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CMPS 102 — Winter 2019 – Homework 3

1. (9 pts)

   a.

   Proof by contradiction:
   Suppose we had the minimum spanning tree T with the set of edge weights w_1, w_2, ... w_n. Suppose T was not a minimum-altitude connected subgraph. There should be a pair of nodes u and v, and two u to v paths $P_1 \neq P_2$, such that P_1 is the path from u to v but P_2 has the smaller altitude. Then, there exists some edge $e = (u', v')$ on P_1 that contains the maximum altitude over all edges in P_1 and P_2. Now, let's consider the set of edges in P_1 and P_2 except e, and suppose we took the following path from u'-v':

   1. P_1 from u' to u
   2. P_2 from u to v
   3. P_1 from v to v'

   It's possible that this path from u'-v' contains a cycle so then, the set of edges in P_1 and P_2 except e must contain some simple path S. And because e is the maximum altitude in T, this would imply that the edges in S and e would also form a cycle. This contradicts the Cycle Property which says that "for any cycle C in the graph, if the weight of an edge e of C is larger than the individual weights of all other edges of C, then this edge cannot belong to an MST". Therefore, we can conclude that T must be a minimum-altitude connected subgraph. $\square$

2. (6 pts)

   a.
   If we choose $\omega = 3$, then we have

$$\Rightarrow \{\omega^0, \omega^1, \omega^2, \omega^3, \omega^4, \omega^5\} = \{3^0, 3^1, 3^2, 3^3, 3^4, 3^5\}$$

$$\Rightarrow \{1, 3, 9, 27, 81, 243\}$$

   In mod 7 we have,

$$= \{1, 3, 2, 6, 4, 5\}$$

   b.

$$F = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 6 & 4 & 5 \\ 1 & 2 & 4 & 1 & 2 & 4 \\ 1 & 6 & 1 & 6 & 1 & 6 \\ 1 & 4 & 2 & 1 & 4 & 2 \\ 1 & 5 & 4 & 6 & 2 & 3 \end{bmatrix}$$

   c.

$$\mathbf{y} = F \cdot \mathbf{a}$$

$$\Rightarrow \qquad \mathbf{y} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 6 & 4 & 5 \\ 1 & 2 & 4 & 1 & 2 & 4 \\ 1 & 6 & 1 & 6 & 1 & 6 \\ 1 & 4 & 2 & 1 & 4 & 2 \\ 1 & 5 & 4 & 6 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ 43 \\ 29 \\ 31 \\ 33 \\ 35 \end{bmatrix}$$

In mod 7 we have,

$$\mathbf{y} = \begin{bmatrix} 4 \\ 1 \\ 1 \\ 3 \\ 5 \\ 0 \end{bmatrix}$$

d.

i.

$$G_n = \frac{1}{6} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3^{-1} & 3^{-2} & 3^{-3} & 3^{-4} & 3^{-5} \\ 1 & 3^{-2} & 3^{-4} & 3^{-6} & 3^{-8} & 3^{-10} \\ 1 & 3^{-3} & 3^{-6} & 3^{-9} & 3^{-12} & 3^{-15} \\ 1 & 3^{-4} & 3^{-8} & 3^{-12} & 3^{-16} & 3^{-20} \\ 1 & 3^{-5} & 3^{-10} & 3^{-15} & 3^{-20} & 3^{-25} \end{bmatrix}$$

$$= \frac{1}{6} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3^{-1} & 2^{-1} & 6^{-1} & 4^{-1} & 5^{-1} \\ 1 & 2^{-1} & 4^{-1} & 1^{-1} & 2^{-1} & 4^{-1} \\ 1 & 6^{-1} & 1^{-1} & 6^{-1} & 1^{-1} & 6^{-1} \\ 1 & 4^{-1} & 2^{-1} & 1^{-1} & 4^{-1} & 2^{-1} \\ 1 & 5^{-1} & 4^{-1} & 6^{-1} & 2^{-1} & 3^{-1} \end{bmatrix}$$

In mod 7 we have,

$$1^{-1} = 1, 2^{-1} = 4, 3^{-1} = 5, 4^{-1} = 2, 5^{-1} = 3, 6^{-1} = 6$$

$$\Rightarrow 6 \cdot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 5 & 4 & 6 & 2 & 3 \\ 1 & 4 & 2 & 1 & 4 & 2 \\ 1 & 6 & 1 & 6 & 1 & 6 \\ 1 & 2 & 4 & 1 & 2 & 4 \\ 1 & 3 & 2 & 6 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 30 & 24 & 36 & 12 & 18 \\ 6 & 24 & 12 & 6 & 24 & 12 \\ 6 & 36 & 6 & 36 & 6 & 36 \\ 6 & 12 & 24 & 6 & 12 & 24 \\ 6 & 18 & 12 & 36 & 24 & 30 \end{bmatrix}$$

Again, in mod 7 we have,

$$G_n = \begin{bmatrix} 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 2 & 3 & 1 & 5 & 4 \\ 6 & 3 & 5 & 6 & 3 & 5 \\ 6 & 1 & 6 & 1 & 6 & 1 \\ 6 & 5 & 3 & 6 & 5 & 3 \\ 6 & 4 & 5 & 1 & 3 & 2 \end{bmatrix}$$

ii.

$$\mathbf{a} = G_n \cdot \mathbf{y}$$

$$\Rightarrow \qquad G_n \cdot \mathbf{y} = \begin{bmatrix} 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 2 & 3 & 1 & 5 & 4 \\ 6 & 3 & 5 & 6 & 3 & 5 \\ 6 & 1 & 6 & 1 & 6 & 1 \\ 6 & 5 & 3 & 6 & 5 & 3 \\ 6 & 4 & 5 & 1 & 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 1 \\ 1 \\ 3 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 84 \\ 57 \\ 65 \\ 64 \\ 75 \\ 51 \end{bmatrix}$$

In mod 7 we have,

$$\mathbf{a} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \\ 5 \\ 2 \end{bmatrix}$$

Sources

https://en.wikipedia.org/wiki/Minimum_spanning_tree#Cycle_property