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CMPS 102 — Winter 2019 – Homework 3 v. 2

Two problems, 15 points, due 11:50 pm Wednesday Feb 13th, see the *Homework Guidelines*

1. (9 pts) Part (a) of Problem 20 in Chapter 4 (Greedy Algorithms). The input is a (undirected) graph where the nodes are towns and the edges represent the roads between them. Each edge is labeled with the maximum altitude along that road, and assume that all these altitudes are different. The height of a path is the maximum of the edge-altitudes over the edges in the path. For each pair of nodes, a winter-optimal path between them is a path from one to the other such that the path height is as small as possible (i.e. no other path between the two nodes has a lower path height). Since there may be many paths, there may be multiple paths with the same path height even though the edge altitudes are all different. Prove that the Minimum Cost Spanning Tree with edge costs equal to the edge altitudes provides winter-optimal paths between the vertices.

   You may use the fact that if a sequence of edges is a path between two vertices, then there is a simple path between the vertices using a subsequence of the edges. (If the path is already a simple path, the subsequence is the original path).

   Solution: The altitudes are different, so the MST where the edge-cost is the altitude is unique. Call this MST T.

   Theorem 1. *For any pair of vertices, the path in T between them is a winter-optimal path between them.*

   Proof. Assume to the contrary that for some pair of vertices v and u, the path between them in T is not a winter-optimal path. Let $v = t_1, t_2, \ldots, t_\ell = u$ be the path between v and u in the tree, and call this path P_T. Let edge (t_i, t_{i+1}) be the highest altitude edge in P_T. Since the altitudes are unique, there is only one edge in P_T with the highest altitude. Since P_T is not winter-optimal, there is another path $v = v_1, v_2, ..., v_k = u$ in the graph, call this path P_G, where every edge on the path has a lower altitude than edge (t_i, t_{i+1}) (and thus (t_i, t_{i+1}) is not on P_G).

   Consider now the path that starts at t_i, goes back along path P_T to v, continues along path P_G to u, and then takes the edges in path P_T back to t_{i+1}. The vertices in this path are $t_i, t_{i-1}, \ldots, t_1 = v = v_1, v_2, \ldots, v_k = u = t_\ell, t_{\ell-1}, t_{\ell-2}, \ldots, t_{i+1}$. Although this path may not be simple (it could repeat vertices), the given fact shows that there is a simple path P from t_i to t_{i-1} using only edges from P_G and edges from P_T other than the (t_i, t_{i+1}) edge. Therefore every edge in P has cost (altitude) less than (t_i, t_{i+1})

   Now, path P is a simple path from t_i to t_{i+1} and adding the edge (t_i, t_{i+1}) to P creates a cycle. The edge (t_i, t_{i+1}) is the most expensive (highest altitude) edge on the cycle and thus is not in any minimum cost spanning tree of the graph (by the *cycle property*). Thus T (which contains (t_i, t_{i+1})) cannot be an MST for the graph, contradicting the assumption that the path between v and u and T is not a winter-optimal path.

   $\square$

2. (6 points) The FFT can also be performed using modular arithmetic and integers rather than complex numbers. This problem will guide you along this path. Note that all calculations should be done modulo 7.

- (1 pt) Find a number ω such that ω^0, ω^1, ω^2, ω^3, ω^4, and ω^5 are all different modulo 7 (i.e. these powers modulo 7 give some permutation of $\{1, 2, 3, 4, 5, 6\}$. The value of "n" will be 6.

  Solution: First try 2, the first powers of 2 are 1, 2, 4, 8, 16, 32, 64, these correspond to 1, 2, 4, 1, 2, 4, 1 modulo 7 and doesn't generate the whole set.
  Next try 3, the first powers of 3 are 1, 3, 9, 27, 81, 243, and taking these mod 7 gives 1, 3, 2, 6, 4, 5, so 3 is a choice of ω where the powers generate the different numbers 1 through 6.

- (1 pt) Write the Fourier Matrix for this ω (see slide 12 of the FFT slides).

  Solution

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 6 & 4 & 5 \\ 1 & 2 & 4 & 1 & 2 & 4 \\ 1 & 6 & 1 & 6 & 1 & 6 \\ 1 & 4 & 2 & 1 & 4 & 2 \\ 1 & 5 & 4 & 6 & 2 & 3 \end{bmatrix}$$

- (2 pts) Use your Fourier Matrix to transform the sequence $\mathbf{a} = (0, 1, 2, 1, 5, 2)$ into the corresponding $\mathbf{y}$ sequence.

  Solution:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 6 & 4 & 5 \\ 1 & 2 & 4 & 1 & 2 & 4 \\ 1 & 6 & 1 & 6 & 1 & 6 \\ 1 & 4 & 2 & 1 & 4 & 2 \\ 1 & 5 & 4 & 6 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \\ 3 \\ 5 \\ 0 \end{bmatrix}$$

- (2 pts) Finally, write down the inverse FFT matrix (G_n on slide 19 of the FFT slides) and compute the inverse FFT transform $G_n \mathbf{y}$ to recover the original sequence $\mathbf{a}$.

  Solution: $\omega = 3$, so $\omega^{-1} = 5$ (as $3 \cdot 5 \mod 7 = 1$). Therefore G_n is just like the Fourier matrix, but with powers of 5 instead of powers of 3. Also, $n = 6$ and $6 \cdot 6 \mod 7 = 1$, so $6 = 6^{-1}$, and working modulo 7:

$$\underbrace{\frac{1}{6}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 5 & 4 & 6 & 2 & 3 \\ 1 & 4 & 2 & 1 & 4 & 2 \\ 1 & 6 & 1 & 6 & 1 & 6 \\ 1 & 2 & 4 & 1 & 2 & 4 \\ 1 & 3 & 2 & 6 & 4 & 5 \end{bmatrix}}_{G_n} \begin{bmatrix} 4 \\ 1 \\ 1 \\ 3 \\ 5 \\ 0 \end{bmatrix} = \underbrace{6\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 5 & 4 & 6 & 2 & 3 \\ 1 & 4 & 2 & 1 & 4 & 2 \\ 1 & 6 & 1 & 6 & 1 & 6 \\ 1 & 2 & 4 & 1 & 2 & 4 \\ 1 & 3 & 2 & 6 & 4 & 5 \end{bmatrix}}_{G_n} \begin{bmatrix} 4 \\ 1 \\ 1 \\ 3 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \\ 5 \\ 2 \end{bmatrix}$$

And the inverse transform takes us back to our starting vector of coefficients.

Note that in mod 7 group, the value x^{-1} is that number z such that $x \cdot z \equiv 1 \pmod 7$, and don't forget the $1/n$ in the definition of G_n.