frac{-2-1.5}{2} = -1.75$

$f(-1.75) < 0$

$\therefore$ new interval $[-1.75, -1.5]$

- $[-1.75, -1.5]$

$c = \frac{-1.75-1.5}{2} = -1.625$

$f(-1.625) = -0.04 < 0$

$\therefore$ new interval $[-1.625, -1.5]$

- $[-1.625, -1.5]$

$c = \frac{-1.625-1.5}{2} = -1.5625$

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$f(-1.5625) = 0.3103 > 0$

$\therefore$ new interval $[-1.625, -1.5625]$

- $c = \dfrac{-1.625 - 1.5625}{2} = -1.59375$

$f(1.59375) = 0.1393 > 0$

$\therefore$ new interval $[-1.625, -1.59375]$

- $c = \dfrac{-1.625 - 1.59375}{2} = -1.609375$

$f(-1.609375) = 0.0503 > 0$

$\therefore$ new interval $[-1.625, -1.609375]$

- $c = \dfrac{-1.625 - 1.609375}{2} = -1.6171875$

- $f(-1.6171875) < 0$

$\therefore$ new interval $[-1.6171875, -1.609375]$

- $c = \dfrac{-1.6171875 - 1.609375}{2} = -1.61328125$

$f(-1.61328125) > 0$

$\therefore$ new interval $[-1.6171875, -1.61328125]$

- $c = \dfrac{-1.6171875 - 1.61328125}{2} = -1.615234375$

$f(c) > 0$

$\therefore$ $[-1.6171875, -1.615234375]$

- $c = \dfrac{-1.6171875 - 1.61523475}{2} = -1.61621125$

$f(c) > 0$

$\therefore$ $[-1.6171875, -1.61621125]$

- $c = \dfrac{-1.6171875 - 1.61621125}{2} = -1.6166995$

$f(c) > 0$

$\therefore$ $[-1.617, -1.617]$ accurate upto 3 dp

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To find no. of intervals

$$\frac{b-a}{2^n} < \varepsilon$$

n : # of iterations

$\log(b-a) - \log(2^n) < \log \varepsilon$

$\log(b-a) - n\log 2 < \log \varepsilon$

$\log(b-a) - \log \varepsilon < n\log 2$

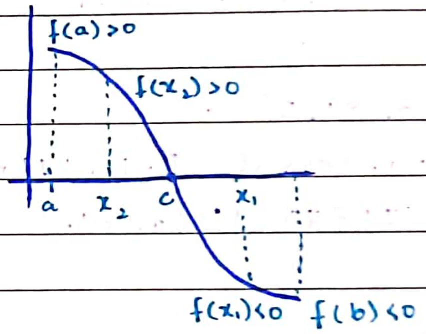
$$\frac{\log(b-a) - \log \varepsilon}{\log 2} < n$$

for correct to 3 dp, put $\varepsilon = 10^{-3}$

→ Week 7: Lecture 1

1. Bisection Method

consider $f(x) = 0$



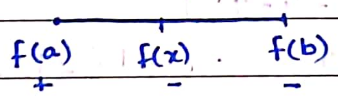
Step 1: $f(a) \cdot f(b) < 0$

Step 2:

$$x_1 = \frac{a+b}{2}$$

$f(x_1) < 0$

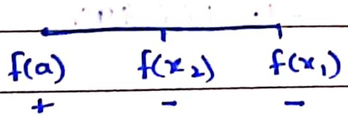
$\therefore [a, x_1]$



$$x_2 = \frac{a+x_1}{2}$$

$f(x_2) < 0$

$\therefore [a, x_2]$



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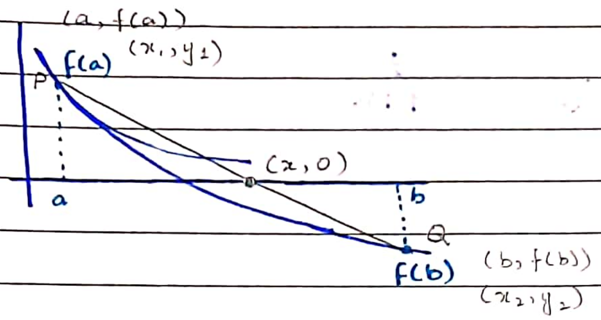
Date:

2. Regular Falsi Method

Consider the equation

$f(x) = 0$



Step 1: $f(b) \cdot f(a) < 0$

Step 2: Find the first approximation to the root x, by joining the chord AC for which cuts axis

$$\frac{b-a}{2^n} < \varepsilon$$

Equation of the chord PQ is

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$$

$x_1 = a$, $x_2 = b$

$y_1 = f(a)$, $y_2 = f(b)$

$x = x$, $y = 0$

$$\frac{-f(a)}{x-a} = \frac{f(b)-f(a)}{b-a}$$

$$-f(a)(b-a) = (f(b)-f(a))(x-a)$$

$$-bf(a) + \cancel{af(a)} = xf(b) - xf(a) - af(b) + \cancel{af(a)}$$

$$-bf(a) = x(f(b)-f(a)) - af(b)$$

$$x = \frac{af(b)-bf(a)}{f(b)-f(a)}$$

Date:

Question: Find the root of $xe^x = 2$, by the regular falsi method:

→ Step 1:

$f(x) = 0$

$f(x) = xe^x - 2 = 0$

$f(0) < 0$

$f(1) > 0$ $[0, 1]$

→ Step 2:

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$= \frac{0 - 1 \cdot f(0)}{f(1) - f(0)}$$

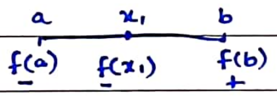
$$= \frac{2}{0.718282 - (-2)}$$

$x_1 = 0.735759$

$f(x_1) < 0$

$\therefore [x_1, b]$

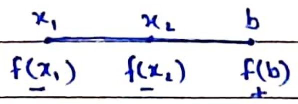
$[0.735759, 1]$



$$x_2 = \frac{x_1 f(b) - bf(x_1)}{f(b) - f(x_1)}$$

$$= \frac{0.735759(0.718282) - 1(-0.4644228)}{0.718282 - (-0.4644228)}$$

$= 0.839521$



$f(x_2) < 0$

$\therefore [x_2, b]$

$[0.839521, 1]$

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Date:

$$x_3 = \frac{x_2 f(b) - b f(x_2)}{f(b) - f(x_2)}$$

$$= \frac{0.839521(0.718282) - 1(\cancel{1.943707}\ {-0.056293})}{0.718282 - \cancel{1.943707}(-0.056293)} = 0.851184$$

$f(x_3) = -0.00617087 < 0$

$\therefore [0.851184, 1]$

$$x_4 = \frac{x_3 f(b) - b f(x_3)}{f(b) - f(x_3)}$$

$$= \frac{0.851184(0.718282) - 1(-0.00617087)}{0.718282 - (-0.00617087)} = 0.852452$$

$f(x_4) = -0.000668827 < 0$

$\therefore [0.852452, 1]$

$$x_5 = \frac{x_4 f(b) - b f(x_4)}{f(b) - f(x_4)}$$

$$= \frac{0.852452(0.718282) - 1(-0.000668827)}{0.718282 - (-0.000668827)} = 0.852589$$

$f(x_5) = -0.000070577594 < 0$

$\therefore [0.852589, 1]$

$$x_6 = \frac{x_5 f(b) - b f(x_5)}{f(b) - f(x_5)}$$

$$= \frac{0.852589(0.718282) - 1(-0.000070578)}{0.718282 - (-0.000070578)} = 0.852603$$

$f(x_6) = -8.776 \times 10^{-6} < 0$

$\therefore [0.852603, 1]$

$$x_7 = \frac{x_6 f(b) - b f(x_5)}{f(b) - f(x_5)}$$

$$= \frac{0.852603(0.718282) - 1(-8.776 \times 10^{-6})}{0.718282 - (-8.776 \times 10^{-6})} = 0.852605 \approx 0.85261$$

$f(x_6) > 0$

$\therefore [0.85260, 0.85261]$

Date:

⇒ Week 7: Lecture 2

Iteration Method or Fixed point iterative method

Step 0: $f(x) = 0$

Step 1: $f(a)f(b) < 0$

Step 2: Rewrite the given function in the form of

$$x = F(x)$$

eg $f(x) = \cos x - 4x + 5 = 0$

$$x = \frac{1}{4}(\cos x + 5)$$

$\downarrow$

$F(x)$

1. $4x = \cos x + 5$

$x = \frac{1}{4} \cos x + 5 \rightarrow F(x)$

OR

2. $\cos x = 4x - 5$

$x = \cos^{-1}(4x-5) \rightarrow F(x)$

$x_1 = F(x_0)$

$x_2 = F(x_1)$

Condition for convergence

$|F'(x)| < 1 \quad [a,b]$

$x_k = F(x_{k-1})$

where $k = 1, 2, \ldots$

Question: $f(x) = x^3 - x - 1 = 0$

$[a,b]$

$f(1) = -ve$

$f(2) = +ve$

$f(1)f(2) < 0$

$\therefore$ interval $[1,2]$

Step 3: $x = F(x)$

$x^3 - x - 1 = 0$

① $x = x^3 - 1 \rightarrow F(x)$

$F'(x) = 3x^2$

Date:

$|F'(x)| = 3x^2$ $[1,2]$

$3(1) = 3 > 1$ hence reject this $F(x)$

(2) $x^3 - x - 1 = 0$

$x = (x+1)^{1/3}$

$\downarrow$

$F(x)$

$|F'(x)| = \frac{1}{3}(x+1)^{-2/3}$

$= \frac{1}{3(x+1)^{2/3}}$

$F'(1) < 1$

$F'(2) < 1$ hence accept this $F(x)$

$F(x) = (x+1)^{1/3}$

Take an initial guess, it can be any middle value, to take extreme point values a few conditions must be specified.

$x_0 = 1.5$

$x_1 = F(x_0)$

$= F(1.5)$

$F(1.5) = (1.5+1)^{1/3} = 1.3572$

$x_2 = F(x_1) = 1.3309$

$x_3 = 1.3259$

$x_4 = 1.3249$

$x_5 = 1.3248...$

$x_6 = 1.3247$

$x_7 = 1.3247$

Date:

Proof

$x = F(x) \rightarrow A$

initial guess $= x_0$

$x_1 = F(x_0) \rightarrow B$

$x_2 = F(x_1)$

$\vdots$

$x_{k+1} = F(x_k) \longrightarrow 1.$

where $k = 0, 1, 2, \ldots$ will give further approximations

Formula (1) is known as the iteration formula

Subtracting B from A

$x_1 - x_0 = F(x_0) - F(x)$

$x - x_1 = F(x) - F(x_0)$

Mean value theorem

$$\frac{F(x) - F(x_0)}{x - x_0} = F'(\varepsilon_0) \qquad \because x < \varepsilon_0 < x_0$$

$$F(\underset{x}{x}) - F(\underset{x_1}{x_0}) = F'(\varepsilon_0)(x - x_0)$$

$2 \Rightarrow x - x_1 = F'(\varepsilon_0)(x - x_0)$

$x - x_2 = F'(\varepsilon_1)(x - x_1)$

$x - x_3 = F'(\varepsilon_2)(x - x_2)$

$\vdots$

$x - x_n = F'(\varepsilon_{n-1})(x - x_{n-1})$

Now multipy all the LHS and RHS values,

$$\cancel{(x - x_1)}\cancel{(x - x_2)}\cancel{(x - x_3)} \cdots (x - x_n) = F'(\varepsilon_0)(x - x_0)\, F'(\varepsilon_1)\cancel{(x - x_1)} \cdots F'(\varepsilon_{n-1})\cancel{(x - x_{n-1})}$$

$$(x - x_n) = (x - x_0)\, F'(\varepsilon_0)\, F'(\varepsilon_1) \cdots F'(\varepsilon_{n-1})$$

all cancel out on RHS + LHS except 1 on eachside.

If the maximum value of $|F'(x)| < 1$, then

$|F'(\varepsilon_0)|, |F'(\varepsilon_1)| \ldots |F'(\varepsilon_{n-1})| < 1$

Date:

⇒ Week 8 : Lecture 1

Sufficient Condition for convergence of fixed point method

Statement : If $x = \alpha$ is exact root of $f(x) = 0$ and I is the interval which contains α. If $F(x)$ and $F'(x)$ are continuous in interval where $x = F(x)$ is equivelant form of $f(x) = 0$.
Now if $|F'(x)| < 1$, $\forall x \in [a,b]$, then the sequence of approximations $x_1, x_2, \dots x_{n+1}$ converges to root exact of α

Proof : To find the root of the equation $f(x) = 0 \to (i)$ by fixed point method
$f(x) = 0$ can be written as

$$x = F(x) \to (ii)$$

Let $x = x_0$ be an initial approximation to a root α, then the 1st approximation x_1 is given

$$x_1 = F(x_0) \to (iii)$$

and

$$x_2 = F(x_1) \to (iv)$$

In general

$$\therefore x_{k+1} = F(x_k) \to (v)$$

where $k = 0, 1, 2, \dots$

Now, subtracting (iii) from (ii) we get

$$x_1 - x = F(x_0) - F(x)$$

$$x - x_1 = F(x) - F(x_0) \to (vi)$$

by mean value theorem

$$\frac{F(x) - F(x_0)}{x - x_0} = F'(\xi_0) \qquad x < \xi_0 < x_0$$

$$\Rightarrow F(x) - F(x_0) = (x - x_0) F'(\xi_0)$$

Date:

(vi) $\rightarrow$ $x - x_1 = (x - x_0) F'(\varepsilon_0)$

$x - x_2 = (x - x_1) F'(\varepsilon_1)$

$x - x_3 = (x - x_2) F'(\varepsilon_2)$

$\vdots$

$x - x_n = (x - x_{n-1}) F'(\varepsilon_{n-1})$

Multiplying all these equations we get

$$x - x_n = (x - x_0) F'(\varepsilon_0) F'(\varepsilon_1) F'(\varepsilon_2) \cdots F'(\varepsilon_{n-1}) \quad \text{--- (vii)}$$

If the maximum value of $|F'(x)| < 1$, then each of the quantities

$|F'(\varepsilon_0)|, |F'(\varepsilon_1)|, |F'(\varepsilon_2)| \cdots < 1$

vii $\rightarrow$ $x - x_n < (x - x_0) \lambda \cdot \lambda \cdot \lambda \cdots \lambda$

$|x - x_n| < |(x - x_0)| \lambda^n$

$\lim_{n \to \infty} (x - x_0) \lambda^n = 0$ $\quad\quad$ $\lim_{n \to \infty} \frac{1}{n} = 0$

$x - x_m \approx 0$

$x \approx x_m$

$\therefore$ The system of approximations $x_0, x_1, x_2, \ldots x_k, x_{k+1}, \cdots$ converges to exact root α

if $|F'(x)| < 1, \ \forall \ [a, b]$

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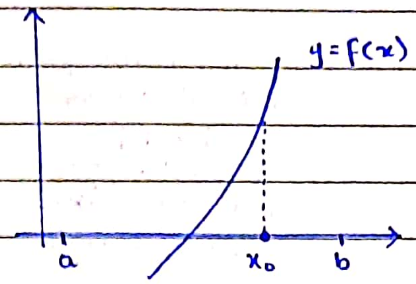
4. $\Rightarrow$ Newton Raphson Method

Let $f(x) = 0$



1. $[a, b]$

   $f(a) \cdot f(b) < 0$

2. x_0, x_1

   ↓

   chosen randomly

   $x_0, f(x_0)$

   $x_1, f(x_1)$

The equation of the tangent at (x_0, y_0)

The tangent line cuts the x axis at $x = x_1$ where $y = 0$

$$y - y_0 = f'(x_0)(x - x_0)$$

$$-y_0 = f'(x_0)(x - x_0)$$

$$\frac{-y_0}{f'(x_0)} = x - x_0$$

$$x_0 - \frac{f(x_0)}{f'(x_0)} = x_1$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$\vdots$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$k = 0, 1, 2, \ldots$

Date:

Question: Find the real root of $3x = \cos x + 1$ by using the Newton Raphson Method.

$f(x) = 0$

$3x - \cos x - 1 = 0$ $\qquad f'(x) = 3 + \sin x$

$[0, 1]$

$f(0) \cdot f(1) < 0$ $\qquad$ -ve (0) +ve (1)

$x_0 = 0.5$ (guess)

∵ Extreme points as Initial guess $[a, b]$
$|f(a)| < |f(b)|$
then a is taken as 1st approximation else b

$f(x_0) = -0.377583$ = -ve

Calculation in Radians

∴ new interval $[0.5, 1]$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.5 - \frac{-0.377583}{3.47943} = 0.60852$$

$f(x_1) = 0.005060$ = +ve

∴ new interval $[0.5, 0.60852]$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.60852 - \frac{0.005061}{3.571654} = 0.607103$$

$f(x_2) = 4.86029 \times 10^{-6}$ = +ve

∴ new interval $[0.5, 0.607103]$

$$x_3 = 0.607103 - \frac{4.86029 \times 10^{-6}}{3.570491} = 0.6071016$$

$f(x_3) = -3.33549 \times 10^{-8}$

∴ new interval $[0.6071016, 0.607103]$

$$x_4 = 0.6071016 + \frac{3.33544 \times 10^{-8}}{3.570489} = 0.607101609$$

∴ interval $[0.607102, 0.607103]$

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Date:

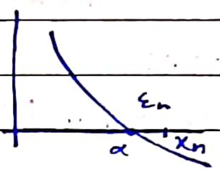
⇒ Week 8 : Lecture 2

Rate of Convergence of Newton - Raphson Method

Let α be the exact value of the root of the equation $f(x)=0$:

$$x_n = \alpha + \varepsilon_n \longrightarrow 1.$$

$$x_{n+1} = \alpha + \varepsilon_{n+1} \longrightarrow 2$$



By Newton's formula

$$1. \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\alpha + \varepsilon_{n+1} = \alpha + \varepsilon_n - \frac{f(\alpha+\varepsilon_n)}{f'(\alpha+\varepsilon_n)}$$

$$\varepsilon_{n+1} = \varepsilon_n - \frac{f(\alpha+\varepsilon_n)}{f'(\alpha+\varepsilon_n)}$$

$$= \varepsilon_n - \frac{f(\alpha) + \varepsilon_n f'(\alpha) + \frac{\varepsilon_n^2}{2!} f''(x) + \cdots}{f'(\alpha) + \varepsilon_n f''(\alpha) + \frac{\varepsilon_n^2}{2!} f'''(\alpha) + \cdots}$$

$$= \varepsilon_n - \frac{f(\alpha) + \varepsilon_n f'(\alpha) + \frac{\varepsilon_n^2 f''(x)}{2}}{f'(\alpha) + \varepsilon_n f''(\alpha)}$$

$f(\alpha) = 0$

$$= \frac{\varepsilon_n f'(\alpha) + \varepsilon_n^2 f''(\alpha) - f(\alpha) - \varepsilon_n f'(\alpha) - \frac{\varepsilon_n^2 f''(\alpha)}{2}}{f'(\alpha) + \varepsilon_n f''(\alpha)}$$

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2 f''(\alpha)}{2[f'(\alpha) + \varepsilon_n f''(\alpha)]}$$

$$= \frac{\varepsilon_n^2 f''(\alpha)}{2f'(\alpha)\left[1 + \underbrace{\frac{\varepsilon_n f''(x)}{f'(\alpha)}}_{\text{neglect this value}}\right]}$$

(Rate at which exact and approximate converge → order of convergence)

$$= K\varepsilon_n^2$$

where $K = \dfrac{f''(\alpha)}{2f'(\alpha)}$

Date:

Condition of convergence of Newton Raphson

Newton's formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = \phi(x_n)$$

where

$x = \phi(x)$

$$\phi(x) = x - \frac{f(x)}{f'(x)} \qquad |F'(x)| < 1$$

$|\phi'(x)| < 1$

$$\phi'(x) = 1 - \frac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2}$$

(quotient rule for derivative)

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

$$|\phi'(x)| = \left|\frac{[\cancel{f'(x)}]^2 - [\cancel{f'(x)}]^2 + f(x)f''(x)}{[f'(x)]^2}\right|$$

$$|\phi'(x)| = \frac{|f(x)f''(x)|}{|[f'(x)]^2} < 1$$

Hence the condition for convergence of Newton Raphson method is

$|\phi'(x)| < 1$

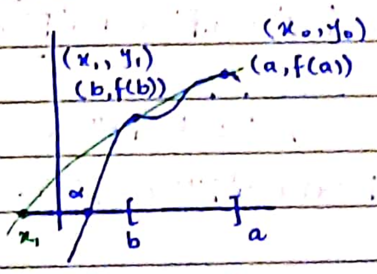
5. ⇒ Secant Method

$f(x) = 0$

$f(a) \cdot f(b) < 0$ ×

$[a, b]$ ✓

$+ \quad +$



3. $x_0, x_1, f(x_0), f(x_1)$

4. $x_2 = ?$

$$\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0}$$

$(x_0, y_0), (x_1, y_1), (x_2, 0)$

$$\frac{0 - y_0}{x_2 - x_0} = \frac{y_1 - y_0}{x_1 - x_0}$$

Date:

$$x_2 = \frac{x_0 y_1 - x_1 y_0}{y_1 - y_0}$$

$$= \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

Date:

→ Numerical Solution of linear system of equations
1. Direct Method
2. Indirect or Iterative Method → Guass Jacobi
↳ Guass Seidel

Indirect or Iterative Method
1: Guass Jacobi Method
⇒ Step 1: Given

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

⇒ Step 2 : Check diagonally dominant condition

$$|a_1| > |b_1| + |c_1|$$
$$|b_2| > |a_2| + |c_2|$$
$$|c_3| > |a_3| + |b_3|$$

(if not alter / rearrange equations)
if still not satisfied, this method can't be used

~~Step 3~~ eg

$$8x - 3y + 2z = 20 \qquad |8| > |-3| + |2| \checkmark$$
$$4x + 11y - z = 33 \qquad |11| > |4| + |-1| \checkmark$$
$$6x + 3y + 12z = 35 \qquad |12| > |6| + |3| \checkmark$$

Step 3:

$$x^{(k+1)} = \frac{1}{a_1}\left(d_1 - b_1 y^{(k)} - c_1 z^{(k)}\right)$$

$$y^{(k+1)} = \frac{1}{b_2}\left(d_2 - a_2 x^{(k)} - c_2 z^{(k)}\right)$$

$$z^{(k+1)} = \frac{1}{c_3}\left(d_3 - a_3 x^{(k)} - b_3 y^{(k)}\right) \qquad k = 0, 1, 2, \ldots$$

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Date:

→ Step 4 :

$k = 0$

$$x^{(1)} = \frac{1}{a_1}\left(d_1 - b_1 y^{(0)} - c_1 z^{(0)}\right)$$

$$y^{(1)} = \frac{1}{b_2}\left(d_2 - a_2 x^{(0)} - c_2 z^{(0)}\right)$$

$$z^{(1)} = \frac{1}{c_3}\left(d_3 - a_3 z^{(0)} - b_3 y^{(0)}\right)$$

eg

$$x^{(1)} = \frac{1}{8}\left(20 + 3y_0^{(0)} - 2z^{(0)}\right) = 2.5 \qquad (x_0, y_0, z_0) = (0,0,0)$$

$$y^{(1)} = \frac{1}{11}\left(33 - 4x^{(0)} + z^{(0)}\right) = 3.0$$

$$z^{(1)} = \frac{1}{12}\left(35 - 6x^{(0)} - 3y^{(0)}\right) = 2.9166$$

Upto 10 iterations required

→ $$x^{(2)} = \frac{1}{8}\left(20 + 3(3) - 2(2.9166)\right) = 2.89585$$

$$y^{(2)} = \frac{1}{11}\left(33 - 4(2.5) + (2.9166)\right) = 2.356054$$

$$z^{(2)} = \frac{1}{12}\left(35 - 6(2.5) - 3(3)\right) = 0.916667$$

→ $$x^{(3)} = \frac{1}{8}\left(20 + 3(2.356054) - 2(0.916667)\right) = 3.1543535$$

$$y^{(3)} = \frac{1}{11}\left(33 - 4(2.89585) + (0.916667)\right) = 2.030297$$

$$z^{(3)} = \frac{1}{12}\left(35 - 6(2.89585) - 3(2.356054)\right) = 0.8797281$$

Date:

$\rightarrow x^{(4)} = \frac{1}{8}(20+3(2.030297)-2(0.8797281)) = 3.0414$

$y^{(4)} = \frac{1}{11}(33-4(3.1543535)+0.8797281) = 1.9329$

$z^{(4)} = \frac{1}{12}(35-6(3.1543535)-3(2.030297)) = 0.8319$

$\rightarrow x^{(5)} = \frac{1}{8}(20+3(1.9329)-2(0.8319)) = 3.0169$

$y^{(5)} = \frac{1}{11}(33-4(3.0414)+(0.8319)) = 1.9697$

$z^{(5)} = \frac{1}{12}(35-6(3.0414)-3(1.9329)) = 0.9127$

$\rightarrow x^{(6)} = \frac{1}{8}(20+3(1.9697)-2(0.9127)) = 3.0104$

$y^{(6)} = \frac{1}{11}(33-4(3.0169)+0.9127) = 1.9859$

$z^{(6)} = \frac{1}{12}(35-6(3.0169)-3(1.9697)) = 0.9158$

$\rightarrow x^{(7)} = \frac{1}{8}(20+3(1.9859)-2(0.9158)) = 3.0158$

$y^{(7)} = \frac{1}{11}(33-4(3.0104)+0.9158) = 1.9886$

$z^{(7)} = \frac{1}{12}(35-6(3.0104)-3(1.9859)) = 0.91496$

$\rightarrow x^{(8)} = \frac{1}{8}(20+3(1.9886)-2(0.91496)) = 3.0169$

$y^{(8)} = \frac{1}{11}(33-4(3.0158)+0.91496 = 1.9865$

$z^{(8)} = \frac{1}{12}(35-6(3.0158)-3(1.9886) = 0.9116$

Date:

$\rightarrow x^{(9)} = 3.0170$

$y^{(9)} = 1.9858$

$z^{(9)} = 0.91156$

$\rightarrow x^{(10)} = 3.0168$

$y^{(10)} = 1.9858$

$z^{(10)} = 0.91169$

Formula

If $x^{(r)}, y^{(r)}, z^{(r)}$ are the r^{th} iterates, then

$$x^{(r+1)} = \frac{1}{a_1}\left(d_1 - a_1 y^{(r)} - c_1 z^{(r)}\right)$$

$$y^{(r+1)} = \frac{1}{b_2}\left(d_2 - a_2 x^{(r)} - c_2 z^{(r)}\right)$$

$$z^{(r+1)} = \frac{1}{c_3}\left(d_3 - a_3 x^{(r)} - b_3 y^{(r)}\right)$$

The process is continued till convergency is secured.

Date:

2. Guass Seidel Method

Step 1 and Step 2 same as Guass Jacobi

Step 3

$$x^{(K+1)} = \frac{1}{a_1}(d_1 - b_1 y^{(K)} - c_1 z^{(K)})$$

$$y^{(K+1)} = \frac{1}{b_2}(d_2 - a_2 x^{(K+1)} - c_2 z^{(K)})$$

$$z^{(K+1)} = \frac{1}{c_3}(d_3 - a_3 x^{(K+1)} - b_3 y^{(K+1)})$$

this will reduce the number of iterations by half
z's value will get accurate first (2 recent values), then y's, then x's

→ Week 9: Lecture 2

Direct Method (exact solution)

1. Guass Elimination Method

Given,

$$a_{11}x_1 + a_{12}x_1 + a_{13}x_1 = d_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = d_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = d_3$$

Make $a_{21} = 0$

$$R_2 - \frac{a_{21}}{a_{11}} R_1 \rightarrow R_2$$

$$a_{21} - \frac{a_{21}}{a_{11}} \cdot a_{11} = 0 \qquad a_{23} - \frac{a_{21}}{a_{11}} \cdot a_{13} = a_{23}^{(2)}$$

$$a_{22} - \frac{a_{21}}{a_{11}} \cdot a_{12} = a_{22}^{(2)} \qquad d_2 - \frac{a_{21}}{a_{11}} \cdot d_1 = d_2^{(2)}$$

Date:

Make $a_{31} = 0$

$$R_3 - \frac{a_{31}}{a_{11}} R_1 \longrightarrow R_3$$

$$\rightarrow a_{31} - \frac{a_{31}}{a_{11}} \cdot a_{11} = 0$$

$$\rightarrow a_{32} - \frac{a_{31}}{a_{11}} \cdot a_{12} = a_{32}^{(2)}$$

$$\rightarrow a_{33} - \frac{a_{31}}{a_{11}} \cdot a_{13} = a_{33}^{(2)}$$

After the above 2 operations

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = d_1$$
$$0 + a_{22}^{(2)} x_2 + a_{23}^{(2)} x_3 = d_2^{(2)}$$
$$0 + a_{32}^{(2)} x_2 + a_{33}^{(2)} x_3 = d_3^{(2)}$$

Now make $a_{32} = 0$

$$R_3 - \frac{a_{32}}{a_{22}} R_2 \longrightarrow R_3$$

$$0 - \frac{a_{32}}{a_{22}} (0) = 0$$

$$a_{32} - \frac{a_{32}}{a_{22}} a_{22}^{(2)} = 0$$

$$a_{33}^{(2)} - \frac{a_{32}}{a_{22}} a_{23}^{(2)} = a_{33}^{(3)}$$

$$d_3^{(2)} - \frac{a_{32}}{a_{22}} d_2^{(2)} = d_3^{(3)}$$

After this

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = d_1$$
$$0 + a_{22}^{(2)} x_2 + a_{23}^{(2)} x_3 = d_2^{(2)}$$
$$0 + 0 + a_{33}^{(3)} x_3 = d_3^{(3)}$$

Date:

Example

$0x_1 + 8x_2 + 2x_3 = -7$

$3x_1 + 5x_2 + 2x_3 = 8$

$6x_1 + 2x_2 + 8x_3 = 26$

Step 1 : $R_2 \leftrightarrow R_1$

$3x_1 + 5x_2 + 2x_3 = 8$

$0 + 8x_2 + 2x_3 = -7$

$6x_1 + 2x_2 + 8x_3 = 26$

Step 2 :

$R_3 - \frac{6}{3} R_1 \rightarrow R_3$

$6 - 2(3) = 0$

$2 - 2(5) = -8$

$8 - 2(2) = 4$

$26 - 2(8) = 10$

$$\left[\begin{array}{ccc:c} 3 & 5 & 2 & 8 \\ 0 & 8 & 2 & -7 \\ 0 & -8 & 4 & 10 \end{array}\right]$$

Step 3 :

$R_3 - \frac{8}{-8} R_2 \rightarrow R_3$

$R_3 + R_2 \rightarrow R_3$

$$\left[\begin{array}{ccc:c} 3 & 5 & 2 & 8 \\ 0 & 8 & 2 & -7 \\ 0 & 0 & 6 & 3 \end{array}\right]$$

Eqs : $3x_1 + 5x_2 + 2x_3 = 8$

$8x_2 + 2x_3 = -7$

$6x_3 = 3$

Solve : $x_3 = 1/2$

$x_2 = -1$

$x_1 = 4$

Date:

→ Week 10: Lecture 1

Guass Elimination Method

Partial pivoting

$$\begin{cases} 3x_1 - 4x_2 + 5x_3 = -1 \\ -3x_1 + 2x_2 + x_3 = 1 \\ \boxed{6x_1} + 8x_2 - x_3 = 35 \end{cases}$$

$R_1 \rightleftarrows R_3$ Biggest value in col 1: check ma

$|3|, |-3|, |6|$

$$6x_1 + 8x_2 - x_3 = 35$$
$$-3x_1 + 2x_2 + x_3 = 1$$
$$3x_1 - 4x_2 + 5x_3 = -1$$

$$\underset{\sim}{R} \left[\begin{array}{ccc:c} 6 & 8 & -1 & 35 \\ 0 & 6 & 0.5 & 37/2 \\ 0 & \boxed{-8} & 5.5 & -37/2 \end{array}\right]$$

$R_2 - \frac{-3}{6} R_1$

$R_2 + \frac{1}{2} R_1$

$R_3 - \frac{1}{2} R_1$

$$\underset{\sim}{R} \left[\begin{array}{ccc:c} 6 & 8 & -1 & 35 \\ 0 & -8 & 5.5 & -37/2 \\ 0 & 6 & 0.5 & 37/2 \end{array}\right]$$

$R_3 + \frac{6}{8} R_2$

$$\underset{\sim}{R} \left[\begin{array}{ccc:c} 6 & 8 & -1 & 35 \\ 0 & -8 & 5.5 & -37/2 \\ 0 & 0 & 37/8 & 37/8 \end{array}\right]$$

Date:

Total Pivoting

$$3x_1 - 4x_2 + 5x_3 = -1$$
$$-3x_1 + 2x_2 + x_3 = 1$$
$$6x_1 + 8x_2 - x_3 = 35$$

$$c = \max_{k \le i \le n} |a_{ik}^{(k)}|$$

$$|a_{kk}^{(k)}| = c$$

$c_1 = 5$	3
$\boxed{c_2 = 3}$	$\boxed{3}$
$c_3 = 8$	6

max value = pivot element

(if no such option, this method fails. shift to partial pivoting if this scenario occurs)

$$-3x_1 + 2x_2 + x_3 = 1$$
$$3x_1 - 4x_2 + 5x_3 = -1$$
$$6x_1 + 8x_2 - x_3 = 35$$

$$\left[\begin{array}{ccc:c} -3 & 2 & 1 & 1 \\ 3 & -4 & 5 & -1 \\ 6 & 8 & -1 & 35 \end{array}\right] \quad \begin{array}{l} \\ R_2 + R_1 \\ R_3 + 2R_1 \end{array}$$

$$\underset{\sim}{R} \left[\begin{array}{ccc:c} -3 & 2 & 1 & 1 \\ 0 & -2 & 6 & 0 \\ 0 & 12 & -1 & 37 \end{array}\right]$$

$c_2 = 6$	2
$\boxed{c_3 = 12}$	$\boxed{12}$

$$\underset{\sim}{R} \left[\begin{array}{ccc:c} -3 & 2 & 1 & 1 \\ 0 & 12 & -1 & 37 \\ 0 & -2 & 6 & 0 \end{array}\right] \quad R_3 + \frac{R_2}{6}$$

$$\underset{\sim}{R} \left[\begin{array}{ccc:c} -3 & 2 & 1 & 1 \\ 0 & 12 & -1 & 37 \\ 0 & 0 & 35/6 & 37/6 \end{array}\right]$$

Date:

→ Ill Conditioning

$x_1 + 3x_2 = 4$

$3x_1 + 9.0000\ x_2 = 12.00001$

Exact Solution

$x_1 = x_2 = 1$

$x_1 = 10,\ x_2 = -2$

If we change the no. of dp and answer changes a lot, then the system is highly sensitive.

→ Week 10 : Lecture 2

Direct Factorization

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$

$\therefore Ax = B$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Step 2 $\quad A = LU$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{23} \end{bmatrix}$$

Date:

Take either $l_{11}, l_{22}, l_{33} = 1$ → DoLittle Method
or $u_{11}, u_{22}, u_{33} = 1$ → Crout's Method

$$A = \begin{bmatrix} l_{11}u_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + l_{22}u_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + l_{33}u_{33} \end{bmatrix}$$

Computing values here
We know that,
$AX = B$ and $A = LU$
$\therefore LUX = B$
Let $UX = Z$ → (i)
then $LZ = B$ (find Z from here and put it in (i) to find X)

⇒ Week 11: Lecture 1

Q. Solve the following system using Direct (Do Little Method) and using Indirect method.

$$4x_1 + 10x_2 + 8x_3 = 44$$
$$10x_1 + 26x_2 + 26x_3 = 128$$
$$8x_1 + 26x_2 + 61x_3 = 214$$

Direct Factorization

Step 1:
$AX = B$
$$\begin{bmatrix} 4 & 10 & 8 \\ 10 & 26 & 26 \\ 8 & 26 & 61 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 44 \\ 128 \\ 214 \end{bmatrix}$$

Step 2:
$A = LU$
$$\begin{bmatrix} 4 & 10 & 8 \\ 10 & 26 & 26 \\ 8 & 26 & 61 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Date:

$$\begin{bmatrix} 4 & 10 & 8 \\ 10 & 26 & 26 \\ 8 & 26 & 61 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

$u_{11} = 4$ $\quad u_{12} = 10$ $\quad u_{13} = 8$

$l_{21} = \frac{10}{4} = 2.5$ $\quad u_{22} = 26 - 25 = 1$ $\quad u_{23} = 26 - 2.5(8) = 6$

$l_{31} = \frac{8}{4} = 2$ $\quad l_{32} = \frac{26 - (2)(10)}{1} = 6$ $\quad u_{33} = 61 - 2(8) - 6(6)$
$= 9$

$$u = \begin{bmatrix} 4 & 10 & 8 \\ 0 & 1 & 6 \\ 0 & 0 & 9 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 2.5 & 1 & 0 \\ 2 & 6 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2.5 & 1 & 0 \\ 2 & 6 & 1 \end{bmatrix} \begin{bmatrix} 4 & 10 & 8 \\ 0 & 1 & 6 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 44 \\ 128 \\ 214 \end{bmatrix}$$

Let $Ux = Z$

$$\begin{bmatrix} 4 & 10 & 8 \\ 0 & 1 & 6 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \cancel{44}\, z_1 \\ \cancel{128}\, z_2 \\ \cancel{214}\, z_3 \end{bmatrix} \longrightarrow (i)$$

$LZ = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2.5 & 1 & 0 \\ 2 & 6 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 44 \\ 128 \\ 214 \end{bmatrix}$$

$z_1 = 44$ $\longrightarrow$ put in (i)
$z_2 = 18$
$z_3 = 18$

$Ux = Z$

$$\begin{bmatrix} 4 & 10 & 8 \\ 0 & 1 & 6 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 44 \\ 18 \\ 18 \end{bmatrix}$$

$x_1 = -8$
$x_2 = 6$
$x_3 = 2$

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Date:

Guass Siedel Method

$$\left[\begin{array}{ccc|c} 4 & 10 & 8 & 44 \\ 10 & 26 & 26 & 128 \\ 8 & 26 & 61 & 214 \end{array}\right]$$

Checking condition

$|a_1| > |b_1| + |c_1|$

this condition doesnt come true for any of the three equations hence we cannot solve this by indirect method.

→ Week 11: Lecture 2

Symmetric Factorization

(Cholesky Method / Square root Method)

$A \neq LU$

Eq 7 pg # 172

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$AX = B \qquad A = LL^T$

$A = UU^T$

$AX = B$

$LL^TX = B$

Let $L^TX = Z$

$LZ = B$

Example 9:

$$\begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Date:

→ Differential Equations

Given $\frac{dy}{dx} = f(x,y)$

$y(x_0) = y_0$

$y(x) = ?$

Single Step Method

1. Taylor series method for 1^{st} ODE, 2^{nd} ODE, system of eq.
2. Picard Method
3. Euler Method, Euler Modified, Euler improved Method
4. RK Method of 1, 2, 3, 4

~~predictor and corrector method~~

Multistep Method

Predictor and Corrector Method

1. Adam Bashforth
2. Milne's Method

→ Taylor Series Method

1^{st} ODE

$\frac{dy}{dx} = f(x,y) \qquad y(x_0) = y_0$

$y(x) = ?$

$y(x) = y_0 + (x - x_0)y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \dots$

$y(x_1) = y(x_0+h) = y(x_0) + hy'(x_0) + \frac{h^2}{2!} y''(x_0) + \dots$

Date:

Week 12: Lecture 1

→ Taylor Series Method

Example

$\frac{dy}{dx} = -xy$, $y(0)=1$, find $y(0.1) = ?$

by using Taylor Series Method

$y' = -xy$, $x_0 = 0$, $y_0 = 1$, $x_1 = 0.1$, $h = 0.1 - 0 = 0.1$

$h = x_1 - x_0$

$y(x) = y_0 + hy'_0 + \frac{h^2}{2!} y''_0 + \frac{h^3}{3!} y'''_0 + \dots$

$y_1 = y(x_1) = 1 + 0.1 y'_0 + \frac{(0.1)^2}{2!} y''_0 + \frac{(0.1)^3}{3!} y'''_0$

(wrt x)

$y' = -xy$

$y'' = -[xy' + y]$

$y''' = -[xy'' + 2y']$

$y' = -xy \rightarrow y'_0 = -x_0 y_0 = 0$

$y'' = -[xy' + y] \rightarrow y''_0 = -[x_0 y'_0 + y_0] = -1$

$y''' = -[xy'' + 2y'] \rightarrow y'''_0 = x_0 y''_0 + 2y'_0$

$= 2(0) = 0$

(3 non zero derivatives required)

Week 13: Lecture 1

Taylor Series Method (system of eq)
(first order)

$$y, z \begin{cases} \frac{dy}{dx} = f(x,y,z) & y(x_0) = y_0 \\ \frac{dz}{dx} = g(x,y,z) & z(x_0) = z_0 \end{cases}$$

Date:

$$y(x) = y_0 + hy'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \dots$$

$$z(x) = z_0 + hz'_0 + \frac{h^2}{2!}z''_0 + \frac{h^3}{3!}z'''_0 + \dots$$

Example

$y(x)$ $\quad \frac{dy}{dx} = x^2z + y \quad ; \quad y(\overset{x_0}{1}) = 1 \qquad y(\overset{x_1}{1.1}) = ?$

$z(x)$ $\quad \frac{dz}{dx} = -xy \quad ; \quad z(\overset{x_0}{1}) = 2 \qquad z(\overset{x_1}{1.1}) = ?$

$z_0 = 2 \qquad z_1 = ?$
$y_0 = 1 \qquad y_1 = ?$
$x_0 = 1 \qquad x_1 = 1.1$

$$y(x) = y_0 + hy'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \dots$$

$$z(x) = z_0 + hz'_0 + \frac{h^2}{2!}z''_0 + \frac{h^3}{3!}z'''_0 + \dots$$

$h = x_1 - x_0$
$= 1.1 - 1$
$= 0.1$

→ for y

$y' = x^2z + y \qquad = 2 + 1 = 3 \qquad (\text{at } x_0\ y_0\ z_0)$

$y'' = 2xz + x^2z' + y' \qquad = 4 + 1^2(-1) + 3 = 6$

$y''' = 2xz' + 2z + 2xz' + x^2z'' + y'' \qquad = 2(1)(-1) + 4 + 2(1)(-1) + (-4) + 6$

$= 2$

→ for z

$z' = -xy \qquad = -1$

$z'' = -xy' + -y \qquad = -1(3) - 1 = -4$

$z''' = -xy'' - y' - y'$

$z_0''' = -xy'' - 2y' \qquad = (-1)(6) - 2(3) = -12$

Date:

Taylor Series 2nd ODE

$y(x) \quad \frac{d^2y}{dx^2} = f(x, y, y')$ convert 2nd ODE to 1st ODE

$y(x_0) = y_0$
$y'(x_0) = y'_0$

Let $\frac{dy}{dx} = z \quad ; \quad y(x_0) = y_0$

$\frac{d^2y}{dx^2} = z' \quad ; \quad z(x_0) = z_0$

$\left.\begin{array}{ll} \frac{dz}{dx} = f(x, y, z) & z(x) \\ \frac{dy}{dx} = z = g(x, y, z) & y(x) \end{array}\right\}$ solve this as 1st ODE
use derivatives of z; no need to find z(x)
only y(x) required

Example

$\frac{d^2y}{dx^2} = xy' + 3y \rightarrow 1. \quad y(0) = 1 \quad y'(0) = 2 \qquad y(0.2) = ?$

$\left.\begin{array}{l} \text{Let } \frac{dy}{dx} = z \\ \frac{d^2y}{dx^2} = z' = \frac{dz}{dx} \end{array}\right\}$ put in eq (1)

$1 \rightarrow \frac{dz}{dx} = xz + 3y$

$\left.\begin{array}{ll} \Rightarrow \frac{dy}{dx} = z & \underset{x_0}{y}(\underset{}{0}) = \underset{y_0}{1} \\ \frac{dz}{dx} = xz + 3y & z(\underset{x_0}{0}) = \underset{z_0}{2} \end{array}\right\}$ system of equations

Date:

$h = 0.2 - 0$
$= 0.2$

$y(x) = y_0 + h y'_0 + \frac{h^2}{2!} y''_0 + \frac{h^3}{3!} y'''_0 + \dots$

$y' = z \quad = 2$ $\qquad z' = xz + 3y \quad = 3$
$y'' = z' \quad = 3$ $\qquad z'' = xz' + z + 3y' \quad = 8$
$y''' = z'' \quad = 8$ $\qquad z''' = z' + z' + xz'' + 3y''$

$y(x) = y_0 + h y'_0 + \frac{h^2}{2!} y''_0 + \frac{h^3}{3!} y'''_0 +$

$= 1 + 0.2(2) + \frac{0.2^2}{2!}(3) + \frac{0.2^3}{3!}(8)$

$= 1.47066$

⇒ Week 13: Lecture 2

Picard Method

Let $\frac{dy}{dx} = f(x,y)$ $\quad ; y(x_0) = y_0$

$\int_{x_0}^{x} \frac{dy}{dx} dx = \int_{x_0}^{x} f(x,y) dx$

$y(x)\Big|_{x_0}^{x} = \int_{x_0}^{x} f(x,y) dx$

$y(x) - y(x_0) = \int_{x_0}^{x} f(x,y) dx$

$y(x) = y(x_0) + \int_{x_0}^{x} f(x,y) dx$

General form

$y(x_n) = y(x_0) + \int_{x_0}^{x} f(x, y_{n-1}) \, dx$

$n = 1, 2, 3 \dots$

Date:__________

Question: Given the equation $\frac{dy}{dx} = 3x^2+y$ with $y(1)=2$
Estimate $y(2)$ by Picard Method

$x_0 \quad y_0$

$$y(x_1) = y_0 + \int_{x_0}^{x} f(t, y_0)\,dt$$

$f(x,y) = 3x^2+y$

$f(t,2) = 3t^2+2$

$$= 2 + \int_{1}^{x} f(t,2)\,dt$$

$$= 2 + \int_{1}^{x} (3t^2+2)\,dt$$

$$= 2 + \left[\frac{3t^3}{3} + 2t\right]_1^x$$

$$= 2 + [t^3+2t]_1^x$$

$$= 2 + [(x^3+2x) - ((1)^3+2(1))]$$

$$= 2 + [x^3+2x-3]$$

$$= x^3+2x-1$$

$n=2$

$$y(x_2) = 2 + \int_{1}^{x} f(t, y_1)\,dt$$

$f(t, y_2) = 3t^2 + y_1$

$3t^2 + x^3 + 2x - 1$

$3t^2 + t^3 + 2t - 1$

$$= 2 + \int_{1}^{2} (3t^2+t^3+2t-1)\,dt$$

$$= 2 + \left[\frac{3t^3}{3} + \frac{t^4}{4} + \frac{2t^2}{2} - t\right]_1^x$$

$$= 2 + \left[t^3 + \frac{t^4}{4} + t^2 - t\right]_1^2$$

$$= 2 + \left[\left(x^3 + \frac{x^4}{4} + x^2 - x\right) - \left(1 + \frac{1}{4} + 1 - 1\right)\right]$$

$$= 2 + \left[\frac{x^4}{4} + x^3 + x^2 - x - \frac{5}{4}\right]$$

$$= \frac{x^4}{4} + x^3 + x^2 - x + \frac{3}{4}$$

Date:

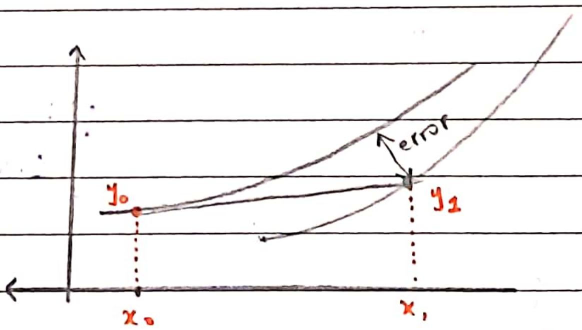
$$y(2) = \frac{2^4}{4} + 2^3 + 2^2 - 2 + 3$$

$$= 14.75$$

Euler Method

Let $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$

$m = \frac{dy}{dx} = f(x,y)$



(x_0, y_0) (x_1, y_1)

$y_1 - y_0 = m(x_1 - x_0)$

$y_1 - y_0 = hf(x_0, y_0)$

$y_1 = y_0 + hf(x_0, y_0)$

$y_2 = y_1 + hf(x_1, y_1)$

$\vdots$

$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$

here $x_1 - x_0 = h$

$f(x_0, y_0) = m$

Date:

Question: Given the equation $\frac{dy}{dx} = 3x^2 + y$, with $y(1) = 2$
Estimate $y(2)$ by Euler Method.

$x_0 \quad y_0$

$y_1 = y_0 + hf(x_0, y_0)$

$\frac{dy}{dx} = f(x,y) = 3x^2 + y$

$y(1) = 2$

$y(2) = ?$

x_1

$f(x_0, y_0) = 3x_0^2 + y_0$

$= 3(1)^2 + 2$

$= 5$

$h = x_1 - x_0 = 1$

$y_1 = 2 + 1(5)$

$= 7$

i) $h = 0.5$

$y_0 = 2 \qquad y_1 = ? \qquad y_2 = ?$

$x_0 = 1 \qquad x_1 = 1.5 \qquad x_2 = 2$

$h = 0.5 \qquad x_n = 2$

$y_1 = 2 + 0.5(5)$

$= 4.5$

$y(1) = 2$

$y(1.5) = 4.5$

$f(x_1, y_1) = 3x_1^2 + y_1$

$= 3(1.5)^2 + 4.5$

$= 11.25$

$y_2 = y_1 + hf(x_1, y_1)$

$= 1.5 + 0.5(11.25)$

$= 7.125$

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Date:

ii) 0.25

$y_0 = 2$ $\quad y_1 = 3.25$

$x_0 = 1 \quad x_1 = 1.25 \quad x_2 = 1.5 \quad x_3 = 1.75 \quad x_4 = 2$

$\rightarrow f(x_0, y_0) = 3x_0^2 + y_0$

$= 3(1)^2 + 2 = 5$

$y_1 = y_0 + hf(x_0, y_0)$

$= 2 + 0.25(5)$

$= 3.25$

$\rightarrow f(x_1, y_1) = 3x_1^2 + y_1$

$= 3(1.25)^2 + 3.25$

$= 7.9375$

$y_2 = y_1 + hf(x_1, y_1)$

$= 3.25 + 0.25(7.9375)$

$= 5.234375$

$\rightarrow f(x_2, y_2) = 3x_2^2 + y_2$

$= 3(1.5)^2 + 5.234375 =$

$= 11.984375$

$y_3 = y_2 + hf(x_2, y_2)$

$= 5.234375 + 0.25(11.984375)$

$= 8.230469$

$\rightarrow f(x_3, y_3) = 3x_3^2 + y_3$

$= 3(1.75)^2 + 8.230469$

$= 17.41797$

$y_4 = y_3 + hf(x_3, y_3)$

$= 8.230469 + 0.25(17.41797)$

$= 12.58496$

$\rightarrow \cancel{f(x_4, y_4) = 3x_4^2 + y_4}$

$\cancel{= 3(2)^2 + y_4}$

Date:

→ Week 14 : Lecture 1

Euler Modified Method

$$y_{n+1} = y_n + h\left\{f\left[x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right]\right\}$$

Euler Improved Method

$$y_{n+1} = y_n + \frac{1}{2}h\left[f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))\right]$$

Question: Given $\frac{dy}{dt} = \frac{y-t}{y+t}$ with the initial condition $y = 1$ at $t = 0$.

Find y approximately at $t = 0.1$ in 5 steps using Euler Improved and Euler Modified

- Euler Method Improved

$$\frac{dy}{dt} = f(t,y) = \frac{y-t}{y+t} \qquad y_0 = 1, t_0 = 0$$

$t_1 = t_0 + h$

$= 0 + 0.02$

$= 0.02$

$$h = \frac{b-a}{n} = \frac{0.1-0}{5}$$

$= 0.02$

$y_0 = 1$	y_1	y_2	y_3	y_4	y_5
$t_0 = 0$	$t_1 = 0.02$	0.04 t_2	0.06 t_3	0.08 t_4	0.1 t_5

- $y_1 = y_0 + \frac{h}{2}\left[f(t_0, y_0) + f\{t_0 + h, y_0 + hf(t_0, y_0)\}\right]$

$$= 1 + \frac{0.02}{2}\left[f(0,1) + f\{0 + 0.02, 1 + 0.02 f(0,1)\}\right]$$

$= 1 + 0.01[1 + f\{0.02, 1.02\}]$

$= 1 + 0.01(1 + 0.9615)$

$= 1.01961$

Date:

- $y_2 = y_1 + \frac{h}{2}[f(t_1, y_1) + f\{t_1+h, y_1 + hf(t_1, y_1)\}]$

$= 1.01961 + 0.01[f(0.02, 1.01961) + f\{0.02+h, 1.01961 + 0.02f(0.02, 1.01961)$

$= 1.02 + 0.01\left[\frac{1.02 - 0.02}{1.02+0.02} + f(0.04,\right.$

$= 1.02 + 0.01[0.96 + 1.0392]$

$= 1.04$

- $y_3 = y_2 + 0.01[f(t_2, y_2) + f\{t_2+h, y_2 + hf(t_2, y_2)\}]$

$= 1.04 + 0.01[f(0.04, 1.04) + f\{0.06, 1.04 + 0.02f(0.04, 1.04)\}]$

$= 1.04 + 0.01[0.93 + f(0.06, 1.059)]$

$= 1.058 \approx 1.06$

- $y_4 = y_3 + 0.01[f(t_3, y_3) + f\{t_3+h, y_3 + hf(t_3, y_3)\}]$

$= 1.06 + 0.01[f(0.06, 1.06) + f(0.08, 1.06 + 0.02f(0.06, 1.06)\}]$

$= 1.06 + 0.01[0.893 + f(0.08, 1.08)]$

$= 1.0775 \approx 1.08$

- $y_5 = y_4 + 0.01[f(t_4, y_4) + f(t_4+h, y_4 + hf(t_4, y_4)\}]$

$= 1.08 + 0.01[f(0.08, 1.08) + f(1, 1.08 + 0.02f(0.08, 1.08)\}]$

$= 1.08 + 0.01[0.862 + f(1, 1.0972)$

$= 1.089$

- Euler Modified Method

(same procedure only change the formula)

Date:

→ RK - Method of Order 1

$$y_{n+1} = y_n + \underbrace{hf(x_n, y_n)}_{k_1}$$ [Same as Euler Method]

→ RK - Method of Order 2

$$y_{n+1} = y_n + \Delta y$$

and $\Delta y = k_2$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_1 = hf(x_n, y_n)$$

$n = 0$

$y_1 = y_0 + \Delta y$

$\Delta y = k_2$

→ RK - Method of Order 3

$$y_{n+1} = y_n + \Delta y$$

$$\Delta y = \frac{1}{6}(k_1 + 4k_2 + k_3)$$

where

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf\left(x_n + \frac{h}{2}; y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf(x_n + h, y_n + 2k_2 - k_1)$$

Date:

→ RK - Method of Order 4

$$y_{n+1} = y_n + \Delta y$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = hf(x_n, y_n)$$
$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf(x_n + h/2, y_n + k_2/2)$$
$$k_4 = hf(x_n + h, y_n + k_3)$$

⇒ Week 14: Lecture 2

→ Question: Using RK Method of Order 4, Find the solution of the given DE at $x = 1.2$, $f(x,y) = y' = \dfrac{2xy + e^x}{x^2 + xe^x}$; $y(1) = 0$

$y_0 = 0$, $x_0 = 1$, $x_1 = 1.2$, $y_1 = ?$
(If h not given then simply x_0 and x_1)

$$y_1 = y_0 + h$$
$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$h = 1.2 - 1$
$= 0.2$

- $k_1 = hf(x_0, y_0)$

$$= 0.2\left[\frac{2(1)(0) + e^1}{1^2 + 1(e)^1}\right]$$

$$= 0.2(0.73106)$$
$$= 0.14621$$

- $k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$$= 0.2 f(1.1, 0.0731)$$
$$= 0.1402$$

Date:

- $k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$
$= 0.2\, f\left(1.1, \frac{0.1402}{2}\right)$
$=$
$= 0.1399$

- $k_4 = hf(x_0 + h, y_0 + k_3)$
$= 0.2\, f(1.2, 0.1399)$
$= 0.1348$

$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= \frac{1}{6}(0.14621 + 2(0.1402) + 2(0.1399) + 0.1348)$

$= 0.1402$

$y_1 = y_0 + \Delta y$
$= 0 + 0.1402 = 0.1402$ $\quad y(1.2) = 0.1402$

→ RK System of Equations

$y(x) \Leftarrow \frac{dy}{dx} = f(x, y, z) \quad , \quad y(x_0) = y_0$ (can be solved using any order of RK)

$z(x) \Leftarrow \frac{dz}{dx} = g(x, y, z) \quad , \quad z(x_0) = z_0$

RK-4

$y_1(x) = y_0 + \Delta y$
$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = hf(x_0, y_0, z_0)$
$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$

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$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$

$k_4 = hf\left(x_0 + h, y_0 + k_3, z_0 + l_3\right)$

$z_1 = z_0 + \Delta z$

$\Delta z = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

$l_1 = hg(x_0, y_0, z_0)$

$l_2 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$ (Solve both k's and l's simultaneously).

$l_3 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$

$l_4 = hg(x_0 + h, y_0 + k_3, z_0 + l_3)$

→ Second Order Differential Equations

$\frac{d^2y}{dx^2} = f(x, y, y')$

$y(x_0) = y_0$

$y'(x_0) = y'_0$

$\frac{dy}{dx} = z = \cancel{g}\, f(x, y, z)$

$\frac{d^2y}{dx^2} = \frac{dz}{dx}$

$\frac{dz}{dx} = g(x, y, z)$

$$\begin{cases} \frac{dy}{dx} = z = \cancel{g}\, f(x, y, z) \\ \frac{dz}{dx} = g(x, y, z) \end{cases}$$

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Date:

Question: Using the RK Method of Order 1,2,3,4, find the solution of the system at $x = 0.2$

$g(x,y,z)$ $\frac{dy}{dx} = x+z$; $y(0) = 0$ $f(x,y,z)$

$\frac{dz}{dx} = x-y$; $z(0) = 1$ $g(x,y,z)$

Using RK-4

$y(0.2)$ $z(0.2)$

$h = x_1 - x_0$

$= 0.2 - 0 = 0.2$

- $k_1 = hf(x_0, y_0, z_0)$
$= 0.2f(0,0,1)$
$= 0.2(1)$
$= 0.2$

- $L_1 = hg(x_0; y_0, z_0)$
$= 0.2(0,0,1)$
$= 0.2(0-0)$
$= 0$

- $k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{L_1}{2})$
$= 0.2f(0.1, 0.1, 1)$
$= 0.22$

- $L_2 = hg(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{L_1}{2})$
$= 0.2g(0.1, 0.1, 1)$
$= 0$

- $k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{L_2}{2})$
$= 0.2f(0.1, 0.11, 1)$
$= 0.22$

- $L_3 = hg(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2} + z_0 + \frac{L_2}{2})$
$= 0.2g(0.1, 0.11, 1)$
$= -0.002$

- $k_4 = hf(x_0 + h, y_0 + k_3, z_0 + L_3)$
$= 0.2f(0.2, 0.22, 0.998)$
$= 0.2396$

- $L_4 = hg(x_0 + h, y_0 + k_3, z_0 + L_3)$
$= 0.2g(0.2, 0.22, 0.998)$
$= -0.004$

$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= \frac{1}{6}(0.2 + 2(0.22) + 2(0.22) + 0.2396) = 0.2194$

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Date:

$y_1 = y_0 + \Delta y$

$= 0 + 0.2199$

$= 0.2199$

$\Delta z = \frac{1}{6}(L_1 + 2L_2 + 2L_3 + L_4)$

$= \frac{1}{6}(0 + 0 + 2(-0.002) - 0.004)$

$= -0.00133$

$z_1 = z_0 + \Delta z$

$= 0.9986$

$y(0.2) = 0.2199$ $z(0.2) = 0.9986$

→ Week 15: Lecture 1

Predictor and Corrector Method

Multistep Method

$y(x_0) = y_0$: y_0', y_1', y_2', y_3'

first use previous methods to find these

1. Milne's Method P-C

$n = 3$

$$y_{4,p} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$$

P: $y_{n+1,p} = y_{n-3} + \frac{4h}{3}(2y_{n-2}' - y_{n-1}' + 2y_n')$

Predictor Corrector pair

C: $y_{n+1,c} = y_{n-1} + \frac{h}{3}(y_{n-1}' + 4y_n' + y_{n+1}')$

$$y_{4,c} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$$

Date:

AnandBa

Question : Given $y' = \frac{1}{x+y}$,

$y(0) = 2$, $y(0.2) = 2.0933$, $y(0.4) = 2.1755$, $y(0.6) = 2.2496$

Find $y(0.8)$ by Milne's Predictor corrector method.

if y's not given calculate through previous methods i.e Taylor ~~...~~ RK.

→ Milne's Predictor

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}(2y'_{n-2} - y'_{n-1} + 2y'_n)$$

$n=3$ $$y_{4,p} = y_0 + \frac{4h}{3}(2y'_1 - y'_2 + 2y'_3) \rightarrow (i)$$

- $y'_1 = \frac{1}{x_1+y_1} = \frac{1}{0.2+2.0933}$

$= 0.43605$

- $y'_2 = \frac{1}{x_2+y_2} = \frac{1}{0.4+2.1755}$

$= 0.38827$

- $y'_3 = \frac{1}{x_3+y_3} = \frac{1}{0.6+2.2496}$

$= 0.35096$

Put in (i)

$$y_{4,p} = 2 + \frac{4(0.2)}{3}(2(0.43605) - 0.38827 + 2(0.35096))$$

$= 2.3161 \longrightarrow$ $y(0.8) = 2.3161 \rightarrow y_4$

→ Milne's Corrector

$$y_{4,c} = y_2 + \frac{h}{3}(y'_2 + 4y'_3 + y'_4) \longrightarrow (ii)$$

$y'_4 = \frac{1}{x_4+y_4} = \frac{1}{0.8+2.3161}$

$= 0.320914$

Date:

put in (ii)

$$y_{4,c} = 2.1755 + \frac{0.2}{3}(0.38827 + 4(0.35096) + 0.320914)$$

$$= 2.3162$$

$y(0.8) = 2.3162$

⇒ Week 15 Lecture 2

→ Adam Bashforth Predictor Corrector Method

$$P: \quad y_{n+1,p} = y_n + \frac{h}{24}\left[55y'_n - 59y'_{n-1}\right]$$

$$C: \quad y_{n+1,c} = y_n + \frac{h}{24}\left[9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}\right]$$

- In these methods, the first four values of y are given, if not they can be calculated using Taylor Series or RK.

→ Finite Difference Method

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \dots$$

optional

$$y(x+h) - y(x) = h(y'(x) + O(h))$$

$$y'(x) = \frac{y(x+h) - y(x)}{h}$$

$$y(x-h) = y(x) - hy'(x) + \frac{h^2}{2!}y''(x) - \frac{h^3}{3!}y'''(x) + \dots$$

$$y'(x) = \frac{y(x) - y(x-h)}{h}$$

$$\frac{y(x+h) - y(x-h)}{2h} = y'(x)$$

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$$y'(x) = \frac{y_{i+1} - y_{i-1}}{2h} \rightarrow (i)$$

$$y''(x) = \frac{y_{i+1} + y_{i-1} - 2y_i}{h^2} \rightarrow (ii)$$

Suppose a boundary value problem

$$y'' + a(x)y' + b(x)y(x) = c(x)$$

together with the boundary conditions

$$y(x_0) = \alpha \quad , y(x_n) = \beta \qquad x \in (x_0, x_n)$$

Put values of $y'(x)$ [i] and $y''(x)$ [ii] in boundary value problem

$$\frac{y_{i+1} + y_{i-1} - 2y_i}{h^2} + a(x)\frac{y_{i+1} - y_{i-1}}{2h} + b(x_i)y(x_i) = c(x_i)$$

Simplifying

$$y_{i+1}\left[1 + \frac{h}{2}a_i\right] + y_i(h^2 b_i - 2) + y_{i-1}\left[1 - \frac{h}{2}a_i\right] = a_i h^2$$

where $i = 1, 2 \dots n-1$

$y_0 = \alpha$, $y_n = \beta$, $a_i = a(x_i)$, $b_i = b(x_i)$, $c_i = c(x_i)$

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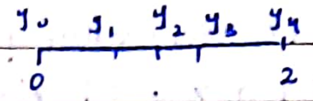
Question: Using the finite difference method, solve $\frac{d^2y}{dx^2} = y$ in $(0,2)$ given $y(0)=0$, $y(2)=3.63$ subdeviding the range of x in four equal parts

$$h = \frac{b-a}{4} = \frac{2}{4} = \frac{1}{2}$$



$$\frac{d^2y}{dx^2} = y_i$$

from (ii) we know

$$y''(x) = \frac{y_{i+1} + y_{i-1} - 2y_i}{h^2} = y_i$$

~~Put~~ Simplify

$$y''(x) = y_{i+1} + y_{i-1} - 2y_i = h^2 y_i$$

$$y_{i+1} + y_{i-1} = y_i(h^2+2)$$

$$y_{i+1} - y_i(h^2+2) + y_{i-1} = 0$$

Put $h = \frac{1}{2}$

$$y_{i+1} - \frac{9}{4} y_i + y_{i-1} = 0 \qquad i = 1,2,3 \rightarrow (i)$$

Hence the equations to be solved

$$y_2 - \frac{9}{4} y_1 + y_0 = 0 \rightarrow (ii)$$

$$y_3 - \frac{9}{4} y_2 + y_1 = 0 \rightarrow (iii)$$

$$y_4 - \frac{9}{4} y_3 + y_2 = 0 \rightarrow (iv)$$

using $y_0 = 0$ $(y(0)=0)$ $y_4 = 3.63$ $(y(2)=3.63)$

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$$y_2 - \frac{9}{4} y_1 = 0 \rightarrow (v)$$

$$y_3 - \frac{9}{4} y_2 + y_1 = 0 \rightarrow (vi)$$

$$3.63 - \frac{9}{4} y_3 + y_2 = 0 \rightarrow (vii)$$

Eliminating y_1 from (v) and (vi)

$(v) \rightarrow \quad y_1 = \frac{4}{9} y_2$

in (vi) $\rightarrow \quad y_3 - \frac{9}{4} y_2 + \frac{4}{9} y_2 = 0$

$$y_3 - \frac{65}{36} y_2 = 0 \rightarrow (viii)$$

form (vii) and (viii)

$(viii) \rightarrow \quad y_3 = \frac{65}{36} y_2$

in (vii) $\rightarrow 3.63 - \frac{9}{4}\left(\frac{65}{36} y_2\right) + y_2 = 0$

$$y_2 = 1.1853$$

put y_2 in (viii)

$$y_3 = \frac{65}{36}(1.1853)$$

$$= 2.1401$$

$(v) \rightarrow y_1 = \frac{4}{9} y_2 = \frac{4}{9}(2.1401)$

$$= 0.5268$$

x	0	0.5	1	1.5	2
y	0	0.5268	1.1853	2.1401	3.63