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NUMERICAL COMPUTING

F I N A L S

Numerical Solution of Ordinary Differential Equations

day / date: 16-11-23

* if we have n^{th} order derivation we need n conditions to get value of constant of integration

these conditions may be initial or bounded or both ←

$$\frac{dy}{dx} = f(x,y) \quad , \quad y(x_0) = y_0$$ ← [we need this condition to complete integration process]

$y\frac{dy}{dx} \Rightarrow$ non-linear

diff order of eq = 4

non-linear eq

$$\frac{d^4y}{dx^4} + 4x\frac{d^2y}{dx^2} + \frac{dy}{dx} + y^5 = 0$$

$y(0) = \alpha \quad , \quad y'(0) = \beta \, , \, y''(0.5) = \gamma \, , \, y'''(1) = \delta$ ← multi-bounded problem

$y(0) = \alpha \quad , \quad y'(0) = \beta \, , \, y''(0) = \gamma \, , \, y'''(0) = \delta$ ← initial value problem

$y(0) = \alpha \quad , \quad y'(0) = \beta \, , \, y''(1) = \gamma \, , \, y'''(1) = \delta$ ← bounded value problem

Taylor Series: ⇒ simplest method

$x_0 \quad x+h = x_1 \quad x+2h = x_2 \quad x+3h = x_3 \ldots$

$$y(x) \leftarrow y = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \ldots$$

decrease in step size will increase the accuracy.

$$y(x_1) = y_0 + \overbrace{(x_1-x_0)}^{h}y_0' + \frac{h^2}{2!}y_0'' + \ldots$$

we need to find y_1 corresponding to x_1 using x_0 and y_0,

" " y_2 " x_2 " x_1 " y_1 and so on.

$$y(x_0+h) = y(x_0) + hy'(x_0) + \frac{h^2}{2}y''(x_0) + \frac{h^3}{3!}y'''(x_0) + \ldots$$ ⇒ since $h = x - x_0 \Rightarrow x = h + x_0$

Example 7.1:

Find $y(1.1)$, given $y' = 2x - y$ and $y(1) = 3$

Solution:

$y' = 2x - y$, $x_0 = 1$, $y_0 = 3$, $x_1 = 1.1$, $h = 0.1$

$y_0' = 2x_0 - y_0 = 2(1) - 3 = -1$

$y'' = 2 - y'$

$y_0'' = 2 - y_0' = 2 - (-1) = 3$

$y''' = -y''$

$y_0''' = -y_0'' = -3$

By Taylor series:

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \ldots$$

putting values, we have

$$y_1 = 3 + (0.1)(-1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(-3) + \ldots$$

$$= 3 - 0.1 + 0.015 - 0.0005 \ldots$$

$$= 2.9145$$

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Example 7.2:

* if we are given $y(0)$ and we have to find both $y(0.1)$ and $y(0.2)$, we will first find y_1 using x_0 and y_0 by Taylor series and then we will find y_2 using x_1 and y_1 by applying Taylor series again.

7.1.1 Taylor Series Method for Simultaneous First Order Diff Eq

* if we have two equations of order 1, we will make two taylor series.

suppose we have $\frac{dy}{dx} = f_1(x,y,z)$ and $\frac{dz}{dx} = f_2(x,y,z)$ with two initial points $y(x_0) = y_0$ and $z(x_0) = z_0$

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \dots$$

$$z_1 = z_0 + hz_0' + \frac{h^2}{2!}z_0 + \frac{h^3}{3!}z_0 + \dots$$

7.1.2 Taylor Series Method for 2nd Order Diff Eq

* if we are given second order diff eq, we would need two values y_0 and y_0'

Example 7.8: Find $y(0.2)$, given $y'' + y = 0$, $y(0) = 1$, $y'(0) = 0$

Solution:

$y'' = -y$, $x_0 = 0$, $y_0 = 1$, $y_0' = 0$, $h = 0.2$

By Taylor Series

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \dots$$

Now

$$y'' = -y \Rightarrow y_0'' = -y_0 = -1$$

$$y''' = -y' \Rightarrow y_0''' = -y_0' = 0$$

$$y^{IV} = -y'' \Rightarrow y_0^{IV} = -y_0'' = -(-1) = 1$$

Taylor series become

$$y_1 = 1 + (0.2)(0) + \frac{(0.2)^2}{2}(-1) + \frac{(0.2)^3}{6}(6) + \frac{(0.2)^4}{24}(1) + \dots$$

$$= 1 - 0.02 + 0.00006667$$

$$= 0.9800667$$

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Q. use taylor series to find approx values of y and z corresponding to $x=0.1$, 0.2

$y(0)=2$, $z(0)=1$ and $\frac{dy}{dx}=x+z$, $\frac{dz}{dx}=x-y^2$

Solution:

$y_0=2$, $z_0=1$, $x_0=0$

$y'=x+z \Rightarrow y_0'=x_0+z_0=1$

$y''=1+z'$

$z'=x-y^2 \Rightarrow z_0'=x_0-y_0^2=-4$

$y_0''=1+z_0'=1+(-4)=-3$

$z''=1-2yy' \Rightarrow z_0''=1-2y_0y_0'=-3$

$y_0'''=z_0''=-3$

Now, by taylor series

$$y_1=y_0+hy_0'+\frac{h^2}{2!}y_0''+\frac{h^3}{3!}y_0'''+\ldots$$

$$=2+0.1(1)+\frac{0.1^2(-3)}{2!}+\frac{0.1^3(-3)}{3!}+\ldots$$

$\boxed{y_1=2.0845}$

$z'''=-2yy''-2y' \Rightarrow z_0'''=-2y_0y_0''-2y_0'=10$

Again by taylor series

$$z_1=z_0+hz_0'+\frac{h^2}{2!}z_0''+\frac{h^3}{3!}z_0'''+\ldots$$

$$=1+0.1(-4)+\frac{0.1^2(-3)}{2!}+\frac{0.1^3(10)}{3!}+\ldots$$

$\boxed{z_1=0.5866}$

Now to find y_2 and z_2

$y_1=2.0845$, $z_1=0.5866$, $x_1=0.1$

$y_1'=x_1+z_1=0.6866$

$z_1'=x_1-y_1^2=0.1-4.3451=-4.2451$

$y_1''=1+z_1'=1-4.2451=-3.2451$

$z_1''=1-2y_1y_1'=1-2(2.0845)(0.6866)=-1.8624$

$y_1'''=z_1''=-1.8624$

$z_1'''=-2y_1y_1''-2y_1'=-2(2.0845)(-3.2451)-2(0.6866)$

$=12.15$

Now, by taylor series

$$y_2=y_1+hy_1'+\frac{h^2}{2!}y_1''+\frac{h^3}{3!}y_1'''+\ldots$$

$$=2.0845+0.1(0.6866)+\frac{0.1^2(-3.2451)}{2!}+\frac{0.1^3(-1.8624)}{3!}+\ldots$$

$\boxed{y_2=2.1366}$

Again by taylor series

$$z_2=z_1+hz_1'+\frac{h^2}{2!}z_1''+\frac{h^3}{3!}z_1'''+\ldots$$

$$=0.5866+0.1(-4.2451)+\frac{0.1^2(-1.8624)}{2!}+\frac{0.1^3(12.15)}{3!}+\ldots$$

$\boxed{z_2=0.1548}$

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How to find error in your result

1. Exact answer − Calculated answer
2. Risidual error (when exact answer not known)

Picard's Method :

$$y_1^{(n)} = y_0 + \int_{x_0}^{x} f(x_1, y_1^{n-1})\,dx$$

Example 7.10:

$y' + y = e^x$, $y(0) = 0$ solve $y(0.1)$ by Picard's method

Solution:

we know that $y = y_0 + \int_{x_0}^{x} f(x,y)\,dx = 0 + \int_0^x (e^x - y)\,dx$

put $y = y_0$

$$y^{(1)} = \int_0^x (e^x - 0)\,dx = e^x - 1$$

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\,dx = \int_0^x (e^x - e^x + 1)\,dx = x$$

$$y^{(3)} = y_0 + \int_{x_0}^{x} f(x, y^{(2)})\,dx = \int_0^x (e^x - x)\,dx = e^x - \frac{x^2}{2} - 1$$

$$y^{(4)} = y_0 + \int_{x_0}^{x} f(x, y^{(3)})\,dx = \int_0^x \left(e^x - e^x + \frac{x^2}{2} + 1\right)dx = \frac{x^3}{6} + x$$

$$y^{(5)} = y_0 + \int_{x_0}^{x} f(x, y^{(4)})\,dx = \int_0^x \left(e^x - \frac{x^3}{6} - x\right)dx = e^x - \frac{x^4}{24} - \frac{x^2}{2} - 1$$

← we put value of x in this eq bcz it's most precise

Now,

Put $x = 0.1$

$$y(0.1) = e^{0.1} - \frac{0.1^4}{24} - \frac{0.1^2}{2} - 1 = 0.1002$$

* the more iterations the more precise answer you'll get

day / date:

Q. Solve by Picard's method and find first-three successive approximations

$$\frac{dy}{dx} = 3e^x + 2y \quad , \quad y(0) = 0$$

Solution:

we know that

$$y = y_0 + \int_{x_0}^{x} f(x,y)\,dx$$

taking $y = y_0$

$$y^{(1)} = \int_0^x (3e^x)\,dx$$

$$= 3(e^x - 1)$$

$$y^{(2)} = \int_0^x f(x, y^{(1)})\,dx = \int_0^x (3e^x + 2(3(e^x - 1)))\,dx$$

$$= \int_0^x (3e^x + 6e^x - 6)\,dx = 3e^x + 6e^x - 6x - 3 - 6 = 9e^x - 6x - 9$$

$$y^{(3)} = \int_0^x f(x, y^{(2)})\,dx = \int_0^x (3e^x + 2(9e^x - 6x - 9))\,dx = \int_0^x (3e^x + 18e^x - 12x - 18)\,dx$$

$$= 3e^x + 18e^x - 12\frac{x^2}{2} - 18x - 3 - 18 = 21e^x - 6x^2 - 18x - 21$$

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Euler's Method: easy but not fast cos order = 1

- step-by-step method

$$\boxed{y_{n+1} = y_n + hf(x_n, y_n)}$$

Modified Euler's Method:

- greater accuracy than euler

$$\boxed{y_{n+1} = y_n + h\left\{f\left[x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right]\right\}}$$

Improved Euler's Method:

$$\boxed{y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f\left[x_n + h, y_n + hf(x_n, y_n)\right]\right]}$$

Example 7.17:

$y' = y + e^x, \; y(0) = 0$

solve $x = 0.2, 0.4$ by improved euler

Solution:

$x_0 = 0, \; y_0 = 0, \; x_1 = 0.2, \; x_2 = 0.4, \; h = 0.2$

By improved Euler's method

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\{x_0 + h, y_0 + hf(x_0, y_0)\}\right]$$

$$= 0 + \frac{0.2}{2}\left[y_0 + e^{x_0} + f\{x_0 + 0.2, y_0 + 0.2(y_0 + e^{x_0})\}\right]$$

$$= 0.1\left[1 + f\{0.2, 0.2\}\right]$$

$$= 0.1\left[1 + 0.2 + e^{0.2}\right]$$

$y(0.2) = 0.24214$

$$y_2 = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f\{x_1 + h, y_1 + hf(x_1, y_1)\}\right]$$

$$= 0.24 + 0.1\left[0.24 + e^{0.2} + f\{0.4, 0.22 + 0.2(0.24 + e^{0.2})\}\right]$$

$$= 0.24 + 0.1\left[1.46354 + f\{0.4, 0.5127\}\right]$$

$$= 0.24 + 0.1\left[1.46354 + 0.5127 + e^{0.4}\right]$$

$y(0.4) = 0.59116$

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Example 7.16 :

using modified euler, find $y(0.1)$, $y(0.2)$ given

$$\frac{dy}{dx} = x^2 + y^2 \quad , \quad y(0) = 1$$

Solution :

$$y' = x^2 + y^2 \quad , \quad x_0 = 0 \, , \; y_0 = 1$$

by modified euler formula

$$y_{n+1} = y_n + hf\left[x_n + \frac{h}{2} \, , \; y_n + \frac{h}{2} f(x_n, y_n)\right]$$

$$y_1 = y_0 + hf\left[x_0 + \frac{h}{2} \, , \; y_0 + \frac{h}{2} f(x_0, y_0)\right]$$

$$= 1 + 0.1 f\left[0.05 \, , \; 1 + 0.05(1)\right]$$

$$= 1 + 0.1(0.05^2 + 1.05^2)$$

$$= 1.1105$$

Now,

$$y_2 = y_1 + hf\left[x_1 + \frac{h}{2} \, , \; y_1 + \frac{h}{2} f(x_1, y_1)\right]$$

$$= 1.1105 + 0.1 f\left[0.1 + 0.05 \, , \; 1.1105 + 0.05 f(0.1, 1.1105)\right]$$

$$= 1.1105 + 0.1 f\left[0.15 \, , \; 1.1105 + 0.05(1.2432)\right]$$

$$= 1.1105 + 0.1 f\left[0.15 \, , \; 1.17266\right]$$

$$= 1.2503$$

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Runge-Kutta Method

→ family of methods
⇒ we gonna study classical method
we sure will

RK-1 and euler are same

$$y_{i+1} = y_i + \Delta y_i$$
$$y_{i+1} = y_i + hf(x_i, y_i) \rightarrow \text{euler}$$

RK-1 : → order 1

$$k_1 = hf(x, y) \qquad \Delta y = k_1$$

* RK-1 ⇒ euler
* RK-2 ⇒ modified euler
* RK-3,4 mai k_1, k_2 are same but k_3 is diff

RK-2 :

$$k_1 = hf(x, y)$$
$$k_2 = hf\left(x + \frac{h}{2}, y + \frac{k_1}{2}\right) \qquad \Delta y = k_2$$

RK-3 :

H.W: 7.4.1

$$k_1 = hf(x, y)$$
$$k_2 = hf\left(x + \frac{h}{2}, y + \frac{k_1}{2}\right) \qquad \Delta y = \frac{1}{6}(k_1 + 4k_2 + k_3)$$
$$k_3 = hf(x + h, y + 2k_2 - k_1)$$

RK-4 :

$$k_1 = hf(x, y)$$
$$k_2 = hf\left(x + \frac{h}{2}, y + \frac{k_1}{2}\right) \qquad \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$k_3 = hf\left(x + \frac{h}{2}, y + \frac{k_2}{2}\right)$$
$$k_4 = hf(x + h, y + k_3)$$

$$y_1 = y_0 + \Delta y$$
$$y_n = y_{n-1} + \Delta y$$

Example 7.18

Q. Use R-K fourth order to solve y at $x = 0.1, 0.2$ for $y' = -y$ given $y(0) = 1$

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Solution:

$y' = -y$, $x_0 = 0, x_1 = 0.1, x_2 = 0.2, y_0 = 1$

$$y_1 = y_0 + \Delta y$$

For R-K fourth-order:

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$k_1 = hf(x_0, y_0)$

$= 0.1(-1) = -0.1$

$k_2 = hf(x_0 + h/2, y_0 + k_1/2)$

$= 0.1(0.05, 0.95) = -0.095$

$k_3 = hf(x_0 + h/2, y_0 + k_2/2)$

$= 0.1(0.05, 0.9525) = -0.09525$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$= 0.1f(0.1, 0.90475) = -0.090475$

$\Delta y = \frac{1}{6}(-0.1 + 2(-0.095) + 2(-0.09525) + (-0.090475))$

$= -0.0951625$

$y_1 = 1 - 0.0951625 = 0.9048$

Now,

$$y_2 = y_1 + \Delta y$$

as, for R-K fourth order:

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$k_1 = hf(x_1, y_1)$

$= 0.1(-0.9048) = -0.09048$

$k_2 = hf(x_1 + h/2, y_1 + k_1/2)$

$= 0.1f(0.15, 0.85956) = -0.085956$

$k_3 = hf(x_1 + h/2, y_1 + k_2/2)$

$= 0.1f(0.15, 0.861822) = -0.0861822$

$k_4 = hf(x_1 + h, y_1 + k_3)$

$= 0.1f(0.2, 0.8186178) = -0.08186178$

$\Delta y = \frac{1}{6}(-0.09048 + 2(-0.085956) + 2(-0.0861822) + (-0.08186178))$

$= -0.08610303$

$y_2 = 0.9048 - 0.08610303 = 0.8187$

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RK for Simultaneous First order diff eq:

$$\frac{dy}{dx} = f(x,y,z) \quad , \quad \frac{dz}{dx} = g(x,y,z)$$

values of x_0, y_0 and z_0 will be given

$$k_1 = hf(x_0, y_0, z_0) \qquad l_1 = hg(x_0, y_0, z_0)$$

$$k_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}, z_0+\frac{l_1}{2}\right) \qquad l_2 = hg\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}, z_0+\frac{l_1}{2}\right)$$

$$k_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, z_0+\frac{l_2}{2}\right) \qquad l_3 = hg\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, z_0+\frac{l_2}{2}\right)$$

$$k_4 = hf(x_0+h, y_0+k_3, z_0+l_3) \qquad l_4 = hg(x_0+h, y_0+k_3, z_0+l_3)$$

$$\Delta y = \frac{1}{6}(k_1+2k_2+2k_3+k_4) \qquad \Delta z = \frac{1}{6}(l_1+2l_2+2l_3+l_4)$$

$$y_1 = y_0 + \Delta y \qquad z_1 = z_0 + \Delta z$$

Example 7.21

Use RK to find solution of the system at $x=(0.1)$

$$\frac{dy_1}{dx} = y_1 - y_2 \quad , \quad \frac{dy_2}{dx} = -y_1 + y_2 \quad , \quad y_1(0)=0 \; , \; y_2(0)=1$$

Solution:

$f(x_1, y_1, y_2) = y_1 - y_2$

$g(x_1, y_1, y_2) = -y_1 + y_2$

$x_{10} = 0 \quad , \quad y_{10} = 0 \quad , \quad y_{20} = 1$

$k_1 = hf(x_{10}, y_{10}, y_{20})$

$= 0.1(0-1) = -0.1$

$l_1 = hg(x_{10}, y_{10}, y_{20})$

$= 0.1(-0+1) = 0.1$

$k_2 = hf\left(x_{10}+\frac{h}{2}, y_{10}+\frac{k_1}{2}, y_{20}+\frac{l_1}{2}\right)$

$= 0.1 f(0.05, -0.05, 1.05)$

$= 0.1(-0.05-1.05) = -0.11$

$l_2 = hg\left(x_{10}+\frac{h}{2}, y_{10}+\frac{k_1}{2}, y_{20}+\frac{l_1}{2}\right)$

$= 0.1(0.05, -0.05, 1.05)$

$= 0.1(+0.05+1.05) = 0.11$

$k_3 = hf\left(x_{10}+\frac{h}{2}, y_{10}+\frac{k_2}{2}, y_{20}+\frac{l_2}{2}\right)$

$= 0.1(0.05, -0.055, 1.055) = -0.111$

$l_3 = hg\left(x_{10}+\frac{h}{2}, y_{10}+\frac{k_1}{2}, y_{20}+\frac{l_1}{2}\right)$

$= 0.1(0.05, -0.055, 1.055) = 0.111$

$k_4 = hf(x_{10}+h, y_{10}+k_3, y_{20}+l_3)$

$= 0.1 f(0.1, -0.111, 1.111)$

$= -0.1222$

$l_4 = hg(x_{10}+h, y_{10}+k_3, y_{20}+l_3)$

$= 0.1 g(0.1, -0.111, 1.111)$

$= 0.1222$

$y_1(0.1) = y_{10} + \Delta y_1 = 0 + \Delta y_1$

$\Delta y_1 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= \frac{1}{6}(-0.1 + 2(-0.11) + 2(-0.111) + (-0.1222))$

$= -0.1107$

$\boxed{y_1(0.1) = -0.1107}$

now,

$y_2(0.1) = y_{20} + \Delta y_2$

$\Delta y_2 = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

$= \frac{1}{6}(0.1 + 2(0.11) + 2(0.111) + 0.1222)$

$\Delta y = 0.1107$

$y_2(0.1) = 1 + 0.1107$

$\boxed{y_2(0.1) = 1.1107}$

RK for Second Order Differential Eq

$$\frac{d^2y}{dx^2} = g\left(x, y, \frac{dy}{dx}\right) \quad \text{— (i)}$$

let $\quad \frac{dy}{dx} = z \quad \Rightarrow \quad y' = z$

then $\quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dz}{dx} = z'$

eq (i) becomes

$$\left.\begin{aligned} \frac{dz}{dx} &= g(x, y, z) \\ \frac{dy}{dx} &= f(x, y, z) \end{aligned}\right]$$

we will solve these equations simultaneously

Example 7.24:

Find $y(0.1)$ from $\frac{d^2y}{dx^2} - y^3 = 0$, $y(0) = 10$, $y'(0) = 50$ by RK

Solution:

$\frac{d^2y}{dx^2} = y^3 \qquad x_0 = 0, y_0 = 10, z_0 = 50$

let

$\frac{dy}{dx} = z \quad \leftarrow f(x,y,z)$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dz}{dx} = y^3 \leftarrow g(x,y,z)$

$k_1 = hf(x_0, y_0, z_0) = 0.1(50) = 5$

$l_1 = hg(x_0, y_0, z_0) = 0.1(10^3) = 100$

$k_2 = hf(x_0 + h/2, y_0 + k_1/2, z_0 + l_1/2)$

$= 0.1f(0.05, 12.5, 100) = 10$

$l_2 = hg(x_0 + h/2, y_0 + k_1/2, z_0 + l_1/2)$

$= 0.1g(0.05, 3.5, 100) = 195.313$

$k_3 = hf(x_0 + h/2, y_0 + k_2/2, z_0 + l_2/2)$

$= 0.1f(0.05, 15, 147.65625) = 14.7656$

$l_3 = hg(x_0 + h/2, y_0 + k_2/2, z_0 + l_2/2)$

$= 0.1g(0.05, 15, 147.65625) = 337.5$

$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$

$= 0.1f(0.1, 24.7656, 387.5)$

$= 38.75$

$y_2 = y_0 + \Delta y_1$

$\Delta y_1 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= \frac{1}{6}(5 + 2(10) + 2(14.7656) + (38.75))$

$= 15.5468$

$y_2 = 10 + 15.5468$

$= 25.5468$

* as we need value at just 1 node, y ki value find kina is enough, but if we need values at more than 1 nodes e.g (0.2), (0.4) etc we'll have to find values of both y and z.

For third order derivative:

example:

$$\frac{d^3y}{dx^3} + xy\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + xe^x = 0 \quad \text{— (i)}$$

$$\frac{dy}{dx} = y_1$$

$$\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = \frac{dy_1}{dx} = y_2$$

$$\Rightarrow \frac{d^2}{dx^2}\left(\frac{dy}{dx}\right) = \frac{d^3y}{dx^3} = \frac{d^2y_1}{dx} = \frac{dy_2}{dx}$$

eq (i) becomes

$$\frac{dy^2}{dx} = xyy_2 - y_1^3 - xe^x$$

$$k_1 = hf(x, y, y_1, y_2)$$
$\vdots$
k_4

$$l_1 = hg(x, y, y_1, y_2)$$
$\vdots$
l_4

$$m_1 = hk(x, y, y_1, y_2)$$
$\vdots$
m_4

Predictor Corrector method:

step 1: predicts the value

step 2: corrects that predicted value

1. Milne's Method:

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\left(2y'_{n-2} - y'_{n-1} + 2y'_n\right) \leftarrow \text{Predictor formula}$$

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left(y'_{n-1} + 4y'_n + y'_{n+1}\right) \leftarrow \text{Corrector formula}$$

* first 4 values of y will be given, in-case not given we can find values of y by Taylor or R-K method.

day / date:

Example 7.25:

$$y' = \frac{1}{x+y} \quad y(0)=2, \quad y(0.2)=2.0933, \quad y(0.4)=2.1755, \quad y(0.6)=2.2493$$

Find $y(0.8)$ by Milne's predictor-corrector method

Solution:

By Milne's predictor formula

$$y_{n+1,P} = y_{n-3} + \frac{4h}{3}(2y'_{n-2} - y'_{n-1} + 2y'_n)$$

taking $n=3$

$$y_{4,P} = y_0 + \frac{4h}{3}(2y'_1 - y'_2 + 2y'_3) \text{ — (i)}$$

$$y' = \frac{1}{x+y}$$

$$y'_1 = \frac{1}{x_1+y_1} = \frac{1}{0.2+2.0933} = 0.43605$$

$$y'_2 = \frac{1}{x_2+y_2} = \frac{1}{0.4+2.1755} = 0.38827$$

$$y'_3 = \frac{1}{x_3+y_3} = \frac{1}{0.6+2.2493} = 0.35096$$

eq (i) becomes

$$y_{4,P} = 2 + \frac{4(0.2)}{3}(2(0.43605) - (0.38827) + 2(0.35096))$$

$$= 2.3161$$

By Milne's corrector formula

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}(y'_{n-1} + 4y'_n + y'_{n+1})$$

taking $n=3$

$$y_{4,c} = y_2 + \frac{h}{3}(y'_2 + 4y'_3 + y'_4) \text{ — (ii)}$$

$$y'_4 = \frac{1}{x_4+y_4} = \frac{1}{0.8+2.3161} = 0.320914$$

eq (ii) becomes

$$y_{4,c} = 2.1755 + \frac{0.2}{3}(0.38827 + 4(0.35096) + 0.320914)$$

$$y(0.8) = 2.3162$$

Adam - Bashforth Method:

$$y_{n+1,P} = y_n + \frac{h}{24}[55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}] \leftarrow \text{Predictor formula}$$

$$y_{n+1,c} = y_n + \frac{h}{24}[9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}] \leftarrow \text{Corrector formula}$$

BVP

day / date:

Boundary Value Problem

pain

By Finite Difference Method

By taylor series:

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \ldots \quad (i)$$

and

$$y(x-h) = y(x) - hy'(x) + \frac{h^2}{2}y''(x) \ldots \quad (ii)$$

subtracting eq (ii) from eq (i)

$$\frac{y(x+h) - y(x-h)}{2h} = y'(x) + \underbrace{O(h^2)}$$ neglecting this cos it's smol

$$\Rightarrow \boxed{y_i' = \frac{y_{i+1} - y_{i-1}}{2h}}$$

Adding eq (i) and eq (ii)

$$y(x+h) + y(x-h) = 2y(x) + h^2 y''(x) + \ldots$$

$$y''(x) = \frac{y(x+h) + y(x-h) - 2y(x)}{h^2} + O(h^2)$$

$$\Rightarrow \boxed{y_i'' = \frac{y_{i-1} + y_{i+1} - 2y_i}{h^2}}$$

day / date:

Q. using finite differences, solve

$xy'' + y = 0$, $y(\underset{a}{1}) = 1$, $y(\underset{b}{2}) = 2$, where $h = \frac{1}{4}$

if h ki jagah n ki value given ho $\Rightarrow h = \frac{b-a}{n}$

Solution:

converting given differential eq into difference eq

* eq mai y' hota tau we wud have replaced that too

$$x_i\left[\frac{y_{i+1} + y_{i-1} - 2y_i}{h^2}\right] + y_i = 0$$

$$x_i y_{i+1} + x_i y_{i-1} - 2x_i y_i + h^2 y_i = 0$$

where $i = 1, 2, 3$ since $h = 0.25$

putting $h = 1/4$

$$x_i y_{i+1} + x_i y_{i-1} - 2x_i y_i + \frac{1}{16} y_i = 0$$

Putting $i = 1, 2$ and 3 we get eqs

$$x_1 y_2 + x_1 y_0 - 2x_1 y_1 + \frac{1}{16} y_1 = 0$$

$$x_2 y_3 + x_2 y_1 - 2x_2 y_2 + \frac{1}{16} y_2 = 0$$

$$x_3 y_4 + x_3 y_2 - 2x_3 y_3 + \frac{1}{16} y_3 = 0$$

Putting $y_0 = 1$ $\because y(1) = 1 = y_0$

$y_4 = 2$ $\because y(2) = 2 = y_4$

$x_1 = x_0 + h = 1.25$, $x_2 = 1.5$, $x_3 = 1.75$

we get eqs

$$1.25y_2 + 1.25 - \frac{39}{16} y_1 = 0 \quad \text{—(i)}$$

$$1.5y_3 + 1.5y_1 - \frac{47}{16} y_2 = 0 \quad \text{—(ii)}$$

$$3.5 + 1.75y_2 - \frac{55}{16} y_3 = 0 \quad \text{—(iii)}$$

eq (i) becomes.

$$\frac{39}{16} y_1 = 1.25 + 1.25y_2$$

$$y_1 = \frac{20}{39} + \frac{20}{39} y_2$$

and eq (iii) becomes

$$\frac{55}{16} y_3 = 3.5 + 1.75y_2$$

$$y_3 = \frac{56}{55} + \frac{28}{55} y_2$$

putting values of y_1 and y_3 in eq (ii)

$$1.5\left(\frac{56}{55} + \frac{28}{55} y_2\right) + 1.5\left(\frac{20}{39} + \frac{20}{39} y_2\right) - \frac{47}{16} y_2 = 0$$

$$\frac{84}{55} + \frac{42}{55} y_2 + \frac{10}{13} + \frac{10}{13} y_2 - \frac{47}{16} y_2 = 0$$

$$-1.4y_2 = -\frac{1642}{715}$$

$$\boxed{y_2 = 1.64}$$

putting this value in eq (i) and eq (iii)

$$1.25(1.64) + 1.25 = 39/16\, y_1$$

$$\boxed{y_1 = 1.35}$$

$$3.5 + 1.75(1.64) = 55/16\, y_3$$

$$\boxed{y_3 = 1.85}$$

x	1	1.25	1.5	1.75	2
y	1	1.35	1.64	1.85	2