CHAPTER 11

Numerical Solution to Ordinary Differential Equations

11.1 INTRODUCTION

Many problems in Engineering and Science can be formulated into ordinary differential equations satisfying certain given conditions. If these conditions are prescribed for one point only, then the differential equation together with the conditions is known as an *initial value problem*. If the conditions are prescribed for two or more points then the problem is termed as *boundary value problem*. Analytically, the solution to an ordinary differential equation in which x is the independent variable and y is the dependent variable means finding an explicit expression for y in terms of a finite number of elementary functions of x. Such a solution is known as the *closed* or *finite* form of solution. But the analytical methods are applicable only to a select class of differential equations. Since the differential equations appearing in the fields of Engineering and Science are due to physical and natural phenomena, they do not belong to any of the above class and hence, cannot have a closed solution. In such cases, we try to approximate a particular solution to the differential equation, i.e. we find numerical values of $y_1, y_2, y_3, \ldots$ corresponding to given numerical values of independent variable values $x_1, x_2, x_3, \ldots$, so that the ordered pairs $(x_1,y_1), (x_2,y_2), (x_3,y_3), \ldots$ satisfy approximately a pre-assigned particular solution. A solution of this type is called *pointwise solution*.

Let us consider the first order differential equation $dy/dx = f(x,y)$ given $y(x_0) = y_0$. Let $y = f(x)$ be the exact solution (smooth curve in Fig.11.1) and $y_1, y_2, y_3, \ldots$ be the pointwise solutions at $x = x_1, x_2, x_3, \ldots$, using a suitable recursive formula (dotted curve in the figure). Computation of these approximate values is known as *Numerical solution* to the differential equation. The difference between the computed value y_i and the true value $f(x_i)$, say, ε_i, is termed as *truncation error* at $x = x_i$. There are many numerical methods for the approximate solution to ordinary differential equations given an initial condition. The most important and frequently used procedures are discussed in this chapter.

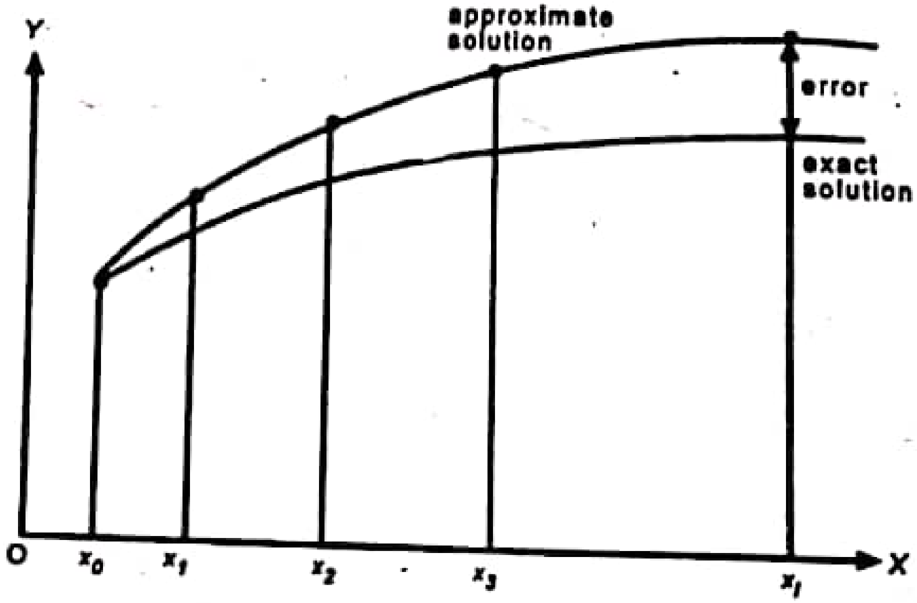

Fig. 11.1

11.2 POWER SERIES SOLUTION

Consider the differential equation

$$\frac{dy}{dx} = y' = f(x, y) \tag{11.1}$$

subject to the condition

$$y(x_0) = y_0 \tag{11.2}$$

This is an initial value problem. Now we can expand the solution to Eqn (11.1), i.e., $y(x)$ in the neighbourhood of $x = x_0$ in power series as

$$y(x) = A_0 + A_1(x - x_0) + A_2(x - x_0)^2 + \cdots \tag{11.3}$$

where $A_0, A_1, A_2, \ldots$ are constants to be determined such that Eqn (11.3) satisfies Eqn (11.1) subject to Eqn (11.2).

To expand $y(x)$ in power series, generally, *Taylor's theorem* or *Maclaurin's theorem* is used. If the boundary condition is at $x_0 (\neq 0)$, then we use *Taylor's series* to expand $y(x)$ about $x = x_0$ and it is

11.4 SOLUTION BY TAYLOR'S SERIES

Let
$$y' = f(x, y) \;;\; y(x_0) = y_0 \tag{11.6}$$
be the differential equation to which the numerical solution is required. Expanding $y(x)$ about $x = x_0$ by Taylor's series,

$$y(x) = y(x_0) + \frac{(x - x_0)}{1!}\, y'(x_0) + \frac{(x - x_0)^2}{2!}\, y''(x_0) + \cdots$$

$$= y_0 + \frac{(x - x_0)}{1!}\, y_0' + \frac{(x - x_0)^2}{2!}\, y_0'' + \cdots$$

Putting $x = x_1 = x_0 + h$, we have

$$y_1 = y(x_1) = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots \tag{11.7}$$

Here, y_0', y_0'', y_0''', ... can be found using Eqn (11.6) and its successive differentiations at $x = x_0$. The series in Eqn (11.7) can be truncated at any stage if h is small. Now, having obtained y_1, we can calculate y_1', y_1'', y_1''' ... from Eqn (11.6) at $x_1 = x_0 + h$.

Now expanding $y(x)$ by Taylor's series about $x = x_1$, we get

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \cdots$$

Proceeding on, we get

$$y_n = y_{n-1} + \frac{h}{1!} y_{n-1}' + \frac{h^2}{2!} y_{n-1}'' + \frac{h^3}{3!} y_{n-1}''' + \cdots \tag{11.8}$$

$$\text{where } y_{n-1}^{r} = \left[\frac{d^r}{dx^r}(y_{n-1})\right]_{(x_{n-1},\, y_{n-1})}$$

By taking sufficient number of terms in the above series, the value of y_n can be obtained without much error.

If we retain the terms upto h^n on the RHS of Eqn (11.8), the error will be proportional to the $(n+1)$th power of step size, i.e. h^{n+1} and Taylor's algorithm is said to be of nth order. By including more number of terms on the RHS of Eqn (11.8), the error can be reduced further.

Example 11.2 Using Taylor's series method, solve $\frac{dy}{dx} = x^2 - y,\; y(0) = 1$ at $x = 0.1, 0.2, 0.3$ and 0.4. Compare the values with the exact solution.

Solution Given : $y' = x^2 - y \;;\; y(0) = 1$

$\therefore\ x_0 = 0,\ y_0 = 1,\ h = 0.1,\ x_1 = 0.1, x_2 = 0.2,\ x_3 = 0.3$ and $x_4 = 0.4$

Now, $y' = x^2 - y \qquad \therefore\ y_0' = x_0^2 - y_0 = -1$

$y'' = 2x - y' \qquad \therefore\ y_0'' = 2x_0 - y_0' = 1$

$y''' = 2 - y'' \qquad \therefore\ y_0''' = 2 - y_0'' = 1$

and $y^{iv} = -y''' \qquad \therefore\ y_0^{iv} = -y_0''' = -1$

By Taylor's series,

$$y_1 = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv} + \cdots$$

$$\therefore \quad y_1 = y(0.1)$$

$$= 1 + (0.1)(-1) + \frac{(0.1)^2}{2!}(1) + \frac{(0.1)^3}{3!}(1) + \frac{(0.1)^4}{4!}(-1) + \ldots$$

$$= 1 - 0.1 + 0.005 + 0.0001667 - 0.0000417 + \ldots$$

$$= 0.905125$$

Now, $y_1' = x_1^2 - y_1 = (0.1)^2 - 0.905125 = -0.895125$

$y_1'' = 2x_1 - y_1' = 2(0.1) - (-0.895125) = 1.095125$

$y_1''' = 2 - y_1'' = 2 - 1.095125 = 0.904875$

$y_1^{iv} = -y_1''' = -0.904875$

and $y_2 = y(0.2) = y_1 + hy_1' + \dfrac{h^2}{2!} y_1'' + \dfrac{h^3}{3!} y_1''' + \dfrac{h^4}{4!} y_1^{iv} + \ldots$

$$\therefore\ y_2 = 0.905125 + (0.1)(-0.895125) + \frac{(0.1)^2}{2!}(1.095125)$$

$$+ \frac{(0.1)^3}{3!}(0.904875) + \frac{(0.1)^4}{4!}(-0.904875) + \cdots$$

$$= 0.8212352$$

Now, $y_2' = x_2^2 - y_2 = (0.2)^2 - 0.8212352 = -0.7812352$

$y_2'' = 2x_2 - y_2' = 2(0.2) - (-0.7812352) = 1.1812352$

$y_2''' = 2 - y_2'' = 2 - 1.1812352 = 0.8187648$

$y_2^{iv} = -y_2''' = -0.8187648$

and $y_3 = y(0.3) = y_2 + hy_2' + \dfrac{h^2}{2!} y_2'' + \dfrac{h^3}{3!} y_2''' + \dfrac{h^4}{4!} y_2^{iv} + \ldots$

$$\therefore\ y_3 = 0.8212352 + (0.1)(-0.7812352) + \frac{(0.1)^2}{2!}(1.1812352)$$

$$+ \frac{(0.1)^3}{3!}(0.8187648) + \frac{(0.1)^4}{4!}(-0.8187648) + \cdots$$

$$= 0.7491509$$

$$\text{Now, } \begin{aligned} y_3' &= x_3^2 - y_3 &&= (0.3)^2 - 0.7491509 &&= -0.6591509 \\ y_3'' &= 2x_3 - y_3' &&= 0.6 - (-0.6591509) = 1.2591509 \\ y_3''' &= 2 - y_3'' &&= 0.740849 \\ y_3^{iv} &= -y_3''' &&= -0.740849 \end{aligned}$$

$$\text{and } y_4 = y(0.4) = y_3 + \frac{h}{1!} y_3' + \frac{h^2}{2!} y_3'' + \frac{h^3}{3!} y_3''' + \frac{y^4}{4!} y_3^{iv} + \dots$$

$$\therefore\ y_4 = 0.7491509 + (0.1)(-0.6591509) + \frac{(0.1)^2}{2!}(1.2591509)$$

$$+ \frac{(0.1)^3}{3!}(0.740849) + \frac{(0.1)^4}{4!}(-0.740849) + \cdots$$

$$= 0.6896519$$

Exact solution : Given : $\dfrac{dy}{dx} = x^2 - y$

or $\dfrac{dy}{dx} + y = x^2$ (a linear equation)

$\therefore$ Solution is $y\, e^{\int dx} = \int (\int dx) x^2\, dx + c$

or $\qquad y\, e^x = \int e^x x^2\, dx + c$

or $\qquad y\, e^x = (x^2 - 2x + 2)e^x + c$

or $\qquad y = [(x-1)^2 + 1] + ce^{-x}$ $\qquad$ (i)

Given that $y(0) = 1 \Rightarrow 1 = 2 + c$ or $c = -1$

Hence, the exact solution is

$y = [(x-1)^2 + 1] - e^{-x}$

$\therefore y_1 = y\,(0.1) = (0.1 - 1)^2 + 1 - e^{-0.1} = 0.9051625$

$y_2 = y\,(0.2) = 0.8212692$, $y_3 = 0.7491817$ and $y_4 = 0.6896799$.

Now we can see that the values coincides upto four decimals and the error is very less.

11.5 TAYLOR'S SERIES METHOD FOR SIMULTANEOUS FIRST ORDER DIFFERENTIAL EQUATIONS

The simultaneous differential equations of the type

$$\frac{dy}{dx} = f(x, y, z) \tag{11.9}$$

and

$$\frac{dz}{dx} = g(x, y, z) \tag{11.10}$$

with initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$ can be solved by Taylor's series method as illustrated below:

If h be the step size, then $y_1 = y(x_0 + h)$, and $z_1 = z(x_0 + h)$.

Now, Taylor's algorithm for Eqns (11.9) and (11.10) gives

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots \tag{11.11}$$

and
$$z_1 = z_0 + hz_0' + \frac{h^2}{2!}z_0'' + \frac{h^3}{3!}z_0''' + \cdots \tag{11.12}$$

Differentiating Eqns (11.9) and (11.10) successively, we get y'', y''', ..., and z_0'', z_0''', ... etc. So the values y_0'', y_0''', ... and z_0'', z_0''', ... are known. Substituting these in Eqns. (11.11) and (11.12), we obtain y_1, z_1 for the next step.

Similarly, we have

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \cdots \tag{11.13}$$

and
$$z_2 = z_1 + hz_1' + \frac{h^2}{2!}z_1'' + \frac{h^3}{3!}z_1''' + \cdots \tag{11.14}$$

Since y_1, z_1 are known, y_1', y_1'', y_1''' ... and z_1', z_1'', $z_1''' + \cdots$ can be calculated. Substituting in Eqns (11.13) and (11.14), we get y_2 and z_2.

Proceeding in the same manner, we get other values of y, step by step.

Example 11.3 Solve $\frac{dy}{dx} = x + z$, $\frac{dz}{dx} = x - y^2$ with $y(0) = 2$, $z(0) = 1$ to get $y(0.1)$, $y(0.2)$, $z(0.1)$ and $z(0.2)$, approximately, by Taylor's algorithm.

Solution: Given that $y' = x + z$; $z' = x - y^2$ with the initial conditions $y(0) = 2$ and $z(0) = 1$.

Now,
$$y' = x + z \qquad z' = x - y^2$$
$$y'' = 1 + z' \qquad z'' = 1 - 2yy'$$
$$y''' = z'' \text{ etc.,} \qquad z''' = -2[yy'' + y'^2] \text{ etc.}$$

Now by Taylor's series for y_1 and z_1, we have

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots \tag{i}$$

$$z_1 = z_0 + hz_0' + \frac{h^2}{2!}z_0'' + \frac{h^3}{3!}z_0''' + \cdots \tag{ii}$$

Here, $y_0 = 2$, $x_0 = 0$, $z_0 = 1$, $h = 0.1$ and

$y_0' = x_0 + z_0 = 1$ $\qquad z_0' = x_0 - y_0^2 = -4$

$y_0'' = 1 + z_0' = 1 - 4 = -3$ $\qquad z_0'' = 1 - 2y_0y_0' = 1 - 2(2)(1) = -3$

$y_0''' = z_0'' = -3$ $\qquad z_0''' = -2[y_0y_0'' + y_0'^2]$

$= -2[2(-3) + (1)^2] = 10$

Subsituting these values in (i) and (ii), we get

$$y_1 = 2 + (0.1)(1) + \frac{(0.1)^2}{2!}(-3) + \frac{(0.1)^3}{3!}(-3) + \cdots$$

$$= 2 + 0.1 - 0.015 - 0.0005 = 2.0845 \text{ (approx.)}$$

$$z_1 = 1 + (0.1)(-4) + \frac{(0.1)^2}{2!}(-3) + \frac{(0.1)^3}{3!}(10) + \cdots$$

$$= 1 - 0.4 - 0.015 + 0.001667 = 0.5867 \text{ (approx.)}$$

$$\therefore y(0.1) = 2.0845 \text{ and } z(0.1) = 0.5867$$

To find out $y(0.2)$ and $z(0.2)$, Taylor's algorithm is

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \cdots \qquad \text{(iii)}$$

$$z_2 = z_1 + hz_1' + \frac{h^2}{2!}z_1'' + \frac{h^3}{3!}z_1''' + \cdots \qquad \text{(iv)}$$

Here, $x_1 = 0.1$, $y_1 = 2.0845$ and $z_1 = 0.5867$ and

$y_1' = x_1 + z_1 = 0.6867$ $\qquad z_1' = x_1 - y_1^2 = -4.2451403$

$y_1'' = 1 + z_1' = -3.2451403$ $\qquad z_1'' = 1 - 2y_1y_1' = -1.8628523$

$y_1''' = z_1'' = -1.8628523$ $\qquad z_1''' = -2[y_1y_1'' + y_1'^2]$

$= 12.585876$

Substuting these values in (iii) and (iv), we get

$$y_2 = 2.0845 + (0.1)(0.6867) + \frac{1}{2!}(0.1)^2(-3.2451403)$$

$$+ \frac{1}{3!}(0.1)^3(-1.8628523) + \cdots$$

$$= 2.1366338 \text{ (approx.)}$$

$$z_2 = 0.5867 + (0.1)(-4.2451403) + \frac{(0.1)^2}{2!}(-1.8628523)$$

$$+ \frac{(0.1)^3}{3!}(12.585876) + \cdots$$

$$= 0.1549693 \text{ (approx.)}$$

$$\therefore y(0.2) = 2.1366338 \text{ and } z(0.2) = 0.1549693$$

11.6 TAYLOR SERIES METHOD FOR HIGHER ORDER DIFFERENTIAL EQUATIONS

Taylor series method can be used to solve numerically second and higher order differential equations. The procedure for solving second order differential equation is illustrated below.

Let $$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$$

i.e. $$y'' = f(x, y, y') \tag{11.15}$$

be the differential equation to be solved numerically subject to the initial conditions

$$y(x_0) = y_0 \tag{11.16}$$

and $$y'(x_0) = y_0' \tag{11.17}$$

where y_0 and y_0' are given constants. Now put

$$y' = p \tag{11.18}$$

so that $y'' = p'$ and hence Eqn (11.15) becomes

$$p' = f(x, y, p) \tag{11.19}$$

The initial conditions, i.e. Eqns (11.16) and (11.17) becomes

$$y(x_0) = y_0 \tag{11.20}$$

and $$p(x_0) = p_0 \tag{11.21}$$

where $y_0' = p_0$.

Hence, we have to solve the two first order differential equations (11.18) and (11.19) subject to the conditions (11.20) and (11.21). The Taylor algorithm for Eqn (11.19) is

$$p_1 = p_0 + hp_0' + \frac{h^2}{2!}p_0'' + \frac{h^3}{3!}p_0''' + \cdots \tag{11.22}$$

Here, h is the step-size and p_1 is the approximate value of p at $x = x_1 = x_0 + h$.

The Taylor algorithm for Eqn (11.18) is

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots$$

$$= y_0 + hp_0 + \frac{h^2}{2!}p_0' + \frac{h^3}{3!}p_0'' + \cdots \tag{11.23}$$

Now equation (11.19) gives p'. Differentiating it successively, we get p'', p''', . . . etc. and hence, the values of p_0', p_0'', p_0''', . . . can be calculated. Substituting them in Eqns (11.22) and (11.23), we get p_1, and y_1. Similarly, for the next interval, we have the algorithms

$$p_2 = p_1 + h\,p_1' + \frac{h^2}{2!}p_1'' + \frac{h^3}{3!}p_1''' + \cdots \qquad (11.24)$$

and
$$y_2 = y_1 + h\,y_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' +$$

$$= y_1 + h\,p_1 + \frac{h^2}{2!}p_1' + \frac{h^3}{3!}p_1'' + \cdots \qquad (11.25)$$

Knowing p_1 and y_1, we can calculate p_1', p_1'', ... at (x_1, y_1) and hence, p_2 and y_2 from Eqns (11.24) and (11.25).

Proceeding on the same lines, we can calculate the other values of y step by step.

The above procedure can be extended to solve higher order differential equations numerically, when sufficient initial conditions are given. But in this chapter, we shall restrict to second order only.

Example 11.4 Evaluate by means of Taylor's series expansion, the following problem at $x = 0.1, 0.2$ to four significant figures.

$y'' - x(y')^2 + y^2 = 0$; $y(0) = 1$, $y'(0) = 0$

Solution Putting $y' = p$, the given equation reduces to

$y' = p$

$$p' - xp^2 + y^2 = 0$$

$p' = y''$

$$\text{i.e. } p' = xp^2 - y^2 \qquad \text{(i)}$$

The initial conditions are

$p(0) = 0$

$$y_0 = y(0) = 1\ ;\ p_0 = y_0' = 0 \qquad \text{(ii)}$$

Now to solve Eqn (i), given $p_0 = p(0) = 0$, we have

$$p_1 = p_0 + h\,p_0' + \frac{h^2}{2!}p_0'' + \frac{h^3}{3!}p_0''' + \cdots \qquad \text{(iii)}$$

From (i), we have

$$\begin{aligned} p' &= xp^2 - y^2 & y' &= p' \\ p'' &= p^2 + 2xpp' - 2yy' & y'' &= p'' \\ p''' &= 2pp' + 2[xpp'' + x(p')^2 + pp'] & y^{iv} &= p''' \\ &\quad - 2[yy'' + (y')^2] \end{aligned}$$

$$\begin{aligned} \therefore\quad p_0' &= x_0p_0^2 - y_0^2 = 0(0)^2 - (1)^2 = -1 \\ p_0'' &= p_0^2 + 2x_0p_0p_0' - 2y_0y_0' \\ &= (0)^2 + 2(0)(0)(-1) - 2(1)(0) = 0 \\ \text{and}\quad p_0''' &= 2. \end{aligned}$$

Substituting these in (iii), we get

$$p_1 = 0 + (0.1)(-1) + \frac{(0.1)^2}{2!}(0) + \frac{(0.1)^3}{3!}(2) + \cdots$$

$$= -0.0997$$

By Taylor's series,

$$y_1 = y(0.1) = y_0 + h\,y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0'' + \cdots$$

$$= 1 + (0.1)p_0 + \frac{(0.1)^2}{2!}(p_0') + \frac{(0.1)^3}{3!}(p_0'') + \frac{(0.1)^4}{4!}(p_0''') + \cdots$$

$$= 1 + (0.1)(0) + \frac{0.01}{2}(-1) + \frac{0.001}{6}(0) + \frac{0.0001}{24}(2) + \cdots$$

$$= 0.9950083 \cong 0.995$$

$$\text{Now } y_2 = y(0.2) = y_1 + h\,y_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \cdots$$

$$= y_1 + h\,p_1 + \frac{h^2}{2!}p_1' + \frac{h^3}{3!}p_1'' + \cdots \qquad \text{(iv)}$$

Here, $y_1 = 0.995, p_1 = -0.0997,$

$p_1' = x_1 p_1^2 - y_1^2 = (0.1)(-0.0997)^2 - (0.995)^2$

$= -0.9890309$

$p_1'' = p_1^2 + 2x_1 p_1 p_1' - 2y_1\,y_1'$

$= -0.1687416.$

Substituting in Eqn (iv), we get

$$y_2 = 0.995 + \frac{(0.1)}{1!}(-0.0997) + \frac{(0.1)^2}{2!}(-0.9890309)$$

$$+ \frac{(0.1)^3}{3!}(-0.1687416) + \cdots$$

$$= 0.9801129 \cong 0.9801$$

$$\therefore \quad y(0.1) = 0.9950 \text{ and } y(0.2) = 0.9801$$

11.7 PICARD'S METHOD OF SUCCESSIVE APPROXIMATIONS

Consider the first order differential equation

$$\frac{dy}{dx} = f(x, y) \tag{11.26}$$

subject to $y(x_0) = y_0$. This equation can be written as

$$dy = f(x, y)\, dx$$

Integrating between the limits, we get

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y) dx$$

or

$$y - y_0 = \int_{x_0}^{x} f(x, y) dx$$

or

$$y = y_0 + \int_{x_0}^{x} f(x, y) dx \tag{11.27}$$

which is an integral equation and can be solved by successive approximation or iteration. Now by Picard's method, for first approximation y_1, we replace y by y_0 in $f(x, y)$ in the RHS of Eqn (11.27),

i.e.

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0) dx \tag{11.28}$$

For second approximation y_2, we replace y by y_1 in $f(x, y)$ on the RHS of Eqn (11.27),

i.e.

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1) dx \tag{11.29}$$

... ...

For nth approximation,

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1}) dx$$

The process is to be stopped when the two values of y, viz. y_n and y_{n-1}, are same to the desired degree of accuracy.

Note : This method is applicable only to a limited class of equations in which the successive integrations can be performed easily.

Example 11.5 Use Picard's method to approximate the value of y when $x = 0.1, 0.2, 0.3, 0.4$ and [illegible] that $y = 1$ at $x = 0$ and $y = 1 + xy$, correct to three decimal places.

Solution Given: $\frac{dy}{dx} = 1 + xy$ ... (1)

$$\therefore \quad f(x, y) = 1 + xy, \; y_0 = 1, x_0 = 0$$

First approximation :

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx = 1 + \int_0^x (1 + x)\,dx$$

$$= 1 + \int_0^x (1+x)\,dx = 1 + x + \frac{x^2}{2} \qquad \text{(i)}$$

Second approximation :

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx = 1 + \int_0^x (1 + xy_1)\,dx$$

$$= 1 + \int_0^x \left\{1 + x\left(1 + x + \frac{x^2}{2}\right)\right\} dx$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} \qquad \text{(ii)}$$

Third approximation :

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\,dx = 1 + \int_0^x (1 + xy_2)\,dx$$

$$= 1 + \int_0^x \left\{1 + x\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}\right)\right\} dx$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48} \qquad \text{(iii)}$$

Fourth approximation :

$$y_4 = y_0 + \int_{x_0}^{x} f(x, y_3)\,dx = 1 + \int_0^x (1 + xy_3)\,dx$$

$$= 1 + \int_0^x \left\{1 + x\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48}\right)\right\} dx$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48} + \frac{x^7}{105} + \frac{x^8}{384} \qquad \text{(iv)}$$

Now at $x = 0.1$,

$y_1 = 1.105$, $y_2 = 1.1053458$, $y_3 = 1.1053465$, $y_4 = y_3$

at $x = 0.2$,

$y_1 = 1.22$, $y_2 = 1.2228667$, $y_3 = 1.2228894$, $y_4 = 1.2228895$

at $x = 0.3$,

$y_1 = 1.345$, $y_2 = 1.35550125$, $y_3 = 1.3551897$, $y_4 = 1.355192$

at $x = 0.4$,

$y_1 = 1.48$, $y_2 = 1.5045333$, $y_3 = 1.5053013$, $y_4 = 1.5053186$

and at $x = 0.5$,

$y_1 = 1.625$, $y_2 = 1.6744792$, $y_3 = 1.6768881$, $y_4 = 1.6769727$

$\therefore$ correct to three decimal places,

$y(0.1) = 1.105$, $y(0.2) = 1.223$, $y(0.3) = 1.355$, $y(0.4) = 1.505$ and $y(0.5) = 1.677$.

Example 11.6 Use Picard's method to approximate the value of y when $x = 0.1$ given that $y = 1$ when $x = 0$ and $\frac{dy}{dx} = \frac{y-x}{y+x}$.

Solution Given : $\frac{dy}{dx} = f(x,y) = \frac{y-x}{y+x}$, $y_0 = 1, x_0 = 0$

First approximation :

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx = y_0 + \int_{x_0}^{x} \frac{y_0 - x}{y_0 + x}\,dx$$

$$= 1 + \int_0^x \frac{1-x}{1+x}\,dx = 1 + \int_0^x \left[\frac{2}{1+x} - 1\right] dx$$

$$= 1 + \left[2\log(1+x) - x\right]_0^x = 1 + 2\log(1+x) - x \qquad \text{(i)}$$

$$\therefore y_1 = 1 - x + 2\log(1+x)$$

Second approximation :

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx = y_0 + \int_{x_0}^{x} \left[\frac{y_1 - x}{y_1 + x}\right] dx$$

$$= 1 + \int_0^x \left\{ \frac{1 - x + 2\log(1+x) - x}{1 - x + 2\log(1+x) + x} \right\} dx$$

$$= 1 + 2\int_0^x \frac{x}{1 + 2\log(1+x)} dx$$

which is quite difficult to integrate.

$\therefore$ We use only first approximation. By putting $x = 0.1$ in (i), we get

$y_1 = 1 - 0.1 + 2\log(1 + 0.1) = 0.9828$

11.8 PICARD'S METHOD FOR SIMULTANEOUS FIRST ORDER DIFFERENTIAL EQUATIONS

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Let $\frac{dy}{dx} = f(x, y, z)$ and $\frac{dz}{dx} = \phi(x, y, z)$ be the simultaneous differential equations with initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$.

Picard's method gives

$$\left.\begin{aligned} y_1 &= y_0 + \int_{x_0}^{x} f(x, y_0, z_0)\,dx \\ z_1 &= z_0 + \int_{x_0}^{x} \phi(x, y_0, z_0)\,dx \end{aligned}\right\} \quad (11.30)$$

$$\left.\begin{aligned} y_2 &= y_0 + \int_{x_0}^{x} f(x, y_1, z_1)\,dx \\ z_2 &= z_0 + \int_{x_0}^{x} \phi(x, y_1, z_1)\,dx \end{aligned}\right\} \quad (11.31)$$

and so on as successive approximations.

Example 11.7 Approximate y and z at $x = 0.1$ using Picard's method for the solution to the equations $\frac{dy}{dx} = z$, $\frac{dz}{dx} = x^3(y + z)$, given that $y(0) = 1$ and $z(0) = \frac{1}{2}$.

Solution Here, $x_0 = 0, y_0 = 1, z_0 = \frac{1}{2}$

and $\dfrac{dy}{dx} = f(x, y, z) = z$

$$\frac{dz}{dx} = \phi(x, y, z) = x^3(y + z)$$

$$\therefore y = y_0 + \int_{x_0}^{x} f(x,y,z)\,dx \qquad \text{and } z = z_0 + \int_{x_0}^{x} \phi(x,y,z)\,dx$$

First approximation :

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0, z_0)\,dx = 1 + \int_0^x \left(\frac{1}{2}\right) dx = 1 + \frac{x}{2}$$

$$z_1 = z_0 + \int_{x_0}^{x} \phi(x, y_0, z_0)\,dx = \frac{1}{2} + \int_0^x x^3\left(1 + \frac{1}{2}\right) dx = \frac{1}{2} + \frac{3x^4}{8}$$

Second approximation :

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1, z_1)\,dx$$

$$= 1 + \int_0^x \left(\frac{1}{2} + \frac{3x^4}{8}\right) dx = 1 + \frac{x}{2} + \frac{3x^5}{40}$$

$$z_2 = z_0 + \int_{x_0}^{x} \phi(x, y_1, z_1)\,dx$$

$$= \frac{1}{2} + \int_0^x x^3\left(1 + \frac{x}{2} + \frac{1}{2} + \frac{3x^4}{8}\right) dx$$

$$= \frac{1}{2} + \frac{3x^4}{8} + \frac{x^5}{10} + \frac{3x^8}{64}$$

Third approximation :

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2, z_2)\,dx$$

$$= 1 + \int_0^x \left(\frac{1}{2} + \frac{3x^4}{8} + \frac{x^5}{10} + \frac{3x^8}{64}\right) dx$$

$$= 1 + \frac{x}{2} + \frac{3x^5}{40} + \frac{x^6}{60} + \frac{x^9}{192}$$

$$z_3 = z_0 + \int_{x_0}^{x} \phi(x, y_2, z_2)\,dx$$

$$= \frac{1}{2} + \int_0^x x^3\left(1 + \frac{x}{2} + \frac{3x^5}{40} + \frac{1}{2} + \frac{3x^4}{8} + \frac{x^5}{10} + \frac{3x^8}{64}\right)dx$$

$$= \frac{1}{2} + \frac{3x^4}{8} + \frac{x^5}{10} + \frac{3x^8}{64} + \frac{7x^9}{360} + \frac{x^{12}}{256}$$

and so on. When $x = 0.1$,

$y_1 = 1.05$; $y_2 = 1.500008$; $y_3 = 1.500008$;

$z_1 = 0.5000375$; $z_2 = 0.5000385$; $z_3 = 0.5000385$;

Hence, $y(0.1) = 1.05$ and $z(0.1) = 0.5$.

11.9 PICARD'S METHOD FOR SECOND ORDER DIFFERENTIAL EQUATIONS

Consider the second order differential equation

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$$

By putting $\frac{dy}{dx} = z$, it can be reduced to two first order simultaneous differential equations :

$$\frac{dy}{dx} = z \quad \text{and} \quad \frac{dz}{dx} = f(x, y, z).$$

These can be solved as explained above.

Example 11.8 Use Picard's method to approximate y when $x = 0.1$ given that $\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + y = 0$ and $y = 0.5$, $\frac{dy}{dx} = 0.1$ when $x = 0$.

Solution Let $\frac{dy}{dx} = z$ so that $\frac{d^2y}{dx^2} = \frac{dz}{dx}$.

Thus the given equation reduces to

$$\frac{dz}{dx}+2xz+y=0\,;\ y(0)=0.5\,,\ z(0)=0.1$$

Now the equations to be solved are

$$\frac{dy}{dx}=f(x,y,z)=z$$

and
$$\frac{dz}{dx}=\phi(x,y,z)=-(2xz+y)$$

with the conditions $y_0=0.5,\ z_0=0.1,$ at $x_0=0$

$$\therefore y=y_0+\int_{x_0}^{x} f(x,y,z)\,dx$$

$$=0.5+\int_0^x z\,dx$$

$$z=z_0+\int_{x_0}^{x}\phi(x,y,z)\,dx$$

$$=0.1-\int_0^x (2xz+y)\,dx$$

First approximation :

$$y_1=0.5+\int_0^x z_0\,dx=0.5+\int_0^x (0.1)\,dx$$
$$=0.5+(0.1)x$$

$$z_1=0.1-\int_0^x (2xz_0+y_0)\,dx=0.1-\int_0^x (0.2x+0.5)\,dx$$
$$=0.1-0.5x-(0.1)x^2$$

Second approximation :

$$y_2=0.5+\int_0^x z_1\,dx$$

$$=0.5+\int_0^x (0.1-0.5x-0.1x^2)\,dx$$

$$= 0.5 + 0.1\,x - \frac{0.5x^2}{2} - \frac{0.1x^3}{3}$$

$$z_2 = 0.1 - \int_0^x (2xz_1 + y_1)\,dx$$

$$= 0.1 - \int_0^x 2x(0.1 - 0.5x - 0.1x^2) + (0.5 + 0.1x)\,dx$$

$$= 0.1 - 0.5x - \frac{0.3x^2}{2} + \frac{x^3}{3} + \frac{0.2x^4}{4}$$

Third approximation :

$$y_3 = 0.5 + \int_0^x z_2\,dx = 0.5 + 0.1x - \frac{0.5x^2}{2} - \frac{0.1x^3}{2} + \frac{x^4}{12} + \frac{0.1x^5}{10}$$

$$z_3 = 0.1 - \int_0^x (2xz_2 + y_2)\,dx$$

$$= 0.1 - 0.5x + \frac{0.3x^2}{2} - \frac{2.5x^3}{6} + 0.2\,x4 + \frac{2x^5}{15} + \frac{0.1x^6}{6}$$

Now at $x = 0.1$,

$y_1 = 0.51,\ y_2 = 0.50746667,\ y_3 = 0.50745933$

$\therefore y(0.1) = 0.5075$ correct to four decimals.

EXERCISE 11.1

1. Using first four terms of the Maclaurin's series find y at $x = 0.1(0.1)(0.6)$ given that $2y' = (1 + x)\,y^2$, $y(0) = 1$. Compare the values with the exact solution.
2. Find the first six terms of the power series solution of $y' = \sin x + y^2$ which passes through the point (0, 1).
3. Given $y' = 3x + \dfrac{y}{2}$ and $y(0) = 1$, find by Taylor's series $y(0.1)$ and $y(0.2)$.
4. Using Taylor's series method solve $y' = xy + y^2$, $y(0) = 1$ at $x = 0.1$, 0.2, 0.3.
5. Solve by Taylor's series method of third order, the problem $y' = (x^3 + xy^2)\,e^{-x}$, $y(0) = 1$ to find y for $x = 0.1, 0.2, 0.3$.

6. Employ Taylor's method to obtain the approximate value of y at $x = 0.2$ for $y' = 2y + 3e^x$, $y(0) = 0$. Compare the numerical solution obtained with exact solution.

7. Solve $y' = y^2 + x$, $y(0) = 1$ using Taylor's series method to compute $y(0.1)$ and $y(0.2)$.

8. Solve $\frac{dy}{dx} = z - x$, $\frac{dz}{dx} = y + x$ with $y(0) = 1$, $z(0) = 1$ to get $y(0.1)$ and $z(0.1)$, using Taylor's method.

9. Given $\frac{dx}{dt} - ty - 1 = 0$ and $\frac{dy}{dt} + tx = 0$, $t = 0$, $x = 0$, $y = 1$, evaluate $x(0.1)$, $y(0.1)$, $x(0.2)$ and $y(0.2)$.

10. Using Taylor's series method, obtain the values of y at $x = (0.1)(0.1)$ 0.3 to four significant figures if y satisfies the equation $\frac{d^2y}{dx^2} + xy = 0$ given that $\frac{dy}{dx} = \frac{1}{2}$ and $y = 1$ when $x = 0$.

11. Evaluate the integral of the following problem to four significant figures at $x = 1.1(0.1)\ 1.3$ using Taylor's series expansion.

$$\frac{d^2y}{dx^2} + y^2\frac{dy}{dx} - x^3 = 0; \left.\frac{dy}{dx}\right]_{x=1} = 1\ ;\ y(1) = 1$$

12. Using Picard's method find $y(0.2)$ given that $y' = x - y$; $y(0) = 1$.

13. Using Picard's method obtain a solution upto the fifth approximation to the equation $y' = y + x$, such that $y(0) = 1$. Check your answer by finding the exact particular solution. Also find $y(0.1)$ and $y(0.2)$.

14. Using Picard's method find $y(0.2)$ and $y(0.4)$ given that $y' = 1 + y^2$ and $y(0) = 0$.

15. Use Picard's method to approximate the value of y when $x = 0.1$ given that $y(0) = 1$ and $y' = 3x + y^2$.

16. Using Picard's method find the approximate values of y and z corresponding to $x = 0.1$ given that $y(0) = 2$, $z(0) = 1$ and

$$\frac{dy}{dx} = x + z,\ \frac{dz}{dx} = x - y^2.$$

17. Using Picards method obtain the second approximation to the solution to $y'' - x^2y' - x^2y = 0$ so that $y(0) = 1$, $y'(0) = 0.5$.

ANSWERS

1.

x	0	0.1	0.2	0.3	0.4	0.5	0.6
Approx. value of y	1	1.055375	1.123	1.205125	1.304	1.421875	1.561
Exact value of y	1	1.055	1.124	1.209	1.316	1.455	1.64

2. $y = 1 + x + \dfrac{3x^2}{2} + \dfrac{4x^3}{3} + \dfrac{11x^4}{8} + \dfrac{23x^5}{15} + \cdots$
3. $y(0.1) = 1.0665$; $y(0.2) = 1.167196$
4. $y(0.1) = 1.1167$, $y(0.2) = 1.2767$, $y(0.3) = 1.5023$
5. $y(0.1) = 1.0047$, $y(0.2) = 1.01812$, $y(0.3) = 1.03995$
6. $y(0.2) = 0.811$, exact value of $y(0.2) = 0.8112$
7. $y(0.1) = 1.1164$, $y(0.2) = 1.2725$
8. $y(0.1) = 1.1003$, $z(0.1) = 1.1102$
9. $x(0.1) = 0.105$, $y(0.1) = 0.9997$
   $x(0.2) = 0.21998$, $y(0.2) = 0.9972$
10. $y(0.1) = 1.050$, $y(0.2) = 1.099$, $y(0.3) = 1.145$
11. $y(1.1) = 1.100$, $y(0.2) = 1.201$, $y(0.3) = 1.306$
12. $y(0.2) = 0.837$
13. $y(0.1) = 1.1103$; $y(0.2) = 1.2428$
14. $y(0.2) = 0.2027$, $y(0.4) = 0.4227$
15. $y(0.1) = 1.127$
16. $y(0.1) = 2.0845$; $z(0.1) = 0.5867$
17. $y_2 = 1 + \dfrac{1}{2}x + \dfrac{3}{40}x^5$

11.10 EULER'S METHOD

Consider the differential equation

$$\frac{dy}{dx} = f(x, y) \tag{11.32}$$

where $y(x_0) = y_0$

Suppose that we wish to find successively $y_1, y_2, \ldots, y_m$, where y_m is the value of y corresponding to $x = x_m$, where $x_m = x_0 + mh$, $m = 1, 2, \ldots$, h being small. Here, we use the property that in a small interval, a curve is nearly a straight line.

Thus, in the interval x_0 to x_1 of x, we approximate the curve by the tangent at the point (x_0, y_0)

Therefore, the equation of the tangent at (x_0, y_0) is

$$y - y_0 = \left(\frac{dy}{dx}\right)_{(x_0, y_0)} (x - x_0)$$

$$= f(x_0, y_0)\,(x - x_0) \quad \text{[from Eqn (11.32)]}$$

or $y = y_0 + (x - x_0) f(x_0, y_0)$

Hence, the value of y corresponding to $x = x_1$ is

$$y_1 = y_0 + (x_1 - x_0)\, f(x_0, y_0)$$

or

$$y_1 = y_0 + h f(x_0, y_0) \tag{11.33}$$

Since the curve is approximated by the tangent in $[x_0, x_1]$, Eqn (11.33) gives the approximated value of y_1.

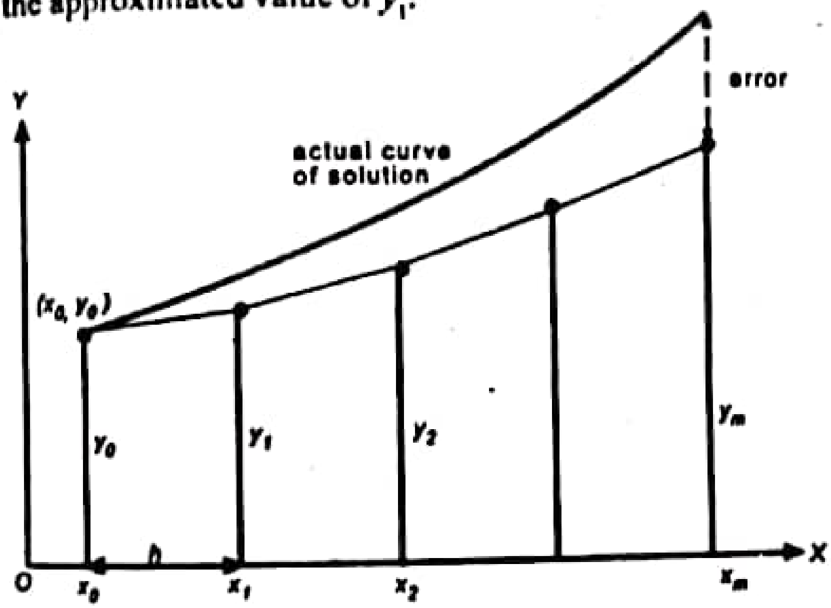

Fig 11.2

Similarly, approximating the curve in the next interval $[x_1, x_2]$ by a line through (x_1, y_1) with slope $f(x_1, y_1)$, we get

$$y_2 = y_1 + h f(x_1, y_1) \tag{11.34}$$

Proceeding on, in general it can be shown that

$$y_{m+1} = y_m + h f(x_m, y_m) \tag{11.35}$$

Remarks In Euler's method, the actual curve of a solution is approximated by a sequence of short lines as shown in Fig 11.2. It is possible that the sequence of lines may deviate from the curve of solution significantly. The process is very slow and to obtain it with reasonable accuracy using Euler's method, we have to take h very small. An improvement over this method is discussed in the following section.

11.11 IMPROVED EULER'S METHOD

Here, we consider a line passing through $A(x_0, y_0)$ whose slope is the average of the slopes at $A(x_0, y_0)$ and $P(x_1, y_1^{(1)})$ such that $y'_1 = y_0 + hf(x_0, y_0)$.

In Fig. 11.3, let AL_1 be the tangent to the curve at $A(x_0, y_0)$ and PL_2 be the line through $P(x_1, y_1^{(1)})$ having the slope $f(x_1, y_1^{(1)})$. Now PM is the line having slope

$$\frac{1}{2}\{f(x_0, y_0) + f(x_1, y_1^{(1)})\}$$

that is, average of two slopes $f(x_0, y_0)$ and $f(x_1, y_1^{(1)})$.

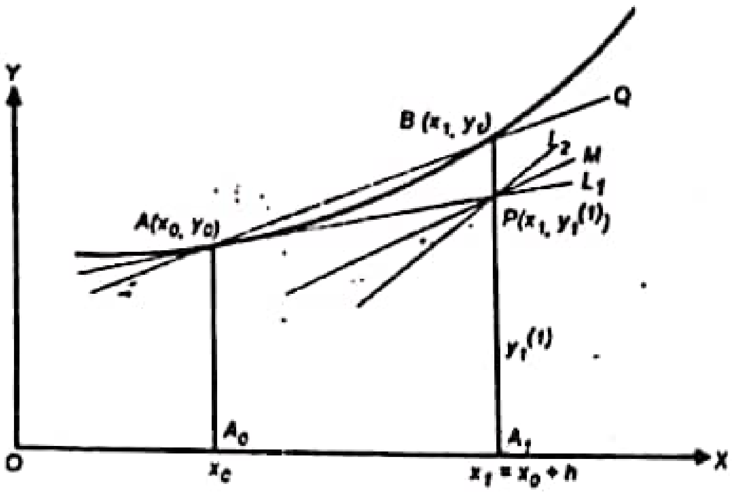

Fig 11.3

Line AQ through (x_0, y_0) and parallel to PM is used to approximate the curve. Then, ordinate of point B will give the value of y_1.

Therefore, equation to ABQ is

$$y - y_0 = (x - x_0)\frac{1}{2}\{f(x_0, y_0) + f(x_1, y_1^{(1)})\} \tag{11.36}$$

As we are assuming that $A_1B = y_1$, coordinates of B will be (x_1, y_1). This point will lie on AQ.

$$\therefore\ y_1 - y_0 = (x_1 - x_0)\frac{1}{2}\{f(x_0, y_0) + f(x_1, y_1^{(1)})\}$$

or

$$y_1 = y_0 + \frac{h}{2}\{f(x_0, y_0) + f(x_1, y_1^{(1)})\}$$

$$= y_0 + \frac{h}{2}\{f(x_0, y_0) + f(x_0 + h, y_0 + hf(x_0, y_0))\} \tag{11.37}$$

In general, we have the formula

$$y_{m+1} = y_m + \frac{h}{2}\{f(x_m, y_m) + f(x_m + h, y_m + hf(x_m, y_m))\} \quad (11.38)$$

where $x_m - x_{m-1} = h$.

11.12 MODIFIED EULER'S METHOD

In this method the curve in the interval (x_0, x_1), where $x_1 = x_0 + h$, is approximated by the line through (x_0, y_0) with slope

$$f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right) \quad (1)$$

that is, the slope at the middle point whose abscissa is the average of x_0 and x_1, i.e. $x_0 + \frac{h}{2}$.

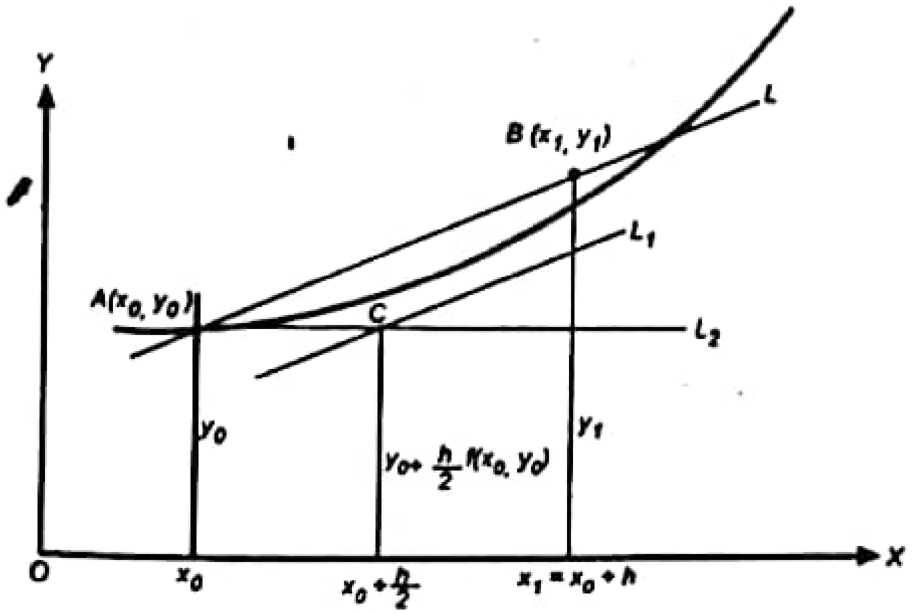

Fig 11.4

Geometrically, line L through (x_0, y_0) which is parallel to L_1, a line through $(x_0 + \frac{h}{2}, \frac{h}{2} f(x_0, y_0))$, with the slope (1) approximates the curve in the interval $[x_0, x_1]$. The ordinate at $x = x_1$, meeting the line L at B, will give the value of y_1.

The equation for line L is

$$y - y_0 = (x - x_0)\left\{f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right)\right\}$$

Putting $x = x_1$, we get

$$y_1 = y_0 + (x_1 - x_0)\{f(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\}$$

$$= y_0 + hf(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0) \quad (11.39)$$

Proceeding in the same way, it can be shown that,

$$y_{m+1} = y_m + hf(x_m + \frac{h}{2}, y_m + \frac{h}{2} f(x_m, y_m) \quad (11.40)$$

Example 11.9 Solve $\frac{dy}{dx} = 1 - y, y(0) = 0$ in the range $0 \le x \le 0.3$ using (i) Euler's method (ii) improved Euler's method, and (iii) modified Euler's method by choosing $h = 0.1$. Compare the answers with exact solution.

Solution Given $\frac{dy}{dx} = 1 - y$ and $y(0) = 0$ and $h = 0.1$. (i)

Now we have to find out the solutions at $x = 0.1, 0.2$ and 0.3.

(i) Euler's method : The algorithm is,

if $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$

then, $\quad y_{m+1} = y_m + hf(x_m, y_m) \quad$ (ii)

Here, $f(x, y) = 1 - y$, $h = 0.1$; $x_0 = 0$, $y_0 = 0$

$\therefore$ From Eqn (ii), $y_{m+1} = y_m + (0.1)(1 - y_m) \quad$ (iii)

or $\quad y_{m+1} = 0.1 + 0.9\, y_m$

Putting $m = 0, 1, 2$ successively, we get

$y_1 = 0.1 + 0.9\, y_0 = 0.1 + (0.9)(0) = 0.1$

$y_2 = 0.1 + 0.9\, y_1 = 0.1 + (0.9)(0.1) = 0.19$

$y_3 = 0.1 + 0.9\, y_2 = 0.1 + (0.9)(0.19) = 0.271$

$\therefore y(0.1) = 0.1$; $y(0.2) = 0.19$; $y(0.3) = 0.271$

(ii) Improved Euler's method : Here, the formula is

$$y_{m+1} = y_m + \frac{h}{2}\{f(x_m, y_m) + f(x_{m+1}, y^{(1)}_{m+1})\}$$

where, $f(x_{m+1}, y^{(1)}_{m+1}) = f\{x_m + h, y_m + hf(x_m, y_m)\}$

Here, $f(x, y) = 1 - y$, $\therefore f(x_m, y_m) = 1 - y_m$

$$\therefore f(x_{m+1}, y^{(1)}_{m+1}) = 1 - \{y_m + hf(x_m, y_m)\}$$

$$= 1 - y_m - h(1 - y_m)$$

$$= (1 - h)(1 - y_m)$$

Substituting it in Eqn (iv), we get

$$y_{m+1} = y_m + \frac{h}{2}\{(1-y_m)+(1-h)(1-y_m)\}$$

$$= y_m + \frac{1}{2}h(2-h)(1-y_m)$$

$$= y_m + 0.095(1-y_m) \quad [\therefore h = 0.1]$$

$$y_{m+1} = 0.095 + 0.095\, y_m \qquad \text{(v)}$$

Putting $m = 0, 1, 2$ successively in Eqn (v), we get

$y_1 = 0.095 + 0.905\, y_0 = 0.095 + (0.905)(0) = 0.095$
$y_2 = 0.095 + 0.905\, y_1 = 0.095 + (0.905)(0.095) = 0.180975$
$y_3 = 0.095 + 0.905\, y_2 = 0.095 + (0.905)(0.180905) = 0.2587823$

(iv) *Modified Euler's method :* The formula is,

$$y_{m+1} = y_m + hf\{x_m + \frac{h}{2}, y_m + \frac{h}{2} f(x_m, y_m)\}$$

$$= y_m + h\{1 - [y_m + \frac{h}{2} f(x_m, y_m)]\}$$

$$= y_m + h\{1 - y_m - \frac{h}{2}(1-y_m)\}$$

$$= y_m + h\{1 - \frac{h}{2})(1-y_m)\}$$

$$= y_m + 0.095(1-y_m) \quad [\because h = 0.1]$$
$$= 0.095 + 0.905\, y_m$$

which is identical to Eqn (vi). Putting $m = 0, 1, 2$ successively in Eqn (vi), we get

$$y_1 = 0.095, y_2 = 0.180975, y_3 = 0.2587823$$

Exact solution : We have

$$\frac{dy}{dx} = 1 - y \text{ or } \frac{dy}{1-y} = dx$$

On integrating, we get

$$-\log(1-y) + \log C = x \text{ or } \frac{C}{1-y} = e^x$$

or

$$e^x(1-y) = C \qquad \text{(vii)}$$

But $y = 0$ at $x = 0$. Therefore, from Eqn (vii), we get $C = 1$ and hence

$$e^x(1-y) = 1 \text{ or } y = 1 - e^{-x} \qquad \text{(viii)}$$

$\therefore \quad y_1 = y(0.1) = 1 - e^{-0.1} = 0.0951625$

$y_2 = y(0.2) = 1 - e^{-0.2} = 0.1812692$

and $\quad y_3 = y(0.3) = 1 - e^{-0.3} = 0.2591817$

Now compare the results in the following table.

x	Euler's method	Improved Euler's method	Modified Euler's method	Exact solution
0	0	0	0	0
0.1	0.1	0.095	0.095	0.0951625
0.2	0.19	0.180975	0.180975	0.1812692
0.3	0.271	0.2587823	0.2587823	0.2591817

Example 11.10 Solve $\dfrac{dy}{dx} = y - \dfrac{2x}{y}$, $y(0) = 1$ in the range $0 \le x \le 0.2$ using (i) Euler's method (ii) improved Euler's method, and (iii) modified Euler's method. Take $h = 0.1$.

Solution Given

$$\frac{dy}{dx} = y - \frac{2x}{y},\ y(0) = 1,\ h = 0.1.$$

Now we have to find out the solutions at $x = 0.1$ and $x = 0.2$.

(i) Euler's method: The algorithm is,

if $\dfrac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$

then $y_{m+1} = y_m + hf(x_m, y_m)$ (i)

Putting $m = 0$, we get

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + h\left(y_0 - \frac{2x_0}{y_0}\right)$$

$$= 1 + (0.1)\left[1 - \frac{2(0)}{1}\right] = 1.1 \quad [\because x_0 = 0, y_0 = 1]$$

tting $m = 1$ in Eqn (i), we get

$$= y_1 + hf(x_1, y_1) = y_1 + 0.1\left(y_1 - \frac{2x_1}{y_1}\right)$$

$$= 1.1 + (0.1)\left[1.1 - \frac{2(0.1)}{1.1}\right] \quad [\because x_1 = 0.1, y_1 = 1.1]$$

$$= 1.1918182$$

(ii) Improved Euler's method : Here,

$$y_{m+1} = y_m + \frac{h}{2}[f(x_m, y_m) + f(x_{m+1}, y^{(1)}_{m+1})] \quad \text{(ii)}$$

where $y^{(1)}_{m+1} = y_m + hf(x_m, y_m)$

Putting $m = 0$ in Eqn (ii), we get

$$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y^{(1)}_1)] \quad \text{(iii)}$$

where $y^{(1)}_1 \quad = y_0 + hf(x_0, y_0)$

$$= 1 + (0.1)\left[1 - \frac{2(0)}{1}\right] = 1.1$$

$\therefore$ From Eqn (iii),

$$y_1 \quad = y_0 + \frac{h}{2}\left[\left(y_0 - \frac{2x_0}{y_0}\right) + \left(y_1^{(1)} - \frac{2x_1}{y_1^{(1)}}\right)\right]$$

$$= 1 + \frac{0.1}{2}\left[\left\{1 - \frac{2(0)}{1}\right\} + \left\{1.1 - \frac{2(0.1)}{1.1}\right\}\right]$$

$$= 1.0959091$$

Putting $m = 1$ in Eqn (ii), we get

$$y_2 = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})$$

where $y^{(1)}_2 = y_1 + hf(x_1, y_1)$

Now, $f(x_1, y_1) = y_1 - \dfrac{2x_1}{y_1}$

$$= 1.0959091 - \frac{2(0.1)}{1.0959091}$$

$$= 0.9134122$$

$y^{(1)}_2 = y_1 + hf(x_1, y_1) = 1.0959091 + (0.1)(0.9134122)$

$= 1.1872503$

$f(x_2, y^{(1)}_2) = [y^{(1)}_2 - 2x_2/y^{(1)}_2]$

$$= 1.1872503 - \frac{2(0.2)}{1.1872503}$$

$$= 0.8503373$$

Substituting all the requisites in Eqn (iv), we get

$$y_2 = 1.0959091 + \frac{0.1}{2}[0.913422 + 0.8503373]$$
$$= 1.1840966$$

(iii) Modified Euler's method: The formula is,

$$y_{m+1} = y_m + hf\{x_m + \frac{h}{2}, y_m + \frac{h}{2} f(x_m, y_m)\} \qquad \text{(v)}$$

Putting $m = 0$ in the above Equation, we get

$$y_1 = y_0 + hf\{x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\} \qquad \text{(vi)}$$

$$f(x_0, y_0) = y_0 - \frac{2x_0}{y_0} = 1$$

$$y_0 + \frac{h}{2} f(x_0, y_0) = 1 + \frac{0.1}{2}(1) = 1.05 \,;\; x_0 + \frac{h}{2} = 0.05$$

$$\therefore f\{x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\} = f\{0.05, 1.05\}$$

$$= 1.05 - \frac{2(0.05)}{1.05} = 0.9547619$$

Hence, from Eqn (vi), we get

$y_1 = 1 + (0.1)(0.09547619) = 1.0954762$

Putting $m = 1$ in Eqn (v), we get

$$y_2 = y_1 + hf\{x_1 + \frac{h}{2}, y_1 + \frac{h}{2} f(x_1, y_1)\} \qquad \text{(vii)}$$

$$x_1 + \frac{h}{2} = 0.1 + \frac{0.1}{2} = 0.15$$

$$f(x_1, y_1) = y_1 - \frac{2x_1}{y_1} = 1.0954762 - \frac{2(0.1)}{1.0954762}$$
$$= 0.9129071$$

$$y_1 + \frac{h}{2} f(x_1, y_1) = 1.0954762 + \frac{0.1}{2}(0.9129071)$$
$$= 1.1411216$$

$$\therefore\ f\{x_1 + \frac{h}{2}, y_1 + \frac{h}{2} f(x_1, y_1)\} = f\{0.15, 1.1411216\}$$

$$= 1.1411216 - \frac{2(0.15)}{1.1411216} = 0.8782223$$

Now substituting the requisites in Eqn (vii), we get
$y_2 = 1.0954762 + (0.1)(0.8782223) = 1.1832984$
The values obtained are shown below in a tabular form.

x	Euler's method	Improved Euler's method	Modified Euler's method
0	1	1	1
0.1	1.1	1.0959091	1.0954762
0.2	1.1918182	1.1840966	1.1832984

EXERCISE 11.2

1. Use Euler's method and Improved Euler's method to approximate y when $x = 0.1$, given that

$$\frac{dy}{dx} = \frac{y-x}{y+x},\ y(0) = 1 \text{ taking } h = 0.2.$$

2. Solve $y' = 3x^2 + y$ in $0 \le x \le 1$ by Euler's method taking $h = 0.1$ given that $y(0) = 4$.
3. Solve $y' = x + y$, $y(0)$ choosing the step length 0.2 for $y(1.2)$ by Euler's method.
4. Using Euler's method solve $y' = x + y$ in $0 \le x \le 1$ with $h = 0.1$, if $y(0) = 1$. Find the exact value of y at $x = 1$, using analytical method.
5. Using Euler's method find $y(0.6)$ of $y' = 1 - 2xy$, given that $y(0) = 0$ taking $h = 0.2$.
6. Solve $y' = -y$; $y(0) = 1$ by (i) Euler's method for $y(0.04)$ and (ii) Modified Euler's method for $y(0.6)$.
7. Solve $y' = x + y + xy$, $y(0) = 1$ for $y(0.1)$ taking $h = 0.025$, using Euler's method.
8. Given that $y' = \log(x + y)$ with $y(0) = 1$. Use (i) Improved Euler's method to find $y(0.2)$, $y(0.5)$, (ii) Modified Euler's method to find $y(0.2)$.

9. Use Euler's method and its Modified form to obtain $y(0.2)$, $y(0.4)$ and $y(0.6)$ correct to three decimal places given that $y' = y - x^2$, $y(0) = 1$.

10. Use Euler's modified method to get $y(0.25)$ given that $y' = 2xy$, $y(0) = 1$.

11. Using Improved Euler's method, solve
$y' = x + |\sqrt{y}|$, $y(0) = 1$ in the range $0 \le x \le 0.6$ taking $h = 0.2$.

12. Given that $y' = 2 + \sqrt{(xy)}$ and $y(1) = 1$. Find $y(2)$ in steps of 0.2 using Improved Euler's method.

13. Given $y^{(1)} = x^2 + y^2$, $y(0) = 1$, determine $y(0.1)$ and $y(0.2)$ by Modified Euler's method.

14. Solve $y^{(1)} = y + e^x$, $y(0) = 0$ for $y(0.2)$, $y(0.4)$ by Improved Euler's method.

15. Solve $y^{(1)} = y + x^2$, $y(0) = 1$ for $y(0.02)$, $y(0.04)$ and $y(0.06)$ using Euler's Modified method.

ANSWERS

1. 1.0928, 1.0932
2. 4.4, 4.843, 5.3393, 5.90023, 6.538253, 7.2670783, 8.1017861, 9.0589647, 9.1039647, 10.257361
3. 1.1831808
4. 1.1, 1.22, 1.362, 1.5282, 1.7210, 1.9431, 2.1974, 2.4871, 2.815, 3.1873 ; exact solution = 3.4366
5. 0.4748
6. 0.9603 ; 0.551368
7. 1.1448
8. 1.0082, 1.0490 ; 1.0095
9. 1.2, 1.432, 1.686 ; 1.218, 1.467, 1.737
10. 1.0625
11. 1.2309, 1.5253, 1.8851
12. 5.051
13. 1.1105, 1.25026
14. 0.24214, 0.59116
15. 1.0202, 1.0408, 1.0619

11.13 RUNGE'S METHOD

Let the differential equation be

$$\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0$$

Let h be the width of the equispaced values of x.
Then the first increment in y is obtained by the following set of formulae:

$$k = f(x_0, y_0) \tag{11.41}$$

$= $ slope at the beginning of the interval (x_0, y_0)

$$k' = f(x_0 + h,\ y_0 + kh) \text{ (say)} \tag{11.42}$$

$$k'' = f(x_0 + h,\ y_0 + k'h) \tag{11.43}$$

$= $ slope at the beginning of the interval (x_0, y_0)

Also, slope at the middle point of interval (x_0, y_0) is

$$k_1 = f(x_0 + \frac{h}{2},\ y_0 + k\frac{h}{2}) \tag{11.44}$$

Now the increment in the value of y in the first interval is given by

$$\Delta y = \frac{h}{6}(k + 4k_1 + k'') \tag{11.45}$$

which can be easily obtained by the Simpson's rule,

$$\Delta y = \int_{x_0}^{x_0+\Delta x} \frac{dy}{dx}\, dx$$

The increment in y in the second interval (x_1, y_1) is obtained by the following formulae :

$$k = f(x_1, y_1)$$
$$k' = f(x_1 + h,\ y_1 + kh)$$
$$k'' = f(x_1 + h,\ y_1 + k'h)$$
$$k_1 = f(x_1 + \frac{h}{2},\ y_1 + k\frac{h}{2})$$

and
$$\Delta y = \frac{h}{6}(k + 4k' + k'').$$

In the same way the increments in y can be obtained very easily in the succeeding intervals.

Working method Given $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$

find out $k_1 = hf(x_0, y_0)$

$$k_2 = hf(x_0 + h,\ y_0 + k_1)$$
$$k_3 = hf(x_0 + h,\ y_0 + k_2)$$

Working method Given $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$

find out $k_1 = hf(x_0, y_0)$

$$k_2 = hf(x_0 + h,\ y_0 + k_1)$$
$$k_3 = hf(x_0 + h,\ y_0 + k_2)$$

$$k_4 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right)$$

Then $k = \frac{1}{6}(k_1 + 4k_4 + k_3)$

$\therefore$ Value of $y = y_0 + k$

Example 11.11 Apply Runge's method to find an approximate value of y when $x = 0.2$, given that $y' = x + y$ and $y(0) = 1$.

Solution Here,

$$f(x, y) = y' = x + y,\ x_0 = 0,\ y_0 = 1,\ h = 0.2$$

$$\therefore \quad f(x_0, y_0) = 0 + 1 = 1$$

$$k_1 = hf(x_0, y_0) = (0.2)(1) = 0.2$$

$$k_2 = hf(x_0 + h,\ y_0 + k_1) = hf(0.2, 1.2)$$
$$= (0.2)(0.2 + 1.2) = 0.28$$

$$k_3 = hf(x_0 + h,\ y_0 + k_2) = hf(0.2, 1.28)$$
$$= (0.2)(0.2 + 1.28) = 0.296$$

$$k_4 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = hf(0.1, 1.1)$$
$$= (0.2)(0.1 + 1.1) = 0.24$$

Then $k = \frac{1}{6}(k_1 + 4k_4 + k_3)$

$$= \frac{1}{6}[0.2 + 4(0.24) + 0.296] = 0.2426666$$

$$\therefore \quad y_1 = y(0.2) = y_0 + k = 1.2426666$$

11.14 RUNGE–KUTTA METHODS

We have seen that solving differential equations numerically using Taylor's series method to determine higher order derivatives is a lengthy process. To overcome this there is a class of methods known as Runge–Kutta methods, which do not require the calculations of higher order derivatives and give greater accuracy. These methods agree with Taylor's series solution upto the term h^r, where r differs from method to method and is known as the order of that method.

First order Runge-Kutta method We have see...
11.10) gives

$y_1 = y_0 + hf(x_0, y_0) = y_0 + hy_0'$ [∵ ...

Now $y_1 = y(x_0 + h)$. Expanding it by Tay...

$$... = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \cdots$$

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$$= \frac{1}{6}[0.2 + 4(0.24) + 0.296] = 0.2426666$$

$$\therefore \quad y_1 = y(0.2) = y_0 + k = 1.2426666$$

11.14 RUNGE–KUTTA METHODS

We have seen that solving differential equations numerically using Taylor's series method to determine higher order derivatives is a lengthy process. To overcome this there is a class of methods known as Runge–Kutta methods, which do not require the calculations of higher order derivatives and give greater accuracy. These methods agree with Taylor's series solution upto the term h^r, where r differs from method to method and is known as the order of that method.

First order Runge–Kutta method We have seen that Euler's method (section 11.10) gives

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + hy_0' \quad [\because y' = f(x, y)] \tag{11.46}$$

Now $y_1 = y(x_0 + h)$. Expanding it by Taylor's series, we get,

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \cdots$$

This implies that Euler's method agrees with Taylor's series solution upto term in h.

Hence, Runge–Kutta method of first order is the Euler's method only.

Runge–Kutta method of second order The Improved Euler's method (section 11.11) gives

$$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_0 + h,\ y_1^{(1)})] \tag{11.47}$$

where $y_1^{(1)} = y_0 + hf(x_0, y_0)$. That is,

$$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f\{x_0 + h,\ y_0 + hf(x_0, y_0)\}] \tag{11.48}$$

Now, $$y_1 = y(x_0 + h) = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots \tag{11.49}$$

(by Taylor's series)

Now expanding $f\{x_0 + h,\ y_0 + hf(x_0, y_0)\}$ by Taylor's series for a function of two variables

$$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_0, y_0) + h\left(\frac{\partial f}{\partial x}\right)_{(x_0, y_0)}$$

$$1 + hf(x_0, y_0) + \left(\frac{\partial f}{\partial y}\right)_{(x_0, y_0)} + \text{terms containing second}$$

and higher powers of h]

$$= y_0 + \frac{1}{2}[2hf(x_0, y_0) + h^2\left\{\left(\frac{\partial f}{\partial x}\right)_{(x_0, y_0)}$$

+ terms containing h^3 onwa

$$= y_0 + hf(x_0, y_0) + \frac{h^2}{2} f'(x_0, y_0) + O(h^3)$$

$$y_0 + \frac{1}{2}\left[2hf(x_0, y_0) + h^2\left\{\left(\frac{\partial f}{\partial x}\right)_{(x_0,y_0)} + f(x_0, y_0)\left(\frac{\partial f}{\partial y}\right)_{(x_0,y_0)}\right\}\right.$$

$$\left. + \text{ terms containing } h^3 \text{ onwards (or } O(h^3))\right]$$

$$= y_0 + hf(x_0, y_0) + \frac{h^2}{2} f'(x_0, y_0) + O(h^3)$$

$$\left[\because f'(x_0, y_0) = \left(\frac{\partial f}{\partial x}\right)_{(x_0,y_0)} + f(x_0, y_0)\left(\frac{\partial f}{\partial x}\right)_{(x_0,y_0)}\right]$$

$$= y_0 + hy_0' + \frac{h^2}{2!} y_0'' + O(h^3) \quad (11.50)$$

Eqns (11.49) and (11.50) imply that the Improved Euler's method agrees with the Taylor's series solution upto the term in h^2. Therefore, the Improved Euler's method is the Runge–Kutta method of second order.

The algorithm to Runge–Kutta method of second order is

$$k_1 = hf(x_0, y_0)$$
$$k_2 = hf(x_0 + h,\ y_0 + k_1)$$
$$k = \frac{1}{2}(k_1 + k_2)$$

and
$$y_1 = y_0 + k$$

11.15 HIGHER ORDER RUNGE–KUTTA METHODS

In this section, we will study the formula for third and fourth order Runge-Kutta methods. Their derivations being tedious and unrequired, the formulae have not been derived here.

Runge–Kutta method of third order: It is defined by the following equations:

$$k_1 = hf(x_0, y_0)$$
$$k_2 = hf(x_0 + h,\ y_0 + k_1)$$
$$k_3 = hf(x_0 + h,\ y_0 + k_2)$$
$$k_4 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right)$$
$$k = \frac{1}{6}(k_1 + 4k_4 + k_3)$$

and
$$y_1 = y_0 + K$$

We can see that this is identical to Runge's method.

Runge–Kutta method of fourth order: It is most commonly known as Runge-Kutta method and the working procedure is as follows. Consider the following equations.

$$\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0$$

To compute y_1, calculate successively

$$k_1 = hf(x_0, y_0)$$
$$k_2 = hf\left(x_0 + \frac{h}{2}, \ldots\right)$$

To compute y_1, calculate successively

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Then $\quad y_1 = y_0 + k$ and $x_1 = x_0 + h$

The increment in y in second interval is computed in a similar manner by means of the formulae

$$k_1 = hf(x_1, y_1)$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Then $\quad y_2 = y_1 + k$ and $x_2 = x_1 + h$

and so on for succeeding intervals.

You can notice that the only change in the formulae for the different intervals is in the values of x and y. Thus, to find k in ith interval, we should have substituted x_{i-1} and y_{i-1} in the expressions for k_1, k_2, k_3 and k_4.

Example 11.12 Given $y' = x^2 - y$, $y(0) = 1$, find $y(0.1)$, $y(0.2)$ using Runge–Kutta methods of (i) second order, (ii) third order and (iii) fourth order.

Solution Given

$$y' = f(x, y) = x^2 - y, x_0 = 0, y_0 = 1$$

$$\therefore \quad f(x_0, y_0) = -1.$$

Let $h = 0.1$

Runge–Kutta method of 2nd order : Here,

$$k_1 = hf(x_0, y_0) = (0.1)(-1) = -0.1$$

$$k_2 = hf(x_0 + h, y_0 + k_1) = hf(0.1, 0.9)$$

$$= (0.1)[(0.1)^2 - 0.9] = -0.089$$

$$\therefore \quad k = \frac{1}{2}(k_1 + k_2) =$$

$$\therefore \quad y_1 = y(0.1) = y_0 +$$

Again, taking $x_1 = 0.1$, $y_1 =$
process,

$$k_1 = hf(x_1, y_1$$

$$= (0.1)[(0.1)^2 - 0.9055] = -0.08955$$

$$k_2 = hf[x_1 + h, y_1 + k_1] = hf(0.2, 0.81595)$$

Runge–Kutta method of 2nd order : Here,

$$k_1 = hf(x_0, y_0) = (0.1)(-1) = -0.1$$
$$k_2 = hf(x_0 + h,\ y_0 + k_1) = hf(0.1, 0.9)$$
$$= (0.1)\,[(0.1)^2 - 0.9] = -0.089$$

$$\therefore \quad k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}[(-0.1) + (0.089)] = -0.0945$$

$$\therefore \quad y_1 = y(0.1) = y_0 + k = 1 - 0.0945 = 0.9055$$

Again, taking $x_1 = 0.1$, $y_1 = 0.9055$ in place of (x_0, y_0) and repeating the process,

$$k_1 = hf(x_1, y_1) = h(x_1^2 - y_1)$$
$$= (0.1)\,[(0.1)^2 - 0.9055] = -0.08955$$
$$k_2 = hf[x_1 + h,\ y_1 + k_1] = hf(0.2, 0.81595)$$
$$= (0.1)\,[(0.2)^2 - 0.81595] = -0.077595$$

$$\therefore k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}[(-0.08955 - 0.077595] = -0.0835725$$

$$\therefore y_2 = y(0.2) = y_1 + k = 0.9055 - 0.0835725$$
$$= 0.8219275$$

Runge–Kutta method of 3rd order : Here,

$$k_1 = hf(x_0, y_0) = -0.1$$
$$k_2 = hf(x_0 + h,\ y_0 + k_1) = -0.089$$
$$k_3 = hf(x_0 + h,\ y_0 + k_2) = hf(0.1, 0.911)$$
$$= (0.1)\,[(0.1)^2 - 0.911] = -0.0901$$
$$k_4 = hf(x_0 + h/2,\ y_0 + k_1/2) = hf(0.05, 0.95)$$
$$= (0.1)\,[(0.05)^2 - 0.95] = -0.09475$$

$$\therefore k = \frac{1}{6}(k_1 + 4k_4 + k_3)$$

$$= \frac{1}{6}[(-0.1) + 4\,(-0.09475) + (-0.0901\,] = -0.09485$$

$$\therefore y_1 = y(0.1) = 0.90515$$

Taking $x_1 = 0.1$, $y_1 = 0.90515$, $h = 0.1$ in place of (x_0, y_0) and repeating the process, we get

$$k_1 = hf(x_1, y_1) = (0.1)\,[(0.1)^2 - 0.90515] = -0.089515$$
$$k_2 = hf(x_1 + h,\ y_1 + k_1) = hf(0.2, 0.815635)$$
$$= (0.1)\,[(0.2)^2 - 0.815635] = -0.0775635$$
$$k_3 = hf(x_1 + h,\ y_1 + k_2) = hf(0.2, 0.8275865)$$
$$= (0.1)\,[(0.2^2 - 0.8275865] = -0.0787586$$

$$k_4 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = hf(0.15, 0.8603925)$$
$$= (0.1)\,[(0.15)^2 - 0.8603925] = -0.0837892$$

$$\therefore k = \frac{1}{6}(k_1 + 4k_4 + k_3)$$

$$= \frac{1}{6}[(-0.089515) + 4\,(-0.0837892)$$

$$\therefore y_2 = y(0.2) = y_1 + k = 0.90515 - 0.0839$$

Runge–Kutta method of 4th order : Here,

$$k_1 = hf(x_0, y_0) = -0.1$$

$$k_4 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k}{2}\right) = hf(0.15, 0.8603925)$$

$$= (0.1)[(0.15)^2 - 0.8603925] = -0.0837892$$

$$\therefore k = \frac{1}{6}(k_1 + 4k_4 + k_3)$$

$$= \frac{1}{6}[(-0.089515) + 4(-0.0837892) - 0.0787586] = -0.0839051$$

$$\therefore y_1 = y(0.2) = y_1 + k = 0.90515 - 0.0839051 = 0.8212449$$

Runge-Kutta method of 4th order : Here,

$$k_1 = hf(x_0, y_0) = -0.1$$

$$k_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right) = (0.1)f[0.05, 0.95]$$

$$= (0.1)[(0.05)^2 - 0.95] = 0.09475$$

$$k_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right) = hf(0.05, 0.952625)$$

$$= (0.1)[(0.05)^2 - 0.952625] = -0.0950125$$

$$k_4 = hf(x_0+h, y_0+k_3) = hf[0.1, 0.9049875]$$

$$= (0.1)[(0.1)^2 - 0.9049875] = -0.0894987$$

$$\text{Now, } k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[-0.1 + 2(-0.09475) + 2(-0.0950125) - 0.0894987]$$

$$= -0.0948372$$

$$\therefore y_1 = y(0.1) = y_0 + k = 1 - 0.0948372 = 0.9051627$$

Taking $x_1 = 0.1$, $y_1 = 0.9051627$ in place of x_0, y_0 and repeating the process, we get

$$k_1 = hf(x_1, y_1) = hf(0.1, 0.9051627)$$

$$= (0.1)[(0.1)^2 - 0.9051627] = -0.0895162$$

$$k_2 = hf\left(x_1+\frac{h}{2}, y_1+\frac{k_1}{2}\right) = hf(0.15, 0.8604046)$$

$$= (0.1)[(0.15)^2 - 0.8604046] = -0.0837904$$

$$k_3 = hf\left(x_1+\frac{h}{2}, y_1+\frac{k_2}{2}\right) = hf(0.15, 0.8632674)$$

$$= (0.1)[(0.15)^2 - 0.8632674] = -0.0840767$$

$$k_4 = hf(x_1+h, y_1+k_3) = hf(0.2, 0.8210859)$$

$$= (0.1)[(0.2)^2 - 0.8210859] = -0.0781085$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[-0.0895162 + 2(-0.0837904) + 2(-0.0840767) - 0.0781085]$$

$$= -0.0838931$$

$$\therefore y_2 = y(0.2) = y_1 + k = 0.9051627 - 0.0838931$$

$$= 0.08212695$$

Example 11.13 Using Runge-
$y(0.1)$, $y(0.2)$ and $y(0.3)$ given th

Solution Here,

$$y' = f(x, y) = xy$$

$$k_1 = hf(x_0, y_0) = (0.1)[(0.1) + (1)^2] = 0.1$$

$= (0.1)\,[(0.15)^2 - 0.8604046] = -0.0837904$

$$k_3 = hf\left(x_1+\frac{h}{2}, y_1+\frac{k_2}{2}\right) = hf(0.15, 0.8632674)$$

$= (0.1)\,[(0.15)^2 - 0.8632674] = -0.0840767$

$$k_4 = hf(x_1+h,\ y_1+k_3) = hf(0.2, 0.8210859)$$
$$= (0.1)\,[(0.2)^2 - 0.8210859] = -0.0781085$$

$$k = \frac{1}{6}(k_1+2k_2+2k_3+k_4)$$

$$= \frac{1}{6}[-0.0895162+2(-0.0837904)+2(-0.0840767)-0.0781085]$$

$= -0.0838931$

$\therefore\ y_2 = y(0.2) = y_1 + k = 0.9051627 - 0.0838931$
$= 0.08212695$

Example 11.13 Using Runge–Kutta method of fourth order, solve for $y(0.1)$, $y(0.2)$ and $y(0.3)$ given that $y' = xy + y^2$, $y(0) = 1$.

Solution Here,

$$y' = f(x, y) = xy + y^2,\ x_0 = 0,\ y_0 = 1,\ h = 0.1$$

$k_1 = hf(x_0, y_0) = (0.1)\,[(0.1)+(1)^2] = 0.1$

$$k_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right) = hf[0.05, 1.05]$$

$= (0.1)\,[(0.05)(1.05)^2 + (1.05)^2] = 0.1155$

$$k_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right) = hf(0.05, 1.05775)$$

$= (0.1)\,[(0.05)(1.05775) + (1.05775)^2] = 0.1171723$

$k_4 = hf(x_0+h,\ y_0+k_3) = hf[0.1, 1.1171723]$

$= (0.1)\,[(0.1)(1.1171723) + (1.1171723)^2] = 0.1359791$

$$k = \frac{1}{6}(k_1+2k_2+2k_3+k_4)$$

$$= \frac{1}{6}[0.1 + 2(0.1155) + 2(0.1171723) + 0.1359791]$$

$= 0.1168873$

$y_1 = y_{(0.1)} = y_0 + k = 1 + 0.1168873 = 1.1168873$

Now, taking $x_1 = 0.1$, $y_1 = 1.1168873$ in place of (x_0, y_0) and repeating the process,

$k_1 = hf(x_1, y_1) = (0.1)\,[(0.1)(1.1168873) + (1.1168873)^2]$
$= 0.1359125$

$$k_2 = hf\left(x_1+\frac{h}{2}, y_1+\frac{k_1}{2}\right) = hf[0.15, 1.1848436]$$

$= (0.1)\,[(0.15)(1.1848436) + (1.1848436)^2] = 0.158158$

$$k_3 = hf\left(x_1+\frac{h}{2}, y_1+\frac{k_2}{2}\right) = hf(0.15, 1.$$

$= (0.1)\,[(0.15)(1.1959663) + (1.1959663)^2] = 0.1609730$

$\ldots + h,\ y_1 + k_3) = hf[0.2, 1.2778603]$

$= 0.1359125$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = hf[0.15, 1.1848436]$$

$$= (0.1)\,[(0.15)(1.1848436) + (1.1848436)^2] = 0.158158$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = hf(0.15, 1.1959663)$$

$$= (0.1)\,[(0.15)(1.1959663) + (1.1959663)^2] = 0.1609730$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = hf[0.2, 1.2778603]$$

$$= (0.1)\,[(0.2)(1.2778603) + (1.2778603)^2] = 0.1888499$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.1359125 + 2(0.158158) + 2(0.1609730) + 0.1888499]$$

$$= 0.160504$$

$$y_2 = y(0.2) = y_1 + k = 1.1168873 + 0.160504 = 1.2773914$$

Taking $x_2 = 0.2$, $y_2 = 1.2773914$ in place of (x_1, y_1) and repeating the process, we get

$$k_1 = hf(x_2, y_2) = (0.1)\,[(0.2)(1.2773914) + (1.2773914)^2]$$

$$= 0.1887207$$

$$k_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right) = hf[0.25, 1.3717517]$$

$$= (0.1)\,[(0.25)(1.3717517) + (1.3717517)^2] = 0.222464$$

$$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right) = hf(0.25, 1.3886234)$$

$$= (0.1)\,[(0.25)(1.3886234) + (1.3886234)^2] = 0.227543$$

$$k_4 = hf(x_2 + h, y_2 + k_3) = (0.1)f[0.3, 1.5049345]$$

$$= (0.1)\,[(0.3)(1.5049345) + (1.5049345)^2] = 0.2716308$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.1887207 + 2(0.222464) + 2(0.227543) + 0.2716308]$$

$$= 0.2267275$$

$$\therefore y_3 = y(0.3) = y_2 + k = 1.2773914 + 0.2267275 = 1.504119$$

$$\therefore y(0.1) = 0.11689,\ y(0.2) = 1.27739;\ y(0.3) = 1.50412$$

11.16 RUNGE–KUTTA M… SIMULTANEOUS …

Consider the simultaneous equa…

$$\frac{dy}{dx} = f_1(x, \ldots$$

with the initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$. Now starting fr…

$= 0.2267275$

$\therefore y_3 = y(0.3) = y_2 + k = 1.2773914 + 0.2267275 = 1.504119$

$\therefore y(0.1) = 0.11689,\ y(0.2) = 1.27739\ ;\ y(0.3) = 1.50412$

11.16 RUNGE–KUTTA METHODS FOR SIMULTANEOUS FIRST ORDER EQUATIONS

Consider the simultaneous equations

$$\frac{dy}{dx} = f_1(x, y, z)\ ;\quad \frac{dz}{dx} = f_2(x, y, z)$$

with the initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$. Now starting from (x_0, y_0, z_0) the increment k and l in y and z are given by the following formulae :

$$k_1 = hf_1(x_0, y_0, z_0)\ ;\ l_1 = hf_2(x_0, y_0, z_0)$$

$$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right);$$

$$l_2 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right);$$

$$l_3 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_4 = hf_1(x_0 + h,\ y_0 + k_3,\ z_0 + l_3);$$

$$l_4 = hf_2(x_0 + h,\ y_0 + k_3,\ z_0 + l_3)$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$l = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

Hence, $\quad y_1 = y_0 + k,\ z_1 = z_0 + l$

To compute y_2, z_2, we simply replace x_0, y_0, z_0 by x_1, y_1, z_1 in the above formulae.

If we consider the second order Runge–Kutta method, then

$$k_1 = hf_1(x_0, y_0, z_0)\ ;\quad l_1 = hf_2(x_0, y_0, z_0)$$

$$k_2 = hf_1(x_0 + h,\ y_0 + k_1,\ z_0 + l_1)\ ;\ l_2 = hf_2(x_0 + h,\ y_0 + k_1,\ z_0 + l_1)$$

$$k = \frac{1}{2}(k_1 + k_2);\qquad l_1 = \frac{1}{2}(l_1 + l_2)$$

$$\therefore y_1 = y_0 + k \quad \text{and} \quad z_1 = z_0 + k$$

Example 11.14 Solve $\frac{dy}{dx} = yz + x$;

$z(0) = -1$ for $y(0.2), z(0.2)$.

$\therefore y_1 = y_0 + k$ and $z_1 = z_0 + k$

Example 11.14 Solve $\frac{dy}{dx} = yz + x$; $\frac{dz}{dx} = xz + y$ given that $y(0) = 1$; $z(0) = -1$ for $y(0.2), z(0.2)$.

Solution Here, $f_1(x, y, z) = yz + x$,
$f_2(x, y, z) = xz + y$, $x_0 = 0$, $y_0 = 1$, $z_0 = -1$.
Let $h = 0.1$.

$$k_1 = hf_1(x_0, y_0, z_0) = (0.1)[(1)(-1) + 0] = -0.1$$
$$l_1 = hf_2(x_0, y_0, z_0) = (0.1)[(0)(-1) + 1] = 0.1$$
$$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = hf_1(0.05, 0.95, -0.95)$$
$$= (0.1)[(0.95)(-0.95) + 0.05] = -0.08525$$
$$l_2 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = hf_2(0.05, 0.95, -0.95)$$
$$= (0.1)[(0.05)(-0.95) + 0.95] = 0.09025$$
$$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) = hf_1(0.05, 0.957375, -0.954875)$$
$$= (0.1)[(0.957375)(-0.954875) + 0.05] = -0.0864173$$
$$l_3 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) = hf_2(0.05, 0.957375, -0.954875)$$
$$= (0.1)[(0.05)(-0.954875) + 0.957375] = -0.0909631$$
$$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + l_3) = hf_1(0.1, 0.9135827, -0.9090369)$$
$$= (0.1)[(0.9135827)(-0.9090369) + 0.1] = -0.073048$$
$$l_4 = hf_2(x_0 + h, y_0 + k_3, z_0 + l_3) = hf_2(0.1, 0.9135827, -0.9090369)$$
$$= (0.1)[(0.1)(-0.9090369) + 0.9135827] = 0.822679$$
$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + + k_4)$$
$$= \frac{1}{6}[0.1 + 2(-0.08525) + 2(-0.0864173) - 0.073048]$$
$$= -0.0860637$$
$$l = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$
$$= \frac{1}{6}[0.1 + 2(0.09025) + 2(0.0909631) - 0.0822679]$$
$$= -0.0907823$$

$\therefore y_1 = y(0.1) = y_0 + k = 1 - 0$
$z_1 = z(0.1) = z_0 + l = -1 +$
Now replacing $x_1 = 0.1$, $y_1 =$ and
repeating the process.
$= -0.0730966$

$$= \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

$$= \frac{1}{6}[0.1 + 2(0.09025) + 2(0.0909631) - 0.0822679]$$

$$= -0.0907823$$

$\therefore y_1 = y(0.1) = y_0 + k = 1 - 0.0860637 = 0.9139363$

$z_1 = z(0.1) = z_0 + l = -1 + 0.0907823 = -0.9092176$

Now replacing $x_1 = 0.1$, $y_1 = 0.9139363$ and $z_1 = -0.9092176$ and repeating the process,

$k_1 = hf_1(x_1, y_1, z_1) = h(y_1z_1 + x_1) = -0.0730966$

$l_1 = hf_2(x_1, y_1, z_1) = h(x_1z_1 + y_1) = -0.08230145$

$k_2 = hf_1(x_1 + h/2, y_1 + k_1/2, z_1 + l_1/2) = hf_1(0.15, 0.877388, -0.8680669)$

$= (0.1)[(0.877388)(-0.8680669) + 0.15] = -0.0611631$

$l_2 = hf_2(x_1 + h/2, y_1 + k_1/2, z_1 + l_1/2) = hf_2(0.15, 0.877388, -0.8680669)$

$= (0.1)[(0.15)(-0.8680669) + 0.877388] = 0.0747177$

$k_3 = hf_1(x_1 + h/2, y_1 - k_2/2, z_1 + l_2/2) = hf_1(0.15, 0.8833547,$
$-0.8718587)$

$= (0.1)[(0.8833547)(-0.8718587) + 0.15] = -0.062016$

$l_3 = hf_2(x_1 + h/2, y_1 + k_2/2, z_1 + l_2/2)$

$= hf_2(0.15, 0.8833547, -0.8718587)$

$= (0.1)[(0.15)(-0.8718587) + 0.8833547] = 0.0750851$

$k_4 = hf_1(x_1 + h, y_1 + k_3, z_1 + l_3) = hf_1(0.2, 0.8519203, -0.8341324)$

$= (0.1)[(0.8519203)(-0.8341324) + 0.2] = -0.0510614$

$l_4 = hf_2(x_1 + h, y_1 + k_3, z_1 + l_3) = hf_2(0.2, 0.8519203, -0.8341324)$

$= (0.1)[(0.2)(-0.8341324) + 0.8519203] = 0.0685093$

$k = 1/6[k_1 + 2k_2 + 2k_3 + k_4]$

$1/6[-0.0730966 + 2(-0.0611631) + 2(-0.062016) - 0.0510614]$

$= -0.0617527$

$l = 1/6[l_1 + 2l_2 + 2l_3 + l_4]$

$= 1/6[0.08230145 + 2(-0.0747177) + 2(0.0750851) + 0.0685093]$

$= 0.0750693$

$\therefore y_2 = y(0.2) = y_1 + k = 0.9139363 - 0.0617527 = 0.8521836$

$\therefore z_2 = z(0.2) = z_1 + l = -0.9092176 + 0.0750693 = -0.8341482$

11.17 RUNGE–KUTTA METHOD FOR SECOND ORDER DIFFERENTIAL EQUATION

Consider the second order differential equation,

$$\frac{d^2y}{dx^2} = \phi\left[x, y, \frac{dy}{dx}\right]; \; y(x_0) = y_0; \; y'x_0 = y_0' \qquad (11.51)$$

Let $\frac{dy}{dx} = z$ then $\frac{d^2y}{dx^2} = \frac{dz}{dx}$

Substituting Eqn (11.51), we get

$\frac{dz}{dx} = \phi[x, y, z]$; $y(x_0) = y_0$; $z(x_0) = z_0$

$\therefore$ The problem reduces to solving the ... equations:

$\frac{dy}{dx} = z = f_1(x, y, z)$ and $dz/dx = z = f_2(x, y, z)$

Consider the second order differential equation,

$$\frac{d^2y}{dx^2} = \phi\left[x, y, \frac{dy}{dx}\right];\ y(x_0) = y_0;\ y'x_0 = y_0' \qquad (11.51)$$

Let $\frac{dy}{dx} = z$ then $\frac{d^2y}{dx^2} = \frac{dz}{dx}$

Substituting Eqn (11.51), we get

$\frac{dz}{dx} = \phi[x, y, z];\ y(x_0) = y_0;\ z(x_0) = z_0$

$\therefore$ The problem reduces to solving the simultaneous equations:

$\frac{dy}{dx} = z = f_1(x, y, z)$ and $dz/dx = z = f_2(x, y, z)$

subject to $y(x_0) = y_0$; $z(x_0) = z_0$ and this can be solved as shown in the previous section.

Example 11.15 Solve $y'' = xy' - y$; $y(0) = 3$, $y'(0) = 0$ to approximate $y(0.1)$.

Solution Given

$$y'' = xy' - y;\ y(0) = 3,\ y'(0) = 0 \qquad (i)$$

Let $y' = z$ then $y'' = z'$

$\therefore$ Eqn(i) reduces to

$y' = z = f_1(x, y, z)$

$z' = xz - y = f_2(x, y, z)$

subject to $y(0) = 3$ and $z(0) = 0$, i.e. $x_0 = 0, y_0 = 3, z_0 = 0$

Now,

$k_1 = hf_1(x_0, y_0, z_0) = h(z_0) = (0.1)(0) = 0$

$l_1 = hf_2(x_0, y_0, z_0) = h(x_0z_0 - y_0) = -0.3$

$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = hf_1(0.05, 3, -0.15)$

$= (0.1)(-0.15) = -0.015$

$l_2 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = hf_2(0.05, 3, -0.15)$

$= (0.1)[(0.05)(-0.15) - 3] = 0.30075$

$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) = hf_1(0.05, 2.9925, -0.150375)$

$= (0.1)(0.150375) = -0.0150375$

$l_3 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) = hf_2(0.05, 2.9925, -0.150375)$

$= (0.1)[(0.05)(-0.150375) - 2.9925] = -0.3000018$

$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + l_3) = hf_1(0.1, 2.9849625, -0.3000018)$

$= (0.1)(-0.3000018) = -0.0300001$

$l_4 = hf_2(x_0 + h, y_0 + k_3, z_0 + l$

$= (0.1)[(0.1)(-0.3000018$ ... $.0014962$

$k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$$l_4 = hf_2(x_0 + h,\ y_0 + k_3,\ z_0 + l_3) = hf_2(0.1,\ 2.9849625,\ -0.3000018)$$
$$= (0.1)\,[(0.1)(-0.3000018) - 2.9849625] = -0.3014962$$

$$k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0 + 2(-0.015) + 2(-0.0150375) - 0.0300001]$$

$$= -0.0150125$$

$$l = \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$$

$$= \frac{1}{6}[-0.3 + 2(-0.30075) + 2(-0.3000018) - 0.3014962]$$

$$= -0.3004999$$

$$\therefore y_1 = y(0.1) = y_0 + k = 3 - 0.0150125 = 2.9849875$$
$$z_1 = z(0.1) = z_0 + l = 0 - 0.3004999 = -0.3004999$$

EXERCISE 11.3

1. Solve $y' = x - y$ given that $y = 0.4$ at $x = 1$ for $y(1.6)$ using Runge's method.
2. Using Runge's method, find y at $x = 1.1$ given
$$\frac{dy}{dx} = 3x + y^2,\ y(1) = 1.2$$
3. Evaluate $y(0.8)$ using Runge's method given
$$y' = \sqrt{x+y}\ ;\ y = 0.41 \text{ at } x = 0.4$$
4. Using second order Runge–Kutta method, find y at $x = 0.1, 0.2$ and 0.3 given $2y' = (1 + x)y^2$; $y(0) = 1$.
5. Find $y(1.2)$ by Runge–Kutta method of fourth order given $y' = x^2 + y^2$; $y(1) = 1.5$. Take $h = 0.1$.
6. If $\dfrac{dy}{dx} = \dfrac{2xy + e^x}{x^2 + xe^x}$; $y(1) = 0$, solve for y at $x = 1.2, 1.4$ using Runge–Kutta method of fourth order.
7. Using Runge–Kutta method of fourth order, find y at $x = 1.1, 1.2$ given that
$$2y' = 2x^3 + y;\ y(1) = 2.$$

8. Find y at $x = 0.1, 0.2$ using fourth order Runge–Kutta algorithm given that
$$y' - yx^2 = 0;\ y(0) = 1.$$
9. Use Runge–Kutta method to evaluat
$$\frac{dy}{dx} - xy = 1;\ y(0) = 2.$$
10. Using Runge–Kutta method of fourt
that

8. Find y at $x = 0.1, 0.2$ using fourth order Runge–Kutta algorithm given that
$y' - yx^2 = 0$; $y(0) = 1$.
9. Use Runge–Kutta method to evaluate y at $x = 0.2, 0.4, 0.6$ given that
$\frac{dy}{dx} - xy = 1$; $y(0) = 2$.
10. Using Runge–Kutta method of fourth order, find $y(0.1), y(0.2)$ given that
$\frac{dy}{dx} - y = -x$; $y(0) = 2$.
11. Solve $10y' = x^2 + y^2, y(0) = 1$ to evaluate $y(0.2)$ and $y(0.4)$ by fourth order R–K algorithm.
12. Given $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$; $x_0 = 0, y_0 = 1, h = 0.2$, find y_1 and y_2 using Runge–Kutta method.
13. Solve $\frac{dy}{dx} = \frac{1}{x+y}$ for $x = 0.5$ to z using R–K method with $x_0 = 0$, $y_0 = 1$ (take $h = 0.5$).
14. Use Runge–Kutta method of order four to find y at $x = 0.1, 0.2$ given that
$x(dy + dx) = y(dx - dy)$; $y(0) = 1$.
15. Solve $y' = x + y, y(0) = 1$ to find y at $x = 0.1, 0.2, 0.3$ using R–K method.
16. Solve the following for $y(0.1), y(0.2)$ using Runge–Kutta algorithms of (i) second order, (ii) third order and (iii) fourth order.

(a) $\frac{dy}{dx} + y = 0$; $y(0) = 1$

(b) $\frac{dy}{dx} + 2y = x$; $y(0) = 1$

17. Use second order Runge–Kutta algorithm to solve $\frac{dy}{dx} + xz = 0$;
$\frac{dy}{dx} - y^2 = 0$ at $x = 0.2, 0.4$ given that $y = 1, z = 1$ at $x = 0$.

18. Solve $\frac{dy}{dx} = 1 + xz$, $\frac{dz}{dx} = -xy$ for $x = 0.3, 0.6, 0.9$ given that $y = 0$, $z = 1$ at $x = 0$ by R–K method.

19. Solve $\frac{dy}{dx} = x + z$, $\frac{dz}{dx} = x$
$z(0) = 1$ by Runge – Kutta
20. Solve $y' = x + z$, $z' = x -$
$x = 0$ by Runge-Kutta me
21. Solve $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 0$ given that $y(0) = 1$, $y'(0) = 0$ for $y(0.1)$

19. Solve $\frac{dy}{dx} = x + z$, $\frac{dz}{dx} = x - y^2$ for $y(0.1)$, $z(0.1)$ given that $y(0) = 2$, $z(0) = 1$ by Runge – Kutta method.
20. Solve $y' = x + z$, $z' = x - y$ for $x = 0.1, 0.2$ given that $y = 0$, $z = 1$ at $x = 0$ by Runge–Kutta method.
21. Solve $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 0$ given that $y(0) = 1$, $y'(0) = 0$ for $y(0.1)$ using Runge–Kutta method.
22. Use Runge–Kutta method to solve $y'' - xy + 4y = 0$; $y(0) = 3$; $y'(0) = 0$ at $x = 0.1$.
23. Apply R–K algorithm to find y at $x = 0.1$ given $\frac{d^2y}{dx^2} = y^3$; $y(0) = 10$; $y'(0) = 5$.
24. Solve $\frac{d^2y}{dx^2} - x\frac{dy}{dx} - y = 0$, $y(0) = 1, y'(0) = 0$ to find $y(0.2)$, $y'(0.2)$ using Runge–Kutta method.
25. Evaluate y(0.2) by R–K method given that $y'' - xy'^2 + y^2 = 0$; $y(0) = 1, y'(0) = 0$.

ANSWERS

1. 0.8176
2. 1.7278
3. 0.8481
4. 1.0552, 1.1230, 1.2073
5. 2.5505
6. 0.1402, 0.2705
7. 2.2213, 2.4914
8. 1.0053, 1.0227
9. 2.243, 2.589, 2.072
10. 2.20517, 2.42139
11. 1.0207, 1.038
12. $y_1 = y(0.2) = 1.19598$; $y_2 = y(0.4) = 1.3751$
13. 1.3571, 1.5837, 1.7555, 1.8956
14. 1.0911, 1.1678,
15. 1.1103, 1.2428, 1.3997
16. (a) 0.905, 0.81901 ; 0.91, 0.82337 ; 0.90484, 0.81873
    (b) 0.825, 0.6905 ; 0.8234, 0.6878 ; 0.82342, 0.6879
17. 0.978, 1.2 ; 0.9003 , 1.382

18. 0.3448, 0.99 ; 0.7738, 0.9121 ; 1.25
19. 2.0845, 0.586
0.1050, 0.9998 ; 0.2199, 0.9986
21. 1.005334

12. $y_1 = y(0.2) = 1.19598$; $y_2 = y(0.4) = 1.3751$
13. 1.3571, 1.5837, 1.7555, 1.8956
14. 1.0911, 1.1678,
15. 1.1103, 1.2428, 1.3997
16. (a) 0.905, 0.81901 ; 0.91, 0.82337 ; 0.90484, 0.81873
    (b) 0.825, 0.6905 ; 0.8234, 0.6878 ; 0.82342, 0.6879
17. 0.978, 1.2 ; 0.9003 , 1.382

18. 0.3448, 0.99 ; 0.7738, 0.9121 ; 1.255, 0.66806
19. 2.0845, 0.586
20. 0.1050, 0.9998 ; 0.2199, 0.9986
21. 1.005334
22. 2.9399
23. 17.42
24. 0.9802, −0.196
25. 0.9801

11.18 PREDICTOR–CORRECTOR METHODS

Consider the differential equation,

$$\frac{dy}{dx} = f(x, y)\,;\; y(x_0) = y_0 \qquad (11.52)$$

To solve the above equation, we use Euler's formula

$$y_{i+1} = y_i + hf'(x_i, y_i), = 0, 1, 2, \ldots \qquad (11.53)$$

$$y_{i+1} = y_i + \frac{1}{2}\, h\{f(x_i, y_i) + f(x_{i+1}, y_{i+1})\} \qquad (11.54)$$

The value of y_{i+1} is first determined using Eqn (11.53), and is substituted on RHS of Eqn (11.54) giving a better approximation of y_{i+1}. Again, this value y_{i+1} is substituted Eqn (11.55) in to calculate a still better approximation of y_{i+1}. The process is repeated till two consecutive values of y_{i+1} are equal upto the desired accuracy. This method of refining an initially crude estimate of y_{i+1} by means of a more accurate formula is known as *Predictor–Corrector method*. In the above, Eqn (11.53) is called *Predictor* and Eqn (11.54) is called *Corrector* of y_{i+1}.

In the following sections we will study two such methods, namely, (i) Milne's method, and (ii) Adam's–Bashforth method.

11.19 MILNE'S METHOD

Here, the equation to be solved numerically is

$$\frac{dy}{dx} = f(x, y)\,;\; y(x_0) = y_0$$

The value $y_0 = y(x_0)$ being given, we calculate $y_1 = y(x_0 + h) = y(x_1)$; $y_2 = y(x_0 + 2h) = y(x_2)$; $y_3 = y(x_0 + 3h) = y(x_3) \ldots$

where h is a suitably chosen spacing.

By Newton's forward interpolation formula, we have

$y_2 = y(x_0 + 2h) = y(x_2)$; $y_3 = y(x_0 + 3h) = y(x_3)\dots$

Where h is a suitably chosen spacing.

By Newton's forward interpolation formula, we have

$$y = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \cdots$$

where $x = x_0 + uh$.

For $y = y'$ the above gives

$$y' = y'_0 + u\Delta y'_0 + \frac{u(u-1)}{2!}\Delta^2 y'_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y'_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y'_0 + \cdots \qquad (11.55)$$

Then to find $y_4 = y(x_0 + 4h)$, we integrate Eqn (11.55) with respect to x_0 over the interval $x_0 + 4h$ to , i.e.

$$\int_{x_0}^{x_0+4h} [y']dx = \int_0^{x_0+4h} \Big\{ y'_0 + u\Delta y'_0 + \frac{1}{2}(u^2 - u)\Delta^2 y'_0 + \frac{1}{6}(u^3 - 3u^2 + 2u)\Delta^3 y'_0 + \frac{1}{24}(u^4 - 6u^3 + 11u^2 - 6u)\Delta^4 y'_0 + \cdots \Big\} dx$$

$$\text{or } [y]_{x_0}^{x_0+4h} = h\int_0^4 \Big\{ y'_0 + u\Delta y'_0 + \frac{1}{2}(u^2 - u)\Delta^2 y'_0 + \frac{1}{6}(u^3 - 3u^2 + 2u)\Delta^3 y'_0 + \frac{1}{24}(u^4 - 6u^3 + 11u^2 - 6u)\Delta^4 y'_0 + \cdots \Big\} du$$

$$[\because x = x_0 + uh \Rightarrow dx = h\,du]$$

This gives, after simplification on both sides,

$$y_4 - y_0 = h\left[4y_0' + 8\Delta y_0' + \frac{20}{3}\Delta^2 y_0' - \frac{8}{3}\Delta^3 y_0' + \frac{14}{45}\Delta^4 y_0' + \cdots\right]$$

Now using $\Delta^n = (1 - E)^n$, $n = 1, 2, 3$ and simplyfying the above, we get

$$y_4 - y_0 = h\Big[4y_0' + 8(y_1' - y_0') + \frac{20}{3}(y_2' - 2y_1' + y_0') +$$

$$\frac{8}{3}(y_3' - 3y_2' + 3y_1' - y_0') + \frac{14}{45}\Delta^4 y$$

$$\text{or } y_4 = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$$

...dering only differences upto third order]

$$[\because x = x_0 + uh \Rightarrow dx = h\,du]$$

This gives, after simplification on both sides,

$$y_4 - y_0 = h\left[4y_0' + 8\Delta y_0' + \frac{20}{3}\Delta^2 y_0' - \frac{8}{3}\Delta^3 y_0' + \frac{14}{45}\Delta^4 y_0' + \cdots\right]$$

Now using $\Delta^n = (1-E)^n$, $n = 1, 2, 3$ and simplyfying the above, we get

$$y_4 - y_0 = h\left[4y_0' + 8(y_1' - y_0') + \frac{20}{3}(y_2' - 2y_1' + y_0') + \frac{8}{3}(y_3' - 3y_2' + 3y_1' - y_0') + \frac{14}{45}\Delta^4 y_0' + \cdots\right] \quad (11.56)$$

$$\text{or } y^4 = y^0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3') \quad (11.57)$$

[by considering only differences upto third order]

Hence, the error in Eqn (11.57) is $= \frac{14}{45}\Delta^4 y_0' + \cdots$ and this can be proved to be $= \frac{14}{45} y^5(\xi)$, where ξ lies in between x_0 and x_4. Hence, Eqn (11.57) can be written as

$$y_4 = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3') + \frac{14h^5}{45} y^5(\xi) \quad (11.58)$$

$\therefore x_0, x_1, x_2, x_3, x_4$ are any five consecutive values of x, the above, in general, can be written as

$$y_{n+1} = y_{n-3} + \frac{4h}{3}(2y_{n-2}' - y_{n-1}' + 2y_n') + \frac{14h^5}{45} y^5(\xi_1) \quad (11.59)$$

where ξ_1 lies between x_{n-3} and x_{n+1}. Eqn (11.59) is known as *Milne's predictor formula.*

To get *Milne's corrector formula*, integrate Eqn (11.55) with respect to x over the interval x_0 to $x_0 + 2h$. Then we have

$$\int_{x_0}^{x_0+2h} [y']dx = h\int_0^2 \Big\{ y_0' + u\Delta y_0' + \frac{1}{2}(u^2 - u)\Delta^2 y_0' + \frac{1}{6}(u^3 - 3u^2 + 2u)\Delta^3 y_0' + \frac{1}{24}(u^4 - 6u^3 + 11u^2 - 6u)\Delta^4 y_0' + \cdots \Big\} du$$

$$\text{or } y_2 - y_0 = h\left[2y_0' + 2\Delta y_0' - \frac{1}{3}\Delta^2 y_0' - \frac{4}{15}\cdot\frac{1}{24}\Delta y_0' + \cdots\right]$$

$$= h\left[2y_0' + 2(y_1 - y_0') + \frac{}{3}(y_2' - 2y_1' + y_0')\right.$$

$$\text{or } y_2 - y_0 = h\left[2y_0' + 2\Delta y_0' - \frac{1}{3}\Delta^2 y_0' - \frac{4}{15}\cdot\frac{1}{24}\Delta y_0' + \cdots\right]$$

$$= h\left[2y_0' + 2(y_1' - y_0') + \frac{1}{3}(y_2' - 2y_1' + y_0') - \frac{h}{90}\Delta^4 y_0' + \cdots\right] \qquad [\text{using } \Delta^n = (E-1)^n, n = 1, 2, 3]$$

or
$$y_2 = y_0 + \frac{h}{3}(y_0' - 4y_1' + y_2') \tag{11.60}$$

Considering only differences upto third order, Eqn (11.60) gives

$$y_2 = y_0 + \frac{h}{3}(y_0' - 4y_1' + y_2') \tag{11.61}$$

$\therefore$ Error $= -\dfrac{h}{90}\Delta^4 y_0' + \cdots$ and this can be proved to be $= -\dfrac{h^5}{90}y^v(\xi)$,
where $x_0 < \xi < x_2$.

$\therefore$ Eqn (11.61) can be written as

$$y_2 = y_0 + \frac{h}{3}(y'_0 - 4y'_1 + y'_2) - \frac{h^5}{90}y^v(\xi) \tag{11.62}$$

Since x_0, x_1, x_2 are any three consecutive values of x, the above can be written in general as

$$y_{n+1} = y_{n-1} + \frac{h}{3}(y_{n-1}' - 4y_n' + y_{n+1}') - \frac{h^5}{90}y^v(\xi_2) \tag{11.63}$$

where ξ_2 lies in between x_{n-1} and x_{n+1}. Eqn (11.63) is called as *Milne's corrector formula*.

Note : This method requires at least four values prior to the required value. If the initial four values are not given, we can obtain them by using Picard's method or Taylor's series method or Euler's method or Runge–Kutta method.

Example 11.16 Given $\dfrac{dy}{dx} = 1/x + y$, $y(0) = 2$, $y(0.2) = 2.0933$, $y(0.4) = 2.1755$, $y(0.6) = 2.2493$, find $y(0.8)$ using Milne's method.

Solution In the usual notation, Milne's predictor formula is

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}(2y_{n-2}' - y_{n-1}' + 2y_n') \tag{i}$$

where $y_{n+1,p}$ denotes the predicted value at y_{n+1}.

In the given problem, $x_0 = 0, x_1 = 0.2, x_2 = 0.4, x_3 = 0.6$, $h = 0.2$, $y_0 = 2$, $y_1 = 2.0933, y_2 = 2.1755, y_3 = 2.2493$

and $y' = \dfrac{1}{x+y}$

Putting $n = 3$ in Eqn (i), the predictor is

$$y_{4,p} = y_0 + \frac{4h}{3}(2y_1' - y \ldots$$

$\ldots = 0.4360528$

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and $y' = \dfrac{1}{x+y}$

Putting $n = 3$ in Eqn (i), the predictor is

$$y_{4,p} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3') \quad \text{(ii)}$$

Now $y_1' = \dfrac{1}{x_1 + y_1} = \dfrac{1}{0.2 + 2.0933} = 0.4360528$

$y_2' = \dfrac{1}{x_2 + y_2} = \dfrac{1}{0.4 + 2.1755} = 0.3882741$

$y_3' = \dfrac{1}{x_3 + y_3} = \dfrac{1}{0.6 + 2.2493} = 0.3509633$

Substituting in Eqn (ii), we get

$y_{4,p} = 2 + 4(0.2)/3\ [2(0.4360528) - 0.3882741 + 2(0.3509633)]$

$= 2.3162022$ (iii)

Now, Milne's corrector formula in general form is

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}(y_{n-1}' + 4y_n' + y_{n+1}') \quad \text{(iv)}$$

where $y_{n+1,c}$ denotes the corrected value of y_{n+1}.

Putting $n = 3$ in above, we get

$$y_{4,c} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4') \quad \text{(v)}$$

From Eqn (iii), $y_{4,p} = 2.3162022$ and $x_4 = 0.8$.

$y_4' = \dfrac{1}{x_4 + y_{4,p}} = \dfrac{1}{0.8 + 2.3162022} = 0.3209034$

Hence, from Eqn (v),

$y_{4,c} = 2.1755 + \dfrac{0.2}{3}[0.3882741 + 4(0.3509633) + 0.3209034]$

$= 2.3163687$

$\therefore y(0.8) = y_4 = 2.3164$ corrected to four decimals.

Example 11.17 Solve $\dfrac{dy}{dx} = (x + y)y,\ y(0) = 1$ using Milne's Predictor–Corrector method for $y(0.4)$. The values for $x = 0.1, 0.2$ and 0.3 should be obtained by Runge–Kutta method of fourth order.

Solution Given $y' = (x + y)y$

Now the values of $y(0.1)$, y

fourth order are

$y(0.1) = 1.11689,\ y(0.2) =$

$y(0.3) = 1.50412$, [refer to

$\therefore x_0 = 0, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3, h = 0.1$

$y_0 = 1, y_1 = 1.11689, y_2 = 1.27739, y_3 = 1.50412$

Solve $\frac{dy}{dx} = (x+y)y,\ y(0) = 1$ using ... Corrector method for $y(0.4)$. The values for $x = 0.1, 0.2$ and 0.3 should be obtained by Runge–Kutta method of fourth order.

Solution Given $y' = (x+y)y$; $y(0) = 1$.

Now the values of $y(0.1), y(0.2), y(0.3)$ using Runge–Kutta method of fourth order are

$y(0.1) = 1.11689$, $y(0.2) = 1.27739$ and

$y(0.3) = 1.50412$, [refer to Ex. 11.13]

$\therefore x_0 = 0, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3, h = 0.1$

$y_0 = 1, y_1 = 1.11689, y_2 = 1.27739, y_3 = 1.50412$

Milne's predictor formula is

$$y_{4,p} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3') \qquad \text{(i)}$$

Now, $y' = (x+y)y$

$\therefore \quad y_1' = (x_1 + y_1)y_1 = (0.1 + 1.11689)(1.11689) = 1.3591323$

$y_2' = (x_2 + y_2)y_2 = (0.2 + 1.27739)(1.27739) = 1.8872032$

$y_3' = (x_3 + y_3)y_3 = (0.3 + 1.50412)(1.50412) = 2.713613$

Substituting in Eqn (i), we get

$$y_{4,p} = 1 + \frac{4(0.1)}{3}[2(1.3591323) - 1.8872032 + 2(2.713613)]$$
$$= 1.8344383 \qquad \text{(ii)}$$

Now Milne's corrector formula is

$$y_{4,c} = y_2 + \frac{h}{3}(y_2' - 4y_3' + 2y_4') \qquad \text{(iii)}$$

From Eqn (ii), $y_{4,p} = 1.8344383, x_4 = 0.4$

$\therefore y_4' = (x_4 + y_{4,p})y_{4,p} = (0.4 + 1.8344383)(1.8344383)$
$= 4.0989392$

Hence, from Eqn (iii),

$$y_{4,c} = 1.27739 + \frac{0.1}{3}[1.8872032 + 4(2.713613) + 4.0989392]$$
$$= 1.8387431$$

$\therefore y_4 = y(0.4) = 1.83874$ correct to five decimals.

11.20 ADAMS–BASHFORTH METHOD

Given

$$\frac{dy}{dx} = f(x, y) \qquad (11.64)$$

and $y(x_0) = y_0$

we compute $y_{-1} = y(x_0 - h),\ y_{-2} = y(x_0 - 2h),\ y_{-3} = y(x_0 - 3h), \ldots$

Now integrating Eqn (11.64), on both sides with respect to x in ... $+ h]$, we get

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we compute $y_{-1} = y(x_0 - h),\ y_{-2} = y(x_0 - 2h),\ y_{-3} = \ldots$

Now integrating Eqn (11.64), on both sides with respect to x in $[x_0, x_0 + h]$, we get

$$y_1 = y_0 + \int_{x_0}^{x_0+h} f(x, y)\,dx \tag{11.65}$$

Replacing $f(x, y)$ by Newton's backward interpolation formula, we get

$$y_1 = y_0 + h\int_0^1 \Big\{ f_0 + u\nabla f_0 + \frac{1}{2}(u^2 + u)\nabla^2 f_0$$

$$\frac{1}{6}(u^3 + 3u^2 + 2u)\nabla^3 f_0 + \cdots \Big\} du$$

$[\because x = x_0 + uh,\ dx = h\,du]$

$$= y_0 + h\left\{ f_0 + \frac{1}{2}\nabla f_0 + \frac{5}{12}\nabla^2 f_0 + \frac{3}{8}\nabla^3 f_0 + \cdots \right\} \tag{11.66}$$

Neglecting the fourth order and higher order differences and using $\nabla f_0 = f_0 - f_{-1},\ \nabla^2 f_0 = f_0 - 2f_{-1} + f_{-2},\ \nabla^3 f_0 = f_0 - 3f_{-1} + 3f_{-2} - f_{-3}$, in Eqn (11.66), we get, after simplification,

$$y_1 = y_0 + \frac{h}{24}\{55 f_0 - 59 f_{-1} + 37 f_{-2} - 9 f_{-3}\}$$

which is known *Adams–Bashforth predictor formula* and is denoted generally as

$$y_{n+1,p} = y_n + \frac{h}{24}\{55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}\}$$

or $$y_{n+1,p} = y_n + \frac{h}{24}\{55 y_n' - 59 y_{n-1}' + 37 y_{n-2}' - 9 y_{n-3}'\} \tag{11.67}$$

Having found y_1, we find $f_1 = f(x_0 + h, y_1)$

Then to find a better value of y_1, we derive a corrector formula by substituting *Newton's backward interpolation formula* at f_1 in place of $f(x, y)$ in Eqn (11.65), i.e.

$$y_1 = y_0 + \int_{x_0}^{x_0+h} \Big\{ f_1 + u\nabla f_1 + \frac{1}{2!}u(u+1)\nabla^2 f_1$$

$$+ \frac{1}{3!}u(u+1)(u+2)\nabla^3 f_1 + \cdots \Big\} dx$$

$$= y_0 + h\int_{-1}^{0} \Big\{ f_1 + u\nabla f_1 \ldots$$

$$+ \frac{1}{3}(u^3 + 3u^2 + 2u)\nabla^3 f_1 + \cdots \Big\} du$$

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$$+\frac{1}{3!}u(u+1)(u+2)\nabla^3 f_1+\cdots\Big\}dx$$

$$= y_0 + h\int_{-1}^{0}\Big\{f_1 + u\nabla f_1 + \frac{1}{2}(u^2+u)\nabla^2 f_1$$

$$+\frac{1}{3}(u^3+3u^2+2u)\nabla^3 f_1+\cdots\Big\}du$$

$$[\because x = x_1 + uh, dx = hdu]$$

$$= y_0 + h\Big\{f_1 - \frac{1}{2}\nabla f_1 - \frac{1}{12}\nabla^2 f_1 - \frac{1}{24}\nabla^3 f_1 \ldots\Big\} \quad (11.68)$$

Neglecting the fourth order and higher order differences and using $\nabla f_1 = f_1 - f_0$, $\nabla^2 f_1 = f_1 - 2f_0 + f_{-1}$, $\nabla^3 f_1 = f_1 - 3f_0 + 3f_{-1} - f_{-2}$ in Eqn (11.68), we get, after simplification,

$$y_1 = y_0 + \frac{h}{24}[9f_1 + 19f_0 - 5f_{-1} + f_{-2}]$$

which is known as *Adams–Bashforth corrector formula* and is denoted, generally,

$$y_{n+1,c} = y_n + \frac{h}{24}\{9f_{n+1} - 19f_n - 5f_{n-1} + f_{n-2}\}$$

or $$y_{n+1,c} = y_n + \frac{h}{24}\{9y_{n+1}' - 19y_n' - 5y_{n-1}' + 9y_{n-2}'\} \quad (11.69)$$

Note : Here also we require at least four values of y prior to the required value of y.

Example 11.18 Using Adams–Bashforth method, find $y(1.4)$ given $y' = x^2(1+y)$, $y(1) = 1$, $y(1.1) = 1.233$, $y(1.2) = 1.548$ and $y(1.3) = 1.979$.

Solution Given

$y' = x^2(1+y)$, $x_0 = 1$, $x_1 = 1.1$, $x_2 = 1.2$, $x_3 = 1.3$, $y_0 = 1$,
$y_1 = 1.233$, $y_2 = 1.548$, $y_3 = 1.979$, $h = 0.1$.

Adams–Bashforth predictor formula is

$$y_{4,p} = y_3 + \frac{h}{24}(55y_3' - 59y_2' - 37y_1' - 9y_0') \quad (i)$$

$y_0' = x_0^2(1+y_0) = (1)^2[1+1] = 2$
$y_1' = x_1^2(1+y_1) = (1.1)^2[1+1.233] = 2.70193$
$y_2' = x_2^2(1+y_2) = (1.2)^2[1+1.548] = 3.66912$
$y_3' = x_3^2(1+y_3) = (1.3)^2[1+1.979] = 5.03451$

$\therefore y_{4,p} = 1.979 + 0.1/24\{55(5.03451) - 59(3.66912) + 37(2.70193) - 9(2)\}$
$= 2.5722974$

Now $y_{4,p}' = x_4^2(1+y_{4,p}) = (1.4)^2\{1+2.57$
$= 7.0017029$

The corrector formula is

$$y_{4,c} = y_3 + \frac{h}{24}(9y_4' + 19y_3' - 5y_2' + y_1')$$

Now $y_{4,p}' = x_4^2(1 + y_{4,p}) = (1.4)^2\{1 + 2.5722974\}$
$= 7.0017029$

The corrector formula is

$$y_{4,c} = y_3 + \frac{h}{24}\{9y_4' + 19y_3' - 5y_2' + y_1'\}$$

$$= 1.979 + \frac{0.1}{24}\{9(7.0017029) + 19(5.03451) - 5(3.66912) + 2.70193\}$$

$$= 2.5749473$$

$\therefore y(0.4) = 2.575$, correct to three decimal places.

Example 11.19 Find $y(0.1), y(0.2), y(0.3)$, from $y' = x^2 - y$; $y(0) = 1$ using Taylor's series method and hence obtain $y(0.4)$ using Adams–Bashforth method.

Solution Given $y' = x^2 - y$; $y(0) = 1$.

From Ex. 11.2 we have

$y(0.1) = 0.905125$, $y(0.2) = 0.8212352$, $y(0.3) = 0.7491509$

i.e. $x_0 = 0, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3$,

$y_0 = 1, y_1 = 0.905125, y_2 = 0.8212352, y_3 = 0.7491509$

Also, $y_0' = -1, y_1' = -0.895125, y_2' = -0.7812352$ and $y_3' = -0.6591509$

Adam's predictor formula is

$$y_{4,p} = y_3 + \frac{h}{24}\{55y_3' - 59y_2' + 37y_1' - 9y_0'\}$$

$$= 0.7491509 + \frac{0.1}{24}[55(-0.6591509) - 59(-0.7812352) + 37(-0.895125) - 9(-1)]$$

$$= 0.6896509$$

Now $y_{4,p}' = x_4^2 - y_{4,p} = (0.4)^2 - 0.6896507 = -0.5296507$

The corrector is,

$$y_{4,c} = y_3 + \frac{h}{24}\{9y_4' + 19y_3' - 5y_2' + y_1'\}$$

$$= 0.7491509 + \frac{0.1}{24}[9(-0.5296507) + 19(-0.6591509) - 5(-0.7812352) + (-0.895125)]$$

$$= 0.6892522$$

$\therefore y(0.4) = 0.6896522$

1. If $y' = 2e^x - y, y(0) = 2,$ ... 2.090, find $y(0.4), y(0.5)$ correct to three decimal places applying Milne's Predictor–Corrector method.

$$= 0.7491509 + \frac{0.1}{24}[9(-0.5296507) + 19(-0.6591509)$$
$$- 5(-0.7812352) + (-0.895125)]$$
$$= 0.6892522$$
$$\therefore y(0.4) = 0.6896522$$

EXERCISE 11.4

1. If $y' = 2e^x - y$, $y(0) = 2$, $y(0.1) = 2.010$, $y(0.2) = 2.040$ $y(0.3) = 2.090$, find $y(0.4)$, $y(0.5)$ correct to three decimal places applying Milne's Predictor–Corrector method.
2. Solve $y' = x^2 - y$ given that $y(0) = 1$, $y(0.1) = 0.9052$, $y(0.2) = 0.8213$, for $y(0.5)$. Here, use Milne's method by computing $y(0.3) = 1$, using Taylor's method.
3. Tabulate the solution to $y' = x + y$ with the initial condition
   (i) $y(0) = 0$ for $0.4 < x\ 1 \le 1.0$, $h = 0.1$
   (ii) $y(0) = 1$ for $0.1 \le x\ 1 \le 0.3$, $h = 0.05$
   using Milne's predictor – Corrector method.
4. Using Taylor's series method, solve $y' = xy + y^2$, $y(0) = 1$ at $x = 0.1$, 0.2, 0.3. Continue the solution at $x = 0.4$ by Milne's method.
5. Solve $y' = 1 + xy^2$, for $y(0.4)$ by Milne's method given that $y(0) = 1$, $y(0.1) = 1.105$, $y(0.2) = 1.223$, $y(0.3) = 1.355$.
6. Use Milne's method to compute $y(0.3)$ from $y' = x^2 + y^2$, $y(0) = 1$. Find the initial values $y(-0.1)$, $y(0.1)$, $y(0.2)$ from the Taylor's series.
7. Solve $y' = x^2 + y^2 - 2$, using Milne's method for $x = 0.3$ given that $y = 1$ at $x = 0$. Compute $y(-0.1)$, $y(0.1)$ $y(0.2)$ using Runge-Kutta method of fourth order.
8. Given $y' + y = 1$, $y(0) = 0$, find $y(0.1)$ by using Euler's method, $y(0.2)$ by modified Euler's method, $y(0.3)$ by Improved Euler's method, and $y(0.4)$ by Milne's method.
9. Solve by Taylor's series of third order, the problem $y' = (x^3 + xy^2)e^{-x}$, $y(0) = 1$ to find y for $x = 0.1$, 0.2, 0.3. Continue the solution at $x = 0.4$ and $x = 0.5$ by Milne's method.
10. Using Adams–Bashforth predictor–corrector method, find $y(1.4)$ given that $x^2y' + xy = 1$; $y(1) = 1$, $y(1.1) = 0.996$, $y(1.2) = 0.986$, $y(1.3) = 0.972$.
11. Using Adams–Bashforth formulae, determine $y(0.4)$ given the equation $y' = 0.5\,xy$; $y(0) = 1$, $y(0.1) = 1.0025$, $y(0.2) = 1.0101$, $y(0.3) = 1.0228$.
12. Using Adams–Bashforth formulae, find $y(0.4)$, $y(0.5)$, if y satisfies $\frac{dy}{dx} = 3e^x + 2y$ with $y(0) = 0$. Compute y at $x = 0.1, 0.2, 0.3$ by means of Runge–Kutta method.

9. Solve by Taylor's series of third order, the problem $y' = (x^3 + xy)$ ... $y(0) = 1$ to find y for $x = 0.1, 0.2, 0.3$. Continue the solution at $x = 0.4$ and $x = 0.5$ by Milne's method.

10. Using Adams–Bashforth predictor–corrector method, find $y(1.4)$ given that $x^2y' + xy = 1$; $y(1) = 1$, $y(1.1) = 0.996$, $y(1.2) = 0.986$, $y(1.3) = 0.972$.

11. Using Adams–Bashforth formulae, determine $y(0.4)$ given the equation $y' = 0.5\,xy$; $y(0) = 1$, $y(0.1) = 1.0025$, $y(0.2) = 1.0101$, $y(0.3) = 1.0228$.

12. Using Adams–Bashforth formulae, find $y(0.4)$, $y(0.5)$, if y satisfies $\frac{dy}{dx} = 3e^x + 2y$ with $y(0) = 0$. Compute y at $x = 0.1, 0.2, 0.3$ by means of Runge–Kutta method.

ANSWERS

1. 2.1621, 2.2546
2. 0.6435
3. (i) 0.0918, 0.1487, 0.2221, 0.3138, 0.4255, 0.5596, 0.7183
   (ii) 1, 1.0525, 1.1105, 1.2312, 1.2604, 1.3265
4. 1.8369
5. 1.45982
6. 1.4392
7. 0.61432
8. 0.3333
9. 1.0709, 1.1103
10. 0.94934
11. 1.0408
12. 2.2089, 3.20798